1. Algebra

Chapter 1 Permutation and Combination

1.1 Permutation

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Basic Formulae and Key Points

1. **Basic Principle of Counting (Fundamental Principles of Counting)**
   i) Additive principle of counting (Rule of sum)
   ii) Multiplicative principle of counting (Rule of product)
   i) **Additive Principle of Counting:** If the first task can be performed in n_1 different ways and the second in n_2 different ways, if these tasks cannot be performed simultaneously, then there are $n_1 + n_2$ ways of doing either task.

Note: The sum or additive rule may be extended to more than two tasks. Thus, if there are n non-simultaneous tasks $t_1, t_2, t_3, ..., t_n$ which can be performed in $m_1, m_2, m_3, ..., m_n$ ways respectively, then the numbers of ways of doing one of these tasks are $m_1 + m_2 + ... + m_n$.

   ii) **Multiplicative Principle of Counting:** If the first task can be performed in n_1 different ways and the second task can be performed in n_2 different ways after the first task is done, then the total number of ways of completing the task is $n_1 \times n_2$.

Note: The product or multiplicative rule also can be extended to more than two tasks. If a task consists of n steps $s_1, s_2, s_3 ...., s_n$ which can be performed simultaneously one after another in $m_1, m_2, m_3, ...,$ m_n ways respectively, then the number of ways of completing the task is $m_1 \times m_2 \times m_3 \times .... \times m_n$.

2. **Factorial Notation**

   Factorial of a natural numbers n is the product of the first n natural numbers. It is denoted by **n!** and defined as

   $n! = n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1.$

   Note: $0! = 1$

3. **Permutation When all Objects are Different**

   Permutation of n different objects taken r at a time is given by

   $${}^nP_r = P(n, r) = \frac{n!}{(n-r)!}, n \geq r.$$

4. **Permutation When all Objects are not Distinct.**

   Permutation of n objects where,

   P_1 objects are of one kind,

   P_2 objects are of second kind,

   P_3 objects are of third kind and so on, is given by $\frac{n!}{P_1! \, P_2! \, P_3! \, ...}$

5. Circular Permutation

There are two cases of circular permutation:

i) Number of circular permutations of n distinct objects taken all at a time is $(n-1)!$

ii) If clock-wise and anti cock-wise arrangements are taken as not different, then the total number of circular permutation is given by $\frac{(n-1)!}{2}$.

6. Permutation When the Repetition of Object is Allowed (Permutation with Repetition)

Number of permutation of n objects taken r at a time when the repetition of objects is allowed is n^r.

7. Properties of Permutation

i) $^nP_n = P(n, n) = n!$

ii) $^nP_1 = P(n, 1) = n$

iii) $^nP_0 = P(n, 0) = 1$

iv) $^nP_{n-1} = P(n, n-1) = n!$

Group 'A' (Multiple Choice Questions and Answers)

1. *In how many ways can 'r' letter be posted in 'n' letter box $(n \geq r)$?* *[2080 GIE]*
   (a) $n \times r$ (b) n^r (c) $\frac{n!}{r!}$ (d) $\frac{n!}{(n-r)!}$

2. *What is the number of arrangements of n different colored beads that can be strung on a necklace?* *[2080 (Optional)]*
   (a) $n \times (n-1) \times (n-2) \times ..... \times 2 \times 1$ (b) $n!$
   (c) $(n-1)!$ (d) $\frac{(n-1)!}{2}$

3. *How many three-digit numbers can be formed by using the integers 4, 5, 6, 8 with repetition?* *[2079 Set J]*
   (a) 64 (b) 27 (c) 24 (d) 6

4. *What is the number of permutations of n different things, taken r at a time while each thing may be repeated any number of times in any permutation?* *[Model 2080]*
   (a) $n!$ (b) n^r (c) $(n-1)!$ (d) $\frac{n!}{(n-r)!}$

5. *The permutation of 'n' things taken 'r' at a time when each things may occur any numbers of times is ....* *[2081 Set V]*
   (a) n ways (b) r^n ways (c) n^r ways (d) $(n \times r)$ ways

6. *What is an arrangement of the n natural numbers called?* *[Model 2079]*
   (a) Induction (b) Permutation (c) Combination (d) Expectation

7. *What is the number of circular permutations of n things taken all at a time?* *[2080 Technical Group]*
   (a) $n!$ (b) $(n-1)!$ (c) $\frac{1}{2}(n-1)!$ (d) $\frac{1}{2}(n)!$

8. *In how many ways can the letters of the word "ELEMENT" be arranged?*
   (a) 640 (b) 740 (c) 840 (d) 1040

9. *The number of ways that 7 beads of different colours can be strung together so as to form a necklace is*
   (a) 5040 (b) 2520 (c) 720 (d) 360

10. *In how many ways 8 people and a host be seated in circular table of a party?*
    (a) 504 (b) 9! (c) 7! (d) 8!

11. *In how many ways 5 persons occupy 3 vacant chairs in a row?*
    (a) 25 (b) 9 (c) 60 (d) 8

12. *Find the number of ways in which the letters A, B, C can be arranged in a row with repetition.*
    (a) 6 (b) 9 (c) 12 (d) 27

13. *If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of occurrence of the event in the given order is*
    (a) $m + n$ (b) $m - n$ (c) $m \times n$ (d) $\frac{m}{n}$

14. $^nP_0$ = ...
    (a) $n!$ (b) 1 (c) $\frac{1}{n!}$ (d) $(n-1)!$

15. *In how many ways 10 identical keys can be arranged in a ring?*
    (a) 9 (b) $\frac{9!}{2}$ (c) 10! (d) $10 \times 9!$

16. *At a dinner party, 6 men and 4 women sit at a round table. In how many ways can they sit?*
    (a) 11! (b) 8! (c) 9! (d) 10!

17. *If clock-wise and anti-clock-wise orders are taken as not different then the total numbers of circular permutations is given by*
    (a) $n!$ (b) $(n-1)!$ (c) $(n-1)! \times 2$ (d) $\frac{1}{2}(n-1)!$

Answer Key

1. b	2. d	3. a	4. b	5. c	6. b	7. b	8. c	9. d	10. d
11. c	12. d	13. c	14. b	15. b	16. c	17. d			

Group 'B' or 'C' (Subjective Questions and Answers)

1. *In how many ways can the letters of the word 'INTERVAL' be arranged so that:* *[Model - 2080]*
   (a) *All vowels are always together?* *[1]*
   (b) *The relative positions of the vowels and consonants are not changed?* *[1]*
   (c) *The vowels may occupy only the odd positions?* *[1]*

Soln: (a) There are 8 letters in the word 'INTERVAL' in which 3 are vowels and 5 are consonants.

If all the vowels are always together then, we have to consider 3 vowels as one, then there are 6 letters (IEA), N, T, R, V, L. They can be arranged in 6! ways.

Also, 3 vowels can be arranged among themselves in 3 ! ways.

$\therefore$ Total no. of arrangements when vowels come always together $= 6! \times 3!$

$= 720 \times 6$

$= 4320$

(b) If the relative positions of the vowels and consonants are not changed then, 3 vowels can be arranged in 3! ways and the 5 consonants can be arranged in 5! ways.

$\therefore$ The total no. of arrangements $= 3! \times 5!$

$= 6 \times 120 = 720$

(c) If the vowels may occupy only the odd positions then,
there are 3 vowels and 4 odd positions for them.

So, they can be arranged in P(4, 3) ways = $\frac{4!}{(4-3)!} = 4!$

Also, the remaining 5 letters (N, T, R, V, L) can be arranged in P(5, 5) ways = 5!

∴ The total no. of arrangements that all vowels may occupy odd positions = 4! × 5!
= 24 × 120 = 2880

2. *From the given integers 2, 3, 4, 5, 6*

(a) Write the formula for finding permutation of n different objects taken r at a time ($r \le n$). ***[1] [2080 GIE Set B Optional]***

(b) How many numbers of three digits can be formed? ***[1]***

(c) How many of them are even by using all the figures only once? ***[1]***

(d) How many of them will be divisible by 5? ***[1]***

Soln: (a) The number of permutation of n different objects taken r at a time is given by

$^nP_r = \frac{n!}{(n-r)!}$ Where, $r \le n$.

(b) Here,
Total no. of integers (n) = 5
No. of digits to be taken (r) = 3

Now, out of 5 integers, 3 integers can be taken in P(5, 3) ways = $\frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$

∴ Total number of three digit numbers = 60.

(c) Here, the numbers formed must be all even. So, the digit in the unit's place must be either 2 or 4 or 6. Hence, the unit place can be filled up in 3 ways and the remaining 2 places can be filled up in P(4, 2) ways.

∴ Total number of even numbers = 3 × P(4, 2)

$= 3 \times \frac{4!}{(4-2)!} = 3 \times \frac{4!}{2!} = 3 \times 12 = 36$ **Ans.**

(d) For the numbers divisible by 5, the unit place can be filled in 1 way and the remaining two places can be filled up in P(4, 2) ways.

∴ The no. of three digit numbers which are divisible by 5 = 1 × P(4, 2)

$= 1 \times \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$ **Ans.**

3. *How many three digit numbers can be formed from the digits 2, 3, 4, 5, 6, 7?* ***[2] [2080 Set G]***

Soln: Here, given digits are 2, 3, 4, 5, 6, 7

∴ Total no. of digits (n) = 6
No. of digits to be taken (r) = 3

Now, the required number of three digits numbers that can be formed = $^6P_3$

$= \frac{6!}{(6-3)!}$

$= \frac{6 \times 5 \times 4 \times 3!}{3!}$

= 120 **Ans.**

4. *Find the number of ways of arranging the letters in the word "objective".* ***[2] [2080 Set I]***

Soln: There are 9 letters in the word 'objective' in which 'e' occurs two times and rest are single.

So, We have,
n = 9, p = 2

∴ Required number of arrangements = $\frac{n!}{p!} = \frac{9!}{2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2!} = 181440$ **Ans.**

5. *There are 7 roads connecting town A and B and 5 roads connecting town B and C.*

(a) In how many ways can a man drive from A to C via B? ***[1]***

(b) In how many ways can he return from C to A, passing through B without driving the same road twice? ***[1]***

(c) If a man is allowed to drive the same road twice then find the number of ways that he can trip from A to C and return, passing through B on both trips. ***[1]***

Soln: (a) A man can drive from town A to B in 7 ways. Again, he can drive from town B to C in 5 ways. So, by basic principle of counting he can drive from town A to C = 7 × 5 = 35 ways.

(b) As he can not use the same road twice, so he can return from town C to town B in 4 ways and from B to A in 6 ways.

∴ The total no. of ways = 4 × 6 = 24 **Ans.**

(c) A man can drive from town A to B in 7 ways and from town B to C in 7 ways. Since, the man can use the same road twice, so without any restriction he can return from town C to B in 5 ways and from town B to A in 7 ways.

∴ The total no. of ways = 7 × 5 × 5 × 7 = 1225 **Ans.**

6. *The letters of the word "COMPLETE" are to be permuted to form different words.*

(a) What is the meaning of ^nP_r? ***[1]***

(b) In how many ways the letters of the word COMPLETE can be arranged so that the repeated letters are always together? ***[2]***

Soln: (a) **Meaning of ^nP_r :** The numbers of permutation of n different things, taking r objects at a time is denoted by ^nP_r and is given by $^nP_r = P(n, r) = \frac{n!}{(n-r)!}$, $n \ge r$.

(b) There are 8 letters in the word 'COMPLETE' in which E repeats 2 times. So, consider 2E's as a single letter, then there are 7 letters. C, O, M, P, L, T, (E, E). They can be arranged in P (7, 7) ways = 7!
= 5040 **Ans.**

7. *The four digits numbers are to be formed from the integers 0, 1, 2, 3, 4 with the help of fundamental principle of counting.*

(a) What is the fundamental principle of counting? ***[1]***

(b) How many such, numbers can be formed? ***[1]***

Soln: (a) The fundamentals principle of counting is a rule to determine the total numbers of possible outcomes for a given situation.

(b) To form 4 digits numbers by using the digits 0, 1, 2, 3, 4, we can not place 0 in the thousand's place. So, there are 4 choices (1, 2, 3 and 4), in thousand's position. Similarly, there are 4 choices in the hundred's position, 3 choices in the ten's position and 2 choices in the unit's position respectively.
So, by fundamental principle of counting the total no. of ways = 4 × 4 × 3 × 2
= 96

∴ The total no. of 4 digits numbers = 96 **Ans.**

(c) If the vowels may occupy only the odd positions then, there are 3 vowels and 4 odd positions for them.

So, they can be arranged in P(4, 3) ways = $\frac{4!}{(4-3)!} = 4!$

Also, the remaining 5 letters (N, T, R, V, L) can be arranged in P(5, 5) ways = 5!

$\therefore$ The total no. of arrangements that all vowels may occupy odd positions = 4! × 5!

$= 24 \times 120 = 2880$

2. *From the given integers 2, 3, 4, 5, 6*

(a) Write the formula for finding permutation of n different objects taken r at a time ($r \le n$). *[1] [2080 GIE Set B Optional]*

(b) How many numbers of three digits can be formed? *[1]*

(c) How many of them are even by using all the figures only once? *[1]*

(d) How many of them will be divisible by 5? *[1]*

Soln: (a) The number of permutation of n different objects taken r at a time is given by

$^nP_r = \frac{n!}{(n-r)!}$ Where, $r \le n$.

(b) Here,

Total no. of integers (n) = 5

No. of digits to be taken (r) = 3

Now, out of 5 integers, 3 integers can be taken in P(5, 3) ways = $\frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$

$\therefore$ Total number of three digit numbers = 60.

(c) Here, the numbers formed must be all even. So, the digit in the unit's place must be either 2 or 4 or 6. Hence, the unit place can be filled up in 3 ways and the remaining 2 places can be filled up in P(4, 2) ways.

$\therefore$ Total number of even numbers $= 3 \times P(4, 2)$

$= 3 \times \frac{4!}{(4-2)!} = 3 \times \frac{4!}{2!} = 3 \times 12 = 36$ **Ans.**

(d) For the numbers divisible by 5, the unit place can be filled in 1 way and the remaining two places can be filled up in P(4, 2) ways.

$\therefore$ The no. of three digit numbers which are divisible by 5 $= 1 \times P(4, 2)$

$= 1 \times \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$ **Ans.**

3. *How many three digit numbers can be formed from the digits 2, 3, 4, 5, 6, 7?* *[2] [2080 Set G]*

Soln: Here, given digits are 2, 3, 4, 5, 6, 7

$\therefore$ Total no. of digits (n) = 6

No. of digits to be taken (r) = 3

Now, the required number of three digits numbers that can be formed = $^6P_3$

$= \frac{6!}{(6-3)!}$

$= \frac{6 \times 5 \times 4 \times 3!}{3!}$

$= 120$ **Ans.**

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(b) As he can not use the same road twice, so he can return from town C to town B in 4 ways and from B to A in 6 ways.

$\therefore$ The total no. of ways = 4 × 6 = 24 **Ans.**

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Soln: (a) The fundamentals principle of counting is a rule to determine the total numbers of possible outcomes for a given situation.

(b) To form 4 digits numbers by using the digits 0, 1, 2, 3, 4, we can not place 0 in the thousand's place. So, there are 4 choices (1, 2, 3 and 4), in thousand's position. Similarly, there are 4 choices in the hundred's position, 3 choices in the ten's position and 2 choices in the unit's position respectively.

So, by fundamental principle of counting the total no. of ways = 4 × 4 × 3 × 2

= 96

$\therefore$ The total no. of 4 digits numbers = 96 **Ans.**

8. *The letters of the word 'STRANGE' are to be shuffled to form different words such that*

(a) The vowels come together. [1]

(b) The arrangements begin with S. [1]

(c) The arrangements begin with S and end with E. [1]

(d) The vowels occupy the odd positions. [1]

Soln: There are 7 letters in the word 'STRANGE', including 2 vowels (A, E) and 5 consonants (S, T, R, N, G).

(a) Let. us consider 2 vowels that always come together as one. Then there are 6 letters which can be arranged in P(6, 6) ways = 6 ! = 720

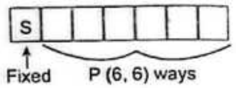
Also, those 2 vowels can be arranged among themselves in P(2, 2) ways = 2 ! = 2

∴ Total no. of words that the vowels come together = 720 × 2

= 1440 Ans.

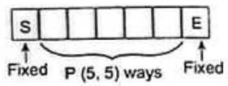
(b) To find the arrangements that begin with S, we have to fix S at first place. Then the remaining 6 letters can be arranged in P(6, 6) ways = 6 !

= 720



(c) To find the arrangements that beginning with S and ending with E, We have to fix S at the first place and E at the last place. So, the remaining 5 letters can be arranged in P(5, 5) ways = 5 !

= 120 **Ans.**

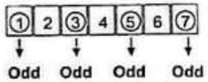


(d) There are 7 letters in the word 'STRANGE' in which 2 are vowels and 4 odd positions (1^{st}, 3^{rd}, 5^{th}, 7^{th}).

So, 2 vowels in 4 odd positions can be arranged in P(4, 2) ways = $\frac{4!}{(4-2)!}$

$= \frac{4!}{2!}$

$= \frac{24}{2} = 12$



Also, the remaining 5 consonants can be arranged in P(5, 5) ways = 5 ! = 120

∴ The total no. of arrangements = 12 × 120 = 1440 ways. **Ans.**

9. *How many 3- digit numbers can be formed from the digits, 1, 2, 3, 4 and 5 assuming that:*

(a) Repetition of the digits is allowed? [1]

(b) Repetition of the digits is not allowed? [1]

(c) All numbers are even? [1]

Soln: (a) Here, the unit's place can be filled by any one of the digits 1, 2, 3, 4 and 5. So, the unit place can be filled in 5 ways. Since, the repetition of digits is allowed, so the ten's and hundred's places can be filled up in 5 ways each respectively.

∴ The total numbers of 3- digits numbers = $5^r = 5^3 = 125$

(b) Here, the unit's place can be filled in 5 ways and since repetition of the digits is not allowed, so there are 4 and 3 choices for the digits in ten's and hundred's places respectively.

∴ Total number of 3- digit numbers = 5 × 4 × 3 = 60

(c) Since, the formed numbers must be even. So, the digit in the unit's place must be either 2 or 4. So, for the digits in unit's place, there are 2 choices and 4 choices in ten's place and 3 choices in hundred's places respectively.

∴ The total numbers of 3- digit even numbers = 2 × 4 × 3 = 24 **Ans.**

10. *In how many ways can four boys and three girls be seated in a row containing seven seats,*

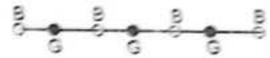
(a) If they may sit anywhere? [1]

(b) If the boys and girls must alternate? [2]

(c) If all three girls are together? [1]

Soln: (a) If the boys and girls may sit anywhere, then there are 7 persons and 7 seats. Without any restriction they can be arranged in P(7, 7) ways = 7 ! = 5040 ways. **Ans.**

(b) If boys and girls must alternates, then 4 boys have to be arranged in 4 odd seats (i.e. 1^{st}, 3^{rd}, 5^{th} and 7^{th}) and 3 girls in 3 even seats (i.e. 2^{nd}, 4^{th} and 6^{th}) as shown in the picture.



Now, 4 boys in 4 seats can be arranged in P(4, 4) ways = 4 ! = 4 × 3 × 2 × 1 = 24 ways

Also, 3 girls in 3 seats can be arranged in P(3, 3) ways = 3 ! = 3 × 2 × 1 = 6 ways

∴ Total no. of arrangements = 24 × 6

= 144 ways **Ans.**

(c) If all three girls sit together in a row, we consider three girls as one. Then there are 4 + 1 = 5 persons altogether.

Now, 5 persons in 5 seats can be arranged in P(5, 5) ways = 5 !

= 5 × 4 × 3 × 2 × 1

= 120 ways

Again,

3 girls in 3 seats can be arranged among themselves in P(3, 3) ways = 3 !

= 3 × 2 × 1

= 6 ways.

∴ Total no. of arrangements = 120 × 6 = 720 **Ans.**

11. *The letters of the word 'MATHEMATICS" are to be arranged to form different words with or without meanings so that,*

(a) How many different words can be formed? [1]

(b) In how many words the vowels occur together? [2]

Soln: (a) There are 11 letters in the word 'MATHEMATICS' in which 'M' occurs 2 times, 'A' occurs 2 times, 'T' occurs 2 times and the rest are single. So,

We have,

n = 11, p = 2, q = 2, r = 2

∴ The required no. of arrangements = $\frac{n!}{p!\,q!\,r!}$

$= \frac{11!}{2!\,2!\,2!}$ **Ans.**

(b) There are 4 vowels (A, E, A, I) and 7 consonants (M, T, H, M, T, C, S).

Now, we have to consider 4 vowels as one, then there are 8 letters in which 'M' comes 2 times and 'T' comes 2 times. So, they can be arranged in $\frac{8!}{2!\,2!}$ = 10080 ways.

Again, 4 vowels can be arranged in $\frac{4!}{2!}$ ways.

= 12 ways.

∴ The total no. of ways that vowels occur together = 10080 × 12 = 120960 **Ans.**

12. ***In how many ways can 4 Art students and 4 Science students be arranged in a circular table if***

(a) They may sit anywhere *[1]*

(b) They sit alternatively *[1]*

(c) All Art students sit together. *[1]*

Soln: (a) If they may sit anywhere then 8 students in a circular table can be arranged in (8 – 1)! ways

$= 7!$

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5040$ **Ans.**

(b) If they have to sit alternatively then 4 Art students in a circular table can be arranged in (4 – 1) ! ways

$= 3!$

$= 3 \times 2 \times 1$

$= 6$ ways

Again, 4 Science students can be arranged in 4 seats in P(4, 4) ways $= 4!$

$= 4 \times 3 \times 2 \times 1 = 24$ ways

∴ Total no. of ways $= 6 \times 24 = 144$ ways. **Ans.**

(c) If all Art students are sit together then we have to consider them as one.

Then there are 5 (i.e. 1 + 4) students altogether.

5 students in a round table can be arranged in (5 – 1) ! ways.

$= 4!$

$= 4 \times 3 \times 2 \times 1$

$= 24$ ways.

Again, those 4 Art students can be arranged among themselves in P(4, 4) ways $= 4!$

$= 4 \times 3 \times 2 \times 1$

$= 24$ ways.

∴ The total no. of ways $= 24 \times 24 = 576$ **Ans.**

13. ***There are 8 boys on the ground and planning to play a game in a circle,***

(a) How many circular permutations are possible with 'n' distinct objects? *[1] [2081 Set E/W]*

(b) In how many ways can they sit in a circle? *[1]*

(c) If their circular positions are not taken as different then in how many ways can they be arranged? *[1]*

Soln: (a) There are (n – 1)! circular permutations for 'n' distinct objects.

(b) Here, n = 8

Now,

The total no. of ways that they can sit in a circle $= (n - 1)!$

$= (8 - 1)!$

$= 7!$

$= 5040$ ways. **Ans.**

(c) If their circular positions are not taken as different then their clockwise and anticlockwise arrangements are considered the same.

So, this can be done in $\frac{(n-1)!}{2}$ ways.

$$= \frac{(8-1)!}{2} = \frac{7!}{2} = \frac{5040}{2} = 2520 \text{ Ans.}$$

14. ***In how many ways can the letters of the word 'LAPTOP' be arranged so that***

(a) The vowels never be separated? *[1]*

(b) All the consonants may not be together? *[1]*

(c) They always begin with L and end with T? *[1]*

(d) They do not begin with L but always end with T? *[1]*

Soln: (a) If the vowels never be separated that means all the vowels of the word 'LAPTOP' should always come together.

Now, considering 2 vowels as one then there are 5 letters L, (AO), P, T, P in which P occurs twice.

They can be arranged in $\frac{5!}{2!} = \frac{120}{2 \times 1} = 60$ ways.

Again, 2 vowels can be arranged among themselves in P(2, 2) ways $= 2!$

$= 2 \times 1$

$= 2$ ways.

∴ Total no. of arrangements that vowels always come together $= 60 \times 2$

$= 120$ **Ans.**

(b) If all the consonants may not be together that means all the consonants of the word 'LAPTOP' should be separated.

Now, considering 4 consonants as one, then there are 3 letters (L, P, T, P), A, O. They can be arranged in P(3, 3) ways $= 3!$

$= 6$

Also, 4 consonants among themselves can be arranged in $\frac{4!}{2!} = \frac{24}{2} = 12$ **[∵ P occurs two times]**

∴ The total no. of arrangements that all consonants come together $= 6 \times 12$

$= 72$

Again,

The total no. of arrangement of the word 'LAPTOP' $= \frac{6!}{2!} = \frac{720}{2} = 360$

Now,

The required no. of arrangements that consonants may not come together $= 360 - 72$

$= 288$ **Ans.**

(c) To find the arrangements that begin with L and end with T. We have to fix L at first position and T at the last positions then the remaining 4 letters can be arranged in $= \frac{4!}{2!}$

$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1}$ **[∵ P occurs 2 times]**

$= 12$ ways.

(d) To find the arrangements that do not begin with 'L' we find the arrangements that begin with 'L'.

For this, we fix 'L' at first place. Then, remaining 5 letters can be arranged in $\frac{5!}{2!}$

$= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$ **[∵ P occurs 2 times]**

$= 60$ ways.

Now, the numbers of arrangements that do not begin with L but end with T $= 60 - 12 = 48$ **Ans.**

1.2 Combination

Basic Formulae and Key Points

1. The total number of combination of n different objects taken r at a time is given by $C(n, r) = \frac{n!}{(n-r)!\,r!}$, $n \geq r$

2. Restricted Combinations

The numbers of combinations of 'n' different things taken 'r' at a time in which

i) p particular things will never occur = ${}^{n-p}C_r$

ii) p particular things will always occur = ${}^{n-p}C_{r-p}$

3. Properties

i) ${}^nC_0 = {}^nC_n = 1$ ii) ${}^nC_1 = n$

iii) ${}^nC_2 = \frac{n(n-1)}{2!}$ iv) ${}^nC_x = {}^nC_y$ then either $x = y$ or $x + y = n$

v) ${}^nC_r = {}^nC_{n-r}$ vi) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Group 'A' (Multiple Choice Questions and Answers)

1. *The number of combination of n things taken 'r' at a time is ......* [2081 Set E/W]

(a) $\frac{n!}{r!}$ (b) $\frac{n!}{(n-r)!}$ (c) $\frac{n!}{r!\,(n-r)!}$ (d) $\frac{n!}{r(n-r)!}$

2. *What is the combination of 'n' objects taken 'r' at a time ($r \leq n$)?* [2080 GIE Set B]

(a) $\frac{n!}{r!}$ (b) $\frac{n!}{(n-r)!}$ (c) $\frac{n!}{(n-r)!\,r!}$ (d) $\frac{n!}{(n-r) \times r!}$

3. *A combination containing 'k' objects chosen from a set of 'n' objects is denoted by C(n, k). C(n, k) exists for all.* [2079 GIE Set B]

(a) $k > n$ (b) $n < k$ (c) $k \leq n$ (d) $n \leq k$

4. *The symbol for the number of combinations when r items are selected from n distinct items is*

(a) ${}^nP_r$ (b) $\frac{n!}{r!}$ (c) ${}^nC_r$ (d) $n!\,r!$

5. *Out of 9 people, a committee of 5 people can be formed in*

(a) 120 ways (b) 126 ways

(c) 70 ways (d) P(9, 5) ways

6. *Find the value of r if ${}^9C_{2r} = {}^9C_{3r-1}$*

(a) 1 or 3 (b) 2 or 3 (c) 3 or 4 (d) 1 or 2

7. *How many different teams of 3 players can be chosen from 10 players?*

(a) 120 (b) 240 (c) 80 (d) 72

8. *20 men handshake with each other without repetition. What is the total number of handshakes made?*

(a) 190 (b) 210 (c) 150 (d) 250

9. *The numbers of ways in which a student can choose 5 courses out of 9 courses if 2 courses are compulsory is*

(a) 32 (b) 33 (c) 34 (d) 35

10. *From a group of 10 persons, in how many ways can a selection of 4 persons to be made such that a particular person is always excluded?*

(a) 120 (b) 320 (c) 126 (d) 30

11. *A man has 5 friends. In how many ways can he invite one or more of them to a dinner?*

(a) ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$ (b) ${}^5C_1 \times {}^5C_2 \times {}^5C_3 \times {}^5C_4 \times {}^5C_5$

(c) C(5, 5) (d) ${}^5P_5$

12. *In an examination, a candidate has to pass in each of the four subjects. In how many ways can the candidates fail?*

(a) 10 (b) 12 (c) 15 (d) 17

13. *A committee is to be chosen from 12 men and 8 women and is to consist of 3 men and 2 women. How many committees can be formed?*

(a) C(12, 3) + C(8, 2) (b) C(12, 3) × C(8, 2)

(c) C (20, 5) (d) C (15, 10)

14. *Find the numbers of ways of selecting 9 balls from 6 red balls 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.*

(a) 40 (b) 400 (c) 100 (d) 2000

Answer Key

1. c	2. c	3. c	4. c	5. b	6. d	7. a	8. a	9. d	10. c
11. a	12. c	13. b	14. d						

Group 'B' or 'C' (Subjective Questions and Answers)

1. *In a group of 12 students, 8 are boys and remaining girls. In how many ways can 5 students be selected for quiz competition so as to include at most three girls?* [3] [2081 Set E/W]

Soln: Here,

Total no. of students = 12

No. of boys = 8

No. of girls = 12 − 8 = 4

The selection of at most three girls can be made as follows:

Girls (4)	Boys (8)	Selection
3	2	C(4, 3) × C(8, 2)
2	3	C(4, 2) × C(8, 3)
1	4	C(4, 1) × C(8, 4)
0	5	C(4, 0) × C(8, 5)

Total no. of ways = C(4, 3) × C(8, 2) + C(4, 2) × C(8, 3) + C(4, 1) × C(8, 4) + C(4, 0) × C(8, 5)

$$= \frac{4!}{1!\,3!} \times \frac{8!}{6!\,2!} + \frac{4!}{2!\,2!} \times \frac{8!}{5!\,3!} + \frac{4!}{3!\,1!} \times \frac{8!}{4!\,4!} + \frac{4!}{4!\,0!} \times \frac{8!}{3!\,5!}$$

$= 4 \times 28 + 6 \times 56 + 4 \times 70 + 1 \times 56$

$= 112 + 336 + 280 + 56$

$= 784$

2. ***In an examination, a candidate has to pass in each of the five subjects. How many ways can he fail?*** [2] [2080 GIE Set B]

Soln: The candidate can fail by failing in 1 or 2 or 3 or 4 or 5 subjects out of 5 in each case.

