

Mathematical Formulae

A Book of High School and Engineering Common Core Mathematical
Formulae

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Part I

Algebra

Chapter 1

Logarithm

1.1 Basic Formulae

For $a^x = b$:

$$\log_a x, \text{ for all } x \leq 0 \text{ is undefined} \quad (1.1)$$

$$\log_a b = x, \text{ } bax \neq 1, a \neq 1 \quad (1.2)$$

$$\log_b a^m = m \log_b a, \text{ for } a^m = b \quad (1.3)$$

$$a^{\log_a x} = x \quad (1.4)$$

$$a^{\log_b c} = c^{\log_b a} \quad (1.5)$$

$$\frac{1}{\log_a b} = \log_b a \quad (1.6)$$

$$\log_c(ab) = \log_c a + \log_c b \quad (1.7)$$

$$\log_c\left(\frac{a}{b}\right) = \log_c a - \log_c b \quad (1.8)$$

$$|\log_a x| = \begin{cases} -\log_a x, & \text{if } 0 < x < 1 \\ \log_a x, & \text{if } 1 \leq x < \infty \end{cases} \quad (1.9)$$

1.2 Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty = \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{x^i}{i!} \quad (1.10)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty = \lim_{n \rightarrow \infty} \sum_{i=1}^n (-1)^{(i-1)} \frac{x^i}{i} \quad (1.11)$$

Chapter 2

Complex Number

2.1 Basic Formulae

For $z = x + iy$,

$$|z| = \sqrt{x^2 + y^2} \quad (2.1)$$

$$\tan \theta = \frac{y}{x} \quad (2.2)$$

$$\bar{z} = x - iy \quad (2.3)$$

2.2 Arithmetic Operation of Complex Number

For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \quad (2.4)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad (2.5)$$

$$|z_1 z_2| = |z_1| |z_2| \quad (2.6)$$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{a_2^2 + b_2^2} \quad (2.7)$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad (2.8)$$

2.3 Euler's Formula

$$z = r e^{i\theta}, \text{ where } r = |z|, e^{i\theta} = \cos \theta + i \sin \theta, \text{ and } \theta = \tan^{-1} \frac{y}{x} \quad (2.9)$$

2.4 Trigonometric Ratios in Complex Form

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta \quad (2.10)$$

$$\Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (2.11)$$

$$e^{i\theta} - e^{-i\theta} = 2 \sin \theta \quad (2.12)$$

$$\Rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \quad (2.13)$$

2.5 De Moivre's Formula

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \quad (2.14)$$

2.6 Application of Euler's and De Moivre's Formula

For $z_1 = |r_1|e^{i\theta_1}$ and $z_2 = |r_2|e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \quad (2.15)$$

2.7 Roots of Unity

$$\sqrt[n]{1} = e^{i \frac{2k\pi}{n}}, \text{ where } k \in [0, n-1] \quad (2.16)$$

2.8 Important Relations of Complex Numbers

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (2.17)$$

$$|z_1 - z_2| \leq |z_1| + |z_2| \quad (2.18)$$

$$|z_1 - z_2| \geq ||z_1| - |z_2|| \quad (2.19)$$

$$|z_1 + z_2| \geq ||z_1| - |z_2|| \quad (2.20)$$

$$|z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) \quad (2.21)$$

Chapter 3

Progression

3.1 Arithmetic Progression (A.P.)

An arithmetic sequence is $a, a + d, a + 2d, \dots \infty$ or $t_n = a + (n - 1)d$, where a is the first term, d is the common difference, and n is the n^{th} -term.

An arithmetic series is $a + (a + d) + (a + 2d) + \dots \infty$.

3.1.1 Sum of A.P. Series

$$\begin{aligned} S_n &= a + (a + d) + \dots + (a + \overline{n - 2}d) + (a + \overline{n - 1}d) \\ S_n &= (a + \overline{n - 1}d) + (a + \overline{n - 2}d + \dots + (a + d) + a \\ &\Rightarrow 2S_n = n(2a + \overline{n - 1}d) \\ &\Rightarrow S_n = \frac{n}{2}(2a + \overline{n - 1}d) \end{aligned} \quad (3.1)$$

3.1.2 Important Relation

If the three terms a, b, c are in A.P., then

$$2b = a + c \quad (3.2)$$

3.2 Geometric Progression (G.P.)

An geometric sequence is $a, ar, ar^2, \dots \infty$ or $t_n = ar^{n-1}$, where a is the first term, r is the common ratio, and n is the n^{th} -term.

An geometric series is $a + ar + ar^2 + \dots \infty$.

3.2.1 The Value of 'r'

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \dots = \frac{t_n}{t_{n-1}} \quad (3.3)$$

3.2.2 Sum of a G.P. Series

For a definite G.P. series, where there are n terms in the series, the sum of the series is:

$$S_n = \frac{a|r^n - 1|}{|r - 1|} \quad (3.4)$$

For an infite G.P. series the sum of the series is defined for $r < 1$. Sum of such a series is:

$$S_\infty = \frac{a}{1 - r} \quad (3.5)$$

3.2.3 Important relations

If the three terms a, b, c are in G.P., then:

$$b^2 = ac \quad (3.6)$$

3.3 Harmonic Progression (H.P.)

If a, b, c are terms of an H.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad (3.7)$$

$$\Rightarrow b = \frac{2ac}{a + c} \quad (3.8)$$

3.4 Arithmetico-Geometric Progression (A.G.P.)

Sequence $a, (a + d)r, (a + 2d)r^2, \dots, (a + \overline{n - 1}d)r^{n-1}$, where $a \rightarrow$ first term of A.G.P., $d \rightarrow$ common difference, and $r \rightarrow$ common ratio.

3.4.1 Sum of A.G.P.:

For an infinite A.G.P. series, the sum is defined for $r < 1$:

$$S_\infty = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} \quad (3.9)$$

3.5 Special Series

For $n \in \mathbb{N}$

$$1 + 2 + 3 + \dots + (n - 1) + n = \frac{n(n - 1)}{2} \quad (3.10)$$

$$1^2 + 2^2 + 3^2 + \dots + (n - 1)^2 + n^2 = \frac{n(n + 1)(2n + 1)}{6} \quad (3.11)$$

$$1^3 + 2^3 + 3^3 + \dots + (n - 1)^3 + n^3 = \left[\frac{n(n - 1)}{2} \right]^2 \quad (3.12)$$

3.5.1 Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (3.13)$$

3.5.2 Riemann's Infinite Series as an Integration

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=r_1}^{r_2} f\left(\frac{i}{n}\right) = \int_{\frac{r_1}{n}}^{\frac{r_2}{n}} f(x) dx \quad (3.14)$$

Chapter 4

Test of Convergence of Infinite Series

If $a_1, a_2, a_3, \dots, a_n$ is a sequence by a_n and their sum of series is S_n , then the following apply.

4.1 Definition

If

$$\lim_{n \rightarrow \infty} S_n = l$$

where l is a finite value, the series S_n is said to converge. A non-convergent series is called a divergent series.

4.2 Tests of Convergence

4.2.1 Comparison Test

If u_n and v_n are two positive series, then:

1. (a) v_n converges
(b) $u_n \leq v_n \forall n$ Then u_n converges.
2. (a) v_n diverges
(b) $u_n \geq v_n \forall n$ Then u_n diverges.

4.2.2 Limit Form

If

$$\lim_{x \rightarrow \infty} \frac{u_n}{v_n} = l$$

where l is a finite quantity $\neq 0$, then u_n and v_n converge and diverge together.

4.2.3 Integral Test or Maclaurin-Cauchy Test

For a series

$$\sum_{i=N}^{\infty} f(x), \text{ where } N \in \mathbb{Z} \quad (4.1)$$

will only converge if the improper integral

$$\int_N^{\infty} f(x)dx \quad (4.2)$$

is finite.

If the improper integral is finite, the upper and lower limit of the infinite series is given by:

$$\int_N^{\infty} f(x)dx \leq \sum_{i=N}^{\infty} f(x) \leq f(N) + \int_N^{\infty} f(x)dx \quad (4.3)$$

4.2.4 Ratio Test

If, for two series $\sum u_n$ and $\sum v_n$:

1. (a) $\sum v_n$ converges
 (b) from or after a particular term $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$, then u_n converges.
2. (a) $\sum v_n$ diverges
 (b) from or after a particular term $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$, then u_n diverges.

4.2.5 D'Alembert's Ratio Test

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lambda \quad (4.4)$$

- series converges if $\lambda < 1$
- series diverges if $\lambda > 1$
- fails if $\lambda = 1$

4.2.6 Rabbe's Test

$$\lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right] = \kappa \quad (4.5)$$

- series converges if $\kappa < 1$
- series diverges if $\kappa > 1$
- fails if $\kappa = 1$

4.2.7 Cauchy's Root Test

$$\lim_{n \rightarrow \infty} |u_n| = \lambda \quad (4.6)$$

- series converges for $\lambda < 1$
- series diverges for $\lambda > 1$
- test fails for $\lambda = 1$

4.2.8 Logarithmic Test

$$\lim_{n \rightarrow \infty} n \log \left(\frac{u_n}{u_{n+1}} \right) = \kappa \quad (4.7)$$

- series converges for $\kappa < 1$
- series diverges for $\kappa > 1$
- test fails for $\kappa = 1$

Chapter 5

Determinants

5.1 Definition

For a determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad (5.1)$$

5.1.1 Minor and Cofactor

For a third order determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the minor of a_{11} is $M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$,

i.e., all the terms of the determinant except those in the same row and columns as the one of which the minor is being calculated.

$$\text{Cofactor } C_{ij} = (-1)^{i+j} M_{ij}$$

5.2 Important Properties

1. Transposing a determinant does not alter its value.
2. If rows and columns are interchanges m times, the value of the new determinant is

$$\Delta' = (-1)^m \Delta \quad (5.2)$$

3. If two parallel lines are equal, then $\Delta = 0$

4. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

5. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

6. For $C_n \rightarrow k_1C_l + k_2C_m + k_3C_n$ or $R_n \rightarrow k_1R_l + k_2R_m + k_3R_n$, $\Delta' = \Delta$

5.3 Cramer's Rule

For a system of equations:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

the following determinants are defined:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The solution of the system of equations is:

$$x = \frac{D_x}{D} \tag{5.3}$$

$$y = \frac{D_y}{D} \tag{5.4}$$

$$z = \frac{D_z}{D} \tag{5.5}$$

5.3.1 Consistency Test

1. If $D \neq 0$, the system is consistent and has unique solutions.
2. If $D = D_x = D_y = D_z = 0$, the system may or may not be consistent and it will have infinite solutions and the system will be dependent.
3. If $D = 0$ and at least one of D_x, D_y, D_z is non zero, the system is inconsistent

Chapter 6

Matrices

For a matrix,

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where I_p is an identity matrix of the p^{th} order, the following relations are applicable.

6.1 Sum of Two Matrices

$$A_{m \times n} + B_{m \times n} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{21} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \quad (6.1)$$

6.2 Multiplication of Two Matrices

If

$$C_{m \times p} = A_{m \times n} \cdot B_{n \times p}$$

then,

$$c_{ik} = \sum_{j=1}^n a_{ij} b_{jk} \quad (6.2)$$

6.2.1 Multiplicative Properties

1. Multiplication of matrices is associative, hence $(AB)C = A(BC)$.

2. $AI = A$
3. $A \cdot A^{-1} = I$
4. $A \cdot (\text{adj} A) = (\text{adj} A) \cdot A = |A|I$
5. $A^{-1} = \frac{1}{|A|}(\text{adj} A)^t$
6. $(AB)^t = B^t A^t$

6.3 Adjoint of a Matrix

$$\text{adj} A = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}^t, \text{ where } M_{ij} \text{ is the minor of } a_{ij} \quad (6.3)$$

6.4 Martin's Rule

For a system of equation,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

The system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (6.4)$$

$$\Rightarrow AX = B \quad (6.5)$$

$$\Rightarrow X = A^{-1}B \quad (6.6)$$

Chapter 7

Binomial Theorem

For a binomial expansion $(a + b)^n$, there are $(n + 1)$ terms and $(a + b + c)^n$ has $\frac{(n + 1)(n + 2)}{2}$ terms.

