Lectures

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1 Rosenblatt's perceptron

1.1 Introduction

Recall: for an input $x \in \mathbb{R}^n$, the parametrized function describing the mapping computed by a perceptron is:

$$F(\omega, x) = F((\omega_1, \omega_2, \dots, \omega_n, \theta), (x_1, x_2, \dots, x_n, -1)) = sgn(\sum_{i=1}^{n} \omega_i x_i - \theta)$$

Or in the vector form:

$$F(\boldsymbol{\omega}, \boldsymbol{x}) = sgn(\boldsymbol{\omega^T x})$$

This perceptron can only classify patterns which are *linearly seprable.*

Theorem 1. *Suppose $C_1 \bigcup C_2 = C$ are linearly separable classes over the training set $z \in Z^T$ with the assumption $z_t = \{\boldsymbol{x_t}, y_t \in C\}$, and perceptron's response r_t with mistake $e_t = y_t - r_t \neq 0$ can be corrected by applying the learning rule to its current state $\boldsymbol{\omega} \in \boldsymbol{\Omega}^T$:*

$$\boldsymbol{\omega_t} = \boldsymbol{\omega_{t-1}} + e_t \boldsymbol{x_t}$$

Then perceptron's error correction algorithm converges in k number of steps with following assumptions: training input is bounded by Euclidean norm $\|\boldsymbol{x_t}\| \leq R$ and $e_t \boldsymbol{\omega_*^T x_t} \geq \gamma$ for $t = 1..T$, where $\gamma > 0$. Initial state $\boldsymbol{\omega_0} = 0$. Note, that γ uses to be sure that some example is classified correctly.

Proof: Multiplying both sides of learning rule equation by some optimal $\boldsymbol{\omega_*^T}$ we will have:

$$\boldsymbol{\omega_*^T \omega_k} = \boldsymbol{\omega_*^T \omega_{k-1}} + e_k \boldsymbol{\omega_*^T x_k} \geq \boldsymbol{\omega_*^T \omega_{k-1}} + \gamma$$

Now we can expand equation above for k steps and keep in mind $\boldsymbol{\omega_0} = 0$ we will get:

$$\boldsymbol{\omega_*^T \omega_k} \geq \boldsymbol{\omega_*^T}(\boldsymbol{\omega_{k-2}} + e_{k-1}\boldsymbol{x_{k-1}}) + \gamma \geq \boldsymbol{\omega_*^T}(\boldsymbol{\omega_{k-3}} + e_{k-2}\boldsymbol{x_{k-2}}) + 2\gamma \geq \dots$$

$$\dots \geq \boldsymbol{\omega_*^T}(\boldsymbol{\omega_0} + e_1 x_1) + (k-1)\gamma \geq k\gamma$$

Let's do one important step which results will be substituted to the final inequality. Suppose we have following Euclidean norm $\|\boldsymbol{\omega_k}\|$ and, as it's known, for squared $L2$ norm the equality holds:

$$\|\boldsymbol{\omega_k}\|^2 = \left\|\boldsymbol{\omega_{k-1}} + e_k \boldsymbol{x_k}\right\|^2 = \left\|\boldsymbol{\omega_{k-1}}\right\|^2 + e_k^2 \|\boldsymbol{x_k}\|^2 + 2e_k \boldsymbol{\omega_{k-1}^T x_k}$$

Since $e_t \neq 0$ there is a misclassification for two possible cases: if for $x_t \in C_1$ the error $e_t > 0$, then $sgn(\cdot) < 0$; if for $x_t \in C_2$ the error $e_t < 0$, then $sgn(\cdot) > 0$. Thus, for any misclassification the signs of error and argument of function sgn are always opposite. Therefore, with $e_k \boldsymbol{\omega_{k-1}^T x_k} < 0$ there is no doubt that:

$$\left\|\boldsymbol{\omega_{k-1}}\right\|^2 + e_k^2 \|\boldsymbol{x_k}\|^2 + 2e_k \boldsymbol{\omega_{k-1}^T x_k} \leq \left\|\boldsymbol{\omega_{k-1}}\right\|^2 + e_k^2 \|\boldsymbol{x_k}\|^2$$

Continuing for k steps we will have:

$$\left\|\boldsymbol{\omega_{k-1}}\right\|^2 + e_k^2 \|\boldsymbol{x_k}\|^2 + 2e_k \boldsymbol{\omega_{k-1}^T x_k} \leq e_k^2 \sum_{j=1}^{k} \left\|\boldsymbol{x_j}\right\|^2$$

Since $\|\boldsymbol{x_t}\| \leq R$:

$$\left\|\boldsymbol{\omega_{k-1}}\right\|^2 + e_k^2 \|\boldsymbol{x_k}\|^2 + 2e_k \boldsymbol{\omega_{k-1}^T x_k} \leq e_k^2 k R^2$$

From *Cauchy-Schwarz inequality*:

$$\left|\boldsymbol{\omega_*^T \omega_k}\right| \leq \left\|\boldsymbol{\omega_*^T}\right\| \cdot \|\boldsymbol{\omega_k}\|$$

$$k\gamma \leq \left\|\boldsymbol{\omega_*^T}\right\| \cdot \|\boldsymbol{\omega_k}\|$$

$$\frac{k\gamma}{\left\|\omega_*^T\right\|} \leq \|\boldsymbol{\omega_k}\|$$

Substitying results obtained earlier:

$$\frac{k^2\gamma^2}{\left\|\omega_*^T\right\|^2} \leq \|\boldsymbol{\omega_k}\|^2 = \left\|\boldsymbol{\omega_{k-1}}\right\|^2 + e_k^2\|\boldsymbol{x_k}\|^2 + 2e_k\boldsymbol{\omega_{k-1}^T}\boldsymbol{x_k} \leq e_k^2 kR^2$$

Finally:

$$\frac{k^2\gamma^2}{\left\|\boldsymbol{\omega_*^T}\right\|^2} \leq e_k^2 kR^2$$

$$\frac{k\gamma^2}{\left\|\boldsymbol{\omega_*^T}\right\|^2} \leq e_k^2 R^2$$

For some finite $\boldsymbol{\omega_*^T}$:

$$k \leq \frac{e_k^2 R^2}{\gamma^2}\left\|\boldsymbol{\omega_*^T}\right\|^2 \blacksquare$$

Note, that the error e_k can be only ± 2, thus:

$$k \leq \frac{4R^2}{\gamma^2}\left\|\boldsymbol{\omega_*^T}\right\|^2$$