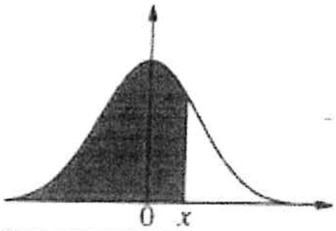
Annexe :

Loi Normale $N(0,1)$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

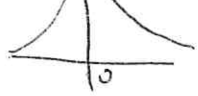
$$\Phi(-x) = 1 - \Phi(x)$$



x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	0.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	0.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	0.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	0.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	0.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	0.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	0.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	0.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	0.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	0.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	0.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	0.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	0.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	0.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	0.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	0.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	0.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	0.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	0.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	0.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	0.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	0.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	0.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	0.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	0.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	0.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	0.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	0.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	0.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	0.9985	.9986	.9986

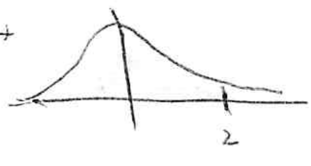
trouver les valeurs suivantes : $X \sim N(0,1)$

$P(X \leq 0) = 0,5$ ou bien on écrit $X(0) = 0,5$ ⟶



$P(X \leq 1) =$ ______ $X(1) =$ ⟶

$P(X \leq 2) =$ ______ $X(2)$ ⟶



$P(X \leq 1,5) =$

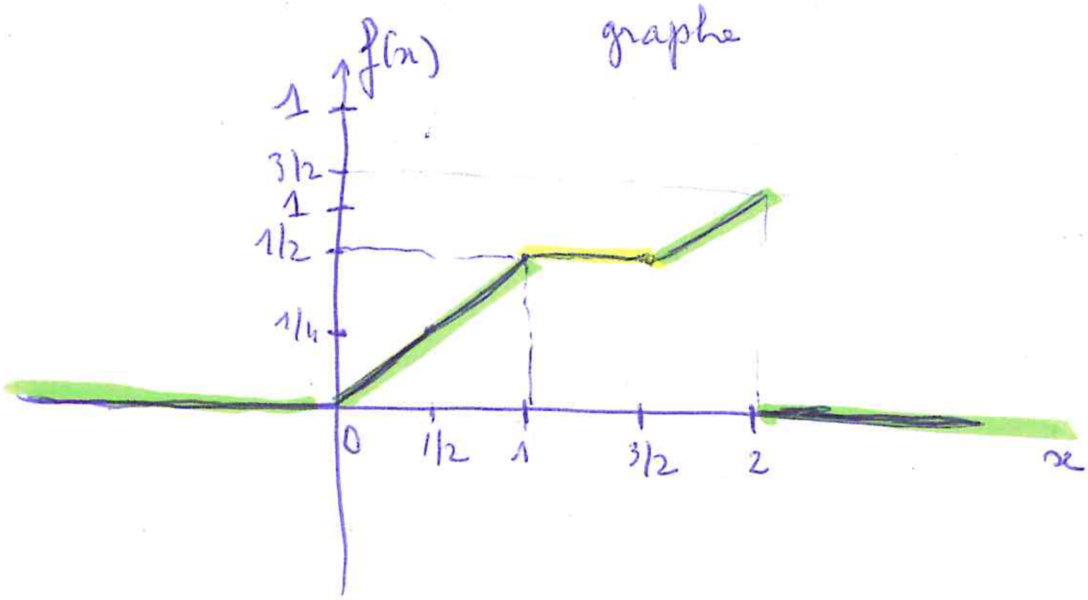
$P(X \leq 2,5) =$

Exercice 1

I

X v.a. de densité $f(x)$ égale à :

- 0 si $x < 0$
- $x/2$ quand $0 \leq x \leq 1$
- $1/2$ — $1 \leq x \leq 3/2$
- $2x - 5/2$ quand $3/2 \leq x \leq 2$
- 0 quand $x > 2$



primitive de $\frac{t}{2} = \frac{t^2}{4}$

* $x \leq 0$ $F_X(x) = \boxed{0}$

* $0 \leq x \leq 1$ $F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \frac{t}{2}\,dt = \left[\frac{t^2}{4}\right]_0^x$

$$= \left(\frac{x^2}{4}\right) - \left(\frac{0^2}{4}\right) = \boxed{\frac{x^2}{4}}$$