7.1 Expansion of a binomial expression

$$\begin{aligned}(a + b)^n &= {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \\ &\quad \dots + {}^nC_n a^0 b^n \quad \forall n \in \mathbb{N} \\ &= \sum_{i=0}^n {}^nC_i a^{n-i} b^i \quad \forall n \in \mathbb{N}\end{aligned}\tag{7.1}$$

$$\begin{aligned}(a + b)^n &= a^n b^0 + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 \\ &\quad + \dots + \frac{n(n-1) \dots 3 \cdot 2 \cdot 1}{n!} a^0 b^n + \dots \infty \quad \forall n \in \mathbb{R}\end{aligned}\tag{7.2}$$

7.2 Trinomial Expansion

For $(a + b + c)^n$:

$$\begin{aligned}(a + b + c)^n &= \sum \frac{n!}{i!j!k!} a^i b^j c^k \\ &\quad \forall (i + j + k) = n; i, j, k, n \in \mathbb{N}\end{aligned}\tag{7.3}$$

7.3 Properties of Coefficients

$$\text{Sum of Co-efficients: } C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n = 2^n \quad (7.4)$$

$$\text{Sum of Odd Co-efficients: } C_0 + C_2 + C_4 + \cdots + C_{2n-3} + C_{2n-1} = 2^{n-1} \quad (7.5)$$

$$C_0 - C_1 + C_2 - \cdots + C_{2n-1} - C_{2n} = 0 \quad (7.6)$$

7.4 Pascal's Rule

For $1 \leq k \leq n$ and $k, n \in \mathbb{N}$:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \quad (7.7)$$

Chapter 8

Boolean Algebra

Let B be a set of a, b, c with operations sum $(+)$ and product (\cdot) . Then B is said to belong to the Boolean Structure if the following conditions are satisfied:

| Property | Name of Property |
|--|------------------|
| $a + b \in B$ $a \cdot b \in B$ | Closure Property |
| $a + b = b + a$ $a \cdot b = b \cdot a$ | Associative Law |
| $a(b + c) = ab + ac$ $a + bc = (a + b)(a + c)$ | Commutative Law |
| $\{0, 1\} \in B$ $a + 0 = a$ $a + 1 = 1$ $a \cdot 0 = 0$ $a \cdot 1 = a$ | Laws of 1 and 0 |
| $a + ab = a$ $a(a + b) = a$ | Absorption Law |
| $(a + b)' = (a'b')$ | De'Morgan's Law |

Table 8.1: Properties of Boolean Algebraic Structure

Chapter 9

Remainder Theorems

9.1 Remainder Theorem

If a function $f(x)$ is divided by a binomial $x - a$, then the remainder is provided by $f(a)$.

$$\frac{f(x)}{x - a} \equiv f(a) \pmod{(x - a)} \quad (9.1)$$

Worked Example

Find the remainder when $f(x) = x^3 - 4x^2 - 7x + 10$ is divided by $(x - 2)$.

The remainder:

$$R = (x^3 - 4x^2 - 7x + 10) \pmod{(x - 2)}$$

is given by:

$$\begin{aligned} R &= f(2) = (2)^3 - 4(2)^2 - 7(2) + 10 \\ &= 8 - 16 - 14 + 10 = -12 \end{aligned}$$

9.2 Euler's Remainder Theorem

According to Euler's Remainder Theorem, if x and n are two co-prime numbers:

$$x^{\varphi(n)} \equiv 1 \pmod{n}, x, n \in \mathbb{Z}^+ \quad (9.2)$$

where, $\varphi(n)$ is Euler's totient function.

9.2.1 Euler's Totient Function

For a number defined as:

$$n = \prod_{i=1}^r a_i^{b_i} \quad (9.3)$$

then Euler's totient function is defined as:

$$\begin{aligned} \varphi(n) &= n \cdot \left[\left(1 - \frac{1}{a_1}\right) \cdot \left(1 - \frac{1}{a_2}\right) \cdot \left(1 - \frac{1}{a_3}\right) \cdots \right] \\ &= n \prod_{i=1}^r \left(1 - \frac{1}{a_i}\right) \end{aligned} \quad (9.4)$$

Worked Example

Find the remainder if 3^{76} is divided by 35.

Since:

$$35 = 5^1 \times 7^1$$

Hence the totient quotient of 35 is:

$$\begin{aligned} \varphi(35) &= 35 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \\ &= 35 \times \frac{4}{5} \times \frac{6}{7} \\ &= 24 \end{aligned}$$

Hence Euler's Theorem yields:

$$\begin{aligned} 3^{24} &\equiv 1 \pmod{35} \\ 3^{76} &\equiv 3^{24 \times 3 + 4} \\ &\equiv (3^{24})^3 \times 3^4 \pmod{35} \\ &\equiv (1)^3 \times 3^4 \pmod{35} \\ &\equiv 81 \pmod{35} \\ &\equiv 11 \pmod{35} \end{aligned}$$

The remainder when 3^{76} is divided by 35 is 11.

9.3 Wilson Theorem

According to Wilson Theorem:

$$(n-1)! \equiv -1 \pmod{n} \quad (9.5)$$

Worked Example

Find the remainder when $28!$ is divided by 31.

By Wilson's Theorem:

$$\begin{aligned}
 30! &\equiv -1 \pmod{31} \\
 \Rightarrow 30 \cdot 29 \cdot 28! &\equiv -1 \pmod{31} \\
 \text{Let } 28! \pmod{31} &= x \\
 \Rightarrow (-1) \cdot (-2) \cdot x &\equiv 30 \pmod{31} \\
 \Rightarrow 2x &= 30 \\
 \Rightarrow x &= 15
 \end{aligned}$$

The remainder when $28!$ is divided by 31 is 15.

Part II

Co-Ordinate Geometry

Chapter 10

2-D Co-ordinate Geometry

For the ordered pairs, $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad (10.1)$$

$$\text{Mid point of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad (10.2)$$

$$\text{Point C, which divides AB in the ratio } m : n = \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n} \right) \quad (10.3)$$

Chapter 11

Triangles

For a triangle defined with three vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and corresponding sides of length a, b, c , then:

$$\text{Centroid of } \triangle ABC = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \quad (11.1)$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad (11.2)$$

For a triangle, the semiperimeter, s , is defined as:

$$s = \frac{a + b + c}{2}$$

Then the radius, r , and centre of incircle, o , is:

$$o = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right) \quad (11.3)$$

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \quad (11.4)$$

The radius, R , and centre, O , of circumcircle is defined as:

$$O = \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right) \quad (11.5)$$

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (11.6)$$

Chapter 12

Straight Line

A straight line can be defined as:

$$y = mx + c \quad (12.1)$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ where } a \text{ and } b \text{ are the intercepts at x and y axes respectively} \quad (12.2)$$

$$x \cos \alpha + y \sin \alpha = p \text{ (Normal Form)} \quad (12.3)$$

$$Ax + By + C = 0 \text{ (General Form)} \quad (12.4)$$

Equation of Straight Line Passing Through (x_0, y_0) and Slope m

$$(y - y_0) = m(x - x_0) \quad (12.5)$$

Distance Between Two Points on a Line

$$\frac{y_1 - y_2}{\sin \theta} = \frac{x_1 - x_2}{\cos \theta} = \gamma \quad (12.6)$$

$$\theta = \tan^{-1} m \quad (12.7)$$

Angle Between Two Lines

For two lines with slopes m_1, m_2 , the angle between them, θ :

$$\theta = \arctan \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) \quad (12.8)$$

Distance of a Point from a Line

Line: $ax + by + c = 0$ Point: (g, h)

$$S = \frac{ag + bh + c}{\sqrt{a^2 + b^2}} \quad (12.9)$$

Angle Bisector of a Line For the two lines: $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the angle bisector is:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (12.10)$$

If the sign of c_1 and c_2 is the same, then the equation obtained is the internal bisector.

Equation of a Straight Line Passing through the Intersection of Two Lines

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \quad \forall k \in \mathbb{R} \quad (12.11)$$

Relative Position of Points w.r.t. a Line For the points (x_1, y_1) and (x_2, y_2) :

$$k_1 = ax_1 + by_1 + c$$

$$k_2 = ax_2 + by_2 + c$$

If k_1 and k_2 have the same sign, they are on the same side of a line, otherwise on opposite sides.

Ratio of Division of Line Segment For any line, $f(x, y) = 0$, the ratio in which it divides (x_1, y_1) and (x_2, y_2) is given by:

$$r = -\frac{f(x_1, y_1)}{f(x_2, y_2)} \quad (12.12)$$

If $\begin{cases} r > 0, \text{ then division is internal} \\ r < 0, \text{ then division is external} \end{cases}$.

Chapter 13

General Theory of Second Degree Equation

For any general equation of the form:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0 \quad (13.1)$$

Δ is defined as:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad (13.2)$$

If $\Delta = 0$ then the equation is a pair of straight lines. If $a + b = 0$ then the lines are \perp .

If the $\Delta \neq 0$:

1. $a = b, h = 0 \rightarrow$ circle
2. $h^2 = ab \rightarrow$ parabola
3. $h^2 < ab \rightarrow$ ellipse
4. $h^2 > ab \rightarrow$ hyperbola

Chapter 14

Conics

The four conic sections are: circle, parabola, ellipse, and hyperbola. Circle has been done separately in the next chapter.

14.1 Parabola

| Property | $y^2 = 4ax$ | $x^2 = 4ay$ |
|--------------------------|-------------|-------------|
| Axis | $y = 0$ | $x = 0$ |
| Eccentricity | 1 | 1 |
| Directrix | $x + a = 0$ | $y + a = 0$ |
| Focus | $(a, 0)$ | $(0, a)$ |
| Vertex | $(0, 0)$ | $(0, 0)$ |
| Length of latus rectum | $ 4a $ | $ 4a $ |
| Equation of latus rectum | $x - a = 0$ | $y - a = 0$ |

Table 14.1: Properties of a Parabola

14.2 Ellipse and Hyperbola

For $a > b$:

| Property | $\frac{x^2}{a} + \frac{y^2}{b} = 1$ Ellipse | $\frac{x^2}{a} - \frac{y^2}{b} = 1$ Hyperbola |
|--------------------------|--|--|
| Length of Major Axis | $2a$ | $2a$ |
| Length of Minor Axis | $2b$ | $2b$ |
| Equation of Major Axis | $x = 0$ | $x = 0$ |
| Equation of Minor Axis | $y = 0$ | $y = 0$ |
| Eccentricity e | $\sqrt{1 - \frac{b^2}{a^2}}$ | $\sqrt{1 + \frac{b^2}{a^2}}$ |
| Vertices | $(\pm a, 0)$ | $(\pm a, 0)$ |
| Foci | $(\pm ae, 0)$ | $(\pm ae, 0)$ |
| Equation of Directrix | $x \pm \frac{a}{e} = 0$ | $x = \pm \frac{a}{e}$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2b^2}{a}$ |
| Equation of latus rectum | $x \pm ae = 0$ | |
| Centre | $(0, 0)$ | $(0, 0)$ |

Table 14.2: Properties of Ellipse and Hyperbola

14.3 Parametric Form of Conics

14.3.1 Hyperbola

$$x = a \sec \theta \quad (14.1)$$

$$y = b \tan \theta \quad (14.2)$$

14.3.2 Ellipse

$$x = a \cos \phi \quad (14.3)$$

$$y = b \sin \phi \quad (14.4)$$

14.3.3 Parabola

$$x = at^2 \quad (14.5)$$

$$y = 2at \quad (14.6)$$

Chapter 15

Circles

15.1 Locus Form

$$(x - g)^2 + (y - h)^2 = r^2 \quad (15.1)$$

where the centre is (g, h) and the radius is r .