∴ The total number of ways in which he can fail = $^5C_1 + {^5C_2} + {^5C_3} + {^5C_4} + {^5C_5}$

$$= \frac{5!}{4!\,1!} + \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} + \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!}$$

$$= 5 + 10 + 10 + 5 + 1$$

$$= 31 \textbf{ Ans.}$$

3. ***A box contains 8 blue socks and 6 red socks. Find the number of ways two socks can be drawn from the box at random if:*** [2]

(a) They can be any color.

(b) They must be same color. [2] [2080 Set-T]

Soln: (a) Here, the total no. of socks in the box = 8 + 6 = 14

Now, 2 socks can be chosen from 14 socks in $^{14}C_2$ ways $= \frac{14!}{(14-2)!\,2!} = \frac{14!}{12!\,2!} = \frac{14 \times 13 \times 12!}{12!\,2!} = 91$

So, there are 91 ways to draw two socks of any color. **Ans.**

(b) There are two cases of selecting 2 socks of the same color :

Case I : Both socks are blue

2 blue socks from 8 blue socks can be chosen in $^8C_2$ ways $= \frac{8!}{6!\,2!} = \frac{8 \times 7 \times 6!}{6!\,2!} = 28$

Case II : Both socks are red

2 red socks from 6 red socks can be chosen in $^6C_2$ ways $= \frac{6!}{4!\,2!} = \frac{6 \times 5 \times 4!}{4!\,2!} = 15$

∴ So, there are 43 ways to draw two socks of the same color. **Ans.**

4. ***A woman has 8 close friends. Find the ways she can invite 3 to dinner where,***

(a) There are no restrictions. [2]

(b) Two of the friends are married to each other and will not sit separately. [2]

(c) Two of the friends are not speaking to each other and will not sit together. [2]

[2079 GIE Set-A]

Soln: (a) If there are no restrictions, then 3 friends can be invited from a group of 8 in $^8C_3$ ways $= \frac{8!}{(8-3)!\,3!}$

$$= 56$$

∴ There are 56 ways to invite 3 friends with no restrictions. **Ans.**

(b) If two of the friends are married to each other and will not sit separately, then we can treat them as either they sit together or they do not attend to the dinner.

So, the total no. of ways = $^2C_2 \times {^6C_1} + {^6C_3}$

$$= 1 \times 6 + \frac{6!}{3!\,3!} = 6 + 20 = 26 \textbf{ Ans.}$$

(c) If women invite two particular friends and if they sit together then the number of ways = $^2C_2 \times {^6C_1}$

$$= 1 \times 6$$

$$= 6$$

Now, the required no. of ways that two of the friends are not speaking to each other and will not sit together = 56 – 6 = 50 **Ans.**

5. ***In how many ways can a cricket team of 11 player be chosen out of a batch of 15 players if,***

(a) There is no restriction on the selection? [1]

(b) A particular player is always chosen? [1]

(c) A particular player is never chosen? [1]

Soln: (a) If there is no restriction on the selection then 11 players can be chosen from 15 players in C(15, 11) ways $= \frac{15!}{(15-11)!\,11!}$

$$= \frac{15!}{4!\,11!} = \frac{15 \times 14 \times 13 \times 12 \times 11!}{4! \times 11!} = 1365 \textbf{ Ans.}$$

(b) If a particular player is always chosen then 11 – 1 = 10 players can be chosen from 15 – 1 = 14 players in C(14, 10) ways $= \frac{14!}{(14-10)!\,10!} = \frac{14!}{4!\,10!} = 1001$

(c) If a particular player is never chosen then, 11 players can be chosen from 15 – 1 = 14 players in C(14, 11) ways $= \frac{14!}{(14-11)!\,11!} = \frac{14!}{3!\,11!} = 364$ **Ans.**

6. ***There are 10 professors and 20 lecturers out of whom a committee of 2 professors and 3 lecturers is to be formed. Find***

(a) In how many ways committee can be formed? [1]

(b) In how many ways a particular professor is included? [1]

(c) In how many ways a particular lecturer is excluded? [1]

Soln: (a) 2 professor can be chosen form 10 professors in C(10, 2) ways.

Also, 3 lecturers can be chosen from 20 lecturers in C(20, 3) ways.

∴ The required no. of committees = C(10, 2) × C(20, 3)

$$= \frac{10!}{(10-2)!\,2!} \times \frac{20!}{(20-3)!\,3!}$$

$$= \frac{10!}{8!\,2!} \times \frac{20!}{17!\,3!}$$

$$= 45 \times 1140 = 51300 \textbf{ Ans.}$$

(b) If a particular professor is included then we have to select 2 – 1 = 1 professor from 10 – 1 = 9 professor in C(9, 1) ways. Also, 3 lecturers can be selected from 20 lecturers in C (20, 3) ways.

∴ The total no. of ways = C(9, 1) × C(20, 3)

$$= \frac{9!}{(9-1)!\,1!} \times \frac{20!}{(20-3)!\,3!} = 9 \times 1140 = 10260 \textbf{ Ans.}$$

(c) If a particular lecturers is excluded then 3 lecturers can be selected from 20 – 1 = 19 lecturers in C(19, 3) ways.

Also, 2 professors can be selected from 10 professors in C(10, 2) ways.

∴ The required no. of ways = C(19, 3) × C (10, 2)

$$= \frac{19!}{(19-3)!\,3!} \times \frac{10!}{(10-2)!\,2!}$$

$$= \frac{19!}{16!\,3!} \times \frac{10!}{8!\,2!}$$

$$= 969 \times 45$$

$$= 43605 \textbf{ Ans.}$$

7. ***There are 7 men and 3 ladies. Find the number of ways in which a committee of 6 persons can be formed if the committee is to have***

(a) ***Exactly three ladies.*** [1]

(b) ***At least one lady.*** [2]

Soln: (a) The selections of committees can be made as follows:

Men (7)	Ladies (3)	Selections
3	3	C(7, 3) × C(3, 3)

∴ Total no. of committees $= C(7,3) \times C(3,3)$

$$= \frac{7!}{(7-3)!\,3!} \times \frac{3!}{(3-3)!\,3!}$$

$$= \frac{7!}{4!\,3!} \times \frac{3!}{0!\,3!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3!} \times 1$$

= 35 **Ans.**

(b) The selections of committees can be made as follows:

Men (7)	Ladies (3)	Selections
5	1	C(7, 5) × C(3, 1)
4	2	C(7, 4) × C(3, 2)
3	3	C(7, 3) × C(3, 3)

∴ Total no. of committees $= C(7,5) \times C(3,1) + C(7,4) \times C(3,2) + C(7,3) \times C(3,3)$

$$= \frac{7!}{(7-5)!\,5!} \times \frac{3!}{(3-1)!\,1!} + \frac{7!}{(7-4)!\,4!} \times \frac{3!}{(3-2)!\,2!} + \frac{7!}{(7-3)!\,3!} \times \frac{3!}{(3-3)!\,3!}$$

$$= \frac{7!}{2!\,5!} \times \frac{3!}{2!\,1!} + \frac{7!}{3!\,4!} \times \frac{3!}{1!\,2!} + \frac{7!}{4!\,3!} \times \frac{3!}{0!\,3!}$$

$= 3 \times 21 + 3 \times 35 + 1 \times 35$

$= 63 + 105 + 35$

= 203 ways **Ans.**

8. ***A student has to answer 8 out of 10 questions in an examination.***

(a) ***How many choice does he have?*** [1]

(b) ***How many choices are there if he must answer first three questions?*** [1]

(c) ***How many choices are there if he must answer at least four of the first five questions?*** [2]

Soln: (a) 8 questions out of 10 questions can be selected in C(10, 8) ways $= \frac{10!}{(10-8)!\,8!}$

$$= \frac{10!}{2!\,8!} = \frac{10 \times 9 \times 8!}{2 \times 1 \times 8!} = 45$$

∴ He has 45 choices. **Ans**

(b) Since, he has to answers of first three questions compulsorily, so he can select 8 − 3 = 5 questions out of 10 − 3 = 7 questions in C(7, 5) ways

$$= \frac{7!}{(7-5)!\,5!} = \frac{7!}{2!\,5!} = \frac{7 \times 6 \times 5!}{2 \times 1 \times 5!} = 21 \text{ Ans.}$$

(c) There are two cases in which the selection can be possible.

Case - I:

Selecting 4 questions from first 5 and rest 4 questions from remaining 5 questions. This can be done in C(5, 4) × C(5, 4) ways.

Case - II:

Selecting 5 questions from first 5 and selecting rest 3 questions from remaining 5 questions. This can be done in C(5, 5) × C(5, 3) ways.

∴ The total no. of ways $= C(5,4) \times C(5,4) + C(5,5) \times C(5,3)$

$$= \frac{5!}{(5-4)!\,4!} \times \frac{5!}{(5-4)!\,4!} + \frac{5!}{(5-5)!\,5!} \times \frac{5!}{(5-3)!\,3!}$$

$$= \frac{5!}{1!\,4!} \times \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!} \times \frac{5!}{2!\,5!}$$

$= 5 \times 5 + 1 \times 10$

$= 25 + 10$

= 35 **Ans.**

Alternatively:

Total no. of questions = 10

No. of questions to answer = 8

First choice (5)	Second choices (5)	Selections
4	4	C(5, 4) × C(5, 4)
5	3	C(5, 5) × C(5, 3)

∴ The total no. of ways $= C(5,4) \times C(5,4) + C(5,5) \times C(5,3)$

$$= \frac{5!}{(5-4)!\,4!} \times \frac{5!}{(5-4)!\,4!} + \frac{5!}{(5-5)!\,5!} \times \frac{5!}{(5-3)!\,3!}$$

$$= \frac{5!}{1!\,4!} \times \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!} \times \frac{5!}{2!\,5!}$$

$= 5 \times 5 + 1 \times 10$

$= 25 + 10$

= 35 **Ans.**

9. ***A bag contains 6 red, 4 white and 8 blue balls. How many three balls can be made so that,***

(a) ***One is red and two are white.*** [1]

(b) ***Two are blue and one is red.*** [1]

(c) ***All three are blue.*** [1]

(d) ***There is one ball of each colour.*** [1]

Soln: (a) 1 red ball can be chosen from 6 red balls in C(6, 1) ways.

Also, 2 white balls can be chosen from 4 white balls in C(4, 2) ways.

∴ Total no. of ways of selecting one is red and two are white = C(6, 1) × C(4, 2)

$= 6 \times 6 = 36$ **Ans.**

(b) 2 blue ball can be chosen from 8 blue balls in C(8, 2) ways.

Also, 1 red ball can be chosen from 6 red balls in C(6, 1) ways.

∴ Total no. of ways of selecting 2 blue balls and 1 red = C(8, 2) × C(6, 1)

$= 28 \times 6 = 168$

(c) 3 blue balls can be chosen from 8 blue balls in C(8, 3) ways = 56 **Ans.**

(d) 1 red ball can be chosen from 6 red balls in C(6, 1) ways.
1 white ball can be chosen from 4 white balls in C(4, 1) ways.
1 blue ball can be chosen from 8 blue balls in C(8, 1) ways.

∴ Total no. of ways of selecting one ball of each color $= C(6, 1) \times C(4, 1) \times C(8, 1)$
$= 6 \times 4 \times 8$
$= 192$ **Ans.**

10. ***In a group of 10 students, 6 are boys. In how many ways can 4 students be selected for mathematical competition so as to include.***

(a) Exactly two boys? [1]
(b) At least two boys? [2]
(c) At most two girls? [2]
(d) No girls [2]

Soln: (a) **Exactly two boys**

Here, total no. of students = 10
No. of boys = 6
No. of girls = 10 – 6 = 4

The selection of exactly two boys can be done as follows:

Boys (6)	Girls (4)	Selections
2	2	C(6, 2) × C(4, 2)

∴ The total no. of ways of selecting exactly two boys = C(6, 2) × C(4, 2)

$$= \frac{6!}{(6-2)!\,2!} \times \frac{4!}{(4-2)!\,2!}$$
$$= \frac{6!}{4!\,2!} \times \frac{4!}{2!\,2!}$$
$$= 15 \times 6$$
$$= 90$$

(b) **At least two boys**

The selection of at least two boys can be made as follows:

Boys (6)	Girls (4)	Selections
2	2	C(6, 2) × C(4, 2)
3	1	C(6, 3) × C(4, 1)
4	0	C(6, 4) × C(4, 0)

Total no. of ways of selecting at least two boys

$= C(6, 2) \times C(4, 2) + C(6, 3) \times C(4, 1) + C(6, 4) \times C(4, 0)$

$$= \frac{6!}{4!\,2!} \times \frac{4!}{2!\,2!} + \frac{6!}{3!\,3!} \times \frac{4!}{3!\,1!} + \frac{6!}{2!\,4!} \times \frac{4!}{4!\,0!}$$
$$= 15 \times 6 + 20 \times 4 + 15 \times 1$$
$$= 90 + 80 + 15$$
$= 185$ **Ans.**

(c) **At most two girls**

The sections of at most two girls can be made as follows:

Girls (4)	Boys (6)	Selections
2	2	C(4, 2) × C(6, 2)
1	3	C(4, 1) × C(6, 3)
0	4	C(4, 0) × C(6, 4)

∴ Total no. of ways of selecting at most two girls

$= C(4, 2) \times C(6, 2) + C(4, 1) \times C(6, 3) + (4, 0) \times C(6, 4)$

$$= \frac{4!}{2!\,2!} \times \frac{6!}{4!\,2!} + \frac{4!}{3!\,1!} \times \frac{6!}{3!\,3!} + \frac{4!}{4!\,0!} \times \frac{6!}{2!\,4!}$$
$$= 6 \times 15 + 4 \times 20 + 1 \times 15$$
$$= 90 + 80 + 15$$
$= 185$ **Ans.**

(d) Since, the group does not include any girl. So, only boys are to be selected.
This can be made as follows:

Girls (4)	Boys (6)	Selection
0	4	C(4, 0) × C(6, 4)

∴ Total no. of ways of selecting no girls = C(4, 0) × C(6, 4)

$$= \frac{4!}{4!\,0!} \times \frac{6!}{2!\,4!}$$
$$= 1 \times 15$$
$= 15$ **Ans.**

□□□

Chapter 2

Binomial Theorem

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2.1 Binomial Theorem

Basic Formulae and Key Points

1. **For any Positive Integers n,**

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \dots + {}^nC_r a^{n-r}x^r + \dots + {}^nC_n x^n.$$

Or

$$C(n, 0)a^n + C(n, 1)\, a^{n-1}x + C(n, 2)a^{n-2}x^2 + \dots + C(n, r)a^{n-r}x^r + \dots + C(n, n)x^n.$$

2. **Some Observations in a Binomial Theorem**
   - i) The total numbers of terms after binomial expansion of $(a+x)^n$ is $n+1$.
   - ii) Sum of the exponent in each term is n.
   - iii) The powers of 'a' go on decreasing and that of 'x' go on increasing by 1 for each term.
   - iv) The first and last terms are a^n and x^n respectively.
   - v) The coefficients of the terms equidistant from the beginning and the end are always equal.
3. **General Term**
   - i) The general term or the $(r+1)^{th}$ term in the expansion of $(a+x)^n$ is given by
     $t_{r+1} = C(n, r)\, a^{n-r}x^r$
   - ii) The general term or the $(r+1)^{th}$ term in the expansion of $(a-x)^n$ is given by
     $t_{r+1} = (-1)^r C(n, r)\, a^{n-r}x^r$
4. **Middle Term / Terms in the Expansion of $(a+x)^n$**

Case (i): If n is even, then the number of terms after the expansion is n + 1 which is odd, then there is only one middle term, given by

$$t_{\frac{n}{2}+1} = C\left(n, \frac{n}{2}\right) a^{\frac{n}{2}} x^{\frac{n}{2}}$$

Case (ii): If n is odd, then the number terms after the expansion is (n + 1) which is even. Therefore, we have two middle terms given by $t_{\frac{n+1}{2}}$ and $t_{\frac{n+1}{2}+1}$.

First middle term: $t_{\frac{n+1}{2}} = t_{\frac{n-1}{2}+1} = C\left(n, \frac{n-1}{2}\right) a^{\frac{n+1}{2}} x^{\frac{n-1}{2}}$

and

Second middle term: $t_{\frac{n+1}{2}+1} = C\left(n, \frac{n+1}{2}\right) a^{n-\frac{n+1}{2}} x^{\frac{n+1}{2}} = C\left(n, \frac{n+1}{2}\right) a^{\frac{n-1}{2}} x^{\frac{n+1}{2}}$

5. **Binomial Coefficients**

The coefficients C(n, 0), C(n, 1) ....., C(n, n) in the expansion of $(a+x)^n$ are known as binomial coefficients.

6. **Properties of Binomial Coefficients**
   - i) The sum of all binomial coefficients is 2^n.
     i.e. $C_0 + C_1 + C_2 + \dots + C_n = 2^n$
   - ii) $C_1 + C_2 + \dots + C_n = 2^n - 1$
   - iii) The sum of the coefficients of the odd terms is equal to sum of the coefficients of the even terms and each is equal to 2^{n-1}.
     i.e. $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

Group 'A' (Multiple Choice Questions and Answers)

1. *The $(k+1)^{th}$ term of $(a+b)^n$ is ....* [2079 GIE SET-A]
   (a) $\binom{n}{k} a^k b^k$ (b) $\binom{n}{k} (ab)^{n-k}$ (c) $\binom{n}{k} a^{n-k} b^k$ (d) $\binom{n}{k} a^{n-k} b^{n-k}$
2. *If n is a positive integers, then the number of terms in the expansion of $(x+a)^{n-1}$.*
   (a) n (b) n + 1 (c) 2n − 1 (d) 2n
3. *The r^{th} term in the expansion of $(a+x)^n$ is*
   (a) ${}^nC_r a^{n-r} x^r$ (b) ${}^nC_{r-1} a^{n-r+1} x^{r-1}$
   (c) ${}^nC_{r-1} a^{n-1}$ (d) ${}^nC_{n+1} a^{n-r-1} x^{r+1}$
4. *How many terms are there in the expansion of $(1+2x+x^2)^{10}$?*
   (a) 11 (b) 20 (c) 21 (d) 30
5. *How many terms are there in the expansion of $\left(1+\frac{2}{x}\right)^9\left(1-\frac{2}{x}\right)^9$?*
   (a) 9 (b) 10 (c) 19 (d) 20
6. *The general term in the expansion of $(x^2-y)^6$ is*
   (a) ${}^6C_r x^{12-2r} y^r$ (b) $(-1)^r\, {}^6C_r x^{12-2r} y^r$
   (c) $(-1)^r\, {}^6p_r x^{12-2r} y^r$ (d) ${}^6p_r x^{12-2r} y^6$
7. *The term containing x^3 in the expansion of $(x-2y)^7$ is*
   (a) 3^{rd} (b) 4^{th} (c) 5^{th} (d) 6^{th}
8. *In the expansion of $(a+b)^n$, if n is odd then the number of middle term is/are*
   (a) 0 (b) 1 (c) 2 (d) more than 2
9. *In the expansion of $\left(x-\frac{1}{2x}\right)^6$, if the $(r+1)^{th}$ term is free from the x, then r is equal to*
   (a) 2 (b) 3 (c) 4 (d) −1
10. *When n is even, the middle term in the expansion of $(a+b)^n$ is*
    (a) $c\left(n, \frac{n+1}{2}\right) a^{\frac{n-1}{2}} x^{\frac{n+1}{2}}$ (b) $c\left(n, \frac{n+1}{2}\right) x^{\frac{n-1}{2}} a^{\frac{x+1}{2}}$
    (c) $c\left(n, \frac{n}{2}\right) a^{\frac{n}{2}} x^{\frac{n}{2}}$ (d) $c\left(n, \frac{n-1}{2}\right) x^{\frac{n+1}{2}} a^{\frac{n-1}{2}}$
11. *In the expansion of $(1+x)^n$, the sum of all the binomial coefficients is*
    (a) 2^{n-1} (b) 2^n (c) $n \times 2^n$ (d) 2^{2n}
12. *If number of terms in the expansion of $(1-2x+x^2)$ is 19, then is*
    (a) 7 (b) 9 (c) 10 (d) 11
13. *What is the coefficient of x^5 in the expansions of $\left(x+\frac{1}{2x}\right)^7$?*
    (a) $\frac{2}{7}$ (b) $\frac{7}{2}$ (c) 14 (d) $\frac{7}{3}$

Answer Key

1. c	2. a	3. b	4. c	5. b	6. b	7. c	8. c	9. b	10. c
11. b	12. b	13. b							

Group 'B' or 'C' (Subjective Questions and Answers)

1. *If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$. Prove that $C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n - \frac{1}{2}(n.2^n) = 0$* [3] [2081 Set V]

Soln: L.H.S. $= C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n - \frac{1}{2}(n.2^n)$

$$= \left\{n + \frac{2n(n-1)}{2!} + \frac{3n(n-1)(n-2)}{3!} + \ldots\ldots + n.1\right\} - n.\frac{2^n}{2}$$

$$= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots\ldots 1\right\} - n.2^{n-1}$$

$$= n(1+1)^{n-1} - n.2^{n-1}$$

$$= n.2^{n-1} = n.2^{n-1}$$

$= 0$ R.H.S. proved.

2. (a) *What is the sum of coefficient of even terms in the expansion of $(1+x)^n$?* [1] [2080 GIE Set A]

(b) *Find the 6th term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$.* [2]

Soln: (a) The sum of the coefficient of even terms in the expansion of $(1+x)^n$ is 2^{n-1}.

i.e. $C_0 + C_2 + C_4 + \ldots\ldots = 2^{n-1}$ **Ans.**

(b) Here,

6th term $(t_6) = t_{5+1}$

$$= C(12, 5)(x^2)^{12-5}.\left(\frac{1}{x}\right)^5$$

$$= \frac{12!}{7!\,5!}(x^2)^7.\frac{1}{x^5}$$

$= 792x^{14-5} = 792x^9$ **Ans.**

3. *If the coefficient of x^{-1} in the expansion of $\left(x + \frac{p}{x^2}\right)^5$ is 90, find the value of p.* [2] [2080 GIE Set B]

Soln: Let, t_{r+1} be the general term which contains x^{-1} in the expansion of $\left(x + \frac{p}{x^2}\right)^5$. Then

$$t_{r+1} = C(n, r)a^{n-r}x^r$$

$$= C(5, r)\,x^{5-r}\left(\frac{p}{x^2}\right)^r$$

$$= C(5, r)\,x^{5-r}\frac{p^r}{x^{2r}}$$

$$= C(5, r)\,p^r x^{5-r-2r}$$

$$= C(5, r)\,p^r x^{5-3r}$$

For the coefficient of x^{-1}, we have

$5 - 3r = -1$

or, $3r = 6$

$\therefore$ $r = 2$

$\therefore$ The coefficient of $x^{-1} = C(5, r)\,p^r$

$= C(5, 2)\,p^2$

By question,

Coefficient of $x^{-1} = 90$

or, $C(5, 2)\,p^2 = 90$

or, $\frac{5!}{3!\,2!}p^2 = 90$

or, $10 \times p^2 = 90$

or, $p^2 = 9$

$\therefore$ $p = \pm 3$ **Ans.**

4. *For what value of r, the coefficient of x^r and x^{r+1} are equal in the expansion of $(1+x)^{11}$?* [3] [2080 GIE Set G]

Soln: Here, $t_{r+1} = C(11, r)\,x^r$

$\therefore$ Coefficient of $x^r = C(11, r)$

Also, $t_{(r+1)+1} = C(11, r+1)\,x^{r+1}$

$\therefore$ Coefficient of $x^{r+1} = C(11, r+1)$

Now,

$C(11, r) = C(11, r+1)$

or, $r + (r+1) = 11$ $[\because C(n, r) = C(n, r') \Rightarrow r = r' \text{ and } r + r' = n]$

or, $2r = 10$

$\therefore$ $r = 5$ **Ans.**

5. *In a binomial expansion $\left(x + \frac{1}{x}\right)^{12}$.* [2079 GIE Set B]

(a) *Find the 7th term.* [2]

(b) *Find the term independent from x.* [2]

(c) *How many terms are there in the expansion?* [1]

Soln: (a) Here,

$t_7 = t_{6+1}$

$$= C(12, 6)\,x^{12-6}\left(\frac{1}{x}\right)^6$$

$$= \frac{12!}{6!\,6!}x^6 \times \frac{1}{x^6}$$

$= 924$

(b) Let, t_{r+1} be the general term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^{12}$. Then

$$t_{r+1} = C(12, r)\,x^{12-r}\left(\frac{1}{x}\right)^r$$

$$= C(12, r)\,x^{12-r-r}$$

$$= C(12, r)\,x^{12-2r}$$

Now, for the term independent of x, we have

$12 - 2r = 0$

or, $2r = 12$

$\therefore$ $r = 6$

$\therefore$ $t_{r+1} = t_{6+1} = t_7$ is the term independent of x. **Ans.**

(c) The no. of terms is the expansion is n + 1 i.e. 12 + 1 = 13 terms. **Ans.**

6. (a) *Write the number of the total terms in the expansion of $\left[\left(x-\frac{1}{x}\right)^2\right]^{25}$*
(b) *Write the middle term in the expansion of $(x + a)^n$ when n is even.*
(c) *What is the sum of binomial coefficients in the expansion of $(1 + x)^n$?* [2081 Set

Soln: (a) Here,
Given binomial expansion is

$$\left[\left(x-\frac{1}{x}\right)^2\right]^{25} = \left(x-\frac{1}{x}\right)^{2\times 25} = \left(x-\frac{1}{x}\right)^{50}$$

$\therefore$ The total no. of terms in the expansion of $\left[\left(x-\frac{1}{x}\right)^2\right]^{25} = 50 + 1 = 51$ **Ans.**

(b) When, n is even, then there is only one middle term.
The middle term in the expansion of $(x + a)^n$ is

$$t_{\frac{n}{2}+1} = C\left(n, \frac{n}{2}\right) a^{\frac{n}{2}} x^{\frac{n}{2}} \text{ Ans.}$$

(c) In the expansion of $(1 + x)^n$, the sum of all binomial coefficients is equal to 2^n.
i.e. $C_0 + C_1 + C_2 + C_3 + ... + C_n = 2^n$ Ans.

7. $\left(x+\frac{1}{2x}\right)^7$ *is a binomial expansion.*
(a) *What is the middle term of the above expansion?*
(b) *What is its general term?*
(c) *Find the sixth term of the expansion.*

Soln: (a) Here, n = 7 which is odd. So, there are two middle terms. They are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ terms respectively.

i.e. $\frac{7+1}{2} = 4^{th}$ and $\left(\frac{7+1}{2}+1\right) = 5^{th}$ term

Now, the first middle term $(t_4) = t_{3+1} = C(7, 3)x^{7-3}\left(\frac{1}{2x}\right)^3$

$$= C(7, 3)\, x^4 \frac{1}{8x^3}$$

$$= \frac{7!}{4!\,3!} \cdot \frac{1}{8} x^{4-3} = \frac{35}{8} x$$

$\therefore$ Second middle term $(t_5) = t_{4+1} = C(7, 4)x^{7-4}\left(\frac{1}{2x}\right)^4$

$$= C(7, 4)x^3 \frac{1}{16x^4} = \frac{35}{16} \frac{1}{x}$$

(b) Let, t_{r+1} be the general term in the expansion of $\left(x+\frac{1}{2x}\right)^7$. Then

$$t_{r+1} = C(7, r)\, x^{7-r}\left(\frac{1}{2x}\right)^r = C(7, r)\, x^{7-r-r} \cdot \frac{1}{2^r} = C(7, r)\frac{1}{2^r} x^{7-2r}$$

(c) Here,
Sixth term $(t_6) = t_{5+1} = C(7, 5)x^{7-5} \cdot \left(\frac{1}{2x}\right)^5$

$$= \frac{7!}{2!\,5!} x^2 \cdot \frac{1}{32x^5} = \frac{21}{32} \times \frac{1}{x^3} \text{ Ans.}$$

8. *In the expansion of $(1 + 2x + x^2)^{50}$, answer the following.*
(a) *Write the total number of terms in the expansion.* [1] [2080 Set I]
(b) *Write the middle term in the expansion.* [1] [2080 Set I]
(c) *What is the coefficient of the middle term?* [1]

Soln: (a) Here,
Given that, $(1 + 2x + x^2)^{50} = [(1 + x)^2]^{50} = (1 + x)^{100}$
$\therefore$ The total no. of terms in the expansion of $(1 + 2x + x^2)^{50}$ is $100 + 1 = 101$ **Ans.**

(b) Here, n = 100 which is even, so there is only one middle term which is $t_{\frac{n}{2}+1}$ i.e. $t_{\frac{100}{2}+1}$ or t_{50+1}
i.e. t_{51}.

