Homework 4

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1 Euclidean Ball

Given

$$\min_{x \in \mathbb{R}^n : ||x-c|| \leq r} ||z - x||$$

Making power of 2 of the objective function and the constraints will not affect the solution, just for simplicity of taking gradients. Thus, the Lagrangian:

$$\mathcal{L}(x, \lambda) = ||x - z||^2 + \lambda(||x - c||^2 - r^2), \ \lambda \text{ is scalar since } ||x - c||^2 - r^2 \text{ is also scalar}$$

The dual function $g(\lambda) = \min_x \mathcal{L}(x, \lambda)$. Thus, the gradient:

$$\nabla_x \mathcal{L}(x, \lambda)|_{x=x^*} = 2(x^* - z) + \lambda(2(x^* - c)) = 0 \implies x^* = \frac{\lambda c + z}{\lambda + 1}$$

The dual problem:

$$d^* = \max_\lambda g(\lambda) = \max_\lambda \{||\frac{\lambda c + z}{\lambda + 1} - z||^2 + \lambda(||\frac{\lambda c + z}{\lambda + 1} - c||^2 - r^2)\}$$

Or

$$d^* = \max_\lambda \{\frac{\lambda^2}{(\lambda+1)^2}||c - z||^2 + \lambda(\frac{1}{(\lambda+1)^2}||c - z||^2 - r^2)\} = \max_\lambda \{\frac{1}{(\lambda+1)^2}||c - z||^2(\lambda^2 + \lambda) - \lambda r^2\} =$$

$$= \max_\lambda \{\frac{\lambda}{\lambda + 1}||c - z||^2 - \lambda r^2\}$$

Therefore,

$$\nabla_\lambda g(\lambda)|_{\lambda=\lambda^*} = \frac{\lambda^* + 1 - \lambda^*}{(\lambda^* + 1)^2}||c - z||^2 - r^2 = 0 \implies (\lambda^* + 1)^2 = \frac{||c - z||^2}{r^2} \implies \lambda^* = \frac{||c - z||}{r} - 1$$

Here we took only positive root since $\lambda \geq 0$. The dual problem d^*: \[
\]

2 projection to the hyperplane

given

$$\min_{x\in q:(a,x)=b} ||x-z||$$

the lagrangian:

$$\updownarrow(x,v) = ||x-z||^2 + v((a,x)-b),\ v \text{ is scalar since } (a,x)=b \text{ is scalar too}$$

$$\nabla_x \updownarrow(x,v)|_{x=x^*} = 2(x^*-z)+va = 0$$

$$x^* = z - v\frac{a}{2}$$

the dual function is:

$$g(v) = \min_x \updownarrow(x,v) = ||z-v\frac{a}{2}-z||^2 + v((a,z-v\frac{a}{2})-b)$$

$$g(v) = (v\frac{a}{2}, v\frac{a}{2}) + v((a,z-v\frac{a}{2})-b) = (v\frac{a}{2}, v\frac{a}{2}) + v(a^tz - a^ta\frac{v}{2}-b) = \frac{v^2}{4}(a,a) + v((a,z)-\frac{v}{2}(a,a)-b)$$

$$g(v) = \frac{v^2}{2}(a,a)(\frac{1}{2}-1) - vb + v(a,z) = -\frac{v^2}{4}(a,a) + v((a,z)-b)$$

the dual problem is:

$$d^* = \max_v g(v)$$

$$\nabla_v g(v)|_{v=v^*} = -\frac{v^*}{2}(a,a) + (a,z) - b = 0 \implies v^* = 2\frac{(a,z)-b}{(a,a)}$$

thus,

$$d^* = g(v^*) = -4\frac{((a,z)-b)^2}{4(a,a)^2}(a,a) + 2\frac{(a,z)-b}{(a,a)}((a,z)-b)$$

$$d^* = -\frac{((a,z)-b)^2}{(a,a)} + 2\frac{((a,z)-b)^2}{(a,a)}$$

$$d^* = \frac{((a,z)-b)^2}{(a,a)}$$

answer: $d^* = \frac{((a,z)-b)^2}{(a,a)}$

3 primal problem

given

$$\min_{(ax,x)\le 1} (c,x)$$

3.1 derivation

the lagrangian:

$$\updownarrow(x,\lambda) = (c,x) + \lambda((ax,x)-1),\ \lambda \text{ is scalar since } (ax,x) \text{ is scalar}$$

the dual function:

$$g(\lambda) = \min_x \updownarrow(x,\lambda)$$

thus, the gradient:

$$\nabla_x \updownarrow(x,\lambda)|_{x=x^*} = c + 2\lambda a x^* = 0 \implies x^* = -\frac{1}{2\lambda}a^{-1}c$$

and the dual function:

$$g(\lambda) = -\frac{1}{2\lambda}(c, a^{-1}c) + \lambda(\frac{1}{4\lambda^2}(aa^{-1}c, a^{-1}c) - 1)$$

$$g(\lambda) = \frac{1}{2\lambda}(c, a^{-1}c)(\frac{1}{2} - 1) - \lambda = -\frac{1}{4\lambda}(c, a^{-1}c) - \lambda$$

the dual problem:

$$d^* = \max_{\lambda} g(\lambda) = \max_{\lambda}\{-\frac{1}{4\lambda}(c, a^{-1}c) - \lambda\}$$

3.2 solution

the gradient over λ:

$$\nabla_\lambda g(\lambda)|_{\lambda=\lambda^*} = \frac{1}{4\lambda^2}(c, a^{-1}c) - 1 = 0 \implies \lambda^* = \frac{1}{2}\sqrt{(c, a^{-1}c)}\text{since } \lambda \geq 0$$

thus, the primal problem has a solution through the dual problem:

$$d^* = -\frac{1}{2\sqrt{(c, a^{-1}c)}}(c, a^{-1}c) - \frac{1}{2}\sqrt{(c, a^{-1}c)} = -\sqrt{(c, a^{-1}c)}$$

answer: $-\sqrt{(c, a^{-1}c)}$