15.2 Diameter Form

$$(x - a)(x - c) + (y - b)(y - d) = 0 \quad (15.2)$$

where (a, b) and (c, d) are the two ends of the diameter.

15.3 General Form

If the equation of a circle is in the form:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (15.3)$$

Then the following is true about the circle:

1. centre of the circle is $(-g, -f)$
2. radius of circle is $\sqrt{g^2 + f^2 - c}$

15.4 Important Relations

1. If the circle passes through the origin, $g = 0, f = 0$.
2. If the circle touches the x-axis $c = g^2$.
3. If the circle touches the y-axis $c = f^2$.

Common for Two Circles

1. The common chord passing between two circles S_1 and S_2 are:

$$S_1 - S_2 = 0 \quad (15.4)$$

2. Circles passing through the intersection of two circles is:

$$S_2 + k(S_1 - S_2) = 0 \quad \forall k \in \mathbb{R} \quad (15.5)$$

Chapter 16

Vectors

Let two vectors be $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$:

16.1 Modulus of a Vector

For a vector \vec{a} , the modulus of the vector is:

$$|\vec{a}| = \sqrt{a^2 + b^2 + c^2} \quad (16.1)$$

16.2 Sum of Vectors

The sum of two vectors is:

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta} \quad (16.2)$$

$$\vec{a} + \vec{b} = (a + x)\hat{i} + (b + y)\hat{j} + (c + z)\hat{k} \quad (16.3)$$

The direction of the resultant vector is:

$$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta} \quad (16.4)$$

where, θ is the angle between the two vectors.

16.3 Product of Vectors

16.3.1 Dot Product

$$\vec{a} \cdot \vec{b} = |a||b|\cos\theta \quad (16.5)$$

$$\vec{a} \cdot \vec{b} = ax + by + cz \quad (16.6)$$

16.3.2 Cross Product

$$\vec{a} \times \vec{b} = |a||b| \sin \theta \hat{n} \quad (16.7)$$

where \hat{n} is a vector $\perp \vec{a}, \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} \quad (16.8)$$

16.4 Test of Co-planarity

Three vectors are called co-planar if:

$$\lambda \vec{a} + \mu \vec{b} = \vec{c} \quad (16.9)$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \quad (16.10)$$

Chapter 17

3-D Geometry

17.1 Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad (17.1)$$

17.2 Section Formula of a Line Segment Divided in the ratio $m : n$

$$P = \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}, \frac{nz_1 + mz_2}{m + n} \right) \quad (17.2)$$

17.3 Centroid of a Triangle

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \quad (17.3)$$

Chapter 18

Line in 3-D Space

For a line which is defined as $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$:

1. Line numbers of the line is

$$< a, b, c > \quad (18.1)$$

2. The line cosines are:

$$< \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} > \quad (18.2)$$

$$=< l, m, n > \quad (18.3)$$

18.1 Angle between Two Lines

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \quad (18.4)$$

$$\Rightarrow \cos \theta = l_1l_2 + m_1m_2 + n_1n_2 \quad (18.5)$$

When two lines are \perp , $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

When two lines are \parallel $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1$.

18.2 Skew and Co-planar Lines

Let there be two lines \vec{r}_1 and \vec{r}_2 ,

$$\vec{r}_1 = \vec{a}_1 + \mu \vec{b}_1, \vec{r}_2 = \vec{a}_2 + \lambda \vec{b}_2 \quad (18.6)$$

18.3 Distances

18.3.1 The shortest distance between r_1 and r_2

$$S = \left| \frac{(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \quad (18.7)$$

If $S = 0$, the lines intersect.

18.3.2 Cartesian Form

For two lines defined as $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$:

$$S = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad (18.8)$$

18.3.3 Distance Between Parallel Lines

$$S = \left| \frac{\vec{b} \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right| \quad (18.9)$$

18.3.4 Distance of a Point to a Line

For a point, (x_1, y_1, z_1) the distance to a line $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$:

$$S = \left(\left| \begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix} \right| + \left| \begin{vmatrix} y_1 - \beta & z_1 - \gamma \\ m & n \end{vmatrix} \right| + \left| \begin{vmatrix} z_1 - \gamma & x_1 - \alpha \\ n & l \end{vmatrix} \right| \right)^{\frac{1}{2}} \quad (18.10)$$

Chapter 19

3-D Plane

A plane in 3-D space can be defined as:

1. Cartesian Form:

$$ax + by + cz + d = 0 \quad (19.1)$$

2. Vectorial Form:

$$\vec{r} \cdot \vec{n} = p \quad (19.2)$$

, where \vec{r} is a line on the plane, \vec{n} is a normal to the plane, and p is perpendicular distance to the plane from the origin.

19.1 Angle Between Two Planes

For two planes, $\vec{r}_1 \cdot \vec{n}_1 = p_1$ and $\vec{r}_2 \cdot \vec{n}_2 = p_2$, the angle between the planes, θ is:

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \quad (19.3)$$

In the Cartesian Form:

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad (19.4)$$

19.2 Distance of a Point from a Plane

19.2.1 Cartesian Form

For the point (p, q, r) and the plane, $ax + by + cz + d = 0$:

$$S = \frac{ap + bq + cr + d}{\sqrt{a^2 + b^2 + c^2}} \quad (19.5)$$

19.2.2 Vector Form

For the point $\vec{g} = p\hat{i} + q\hat{j} + r\hat{k}$ and the plane $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) + d = 0$:

$$S = \frac{(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (p\hat{i} + q\hat{j} + r\hat{k})}{\sqrt{a^2 + b^2 + c^2}} \quad (19.6)$$

$$\Rightarrow S = \frac{(a\hat{i} + b\hat{j} + c\hat{k}) \cdot \vec{g}}{|a\hat{i} + b\hat{j} + c\hat{k}|} \quad (19.7)$$

Part III

Statistics

Chapter 20

Descriptive Statistics

20.1 Measure of Location

20.1.1 Mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (20.1)$$

20.1.2 Median

For odd number of elements in a dataset:

$$\tilde{x} = x_{\frac{n+1}{2}} \quad (20.2)$$

For even number of elements in a dataset:

$$\tilde{x} = \frac{x_{\frac{n}{2}} + x_{(\frac{n}{2}+1)}}{2} \quad (20.3)$$

20.1.3 Mode

$$Mo(x) = \max(f(x_i)) \quad (20.4)$$

20.1.4 Quartile

Measure of percentage of elements less than or equal to a term

20.2 Measure of Spread

20.2.1 Variance

Variance measured on the whole population

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (20.5)$$

20.2.2 Sample Variance

Variance measured on a sample population

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (20.6)$$

20.2.3 Standard Deviation and Sample Standard

$$\sigma = \sqrt{\sigma^2} \quad (20.7)$$

$$s = \sqrt{s^2} \quad (20.8)$$

20.2.4 Co-efficient of Variance

$$v = \frac{s}{\bar{x}} \quad (20.9)$$

20.3 Skewness

20.3.1 Types of Skewness

| Name | Other Name | Characteristic |
|------------------------|---------------------|--------------------------------------|
| Right Skew | Positive Skew | Data concentrated on the lower side |
| Symmetric Distribution | Normal Distribution | Data distributed evenly |
| Left Skew | Negative Skew | Data concentrated on the higher side |

20.3.2 Measure of Skewness

Skewness is measured by the Moment Co-efficient of Skewness.

$$g_m = \frac{m_3}{s^3}, \text{ where} \quad (20.10)$$

$$m_3 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^3 \quad (20.11)$$

Type of Skewness

The type of skewness from the value is g_m is:

| Value of g_m | Type |
|----------------|---------------|
| $g_m = 0$ | Symmetric |
| $g_m > 0$ | Positive Skew |
| $g_m < 0$ | Negative Skew |

Degree of Skewness

The degree of skewness from the value is g_m is:

| Value of g_m | Degree |
|----------------------|-------------------|
| $ g_m > 1$ | High Skewness |
| $0.5 < g_m \leq 1$ | Moderate Skewness |
| $ g_m \leq 0.5$ | Low Skewness |

20.4 Kurtosis

Kurtosis is the measure of peakedness of data. Fisher's kurtosis measure is defined as:

$$\gamma = \frac{m_4}{s^4}, \text{ where} \quad (20.12)$$

$$m_4 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^4 \quad (20.13)$$

20.4.1 Type of Kurtosis

The types of kurtosis from the value of γ are:

| Value of γ | Type |
|-------------------|-----------------------------------|
| $\gamma = 0$ | Normal Distribution or Mesokurtic |
| $\gamma < 0$ | Flattened or Platykurtic |
| $\gamma > 0$ | Peaked or Lepokurtic |

Chapter 21

Hypothesis Testing

21.1 T-Test

$$T = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \quad (21.1)$$

where:

\bar{X} = Sample Mean

μ = Assumed Mean

s = Number of Samples

n = Number of observations

If $T < t_c$ the H_0 is not rejected. t_c is a functions of level of significance (α) and degrees of freedom ($v = n - 1$).

21.2 χ^2 Test

$$\chi^2 = \sum_i \sum_j \frac{(h_{ij}^o - h_{ij}^e)^2}{h_{ij}^e} \quad (21.2)$$

where:

h_e = Expected Value

h_o = Actual Value

If $\chi^2 < \chi_c^2$ then H_0 is not rejected. χ_c is a functions of level of significance (α) and degrees of freedom ($v = (i - 1)(j - 1)$).