$\therefore$ Middle term $= t_{51} = t_{50+1} = C(100, 50).\, 1^{100-50}.\, x^{50}$
$= C(100, 50)x^{50}$ **Ans.**

(c) The coefficient of the middle term is C(100, 50) **Ans.**

9. *For any positive integer n, $(a + x)^n = C_0 a^n + C_1 a^{n-1}x + C_2 a^{n-2}x^2 ..... + C_n x^n$.*
(a) *How many terms are there in the expansion?* [1]
(b) *Write the binomial coefficients in the expansion.* [1]
(c) *Write the general term in the expansion.* [1]
(d) *Write the relation among C(n, r − 1), C(n + 1, r) and C(n, r)* [1]
(e) *What is the value of $C_0 + C_1 + C_2 + ... + C_n$?* [1]
[Model 2079]

Soln: (a) There are n + 1 terms in the expansion of $(a + x)^n$.
(b) The coefficient $C_0, C_1, C_2, ..., C_n$ in the expansion of $(a + x)^n$ are binomial coefficients.
(c) The general term in the expansion of $(a + x)^n$ is given by
$t_{r+1} = C(n, r)\, a^{n-r} x^r$.
(d) The relation among given binomial coefficients is C(n, r − 1) + C(n, r) = C(n + 1, r).
(e) We know that the sum of all binomial coefficients is 2^n.
i.e. $C_0 + C_1 + C_2 + ... + C_n = 2^n$ **Ans.**

10. *In the expansion of $(x + a)^n$, n is a positive integer.*
(a) *Write the general term of the expansion.* [1]
(b) *If a is replaced by (−a), what is the general term of the expansion?* [1]
(c) *How many terms are there in the expansion of $(x + a)^n$?* [1]
(d) *What is the value of $C_1 + C_2 + C_3 + ... + C_n$?* [1]
[Model 2080]

Soln: (a) The general term in the expansion of $(x + a)^n$ is given by
$t_{r+1} = C(n, r)\, x^{n-r} a^r$.
(b) If a is replaced by (−a) then the binomial expansion will be $(x - a)^n$. In this case the general term is given by
$t_{r+1} = (-1)^r\, C(n, r)x^{n-r}a^r$.
(c) There are n + 1 terms in the expansions of $(x + a)^n$.
(d) The value of $C_1 + C_2 + C_3 ...... + C_n = 2^n - 1$ [$\because$ **By the property of binomial coefficient.**]

11. *In the expansion of* $\left(x^2-\frac{1}{3x^2}\right)^{12}$, *answer the following questions.*

(a) *Find the term independent of x in the expansion.* [2]

(b) *Find the coefficient of x^8 in the expansion.* [1]

(c) *How many middle term/terms is /are there in the expansion? What term is the middle term? Justify your answer.* [1]

Soln: (a) Let, t_{r+1} be the term independent of x. Then

$$t_{r+1} = (-1)^r C(12, r)\,(x^2)^{12-r}.\left(\frac{1}{3x^2}\right)^r$$

$$= (-1)^r C(12, r)\,x^{24-2r}\frac{1}{3^r}.\frac{1}{x^{2r}}$$

$$= (-1)^r C(12, r).\frac{1}{3^r}.x^{24-2r-2r}$$

$$= (-1)^r C(12,r).\frac{1}{3^r}.x^{24-4r}$$

For the term independent of x, we have

$24-4r=0$

or, $4r=24$

$\therefore$ $r=6$

$\therefore$ $t_{r+1}=t_{6+1}=t_7$ is the term independent of x.

Now, $t_7 = t_{6+1} = (-1)^6.\ C(12,6).\frac{1}{3^6}$

$$= \frac{12!}{6!\,6!}\times\frac{1}{3^6} = \frac{308}{243}\text{ Ans.}$$

(b) For the coefficient of x^8, we have

$24-4r=8$

or, $4r=24-8$

or, $4r=16$

$\therefore$ $r=4$

Here, $t_{r+1}=t_{4+1}=t_5$ contains x^8

$\therefore$ Coefficient of $x^8 = (-1)^4\,C(12, 4).\frac{1}{3^4}$

$$= \frac{12!}{8!\,4!}\cdot\frac{1}{81}$$

$$= \frac{55}{9}\text{ Ans.}$$

(c) Here, n = 12 which is even. So, there is only one middle term in the expansion of $\left(x^2-\frac{1}{3x^2}\right)^{12}$.

The middle term is given by $t_{\frac{n}{2}+1} = t_{\frac{12}{2}+1} = t_{6+1} = t_7$

$\therefore$ t_7 is the middle term in the expansion of $\left(x^2-\frac{1}{3x^2}\right)^{12}$.

12. *In the expansion of $(1+x)^{2n}$,*

(a) *How many middle terms are there in the expansion?* [1]

(b) *Show that the middle term is $\frac{1.3.5.\,.....\,(2n-1)}{n!}2^nx^n$.* [2]

(c) *Write the middle terms in the expansion of $(a+x)^n$, when n is odd.* [1]

Soln: (a) Here, the exponent of binomial expansion is 2n which is even. So, there is only one middle term, which is given by $t_{\frac{2n}{2}+1}$ i.e. t_{n+1}.

(b) Here, the middle term = t_{n+1} = $C(2n, n)\,1^{2n-n}.\,x^n$

$$= \frac{(2n)!}{(2n-n)!\,n!}x^n$$

$$= \frac{[1.2.3.4.\,.....\,(2n-2).(2n-1).2n]\ x^n}{n!\,n!}$$

$$= \frac{[1.3.5....(2n-1)]\,[2.4.6.....(2n-2).2n]\,x^n}{n!\,n!}$$

$$= \frac{[1.3.5.......(2n-1)]\,2^n\,[1.2.3.\,.....\,(n-1)]x^n}{n!\,n!}$$

$$= \frac{[1.3.5.\,......\,(2n-1)].2^n.n!\,x^n}{n!\,n!}$$

$$= \frac{1.3.5.\,.....\,(2x-1)}{n!}\,2^n.x^n$$ **Hence Proved.**

(c) When, n is odd, there will be two middle terms. They are $t_{\frac{n+1}{2}}$ and $t_{\left(\frac{n+1}{2}+1\right)}$ terms respectively.

Middle terms are given by.

$$t_{\frac{n+1}{2}} = t_{\left(\frac{n-1}{2}+1\right)} = C\left(n, \frac{n-1}{2}\right)a^{\frac{n+1}{2}}.x^{\frac{n-1}{2}}\text{ and}$$

$$t_{\left(\frac{n+1}{2}+1\right)} = C\left(n, \frac{n+1}{2}\right)a^{\frac{n-1}{2}}.x^{\frac{n+1}{2}}\text{ Ans.}$$

13. *If C_0, C_1, C_2 ......, C_n are the coefficients in the expansion of $(1+x)^n$,*

(a) *Prove that the sum of the coefficient of even terms is equal to the sum of the coefficients of odd terms and each is equal to 2^{n-1}.* [2]

(b) *Define general term of the binomial expansion.* [1]

(c) *What is the value of the coefficient C_n?* [1]

(d) *Write the coefficient of x^n in the expansion of $(1+x)^{2n}$.* [1]

Soln: (a) Proof: We know that,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \,.......\, + C_nx^n \,........\, \text{(i)}$$

Putting x = 1 in (i), we get

$$(1+1)^n = C_0 + C_1 + C_2 + C_3 + \,.........\, + C_n$$

$\therefore$ $2^n = C_0 + C_1 + C_2 + C_3 + \,............\, + C_n \,............\, \text{(ii)}$

Putting x = −1 in (i), we get

$$(1-1)^n = C_0 - C_1 + C_2 - C_3 + \,...........\, (-1)^nC_n$$

i.e. $0 = C_0 - C_1 + C_2 - C_3 + \,...........\, (-1)^n C_n \,.................\, \text{(iii)}$

Adding (ii) & (iii) we get

$2^n = 2C_0 + 2C_2 + 2C_4 + \dots\dots$

or, $2^n = 2(C_0 + C_2 + C_4 + \dots\dots)$

$\therefore$ $C_0 + C_2 + C_4 + \dots\dots = \frac{2^n}{2} = 2^{n-1}$

i.e. Sum of the coefficients of even terms $= 2^{n-1}$

Also, Subtracting (iii) from (ii), we get

$2^n = 2C_1 + 2C_3 + 2C_5 + \dots\dots$

or, $2^n = 2(C_1 + C_3 + C_5 + \dots\dots)$

$\therefore$ $C_1 + C_3 + C_5 + \dots\dots = \frac{2^n}{2} = 2^{n-1}$

i.e. Sum of the coefficient of odd terms $= 2^{n-1}$

Hence, the sum of the coefficients of even terms is equal to the sum of the coefficients of odd terms and each is equal to 2^{n-1} **Proved.**

(b) **General term:** The $(r+1)^{th}$ term in the expansion of $(a+x)^n$ is called the general term. It is denoted by t_{r+1} and is given by $t_{r+1} = C(n,r)\, a^{n-r}x^r$.

(c) We have, $C_n = {}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = \frac{1}{0!}$

$= \frac{1}{1}$ $\quad [\because 0! = 1]$

$= 1$

$\therefore$ The value of the coefficient $C_n = 1$ **Ans.**

(d) Here, the binomial expansion is $(1+x)^{2n}$.

Let, t_{r+1} be the general term, then

$t_{r+1} = C(2n, r).\,1^{2n-r}.\,x^r$

$= C(2n, r)\,x^r$

For the coefficient of x^n, we have

$n = r$

$\therefore$ $r = n$

$\therefore$ The coefficient of $x^n = C(2n, n)$ **Ans.**

14. (a) ***If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_nx^n$, Prove that: $C_0C_n + C_1C_{n-1} + \dots\dots + C_nC_0 = \frac{2n!}{n!\,n!}$.*** [3]

(b) ***In the binomial expansion of $(a+x)^n$, what is the sum of powers of a and x in each term of expansion?*** [1]

Soln: (a) Here, given that

$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_{n-1}x^{n-1} + C_nx^n \dots\dots$ (i)

Also,

$(1+x)^n = C_nx^n + C_{n-1}x^{n-1} + \dots\dots + C_1x + C_0 \dots\dots$ (ii)

Multiplying (i) & (ii), we get

$(1+x)^{2n} = C_0C_nx^n + C_1C_{n-1}x^n + \dots\dots + C_nC_0x^n$

i.e. $(1+x)^{2n} = (C_0C_n + C_1C_{n-1} + \dots\dots C_nC_0)\,x^n \dots\dots$ (iii)

Since, (iii) is an identity, the coefficient of any powers of x of the L.H.S. should be equal to the coefficient of the same powers of x of the R.H.S. of (iii).

Now, coefficient of x^n in the expansion of $(1+x)^{2n} = C(2n, n) \dots\dots$ (iv)

Equating the coefficient of x^n from (iii) and (iv), we get

$C_0C_n + C_1C_{n-1} + \dots\dots + C_nC_0 = C(2n, n) = \frac{2n!}{n!\,n!}$. **Hence Proved.**

(b) In the binomial expansion of $(a+x)^n$, the sum of the powers of a and x in each term of expansion is n.

15. ***Answer the following questions.***

(a) ***State Binomial theorem.*** [1]

(b) ***In the expansion of $(1+x)^{21}$, the coefficient of $(2r+1)^{th}$ term is equal to the coefficient of $(3r+2)^{th}$ term. Find r.*** [2]

(c) ***If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_nx^n$, prove that $C_1 - 2C_2 + 3C_3 - \dots\dots + n(-1)^{n-1}C_n = 0$*** [2]

Soln: (a) Binomial Theorem states that for any positive integer n,

$(a+x)^n = C(n,0)a^n + C(n,1)a^{n-1}x + C(n,2)a^{n-2}x^2 + \dots\dots + C(n,r)a^{n-r}x^r + \dots\dots + C(n,n)x^n$.

(b) Here, n = 21, a = 1

$(2r+1)^{th}$ term $= t_{2r+1} = C(21, 2r)x^{2r}$

$\therefore$ Coefficient of $(2r+1)^{th}$ term $= C(21, 2r)$

Also, $(3r+2)^{th}$ term $= t_{(3r+2)} = t_{(3r+1)+1} = C(21, 3r+1)x^{3r+1}$

$\therefore$ Coefficient of $(3r+2)^{th}$ term $= C(21, 3r+1)$

Now, by question,

$C(21, 2r) = C(21, 3r+1)$

or, $2r + (3r+1) = 21$ $\quad [\because$ If $C(n, r) = n(n, r')$ then $r + r' = n]$

or, $5r = 20$

$\therefore$ $r = 4$

(c) L.H.S. $= C_1 - 2C_2 + 3C_3 - \dots\dots + n(-1)^{n-1}.C_n$

$= n - 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} - \dots\dots + n(-1)^{n-1}.1$

$= n\left[1 - (n-1) + \frac{(n-1)(n-2)}{2!} - \dots\dots + (-1)^{n-1}\right]$

$= n(1-1)^{n-1}$

$= n \times 0$

$= 0$ R. H.S. **Proved.**

16. (a) ***If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_nx^n$, prove that $C_0 + 3C_1 + 5C_2 + \dots\dots + (2n+1)C_n = (n+1)2^n$.*** [2]

(b) ***In the expansion of $(1+x)^{2n+1}$, the coefficient of x^r and x^{r+1} are equal. Find r.*** [2]

(c) ***What is the r^{th} term in the expansion of $(a+x)^n$?*** [1]

Soln: (a) L.H.S. $= C_0 + 3C_1 + 5C_2 + \dots\dots + (2n+1)C_n$

$= C_0 + (C_1 + 2C_1) + (C_2 + 4C_2) + \dots\dots + (2nC_n + C_n)$

$= (C_0 + C_1 + C_2 + \dots\dots + C_n) + (2C_1 + 4C_2 + \dots\dots + 2nC_n)$

$= 2^n + 2(C_1 + 2C_2 + \dots\dots + nC_n)$

$= 2^n + 2\left\{n + 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + \dots\dots + n.1\right\}$

$= 2^n + 2n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \dots\dots + 1\right\}$

$= 2^n + 2n(1+1)^{n-1}$

$= 2^n + n.2^n$

$= (n+1)2^n$ **R.H.S. Proved.**

(b) Here,

The $(r+1)^{th}$ term $= t_{r+1} = C(2n+1, r).\,x^r$

$\therefore$ Coefficient of $x^r = C(2n+1, r)$

Also, the $(r+2)^{th}$ term $= t_{(r+1)+1} = C(2n+1, r+1)x^{r+1}$

$\therefore$ Coefficient of $x^{r+1} = C(2n+1, r+1)$

Now, by questions,

$C(2n+1, r) = C(2n+1, r+1)$

or, $r + (r+1) = 2n+1$

or, $2r = 2n$

$\therefore$ $r = n$ **Ans.**

(c) The r^{th} term is $t_{(r-1)+1}$ and is given by

$t_{(r-1)+1} = {}^nC_{r-1}\, a^{n-r+1} . x^{r-1}$ **Ans.**

2.2 Application of Binomial Theorem

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Basic Formulae and Key Points

Let, n be any rational number, either positive or negative integer or fractional and x be any real number such that |x| < 1, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots\ldots \text{ to } \infty.$$

This expansion is valid only when |x| < 1, i.e. $-1 < x < 1$.

1. Particular Cases:

i) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$ to ∞.

ii) $(1-x)^{n} = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$ to ∞.

iii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$ to ∞.

2. Exponential and Logarithmic Series:

i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$ to ∞ ii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$ to ∞

When, n = 1, from (1) & (2) we get ,

$e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots$

iii) $a^x = 1 + \frac{x}{1!}\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \ldots\ldots$ to ∞

Where, a is any positive integer

iv) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$ $(-1 < x \le 1)$

v) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$ $(-1 \le x < 1)$

vi) $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$

3. Some Important Notes:

i) $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$ ii) $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\ldots$

iii) $\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$ iv) $\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots$

v) The value of e lies between **2** and **3** and **e** is approximately equal to **2.7182818285.........**

Note: i) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$

ii) $e - 1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$

iii) $e - 1 - \frac{1}{1!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$

i.e. $e - 2 = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$

Group 'A' (Multiple Choice Questions and Answers)

1. *The expansion of $(1+x)^n$, when n is other than positive integer and |x|<1 is*
(a) $C_0 + C_1x + C_2x^2 + C_2x^3 + \ldots\ldots + C_{n-1}x^{n-1} + C_nx^n$
(b) $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$
(c) $1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$
(d) $1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$

2. *For any rational index n and for any real x the expansion of $(1+x)^{-n}$ is possible if*
(a) $1 < x < \infty$ (b) $-\infty < x < -1$ (c) $-\infty < x < \infty$ (d) $-1 < x < 1$

3. *For any real number x such that |x| < 1, the infinite series $1 - x + x^2 - x^3 + x^4 - \ldots\ldots (-1)^r x^r + \ldots\ldots \infty$ is the expansion of*
(a) $(1-x)^{-1}$ (b) $(1+x)^{-1}$ (c) $\sqrt{1+x}$ (d) $\frac{1}{\sqrt{1+x}}$

4. *The expansion of e^x is*
(a) $\sum_{r=0}^{\infty}\frac{x^r}{r}$ (b) $\sum_{r=0}^{\infty}\frac{x^r}{r!}$ (c) $\sum_{r=0}^{\infty}\frac{x^{r+1}}{r+1}$ (d) $\sum_{r=0}^{\infty}\frac{x^{r+1}}{(r+1)!}$

5. *e is the irrational number such that 2< e < 3. Then e = ?*
(a) 2.5 (b) 2.9 (c) 2.71828..... (d) 2.14

6. *For any real number x, the infinite series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$ to ∞ is the expansion of*
(a) $\log_e(1+x)$ (b) $\log_e(1-x)$ (c) e^x (d) e^{-x}

7. *Which of the following series can be obtained for the exponential series e^x when x = 1*
(a) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$ (b) $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$
(c) $-1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} - \ldots\ldots$ (d) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$

8. *For $-1 < x \le 1$, the infinite series: $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots$ is the expansion of*
(a) $\log_e(1+x)$ (b) $\log_e(1-x)$ (c) e^x (d) e^{-x}

9. *If $e^y = 1 + x$ then y = ...*

(a) $x + \frac{x}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots$

(b) $-x - \frac{x}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$

(c) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

(d) $\frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots$

10. *The sum of the series $1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\ldots$ is equal to*

(a) e (b) $\frac{1}{e}$ (c) $\frac{1}{2}(e + e^{-1})$ (d) $(e - e^{-1})$

11. *The value of $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ is*

(a) 0 (b) 1 (c) e (d) ∞

12. *The expansion of $\log_e(1 - x)$ is valid for all x satisfying*

(a) $-1 < x < 1$ (b) $-1 \le x < 1$ (c) $-1 \le x \le 1$ (d) $-1 < x \le 1$

13. *For every real value of x, e^x lies on the interval*

(a) (0, 2) (b) (–1, 1) (c) (2, 3) (d) $(0, \infty)$

14. *Which one of the following series is obtained by the operation $\frac{1}{2}(e - e^{-1})$?*

(a) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$ to ∞

(b) $\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots\ldots$ to ∞

(c) $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots\ldots$ to ∞

(d) $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots$ to ∞

15. *The sum of the series $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots$ to ∞ is :*

(a) e (b) $\frac{1}{e}$ (c) 2e (d) $2e^{-1}$

Answer Key

1. b	2. d	3. b	4. b	5. c	6. c	7. a	8. a	9. c	10. c
11. c	12. b	13. d	14. b	15. b					

Group 'B' or 'C' (Subjective Questions and Answers)

1. *If $x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$, show that $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$* [2] [2080 Set G]

Soln: Here, $x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$

Adding 1 on both sides, we get

$$1 + x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

or, $1 + x = e^y$

or, $y = \log_e(1 + x)$

$\therefore$ $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ **Hence Proved.**

2. (a) *Define exponential function with an example.* [1]

(b) *Write the series of $\log_e(1 - x)$, $(x < 1)$.* [1] [2081 Set E/W]

(c) *Show that: $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots\ldots$ to $\infty = e$* [2]

(d) *Write e^{-x} in series from.* [1] [2081 Set V]

Or,

Express e^{-x} in an infinite series. [1] [2080 GIE Set A]

Note: Both questions have the same meaning.

Soln: (a) A function in which a constant quantity is raised to a variable power is called an exponential function. The function $y = 2^x$ is an example of exponential function where 2 is constant and the power x is the variable.

(b) Here, $\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ (-1 \le x < 1)$.

(c) L.H.S. $= \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots\ldots$ to ∞

$$= \frac{1+1}{1!} + \frac{3+1}{3!} + \frac{5+1}{5!} + \ldots\ldots \text{ to } \infty$$

$$= \frac{1}{1!} + \frac{1}{1!} + \frac{3}{3!} + \frac{1}{3!} + \frac{5}{5!} + \frac{1}{5!} + \ldots\ldots$$

$$= \frac{1}{1!} + \frac{1}{1!} + \frac{3}{3 \times 2!} + \frac{1}{3!} + \frac{5}{5 \times 4!} + \frac{1}{5!} + \ldots\ldots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

= e **R.H.S. Proved.**

(d) We have,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots \text{ to } \infty \ldots\ldots\text{(i)}$$

When x is replaced by –x, then (i) becomes

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots \text{ to } \infty.$$

3. *The value of $\log_e(1 + x)$ is given by $\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + (-1)^{r-1}\frac{x^r}{r} + \ldots\ldots$.* [2079 Optional]

(a) *Name the above series.* [1]

(b) *Find the value of $\log_e\left(\frac{1+x}{1-x}\right)$.* [1]

Soln: (a) The name of the above series is logarithmic series.

(b) We know that

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \text{(i)}$$

$$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots \text{(ii)}$$

Subtracting (ii) from (i), we get

$$\log_e(1 + x) - \log_e(1 - x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots\ldots$$

$\therefore$ $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$ **Ans.**

4. ***The expansion of e^x is given by $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^r}{r!} + \ldots\ldots$ to ∞.***

(a) What will the series be if we replace x with 1 in the expansion of e^x? [1]

(b) Name the expansion of above series. [1]

(c) How e is defined as? [1]

(d) What is the $(r+1)^{th}$ term of the above expansion? [1]

Soln: (a) If we replace x with 1 in the expansion of e^x, then $e^1 = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \ldots\ldots$ to ∞.

i.e. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$ to ∞. **Ans.**

(b) The expansion is known as an exponential series.

(c) e is defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

(d) The $(r+1)^{th}$ term is $\frac{x^r}{r!}$.

5. ***In a binomial expansion for a rational n, and |x| < 1, the expansion of $(1 + x)^n$ is given by $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$***

(a) What will be the expansion of $(1 + x)^n$ be if we replace n with (–n)? [1]

(b) Expand $(1 + x)^{-1}$ up to 5^{th} terms. [1]

(c) When does above expansion hold? [1]

Soln: (a) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$ to ∞. **Ans.**

(b) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \ldots\ldots$ to ∞. **Ans.**

(c) The above expansion holds only for |x| < 1. **Ans.**

6. ***Answer the following questions:***

(a) Find the value of $\left(e+\frac{1}{e}\right)$. [2]

(b) Show that: $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots\ldots = e^{-1}$ [2] [2079 GIE Set A]

Soln: (a) We know,

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$

Putting x = 1 and x = –1, we get

$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots$ (i)

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots$ (ii)

Adding (i) & (ii) we get

$e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \ldots\ldots$

∴ $e + \frac{1}{e} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots\right)$ **Ans.**

(b) L.H.S. $= \frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots\ldots$

$= \frac{3-1}{3!} + \frac{5-1}{5!} + \frac{7-1}{7!} + \ldots\ldots$

$= \frac{3}{3!} - \frac{1}{3!} + \frac{5}{5!} - \frac{1}{5!} + \ldots\ldots$

$= \frac{3}{3\times 2!} - \frac{1}{3!} + \frac{5}{5\times 4!} - \frac{1}{5!} + \ldots\ldots$

$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots\ldots$

$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots\ldots$

$= e^{-1}$ **R.H.S. Proved.**

7. *(a) Write the logarithmic series for the expansion of $\log_e(1 + x)$.* [1] [2080 Optional]

(b) If x is replaced by (–x), then write the logarithmic series. [1]

(c) By using the expansion of $\log_e(1 + x)$ prove that : $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots\ldots = 1 - \log_e 2$. [2]

(d) Write the single relation of $\log_e\left(\frac{1+x}{1-x}\right)$. [1]

Soln: (a) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

(b) If x is replaced by (–x) then we get $\log_e(1 - x)$ and the expansion is given by

$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$

(c) We have,

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \ldots\ldots$

When, x = 1, we get

$\log_e(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots\ldots$

or, $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots\ldots$

or, $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \ldots\ldots = 1 - \log_e 2$

or, $\frac{3-2}{2.3} + \frac{5-4}{4.5} + \frac{7-6}{6.7} + \ldots\ldots = 1 - \log_e 2$

∴ $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots\ldots = 1 - \log_e 2$ **Hence Proved.**

(d) We know that,

$\log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$

i.e. $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$ **Ans.** $\left[\because \log_e a - \log_e b = \log_e\left(\frac{a}{b}\right)\right]$

8. (a) *If* $y = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots$, *show that* $x = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots\ldots$ [2]

(b) *Prove that:* $\dfrac{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots}{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots} = \dfrac{e+1}{e-1}$ [3]

Soln: (a) Here, given

$$y = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots$$

or, $-y = -\left(\frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots\right)$ [∵ Multiplying both sides by −1]

$$-y = -\frac{x}{1!} + \frac{x^3}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\ldots$$

Adding 1 on both sides, we get

$$1 - y = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\ldots$$

or, $1 - y = e^{-x}$

or, $-x = \log_e(1-y)$ [∵ If $y = e^x \Rightarrow x = \log_e y$]

or, $-x = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots\ldots$

∴ $x = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots\ldots$ **Hence Proved.**

(b) We know,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$

When, $x = 1$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots \quad \text{(i)}$$

When, $x = -1$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots \quad \text{(ii)}$$

Adding (i) & (ii), we get

$$e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots\right)$$

∴ $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots$

Similarly, subtracting (i) & (ii), we get

$$e - e^{-1} = 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\right)$$

∴ $\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots$

$$\text{L.H.S.} = \frac{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots}{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots}$$

$$= \frac{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots}{\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots\right) - 1}$$

$$= \frac{\frac{e - e^{-1}}{2}}{\frac{e + e^{-1}}{2} - 1}$$

$$= \frac{\frac{e - e^{-1}}{2}}{\frac{e + e^{-1} - 2}{2}}$$

$$= \frac{e - e^{-1}}{e + e^{-1} - 2}$$

$$= \frac{e - \frac{1}{e}}{e + \frac{1}{e} - 2}$$

$$= \frac{\frac{e^2 - 1}{e}}{\frac{e^2 + 1 - 2e}{e}} = \frac{(e-1)(e+1)}{(e-1)^2} = \frac{e+1}{e-1} \text{ R.H.S. Proved.}$$

9. *For any real number x such that |x| < 1, the infinite series for the expansion of $(1 + x)^n$, where n is any rational index other than positive integer.*

(a) *Use the above concept to prove* $1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \ldots\ldots$ *to* $\infty = 2^{\frac{2}{3}}$ [3]

(b) *Write the expansion of* $\sqrt{1+x}$ *up to four terms.* [2]

(c) *Write the coefficient of* x^3 *in the expansion of* $\sqrt{1+x}$. [1]

(d) *What mathematical concept is the series expansion of* $(1 + x)^n$ *based on?* [1]

Soln: (a) In a binomial expansion for a rational index n, and |x| < 1, the expansion of $(1 + x)^n$ is given by

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{(n-1)(n-2)}{3!}x^3 + \ldots\ldots \text{ to } \infty.$$

Let, $1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \ldots\ldots$ to $\infty = (1 + x)^n$ .... (i)

Then,

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots = 1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots\ldots$$

Equating the second and third terms,

$$nx = \frac{1}{4} \text{ and } \frac{n(n-1)}{2!}x^2 = \frac{1.4}{4.8}$$

$$n^2x^2 = \frac{1}{16} \text{ .... (ii) and } n(n-1)x^2 = \frac{1}{4} \text{ .... (iii)}$$

Dividing eq^n. (iii) by eq^n. (ii), we get

$$\frac{n(n-1)x^2}{n^2x^2} = \frac{1}{4} \times \frac{16}{1}$$

or, $n - 1 = 4n$

$\therefore \quad n = \frac{-1}{3}$

Also, $nx = \frac{1}{4}$

or, $\left(\frac{-1}{3}\right)x = \frac{1}{4}$

$\therefore \quad x = \frac{-3}{4}$

Thus, from (i),

$$1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \text{..........} = \left(1 - \frac{3}{4}\right)^{\frac{-1}{3}}$$

$$= \left(\frac{1}{4}\right)^{\frac{-1}{3}}$$

$$= (4)^{\frac{1}{3}}$$

$$= (2)^{\frac{2}{3}} \text{ R.H.S. Proved}$$

(b) Here, $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$, where $n = \frac{1}{2}$

We know that,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \text{.........}$$

Putting $n = \frac{1}{2}$, we get

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \text{.........}$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \text{.........}$$

(c) The coefficient of x^3 in the expansion of $\sqrt{1+x}$ is $\frac{1}{16}$. **Ans.**

(d) The series expansion of $(1 + x)^n$ is based on the binomial theorem for any rational index other than positive integer. **Ans.**

10. ***Prove that:*** $\frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+3}{4!} + \text{.......} = \frac{e}{2}$ *[4] [2080 Set I/GIE]*

Soln: Let, t_n be the nth term of the given series.