* $1 \leq x \leq \frac{3}{2}$

x démarre de 0

$F_X(x) = \int_0^1 f_X(t)\,dt + \int_1^x f_X(t)\,dt$

$= \frac{1^2}{4} + \int_1^x \frac{1}{2}\,dt = \frac{1}{4} + \left[\frac{t}{2}\right]_1^x$

$$\frac{1}{4} + \left[\frac{x}{2} - \frac{1}{2}\right] = \boxed{\frac{x}{2} - \frac{1}{4}}$$

* $\frac{3}{2} \leq x \leq 2$

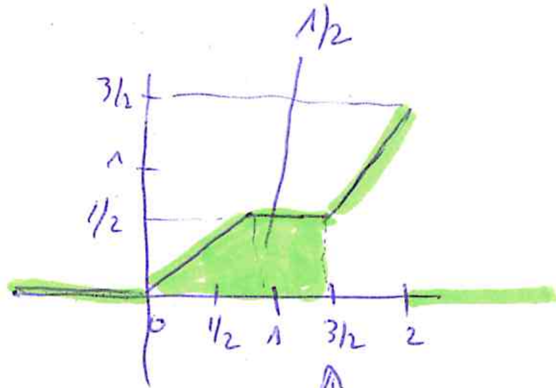
$$F_X(x) = \int_0^{3/2} f_X(t)\,dt + \int_{3/2}^{x} 2t - \frac{5}{2}\,dt$$

$$= \left[\cancel{\tfrac{3}{4}}\ \frac{x}{2} - \frac{1}{4}\right]_0^{3/2} + \left[2 \times \frac{t^2}{2} - \frac{5t}{2}\right]_{3/2}^{2}$$

$$= \frac{3}{4} - \frac{1}{4} + \left[t^2 - \frac{5t}{2}\right]_{3/2}^{2} = \left(x^2 - \frac{5}{2}x + 2\right)$$

$$= \frac{1}{2} + \left[\left(2^2 - \frac{5 \times 2}{2}\right) - \left((3/2)^2 - \frac{5 \times 3/2}{2}\right)\right] = \boxed{x^2 - \frac{5}{2}x + 2}$$

3) $P(X \leq x) = \frac{3}{4}$ trouver x



x ?

tel que proba sous la courbe $= 3/4$

d'après ce qui précède $P(X \leq 3/2) = \frac{1}{2}$

donc $x > 3/2$ et donc

$$P(X \leq x) = F_X(x) = x^2 - \frac{5}{2}x + 2$$

Il faut résoudre : $x^2 - \frac{5}{2}x + 2 = \frac{3}{4}$

$$x^2 - \frac{5}{2}x + \frac{5}{4} = 0$$

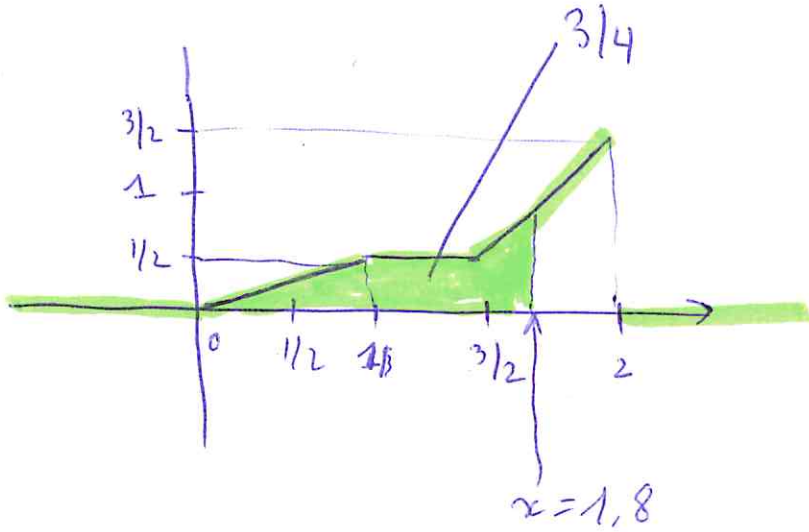
$$\Leftrightarrow \quad 4x^2 - 10x + 5 = 0$$