Chapter 22

Research and Survey Design

22.1 Population Covariance

$$\text{Cov}(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) \quad (22.1)$$

22.2 Sample Covariance

$$\text{Cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \quad (22.2)$$

22.3 Bravais-Pearson Correlation Co-efficient

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \quad (22.3)$$

$$= \frac{\text{Cov}(x, y)}{\sqrt{\text{Var}(x) \cdot \text{Var}(y)}} \quad (22.4)$$

$$= \frac{\text{Cov}(x, y)}{\sigma_x \cdot \sigma_y} \quad (22.5)$$

22.4 Spearman's Rank Correlation Co-efficient

$$r_s = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)} \quad (22.6)$$

$$d_i = R(X_i) - R(Y_i) \quad (22.7)$$

Chapter 23

Estimation of Regression Function

For the regression functions:

$$Y_i = \beta_0 + \beta_1 X_1 \quad (23.1)$$

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 \quad (23.2)$$

$$(23.3)$$

where Y_i is the observed dependent variable (DV), \hat{Y}_i is the estimated DV, and X_i is the independent variable (IV).

$$u_i = Y_i - \hat{Y}_i \quad (23.4)$$

$$\Rightarrow Y_i = \hat{Y}_i + u_i \quad (23.5)$$

$$\Rightarrow Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 + u_i \quad (23.6)$$

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \quad (23.7)$$

The objective function is:

$$\min_{u_i} \sum u_i = \min \sum_i \left[Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \right]^2$$

Since the regression function passes through: (\bar{X}, \bar{Y})

$$\beta_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$\begin{aligned} \min_{u_i} \sum u_i &= \min \sum_i \left[Y_i - \bar{Y} + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 X_i \right]^2 \\ &= \min \sum_i \left[(Y_i - \bar{Y}) - \hat{\beta}_1 (X_i - \bar{X}) \right]^2 \\ &= \min \sum_i \left[(Y_i - \bar{Y})^2 - 2 \cdot (Y_i - \bar{Y}) \cdot \hat{\beta}_1 (X_i - \bar{X}) + \hat{\beta}_1^2 (X_i - \bar{X})^2 \right] \\ &= \min \left[\sum_i (Y_i - \bar{Y})^2 - 2 \cdot \hat{\beta}_1 \sum_i (Y_i - \bar{Y}) \cdot (X_i - \bar{X}) + \hat{\beta}_1^2 \sum_i (X_i - \bar{X})^2 \right] \\ &\Rightarrow u_i^{\beta_1} = -2 \sum_i (Y_i - \bar{Y}) + 2 \hat{\beta}_1 (X_i - \bar{X})^2 = 0 \end{aligned}$$

(For optima Conditions)

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_i (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_i (X_i - \bar{X})^2}$$

$$\Rightarrow \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

23.1 Sum of Squares Error

$$TSS = \sum_i (Y_i - \bar{Y})^2 \quad (23.8)$$

$$= \underbrace{\sum_i (\hat{Y}_i - \bar{Y})}_{\text{Explained Sum of Square Error (ESS)}} + \underbrace{\sum_i u_i^2}_{\text{Residual Sum of Squares Error (RSS)}} \quad (23.9)$$

23.1.1 R^2 : Coefficient of Determination

$$R^2 = \frac{\text{ESS}}{\text{TSS}} \quad (23.10)$$

$$= 1 - \frac{\text{RSS}}{\text{TSS}} \quad (23.11)$$

$$= 1 - \frac{\sum_i u_i^2}{\sum_i (Y_i - \bar{Y})^2} \quad (23.12)$$

For a regression analysis with single IV:

$$\sqrt{R^2} = v$$

23.1.2 \bar{R}^2 : Coefficient of Determination

$$\bar{R}^2 = 1 - \frac{\frac{\sum_i u_i^2}{(N - K - 1)}}{\frac{\sum_i (Y_i - \bar{Y})^2}{(N - 1)}} \quad (23.13)$$

where, N is the number of observations and K is the number of independent variables.

23.2 T-Test

Test for statistical significance of a single IV.

$$T = \frac{\hat{\beta}_1 - 0}{S_e(\hat{\beta}_1)} \quad (23.14)$$

23.3 F-Test

Test for statistical significance of all IVs together.

$$F = \frac{\frac{\text{ESS}}{(K - 1)}}{\frac{\text{RSS}}{(N - K)}} \quad (F \geq F_c, H_0 \text{ is rejected})$$

23.4 Test for Heteroskedasticity

23.4.1 Definition

$$\sigma_{\epsilon_i} \forall \epsilon_i \in [X_a, X_b] = \sigma_{\epsilon_i} \forall \epsilon_i \in [X_{b+1}, X_c]$$

23.4.2 Durbin-Watson Test

$$d_e = \frac{\sum_{t=2}^n (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=2}^n \hat{u}_t^2} \quad (23.15)$$

For the H_0 : No autocorrelation:

| d | H_0 |
|---|--------------------|
| $0 \leq d_e \leq d_L \text{ \& } (4 - d_L) \leq d_e \leq 4$ | Rejected |
| $d_L < d_e \leq d_U \text{ \& } (4 - d_U) < d_e \leq (4 - d_L)$ | Decision Free Zone |
| $d_L < d_e < D_U$ | Not rejected |

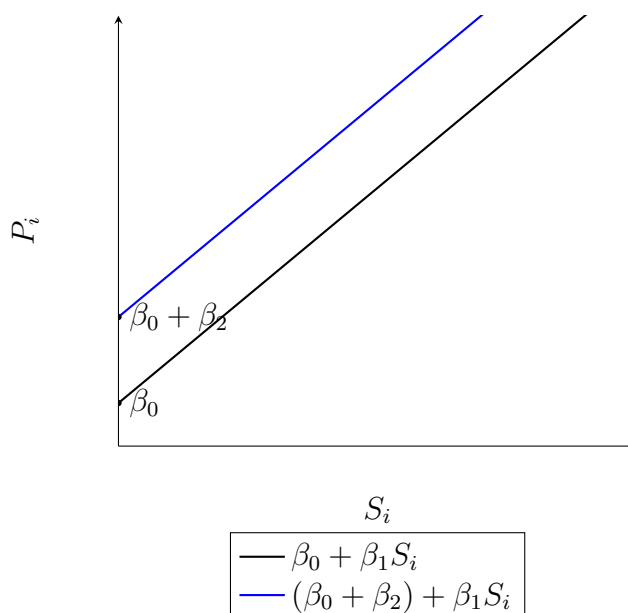
Chapter 24

Dummy Variables

24.1 Dummy Variable

$$P_i = \beta_0 + \beta_1 S_i + \beta_2 D_i + \epsilon_i \quad (24.1)$$

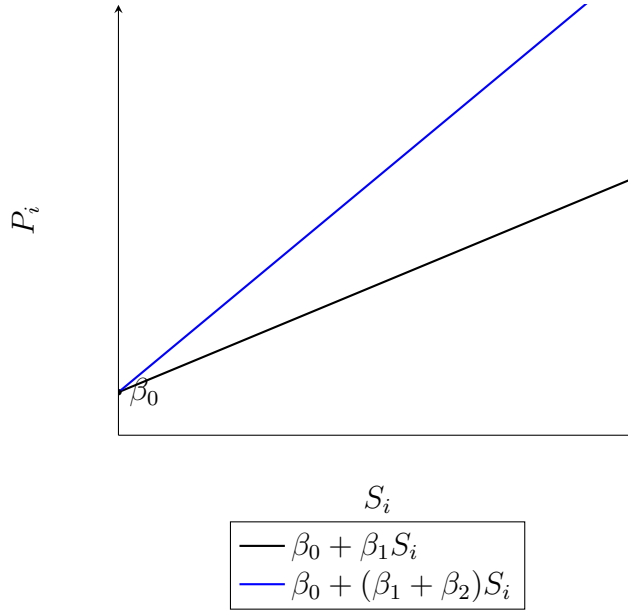
$$E(P_i) = \begin{cases} (\hat{\beta}_0 + \hat{\beta}_2) + \hat{\beta}_1 S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \quad (24.2)$$



24.2 Slope Dummy Variable

$$P_i = \beta_0 + \beta_1 S_i + \beta_2 (S_i \cdot D_i) + \epsilon_i \quad (24.3)$$

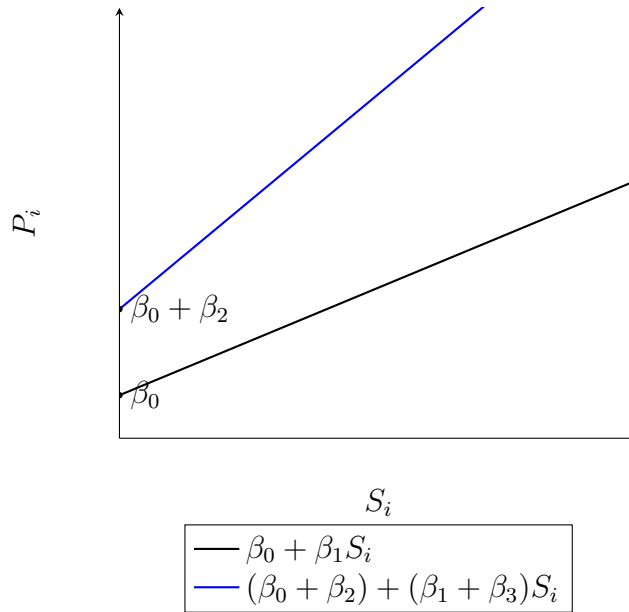
$$E(P_i) = \begin{cases} \hat{\beta}_0 + (\hat{\beta}_1 + \hat{\beta}_2) S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \quad (24.4)$$



24.3 Slope & Dummy Variable

$$P_i = \beta_0 + \beta_1 S_i + \beta_2 D_i + \beta_3 S_i D_i + \epsilon_i \quad (24.5)$$

$$E(P_i) = \begin{cases} (\hat{\beta}_0 + \hat{\beta}_2) + (\hat{\beta}_1 + \hat{\beta}_3) S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \quad (24.6)$$



24.4 Multi-Categories Dummy Variable

$$P_0 = b_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (24.7)$$

Leads to Perfect Multicollinearity

Alternatives

- B_n captures the mean of each category, but F-Test is impossible

$$y = \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} \quad (24.8)$$

- Computer drops automatically drops a variable

$$y = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} \quad (24.9)$$

- Manually dropping a variable

$$y = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} \quad (24.10)$$

Chapter 25

Logistic Regression

For $Y_i \in \{0, 1\}$:

$$z_k = \beta_0 + \sum_{j=1}^n \beta_{jk} x_j + \epsilon_k, \beta_j \rightarrow \text{Logit Coefficient} \quad (25.1)$$

$$p = \frac{\exp^k}{1 + \exp^k} = \frac{1}{1 + \exp^{-k}} \quad (25.2)$$

where p is the probability of $y = 1$.

Part IV

Trigonometry

Chapter 26

Circular Trigonometric Functions

| θ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ |
|------------|---------------------------------|---------------------------------|--|
| 0° | 0 | 1 | 0 |
| 15° | $\frac{1}{4}$ | $\frac{1}{4(2-\sqrt{3})}$ | $2 - \sqrt{3}$ |
| 18° | $\frac{\sqrt{5}-1}{4}$ | $\frac{\sqrt{10+2\sqrt{5}}}{4}$ | $\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$ |
| 30° | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 36° | $\frac{\sqrt{5}+1}{4}$ | $\frac{\sqrt{10-2\sqrt{5}}}{4}$ | $\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$ |
| 45° | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| 60° | $\frac{1}{\sqrt{3}}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 72° | $\frac{\sqrt{10+2\sqrt{5}}}{4}$ | $\frac{\sqrt{5}-1}{4}$ | $\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$ |
| 90° | 1 | 0 | ∞ |

Table 26.1: Trigonometric Ratios of Standard Angles

For any given triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \quad (26.1)$$

, where $2R$ is the radius of circumcircle.