Then

$$t_n = \frac{1+2+3+\text{...........}+n}{(n+1)!}$$

$$= \frac{\frac{n(n+1)}{2}}{(n+1)!} \qquad \left[\because \text{Sum of first n natural numbers } (S_n) = \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{2(n+1).n!}$$

$$= \frac{n}{2n!}$$

$$= \frac{n}{2n(n-1)!}$$

$$= \frac{1}{2(n-1)!}$$

Now, putting n = 1, 2, 3, ..., we get

$t_1 = \frac{1}{2.0!}$

$t_2 = \frac{1}{2.1!}$

$t_3 = \frac{1}{2.2!}$

$t_4 = \frac{1}{2.3!}$

....

Now, by addition, the sum of given series is

$$t_1 + t_2 + t_3 + t_4 + \text{.........} = \frac{1}{2}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{........}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{........}\right)$$

$$= \frac{1}{2} \times e = \frac{e}{2} \text{ R.H.S. Proved.}$$

11. ***Sum to infinity the series:*** $1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \text{..............}$ *[4]*

Soln: Let, t_n be the nth term of the given series. Then

t_n of the numerator = (nth term of 1, 3, 5, 7.......)

= [a + (n − 1) d]

= [1 + (n − 1) 2]

= [1 + 2n − 2]

= 2n − 1

Also, t_n of the denominator = (n^{th} term of 0, 1, 2, 3, ...)

$$= [a + (n-1).d]$$
$$= [0 + (n-1).1]$$
$$= 0 + n - 1$$
$$= n - 1 \quad [\because \text{ denominators of each term are having with } 0!, 1!, 2!, 3!.....]$$

$$\therefore \quad t_n = \frac{2n-1}{(n-1)!}$$
$$= \frac{2(n-1)+1}{(n-1)!}$$
$$= \frac{2(n-1)}{(n-1)!} + \frac{1}{(n-1)!} = \frac{2(n-1)}{(n-1)(n-2)!} + \frac{1}{(n-1)!} = \frac{2}{(n-2)!} + \frac{1}{(n-1)!}$$

Now, putting n = 1, 2, 3, ..., we get

$$t_1 = 0 + \frac{1}{0!}$$
$$t_2 = \frac{2}{0!} + \frac{1}{1!}$$
$$t_3 = \frac{2}{1!} + \frac{1}{2!}$$
$$t_4 = \frac{2}{2!} + \frac{1}{3!}$$

....

Now, by addition, the sum of given series is

$$\therefore \quad t_1 + t_2 + t_3 + t_4 + ... = \left(\frac{2}{0!} + \frac{2}{1!} + \frac{2}{2!} + ...\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$
$$= 2\left(1 + \frac{1}{1!} + \frac{1}{2!} + ...\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$
$$= 2e + e = 3e \textbf{ Ans.}$$

12. ***Show that:*** $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!} = e - 1.$ [4]

Soln: Let, t_n be the n^{th} term of the given series. Then

$$t_n = \frac{n^2}{(n+1)!}$$
$$= \frac{n^2 - 1 + 1}{(n+1)!}$$
$$= \frac{n^2-1}{(n+1)!} + \frac{1}{(n+1)!}$$
$$= \frac{(n+1)(n-1)}{(n+1)!} + \frac{1}{(n+1)!}$$
$$= \frac{(n+1)(n-1)}{(n+1)n!} + \frac{1}{(n+1)!}$$
$$= \frac{n-1}{n!} + \frac{1}{(n+1)!}$$
$$= \frac{n}{n!} - \frac{1}{n!} + \frac{1}{(n+1)!}$$
$$= \frac{1}{(n-1)!} - \frac{1}{n!} + \frac{1}{(n+1)!}$$

Now, putting n = 1, 2, 3, ..., we get

$$t_1 = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!}$$
$$t_2 = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!}$$
$$t_3 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

....

Adding all the terms, we get

$$\therefore \quad t_1 + t_2 + t_3 + ... = \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\right)$$
$$= e - (e-1) + (e-2)$$
$$= e - e + 1 + e - 2$$
$$= e - 1 \textbf{ R.H.S. Proved.}$$

13. ***(a)*** ***If*** $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + ...$, ***show that:*** $y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

(b) ***Prove that:*** $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + ... = 2(1 - ln2)$

Soln: (a) Given that,

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + ...$$

or, $x = \log_e(1+y)$

or, $1 + y = e^x$

or, $1 + y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

$\therefore \quad y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$ **Hence Proved.**

(b) L.H.S. $= \frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + ...$

$$= 2\left(\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \frac{1}{8.9} + ...\right)$$
$$= 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{8} - \frac{1}{9}\right) + ...\right]$$
$$= 2\left[1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - ....\right)\right]$$
$$= 2[1 - ln(1+1)]$$
$$= 2(1 - ln2) \textbf{ R.H.S.}$$

□□□

Chapter 3

Complex Number

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3.1 Complex Number

Basic Formulae and Key Points

1. **Complex Number:** An ordered pair (a, b) of two real numbers a and b is said to be a complex number. The complex number (a, b) can be written as a + ib, where $i^2 = -1$.
2. **Conjugate of a Complex Number:** If $z = a + ib$ then, it's conjugate $\bar{z} = a - ib$.
3. **Absolute Value of a Complex Number:** If $z = a + ib$ then, its absolute value $|z| = \sqrt{a^2 + b^2}$
4. **Polar (Trigonometric) Form of a Complex Number:**

   If $z = a + ib$ then its polar form is $z = r(\cos\theta + i\sin\theta)$ Where, $r = |z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, $x \neq 0$.

> **Note:** *θ is an amplitude or argument of z, and written as* ***amp(z) or arg(z)***.

5. **For θ :**

   (i) If x, y > 0, then θ lies in 1st quadrant.
   (ii) If x < 0, y > 0, then θ lies in 2nd quadrant.
   (iii) If x, y < 0, then θ lies in 3rd quadrant.
   (iv) If x > 0, y < 0 then θ lies in 4th quadrant.
   (v) The general form of z = r (cos θ + i sin θ) is $z = r[\cos(\theta + 360°.n) + i\sin(\theta + 360°.n)]$, Where, n ∈ I.

$180° - \theta$	$0 \le \theta \le 90°$
$180° + \theta$	$360° - \theta$

6. **Euler's Formula:**

   The Euler's formula is defined by $e^{i\theta} = \cos\theta + i\sin\theta$.
   If $z = r(\cos\theta + i\sin\theta)$ then it's Euler's form is $z = re^{i\theta}$.
7. **Product and Quotient of Complex Number in Polar Forms:**

   If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then,

   $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

   $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

   Where, $amp(z_1z_2) = \theta_1 + \theta_2 = amp(z_1) + amp(z_2)$ and $amp\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = amp(z_1) - amp(z_2)$.
8. **De Moivre's Theorem:**

   If n is any positive integer, then $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
9. **Cube Roots of Unity:**

   The three cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ denoted by 1, ω and ω^2.

   (i) $\omega^3 = 1$ (ii) $1 + \omega + \omega^2 = 0$

Group 'A' (Multiple Choice Questions and Answers)

Polar Form, Euler's Form, De Moiver's Theorem and its Application

1. *It is given that $z = r(\cos\theta + i\sin\theta)$ and n be a positive integer, which one of the following represent z^n?* **[2081 Supp. Set A]**

   (a) $r(\cos n\theta + i\sin n\theta)$ (b) $r^n(\cos n\theta - i\sin n\theta)$
   (c) $r^n(\cos n\theta + i\sin n\theta)$ (d) $r(\cos n\theta - i\sin n\theta)$
2. *Which one of the following is amplitude of the complex number 1 – i ?* **[2081 Supp. Set B]**

   (a) 45° (b) 135° (c) 225° (d) 315°
3. *The cartesian form of complex number 2(cos 30° + i sin 30°) is ....*

   (a) $\sqrt{3} + i$ (b) $-\sqrt{3} + i$ (c) $\sqrt{3} - i$ (d) $-\sqrt{3} - i$
4. *The cartesian form of complex number 3 (cos 120° + i sin 120°) is .....*

   (a) $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ (b) $\frac{3}{2} - \frac{3\sqrt{3}}{2}i$ (c) $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ (d) $-\frac{3}{2} - \frac{3}{2}\sqrt{3}i$
5. *The Euler's form of complex number 3(cos 60° + i sin 60°) is*

   (a) $3e^{\frac{i\pi}{6}}$ (b) $3e^{-\frac{i\pi}{3}}$ (c) $3e^{\frac{i\pi}{3}}$ (d) $3e^{-\frac{i\pi}{6}}$
6. *The Euler's form of complex number 2(cos 150° + i sin 150°) is*

   (a) $2e^{\frac{i\pi}{6}}$ (b) $2e^{\frac{i5\pi}{6}}$ (c) $2e^{\frac{i\pi}{3}}$ (d) $2e^{\frac{i\pi}{4}}$
7. *Which one of the following is Euler's form of complex number – i ?* **[2081 Set V]**

   (a) $e^{\frac{\pi i}{6}}$ (b) $e^{\frac{3\pi i}{2}}$ (c) $e^{\frac{3\pi i}{4}}$ (d) $e^{\frac{\pi i}{2}}$
8. *What is the amplitude of a complex number i ?* **[2079 Set J]**

   (a) –1 (b) 90° (c) 270° (d) –90°
9. *The argument of complex number $i - \sqrt{3}$ is*

   (a) 30° (b) 150° (c) 210° (d) 330°
10. *The polar form of complex number $\sqrt{3} + i$ is ....*

    (a) 2 (cos 60° + i sin 60°) (b) 2 (cos 150° + i sin 150°)
    (c) 2 (cos 30° + i sin 30°) (d) 2 (sin 30° + i cos 30°)
11. *The polar form of complex number $\frac{i}{1-i}$ is ....*

    (a) $\frac{1}{\sqrt{2}}(\cos 45° + i\sin 45°)$ (b) $\frac{1}{\sqrt{2}}(\cos 135° + i\sin 135°)$
    (c) $\frac{1}{\sqrt{2}}(\cos 225° + i\sin 225°)$ (d) $\frac{1}{\sqrt{2}}(\cos 315° + i\sin 315°)$
12. *The polar form of complex number $\frac{1-i}{1+i}$ is ....* **[2081 Set W]**

    (a) cos 0° + i sin 0° (b) cos 90° + i sin 90°
    (c) cos 120° + i sin 120° (d) cos 270° + i sin 270°
13. *Find the value of (cos 25° + i sin 25°) (cos 20° + i sin 20°)*

    (a) $\frac{1}{\sqrt{2}}(1 + i)$ (b) $\frac{1}{\sqrt{2}}(-1 + i)$ (c) $-\frac{1}{\sqrt{2}}(1 + i)$ (d) $\frac{1}{\sqrt{2}}(1 - i)$

14. *Find the value of* $\frac{\cos 70° + i \sin 70°}{\cos 10° + i \sin 10°}$

(a) $\frac{1}{2}(-1+\sqrt{3}i)$ (b) $-\frac{1}{2}(1+\sqrt{3}i)$ (c) $\frac{1}{2}(1+\sqrt{3}i)$ (d) $\frac{1}{2}(1-\sqrt{3}i)$

15. *The value of* $[2(\cos 15° + i \sin 15°)]^6$ *is*

(a) 64 (b) −64 (c) 64i (d) −64i

16. *The value of* $(\cos 18° + i \sin 18°)^{20}$ *is*

(a) 1 (b) −1 (c) i (d) −i [2079 Set A]

17. *What is the magnitude of the complex number* $(1 - i)$?

(a) −1 (b) 1 (c) $\sqrt{2}$ (d) 2

Answer Key

1. c	2. d	3. a	4. c	5. c	6. b	7. b	8. b	9. b	10. c
11. b	12. d	13. a	14. c	15. c	16. a	17. b			

Group 'B' or 'C' (Subjective Questions and Answers)

1. *For any positive integer n,* $[r(\cos\theta + i\sin\theta)]^n = r^n[\cos n\theta + i\sin n\theta]$ [2080 Set G] [1]
   (a) *Name the theorem for positive integer n.* [1]
   (b) *Reduce the given statement when we replace n by (−n)?* [1]
   (c) *What does 'r' represent in the above statement?* [1]
   (d) *What does 'θ' represent in the above statement?* [1]
   (e) *Write any one application of the above statement?* [1]

Soln: (a) The name of given theorem is De Moivre's theorem.

(b) When, n is replaced by −n then given theorem becomes,
$[r(\cos\theta + i\sin\theta)]^{-n} = r^{-n}[\cos(-n).\theta + i\sin(-n).\theta] = r^{-n}[\cos n\theta - i\sin\theta]$

(c) In the above statement 'r' represent the modulus of complex number.

(d) In the above statement 'θ' represent the argument or amplitude of complex number.

(e) One application of De Moivre's theorem is in find n^{th} roots of complex number.

2. *If* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ [2080 Set I]
   (a) *What is the argument of* z_1z_2? [1]
   (b) *What is the modules of* z_1z_2? [1]
   (c) *What is the polar form of* z_1z_2? [1]

Soln: Given, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = z_2(\cos\theta_2 + i\sin\theta_2)$

(a) The argument of z_1z_2 is $\theta_1 + \theta_2$.

(b) The modules of z_1z_2 is r_1r_2.

(c) The polar form of z_1z_2 is $r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_1 + i\sin\theta_2)$
$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

3. (a) *What do you mean by argument of complex number?* [1]
   (b) ***Express the complex number in cartesian form whose modulus is 6 and amplitude is 60°.*** [1]
   (c) *Simplify:* $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16}$ [1]

Soln: (a) Let, $z = r(\cos\theta + i\sin\theta)$ be a complex number in polar form then θ is called the argument of z and is denoted by arg (z).

(b) Here, r = 6 and θ = 60°.
Then required complex number is $r(\cos\theta + i\sin\theta)$

$= 6(\cos 60° + i\sin 60°) = 6\left(\frac{1}{2} + i.\frac{\sqrt{3}}{2}\right) = 3 + 3\sqrt{3}i$

(c) $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16}$

$= 3^{16}\left[\cos\left(\frac{\pi}{4}\times 16\right) + i\sin\left(\frac{\pi}{4}\times 16\right)\right]$ [∵ By De Moivre's theorem]

$= 3^{16}[\cos 4\pi + i\sin 4\pi]$

$= 3^{16}(1 + i\times 0) = 3^{16}$

4. *If* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *be two complex numbers, prove that*
   (a) $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ [2]
   (b) $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$ [2]

Soln: Here, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1}$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2}$

(a) $z_1z_2 = r_1e^{i\theta_1}.r_2e^{i\theta_2}$
$= r_1r_2e^{i\theta_1 + i\theta_2}$
$= r_1r_2e^{i(\theta_1+\theta_2)}$
$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

(b) $\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}$
$= \frac{r_1}{r_2}\times e^{i\theta_1 - i\theta_2}$
$= \frac{r_1}{r_2}\times e^{i(\theta_1-\theta_2)}$
$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

5. (a) *State De Moivre's theorem.* [1]
   (b) *Write the amplitude of* $z = r(\cos\theta + i\sin\theta)$. [1]
   (c) *Express* $\frac{i}{i+1}$ *in the polar form.* [2]

Soln: (a) De Moivre's theorem
If n is any positive integer, then $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

(b) The amplitude of z is θ.

(c) Let, $x + iy = \frac{i}{1+i}$
$= \frac{i}{1+i}\times\frac{1-i}{1-i}$
$= \frac{i-i^2}{1-i^2} = \frac{i+1}{1+1} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$

∴ $x = \frac{1}{2}, y = \frac{1}{2}$

We have, $r = \sqrt{x^2+y^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

$$\tan\theta = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 = \tan 45°$$

$\therefore \quad \theta = 45°$

Now, the polar form of $\frac{i}{1+i}$ is $r(\cos\theta + i\sin\theta) = \frac{1}{\sqrt{2}}(\cos 45° + i\sin 45°)$.

6. (a) *Express the complex number $\sqrt{\frac{1-i}{1+i}}$ into Euler's form.* [2]

(b) *If $z = \cos\theta + i\sin\theta$, prove that* [2]

(i) $z^n + \frac{1}{z^n} = 2\cos n\theta$ (ii) $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Soln: (a) Let, $x + iy = \sqrt{\frac{1-i}{1+i}}$

$$= \sqrt{\frac{1-i}{1+i}\times\frac{1-i}{1-i}} = \sqrt{\frac{(1-i)^2}{1-i^2}} = \frac{1-i}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} + \left(\frac{-1}{\sqrt{2}}\right)i$$

$\therefore \quad x = \frac{1}{\sqrt{2}}, \quad y = -\frac{1}{\sqrt{2}}$

Now,

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{\frac{2}{2}} = 1$$

Again,

$$\tan\theta = \frac{y}{x}$$
$$= \frac{\frac{-1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1 = \tan\left(2\pi - \frac{\pi}{4}\right) \qquad [\because x > 0 \text{ and } y < 0]$$

$\therefore \quad \theta = \frac{7\pi}{4}$

Hence, the Euler's form of given complex number is $re^{i\theta} = 1e^{\frac{i7\pi}{4}} = e^{\frac{7\pi i}{4}}$

(b) Here, $z = \cos\theta + i\sin\theta$, then

$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$

Also,

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n)\theta + i\sin(-n)\theta = \cos n\theta - i\sin n\theta$$

Now,

(i) $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= 2\cos n\theta$

(ii) $z^n - \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$
$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$
$= 2i\sin n\theta$

7. (a) *Find the value of $(\sqrt{3}-i)^{14}$.* [2]

(b) *If $\bar{z}$ be the conjugate of the complex number z, prove that $Arg(\bar{z}) = 2\pi - Arg(z)$.* [2] [2079 Set A]

Soln: (a) Let, $x + iy = \sqrt{3} - i$

$\therefore \quad x = \sqrt{3}, y = -1$

We have,

$$r = \sqrt{x^2+y^2} = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{4} = 2$$

Also,

$$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}} = \tan(360° - 30°) \qquad [\because x > 0, y < 0]$$

$\therefore \quad \theta = 330°$.

Then, $\sqrt{3} - i = r(\cos\theta + i\sin\theta) = 2(\cos 330° + i\sin 330°)$

Now,

$$(\sqrt{3}-i)^{14} = [2(\cos 330° + i\sin 330°)]^{14}$$
$$= 2^{14}[\cos(330° \times 14) + i\sin(330° \times 14)]$$
$$= 2^{14}[\cos 4620° + i\sin 4620°]$$
$$= 2^{14}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$= \frac{2^{14}}{2}(1 - \sqrt{3}i) = 2^{13}(1 - \sqrt{3}i).$$

(b) Let, $z = r(\cos\theta + i\sin\theta)$ be any complex number then $Arg(z) = \theta$.

We have, $\bar{z} = r(\cos\theta - i\sin\theta)$
$= r[\cos(2\pi - \theta) + i\sin(2\pi - \theta)]$

$\therefore \quad Arg(\bar{z}) = 2\pi - \theta$
$= 2\pi - Arg(z)$.

8. (a) *Apply De Moivre's theorem to compute $(-1-\sqrt{3}i)^4$.* [2] [2079 GIE A]

(b) *If z be a complex number, prove that $amp\left(\frac{1}{z}\right) = -amp(z)$.* [2]

Soln: (a) Let, $x + iy = -1 - \sqrt{3}i$

$\therefore \quad x = -1, y = -\sqrt{3}$

We have,

$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$$

Also,

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$
$$= \tan(180^\circ + 60^\circ) \qquad [\because x < 0, y < 0]$$

$\therefore \quad \theta = 240^\circ$

Then, $-1 - \sqrt{3}i = r(\cos\theta + i\sin\theta) = 2(\cos 240^\circ + i\sin 240^\circ)$

Now, $(-1-\sqrt{3}i)^4 = [2(\cos 240^\circ + i\sin 240^\circ)]^4$
$$= 2^4[\cos 240^\circ \times 4 + i\sin 240^\circ \times 4]$$
$$= 16(\cos 960^\circ + i\sin 960^\circ)$$
$$= 16\left[\frac{-1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -8 - 8\sqrt{3}i$$

(b) Let, $z = r(\cos\theta + i\sin\theta)$ be any complex number then $\arg(z) = \theta$.

Now,
$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$
$$= \frac{1}{r(\cos\theta + i\sin\theta)} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$
$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta - i^2\sin^2\theta)}$$
$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$$

$\therefore \quad \arg\left(\frac{1}{z}\right) = -\theta$
$$= -\arg(z)$$

9. (a) If $z = \cos\theta + i\sin\theta$, find the value of $z^n + \frac{1}{z^n}$ by using De Moivre's theorem. [2, 2081 Set V]

(b) Find the square roots of $1 - \sqrt{3}i$ using De Moivre's theorem. [2, 2081 Set I]

Soln: (a) Here, $z = \cos\theta + i\sin\theta$, then
$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$

Also,
$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$$

Now,
$$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$$

(b) Let, $x + iy = 1 - \sqrt{3}i$

$\therefore \quad x = 1, y = -\sqrt{3}$

We have,
$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

and $\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3} = \tan(360^\circ - 60^\circ)$

$\therefore \quad \theta = 300^\circ$

Hence, $1 - \sqrt{3}i = r(\cos\theta + i\sin\theta) = 2(\cos 300^\circ + i\sin 300^\circ)$

Let, $z^2 = 2(\cos 300^\circ + i\sin 300^\circ)$
$$= 2[\cos(300^\circ + k.360^\circ) + i\sin(300^\circ + k.360^\circ)]$$

$\therefore \quad z_k = [2\{\cos(300^\circ + k.360^\circ) + i\sin(300^\circ + k.360^\circ)\}]^{\frac{1}{2}}$
$$= 2^{\frac{1}{2}}\left[\cos\left(\frac{300^\circ + k.360^\circ}{2}\right) + i\sin\left(\frac{300^\circ + k.360^\circ}{2}\right)\right]$$
$$= \sqrt{2}[\cos(150^\circ + k.180^\circ) + i\sin(150^\circ + k.180^\circ)], \ k = 0, 1$$

When, $k = 0$
$$z_0 = \sqrt{2}(\cos 150^\circ + i\sin 150^\circ) = \sqrt{2}\left(\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{\sqrt{2}}{2}(-\sqrt{3} + i) = \frac{1}{\sqrt{2}}(-\sqrt{3} + i)$$

When, $k = 1$,
$$z_1 = \sqrt{2}(\cos 330^\circ + i\sin 330^\circ)$$
$$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = \frac{\sqrt{2}}{2}(\sqrt{3} - i) = \frac{1}{\sqrt{2}}(\sqrt{3} - i)$$

Hence, the required square roots are $\pm\frac{1}{\sqrt{2}}(\sqrt{3} - i)$.

10. *The complex number for two real number -1 and $\sqrt{3}$ is given as $-1 + \sqrt{3}i$.*

(a) *State De Moivre's theorem.*

(b) *Find the amplitude of given complex number.*

(c) *Express the complex number in polar form.*

(d) *Find the square roots of given complex number.*

Soln: (a) De Moivre's theorem:

If n is any positive integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(b) Let, $x + iy = -1 + \sqrt{3}i$

$\therefore \quad x = -1$ and $y = \sqrt{3}$

We have, $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

or, $\quad \tan\theta = \tan(180^\circ - 60^\circ) \qquad [\because x < 0, y > 0]$

$\therefore \quad \theta = 120^\circ$

i.e. amplitude is 120°

(c) Also, we have $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\therefore$ Required polar form is $r(\cos\theta + i\sin\theta) = 2(\cos 120^\circ + i\sin 120^\circ)$

(d) Let, $z^2 = 2(\cos 120^\circ + i\sin 120^\circ)$
$$= 2[\cos(120^\circ + k.360^\circ) + i\sin(120^\circ + k.360^\circ)]$$

$\therefore \quad z_k = [2\{\cos(120^\circ + k.360^\circ) + i\sin(120^\circ + k.360^\circ)\}]^{\frac{1}{2}}$
$$= 2^{\frac{1}{2}}\left[\cos\left(\frac{120^\circ + k.360^\circ}{2}\right) + i\sin\left(\frac{120^\circ + k.360^\circ}{2}\right)\right]$$
$$= \sqrt{2}[\cos(60^\circ + k.180) + i\sin(60 + k.180)],$$

Putting k = 0, 1

When, k = 0

$$z_0 = \sqrt{2}\ (\cos 60° + i \sin 60°) = \sqrt{2}\left(\frac{1}{2} + i.\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{2}(1 + \sqrt{3}i) = \frac{1}{\sqrt{2}}(1 + \sqrt{3}i)$$

When, k = 1,

$$z_1 = \sqrt{2}\,[\cos (60° + 180°) + i \sin (60° + 180°)]$$
$$= \sqrt{2}\,[\cos 240° + i \sin 240°) = \sqrt{2}\left(-\frac{1}{2} - i.\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-\sqrt{2}}{2}(1 + \sqrt{3}i) = \frac{-1}{\sqrt{2}}(1 + \sqrt{3}i\,)$$

Hence, the required square roots are $\pm\frac{1}{\sqrt{2}}(1 + \sqrt{3}i)$

[3]

11. *Using De Moivre's theorem, find the cube roots of the complex number −27i.*

Soln: Let, $x + iy = 0 - 27i$

$\therefore$ $x = 0, y = -27$

We have, $r = \sqrt{x^2 + y^2}$
$= \sqrt{0^2 + (-27)^2} = \sqrt{729} = 27$

and $\tan\theta = \frac{y}{x} = \frac{-27}{0} = \tan 270°$

$\therefore$ $\theta = 270°$

Hence, $-27i = r(\cos\theta + i \sin\theta)$
$= 27 (\cos 270° + i \sin 270°)$

Let, $z^3 = 27 (\cos 270° + i \sin 270°)$
$= 27 \{\cos (270° + k.\ 360°) + i \sin (270° + k.\ 360°)\}$

$$\therefore\quad z_k = [27 \{\cos (270° + k.\ 360°) + i \sin (270° + k.\ 360°)\}]^{\frac{1}{3}}$$
$$= (27)^{\frac{1}{3}}\left[\cos\left(\frac{270° + k.360°}{3}\right) + i \sin\left(\frac{270° + k.\ 360°}{3}\right)\right]$$
$$= 3[\cos (90° + k.\ 120°) + i \sin (90° + k.\ 120°)],\ k = 0, 1, 2$$

When, k = 0,

$$z_0 = 3(\cos 90° + i \sin 90°) = 3\,(0 + i \times 1) = 3i$$

When, k = 1,

$$z_1 = 3\,[\cos (90° + 120°) + i \sin (90° + 120°)]$$
$$= 3\,(\cos 210° + i \sin 210°)$$
$$= 3\left\{-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right\} = -\frac{3}{2}(\sqrt{3} + i)$$

When, k = 2,

$$z_2 = 3[\cos (90° + 240°) + i \sin (90° + 240°)]$$
$$= 3\,(\cos 330° + i \sin 330°)$$
$$= 3\left(\frac{\sqrt{3}}{2} - i.\frac{1}{2}\right) = \frac{3}{2}(\sqrt{3} - i)$$

$\therefore$ Required cube roots are $3i$, $\frac{-3}{2}(\sqrt{3} + i)$ and $\frac{3}{2}(\sqrt{3} - i)$.

