COMPUTING THE EXPONENTIAL MAP ON SO(3)

ADAM WILLIAMS

1. Overview

This document details how to compute various quantities related to the Lie group SO(3) and its Lie algebra $\mathfrak{so}(3)$, for practical engineering purposes. Recall that SO(3) is the set of orthogonal, orientation-preserving linear transformations on $\mathbb{R}^3$ and can be realized as a matrix group:

$$\mathrm{SO}(3) = \{R \in \mathrm{GL}(3) \mid RR^\top = I\}.$$

2. Characterizing the Lie Algebra $\mathfrak{so}(3)$

2.1. **Skew-Symmetric Matrix Representation.** Differentiating a curve R in SO(3) with $R(0) = I$ yields

$$R'(0)^\top + R'(0) = 0,$$

meaning that the Lie algebra $\mathfrak{so}(3)$ can be realized as the vector space of 3×3 skew-symmetric matrices, with Lie bracket given (as with all matrix groups) by the commutator: $[X, Y] = XY - YX$. Hence we can choose the basis $\mathcal{G} = (G_1, G_2, G_3)$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as a choice of exponential coordinates (or infinitesimal generators) on SO(3), and do computations with respect to this basis.

2.2. **Axis-Angle Vector Representation.** An alternative realization of $\mathfrak{so}(3)$ is the vector space $\mathbb{R}^3$ with Lie bracket given by the cross product:

$$[\omega, \eta] = \omega \times \eta.$$

The isomorphism is given by the mapping

$$\begin{aligned}\mathbb{R}^3 &\cong \mathfrak{so}(3)\\ \omega &\mapsto \omega_\times\end{aligned}$$

taking $\omega \in \mathbb{R}^3$ to its *skew matrix*

$$\omega_\times = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \omega_x G_1 + \omega_y G_2 + \omega_z G_3.$$

The **axis-angle** vector ω has the following geometric interpretation: Let

$$\begin{aligned}\hat{\omega} &= \omega/\|\omega\|\\ \theta &= \|\omega\|.\end{aligned}$$

Then $\exp(\omega)$ is a right-handed rotation of angle θ around the axis $\hat{\omega} \in \mathbb{R}^3$. Hence the infinitesimal generators G_1, G_2, and G_3 that we have chosen are perturbations around the x, y, and z axis, respectively. And the one-parameter subgroups $\theta \mapsto \exp(\theta\hat{\omega})$ compound infinitesimal rotations around a given axis for an angle of θ.

3. Exponential and Logarithm

3.1. **Exponential.** The goal of this section is to compute the exponential map, taking an axis-angle vector to its rotation matrix in SO(3), using the matrix exponential:

$$\exp(\omega) = \sum_{k=0}^{\infty} \frac{\omega_\times^k}{k!}.$$

Let $\hat{\omega} = \omega/\|\omega\|$, and $\theta = \|\omega\|$. Using properties of the cross product, we have $\hat{\omega}_\times^3 = -\hat{\omega}_\times$, and so higher powers of $\hat{\omega}_\times$ in the power series collapse:

$$\begin{aligned}\hat{\omega}_\times^3 &= -\hat{\omega}_\times\\ \hat{\omega}_\times^4 &= -\hat{\omega}_\times^2\\ \hat{\omega}_\times^5 &= \hat{\omega}_\times.\end{aligned}$$

Hence we can group all of the power series terms into a $\hat{\omega}_\times$ term and a $\hat{\omega}_\times^2$ term:

$$\begin{aligned}\exp(\omega) &= I + \left(1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\hat{\omega}_\times + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!}\right)\hat{\omega}_\times^2\\ &= I + \left(\frac{\sin\theta}{\theta}\right)\omega_\times + \left(\frac{1-\cos\theta}{\theta^2}\right)\omega_\times^2.\end{aligned}$$

Hence we have a practical formula for the exponential of a skew matrix. Note that, in practice, the trigonometric coefficients should be computed using a Taylor series approximation for sufficiently small θ:

$$\frac{\sin(\theta)}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} + O\left(\theta^6\right)$$
$$\frac{1-\cos\theta}{\theta^2} = \frac{1}{2} - \frac{\theta^2}{24} + \frac{\theta^4}{720} + O\left(\theta^6\right)$$

3.2. **Logarithm.** In order to derive a formula for the (partial) inverse of exp, we must

(1) Determine the **normal neighborhood** $U \ni 0 \in \mathfrak{so}(3)$ on which exp is invertible.
(2) Compute a formula for $\log : \exp(U) \cong U$.

First, to compute the angle of rotation $\theta = \|\omega\|$. If we take the trace of $R = \exp(\omega)$, then applying linearity and the fact that $\omega_\times$ is traceless, we obtain

$$\begin{aligned} \operatorname{tr} R &= \operatorname{tr} I + 0 + \left(\frac{1-\cos\theta}{\theta^2}\right) \operatorname{tr} \omega_\times^2 \\ &= 3 + \left(\frac{1-\cos\theta}{\theta^2}\right)(-2\|\omega\|) \end{aligned}$$

and thus

$$\cos\theta = \frac{\operatorname{tr} R - 1}{2}.$$

Restricting $\theta \in [0, \pi]$ then gives the unique solution

$$\theta = \arccos \frac{\operatorname{tr} R - 1}{2}.$$

Now to compute the axis of rotation. we first recall the fact that the set of $n \times n$ matrices $M_n(\mathbb{R})$ is the direct (vector space) sum of the set of skew-symmetric matrices and the set of symmetric matrices:

$$M_n(\mathbb{R}) = \operatorname{Skew}_n(\mathbb{R}) \oplus \operatorname{Sym}_n(\mathbb{R}).$$