26.1 Negative Angle Formula

$$\sin(-\theta) = -\sin \theta \quad (26.2)$$

$$\cos(-\theta) = \cos \theta \quad (26.3)$$

$$\tan(-\theta) = -\tan \theta \quad (26.4)$$

$$\csc(-\theta) = -\csc \theta \quad (26.5)$$

$$\sec(-\theta) = \sec \theta \quad (26.6)$$

$$\cot(-\theta) = -\cot \theta \quad (26.7)$$

26.2 Sum of Angles Formula

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad (26.8)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (26.9)$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad (26.10)$$

26.3 Difference of Angles Formula

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad (26.11)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (26.12)$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad (26.13)$$

26.4 Multiples and Sub-multiples of π and $\frac{\pi}{2}$

$$\forall k \in \mathbb{Z}$$

$$\sin\left((4k + 1)\frac{\pi}{2}\right) = 1 \quad (26.14)$$

$$\sin\left((4k - 1)\frac{\pi}{2}\right) = -1 \quad (26.15)$$

$$\sin k\pi = 0 \quad (26.16)$$

$$\sin\left((2k + 1)\frac{\pi}{2}\right) = 0 \quad (26.17)$$

$$\sin\left((2k - 1)\frac{\pi}{2}\right) = 0 \quad (26.18)$$

$$\sin k\pi = (-1)^k \quad (26.19)$$

26.5 $\left(\frac{\pi}{2} \pm \theta\right)$ Formula

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad (26.20)$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta \quad (26.21)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \quad (26.22)$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta \quad (26.23)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \quad (26.24)$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot \theta \quad (26.25)$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta \quad (26.26)$$

$$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan \theta \quad (26.27)$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta \quad (26.28)$$

$$\csc\left(\frac{\pi}{2} + \theta\right) = \sec \theta \quad (26.29)$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta \quad (26.30)$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\csc \theta \quad (26.31)$$

26.6 $\left(\frac{\pi}{4} \pm \theta\right)$ Formula

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan \theta}{1 - \tan \theta} \quad (26.32)$$

$$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan \theta}{1 + \tan \theta} \quad (26.33)$$

26.7 Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (26.34)$$

$$\tan^2 \theta + 1 = \sec^2 \theta \quad (26.35)$$

$$\cot^2 \theta + 1 = \csc^2 \theta \quad (26.36)$$

26.8 Double Angle Formula

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ &= \frac{2 \tan \theta}{1 + \tan^2 \theta}\end{aligned}\tag{26.37}$$

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \\ &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}\end{aligned}\tag{26.38}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}\tag{26.39}$$

26.9 Triple Angle Formula

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta\tag{26.40}$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta\tag{26.41}$$

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}\tag{26.42}$$

26.10 Sum and Product of Two Ratios

For $A > B$:

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \quad (26.43)$$

$$\sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \quad (26.44)$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B) \quad (26.45)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B) \quad (26.46)$$

$$\cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \quad (26.47)$$

$$\cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \quad (26.48)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B) \quad (26.49)$$

$$2 \cos A \sin B = \cos(A+B) - \cos(A-B) \quad (26.50)$$

$$\sin(A-B) \sin(A+B) = \sin^2 A - \sin^2 B \quad (26.51)$$

$$\cos(A-B) \cos(A+B) = \cos^2 A - \sin^2 B \quad (26.52)$$

$$\tan(A-B) \tan(A+B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} \quad (26.53)$$

26.11 General Solutions

If $\sin \theta = \sin \alpha$:

$$\theta = n\pi + (-1)^n \alpha \quad (26.54)$$

$n \in \mathbb{Z}$

If $\cos \theta = \cos \alpha$:

$$\theta = 2n\pi \pm \alpha \quad (26.55)$$

$n \in \mathbb{Z}$

If $\tan \theta = \tan \alpha$:

$$\theta = n\pi \pm \alpha \quad (26.56)$$

$n \in \mathbb{Z}$

26.12 Taylor Series Expansion of Trigonometric Ratios

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^{2i-1}}{(2i-1)!} \quad (26.57)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} \quad (26.58)$$

Chapter 27

Inverse Circular Trigonometric Function

27.1 Definition of Inverse Circular Trigonometric Function

27.1.1 For $\sin x$

$y = \arcsin x$ iff $x = \sin y$, then:

1. $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. domain of $x \in [-1, 1]$
3. $\sin(\arcsin x) = x, \forall x \in [-1, 1]$
4. $\arcsin(\sin y) = y, \forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
5. $\sin x$ is a strictly increasing in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and one-one.

27.1.2 For $\cos x$

$y = \arccos x$ iff $x = \cos y$, then:

1. $y \in [0, \pi]$
2. domain of $x \in [-1, 1]$
3. $\cos(\arccos x) = x, \forall x \in [-1, 1]$
4. $\arccos(\cos y) = y, \forall y \in [0, \pi]$
5. $\cos x$ is a strictly decreasing in the domain $[0, \pi]$ and one-one.

27.1.3 For $\tan x$

$y = \arctan x$ iff $x = \tan y$, then:

1. $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. domain of $x \in \mathbb{R}$
3. $\tan(\arctan x) = x, \forall x \in \mathbb{R}$
4. $\arctan(\tan y) = y, \forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
5. $\tan x$ is a strictly increasing in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and one-one.

27.1.4 For $\cot x$

$y = \cot^{-1} x$ iff $x = \cot y$, then:

1. $y \in (0, \pi)$
2. domain of $x \in \mathbb{R}$
3. $\cot(\cot^{-1} x) = x, \forall x \in \mathbb{R}$
4. $\cot^{-1}(\cot y) = y, \forall y \in (0, \pi)$
5. $\cot x$ is a strictly decreasing in the domain $(0, \pi)$ and one-one.

For $\sec x$

$y = \sec^{-1} x$ iff $x = \sec y$

1. $y \in \{[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\}$
2. $x \in \{(-\infty, -1] \cup [1, \infty)\}$
3. $\sec(\sec^{-1} x) = x, \forall |x| \geq 1$
4. $\sec^{-1}(\sec y) = y, \forall y \in \{[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\}$

27.1.5 For $\csc x$

$y = \csc^{-1} x$ iff $x = \csc y$

1. $y \in \{[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\}$
2. $x \in \{(-\infty, -1] \cup [1, \infty)\}$
3. $\csc(\csc^{-1} x) = x, \forall |x| \geq 1$
4. $\csc^{-1}(\csc y) = y, \forall y \in \{[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\}$

27.2 Negative Arguments

$$\arcsin(-x) = -\arcsin x \quad (27.1)$$

$$\arctan(-x) = -\arctan x \quad (27.2)$$

$$\csc^{-1}(-x) = -\csc^{-1} x \quad (27.3)$$

$$\arccos(-x) = \pi - \arccos x \quad (27.4)$$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x \quad (27.5)$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x \quad (27.6)$$

27.3 Reciprocal Relations

$$\csc^{-1} x = \arcsin \frac{1}{x} \quad (27.7)$$

$$\sec^{-1} x = \arccos \frac{1}{x} \quad (27.8)$$

$$\sec^{-1} x = \begin{cases} \arctan \frac{1}{x}, & x > 0 \\ \pi + \arctan \frac{1}{x}, & x < 0 \end{cases} \quad (27.9)$$

27.4 I.T.F. Identities

$$\arcsin x + \arccos x = \frac{\pi}{2}, |x| \leq 1 \quad (27.10)$$

$$\arctan x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathbb{R} \quad (27.11)$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}, |x| \geq 1 \quad (27.12)$$

27.5 Sum of Two Angles

$$\arctan x + \arctan y = \arctan \left(\frac{x+y}{1-xy} \right) \quad (27.13)$$

$$\arcsin x + \arcsin y = \arcsin(y\sqrt{1-x^2} + x\sqrt{1-y^2}) \quad (27.14)$$

$$\arccos x + \arccos y = \arccos(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \quad (27.15)$$

27.6 Difference of Two Angles

$$\arctan x - \arctan y = \arctan \left(\frac{x - y}{1 + xy} \right) \quad (27.16)$$

$$\arcsin x - \arcsin y = \arcsin(x\sqrt{1 - y^2} - y\sqrt{1 - x^2}) \quad (27.17)$$

$$\arccos x - \arccos y = \arccos(xy + \sqrt{1 - x^2}\sqrt{1 - y^2}) \quad (27.18)$$

27.7 Interconversion of Ratios

$$\begin{aligned} \arcsin x &= \arccos \sqrt{1 - x^2} \\ &= \arctan \left(\frac{x}{\sqrt{1 - x^2}} \right) \end{aligned} \quad (27.19)$$

$$\begin{aligned} \arccos x &= \arcsin \sqrt{1 - x^2} \\ &= \arctan \left(\frac{\sqrt{1 - x^2}}{x} \right) \end{aligned} \quad (27.20)$$

$$\begin{aligned} 2 \arctan x &= \arcsin \left(\frac{2x}{1 + x^2} \right) \\ &= \arccos \left(\frac{1 - x^2}{1 + x^2} \right) \\ &= \arctan \left(\frac{2x}{1 - x^2} \right) \end{aligned} \quad (27.21)$$

27.8 Miscellaneous Relations

$$\cos(\arcsin x) = \sin(\arccos x) = \sqrt{1 - x^2} \quad (27.22)$$

$$\arctan x = \frac{\pi}{2} - \arctan \left(\frac{1}{x} \right), x > 1 \quad (27.23)$$

Chapter 28

Hyperbolic Trigonometric Function

28.1 Definition

Hyperbolic trigonometric functions are defined such that $(\cosh t, \sinh t)$ form the right half of an equilateral hyperbola. The functions are defined as follows:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2} \quad (28.1)$$

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2} \quad (28.2)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)} \quad (28.3)$$

$$\coth x = \frac{1}{\tanh x} = \frac{\exp(x) + \exp(-x)}{\exp(x) - \exp(-x)} \quad (28.4)$$

$$csch x = \frac{1}{\sinh x} = \frac{2}{\exp(x) - \exp(-x)} \quad (28.5)$$

$$sech x = \frac{1}{\cosh x} = \frac{2}{\exp(x) + \exp(-x)} \quad (28.6)$$

28.2 Identities

$$\coth^2 x - \sinh^2 x = 1 \quad (28.7)$$

$$\tanh^2 x + sech^2 x = 1 \quad (28.8)$$

$$\coth^2 x - csch^2 x = 1 \quad (28.9)$$

28.3 Inverse Hyperbolic Function

$$\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1}) \quad (28.10)$$

$$\cosh^{-1} z = \ln(z \pm \sqrt{z^2 - 1}) \quad (28.11)$$

$$\tanh^{-1} z = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right) \quad (28.12)$$

$$\coth^{-1} z = \frac{1}{2} \ln \left(\frac{z+1}{z-1} \right) \quad (28.13)$$

$$\operatorname{csch}^{-1} z = \ln \left(\frac{1 \pm \sqrt{z^2 + 1}}{z} \right) \quad (28.14)$$

$$\operatorname{sech}^{-1} z = \ln \left(\frac{1 \pm \sqrt{1 - z^2}}{2} \right) \quad (28.15)$$

28.4 Relation to Circular Trigonometric Functions

$$\sinh(z) = -i \sin(iz) \quad (28.16)$$

$$\cosh(z) = \cos(iz) \quad (28.17)$$

$$\tanh(z) = -i \tan(iz) \quad (28.18)$$

$$\operatorname{csch}(z) = i \csc(iz) \quad (28.19)$$

$$\operatorname{sech}(z) = \sec(iz) \quad (28.20)$$

$$\coth(z) = i \cot(iz) \quad (28.21)$$

Part V

Calculus

Chapter 29

Limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (29.1)$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad (29.2)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (29.3)$$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \quad (29.4)$$

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) \quad (29.5)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \lim_{x \rightarrow a} g(x) \neq 0 \quad (29.6)$$

$$\lim_{x \rightarrow 0} \exp(x) = 1 \quad (29.7)$$

$$\lim_{x \rightarrow a} \exp(x) = \exp(c) \quad (29.8)$$

$$\lim_{x \rightarrow 0} \frac{\exp(x) - 1}{x} = 1 \quad (29.9)$$

$$\lim_{x \rightarrow a} c^x = c^a \quad (29.10)$$

$$\lim_{x \rightarrow a} \ln x = \ln a \quad (29.11)$$

$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \quad (29.12)$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \quad (29.13)$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \quad (29.14)$$

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x \in \mathbb{R} \quad (29.15)$$