$\Delta = b^2 - 4ac = 100 - 80 = 20$

$x_1 = \dfrac{10 + 2\sqrt{5}}{8}$; $x_2 = \dfrac{10 - 2\sqrt{5}}{8}$

$x_1 = 0{,}69$ ou $x_2 = 1{,}8$

Puisque $x > 3/2 \Rightarrow$ la solution est $x = 1{,}8$



Exercice 2

taille moyenne d'un homme de 25 ans $\Rightarrow$ v.a réelle T

à densité normale de moyenne 175 et écart-type 6

donc $T \leadsto \mathcal{N}(175 ; 6)$

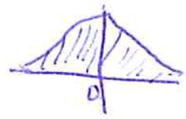
1. $Y = \alpha X + \beta$

$E(Y) = E(\alpha X + \beta) = \alpha E(X) + \beta$

$var(Y) = var(\alpha X + \beta) = E\left(\left((\alpha X + \beta) - E(\alpha X + \beta)\right)^2\right)$

$= \alpha^2 var(X)$

$\sigma(Y) = \sqrt{var(Y)} = \alpha \sigma(X) = \sigma(Y)$

2. Soit $X = T$, α et β pour que Y suive une loi normale réduite c'est à dire $Y \rightsquigarrow \mathcal{N}(0,1) \rightarrow$ 

$$\left.\begin{aligned} E(X) &= 175 \\ \sigma(X) &= 6 \end{aligned}\right\} \text{énoncé}$$

Y suit une loi normale centrée réduite $\xrightarrow{\text{si}}$ $m = 0$ et $\sigma^2 = 1$

$$E(Y) = 0 \rightarrow \alpha\, E(X) + \beta = 0$$

$$Var(Y) = \sigma(Y) = 1 \rightarrow \alpha\, \sigma(X) = 1$$

$$\begin{cases} 175\alpha + \beta = 0 \\ 6\alpha = 1 \end{cases} \Rightarrow \quad \begin{aligned} \alpha &= 1/6 \\ \beta &= -\frac{175}{6} \end{aligned}$$

3. Exprimer $P(T \geqslant 185)$ en fonction d'une proba sur Y

$$P(T \geqslant 185) = P\left(\frac{Y - \beta}{\alpha} \geqslant 185\right) \longrightarrow \begin{aligned} X &= T \\ Y &= \alpha X + \beta \\ X &= \frac{Y - \beta}{\alpha} \end{aligned}$$

$$= P\left(\frac{Y + 175/6}{1/6} \geqslant 185\right)$$

$$= P\left(Y \geqslant \frac{185 - 175}{6}\right) = P\left(Y \geqslant \frac{10}{6}\right)$$

$$\boxed{P(T \geqslant 185) = P\left(Y \geqslant \frac{5}{3}\right)}$$

4. Exprimer pour α quelconque, $P(Y \geqslant \alpha)$ en fonction d'un certain $F(u)$

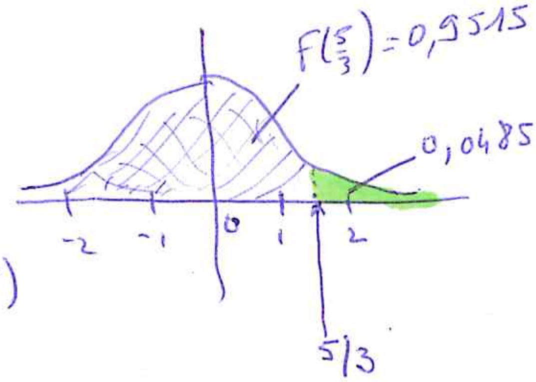
$P(Y \geqslant \alpha) = 1 - P(Y < \alpha)$; si F est continue,

$P(Y < \alpha) = P(Y \leq \alpha) = F(\alpha)$

donc $P(Y \geqslant \alpha) = 1 - F(\alpha)$

5. $P(T \geqslant 185)$ avec la table

$P(T \geqslant 185) = P(Y \geqslant \frac{5}{3}) = 1 - F(5/3)$



$F(\frac{5}{3}) = 0,9515$

donc $P(Y \geqslant \frac{5}{2}) = 0,0485 \; (= 1 - 0,9515)$

$P(T \geqslant 185) = 0,0485$ (~ 5% de la population)