12. *Use De Moivre's theorem to solve $z^3 + 1 = 0$*

OR

Using De Moivre's theorem compute cube roots of −1. [2]

Soln: Here, $z^3 + 1 = 0$

or, $z^3 = -1$
$= -1 + 0 \times i$
$= \cos 180° + i \sin 180°$
$= \cos (180° + k.\ 360°) + i \sin (180° + k.\ 360°)$

$$z_k = [\cos (180° + k\ .\ 360°) + i \sin (180° + k\ .\ 360°)]^{\frac{1}{3}}$$
$$\therefore\quad z_k = \cos\left(\frac{180° + k.\ 360°}{3}\right) + i \sin\left(\frac{180° + k.\ 360°}{3}\right)$$
$$= \cos (60° + k.\ 120°) + i \sin (60° + k.\ 120°),\ k = 0, 1, 2$$

When, k = 0,

$$z_0 = \cos 60° + i \sin 60° = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1 + \sqrt{3}i}{2}$$

When, k = 1,

$$z_1 = \cos (60° + 120°) + i \sin (60° + 120°)$$
$$= \cos (180°) + i \sin 180° = -1 + i \times 0 = -1$$

When, k = 2,

$$z_2 = \cos (60° + 240°) + i \sin (60° + 240°)$$
$$= \cos 300° + i \sin 300° = \frac{1}{2} - \frac{\sqrt{3}}{2}i = \frac{1 - \sqrt{3}i}{2}$$

$\therefore$ The required cube roots are -1, $\frac{1 \pm \sqrt{3}i}{2}$

13. *Using De Moivre's theorem, find the fourth roots of $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$* [3]

Soln: Let, $x + iy = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore$ $x = \frac{-1}{2}$ and $y = \frac{\sqrt{3}}{2}$

We have,

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

and $\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}} = -\sqrt{3}$

or, $\tan\theta = \tan (180° - 60°)$ $\qquad (\because x < 0, y > 0)$

$\therefore$ $\theta = 120°$

Hence, $\frac{-1}{2} + \frac{3}{2}i = r(\cos\theta + i \sin\theta)$
$= \cos 120° + i \sin 120°$

Let, $z^4 = \cos 120° + i \sin 120°$

$= \cos(120° + k.360°) + i \sin(120° + k.360°)$

$\therefore\ z_k = [\cos(120° + k.360°) + i \sin(120° + k.360°)]^{\frac{1}{4}}$

$= \cos\left(\frac{120° + k.360°}{4}\right) + i \sin\left(\frac{120° + k.360°}{4}\right)$

$= \cos(30° + k.90°) + i \sin(30° + k.90°),\ k = 0, 1, 2, 3$

When, k = 0,

$z_0 = \cos 30° + i \sin 30° = \frac{\sqrt{3}}{2} + i.\frac{1}{2} = \frac{\sqrt{3}+i}{2}$

When, k = 1

$z_1 = \cos(30° + 90°) + i \sin(30° + 90°)$

$= \cos 120° + i \sin 120° = \frac{-1}{2} + \frac{\sqrt{3}}{2}i = \frac{-(1-\sqrt{3}i)}{2}$

When, k = 2,

$z_2 = \cos(30° + 180°) + i \sin(30° + 180°)$

$= \cos 210° + i \sin 210° = \frac{-\sqrt{3}}{2} - i.\frac{1}{2} = -\frac{(\sqrt{3}+i)}{2}$

When, k = 3,

$z_3 = \cos(30° + 270°) + i \sin(30° + 270°)$

$= \cos 300° + i \sin 300° = \frac{1}{2} - i.\frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}i}{2}$

Hence, the required root are $\pm\left(\frac{\sqrt{3}+i}{2}\right), \pm\left(\frac{1-\sqrt{3}i}{2}\right)$.

14. ***Using De Moivre's theorem solve $z^6 = 1$.***

Or

Using De Moivre's theorem, compute sixth roots of 1.

Soln: Given, $z^6 = 1 = 1 + 0 \times i = \cos 0 + i \sin 0$

$= \cos(0 + k.360°) + i \sin(0 + k.360°)$

$= \cos(360°k) + i \sin(360°k)$

$\therefore\ z_k = [\cos(360k) + i \sin(360k)]^{\frac{1}{6}}$

$= \cos\left(\frac{360°k}{6}\right) + i \sin\left(\frac{360°k}{6}\right) = \cos(60°k) + i \sin(60°k),\ k = 0, 1, 2, 3, 4, 5$

When, k = 0,

$z_0 = \cos 0° + i \sin 0° = 1 + i \times 0 = 1$

When, k = 1,

$z_1 = \cos 60° + i \sin 60° = \frac{1}{2} + i.\frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}i}{2}$

When, k = 2,

$z_2 = \cos 120° + i \sin 120° = \frac{-1}{2} + i.\frac{\sqrt{3}}{2} = -\left(\frac{1-\sqrt{3}i}{2}\right)$

When, k = 3,

$z_3 = \cos 180° + i \sin 180° = -1 + i \times 0 = -1$

When, k = 4,

$z_4 = \cos 240° + i \sin 240° = \frac{-1}{2} + i\left(\frac{-\sqrt{3}}{2}\right) = -\left(\frac{1+\sqrt{3}i}{2}\right)$

When, k = 5,

$z_5 = \cos 300° + i \sin 300° = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}i}{2}$

$\therefore$ Required roots are $\pm 1, \pm\frac{1}{2}(1 + \sqrt{3}i)$ and $\pm\frac{1}{2}(1 - \sqrt{3}i)$.

15. ***State and prove De Moivre's theorem. Is the theorem true for any integers? Justify your answer.*** ***[1+2+1]***

Soln: De Moivre's theorem:

If n is any positive integer, then $[r(\cos\theta + i \sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$.

Proof:

When, n = 1, $[r(\cos\theta + i \sin\theta)]^1 = r(\cos\theta + i \sin\theta) = r^1(\cos 1.\theta + i \sin 1.\theta)$

Thus, the theorem is true for n = 1.

Assume that the theorem is true for n = k.

i.e. $[r(\cos\theta + i \sin\theta)]^k = r^k[\cos k\theta + i \sin k\theta]$ .... (i)

We shall show that the theorem is true for n = k + 1, whenever n = k.

For this, $[r(\cos\theta + i \sin\theta)]^{k+1} = [r(\cos\theta + i \sin\theta)]^k\, r(\cos\theta + i \sin\theta)$

$= r^k[\cos k\theta + i \sin k\theta].\, r(\cos\theta + i \sin\theta)$ [$\because$ using (i)]

$= r^{k+1}[\cos(k\theta + \theta) + i \sin(k\theta + \theta)]$

$= r^{k+1}[\cos(k+1)\theta + i \sin(k+1)\theta]$

This shows that the theorem is true for n = k + 1, whenever it is true for n = k. Then by principle of mathematical induction, the theorem is true for all positive integer n.

"Last Part"

[3 When, n = 0, $[r(\cos\theta + i \sin\theta)]^0 = r^0[\cos 0° + i \sin 0°]$

i.e. $1 = 1(1 + i \times 0) = 1$ (True).

Again, when $n = -k$, $(k > 0)$.

Then, $[r(\cos\theta + i\sin\theta)]^{-k} = \frac{1}{[r(\cos\theta + i \sin\theta)]^k}$

$= \frac{1}{r^k(\cos k\theta + i \sin k\theta)}$

$= \frac{r^{-k}}{\cos k\theta + i \sin k\theta} \times \frac{\cos k\theta - i \sin k\theta}{\cos k\theta - i \sin k\theta}$

$= \frac{r^{-k}(\cos k\theta - i \sin k\theta)}{\cos^2 k\theta - i^2 \sin^2 k\theta}$

$= \frac{r^{-k}\{\cos(-k\theta) + i \sin(-k\theta)\}}{\cos^2 k\theta + \sin^2 k\theta}$

$= \frac{r^{-k}\{\cos(-k).\theta + i \sin(-k).\theta\}}{1}$

$= r^{-k}\{\cos(-k).\theta + i \sin(-k).\theta\}$.

This shows that the theorem is true for $n = -k$, $(k > 0)$.

Hence, this theorem is true for any integer.

3.2 Cube Roots of Unity

Group 'A' (Multiple Choice Questions and Answers)

1. *If 1, ω, ω^2 are the cube roots of unity, then*
   (a) $1 + \omega + \omega^2 = 0$ (b) $1 + \omega^3 = 0$ (c) $\omega = -\omega^2$ (d) $\omega + \omega^2 = 1$
2. *If ω be the one of the complex cube roots of unity, then*
   (a) $\omega^3 = 1$ (b) $1 + \omega + \omega^2 = 0$ (c) $\omega = \frac{1}{\omega^2}$ (d) All of them.
3. *If ω be the one of the complex cube roots of unity, then $\omega\bar{\omega} =$*
   (a) -1 (b) 0 (c) 1 (d) ω^2
4. *If ω be the one of the complex cube roots of unity, then $\omega + \bar{\omega} =$*
   (a) -1 (b) 0 (c) 1 (d) ω^2
5. *The value of $(1 - \omega + \omega^2)^5$ is*
   (a) 32 (b) 48 (c) 64 (d) 128
6. *The value of $\omega^7 + \omega^8 + \omega^9$ is*
   (a) -1 (b) 0 (c) 1 (d) ω
7. *The value of $\frac{a + b\omega + c\omega^2}{c + b\omega^2 + a\omega}$ is*
   (a) 0 (b) 1 (c) ω (d) ω^2
8. *If $x = a + b$, $y = a\omega + b\omega^2$ and $z = a\omega^2 + b\omega$ then $x + y + z =$*
   (a) 0 (b) 1 (c) -1 (d) w
9. *If ω is an imaginary cube roots of unity, then $|\omega| + |\omega^2| =$*
   (a) 0 (b) 1 (c) 2 (d) 2
10. *If ω is an imaginary cube roots of unity, then which one is correct?*
    (a) $|\omega| > |\omega^2|$ (b) $|\omega| < |\omega^2|$ (c) $1 + |\omega| + |\omega^2| = 0$ (d) $|\omega| = |\omega^2|$

Answer Key

1. a	2. d	3. c	4. a	5. c	6. b	7. d	8. a	9. c	10. d

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Using De Moivre's theorem, find the cube roots of unity. If $\omega = \frac{-1 + \sqrt{3}i}{2}$ be complex cube ro[ots] of unity, show that $\omega^2 = \frac{-1 - \sqrt{3}i}{2}$* [2+...]

   (b) *If 1, ω, ω^2 are cube roots of unity, find the value of $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$.* [2080 Option...]

Soln: (a) Let, z be the cube roots of unity, then $z^3 = 1 = 1 + 0.i$

$= \cos 0^\circ + i \sin 0^\circ$

$= \cos (0 + k.\,360^\circ) + i \sin (0 + k.\,360^\circ)$

$\therefore \quad z_k = [\cos (360^\circ k) + i \sin (360^\circ k)]^{\frac{1}{3}}$

$= \cos\left(\frac{360^\circ k}{3}\right) + i \sin\left(\frac{360^\circ k}{3}\right) = \cos (120^\circ k) + i\sin (120^\circ k),\ k = 0, 1, 2$

When, k = 0,

$z_0 = \cos 0^\circ + i \sin 0^\circ = 1 + i \times 0 = 1$

When, k = 1,

$z_1 = \cos 120^\circ + i \sin 120^\circ = \frac{-1}{2} + i.\frac{\sqrt{3}}{2} = \frac{-1 + \sqrt{3}i}{2}$

When, k = 2,

$z_2 = \cos 240^\circ + i \sin 240^\circ = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = \frac{-1 - \sqrt{3}i}{2}$

$\therefore$ The cube roots of unity are 1, $\frac{-1 + \sqrt{3}i}{2}$ and $\frac{-1 - 1\sqrt{3}i}{2}$.

Next Part:

Given, $\omega = \frac{-1 + \sqrt{3}i}{2}$, then

$$\omega^2 = \left(\frac{-1 + \sqrt{3}i}{2}\right)^2$$

$$= \frac{(-1)^2 + 2(-1).\sqrt{3}i + (\sqrt{3}i)^2}{4}$$

$$= \frac{1 - 2\sqrt{3}i - 3}{4} = \frac{-2 - 2\sqrt{3}i}{4} = \frac{-1 - \sqrt{3}i}{2}.$$

(b) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega^3.\,\omega)\{1 - (\omega^3)^2.\,\omega^2\}$

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= (1 - \omega)^2 (1 - \omega^2)^2$

$= \{(1 - \omega)(1 - \omega^2)\}^2$

$= (1 - \omega^2 - \omega + \omega^3)^2$

$= \{1 - (\omega^2 + \omega) + 1\}^2$

$= \{1 - (-1) + 1\}^2 \qquad (\because 1 + \omega + \omega^2 = 0)$

$= 3^2$

$= 9$

2. *Using De Moivre's theorem find the cube roots of unity. Also establish the properties of the cube roots of unity.* [2+3]

Soln: First Part:

See Q. No. (1) (a).

Second part:

Properties of cube roots of unity.

Each imaginary cube root of unity is the square of the other.

Proof:

$$\left(\frac{-1 + \sqrt{3}i}{2}\right)^2 = \frac{1 - 2\sqrt{3}i + 3i^2}{4}$$

$$= \frac{1 - 2\sqrt{3}i - 3}{4} = \frac{-2 - 2\sqrt{3}i}{4} = \frac{-1 - \sqrt{3}i}{2}$$

Again,

$$\left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \frac{1+2\sqrt{3}i+3i^2}{4}$$
$$= \frac{1+2\sqrt{3}i-3}{4} = \frac{-2+2\sqrt{3}i}{4} = \frac{-1+\sqrt{3}i}{2}$$

The sum of the three cube roots of unity is zero.

Proof:

$$1+\left(\frac{-1+\sqrt{3}i}{2}\right)+\left(\frac{-1-\sqrt{3}i}{2}\right) = \frac{2-1+\sqrt{3}i-1-\sqrt{3}i}{2}$$
$$= \frac{2-2}{2} = 0$$

The product of two imaginary cube roots of unity is 1.

Proof:

$$\left(\frac{-1+\sqrt{3}i}{2}\right)\times\left(\frac{-1-\sqrt{3}i}{2}\right) = \frac{(-1)^2-\left(\sqrt{3}i\right)^2}{4}$$
$$= \frac{1-3i^2}{4}$$
$$= \frac{1+3}{4} = 1$$

3. (a) *If ω is a complex cube root of unity, show that $(1-\omega+\omega^2)^4(1+\omega-\omega^2)^4 = 256$.* [2]

(b) *Show that $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = \omega$, where ω and ω^2 are cube root of unity.* [2] [2081 Set W]

Soln: (a)
$$(1-\omega+\omega^2)^4(1+\omega-\omega^2)^4$$
$$= (1+\omega^2-\omega)^4(1+\omega-\omega^2)^4$$
$$= (-\omega-\omega)^4(-\omega^2-\omega^2)^4 \qquad (\because 1+\omega+\omega^2=0)$$
$$= (-2\omega)^4(-2\omega^2)^4$$
$$= (4\omega^3)^4 = 4^4 = 256$$

(b)
$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}$$
$$= \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\times\frac{\omega}{\omega}$$
$$= \frac{a+b\omega+c\omega^2}{b\omega+c\omega^2+a\omega^3}\times\frac{\omega}{1} = \frac{a+b\omega+c\omega^2}{a+b\omega+c\omega^2}\times\omega = \omega.$$

4. (a) *Find the value of $(1-\omega+\omega^2)^4+(1+\omega-\omega^2)^4$, where ω and ω^2 are imaginary cube roots of unity.* [2] [2081 Set V]

(b) *If $x = a+b$, $y = a\omega+b\omega^2$ and $z = a\omega^2+b\omega$, prove that $xyz = a^3+b^3$.* [2]

Soln: (a)
$$(1-\omega+\omega^2)^4+(1+\omega-\omega^2)^4$$
$$= (1+\omega^2-\omega)^4+(1+\omega-\omega^2)^4$$
$$= (-\omega-\omega)^4+(-\omega^2-\omega^2)^4 \qquad (\because 1+\omega+\omega^2=0)$$
$$= (-2\omega)^4+(-2\omega^2)^4$$
$$= 16\omega^4+16\omega^8$$
$$= 16\omega^3.\omega+16(\omega^3)^2.\omega^2$$
$$= 16\omega+16\omega^2 \qquad (\because \omega^3=1)$$
$$= 16(\omega+\omega^2)$$
$$= 16(-1) = -16.$$

(b)
$$xyz = (a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$$
$$= (a+b)(a^2\omega^3+ab\omega^2+ab\omega^4+b^2\omega^3)$$
$$= (a+b)(a^2+ab\omega^2+ab\omega^3.\omega+b^2)$$
$$= (a+b)\{a^2+ab(\omega^2+\omega)+b^2\}$$
$$= (a+b)(a^2-ab+b^2)$$
$$= a^3+b^3. \qquad (\because 1+\omega+\omega^2=0)$$

5. (a) *If $\alpha = \frac{1}{2}(-1+\sqrt{-3})$, $\beta = \frac{1}{2}(-1-\sqrt{-3})$, show that $\alpha^4+\alpha^2\beta^2+\beta^4=0$* [2] [2081 Set V]

(b) *If 1, ω, ω^2 be the cube roots of unity, prove that*

(i) $(1+\omega^2)^3-(1+\omega)^3=0$ [1]

(ii) $(2+\omega)(2+\omega^2)(2-\omega^2)(2-\omega^4)=21$ [2]

Soln: (a) Given, $\alpha = \frac{1}{2}(-1+\sqrt{-3}) = \frac{-1+\sqrt{3}i}{2} = \omega$

and $\beta = \frac{1}{2}(-1-\sqrt{-3}) = \frac{-1-\sqrt{3}i}{2} = \omega^2$ (say).

Now,
$$\alpha^4+\alpha^2\beta^2+\beta^4 = \omega^4+\omega^2(\omega^2)^2+(\omega^2)^4$$
$$= \omega^4+\omega^2.\omega^4+\omega^8$$
$$= \omega^3.\omega+(\omega^3)^2+(\omega^3)^2.\omega^2$$
$$= \omega+1+\omega^2 \qquad (\because \omega^3=1)$$
$$= 0$$

(b) (i)
$$(1+\omega^2)^3-(1+\omega)^3$$
$$= (-\omega)^3-(-\omega^2)^3$$
$$= -\omega^3+\omega^6$$
$$= -1+(\omega^3)^2 = -1+1^2 = 0$$

(ii)
$$(2+\omega)(2+\omega^2)(2-\omega^2)(2-\omega^4)$$
$$= (2+\omega)(2+\omega^2)(2-\omega^2)(2-\omega^3.\omega)$$
$$= (2+\omega)(2-\omega)(2+\omega^2)(2-\omega^2)$$
$$= (4-\omega^2)(4-\omega^4)$$
$$= (4-\omega^2)(4-\omega^3.\omega)$$
$$= (4-\omega^2)(4-\omega)$$
$$= 16-4\omega-4\omega^2+\omega^3$$
$$= 16-4(\omega+\omega^2)+1$$
$$= 17-4(-1) \qquad (\because 1+\omega+\omega^2=0)$$
$$= 17+4$$
$$= 21$$

6. (a) *If α, β are the imaginary cube roots of unity, prove that*

(i) $\alpha^2+\beta^2-\alpha\beta=-2$ [1]

(ii) $\alpha^4+\beta^4+\dfrac{1}{\alpha\beta}=0$ [2]

(b) *If $x = a+b$, $y = a\omega+b\omega^2$ and $z = a\omega^2+b\omega$ prove that $x^2+y^2+z^2=6ab$.* [2]

Soln: (a) Here, α and β are the imaginary cube roots of unity, so let $\alpha=\omega$ and $\beta=\omega^2$.

(i)
$$\alpha^2+\beta^2-\alpha\beta = \omega^2+(\omega^2)^2-\omega.\omega^2$$
$$= \omega^2+\omega^4-\omega^3$$
$$= \omega^2+\omega^3.\omega-1$$

$= \omega^2 + \omega - 1$ $\quad (\because \omega^3 = 1)$

$= -1 - 1$ $\quad (\because 1 + \omega + \omega^2 = 0)$

$= -2$

(ii) $\alpha^4 + \beta^4 + \frac{1}{\alpha\beta} = \omega^4 + (\omega^2)^4 + \frac{1}{\omega.\omega^2}$

$= \omega^4 + \omega^8 + \frac{1}{\omega^3}$

$= \omega^3.\omega + (\omega^3)^2.\omega^2 + \frac{1}{1}$

$= \omega + \omega^2 + 1$

$= 0$

(b) $x^2 + y^2 + z^2 = (a+b)^2 + (a\omega + b\omega^2)^2 + (a\omega^2 + b\omega)^2$

$= a^2 + 2ab + b^2 + a^2\omega^2 + 2ab\omega^3 + b^2\omega^4 + a^2\omega^4 + 2ab\omega^3 + b^2\omega^2$

$= a^2 + 2ab + b^2 + a^2\omega^2 + 2ab + b^2\omega^3.\omega + a^2\omega^3.\omega + 2ab + b^2\omega^2$

$= 6ab + a^2 + b^2 + a^2\omega^2 + b^2\omega + a^2\omega + b^2\omega^2$

$= 6ab + a^2(1 + \omega + \omega^2) + b^2(1 + \omega + \omega^2)$

$= 6ab + a^2 \times 0 + b^2 \times 0$

$= 6ab.$

7. (a) *Show that* $\sqrt{-1-\sqrt{-1-\sqrt{-1-\sqrt{-1-\ldots}}}}$ *to* $\infty = \omega$ *or* ω^2. [2]

(b) *If* $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$ *prove that* $x^3 + y^3 + z^3 = 3(a^3 + b^3)$. [3]

Soln: (a) Let, $\sqrt{-1-\sqrt{-1-\sqrt{-1-\ldots}}}$ to $\infty = z$ ......(i)

Then $\sqrt{-1-z} = z$ $\quad$ ($\because$ **Using (i)**)

Squaring, $-1 - z = z^2$

or, $z^2 + z + 1 = 0$

$\therefore \quad z = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$= \frac{-1 + \sqrt{3}i}{2}$ or $\frac{-1 - \sqrt{3}i}{2}$

$= \omega$ or ω^2.

(b) $x^3 + y^3 + z^3 = (a+b)^3 + (a\omega + b\omega^2)^3 + (a\omega^2 + b\omega)^3$

$= (a+b)^3 + \omega^3(a + b\omega)^3 + \omega^3(a\omega + b)^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 + a^3 + 3a^2b\omega + 3ab^2\omega^2 + b^3\omega^3 + a^3\omega^3 + 3a^2b\omega^2 + 3ab^2\omega + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 + a^3 + 3a^2b\omega + 3ab^2\omega^2 + b^3 + a^3 + 3a^2b\omega^2 + 3ab^2\omega + b^3$

$= 3a^3 + 3b^3 + 3a^2b(1 + \omega + \omega^2) + 3ab^2(1 + \omega + \omega^2)$

$= 3a^3 + 3b^3 + 3a^2b \times 0 + 3ab^2 \times 0$

$= 3(a^3 + b^3).$

8. (a) *Prove that* $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^6 = 2.$ [2]

(b) *Prove that* $\left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n = \begin{cases} 2 \text{ if n is multiple of 3} \\ -1 \text{ if n is any other integer.} \end{cases}$ [1+2]

Soln: Since, $\frac{-1+\sqrt{-3}}{2} = \frac{-1+\sqrt{3}i}{2} = \omega$ and $\frac{-1-\sqrt{-3}}{2} = \frac{-1-\sqrt{3}i}{2} = \omega^2$

(a) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^6 = \omega^9 + (\omega^2)^6$

$= (\omega^3)^3 + (\omega^3)^4$

$= 1 + 1$

$= 2$

(b) **First Part:**

If n is multiple of 3, then n = 3k, k = 0, 1, 2, ...

Now,

$\left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n = (\omega)^{3k} + (\omega^2)^{3k}$

$= (\omega^3)^k + (\omega^3)^{2k}$

$= 1^k + 1^{2k}$

$= 1 + 1 = 2$

Second Part:

If n is not multiple of 3, then n = 3k + 1 and n = 3k + 2, k = 0, 1, 2, 3 ......

When, n = 3k + 1,

$\left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n = (\omega)^{3k+1} + (\omega^2)^{3k+1}$

$= (\omega^3)^k.\omega + (\omega^3)^{2k}.\omega^2$

$= 1.\omega + 1.\omega^2$ $\quad (\because \omega^3 = 1)$

$= \omega + \omega^2$

$= -1$ $\quad (\because 1 + \omega + \omega^2 = 0)$

When, n = 3k + 2

$\left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n = \omega^{3k+2} + (\omega^2)^{3k+2}$

$= (\omega^3)^k.\omega^2 + (\omega^3)^{2k}.\omega^4$

$= 1.\omega^2 + 1.\omega^3.\omega$ $\quad (\because \omega^3 = 1)$

$= \omega^2 + \omega$

$= -1$ $\quad (\because 1 + \omega + \omega^2 = 0)$

□□□

Chapter 4

Sequence and Series

4.1 Sequence and Series

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Basic Formulae and Key Points

1. **Sum of the Finite Natural Numbers**
   i) Sum of the first **n** natural numbers
   i.e. $1+2+3+...+n = \frac{n(n+1)}{2}$
   $\therefore \; S_n = \frac{n(n+1)}{2}$
   ii) Sum of the first **n odd** natural numbers
   i.e. $1+3+5+7+...$ to n terms $= n^2$
   $\therefore \; S_n = n^2$
   iii) Sum of the first **n even** natural numbers:
   i.e. $2+4+6+....$ to n terms $= n(n+1)$
   $\therefore \; S_n = n(n+1)$
2. **Sum of the Squares of First n Natural Numbers:**
   i.e. $1^2+2^2+3^2+4^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$
   $\therefore \; S_n = \frac{n(n+1)(2n+1)}{6}$
3. **Sum of the Cubes of First n Natural Numbers:**
   i.e. $1^3+2^3+3^3+...+n^3 = \left[\frac{n(n+1)}{2}\right]^2$
   $\therefore \; S_n = \frac{n^2(n+1)^2}{4}$

Remarks: i) $\sum 1 = n$ ii) $\sum n = \frac{n(n+1)}{2}$

iii) $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ iv) $\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

➤ Recap

4. **In Arithmetic Sequence:**
   i) Common difference (d) $= t_2 - t_1 = t_3 - t_2 = ... = t_n - t_{n-1}$
   ii) General Term $(t_n) = a + (n-1)d$
   iii) Sum of n terms $(S_n) = \frac{n}{2}[2a + (n-1)d]$
   iv) $S_n = \frac{n}{2}(a + l)$

5. **In Geometric Sequence:**
   i) Common ratio (r) $= \frac{t_2}{t_1} = \frac{t_3}{t_2} = ... = \frac{t_n}{t_{n-1}}$
   ii) General Term $(t_n) = ar^{n-1}$
   iii) Sum of n terms $(S_n) = \frac{a(1-r^n)}{1-r}$ if $r<1$
   - $S_n = \frac{a(r^n-1)}{r-1}$ if $r>1$
   - $S_n = \frac{lr-a}{r-1}$

Group 'A' (Multiple Choice Questions and Answers)

1. *The sum of first 20 natural numbers is*
   (a) 200 (b) 210 (c) 400 (d) 420
2. *The sum of the first 12 even natural numbers is*
   (a) 156 (b) 126 (c) 108 (d) 112
3. *The sum of the first n odd natural numbers is*
   (a) $\frac{n(n+1)}{2}$ (b) $\frac{n+1}{2}$ (c) n^2 (d) n^3
4. *The n^{th} term of the series $1 + (1+3) + (1+3+5) + ...$ is*
   (a) $\frac{n^2(n+1)^2}{4}$ (b) n^3 (c) n^2 (d) $(2n-1)$
5. *The sum of the squares of first n natural numbers is*
   (a) n^2 (b) $\frac{n(n+1)}{2}$ (c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\left[\frac{n(n+1)}{2}\right]^2$
6. *The n^{th} term of the series $2+6+12+20+...$ is*
   (a) $\frac{n+3}{2}$ (b) $\frac{n(n+4)}{2}$ (c) $n(n+1)$ (d) $n(3n-1)$
7. *The t_n term of the series: $1.2+2.3+3.4+4.5+...$ to n terms is*
   (a) n^2+1 (b) $n(n+1)$ (c) $n^2(n+1)$ (d) n^3+n
8. *What is the n^{th} term of the series; $1.2^2+2.3^2+3.4^2+........$?*
   (a) $(n+1)^2$ (b) $n^2(n+1)$ (c) $n(n+1)^2$ (d) n^2+2n
9. *What is the sum of first 5 odd natural numbers?*
   (a) 15 (b) 25 (c) 42 (d) 255
10. *The sum of n terms of series: $1.1+2.2+3.3+4.4+............$ is*
    (a) $\frac{n(n+1)}{2}$ (b) $\left[\frac{n(n+1)}{2}\right]^2$ (c) n^2 (d) $\frac{n(n+1)(2n+1)}{6}$

Answer Key

1. b	2. a	3. c	4. c	5. c	6. c	7. b	8. c	9. b	10. d

Group 'B' or 'C' (Subjective Questions and Answers)

1. The numbers 1, 2, 3, 4, ... ∞ are said to be natural numbers. These numbers are used to form various series.
   (a) Write an expression for the sum of first $(n + 2)$ natural numbers.
   (b) Write the formula for the sum of first $(n + 1)$ even natural numbers.
   (c) Write the formula for the general term of the sequence formed by the first 'n' odd natural numbers.
   (d) Write the formula for the sum of squares of first $(n - 1)$ natural numbers.
   (e) Write the formula for the sum of cubes of first 'n' natural numbers.
   [5×1=5] [2079 Set D]

Soln: (a) We know that, the sum of first n natural numbers is given by

$$S_n = \frac{n(n+1)}{2}$$

Hence, the sum of first $(n + 2)$ natural numbers is $S_{n+2} = \frac{(n+2)(n+2+1)}{2} = \frac{(n+2)(n+3)}{2}$ **Ans.**

(b) The sum of first $(n + 1)$ even natural numbers

$S_{n+1} = (n+1)(n+1+1)$ [∵ $S_n = n(n+1)$]

$= (n+1)(n+2)$ **Ans.**

(c) Let, the sequence formed by first n odd natural numbers be 1, 3, 5, 7, ... to n terms. Then

$t_n = a + (n-1)d$
$= 1 + (n-1).2$
$= 1 + 2n - 2$
$= 2n - 1$

General term $(t_n) = 2n - 1$ **Ans.**

(d) We know

The sum of the squares of the first n natural number is $S_n = \frac{n(n+1)(2n+1)}{6}$

To find the sum of square of first $(n - 1)$ natural numbers, we substitute n by $(n - 1)$, we get

$$S_{n-1} = \frac{(n-1)(n-1+1)\{2(n-1)+1\}}{6}$$

$$= \frac{(n-1).n.(2n-2+1)}{6} = \frac{n(n-1)(2n-1)}{6}$$ **Ans.**

(e) The sum of cubes of first n natural numbers is given by $(S_n) = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$ **Ans.**

2. **The series is given as 2.3 + 3.4 + 4.5 + ____**
   (a) **Write the general term t_n for this series.** [1]
   (b) **How does the series change if the series starts at 3.4 instead of 2.3?** [1]
   (c) **Write the general term t_n after the series is modified.** [1]
   (d) **Find the sum of the first n term of the given series.** [2]

Soln: (a) Here

t_n = (nth term of 2, 3, 4, ...) × (nth term of 3, 4, 5, ...)
$= [2 + (n-1).1] \times [3 + (n-1).1]$
$= [2 + n - 1] \times [3 + n - 1]$
$= (n+1)(n+2)$
$= n^2 + 3n + 2$ **Ans.**

(b) If the series starts at 3.4 instead of 2.3, the series will be: 3.4 + 4.5 + 5.6 + ____ **Ans.**

(c) After the series is modified, the general term t_n will be:

t_n = (nth term of 3, 4, 5, ...) × (nth term of 4, 5, 6, ...)
$= [3 + (n-1).1] \times [4 + (n-1).1]$
$= (3 + n - 1) \times (4 + n - 1)$
$= (n+2)(n+3)$
$= n^2 + 5n + 6$

(d) We have,

$t_n = n^2 + 3n + 2$

Now, $S_n = \sum t_n$

$= \sum(n^2 + 3n + 2)$
$= \sum n^2 + 3\sum n + 2\sum 1$
$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n$
$= n\left[\frac{(n+1)(2n+1)}{6} + \frac{3(n+1)}{2} + 2\right]$
$= n\left[\frac{2n^2 + 3n + 1 + 9n + 9 + 12}{6}\right]$
$= \frac{n(2n^2 + 12n + 22)}{6} = \frac{2n(n^2 + 6n + 11)}{6} = \frac{n(n^2 + 6n + 11)}{3}$ **Ans.**

3. (a) *Find the sum of square of first 20 natural numbers.* [1]
   (b) *Find the sum of n terms of the series whose nth term is n(n + 3).* [2]
   (c) *Use the summation symbol to express the series given in (b), for the first n terms.* [1]
   (d) *Write the formula for sum of first n even numbers.* [1] [2080 GIE Set B]

Soln: (a) We know that the sum of squares of first n terms is $S_n = \frac{n(n+1)(2n+1)}{6}$

When, n = 20 then

$S_{20} = \frac{20(20+1)(2 \times 20 + 1)}{6}$

$= \frac{20 \times 21 \times 41}{6} = \frac{17220}{6} = 2870$ **Ans.**

(b) Here, $t_n = n(n + 3)$

i.e. $t_n = n^2 + 3n$

Now, the sum to n terms,

$S_n = \sum t_n$
$= \sum(n^2 + 3n)$
$= \sum n^2 + 3\sum n$
$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$
$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 3\right]$
$= \frac{n(n+1)}{2} \cdot \left[\frac{2n+1+9}{3}\right]$
$= \frac{n(n+1)}{2} \cdot \frac{2n+10}{3} = \frac{n(n+1)}{2} \cdot \frac{2(n+5)}{3} = \frac{n(n+1)(n+5)}{3}$ **Ans.**

(c) Here, $t_n = n^2 + 3n$

The series with the n^{th} term can be written in summation notation as:

$$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} (r^2 + 3r)$$

(d) The sum of first n even natural numbers = n(n + 1). **Ans.**

4. ***The series is given as: 2 + (2 + 3) + (2 + 3 + 4) + ...*** *[1] [2080 GIE Set B]*

(a) What is the sum of first n natural numbers? *[1]*

(b) Find the general term of the series. *[2]*

(c) Find the sum of the first n terms of the series. *[1]*

(d) Write the formula for the sum of squares of first (n − 2) natural numbers.