29.1 L'Hospital Rule

If:

$$L = \lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

is such that $f(a) = 0$ and $g(a) = 0$, or $f(a) = \infty$ and $g(a) = \infty$, then:

$$L = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Chapter 30

Differentiation

30.1 Differentiation by First Principle

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (30.1)$$

30.2 Standard Differentiation Formulae

$$\frac{dk}{dx} = 0 \quad (30.2)$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad (30.3)$$

$$\frac{da^x}{dx} = \ln a \cdot a^x \quad (30.4)$$

$$\frac{d \exp(x)}{dx} = \exp(x) \quad (30.5)$$

$$\frac{d \ln x}{dx} = \frac{1}{x} \quad (30.6)$$

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}} \quad (30.7)$$

$$(30.8)$$

30.2.1 Circular Trigonometric Functions

$$\frac{d \sin x}{dx} = \cos x \quad (30.9)$$

$$\frac{d \cos x}{dx} = -\sin x \quad (30.10)$$

$$\frac{d \tan x}{dx} = \sec^2 x \quad (30.11)$$

$$\frac{d \sec x}{dx} = \sec x \tan x \quad (30.12)$$

$$\frac{d \csc x}{dx} = -\csc x \cot x \quad (30.13)$$

$$\frac{d \cot x}{dx} = -\csc^2 x \quad (30.14)$$

30.2.2 Inverse Circular Trigonometric Functions

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}, |x| \leq 1 \quad (30.15)$$

$$\frac{d \arccos x}{dx} = -\frac{1}{\sqrt{1-x^2}}, |x| \leq 1 \quad (30.16)$$

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2} \quad (30.17)$$

$$\frac{d \cot^{-1} x}{dx} = -\frac{1}{1+x^2} \quad (30.18)$$

$$\frac{d \sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}}, |x| \geq 1 \quad (30.19)$$

$$\frac{d \csc^{-1} x}{dx} = -\frac{1}{x\sqrt{x^2-1}}, |x| \geq 1 \quad (30.20)$$

30.2.3 Hyperbolic Trigonometric Function

$$\frac{d \sinh x}{dx} = \cosh x \quad (30.21)$$

$$\frac{d \cosh x}{dx} = \sinh x \quad (30.22)$$

$$\frac{d \tanh x}{dx} = 1 - \tanh^2 x = \operatorname{sech}^2(x) \quad (30.23)$$

$$\frac{d \coth x}{dx} = 1 - \coth^2 x = -\operatorname{csch}^2(x) \quad (30.24)$$

$$\frac{d[\operatorname{sech}(x)]}{dx} = -\tanh x \operatorname{sech} x \quad (30.25)$$

$$\frac{d[\operatorname{csch}(x)]}{dx} = -\coth x \operatorname{csch} x \quad (30.26)$$

30.2.4 Inverse Hyperbolic Trigonometric Function

$$\frac{d \sinh x}{dx} = \frac{1}{\sqrt{x^2 + 1}} \quad (30.27)$$

$$\frac{d \cosh x}{dx} = \frac{1}{\sqrt{x^2 - 1}} \quad (30.28)$$

$$\frac{d \tanh x}{dx} = \frac{1}{1 - x^2} \quad (30.29)$$

$$\frac{d \coth x}{dx} = \frac{1}{1 - x^2} \quad (30.30)$$

$$\frac{d[\operatorname{sech}(x)]}{dx} = \frac{1}{x\sqrt{1 - x^2}} \quad (30.31)$$

$$\frac{d[\operatorname{csch}(x)]}{dx} = \frac{1}{|x|\sqrt{1 + x^2}} \quad (30.32)$$

30.3 Rules of Differentiation

$$\frac{d[cf(x)]}{dx} = c \frac{df(x)}{dx} \quad (30.33)$$

$$\frac{d[f_1(x) + f_2(x)]}{dx} = \frac{d[f_1(x)]}{dx} + \frac{d[f_2(x)]}{dx} \quad (30.34)$$

$$\frac{d[f_1 f_2]}{dx} = f_1 f_2' + f_2 f_1' \quad (30.35)$$

$$\frac{d\left(\frac{f_1}{f_2}\right)}{dx} = \frac{f_2 f_1' - f_1 f_2'}{f_2^2} \quad (30.36)$$

30.4 Chain Rule

If two functions are defined as $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \quad (30.37)$$

If two functions are defined as $x = f(\theta)$ and $y = g(\theta)$:

$$\frac{d^2 y}{dx^2} = \left[\frac{d}{d\theta} \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \right) \right] \frac{d\theta}{dx} \quad (30.38)$$

Chapter 31

Successive Differentiation

$$D^n(ax+b)^m = m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n} \quad (31.1)$$

$$D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}} \quad (31.2)$$

$$D^n \ln(ax+b) = \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}, n \geq 2 \quad (31.3)$$

$$D^n(a^{mx}) = m^n (\ln a)^n a^{mx} \quad (31.4)$$

$$D^n(e^{mx}) = m^n e^{mx} \quad (31.5)$$

$$D^n \sin(ax+b) = a^n \sin(ax+b+n\frac{\pi}{2}) \quad (31.6)$$

$$D^n \cos(ax+b) = a^n \cos(ax+b+n\frac{\pi}{2}) \quad (31.7)$$

$$D^n[e^{ax} \sin(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin(bx+c+n \arctan \frac{b}{a}) \quad (31.8)$$

$$D^n[e^{ax} \cos(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos(bx+c+n \arctan \frac{b}{a}) \quad (31.9)$$

31.1 Leibnitz's Theorem

For two functions u and v of x , the successive differentiation of their product is defined as:

$$\begin{aligned} (uv)_n &= {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + \cdots + {}^nC_n u v_n \\ &= \sum_{i=0}^n {}^nC_i u_{n-i} v_i \end{aligned} \quad (31.10)$$

Chapter 32

Partial Derivative

If $f(x, y)$ is a function of (x, y) , then $\frac{\delta f(x, y)}{\delta x}$ is the differentiation of $f(x, y)$ w.r.t. x , keeping all other parameters constant.

32.1 Chain Rule

If f is a function of u and v , which are functions of x and y , then:

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u} \frac{\delta u}{\delta x} + \frac{\delta f}{\delta v} \frac{\delta v}{\delta x} \quad (32.1)$$

$$\frac{\delta f}{\delta y} = \frac{\delta f}{\delta u} \frac{\delta u}{\delta y} + \frac{\delta f}{\delta v} \frac{\delta v}{\delta y} \quad (32.2)$$

If f is a function of x and y , which are functions of t , then:

$$\frac{df}{dt} = \frac{\delta f}{\delta x} \frac{dx}{dt} + \frac{\delta f}{\delta y} \frac{dy}{dt} \quad (32.3)$$

32.2 Euler's Theorem

For a homogeneous function ¹, $f(x_i)$ of degree n :

$$\sum x_i \frac{\delta f}{\delta x_i} = n f(x_i) \quad (32.4)$$

¹Homogeneous functions are defined as $f(ax, ay) = a^\kappa f(x, y)$, where κ is the degree of homogeneity. E.g. $f(x, y) = x^2 + y^2$, then $f(tx, ty) = t^2(x^2 + y^2)$, and the degree of homogeneity is 2.

Chapter 33

Application of Differentiation

33.1 Rolle's Theorem

For a function $f(x)$:

1. is continuous in $[a, b]$
2. is differentiable in (a, b)
3. $f(a) = f(b)$,

then there exists a point $x = c$ such that $f'(c) = 0$, $c \in (a, b)$

33.2 Mean Value Theorem or LaGrange's Theorem

For a function $f(x)$:

1. is continuous in $[a, b]$
2. is differentiable in (a, b) ,

then there exists a point $x = c$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$, $c \in (a, b)$,

i.e., the tangent is parallel to the line joining the the points $(a, f(a))$ and $(b, f(b))$.

33.3 Cauchy's Mean Value Theorem

For a function $f(x)$ and $g(x)$:

1. are continuous in $[a, b]$
2. are differentiable in (a, b)
3. $g'(x) \neq 0$ in (a, b) ,

then there exists a point $c \in (a, b)$, such that $\frac{f(x)}{g(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

33.4 Maxima and Minima

33.4.1 Maxima

For the local maxima of a function $f(x)$:

1. $f'(c) = 0$ and

$$\lim_{\epsilon \rightarrow c^-} f'(\epsilon) > 0$$

$$\lim_{\epsilon \rightarrow c^+} f'(\epsilon) < 0$$

OR

2. $f'(c) = 0$ and $f''(x) < 0$,

then $f(c)$ is the local maxima point of the function $f(x)$.

33.4.2 Minima

For the local minima of a function $f(x)$:

1. $f'(c) = 0$ and

$$\lim_{\epsilon \rightarrow c^-} f'(\epsilon) < 0$$

$$\lim_{\epsilon \rightarrow c^+} f'(\epsilon) > 0$$

OR

2. $f'(c) = 0$ and $f''(x) > 0$,

then $f(c)$ is the local minima point of the function $f(x)$.

33.5 Taylor's Theorem

For a function which is differentiable n times:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{x!}R_n \quad (33.1)$$

where R_n is the remainder term.

33.5.1 Remainder Term

LeGrange's Form

$$R_n = f^n(a + \theta h), \theta \in (0, 1) \quad (33.2)$$

Cauchy's Form

$$R_n = n(1 - \theta)^{n-1} f^n(a + \theta h), \theta \in (0, 1) \quad (33.3)$$

33.5.2 Conditions for Validity of Expansion

For validity of Taylor Expansion, the condition

$$\lim_{n \rightarrow \infty} R_n = 0 \quad (33.4)$$

needs to be satisfied either where R_n is the remainder term in either LeGrange's Form or Cauchy's Form. If the condition is satisfied in a certain domain, then the expansion is valid within that domain only.

33.5.3 Taylor's Theorem for Two Variables

$$\begin{aligned} f(a + x, b + y) = & f(x, y) + \left(a \frac{\delta}{\delta x} + b \frac{\delta}{\delta y} \right) f(x, y) + \\ & \frac{1}{2!} \left(a^2 \frac{\delta^2}{\delta x^2} + b^2 \frac{\delta^2}{\delta y^2} \right) f(x, y) + \cdots + \\ & \frac{1}{n!} \left(a^n \frac{\delta^n}{\delta x^n} + b^n \frac{\delta^n}{\delta y^n} \right) f(x + \theta a, y + \theta b), \theta \in (0, 1) \end{aligned} \quad (33.5)$$

33.6 Maclaurin's Series

$$\begin{aligned} f(x) = & f(0) + x f'(0) + \frac{1}{2!} x^2 f''(0) + \frac{1}{3!} x^3 f'''(0) + \cdots \infty \\ = & \sum_{i=0}^{\infty} \frac{1}{i!} x^i f^i(0) \end{aligned} \quad (33.6)$$

33.6.1 Maclaurin's Series with Two Variables

$$\begin{aligned} f(a, b) = & f(0, 0) + \left(a \frac{\delta}{\delta x} + b \frac{\delta}{\delta y} \right) f(0, 0) + \\ & \frac{1}{2!} \left(a^2 \frac{\delta^2}{\delta x^2} + b^2 \frac{\delta^2}{\delta y^2} \right) f(0, 0) + \cdots \infty \\ = & \sum_{i=0}^{\infty} \frac{1}{n!} \left(a^i \frac{\delta^i}{\delta x^i} + b^i \frac{\delta^i}{\delta y^i} \right) f(0, 0) \end{aligned} \quad (33.7)$$

33.7 Curvature

Curvature is the rate of change of direction w.r.t. arc. Mathematically:

$$\text{Curvature} = \frac{d(\text{direction})}{d(\text{arc})}$$

$$\lim_{\Delta s \rightarrow 0} \frac{\Delta \psi}{\Delta s} = \frac{d\psi}{ds} \quad (33.8)$$

33.7.1 Radius of Curvature

Cartesian Form

For a curve $y = f(x)$:

$$\rho = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \quad (33.9)$$

However, this formula fails for $y' \rightarrow \infty$.