6. $P(160 < T < 190)$

Remarque $<$ ou $\leq$ est identique (cf. Cours) car F continue,

$P(Y = y) = 0$

$P(T < 190) = P(Y < \frac{190 - 175}{6}) = P(Y < 2,5)$

$P(T < 160) = P(Y < \frac{160 - 175}{6}) = P(Y < -2,5)$

donc

$P(160 < T < 190) = P(T < 190) - P(T < 160))$

$= P(Y < 2,5) - P(Y < -2,5)$

$= F(2,5) - F(-2,5)$

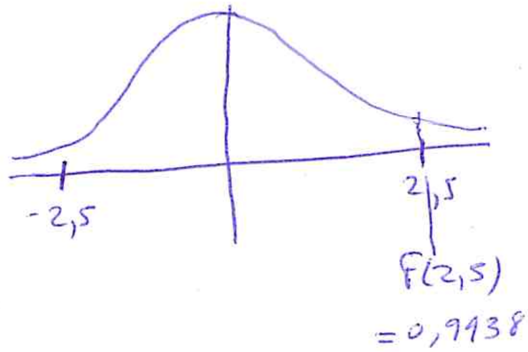
$= F(2,5) - (1 - F(2,5))$

$= 2\underbrace{F(2,5)}_{} - 1$ d'après la table

$= 2 \times \underbrace{0,9938}_{} - 1$

$= 0,9876$

presque 99% de la population a une taille entre 160 et 190.



7. Exprimer $P(175 - \varepsilon \leq T \leq 175 + \varepsilon)$ en fonction de F, puis

$$P(175 - \varepsilon \leq T \leq 175 + \varepsilon) = 2F\left(\frac{\varepsilon}{6}\right) - 1$$

$\Downarrow$

car $P\left(\frac{175 - \varepsilon - 175}{6} \leq Y \leq \frac{175 + \varepsilon - 175}{6}\right) = P\left(-\frac{\varepsilon}{6} \leq Y \leq \frac{\varepsilon}{6}\right)$

$= F\left(\frac{\varepsilon}{6}\right) - F\left(-\frac{\varepsilon}{6}\right) = F\left(\frac{\varepsilon}{6}\right) - \left(1 - F\left(\frac{\varepsilon}{6}\right)\right) = 2F\left(\frac{\varepsilon}{6}\right) - 1$

On cherche $2F\left(\frac{\varepsilon}{6}\right) - 1 \geq 0,9 \Rightarrow 2F\left(\frac{\varepsilon}{6}\right) \geq 1,9$

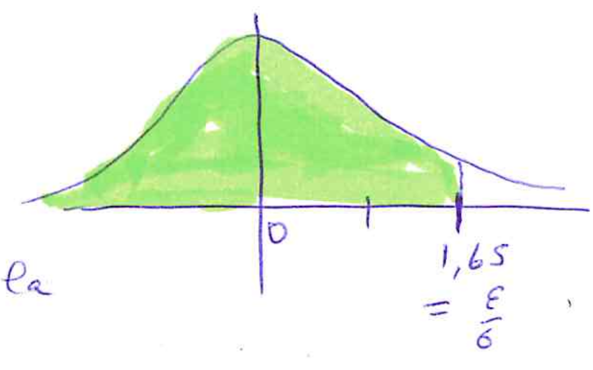
$F\left(\frac{\varepsilon}{6}\right) \geq 0,95$

D'après la table, il faut que

$\frac{\varepsilon}{6} \geq 1,65$, c'est à dire

$\varepsilon \geq 9,9$

Soit environ 10 cm, cohérent avec la question précédente



Exercice 3

Vecteurs aléatoires et somme de variables aléatoires

Couple de variables aléatoires réelles (X,Y) de loi uniforme $[a,b] \times [a,b]$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{si } a \leq x \leq b \\ 0 & \text{sinon} \end{cases}$$

1. a) Densité de f

$$f(x,y) = \frac{1}{b-a} \cdot \frac{1}{b-a} = \begin{cases} \frac{1}{(b-a)^2} & , \text{si } a \leq x, y \leq b \\ 0 & \text{sinon} \end{cases}$$

b. $a = 0 \quad b = 1$ et $Z = X + Y$

Calcul de la fonction de répartition de Z

$$= \int_{-\infty}^{z} f(z)\, dz$$

$$F(z) = \iint_{x+y \leq z} f(x,y)\, dx\, dy$$

x et y varient entre 0 et 1, pour que F soit supérieure à 0 il faut que z soit positif