Soln: (a) The sum of first n natural numbers $(S_n) = \frac{n(n+1)}{2}$ **Ans.**

(b) Let, t_n be the n^{th} term of the given series.

Then,

$$t_n = (2 + 3 + 4 + 5 + \dots \text{ to n terms})$$

$$= \frac{n}{2}[2 \times 2 + (n-1).1] = \frac{n}{2}[4 + n - 1] = \frac{n}{2}(n+3)$$

(c) If S_n be the sum of first n terms of the series then, we have

$$t_n = \frac{n}{2}(n+3)$$

Now, $S_n = \sum t_n$

$$= \sum\left[\frac{n}{2}(n+3)\right]$$

$$= \sum\left[\frac{n^2}{2} + \frac{3n}{2}\right]$$

$$= \frac{1}{2}\sum n^2 + \frac{3}{2}\sum n$$

$$= \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{3}{2}\cdot\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{6} + \frac{3}{2}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1+9}{6}\right] = \frac{n(n+1).2.(n+5)}{12} = \frac{n(n+1)(n+5)}{6} \textbf{ Ans.}$$

(d) We know that,

The sum of squares of first n natural numbers is given by

$S_n = \frac{n(n+1)(2n+1)}{6}$, then the sum of squares of first (n − 2) terms is given by

$$S_{n-2} = \frac{(n-2)(n-2+1)[2(n-2)+1]}{6}$$

$$= \frac{(n-2)(n-1)(2n-4+1)}{6}$$

$$\therefore\ S_{n-2} = \frac{(n-2)(n-1)(2n-3)}{6} \textbf{ Ans.}$$

5. ***The series is given as $1^2 + 2^2 + 3^2 + \dots\dots$***

(a) Identify the type of sequence represented by this series. *[1]*

(b) Which formula would be appropriate to find the sum of the first n terms of this series? *[1]*

(c) If each term of this series is multiplied by the consecutive natural numbers starting from 2, write expression for the new series. So formed. *[1]*

(d) Write the general term t_n after the new series is formed. *[1]*

(e) Write the sum of first n terms of this new series. *[1]*

Soln: (a) The given series is a sequence of the squares of natural numbers.

(b) The sum of squares of the first n natural numbers is given by $S_n = \frac{n(n+1)(2n+1)}{6}$

(c) If each term is multiplied by consecutive natural numbers starting from 2, the new series will be: $1^2.2 + 2^2.3 + 3^2.4 + \dots$ **Ans.**

(d) Here, given series is $1^2.2 + 2^2.3 + 3^2.4 + \dots$

$$t_n = [n^{th} \text{ term of } 1, 2, 3 \dots]^2 \times [n^{th} \text{ term of } 2, 3, 4, \dots]$$

$$= [1 + (n-1).1]^2 \times [2 + (n-1).1]$$

$$= (1 + n - 1)^2 \times (2 + n - 1)$$

$$= n^2(n+1) = n^2 + n^3 \textbf{ Ans.}$$

(e) Here, $t_n = n^2 + n^3$

If S_n be the sum of firs t_n terms of the given series. Then,

$S_n = \sum t_n$

$$= \sum(n^2 + n^3)$$

$$= \sum n^2 + \sum n^3$$

$$= \frac{n(n+1)(2n+1)}{6} + \left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{n(n+1)(2n+1)}{6}\ \frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{4n+2+3n^2+3n}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{(3n+1)(n+2)}{6}\right] = \frac{n(n+1)(n+2)(3n+1)}{12} = \frac{1}{12}n(n+1)(n+2)(3n+1) \textbf{ Ans.}$$

6. ***(a) Find the sum of the first n terms of the series whose n^{th} term is $n - \frac{1}{2}$*** *[1]*

(b) Find the general term of the series 1.n + 2.(n − 1) + 3.(n − 2) + ... *[1]*

(c) Find the sum of first n term of the series. *[2]*

(d) Write the formula for the sum of squares of first (n + 1) natural numbers. *[1]*

Soln: (a) Here, given that,

$$t_n = n - \frac{1}{2}$$

If S_n be the sum of the first n terms of the series. Then

$S_n = \sum t_n$

$= \sum\left(n - \frac{1}{2}\right)$

$= \sum n - \frac{1}{2}\sum 1$

$= \frac{n(n+1)}{2} - \frac{1}{2}.n$

$= \frac{n(n+1-n)}{2} = \frac{1}{2}n$

(b) Here, given series is $1.n + 2.(n-1) + 3.(n-2) + ...$

Let, t_r be the general term and S_n the sum of first n terms of the given series. Then

$t_r = [r^{th}$ term of $1, 2, 3 ...] \times [r^{th}$ term of $n, n-1, n-2, ...]$

$= [1 + (r-1).1] \times [n + (r-1).(-1)]$ **[$\because$ d = $t_2 - t_1$ = n − 1 − n = −1]**

$= (1 + r - 1)(n - r + 1)$

$= r(n - r + 1)$

$= nr - r^2 + r$ **Ans.**

(c) If S_n the sum of the first n terms of the series.

Then,

$S_n = \sum_{r=1}^{n} t_r$

$= \sum_{r=1}^{n} (nr - r^2 + r)$

$= n\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= n.\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[n - \frac{2n+1}{3} + 1\right]$

$= \frac{n(n+1)}{2}\frac{(3n-2n-1+3)}{3} = \frac{n(n+1)(n+2)}{6}$ **Ans.**

(d) We know, that the sum of squares of first n natural numbers is given by

$S_n = \frac{n(n+1)(2n+1)}{6}$, Then

$S_{n+1} = \frac{(n+1)\{(n+1)+1\}\{2(n+1)+1\}}{6}$

$= \frac{(n+1)(n+2)(2n+2+1)}{6}$

$\therefore$ $S_{n+1} = \frac{(n+1)(n+2)(2n+3)}{6}$ which is the formula for the sum of squares of first (n + 1) natural numbers. **Ans.**

7. ***The series is given as 2 + 6 + 12 + 20 + ...***

(a) ***Find the general term (t_n) of the series.*** *[2]*

(b) ***Find the sum of the first n terms of the series.*** *[2]*

(c) ***Find the sum of first 10 even natural numbers.*** *[1]*

Soln: (a) Let, $S_n = 2 + 6 + 12 + 20 + ... + t_{n-1} + t_n$ ..... (i)

$S_n = \quad 2 + 6 + 12 + ... + t_{n-1} + t_n$ ..... (ii)

Subtracting eqn. (ii) from eq^n. (i), we get

$S_n = 2 + 6 + 12 + 20 + ... + t_{n-1} + t_n$

$S_n = \quad 2 + 6 + 12 + ... \quad ... + t_{n-1} + t_n$

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$0 = 2 + 4 + 6 + 8 + ... + (t_n - t_{n-1}) - t_n$

or, $t_n = 2 + 4 + 6 + 8 + ...$ to n terms

$\therefore$ $t_n = n(n+1)$ **[$\because$ Sum of first n even natural numbers = n(n + 1)]**

(b) Here,

$t_n = n(n+1)$

i.e. $t_n = n^2 + n$

If S_n the sum of first n terms of the series.

Then,

$S_n = \sum t_n$

$= \sum(n^2 + n)$

$= \sum n^2 + \sum n$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 1\right]$

$= \frac{n(n+1)(2n+1+3)}{6}$

$= \frac{n(n+1)(2n+4)}{6}$

$= \frac{2n(n+1)(n+2)}{6}$

$= \frac{n(n+1)(n+2)}{3}$ **Ans.**

(c) We know,

The sum of first n even natural numbers is

$S_n = n(n+1)$

When, n = 10

$S_{10} = 10 \times (10 + 1)$

$= 10 \times 11$

$= 110$

$\therefore$ The sum of first 10 even natural numbers = 110 **Ans.**

8. *Find the n^{th} term and the sum of n terms of the series: $3 + 7 + 13 + 21 + 31 + ...$* [4]

Soln: $S_n = 3 + 7 + 13 + 21 + 31 + ... + t_{n-1} + t_n$ .... (i)

$S_n = \quad 3 + 7 + 13 + 21 + ... + t_{n-1} + t_n$ .... (ii)

Subtracting (ii) from (i), we get

$S_n = 3 + 7 + 13 + 21 + 31 + ... + t_{n-1} + t_n$

$S_n = \quad 3 + 7 + 13 + 21 + ... + t_{n-1} + t_n$

$(-) \quad (-)(-)(-) \quad (-) \quad (-) \qquad (-) \quad (-)$

$0 = 3 + 4 + 6 + 8 + 10 + ... + (t_n - t_{n-1}) - t_n$

$t_n = 3 + 4 + 6 + 8 + ... \text{ to n terms.}$

$= 3 + [\underbrace{4 + 6 + 8 + 10 + ...}_{\text{A.P.}} \text{ to } (n-1) \text{ terms}]$

$= 3 + \left[\frac{n-1}{2}\{2.4 + (n-1-1).2\}\right] \qquad \left[\because S_n = \frac{n}{2}[(2a + (n-1)d)]\right]$

$= 3 + \left[\frac{n-1}{2}(8 + 2n - 4)\right]$

$= 3 + \left[\frac{n-1}{2}(2n + 4)\right]$

$= 3 + \frac{n-1}{2} \times 2(n + 2)$

$= 3 + (n-1)(n+2)$

$= 3 + n^2 + 2n - n - 2$

$= n^2 + n + 1$

$\therefore t_n = n^2 + n + 1$ Ans.

Again, if S_n be the sum of first n terms of the series, then

$S_n = \sum t_n$

$= \sum(n^2 + n + 1)$

$= \sum n^2 + \sum n + \sum 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$

$= n\left[\frac{(n+1)(2n+1)}{6} + \frac{n+1}{2} + 1\right]$

$= n\left[\frac{2n^2 + n + 2n + 1 + 3n + 3 + 6}{6}\right]$

$= n\left[\frac{2n^2 + 6n + 10}{6}\right]$

$= \frac{2n(n^2 + 3n + 5)}{6}$

$= \frac{n(n^2 + 3n + 5)}{3}$ **Ans.**

9. *The series is defined as follows:*

$1.2^2 + 3.4^2 + 5.6^2 + ...$

(a) *Write the sum of the series $1^2 + 2^2 + 3^2 + ... + n^2$.* [1] [2081 Optional]

(b) *Find the sum up to n terms of the given series.* [3]

Soln: (a) The sum of the series $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ **Ans.**

(b) Let, t_n be the n^{th} term and S_n the sum of first n terms of the series. Then

$t_n = [n^{th} \text{ term of } 1, 3, 5, ...] \times [n^{th} \text{ term of } 2, 4, 6, ...]^2$

$= [1 + 2n - 2] \times [2 + 2n - 2]^2$

$= (2n - 1)(2n)^2$

$= 4n^2(2n - 1)$

$= 8n^3 - 4n^2$

Again,

$S_n = \sum t_n$

$= \sum(8n^3 - 4n^2)$

$= 8\sum n^3 - 4\sum n^2$

$= 8\left[\frac{n(n+1)}{2}\right]^2 - 4.\frac{n(n+1)(2n+1)}{6}$

$= \frac{8n^2(n+1)^2}{4} - \frac{2n(n+1)(2n+1)}{3}$

$= 2n^2(n+1)^2 - \frac{2n(n+1)(2n+1)}{3}$

$= 2n(n+1)\left[n(n+1) - \frac{2n+1}{3}\right]$

$= 2n(n+1)\left[\frac{3n^2 + 3n - 2n - 1}{3}\right]$

$= \frac{2}{3}n(n+1)(3n^2 + n - 1)$ **Ans.**

4.2 Principle of Mathematical Induction

Basic Formulae and Key Points

1. Statement of the Principle of Mathematical Induction.

The principle of mathematical induction states that, if P(n) be the statement and if

i) P(1) is true

ii) P(k + 1) is true whenever P(k) is true then the statement P(n) is true for all $n \in N$.

2. Working Rules for the use of Principle of Mathematical Induction.

While solving the problem by using the principle of mathematical induction, the following steps are used.

Step – 1: Denote the given statement by P(n).

Step – 2: Prove that the given statement is true for n = 1 i.e. P(1) is true.

Step – 3: Suppose that the given statement is true for some natural number n = k i.e. suppose P(k) is true.

Step – 4: Prove that the given statement is also true for n = k + 1, when it is true for n = k i.e. P(k + 1) is true whenever P(k) is true.

Step – 5: Draw a conclusion that the statement is true for all natural number $n \in N$.

Group 'A' (Multiple Choice Questions and Answers)

1. *Let, P(n) be a statement such that P(1) is true, P(k + 1) is true whenever P(k) is true. Then P(n) is true for all n, where n is a/an*
   (a) Integer (b) Natural number
   (c) Both (d) Whole number
2. *Let, P(n): $2^n < n!$ then smallest positive integer for which P(n) is true is*
   (a) 1 (b) 2 (c) 3 (d) 4
3. *If P(n) is the statement "n(n + 2)" is an odd number, then*
   (a) P(1) is false, P(2) is true (b) P(1) is true P(2) is false
   (c) both of P(1) and P(2) are false (d) both of P(1) and P(2) are true
4. *Consider the statement P(n): $n^2 - n + 41$ is prime. Then which of the following is true?*
   (a) both P(3) and P(5) are true (b) P(3) is true but P(5) is false.
   (c) both P(3) and P(5) are false (d) P(3) is false but P(5) is true.
5. *Given the statement P(n): $3^n > 11n$, choose the true option.*
   (a) P(1) (b) P(2) (c) P(3) (d) P(4)
6. *According to principle of mathematical induction, if $P(k + 1) = m^{k+1} + 5$ is true then ___ must be true*
   (a) $P(k) = 3m^k$ (b) $P(k) = m^k + 5$
   (c) $P(k) = m^{k+2} + 5$ (d) $P(k) = m^k$
7. *Let, P(n): $n^2 + 3n$ is always divisible by which of the following number, provided n is an integer?*
   (a) 2 (b) 3 (c) 4 (d) 5
8. *What will be P(k + 1) for $P(n) = n^2(n + 1)$?*
   (a) $(k + 1)^4$ (b) $k^4 + 5k^3 + 9k^2 + 7k + 2$
   (c) $k^4 + 6k^3 + 9k^2 + 7k + 2$ (d) $k^4 + 3k^3 + 9k^2 + 6k + 2$
9. *$n^2 < 2^n$ is true for all natural numbers, if*
   (a) $n \geq 5$ (b) $n < 5$ (c) $n \leq 3$ (d) for all cases
10. *P(n) : $7^n - 3^n$ is divisible by ___ for every positive integer n.*
    (a) 7 (b) 5 (c) 4 (d) more than one of the above

Answer Key

1. b	2. d	3. b	4. a	5. d	6. b	7. a	8. b	9. a	10. c

Group 'B' or 'C' (Subjective Questions and Answers)

1. ***Prove by the method of principle of mathematical induction that $n^3 + 2n$ is divisible by 3.*** **[3] [2081 Set E/W]**

Soln:

Let, P(n) be the given statement. Then,

P(n) : $n^3 + 2n$ is divisible by 3.

When, n = 1, P(1): $1^3 + 2 \times 1 = 3$ which is divisible by 3.

∴ P(1) is true.

Let, us suppose that P(k) is true for all k∈N.

P(k): $k^3 + 2k$ is divisible by 3.

Now, we have to show that P(k + 1) is true whenever P(k) is true. Then

$$
\begin{aligned}
P(k+1) &: (k+1)^3 + 2(k+1) \\
&= (k+1)[(k+1)^2 + 2] \\
&= (k+1)[k^2 + 2k + 1 + 2] \\
&= (k+1)[k^2 + 2k + 3] \\
&= k^3 + 2k^2 + 3k + k^2 + 2k + 3 \\
&= (k^3 + 2k) + 3k^2 + 3k + 3 \\
&= (k^3 + 2k) + 3(k^2 + k + 1)
\end{aligned}
$$

is divisible by 3 as $(k^3 + 2k)$ and $3(k^2 + k + 1)$ both are divisible by 3.

∴ P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all n∈N.

2. ***Use principle of mathematical induction to prove that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$.*** **[3] [2081 Set V]**

Soln: Let. P(n) : $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$

When, n = 1, L.H.S. = 1

R.H.S. = $1^2 = 1$

∴ P(1) is true.

Let, us suppose that P(k) is true for k∈N. Then

P(k): $1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$ .... (i)

Now, we have to show that P(k + 1) is true when P(k) is true. For this adding $2(k + 1) - 1 = 2k + 1$ on both sides of eqn (i), we get

$$
\begin{aligned}
1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\
&= k^2 + 2k + 1 = (k + 1)^2
\end{aligned}
$$

∴ P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all n∈N.

3. ***Using principle of Mathematical induction, show that n(n + 1) (2n + 1) is divisible by 6 for all n∈N.*** **[3] [2080 GIE Set A]**

Soln: Let. P(n) be the given statement. Then

P(n): n(n + 1) (2n + 1) is divisible by 6.

When, n = 1, P(1) : 1 (1 + 1) (2 × 1 + 1) = 6 which is divisible by 6.

∴ P(1) is true.

Let, us suppose that P(k) is true for all k∈N. Then

P(k) = k (k + 1) (2k + 1) is divisible by 6 .... (i)

Now, we have to show that P(k + 1) is true when P(k) is true.

P(k + 1) : (k + 1) (k + 2) (2 (k + 1) + 1) is divisible by 6.

Now,

$$
\begin{aligned}
P(k+1) &: (k+1)(k+2)(2k+2+1) \\
&= (k+1)(k+2)(2k+3) \\
&= k(k+1)(2k+3) + 2(k+1)(2k+3) \\
&= k(k+1)\{(2k+1)+2\} + 2(k+1)(2k+3) \\
&= k(k+1)(2k+1) + 2k(k+1) + 2(k+1)(2k+3) \\
&= k(k+1)(2k+1) + (k+1)(2k+4k+6) \\
&= k(k+1)(2k+1) + (k+1)6(k+1) \\
&= k(k+1)(2k+1) + 6(k+1)^2
\end{aligned}
$$

Which is divisible by 6 as k(k + 1) (2k + 1) is divisible by 6 which is shown in (i) and $6(k + 1)^2$ is multiple of 6 and hence divisible by 6.

This shows that P(k + 1) is true whenever P(k) is true. Hence by principle of mathematical induction P(n) is true for all natural numbers n.

4. ***Using mathematical induction, prove that $n! \geq 2^n$ for $n \geq 4$.*** **[2] [2080 Set G]**

Sol^n: Let, P(n) be the given statement. Then,

$P(n) : n! \geq 2^n$ for $n \geq 4$.

When, n = 4

L.H.S. = n! = 4! = 24

R.H.S. = $2^n = 2^4 = 16$

Here, 24 > 16

Hence P(4) is true for n = 4.

Let, as suppose that P(k) is true for some $k \in N$, $k \geq 4$. Then,

$P(k) : k! \geq 2^K$ .... (i)

Now, we have to show that P(k + 1) is true when P(k) is true.

$P(k+1) : (k+1)!$

$= (k+1).k!$

$\geq (k+1).2^K$ **[∵ From (i), $k! \geq 2^K$]**

$> 2.2^K$ **[∵ $k+1 > 2$ for $k \geq 4$]**

$= 2^{K+1}$

This shows that P(k + 1) is true whenever P(k) is true.

Hence, P(n) is true for all $n \geq 4$.

5. ***If the Statement is given as $P(n) : n^3 \geq 2^n$***

(a) Show that P(1) is false and P(2) is true.

(b) If the statement is modified to $P(n) : 2^n \leq n^3$, will P(2) still hold true?

(c) Is the statement $P(n): n^3 \geq 2^n$ true for all natural number $n \geq 3$? Justify your answer.

(d) To prove P(k + 1) is true, using mathematical induction what must be assumed earlier to be true as mandatory?

Soln: (a) Here, given statement is $P(n) : n^3 \geq 2^n$

For n = 1, $P(1) : 1^3 \geq 2^1$ which is false.

For n = 2, $P(2) : 2^3 \geq 2^2$ i.e. $8 \geq 4$ which is true.

Hence, P(1) is false and P(2) is true. **Proved.**

(b) If the statement is modified to $P(n): 2^n \leq n^3$ then,

$P(2) : 2^2 \leq 2^3$ i.e. $4 \leq 8$ which is true.

∴ The statement P(2) remains true even after the modification. **Ans.**

(c) Let, us test a few values of n:

For n = 3, $P(3) = 3^3 \geq 2^3$ i.e. $27 \geq 8$ which is true.

For n = 4, $P(4) = 4^3 \geq 2^4$ i.e. $64 \geq 16$ which is true.

From above we can conclude that as n increases, n^3 increases bigger than 2^n. So, P(n) is true for all natural numbers $n \geq 3$.

(d) To make the statement true for (k + 1) using mathematical induction, we must assume $P(k): k^3 \geq 2^k$ true as mandatory.

6. ***(a) State principle of mathematical induction.*** **[2]**

(b) Explain first and last step as the working rules for the use of principle of mathematical induction. **[2]**

(c) Is the statement $P(n) : n^2 + n$ is even number for all natural numbers n? Justify your answer. **[1]**

Soln: (a) The principle of mathematical induction states that,

If P(n) be the statement and if

i) P(1) is true.

ii) P(k + 1) is true whenever P(k) is true then the statement is true for all $n \in N$.

(b) The first and last working steps are as follows:

First step – Denote the given statement by P(n).

Last step – Draw a conclusion that the statement is true for all natural numbers $n \in N$.

(c) Here, the given statement is

$P(n) = n^2 + n$

i.e. $P(n) = n^2 + n = n(n+1)$

Since, n and n + 1 are both two consecutive integers. The product of two consecutive integers is always even.

∴ n(n + 1) is always even number.

Hence, the statement $P(n) : n^2 + n$ is even number for all natural numbers n.

7. ***Given the statement $P(n) : (n+3)^2 > 2n + 7$:***

(a) What is the hypothetical inductive step required to prove this statement using the principle of mathematical induction? Write it. **[1]**

(b) Use mathematical induction to prove the given statement. **[3]**

(c) If for all integer n, $x^n - 1$ is divisible by x – k, then what is the value of k? **[1]**

Soln: (a) **Hypothetical inductive step:** We have to suppose that the statement is true for some natural number n = k i.e. $P(k): (k+3)^2 > 2k + 7$ is true.

(b) Let, $P(n): (n+3)^2 > 2n + 7$

For n = 1, L.H.S. = $(1+3)^2 = 4^2 = 16$

R.H.S. = 2 × 1 + 7 = 9

i.e. 16 > 9

∴ L.H.S > R.H.S.

Hence P(1) is true.

Let, us suppose that P(k) is true for $k \in N$.

$P(k) : (k+3)^2 > 2k + 7$ .... (i)

Now, We have to show that P(k + 1) is true when P(k) is true. Then

$P(k+1): [(k+1)+3]^2 = (k+4)^2$

$= k^2 + 8k + 16$

$= (k^2 + 6k + 9) + 2k + 7$

$= (k+3)^2 + 2K + 7$

$> 2k + 7 + 2k + 7$ **[∵ Using (i)]**

$= 2k + 2 + 5 + 2k + 7$

$= 2(k+1) + 7 + 2k + 5$

$> 2(k+1) + 7$ **[∵ k is a positive integer ⇒ 2k + 5 > 0]**

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

(c) Let, P(n) : $x^n - 1$ is divisible by $x - k$.
For n = 1, P(1) = $x^1 - 1$ is divisible by $x - k$.
So, $x - 1$ is divisible by $x - k$ this implies that k = 1. **Ans.**

8. ***Given the statement $P(n) = 2^{3n} - 1$ is divisible by 7.***
(a) Give an example of a statement P(n) which is true for all $n \in N$. [1]
(b) Use mathematical induction to prove the given statement. [3]

Soln: (a) An example of a statement, P(n): $1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$ is true for all $n \in N$.

(b) Let, P(n): $2^{3n} - 1$ is divisible by 7.
When, n = 1, P(1) : $2^{3\times1} - 1 = 2^3 - 1 = 8 - 1 = 7$, which is divisible by 7.
∴ P(1) is true.
Let us suppose that P(k) is true for all $K \in N$.
P(k) : $2^{3k} - 1$ is divisible by 7.
Now, we have to show that P(k + 1) is true when p(k) is true.

$$P(k+1) : 2^{3(k+1)} - 1 = 2^{3k+3} - 1$$
$$= 2^{3k}.2^3 - 1$$
$$= 2^{3k}.8 - 8 + 8 - 1$$
$$= 8(2^{3k} - 1) + 7$$

Which is divisible by 7 as $(2^{3k} - 1)$ is divisible by 7 as shown in P(k) and multiplied by 8 also divisible by 7 and 7 is divisible by 7 itself and their sum is also divisible by 7.
∴ P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

9. ***Let, P(n) be the statement "$3^n > n$".***
(a) Is P(1) true? [1]
(b) what is P(k + 1)? [1]
(c) If P(k) is true, then prove that P(k + 1) is true. [2]

Soln: Let, P(n) : $3^n > n$
(a) When, n = 1,
L.H.S. = $3^1 = 3$
R.H.S. = 1
i.e. 3 > 1
∴ L.H.S. > R.H.S.
Here, P(1) is true.