Parametric Form

For a curve defined as $x = \phi(t)$ and $y = \psi(t)$:

$$\rho = \frac{(\ddot{x}^2 + \ddot{y}^2)^{\frac{3}{2}}}{x\ddot{y} - y\ddot{x}} \quad (33.10)$$

33.7.2 Newton's Formula

1. If the curve passes through origin, and the tangent at origin is the x-axis:

$$\rho = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2}{2y} \quad (33.11)$$

2. If the curve passes through origin, and the tangent at origin is the y-axis:

$$\rho = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{y^2}{2x} \quad (33.12)$$

3. If the curve passes through origin and $ax + by + c = 0$ is the tangent at origin:

$$\rho(0,0) = \frac{1}{2} \sqrt{a^2 + b^2} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{a^2 + y^2}{ax + by} \quad (33.13)$$

33.7.3 Tangent at Origin

For a curve

$$\sum c_i x^j y^k = 0, i \in \mathbb{N} \text{ and } j, k \in \mathbb{Z} - \{0\} \quad (33.14)$$

The curve passes through origin $\because c = 0$. Then the lowest degree term equated to x gives the tangent at origin.

33.8 Asymptotes

If the distance between a line P and a curve $f(x)$, s is such that $s \rightarrow 0$, as $x \rightarrow \infty$, then P is the asymptote of $f(x)$. For asymptotes not parallel to x-axis:

Let $y = mx + c$ be the asymptote of the function $y = f(x)$, then:

$$m = \lim_{x \rightarrow \infty} \frac{y}{x} \quad (33.15)$$

$$c = \lim_{x \rightarrow \infty} (y - mx) \quad (33.16)$$

33.8.1 Asymptote of Algebraic Curves

For an algebraic curve, passing through origin, defined as:

$$\begin{aligned} & (a_0 x^n + a_1 x^{n-1} y^1 + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n) + \\ & (b_0 x^{n-1} + b_1 x^{n-2} y^1 + b_2 x^{n-3} y^2 + \cdots + b_{n-1} x y^{n-2} + a_n y^{n-1}) + \\ & \quad \quad \quad \dots = 0 \end{aligned}$$

$$\Rightarrow x^n \phi_n \left(\frac{y}{x} \right) + x^{n-1} \phi_{n-1} \left(\frac{y}{x} \right) + \cdots + x \phi_1 \left(\frac{y}{x} \right) = 0$$

The asymptote(s) defined as $y = mx + c$,

1. m is the solution for the equation

$$\phi_n(m) = 0 \quad (33.17)$$

- 2.

$$c = -\frac{\phi_{n-1}(m)}{\phi_n(m)} \quad (33.18)$$

where c is a finite value.

Chapter 34

Integration

34.1 General Formulae

¹

$$\int nx^{n-1}dx = x^n + A \quad (34.1)$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + A \quad (34.2)$$

$$\int e^x dx = e^x + A \quad (34.3)$$

$$\int \frac{1}{x} dx = \ln x + A \quad (34.4)$$

$$\int \ln x dx = x(\ln x - 1) + A \quad (34.5)$$

¹A is the constant of integration in all cases

34.2 Circular Trigonometric Functions

$$\int \sin x dx = -\cos x + A \quad (34.6)$$

$$\int \cos x dx = \sin x + A \quad (34.7)$$

$$\int \sec^2 x dx = \tan x + A \quad (34.8)$$

$$\int \csc^2 x dx = -\cot x + A \quad (34.9)$$

$$\int \sec x \tan x dx = \sec x + A \quad (34.10)$$

$$\int \csc x \cot x dx = -\csc x + A \quad (34.11)$$

$$\int \sec x dx = \ln(\sec x + \tan x) + A \quad (34.12)$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + A \quad (34.13)$$

$$\begin{aligned} \int \tan x dx &= -\ln(\cos x) + A \\ &= \ln(\sec x) + A \end{aligned} \quad (34.14)$$

$$\int \cot x dx = \ln(\sin x) + A \quad (34.15)$$

34.3 Inverse Circular Trigonometric Function

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + A \quad (34.16)$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + A \quad (34.17)$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + A \quad (34.18)$$

$$\int \frac{-1}{1+x^2} dx = \cot^{-1} x + A = -\tan^{-1} x + A \quad (34.19)$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + A = -\csc^{-1} x + A \quad (34.20)$$

$$\int \frac{-1}{x\sqrt{x^2-1}} dx = \csc^{-1} x + A = -\sec^{-1} x + A \quad (34.21)$$

34.4 Standard Integrals

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + A \quad (34.22)$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + A \quad (34.23)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + A \quad (34.24)$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln(x + \sqrt{x^2 - a^2}) + A \quad (34.25)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + A \quad (34.26)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + A \quad (34.27)$$

$$\int \sqrt{a^2 + x^2} dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + A \quad (34.28)$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln(x + \sqrt{a^2 - x^2}) + A \quad (34.29)$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + A \quad (34.30)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + A \quad (34.31)$$

2

34.5 Special Forms

For a function $f(x)$:

$$\int [f(x)]^n f'(x) dx = \begin{cases} \frac{[f(x)]^{n+1}}{n+1} + A, n \neq -1 \\ \ln|f(x)| + A, n = -1 \end{cases} \quad (34.32)$$

² a is a constant $\in \mathbb{R}$

34.5.1 Integration by Part

For two functions $u(x)$ and $v(x)$:

$$\int u(x)v(x)dx = u(x) \left[\int v(x)dx \right] - \int \left[\frac{du(x)}{dx} \left(\int v(x)dx \right) dx \right] \quad (34.33)$$

Chapter 35

Definite Integral

35.1 Definition

For a function $f(x)$ for which $\int f(x)dx = F(x) + A$,

$$\int_a^b f(x)dx = F(b) - F(a) \quad (35.1)$$

35.2 Properties of Definite Integration

$$\int_a^b f(x)dx = \int_a^b f(t)dt \quad (35.2)$$

$$\int_a^b f(x)dx = - \int_b^a f(x)dx \quad (35.3)$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad (35.4)$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx \quad (35.5)$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad (35.6)$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2 \int_0^a f(x)dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = f-(x) \end{cases} \quad (35.7)$$

$$\int_{-a}^a f(x)dx = \begin{cases} 2 \int_0^a f(x)dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd} \end{cases} \quad (35.8)$$

35.3 Approximation

$$f(a)(b-a) \leq \int_a^b f(x)dx \leq f(b)(b-a) \quad (35.9)$$

35.4 Sum of Infinite Series as a Definite Integral

Refer to 3.5.2.

Chapter 36

Reduction Formulae

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx \quad (36.1)$$

$$\int \cos^n x dx = -\frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx \quad (36.2)$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{(n-1) \cdot (n-3) \cdots 3 \cdot 1}{n \cdot (n-2) \cdots 4 \cdot 2} \left(\frac{\pi}{2}\right), n \rightarrow \text{even} \\ \frac{(n-1) \cdot (n-3) \cdots 4 \cdot 2}{n \cdot (n-2) \cdots 3 \cdot 1}, n \rightarrow \text{odd} \end{cases} \quad (36.3)$$

$$\int \sin^m x \cos^n x dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x dx \quad (36.4)$$

For $I(m, n) = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$:

When m and n are both even:

$$I(m, n) = \frac{[(m-1).(m-3) \cdots 3.1][(n-1).(n-3) \cdots 3.1]}{(m+n).(m+n-1) \cdots (4).(2)} \cdot \frac{\pi}{2} \quad (36.5)$$

Otherwise:

$$I(m, n) = \frac{[(m-1).(m-3) \cdots (2 \text{ or } 1)][(n-1).(n-3) \cdots (2 \text{ or } 1)]}{(m+n).(m+n-1) \cdots (2 \text{ or } 1)} \quad (36.6)$$

$$\begin{aligned} I_n &= \int \tan^n x dx \\ \Rightarrow I_n &= \frac{\tan^{n-2} x}{n-1} - I_{n-2} \end{aligned} \quad (36.7)$$

$$\begin{aligned} I_n &= \int \cot^n x dx \\ \Rightarrow I_n &= -\frac{\cot^{n-2} x}{n-1} - I_{n-2} \end{aligned} \quad (36.8)$$

$$\begin{aligned} I_n &= \int \sec^n x dx \\ \Rightarrow I_n &= \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \end{aligned} \quad (36.9)$$

$$\begin{aligned} I_n &= \int \csc^n x dx \\ \Rightarrow I_n &= -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2} \end{aligned} \quad (36.10)$$

$$I_n = \int x^n e^{ax} dx \quad (36.11)$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-2} \quad (36.12)$$

$$I(m, n) = \int x^m (\ln x)^n dx \quad (36.13)$$

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} I_{m, n-1} \quad (36.14)$$

Chapter 37

Multiple Integrals

37.1 Two Variables

For

$$I = \iint_R f(x, y) dx dy \quad (37.1)$$

The following substitution are made:

$$x = g(r, \theta) \quad (37.2)$$

$$y = h(r, \theta) \quad (37.3)$$

$$\therefore dx dy = |J| dr d\theta \quad (37.4)$$

Where J is the Jacobian, defined as:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta y}{\delta r} \\ \frac{\delta x}{\delta \theta} & \frac{\delta y}{\delta \theta} \end{vmatrix} \quad (37.5)$$

The equivalent integral is:

$$I = \iint_{R_1} f(g(r, \theta), h(r, \theta)) |J| dr d\theta \quad (37.6)$$

37.2 Three Variables

For

$$I = \iiint_R f(x, y, z) dx dy dz \quad (37.7)$$

The following substitution are made:

$$x = g(r, \theta, \phi) \quad (37.8)$$

$$y = h(r, \theta, \phi) \quad (37.9)$$

$$z = k(r, \theta, \phi) \quad (37.10)$$

$$\therefore dx dy dz = |J| dr d\theta d\phi \quad (37.11)$$

Where J is the Jacobian, defined as:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta y}{\delta r} & \frac{\delta z}{\delta r} \\ \frac{\delta x}{\delta \theta} & \frac{\delta y}{\delta \theta} & \frac{\delta z}{\delta \theta} \\ \frac{\delta x}{\delta \phi} & \frac{\delta y}{\delta \phi} & \frac{\delta z}{\delta \phi} \end{vmatrix} \quad (37.12)$$

The equivalent integral is:

$$I = \iiint_{R_1} f(g(r, \theta, \phi), h(r, \theta, \phi), k(r, \theta, \phi)) |J| dr d\theta d\phi \quad (37.13)$$

Chapter 38

Differential Equation

38.1 1st Order, 1st Degree Differential Equation

For the equation:

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (38.1)$$

Then an Integral Function (I.F.) is defined as:

$$I.F. = e^{\int P(x)dx} \quad (38.2)$$

Then the solution of the equation 38.1 is given by:

$$y(I.F.) = \int Q(I.F.)dx \quad (38.3)$$

38.2 2nd Order, 1st Degree Differential Equation

For the equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \quad (38.4)$$

OR

$$y'' + ay' + by = 0 \quad (38.5)$$

By substituting $y = e^{\lambda x}$, the equation obtained is:

$$\begin{aligned} \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + b e^{\lambda x} &= 0 \\ \therefore e^{\lambda x} &\neq 0 \\ \Rightarrow \lambda^2 + a\lambda + b &= 0 \end{aligned} \quad (38.6)$$

If α and β are the solutions of the equation 38.6, then the solution of 38.4 can be:

1. If $\alpha = \beta$ and $\alpha, \beta \in \mathbb{R}$:

$$y = (c_1 + c_2x)e^{\alpha x} \quad (38.7)$$

2. If $\alpha \neq \beta$ and $\alpha, \beta \in \mathbb{R}$:

$$y = c_1e^{\alpha x} + c_2e^{\beta x} \quad (38.8)$$

3. If $\lambda = \alpha + i\beta$:

$$y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)] \quad (38.9)$$

38.3 Special Cases of Differential Equation

38.3.1 Definition of Inverse Operator

The operator D is equivalent to $\frac{d}{dx}$. If $Df(x) = X$, then $f(x) = \frac{1}{D}X = \int X dx$.