- $F(z) = 0$ si $z < 0$
- si $z \geq 0$ $\quad F(z) = \iint_{x+y \leq z} \left(\frac{1}{1-0}\right)^2 dx\, dy$

$$F(z) = \int_0^{z} \int_0^{z-x} 1 \, dx\, dy = \int_0^{z} \left[y \right]_0^{z-x} dx$$

$$= \int_0^{z} \left[(z-x) - 0 \right] dx = \int_0^{z} (z-x)\, dx$$

$$= \left[xz - \frac{x^2}{2} \right]_0^{z} = \left(z^2 - \frac{z^2}{2}\right) - (0)$$

$$= z^2 - \frac{z^2}{2} = \frac{z^2}{2}$$

$$f(x,y) = \begin{cases} cxy, & \text{si } 0 < x < 4 \text{ et } 1 < y < 5 \\ 0 & \text{sinon} \end{cases}$$

1. Calculer c

$$\int_0^4 \int_1^5 cxy \, dx\, dy = c \left[\frac{x^2}{2} \right]_0^4 \times \left[\frac{y^2}{2} \right]_1^5$$

$$= c \times \left(\frac{16}{2} - 0 \right) \times \left(\frac{25}{2} - \frac{1}{2} \right) = c \times 8 \times 12 = c \times 96$$

$$c \times 96 = 1 \Rightarrow c = 1/96$$

2. Calculer $P(1 < X < 2,\ 2 < Y < 3)$

$$\int_1^2 \int_2^3 cxy \, dx\, dy = \frac{1}{96} \left[\frac{x^2}{2} \right]_1^2 \times \left[\frac{y^2}{2} \right]_2^3$$

$$= \frac{1}{96} \left(2 - \frac{1}{2} \right) \left(\frac{9}{2} - 2 \right) = \frac{1}{\cancel{96}_{32}} \times \frac{\cancel{3}}{2} \times \frac{5}{2} = \frac{5}{128}$$

3. Calculer les lois marginales de X et Y

Pour la marginale de X qu'on appelle $h(x)$ on intègre par rapport à y donc

$$h(x) = \int_1^5 c\,x\,y \times 1_{0 \leq x \leq 4}\, \hat{dx}\, dy = c\,x \left[\frac{y^2}{2}\right]_1^5 \times 1_{0 \leq x \leq 4}$$

$$= \frac{1}{96} \times x \times \left(\frac{25}{2} - \frac{1}{2}\right) = \frac{12}{96} x = \frac{1}{8} x$$

$$h(x) = \frac{1}{8} x$$

pour Y, on intègre par rapport à x

$$g(y) = \int_0^4 c\,x\,y \times 1_{1 \leq y \leq 5}\, dx\, \hat{dy}$$

$$= \frac{1}{96} \times y \times \left[\frac{x^2}{2}\right]_0^4 = \frac{1}{96} y \times \frac{16}{2}$$

$$g(y) = \frac{1}{12} y$$

4. Fonction de répartition F de (X, Y), F_X de X et F_Y de Y

Pas de problème on trouve $F_{X,Y} = F_X \times F_Y =$

$$F_{X,Y} = \int_0^x \int_1^y \frac{1}{96}\, x y \, dx\, dy = \frac{1}{96} \left[\frac{x^2}{2}\right]_0^x \times \left[\frac{y^2}{2}\right]_1^y$$

$$= \frac{1}{96} \left(\frac{x^2}{2}\left(\frac{y^2}{2} - \frac{1}{2}\right)\right)$$

10

$$F_X = \int_0^x \frac{1}{8} x \, dx = \frac{1}{8} \left[\frac{x^2}{2} \right]_0^x = \frac{1}{8} \frac{x^2}{2}$$

$$F_Y = \int_1^y \frac{1}{12} y \, dy = \frac{1}{12} \left[\frac{y^2}{2} \right]_1^y = \frac{1}{12} \left(\frac{y^2}{2} - \frac{1}{2} \right)$$

$$F_{X,Y} = F_X \cdot F_Y \quad \text{car} \quad \frac{1}{96} = \frac{1}{8} \times \frac{1}{12}$$