(b) P (k + 1) : $3^{k+1} > (k + 1)$ **Ans.**

(c) If P(k) is true then we have to show that P(k + 1) is true.
Now, P(k) : $3^k > k$ is true ......(i)
for P(k + 1) : $3^{k+1} = 3^k.3^1$
$> k.3$
$= 3k$
$> k + 1$ [∵ k ≥ 1 and $k \in N$]

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction P(n) is true for all $n \in N$. **Hence Proved.**

10. ***Using the principle of mathematical induction, show that: $1 + 2 + 3 \ldots\ldots\ldots + n < \frac{1}{8}(2n + 1)^2$*** [3] [Model 2079]

Soln: Let, P(n) be the given statement. Then
P(n) : $1 + 2 + 3 + \ldots\ldots\ldots + n < \frac{1}{8}(2n + 1)^2$
When, n = 1
L.H.S. n = 1, R.H.S. = $\frac{1}{8}(2 \times 1 + 1)^2 = \frac{1}{8} \times 9 = \frac{9}{8}$
Since, $1 < \frac{9}{8}$
i.e. L.H.S.< R.H.S.
∴ P(1) is true.
Let, us suppose that P(k) is true for all $k \in N$.
Then,

$$P(k) : 1 + 2 + 3 + \ldots\ldots + k < \frac{1}{8}(2k + 1)^2 \ldots\ldots\ldots(i)$$

Now, we have to prove that P(k + 1) is true whenever P(k) is true. For this adding (k + 1) on both sides of equation (i), we get

$$1 + 2 + 3 + \ldots\ldots + k + (k + 1) < \frac{1}{8}(2k + 1)^2 + 8(k + 1)$$
$$= \frac{1}{8}[(4k^2 - 4k + 1) + 8(k + 1)]$$
$$= \frac{1}{8}[4k^2 + 4k + 1 + 8k + 8]$$
$$= \frac{1}{8}[4k^2 + 12k + 9]$$
$$= \frac{1}{8}[2k + 3]^2$$
$$= \frac{1}{8}[2(k + 1) + 1]^2$$

i.e. $1 + 2 + 3 + \ldots\ldots + k + (k + 1) < \frac{1}{8}[2(k + 1) + 1]^2$
Thus, P(k + 1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

11. ***Using the principle of mathematical induction prove that: $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.*** [3]

Soln: Let, P(n) : $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.
When, n = 1, L.H.S. = $1^3 = 1$ and R.H.S. = $\left\{\frac{n(n+1)}{2}\right\}^2$
$= \left(\frac{2}{2}\right)^2 = 1$
Here, L.H.S. = R.H.S.
∴ P(1) is true.

(c) Let, $P(n): x^n - 1$ is divisible by $x - k$.

For $n = 1$, $P(1) = x^1 - 1$ is divisible by $x - k$.

So, $x - 1$ is divisible by $x - k$ this implies that $k = 1$. **Ans.**

8. ***Given the statement $P(n) = 2^{3n} - 1$ is divisible by 7.***

(a) ***Give an example of a statement P(n) which is true for all $n \in N$.***

(b) ***Use mathematical induction to prove the given statement.***

Soln: (a) An example of a statement, $P(n): 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$ is true for all $n \in N$.

(b) Let, $P(n): 2^{3n} - 1$ is divisible by 7.

When, $n = 1$, $P(1): 2^{3\times1} - 1 = 2^3 - 1 = 8 - 1 = 7$, which is divisible by 7.

$\therefore$ P(1) is true.

Let, us suppose that P(k) is true for all $K \in N$.

$P(k): 2^{3k} - 1$ is divisible by 7.

Now, we have to show that P(k + 1) is true when p(k) is true.

$$\begin{aligned} P(k+1): 2^{3(k+1)} - 1 &= 2^{3k+3} - 1 \\ &= 2^{3k}.2^3 - 1 \\ &= 2^{3k}.8 - 8 + 8 - 1 \\ &= 8(2^{3k} - 1) + 7 \end{aligned}$$

Which is divisible by 7 as $(2^{3k} - 1)$ is divisible by 7 as shown in P(k) and multiplied by 8 also divisible by 7 and 7 is divisible by 7 itself and their sum is also divisible by 7.

$\therefore$ P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

9. ***Let, P(n) be the statement "$3^n > n$".***

(a) ***Is P(1) true?***

(b) ***what is P(k + 1) ?***

(c) ***If P(k) is true, then prove that P(k + 1) is true.***

Soln: Let, $P(n): 3^n > n$

(a) When, $n = 1$,

L.H.S. $= 3^1 = 3$

R.H.S. $= 1$

i.e. $3 > 1$

$\therefore$ L.H.S. > R.H.S.

Here, P(1) is true.

(b) $P(k + 1): 3^{k+1} > (k + 1)$ **Ans.**

(c) If P(k) is true then we have to show that P(k + 1) is true.

Now, $P(k): 3^k > k$ is true ....(i)

for $P(k+1): 3^{k+1} = 3^k.3^1$

$$\begin{aligned} &> k.3 \\ &= 3k \\ &> k + 1 \quad [\because k \geq 1 \text{ and } k \in N] \end{aligned}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction P(n) is true for all $n \in N$. **Hence Proved.**

10. ***Using the principle of mathematical induction, show that: $1 + 2 + 3 \ldots\ldots\ldots + n < \frac{1}{8}(2n + 1)^2$*** ***[3] [Model 2079]***

Soln: Let, P(n) be the given statement. Then

$P(n): 1 + 2 + 3 + \ldots\ldots\ldots\ldots + n < \frac{1}{8}(2n + 1)^2$

When, $n = 1$

L.H.S. $n = 1$, R.H.S. $= \frac{1}{8}(2 \times 1 + 1)^2 = \frac{1}{8} \times 9 = \frac{9}{8}$

Since, $1 < \frac{9}{8}$

i.e. L.H.S. < R.H.S.

$\therefore$ P(1) is true.

Let, us suppose that P(k) is true for all $k \in N$.

Then,

$P(k): 1 + 2 + 3 + \ldots\ldots\ldots + k < \frac{1}{8}(2k + 1)^2$ ....(i)

Now, we have to prove that P(k + 1) is true whenever P(k) is true. For this adding (k + 1) on both sides of equation (i), we get

$$\begin{aligned} 1 + 2 + 3 + \ldots\ldots\ldots + k + (k+1) &< \frac{1}{8}(2k+1)^2 + 8(k+1) \\ &= \frac{1}{8}[(4k^2 - 4k + 1) + 8(k + 1)] \\ &= \frac{1}{8}[4k^2 + 4k + 1 + 8k + 8] \\ &= \frac{1}{8}[4k^2 + 12k + 9] \\ &= \frac{1}{8}[2k + 3]^2 \\ &= \frac{1}{8}[2(k + 1) + 1]^2 \end{aligned}$$

i.e. $1 + 2 + 3 + \ldots\ldots\ldots\ldots + k + (k+1) < \frac{1}{8}[2(k + 1) + 1]^2$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

11. ***Using the principle of mathematical induction prove that: $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$. [3]***

Soln: Let, $P(n): 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

When, $n = 1$, L.H.S. $= 1^3 = 1$ and R.H.S. $= \left\{\frac{n(n+1)}{2}\right\}^2$

$$= \left(\frac{2}{2}\right)^2 = 1$$

Here, L.H.S. = R.H.S.

$\therefore$ P(1) is true.

Let, us suppose that P(k) is true for some $k \in N$. Then,

$$1^3 + 2^3 + 3^3 + \ldots\ldots + k^3 = \left\{\frac{k(k+1)}{2}\right\}^2 \quad \ldots\ldots(i)$$

Now, we have to show that P(k + 1) is true when P(k) is true. For this adding $(k+1)^3$ on both sides of equation (i), we get

$$1^3+2^3+3^3+\ldots\ldots+k^3+(k+1)^3 = \left\{\frac{k(k+1)}{2}\right\}^2 + (k+1)^3$$

$$= (k+1)^2\left(\frac{k^2}{4}+k+1\right)$$

$$= (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)^2\{(k+1)+1\}}{2}\right]^2$$

Which shows that P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction P(n) is true for all $n \in N$.

12. ***Using the principle of mathematical induction, prove that $3^{2n}-1$ is divisible by 8.***

Soln: Let, P(n) : $3^{2n} - 1$ is divisible by 8.

When, n = 1,

P(1) : $3^{2\times1} - 1 = 9 - 1 = 8$ which is divisible by 8.

i.e. P(1) is true.

Let, us suppose that P(k) is true for some $k \in N$. Then

P(k) : $3^{2k} - 1$ is divisible by 8........(i)

Now, we have to show that P(k+1) is true when P(k) is true.

Now,

P(k+1): $3^{2(k+1)} - 1$

$= 3^{2k+2} - 1$

$= 3^{2k}.3^2 - 1$

$= 3^{2k}.9 - 9 + 9 - 1$

$= 9(3^{2k} - 1) + 8$ Which is divisible by 8 as first part is divisible by 8 as shown in (i).

This shows that P(k + 1) is true whenever P(k) is true. Then by the principle of mathematical induction P(n) is true for all $n \in N$.

□□□

Chapter 5 Matrix Based System of Linear Equations

5.1 Matrix Based System of Linear Equations

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Basic Formulae and Key Points

1. **A System of Linear Equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ is**
   - (i) **Consistent and Independent** if it has unique solution or exactly one solution.
   - (ii) **Consistent and Dependent** if it has infinitely many solutions.
   - (iii) **Inconsistent and Independent** if it has no solutions.
2. **Quick Reference Techniques for Justifying Answers**

The system has:

1. a unique solution if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (Consistent and independent)
2. an infinite number of solutions if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Consistent and dependent)
3. no solution if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (Inconsistent and independent)

Group 'A' (Multiple Choice Questions and Answers)

1. ***The condition for the pair of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, to have a unique solutions is***

   (a) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (b) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

   (c) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (d) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

2. ***If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ has***

   (a) No solution (b) Unique solution

   (c) Infinitely many solutions (d) None of these

3. ***The value of 'k' for which the system of equations $kx - y = 2$, $6x - 2y = 3$ has unique solution is***

   (a) k = 3 (b) $k \neq 3$ (c) $k \neq 0$ (d) k = 0

4. ***The pair of linear equations $3x - 5y = 7$ and $-6x + 10y = 7$ will have***

   (a) a unique solution (b) infinitely many solutions

   (c) no solution (d) both (a) and (b)

5. ***Find the value of 'k' for which the system $kx + 2y = 5$, $3x + y = 1$ has no solution.***

   (a) 2 (b) 3 (c) 5 (d) 6

6. ***A pair of equations $2x + 5y = 20$ and $4x + 10y = 40$ have***

   (a) Unique solution (b) No solution

   (c) Infinitely many solutions (d) Two solutions

7. **For what value of k, do the equations $x - y + 8 = 0$ and $3x - ky = -24$ represent coincident lines?**
(a) $\frac{1}{3}$ (b) 3 (c) -3 (d) 4

8. **If the graph of the two linear equations in two variables intersect at a point, then the system is**
(a) consistent and dependent (b) consistent and independent
(c) inconsistent and independent (d) inconsistent and dependent

9. **In a consistent dependent system..........**
(a) There are infinitely many solutions (b) There is only one solution
(c) There is not solution (d) There are two solutions

10. **The system of equations $9x - 2y = -4$, $3x + 4y = 1$ is**
(a) consistent and dependent (b) consistent and independent
(c) inconsistent and independent (d) inconsistent and dependent

11. **The system of equations represents geometrically parallel lines, then it is:**
(a) consistent and dependent (b) consistent and independent
(c) inconsistent and independent (d) inconsistent and dependent

12. **Which pair of words describes this system of equations?**
$3y = 9x - 6$
$2y - 6x = 4$
(a) inconsistent and dependent (b) inconsistent and independent
(c) consistent and dependent (d) consistent and independent.

Answer Key

1. a	2. b	3. b	4. c	5. d	6. c	7. b	8. b	9. a	10. b
11. b	12. c								

5.2 Cramer's Rule

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Basic Formulae and Key Points

1. **For the system of equations $a_1x + b_1y = k_1$ and $a_2x + b_2y = k_2$, the solution (x, y) can be determined by**

$x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$

Where, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficient with the constants.

2. **For the system of equations: $a_1x + b_1y + c_1z = k_1$**
$a_2x + b_2y + c_2z = k_2$
$a_3x + b_3y + c_3z = k_3$,

The solution (x, y, z) can be determined by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$

Where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$ replace the y coefficients with the constants.

$D_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$ replace the z coefficients with the constants.

Note: i) If there are n variables and n equations, we have to compute (n + 1) determinants.
ii) Cramer's rule can give the solutions only when $D \neq 0$.
iii) Cramer's rule is not applicable when D = 0. i.e. when the determinant of the coefficient matrix is 0, we can not find the solution of the system of equations using Cramar's rule
iv) When the value of D = 0 and D_x, D_y and D_z are all zero, the system is consistent and dependent and there are infinitely many solutions.
v) When the value of D = 0 and D_x, D_y and D_z are not all zero, the system is inconsistent and there is no solutions.

Group 'A' (Multiple Choice Questions and Answers)

1. **Cramer's rule is also known as ....**
(a) Inverse matrix method (b) Matrix method
(c) Determinant method (d) Inverse method

2. **Cramer's rule fails for...................................**
(a) Determinant > 0 (b) Determinant < 0
(c) Determinant = 0 (d) Determinant = non – real

3. **For a system of m linear equations in n variable, the Cramer's rule is applicable when:**
(a) m = n only
(b) $m \neq n$ only
(c) m = n and the coefficient matrix is non-singular.
(d) $M \neq n$ and the coefficient matrix is singular.

4. **To find the determinant D from the system of two linear equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ we obtain as:**
(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (b) $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ (c) $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ (d) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

5. **The determinant Dy formed from the system of equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ is**
(a) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ (b) $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ (c) $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ (d) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

6. **In Cramer's rule of solving equations with three variables x, y and z the value of y can be obtained by using the relation**
a) $\frac{D_x}{D_y}$ (b) $\frac{D_y}{D_x}$ (c) $\frac{D_z}{D}$ (d) $\frac{D_y}{D}$

7. For simultaneous equation in variable x and y, Dx = 49, Dy = − 63, D = 7 then what is x?

(a) 7 (b) − 7 (c) $\frac{1}{7}$ (d) $-\frac{1}{7}$

8. Solving system of linear equations by Cramer's rule If the values of determinants D = 0 and $D_x = D_y = D_z = 0$, then the system has

(a) A unique solution (b) three solutions
(c) infinitely many solutions (d) at most two solutions

9. The solution of the system of equations 2x + y = 7 and x + 3y = 11 by Cramer's rule is

(a) (2, 3) (b) (3, 2) (c) (−2, 3) (d) (2, − 3)

10. When the value of determinant D = 0 and D_x, D_y and D_z are not zero, then the system is called........

(a) consistent (b) inconsistent
(c) independent (d) dependent

11. The determinant of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are real numbers, is

(a) bc − ad (b) ac − bd (c) ab − cd (d) ad − bc

Answer Key

1. c	2. c	3. c	4. a	5. c	6. d	7. a	8. c	9. a	10. b
11. d									

5.3 Row Equivalent Matrix Method

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Basic Formulae and Key Points

1. Elementary Row Operations.
   i) The two rows of the augmented matrix can be interchanged i.e. $R_i \leftrightarrow R_j$
   ii) The elements of a particular row can be multiplied or divided with a constant. i.e. $R_i \to k(R_i)$
   iii) The multiple of a row can be added to another row of the matrix i.e. $R_i \to R_i + kR_j$
   iv) The particular row can be added and subtracted to other rows of the matrix. i.e. $R_i \to R_i + k.R_j$ & $R_i \to R_i - kR_j$

Group 'A' (Multiple Choice Questions and Answers)

1. Write the equations 4x + 5y = 12 and 3x + 7y = 11 as an augmented matrix.

(a) $\begin{bmatrix} x & 4 & : & 5 \\ y & 3 & : & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 & : & 12 \\ 3 & 7 & : & 11 \end{bmatrix}$ (c) $\begin{bmatrix} 12 & x & : & 4 \\ 11 & y & : & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 12 & 4 & : & 5 \\ 11 & 3 & : & 7 \end{bmatrix}$

2. Which one of the following is not an elementary row operation? [2080 GIE Set I]

(a) Interchange of two rows
(b) Multiply a row by a non−zero scalar
(c) Divide a row by a non−zero scalar
(d) Interchange of two columns

3. Which augmented matrix represents the following system of equations?

x + 2y = 3
4y + 5x = 6

(a) $\begin{bmatrix} 0 & 2 & : & 3 \\ 4 & 5 & : & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & : & 3 \\ 5 & 4 & : & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & : & 3 \\ 5 & 4 & : & 6 \end{bmatrix}$

4. The system of equations corresponding to the augmented matrix $\begin{bmatrix} 3 & -4 & : & 6 \\ 2 & 3 & : & 10 \end{bmatrix}$ is

(a) 3x + 2y = 6
 − 4x + 3y = 10
(b) − 4x + 3y = 6
 3x + 2y = 10
(c) − 4x + 3y = 6
 2x + 3y = 10
(d) 3x − 4y = 6
 2x + 3y = 10

5. What is the solution to the system of equation represented with this augmented matrix? (Assume that the variables are x, y, z in that order.)

$$\begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & -1 \end{bmatrix}$$

(a) x = 1, y = 0, z = 0 (b) x = 2, y = 3, z = − 1
(c) There is no solution (d) can not decide without system of equations.

6. If the reduced form of the augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & : & 4 \\ 0 & 1 & 0 & : & -2 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$, then the solution of corresponding system of equations is

(a) unique solution: x = 4, y = − 2, z = 2
(b) unique solution: x = 4, y = −2, z = 0
(c) infinitely many solutions
(d) having no solution.

7. When the augmented matrix $\begin{bmatrix} 2 & 6 & : & -2 \\ 4 & 8 & : & 3 \end{bmatrix}$ is operated as $R_1 \to R_1 + 2R_2$ then the equivalent matrix is

(a) $\begin{bmatrix} 4 & 12 & : & -4 \\ 4 & 8 & : & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 6 & : & -2 \\ 8 & 20 & : & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 10 & 22 & : & 4 \\ 4 & 8 & : & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 8 & 6 & : & -2 \\ 20 & 8 & : & 3 \end{bmatrix}$

8. The augmented matrix $\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 2 & -1 & 3 & : & 9 \\ 1 & -1 & 1 & : & 2 \end{bmatrix}$ is reduced to $\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -3 & 1 & : & -3 \\ 1 & -1 & 1 & : & 2 \end{bmatrix}$ which one of the following row operation was performed ?

(a) $R_1 \to R_1 - 2R_2$ (b) $R_2 \to R_2 - 2R_1$
(c) $R_2 \to \frac{1}{2} R_2$ (d) $R_2 \to R_2 - R_1$

9. The new matrix after applying the elementary operation $R_2 \to 2R_2 + 3R_1$ on the matrix $A = \begin{bmatrix} 2 & 5 & 4 \\ 5 & 2 & 6 \\ 7 & 2 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 2 & 5 & 4 \\ 16 & 19 & 24 \\ 7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 5 & 4 \\ 19 & 19 & 24 \\ 7 & 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -5 & 4 \\ 16 & 19 & 24 \\ 7 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 5 & 4 \\ 16 & 19 & 24 \\ 7 & 2 & 1 \end{bmatrix}$

Answer key

1. b	2. d	3. c	4. d	5. b	6. d	7. c	8. b	9. a

5.4 Inverse Matrix Method

Basic Formulae and Key Points

1. **Singular Matrix:** A square matrix A is said to be singular, if $|A| = 0$.
2. **Non-singular Matrix:** A square matrix A is said to be non-singular if $|A| \neq 0$.
3. **Adjoint of a Matrix:** The adjoint of a square matrix A is defined as the transpose of a cofactor matrix. Adjoint of the matrix A is denoted by Adj. A and is given by Adj. $A = [A_{ij}]^T$ where $[A_{ij}]$ is the cofactor matrix of A. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 matrix, and if A_{ij} be the cofactor of a_{ij}, then the adjoint matrix of matrix A is

$$\text{Adj. } A = \text{transpose of cofactor matrix of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

4. **Determination of Inverse of a Matrix**

Let, A be any non-singular square matrix. Then its inverse can be obtained by the formula:

$$A^{-1} = \frac{\text{Adj.}A}{|A|}$$

> **Note:** i) The inverse of a matrix, if it exists, is uniquely determined.
> ii) A square matrix A is invertible if and only if $|A| \neq 0$.

Group 'A' (Multiple Choice Questions and Answers)

1. *The adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is*

(a) $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

2. *For what value of x, matrix $\begin{bmatrix} 6-x & 4 \\ 3-x & 1 \end{bmatrix}$ is a singular matrix?*

(a) 1 (b) 2 (c) –1 (d) –2

3. *If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is*

(a) 3 (b) ± 3 (c) ± 6 (d) 6

4. *The square matrix A is said to be inverse of square matrix B if*

(a) AB = 0 = BA (b) AB = BA = I (c) A'B = BA' = I (d) AB' = B'A = I

5. *The inverse of a square matrix A i.e. A^{-1} exists if and only if*

(a) $|A| = 0$ (b) $|A| \neq 0$
(c) A is non-singular matrix (d) both (B) & (C)

6. *The inverse of a square matrix A i.e. A^{-1} using adjoint method is given by*

(a) A^{-1} = adj. (A) (b) $A^{-1} = \frac{1}{|A|}$ adj.(A)
(c) $A^{-1} = \frac{1}{A}$ (d) $A^{-1} = |A|$ adj.(A)

7. *The inverse of square matrix $A = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ is given by*

(a) $\frac{1}{2}\begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$ (c) $\frac{1}{2}\begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & -\frac{3}{2} \end{bmatrix}$ (d) $\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

8. *If $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$, then A^{-1} is*

(a) $\frac{1}{17}\begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix}$ (b) $\frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$ (c) $\frac{-1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$ (d) $\frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$

9. *The system of linear equations in matrix form is AX = B. Then $X = A^{-1}B$ exists if*

(a) $|A| \neq 0$ (b) $|A| = 0$ (c) $|B| \neq 0$ (d) $|B| = 0$

10. *The equations 3x + 2y = 5 and 7x + 5y = 12 expressed in matrix equation form is*

(a) $\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 12 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
(c) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

11. *If $\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, then the values of x and y are*

(a) 2 and 1 (b) 3 and – 2 (c) –1 and 5 (d) 3 and – 1

Answer Key

1. b	2. b	3. c	4. b	5. d	6. b	7. d	8. b	9. a	10. c
11. d									

Group 'B' or 'C' (Subjective Questions and Answers)

1. *Solve the following system of equation by row equivalent matrix method.* *[3] [2081 Set E/W]*
4x – 5y + 2z = 1, 3x = 4z – 10, 2y = 3z – 6

Soln: The augmented matrix of the given system of linear equations is

$$\left[\begin{array}{ccc:c} 4 & -5 & 2 & 1 \\ 3 & 0 & -4 & -10 \\ 0 & 2 & -3 & -6 \end{array}\right]$$

Applying $R_2 \rightarrow \frac{1}{4}R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{-5}{4} & \frac{2}{4} & \frac{1}{4} \\ 3 & 0 & -4 & -10 \\ 0 & 2 & -3 & -6 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{-5}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{15}{4} & \frac{-11}{2} & \frac{-43}{4} \\ 0 & 2 & -3 & -6 \end{array}\right]$$

Applying $R_2 \to \frac{4}{15} R_2$

$$\sim \begin{bmatrix} 1 & \frac{-5}{4} & \frac{1}{2} & : & \frac{1}{4} \\ 0 & 1 & \frac{-22}{15} & : & \frac{-43}{15} \\ 0 & 2 & -3 & : & -6 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{5}{4} R_2$ and $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-4}{3} & : & \frac{-10}{3} \\ 0 & 1 & \frac{-22}{15} & : & \frac{-43}{15} \\ 0 & 0 & \frac{-1}{15} & : & \frac{-4}{15} \end{bmatrix}$$

Applying $R_3 \to (-15) R_3$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-4}{3} & : & \frac{-10}{3} \\ 0 & 1 & \frac{-22}{15} & : & \frac{-43}{15} \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

Applying $R_2 \to R_2 + \frac{22}{15} R_3$ and $R_1 \to R_1 + \frac{4}{3} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$\therefore$ $x = 2$, $y = 3$ and $z = 4$ **Ans.**

2. ***Using Row-equivalent matrix method, solve the following system of linear equations:***
$2x + y - z = 9$, $3x - y + 2z = -1$, $4x + y - 3z = 17$ ***[4] [2080 GIE Set A]***

Soln: The augmented matrix of the given system of linear equations is

$$\begin{bmatrix} 2 & 1 & -1 & : & 9 \\ 3 & -1 & 2 & : & -1 \\ 4 & 1 & -3 & : & 17 \end{bmatrix}$$

Applying $R_1 \to \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & : & \frac{9}{2} \\ 3 & -1 & 2 & : & -1 \\ 4 & 1 & -3 & : & 17 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & : & \frac{9}{2} \\ 0 & \frac{-5}{2} & \frac{7}{2} & : & \frac{-29}{2} \\ 0 & -1 & -1 & : & -1 \end{bmatrix}$$

Applying $R_2 \to \left(\frac{-2}{5}\right) R_2$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & : & \frac{9}{2} \\ 0 & 1 & \frac{-7}{5} & : & \frac{29}{5} \\ 0 & -1 & -1 & : & -1 \end{bmatrix}$$

Applying $R_1 \to R_1 - \frac{1}{2} R_2$ and $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{5} & : & \frac{8}{5} \\ 0 & 1 & \frac{-7}{5} & : & \frac{29}{5} \\ 0 & 0 & \frac{-12}{5} & : & \frac{24}{5} \end{bmatrix}$$

Applying $R_3 \to \left(\frac{-5}{12}\right) R_3$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{5} & : & \frac{8}{5} \\ 0 & 1 & \frac{-7}{5} & : & \frac{29}{5} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Applying $R_1 \to R_1 - \frac{1}{5} R_3$ and $R_2 \to R_2 + \frac{7}{5} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$\therefore$ $x = 2$, $y = 3$ and $z = -2$ **Ans.**

3. ***Solve the following system of equations using inverse matrix method.*** ***[3] [2081 Set V]***
$x + 2y + 3z = 20$, $5x = 2y + 4$, $3z = 4x + 4$

Soln: Given system of equations are

$x + 2y + 3z = 20$

$5x - 2y = 4$

$-4x + 3z = 4$

Writing the above system of equations in matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & -2 & 0 \\ -4 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \\ 4 \end{bmatrix}$$

or, $AX = B$

$\therefore$ $X = A^{-1}B$ ....(i)

Where,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -2 & 0 \\ -4 & 0 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 20 \\ 4 \\ 4 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -2 & 0 \\ -4 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} -2 & 0 \\ 0 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & 0 \\ -4 & 3 \end{vmatrix} + 3\begin{vmatrix} 5 & -2 \\ -4 & 0 \end{vmatrix}$$

$$= 1(-6-0) - 2(15-0) + 3(0-8)$$

$$= -6 - 30 - 24$$

$$= -60 \neq 0$$

So, A^{-1} exists.

The cofactors are:

$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ 0 & 3 \end{vmatrix} = -6$ $\quad$ $A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 0 \\ -4 & 3 \end{vmatrix} = -(15-0) = -15$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & -2 \\ -4 & 0 \end{vmatrix} = 0 - 8 = -8$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} = -(6-0) = -6$ $\quad$ $A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -4 & 3 \end{vmatrix} = 3 + 12 = 15$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix} = -(0+8) = -8$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$ $\quad$ $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = -(0-15) = 15$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} = -2 - 10 = -12$

$\therefore$ Matrix of cofactors $= \begin{bmatrix} -6 & -15 & -8 \\ -6 & 15 & -8 \\ 6 & 15 & -12 \end{bmatrix}$

$$\text{Adj. } A = \begin{bmatrix} -6 & -15 & -8 \\ -6 & 15 & -8 \\ 6 & 15 & -12 \end{bmatrix}^T$$

$$= \begin{bmatrix} -6 & -6 & 6 \\ -15 & 15 & 15 \\ -8 & -8 & -12 \end{bmatrix}$$

We know,

$$A^{-1} = \frac{1}{|A|} \times \text{adj. } A$$

$$= -\frac{1}{60}\begin{bmatrix} -6 & -6 & 6 \\ -15 & 15 & 15 \\ -8 & -8 & -12 \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 6 & 6 & -6 \\ 15 & -15 & -15 \\ 8 & 8 & 12 \end{bmatrix}$$

Now, from (i) we have

$$X = A^{-1}B$$

$$= \frac{1}{60}\begin{bmatrix} 6 & 6 & -6 \\ 15 & -15 & -15 \\ 8 & 8 & 12 \end{bmatrix}\begin{bmatrix} 20 \\ 4 \\ 4 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} 6 \times 20 + 6 \times 4 - 6 \times 4 \\ 15 \times 20 - 15 \times 4 - 15 \times 4 \\ 8 \times 20 + 8 \times 4 + 12 \times 4 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} 120 \\ 180 \\ 240 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\therefore$ $x = 2, y = 3$ and, $z = 4$ **Ans.**

4. ***Given a matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$***

   (a) ***Find the adjoint of a matrix A.*** [1]

   (b) ***Under what conditions does A^{-1} exist?*** [1]

   (c) ***Write the formula to find A^{-1}.*** [1]

   (d) ***Define singular and non-singular matrix.*** [1]

   (e) ***Examine whether the matrix $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ is singular or non-singular. Justify your answer.*** [1]

Soln: (a) The adjoint of a 2×2 matrix A is given by

$$\text{adj. } (A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(b) The A^{-1} exists if and only if the determinant of A is non-zero i.e. $|A| \neq 0$.