38.3.2 Special Cases

- 1.

$$f(x) = \frac{1}{D-a}X = e^{ax} \int X e^{-ax} dx \quad (38.10)$$

- 2.

$$\frac{1}{f(D)}e^{ax} = \begin{cases} \frac{e^{ax}}{f(a)}, f(a) \neq 0 \\ x \frac{e^{ax}}{f'(a)}, f(x) = 0 \text{ and } f'(a) \neq 0 \\ x^2 \frac{e^{ax}}{f''(a)}, f(x) = 0 \text{ and } f'(a) = 0 \end{cases} \quad (38.11)$$

- 3.

$$\frac{1}{f(D)}x^m = [f(D)]^{-1}x^m \quad (38.12)$$

$[f(D)]^{-1}$ is expanded and arranged in terms of ascending powers of D and operated on x^m .

4. (a)

$$\begin{aligned} \frac{1}{f(D)} \sin(ax) &= \frac{1}{\phi(D^2)} \sin(ax) \\ &= \frac{1}{\phi(-a^2)} \sin(ax) \end{aligned} \quad (38.13)$$

(b)

$$\begin{aligned}
 \frac{1}{f(D)} \cos(ax) &= \frac{1}{\phi(D^2)} \cos(ax) \\
 &= \frac{1}{\phi(-a^2)} \cos(ax)
 \end{aligned}
 \tag{38.14}$$

5. (a)

$$\begin{aligned}
 \frac{1}{f(D)} \sin(ax) &= \frac{1}{\phi(D^2, D)} \sin(ax) \\
 &= \frac{1}{\phi(-a^2, D)} \sin(ax)
 \end{aligned}
 \tag{38.15}$$

(b)

$$\begin{aligned}
 \frac{1}{f(D)} \cos(ax) &= \frac{1}{\phi(D^2, D)} \cos(ax) \\
 &= \frac{1}{\phi(-a^2, D)} \cos(ax)
 \end{aligned}
 \tag{38.16}$$

6. (a)

$$\begin{aligned}
 \frac{1}{f(D)} \sin(ax) &= \frac{\psi(D)}{\phi(D^2)} \sin(ax) \\
 &= \frac{\psi(D)}{\phi(-a^2)} \sin(ax)
 \end{aligned}
 \tag{38.17}$$

(b)

$$\begin{aligned}
 \frac{1}{f(D)} \cos(ax) &= \frac{\psi(D)}{\phi(D^2)} \cos(ax) \\
 &= \frac{\psi(D)}{\phi(-a^2)} \cos(ax)
 \end{aligned}
 \tag{38.18}$$

7. (a)

$$\frac{1}{f(D)} \sin(ax) = x \frac{1}{f'(D)} \sin(ax)
 \tag{38.19}$$

(b)

$$\frac{1}{f(D)} \cos(ax) = x \frac{1}{f'(D)} \cos(ax)
 \tag{38.20}$$

38.4 Method of Variation of Parameters

If the equation is of the form:

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + by = f
 \tag{38.21}$$

where a, b, f are functions of x . The solution for 38.21 is obtained by solving for:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \quad (38.22)$$

If y_1 and y_2 are the two independent solution of equation 38.22. Then the general solution of the equation is:

$$y = c_1y_1 + c_2y_2 \quad (38.23)$$

where c_1 and c_2 are the constants.

The particular solution of equation 38.22 will be:

$$y = y_1 \left(\int \frac{y_2(-f)}{W} dx \right) + y_2 \left(\int \frac{y_1 f}{W} dx \right) \quad (38.24)$$

W is the Wronskian, which is defined by:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad (38.25)$$

38.5 Singular and Ordinary Point

For a differential equation:

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = R(x) \quad (38.26)$$

where $P_0 \cdots P_n$ are functions of x .

If at a point $x = x_0$:

1. $P_0(x_0) \neq 0$, x_0 is an ordinary point.
2. $P_0(x_0) = 0$, x_0 is an singular point:

(a)

$$\lim_{x \rightarrow x_0} (x - x_0) P_1(x) = c_1 \quad (38.27)$$

$$\lim_{x \rightarrow x_0} (x - x_0)^2 P_2(x) = c_2 \quad (38.28)$$

$$(38.29)$$

where c_1 and c_2 are both finite quantities x_0 is a regular singular point.

- (b) otherwise it is an irregular singular point.

Chapter 39

Beta and Gamma Functions

For $m, n > 0$:

$$\begin{aligned}\beta(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ &= 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x dx\end{aligned}\tag{39.1}$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx\tag{39.2}$$

39.1 Important Relations between $\beta(m, n)$ and $\Gamma(n)$ Functions

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}\tag{39.3}$$

$$\Gamma(1) = 1\tag{39.4}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \approx 1.772\tag{39.5}$$

$$\Gamma(n+1) = n!, n \in \mathbb{N}\tag{39.6}$$

$$\Gamma(m)\Gamma\left(m + \frac{1}{2}\right) = \sqrt{\pi}\Gamma(2m)\tag{39.7}$$

$$\Gamma(m)\Gamma(m-1) = \pi \csc(m\pi)\tag{39.8}$$

$$\beta(m, n) = \beta(n, m) \quad (39.9)$$

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (39.10)$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi \quad (39.11)$$

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x = \frac{1}{2} \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q}{2})} \quad (39.12)$$

$$(39.13)$$

Chapter 40

Laplace Transformations

The Laplace Transformation of a function $f(t)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{x \rightarrow \infty} \int_0^x e^{-st} f(t) dt \tag{40.1}$$

40.1 Basic Transformations

| $f(t)$ | $F(s)$ |
|------------------|---------------------------------|
| $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| 1 | $\frac{1}{s}$ |
| t | $\frac{1}{s^2}$ |
| t^n | $\frac{n!}{s^{n+1}}$ |
| e^{at} | $\frac{1}{s - a}$ |
| $\cos(\omega t)$ | $\frac{s}{s^2 + \omega^2}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2 + \omega^2}$ |
| $\sinh at$ | $\frac{s}{s^2 - a^2}$ |
| $\cosh at$ | $\frac{s}{s^2 - a^2}$ |

Table 40.1: Table of Laplace Transformations

40.2 Important Relations

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \quad (40.2)$$

$$\mathcal{L}\{tf(t)\} = -F'(s) \quad (40.3)$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \quad (40.4)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \lim_{x \rightarrow \infty} \int_s^x F(u) du \quad (40.5)$$

$$\mathcal{L}\left\{\frac{f(t)}{t^n}\right\} = \lim_{x \rightarrow \infty} \int_1 \int_2 \cdots \int_s^x F(u) du \cdots du \quad (40.6)$$

40.3 Convolution

For two functions $f(t)$ and $g(t)$ be given such that their Laplace transforms are $F(s)$ and $G(s)$, then:

$$\mathcal{L}\{f(t) \star g(t)\} = F(s)G(s) \quad (40.7)$$

where $f(t) \star g(t)$ is defined as:

$$\int_0^t f(u)g(t-u)du \quad (40.8)$$

40.4 Laplace Transforms of Differentials

If the Laplace Transform of $f(t)$ is $F(s)$ ¹:

$$\mathcal{L}\{f'(t)\} = sF(s) - y(0) \quad (40.9)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - [sy(0) + y'(0)] \quad (40.10)$$

$$\vdots$$

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - \left[\sum_{i=0}^{n-1} s^{n-i} y^{(i)}(0) \right] \quad (40.11)$$

¹Used in initial value problems

Part VI

Operations Research

Chapter 41

Linear Programming Problems

41.1 Basic Feasible Solution

The standard LPP problem has an objective function and conditions.

$$\begin{aligned}
 Z &= a_1x_1 + a_2x_2 + \cdots + a_nx_n \\
 &\text{Subject to:} \\
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
 \end{aligned}$$

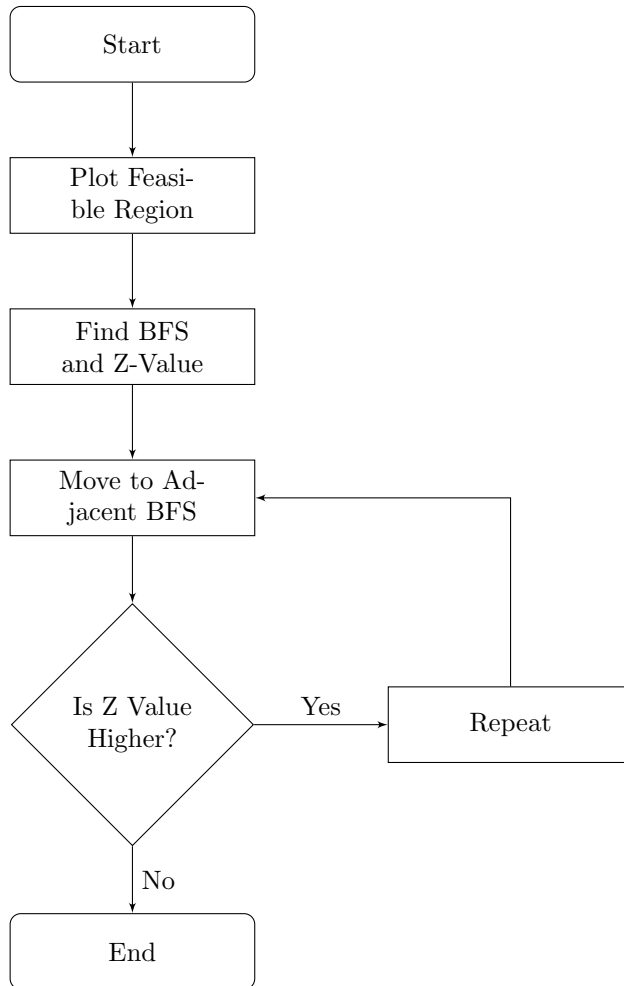
For a system with n variables and m conditions, the number of basic solutions are: $\binom{n}{m}$. For any $n - m$ system there are $n - m$ non-basic variables (NBV) and m basic variables (BV).

For the above system, the basic solutions are obtained by:

| NBV | BV | BFS |
|---------------------------------|--------------------------------------|--|
| $x_1, x_2, \cdots, x_{n-m} = 0$ | $x_{n-m+1} = c_1, \cdots, x_n = c_n$ | If $x_{n-m+1}, \cdots, x_n \geq 0$ then it is a basic feasible solution. |
| \vdots | \vdots | |

41.1.1 Adjacent Basic Feasible Solutions

If two adjacent BFS share $m - 1$ BV then they are called adjacent variables. The optimal solution is always an extreme point. Thus, graphically:



41.2 Simplex Method