That is, any square matrix can be written *uniquely* as a skew-symmetric matrix and a symmetric matrix. In fact, this decomposition has a simple formula:

$$A = \frac{A - A^\top}{2} + \frac{A + A^\top}{2}.$$

Now, let $R = \exp(\omega)$. Observe that the formula for exp from the previous section is actually the skew-symmetric/symmetric decomposition of R:

$$R = \underbrace{\left(\left(\frac{\sin\theta}{\theta}\right)\omega_\times\right)}_{\frac{R-R^\top}{2}} + \underbrace{\left(I + \left(\frac{1-\cos\theta}{\theta^2}\right)\omega_\times^2\right)}_{\frac{R+R^\top}{2}}.$$

Hence, if $\sin\theta \neq 0$, we can use the skew-symmetric part to solve for $\omega_\times$:

$$\omega_\times = \left(\frac{\theta}{2\sin\theta}\right)\left(R - R^\top\right).$$

And so we simply choose ω such that

$$\omega_\times = R - R^\top.$$

Note that, when θ is small, it is best to compute the coefficient using a Taylor series:

$$\frac{\theta}{2\sin\theta} = \frac{1}{2} + \frac{\theta^2}{12} + \frac{7\theta^4}{720} + O(\theta^6).$$

Now, this method breaks down when $\theta > 0$ and $\sin(\theta) = \pi$ (note that if $\theta = 0$ then we choose $\omega = 0$, trivially). So, we have a unique log for all ω within the interior of a ball of radius π. At the boundary $\theta = \pi$, however, R is purely symmetric (hence $R = R^\top = R^{-1}$) and so the skew matrix provides no information. Note that this is geometrically intuitive: A rotation of angle π is its own inverse. Hence we must use the symmetric part to solve for $\pm\omega$, and make an arbitrary choice of sign: setting $\theta = \pi$, we have

$$R = I + 2\hat{\omega}_\times^2.$$

Note that $\hat{\omega}_\times^2 = \hat{\omega}\hat{\omega}^\top - I$, and hence

$$\hat{\omega}\hat{\omega}^\top = \frac{1}{2}(R + I).$$

we thus have

$$\begin{aligned}
\hat{\omega}_x^2 &= \frac{1}{2}(r_{11} + 1)\\
\hat{\omega}_y^2 &= \frac{1}{2}(r_{22} + 1)\\
\hat{\omega}_z^2 &= \frac{1}{2}(r_{33} + 1),
\end{aligned}$$

and the relative signs of the entries of $\hat{\omega}$ can be determined by the off-diagonal components. This determines ω up to a factor of ± 1.

3.3. **Adjoint Representation of the Group.** The adjoint representation of a group element $R \in \mathrm{SO}(3)$ is the linear mapping

$$\omega_\times \mapsto R\omega_\times R^\top.$$

from $\mathfrak{so}(3) \to \mathfrak{so}(3)$. We would like to write this linear transformation with respect to the basis $\mathcal{G}$, i.e. express it as a left multiplication by a 3×3 matrix

$$\omega \mapsto \mathrm{Ad}_R\,\omega.$$

This is straightforward, using the fact that the cross product is preserved by rotation: $Rx \times Ry = R(x \times y)$. Let $y \in \mathbb{R}^3$ be arbitrary, and observe

$$\begin{aligned}
\left[\mathrm{Ad}_R\, \omega\right]_\times Ry &= \left[R\omega_\times R^\top\right] Ry \\
&= R(\omega \times y) \\
&= R\omega \times Ry \\
&= \left[R\omega\right]_\times Ry.
\end{aligned}$$

Hence $\left[\mathrm{Ad}_R\, \omega\right]_\times = \left[R\omega\right]_\times$, meaning

$$\mathrm{Ad}_R = R.$$

We say that SO(3) is *self-adjoint*.

3.4. **Adjoint Representation of the Algebra.**

The adjoint representation of a Lie algebra element $\omega \in \mathfrak{so}(3)$ is the linear mapping

$$\eta_\times \mapsto \omega_\times \eta_\times - \eta_\times \omega_\times$$

from $\mathfrak{so}(3) \to \mathfrak{so}(3)$. We can derive the matrix for this mapping with respect to $\mathcal{G}$ by applying the *Jacobi identity* for the cross product: For any $x \in \mathbb{R}^3$, and any vectors $\omega, \eta \in \mathfrak{so}(3)$,

$$\omega \times (\eta \times x) + \eta \times (x \times \omega) + x \times (\omega \times \eta) = 0.$$

Applying anti-commutativity,

$$\omega \times (\eta \times x) - \eta \times (\omega \times x) = (\omega \times \eta) \times x.$$

Hence

$$\begin{aligned}
\left[\mathrm{ad}_\omega\, \eta\right]_\times x &= \omega_\times \eta_\times x - \eta_\times \omega_\times x \\
&= \omega \times (\eta \times x) - \eta \times (\omega \times x) \\
&= (\omega \times \eta) \times x \\
&= \left[\omega \times \eta\right]_\times x.
\end{aligned}$$

Hence $\left[\mathrm{ad}_\omega\, \eta\right]_\times = \left[\omega \times \eta\right]_\times = \left[\omega_\times \eta\right]_\times$, meaning

$$\mathrm{ad}_\omega = \omega_\times.$$