(c) Let, A be a non-singular matrix. Then its inverse can be obtained by using the formula:

$$A^{-1} = \frac{1}{|A|} \times \text{adj. } (A)$$

Where, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and adj. $(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(d) A square matrix A is said to be singular, if $|A| = 0$ and non-singular if $|A| \neq 0$.

(e) Here, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$$|B| = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 \neq 0$$

Since, the determinant of matrix B is 5 which is non-zero.

Hence, the matrix B is non-singular.

5. (a) ***How can we apply elementary row operations? Write.*** [2]

(b) ***Use elementary row operations to solve: $3x - 2y = 8$; $5x + 3y = 7$*** [2]

Soln: (a) We can apply three types of elementary row operations:

i) We can interchange any two rows.

ii) We can multiply/divide any row by a non-zero number.

iii) We can multiply/divide a row by a non-zero number and add/subtract it to another row.

(b) The augmented matrix of the given system of linear equations is

$$\left[\begin{array}{cc:c} 3 & -2 & 8 \\ 5 & 3 & 7 \end{array}\right]$$

Applying $R_1 \to \frac{1}{3} R_1$

$$\sim \left[\begin{array}{cc:c} 1 & \frac{-2}{3} & \frac{8}{3} \\ 5 & 3 & 7 \end{array}\right]$$

Applying $R_2 \to R_2 - 5R_1$

$$\sim \left[\begin{array}{cc:c} 1 & \frac{-2}{3} & \frac{8}{-3} \\ 0 & \frac{19}{3} & \frac{-19}{3} \end{array}\right]$$

Applying $R_2 \to \left(\frac{3}{19}\right) R_2$

$$\sim \left[\begin{array}{cc:c} 1 & \frac{-2}{3} & \frac{8}{3} \\ 0 & 1 & -1 \end{array}\right]$$

Applying $R_1 \to R_1 + \frac{2}{3} R_2$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$\therefore$ $x = 2$; $y = -1$ **Ans.**

6. (a) ***Determine the nature of the solutions for the given system of linear equations:***

$x - 2y = -1$,

$2x - y = 4$

(b) ***Solve the system of equations using Cramer's rule.***

Soln: (a) The given equations are

$x - 2y = -1$ ....(i)

$2x - y = 4$ ....(ii)

Comparing equations (i) and (ii) with $a_1x + b_1y = c_1$ & $a_2x + b_2y = c_2$, we get

$a_1 = 1$, $b_1 = -2$, $c_1 = -1$ and $a_2 = 2$, $b_2 = -1$, $c_2 = 4$

Here, $\frac{a_1}{a_2} = \frac{1}{-2}$

$\frac{b_1}{b_2} = \frac{-2}{-1} = 2$

Since, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Hence, the given system has unique solution.

(b)

Coeff. of x	Coeff. of y	Constant term
1	−2	−1
2	−1	4

Then, $D = \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -1 + 4 = 3$

$D_x = \begin{vmatrix} -1 & -2 \\ 4 & -1 \end{vmatrix} = 1 + 8 = 9$

$D_y = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4 + 2 = 6$

Now, using Cramer's Rule, we have,

$x = \frac{D_x}{D} = \frac{9}{3} = 3$

$y = \frac{D_y}{D} = \frac{6}{3} = 2$

$\therefore$ $x = 3$, $y = 2$ **Ans:**

7. ***Consider the system of linear equations:***

$a_1x + b_1y = c_1$,

$a_2x + b_2y = c_2$.

(a) ***Under what conditions is a system of linear equations considered consistent and dependent?*** [1]

(b) ***What does it mean if $D = 0$ and either $D_1 \neq 0$ or $D_2 \neq 0$ in a system of linear equations?*** [1]

(c) ***Form the coefficient matrix for the given system of equations.*** [1]

(d) ***Construct the augmented matrix for the system.*** [1]

Soln: (a) A system of linear equations is considered consistent and dependent if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(b) If $D = 0$ and either $D_1 \neq 0$ or $D_2 \neq 0$, then the system of equation is inconsistent and independent.

(c) The coefficient matrix A for the given system of equations is: $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

(d) Augmented matrix $= \begin{pmatrix} a_1 & b_1 & : & c_1 \\ a_2 & b_2 & : & c_2 \end{pmatrix}$

8. (a) **Define the adjoint matrix.** [1]

(b) **When does a square matrix fail to have an inverse?** [1]

(c) **Solve the following system of equations using the inverse matrix method: $x - y = 2$, $2x + 3y = 9$.** [1]

Soln: (a) Let, A be a square matrix of order n. Then the adjoint matrix of A is defined as the transpose of the matrix of cofactors of A. It is denoted by adj.(A).

(b) A square matrix fail to have an inverse when its determinant is zero.

(c) Here, $x - y = 2$ ....(i)

$2x + 3y = 9$ ....(ii)

Writing the above system of equations in matrix form

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

or, $AX = B$

$\therefore$ $X = A^{-1}B$ ....(iii)

Where, $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ & $B = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5 \neq 0$

So, A^{-1} exists.

Also, $A_{11} = 3$, $A_{12} = -2$

$A_{21} = -(-1) = 1$, $A_{23} = 1$

Matrix of cofactors $= \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

$\therefore$ Adj. $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}^T$

$= \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$

Now, $A^{-1} = \frac{1}{|A|} \times \text{adj. } A$

$= \frac{1}{5}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$

Hence, from equation (iii), we have

$X = A^{-1}B$

or, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix}$

or, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 6+9 \\ -4+9 \end{pmatrix}$

or, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ 5 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\therefore$ $x = 3$ and $y = 1$ **Ans.**

9. ***Applying row equivalent matrix method, solve the following equations.***
$x + 4y + 3z = 6$, $3x + 9y = 18$, $-5x - 6y + 2z = -5$.

Soln: The augmented matrix is

$$\begin{bmatrix} 1 & 4 & 3 & : & 6 \\ 3 & 9 & 0 & : & 18 \\ -5 & -6 & 2 & : & -5 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ & $_3 \to R_3 + 5R_1$

$$\sim \begin{bmatrix} 1 & 4 & 3 & : & 6 \\ 0 & -3 & -9 & : & 0 \\ 0 & 14 & 17 & : & 25 \end{bmatrix}$$

Applying $R_2 \to \left(-\frac{1}{3}\right) R_2$

$$\sim \begin{bmatrix} 1 & 4 & 3 & : & 6 \\ 0 & 1 & 3 & : & 0 \\ 0 & 14 & 17 & : & 25 \end{bmatrix}$$

Applying $R_1 \to R_1 - 4R_2$ & $R_3 \to R_3 - 14R_2$

$$\sim \begin{bmatrix} 1 & 0 & -9 & : & 6 \\ 0 & 1 & 3 & : & 0 \\ 0 & 0 & -25 & : & 25 \end{bmatrix}$$

Applying $R_3 \to \left(-\frac{1}{25}\right) R_3$

$$\sim \begin{bmatrix} 1 & 0 & -9 & : & 6 \\ 0 & 1 & 3 & : & 0 \\ 0 & 0 & 1 & : & -1 \end{bmatrix}$$

Applying $R_1 \to R_1 + 9R_3$ & $R_2 \to R_2 - 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -3 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & -1 \end{bmatrix}$$

$\therefore$ $x = -3$, $y = 3$ & $z = -1$ **Ans:**

10. ***Solve the following system of equations by inverse matrix method.*** [3]
$x - y = 1$, $z + x = -6$, $x + y - 2z = 3$

Soln: Writing the above system of equations of equations in matrix form

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $AX = B$

$\therefore$ $X = A^{-1}B$ ....(i)

Where,

$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -6 \\ 3 \end{bmatrix}$

Now, for A^{-1}

$|A| = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 0$

$= 1(0-1) + 1(-2-1)$

$= -1 - 3$

$= -4 \neq 0$

So, A^{-1} exists.

The cofactors are:

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 0 - 1 = -1$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-2-1) = 3$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$

$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} = -(2-0) = -2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = -2 - 0 = -2$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(1+1) = -2$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -(1-0) = -1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 0 + 1 = 1$$

$\therefore$ Matrix of cofactors $= \begin{bmatrix} -1 & 3 & 1 \\ -2 & -2 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$$\text{Adj. } A = \begin{bmatrix} -1 & 3 & 1 \\ -2 & -2 & -2 \\ -1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{1}{|A|} \times \text{adj. } A$$

$$= \frac{1}{-4}\begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

From (i), we have

$$X = A^{-1} B$$

or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-4}\begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & -1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \\ 3 \end{bmatrix}$

or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-4}\begin{bmatrix} -1+12-3 \\ 3+12-3 \\ 1+12+3 \end{bmatrix}$

or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-4}\begin{bmatrix} 8 \\ 12 \\ 16 \end{bmatrix}$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$

$\therefore$ $x = -2, y = -3, z = -4$ **Ans:**

11. ***Solve the following system of equations by using Cramer's Rule.***
$3x + 5y = 2, 2x - 3z = -7, 4y + 2z = 2$

Soln: Writing the coefficients of variables and constants in order as follows;

Coeff. of x	Coeff. of y	Coeff. of y	Constant term
3	5	0	2
2	0	-3	-7
0	4	2	2

Then,

$$D = \begin{vmatrix} 3 & 5 & 0 \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} + 0$$

$$= 3(0+12) - 5(4-0)$$
$$= 36 - 20$$
$$= 16$$

$$D_x = \begin{vmatrix} 2 & 5 & 0 \\ -7 & 0 & -3 \\ 2 & 4 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} -7 & -3 \\ 2 & 2 \end{vmatrix} + 0$$

$$= 2(0+12) - 5(-14+6)$$
$$= 24 + 40$$
$$= 64$$

$$D_y = \begin{vmatrix} 3 & 2 & 0 \\ 2 & -7 & -3 \\ 0 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} -7 & -3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} + 0$$

$$= 3(-14+6) - 2(4-0)$$
$$= -24 - 8$$
$$= -32$$

$$D_z = \begin{vmatrix} 3 & 5 & 2 \\ 2 & 0 & -7 \\ 0 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & -7 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & -7 \\ 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= 3(0+28) - 5(4-0) + 2(8-0)$$
$$= 84 - 20 + 16$$
$$= 80$$

Now, by Cramer's Rule

$$x = \frac{D_x}{D} = \frac{64}{16} = 4$$

$$y = \frac{D_y}{D} = \frac{-32}{16} = -2$$

$$z = \frac{D_z}{D} = \frac{80}{16} = 5$$

$\therefore$ $x = 4, y = -2$ and $z = 5$ **Ans.**

12. **(a)** ***For what value of P the pair of linear equation $Px + 2y - 5 = 0$ and $3x + y - 1 = 0$ has unique solution?*** **[2]**

(b) ***Give reason why the system of equations $x + 2y = 5$ and $3x + 6y = 12$ are not solvable by Cramer's rule.*** **[2]**

Soln: (a) Here, given equations are

$Px + 2y - 5 = 0$ .... (i)

$3x + y - 1 = 0$ ....(ii)

Comparing these two equations with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$. Then we have

$a_1 = P$, $b_1 = 2$, $c_1 = -5$ and $a_2 = 3$, $b_2 = 1$ & $c_2 = -1$

The given pair of linear equations has unique solution if

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

$\Rightarrow$ $\frac{P}{3} \neq \frac{2}{1}$

$\Rightarrow$ $P \neq 6$

Hence, the given pair of linear equations has unique solution for all real values of P except 6. **Ans.**

(b) Here, given system of equations are,

$x + 2y = 5$ and $3x + 6y = 12$

Writing the coefficients of variables and constants in order, we have

Coeff. of x	Coeff. of y	Constant term
1	2	5
3	6	12

Now, $D = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$

Since, the determinant of the coefficients matrix D = 0, the given system of linear equation are not solvable by Cramer's rule. **Ans.**

13. ***Use determinant method to solve the equations $x = 2y$ and $3x + 2y = 8$*** [3]

> **Note:** The determinant method for solving a system of linear equations is known as **Carmer's Rule.**

Soln: Here, the given system of equations are,

$x - 2y = 0$ .... (i)

$3x + 2y = 8$ .... (ii)

Writing the coefficients of variables and constants in order as follows;

Coeff. of x	Coeff. of y	Constant term
1	–2	0
3	2	8

Then,

$D = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 2 + 6 = 8$

$D_x = \begin{vmatrix} 0 & -2 \\ 8 & 2 \end{vmatrix} = 0 + 16 = 16$

$D_y = \begin{vmatrix} 1 & 0 \\ 3 & 8 \end{vmatrix} = 8 - 0 = 8$

Now, by using Cramer's rule, we have

$x = \frac{D_x}{D} = \frac{16}{8} = 2$

$y = \frac{D_y}{D} = \frac{8}{8} = 1$

$\therefore$ $x = 2$ and $y = 1$ **Ans:**

5.5 Language (word) Problems

1. ***The cost of 2 pen and 3 exercise book is Rs. 420 and the cost of 3 pen and 5 exercise book is Rs. 680. Find the cost of one pen and one exercise book using Carmer's rule.*** ***[3] [2081 – GIE]***

Soln: Let, the cost of one pen and one exercise book be Rs. x and Rs. y respectively.

According to the given question,

$2x + 3y = 420$ .... (i)

$3x + 5y = 680$ .... (ii)

Writing the coefficients of variables and constants in order

Coeff. of x	Coeff. of y	Constant term
2	3	420
3	5	680

Then, $D = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$

$D_x = \begin{vmatrix} 420 & 3 \\ 680 & 5 \end{vmatrix} = 2100 - 2040 = 60$

$D_y = \begin{vmatrix} 2 & 420 \\ 3 & 680 \end{vmatrix} = 1360 - 1260 = 100$

By using Cramer's rule,

$x = \frac{D_x}{D} = \frac{60}{1} = 60$

$y = \frac{D_y}{D} = \frac{100}{1} = 100$

$\therefore$ The cost of one Pen (x) = Rs. 60

The cost of one exercise book (y) = Rs. 100 **Ans.**

2. ***There are 50 workers employed in a sugar factory. If the total daily wage of the employees is Rs. 5,800 when a man gets Rs. 120 and a woman gets Rs. 100 a day. Find the number of men and women employed in the factory by using Cramer's rule.*** [3]

Soln: Let, x and y be the number of men and women employed respectively. Then

$x + y = 50$ .... (i)

$120x + 100y = 5800$

or, $6x + 5y = 290$ .... (ii)

Writing the coefficients of variables and constants in order as follows;

Coeff. of x	Coeff. of y	Constants
1	1	50
6	5	290

Then, $D = \begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} = 1 \times 5 - 6 \times 1 = 5 - 6 = -1$

$D_x = \begin{vmatrix} 50 & 1 \\ 290 & 5 \end{vmatrix} = 50 \times 5 - 290 \times 1 = 250 - 290 = -40$

$D_y = \begin{vmatrix} 1 & 50 \\ 6 & 290 \end{vmatrix} = 1 \times 290 - 6 \times 50 = 290 - 300 = -10$

Now, by Cramer's rule, we have,

$x = \frac{D_x}{D} = \frac{-40}{-1} = 40$

$y = \frac{D_y}{D} = \frac{-10}{-1} = 10$

$\therefore$ Number of men = 40

Number of women = 10 **Ans.**

3. ***The cost of 3 Kg tomato and 2 kg potato is Rs. 69. The cost of 4 kg tomato and 3 kg potato is Rs.***
Find the rate of each kind of item.

Soln: Let, Rs. x and Rs. Y be the cost of 1 kg of tomato and potato respectively. Then,

$3x + 2y = 69$

$4x + 3y = 96$

Writing the coefficients of variables and constants in order as follows;

Coeff. of x	Coeff. of y	Constant
3	2	69
4	3	96

Then, $D = \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = 9 - 8 = 1$

$D_x = \begin{vmatrix} 69 & 2 \\ 96 & 3 \end{vmatrix} = 207 - 192 = 15$

$D_y = \begin{vmatrix} 3 & 69 \\ 4 & 96 \end{vmatrix} = 288 - 276 = 12$

Using Cramer's rule,

$x = \frac{D_x}{D} = \frac{15}{1} = 15$

$y = \frac{D_y}{D} = \frac{12}{1} = 12$

$\therefore$ The rates of tomato and potato are Rs. 15 and Rs. 12 respectively. **Ans.**

4. ***Ashish purchases 5 copies and 4 books and pays Rs. 560. From the same shop, Raman purchas 4 copies and 5 books and pays Rs. 580. Find the rate of one copy and one book. (Use Cramer rule).***

Soln: Let, Rs. x and Rs. y be the cost of 1 copy and 1 book respectively. Then by question,

$5x + 4y = 560$.........(i)

$4x + 5y = 580$ ....(ii)

Writing the coefficients of variables and constants in order as follows;

Coeff. of - x	Coeff. of - y	Constant term
5	4	560
4	5	580

Then,

$D = \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 25 - 20 = 5$

$D_x = \begin{vmatrix} 560 & 4 \\ 580 & 5 \end{vmatrix} = 2800 - 2320 = 480$

$D_y = \begin{vmatrix} 5 & 560 \\ 4 & 580 \end{vmatrix} = 2900 - 2240 = 660$

By using Cramer's rule,

$x = \frac{D_x}{D} = \frac{480}{5} = 96$

$y = \frac{D_y}{D} = \frac{660}{5} = 132$

$\therefore$ The cost of one copy and book are Rs. 96 and Rs. 132 respectively. **Ans.**

5. ***40 people are employed in a certain factory. If the daily total bill of factory is Rs. 3625 when a man gets Rs. 100 and a woman gets Rs. 75 a day, find the number of men and women employed in the factory by using Cramer's rule.*** [3]

Soln: Let, x and y be the number of men and women employed in the factory. Then by question,

$x + y = 40$ ....(i)

Since, each man earns Rs. 100 a day and each women earns Rs. 75 a day. So, we have

$100x + 75y = 3625$

$\therefore$ $4x + 3y = 145$ ....(ii)

Now, we have the following system of linear equations

$x + y = 40$ ....(i)

$4x + 3y = 145$ ......(ii)

Writing the coefficients of variables and constants in order as follows;

Coeff. of – x	Coeff. of – y	Constant term
1	1	40
4	3	145

$D = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 3 - 4 = -1$

$D_x = \begin{vmatrix} 40 & 1 \\ 145 & 3 \end{vmatrix} = 120 - 145 = -25$

$D_y = \begin{vmatrix} 1 & 40 \\ 4 & 145 \end{vmatrix} = 145 - 160 = -15$

By using Cramer's rule,

$x = \frac{D_x}{D} = \frac{-25}{-1} = 25$

$y = \frac{D_y}{D} = \frac{-15}{-1} = 15$

$\therefore$ There are 25 men and 15 women employed in the factory.

Unit-wise Model Questions For Algebra

Set 1

Group 'A'

1. The permutation of 'n' things taken 'r' at a time when each thing may occur any numbers of times is ....
(a) n ways (b) r^n ways (c) n^r ways (d) $(n \times r)$ ways **[Ans: c]**

2. Which one of the following is Euler's form of complex number –i?
(a) $e^{\frac{\pi i}{4}}$ (b) $e^{\frac{3\pi i}{2}}$ (c) $e^{\frac{3\pi i}{4}}$ (d) $e^{\frac{\pi i}{2}}$ **[Ans: b]**

Group 'B'

12. (a) Write the number of the total terms in the expansion of $\left[\left(x - \frac{1}{x}\right)^2\right]^{25}$. **[1] [Ans: 51]**

(b) Write the middle term in the expansion of $(x + a)^n$ when n is even. **[1] [Ans: $C(n, \frac{n}{2})\, x^{n/2} a^{n/2}$]**

(c) What is the sum of binomial coefficient in the expansion $(1 + x)^n$? **[1] [Ans: 2^n]**

(d) Write $\log_e (1 + x)$ in series form. $[-1 < x \le 1]$ **[1] [Ans: $x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \dots$ to ∞]**

(e) Write e^{-x} in series form. **[1] [Ans: $1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$ to ∞]**

13. (a) Find the value of $(1-\omega+\omega^2)^4+(1+\omega-\omega^2)^4$, where ω and ω^2 are imaginary cube roots of unity. [2] [Ans: -16

(b) Solve the following system of equations using inverse matrix method. $x+2y+3z=20$, $5x=2y+4$, $3z=4x+4$ [3] [Ans: $x=2, y=3, z=4$

Group 'C'

20. (a) If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots\ldots+C_nx^n$, prove that $C_1+2C_2+3C_3+\ldots\ldots+nC_n-\frac{1}{2}(n.2^n)=0$ [3

(b) Find the square root of $1-\sqrt{3}\,i$ using De-Moivre's theorem. [2] [Ans: $\pm\frac{1}{\sqrt{2}}(\sqrt{3}-i)$]

(c) Use the principle of mathematical induction to prove that $1+3+5+7+\ldots\ldots+(2n-1)=n^2$. [3

Set 2

Group 'A'

1. The number of combination of n things taken 'r' at a time is ....

(a) $\frac{n!}{r!}$ (b) $\frac{n!}{(n-r)!}$ (c) $\frac{n!}{r!(n-r)!}$ (d) $\frac{n!}{r(n-r)!}$ [Ans: c

2. The polar form of complex number $\frac{1-i}{1+i}$ is ....

(a) $\cos 0^\circ+i\sin 0^\circ$ (b) $\cos 90^\circ+i\sin 90^\circ$

(c) $\cos 120^\circ+i\sin 120^\circ$ (d) $\cos 270^\circ+i\sin 270^\circ$ [Ans: d

Group 'B'

12. (a) Write the total number of permutations of set of n objects arranged in a circle. [1] [Ans: (n-1)!

(b) Write the general term in the expansion $(a+x)^n$. [1] [Ans: $C(n,r)\,a^{n-r}x^r$

(c) Write the series of $\log_e(1-x)$ [x<1]. [1] [Ans: $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-$

(d) Write the complex number $(\cos\theta+i\sin\theta)$ in Euler's form. [1] [Ans: $e^{i\theta}$

(e) Write the sum of cube of first n natural numbers. [1] [Ans: $\frac{n^2(n+1)^2}{4}$

13. (a) Show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}=\omega$, where, ω and ω^2 are cube root of unity. [2

(b) Solve the following system of equations by row equivalent matrix method. $4x-5y+2z=1$, $3x=4z-10$, $2y=3z-6$. [3] [Ans: $x=2, y=3, z=4$

Group 'C'

20. (a) If $z=\cos\theta+i\sin\theta$, find the value of $z^n+\frac{1}{z^n}$ by using De-moivre's theorem. [2] [Ans: 2Cosnθ

(b) In a group of 12 students, 8 are boys and remaining girls. In how many ways can 5 students be selected for quiz competition so as to include at most three girls. [3] [Ans: 784

(c) Prove by the method of principle of mathematical induction that n^3+2n is divisible by 3. [3

Set 3

Group 'A'

1. It is given that $Z=r(\cos\theta+i\sin\theta)$ and n be a positive integer. Which one of the following represent Z^n

(a) $r[\cos n\theta+i\sin n\theta]$ (b) $r^n[\cos n\theta-i\sin n\theta]$

(c) $r^n[\cos n\theta+i\sin n\theta]$ (d) $r[\cos n\theta-i\sin n\theta]$

2. $C(n+1, r)$, $C(n, r)$ and $C(n, r-1)$ are binomial coefficients and satisfies the condition $C(n+1, r)=C(n, r)+C(n, r-1)$, which one of the following exist?

(a) $C(n-1, r)+C(n-1, r-1)=C(n, r-1)$ (b) $C(n-1, r)+C(n-1, r+1)=C(n, r-1)$

(c) $C(n-1, r-1)+C(n-1, r-2)=C(n, r-1)$ (d) $C(n-1, r)+C(n-1, r-2)=C(n, r-1)$

Group 'B'

12. (a) Write the number of circular permutations of n things. [1] [Ans: (n-1)!]

(b) Write the general term in the expansion $(x+a)^n$. [1] [Ans: $C(n, r)a^{n-r}x^r$]

(c) Write the two complex cube roots of unity. [1] $\left[\text{Ans: } \frac{-1+\sqrt{3}i}{2} \text{ and } \frac{-1-\sqrt{3}i}{2}\right]$

(d) Write the sum of $1^2+2^2+3^2+\ldots+n^2$ in terms of n. [1] $\left[\text{Ans: } \frac{n(n+1)(2n+1)}{6}\right]$

(e) Write the augmented matrix for system of three simultaneous equations $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$ and $a_3x+b_3y+c_3z=d_3$. [1] $\left[\text{Ans: } \begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}\right]$

13. (a) It is given that the permutation $P(n, r)=240$ and combination $C(n, r)=120$. Find the value of r. [3] [Ans: 2]

(b) Show that in the expansion of $\left(ax^2+\frac{b}{x}\right)^{25}$, there is no term independent of x. [2]

Group 'C'

20. (a) Using De' Moivre's theorem, evaluate $(1+\sqrt{3}i)^4$. [2] [Ans: $-8-8\sqrt{3}i$]

(b) Prove that $1+4+7+10+\ldots+(3n-2)=\frac{n(3n-1)}{2}$ by using principle of mathematical induction. [3]

(c) The cost of 2 pen and 3 exercise book is Rs. 420 and the cost of 3 pen and 5 exercise book is Rs. 680. Find the cost of one book and one exercise book using Cramer's rule. [3] [Ans: Rs. 60, Rs. 100]

Set 4

Group 'A'

1. Which one of the following is the expansion of $\ln(1+x)$ $(x<1)$?

(a) $x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^3}{2}+\ldots\ldots$ (b) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

(c) $x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots$ (b) $x-\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots 1$ [Ans: b]

2. Which one of the following is amplitude of the complex number $1-i$?

(a) 45° (b) 135° (c) 225° (d) 315° [Ans: b]

Group 'B'

12. (a) State De-Moivre's the theorem. [1] [Ans: $[r(\cos\theta+i\sin\theta)]^n=r^n(\cos n\theta+i\sin n\theta)$]

(b) Write the relation between P(n,r) and C(n,r). [1] [Ans: $P(n, r)=C(n, r)\times r!$]

(c) Write the series of e^{-x}. [1] $\left[\text{Ans: } 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots \text{ to } \infty\right]$

(d) What is the sum of first 'n' natural numbers? [1] $\left[\text{Ans: } \frac{n(n+1)}{2}\right]$

(e) What does n represent in $(a+b)^n$? [1] [Ans: Positive integer]

13. (a) In how many ways the words of "ASSESSMENT" are arranged? [2] $\left[\text{Ans: } \frac{10!}{2!\,4!}\right]$

(b) Using mathematical induction, prove that $1+3+5+\ldots..+2n-1=n^2$. [3]

Group 'C'

20. (a) There are 6 boys and 5 girls in a class. In how many ways the committee of 5 be made such that there are 2 girls and 3 boys are included. [2] [Ans: 200]

(b) Prove that the middle term of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)(2n-5)}{n!}x^n$ [3]

(c) Solve the following linear equations by using Row equivalent matrix methods $2x+3y=10$, $3x-2y=0$. [3] $\left[\text{Ans: } \left(\frac{20}{13}, \frac{30}{13}\right)\right]$

□□□

2. Trigonometry

Chapter 6 Properties of Triangle

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6.1 Properties of Triangle

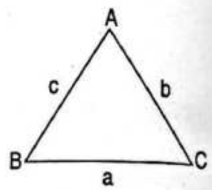
Basic Formulae and Key Points

In ΔABC, we denote $\angle BAC$ by A, $\angle ABC$ by B and $\angle ACB$ by C and the lengths of the sides opposite to the angles A, B and C are denoted by a, b, c respectively.

i.e. AB = c, BC = a and AC = b.

let the semi–perimeter of the triangle is denoted by s then $2s = a + b + c$.

The circum-radius and area of the triangle ABC are denoted by R and Δ respectively.



1. The Cosine Law:

In any ΔABC

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ i.e, $a^2 = b^2 + c^2 - 2bc \cos A$

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ i.e, $b^2 = c^2 + a^2 - 2ca \cos B$

(iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ i.e, $c^2 = a^2 + b^2 - 2ab \cos C$

2. The Sine Law:

In any ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circum-radius.

3. The Half Angle Laws:

In any ΔABC,

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$, $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

4. The Projection Law:

In any ΔABC,

(i) $a = b\cos C + c\cos B$

(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

5. The Area of a Triangle:

In any ΔABC,

(i) $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

(ii) $\Delta = \frac{abc}{4R}$

(iii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

(iv) $\Delta = \frac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$

6. The Tangent Law:

In any ΔABC,

(i) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

(ii) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$.

Note: In ΔABC, we have $A + B + C = \pi^c$, then

(a) (i) $\sin(A+B) = \sin C$,

(ii) $\cos(A+B) = -\cos C$

(iii) $\tan(A+B) = -\tan C$

(b) (i) $\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$

(ii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

(iii) $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

Group 'A' (Multiple Choice Questions and Answers)

1. *In any triangle ABC, a, b, c are sides of the ΔABC and s is semi-perimeter of the triangle. Which one of the following is equal to $\cos\frac{A}{2}$?* [2080 Optional]

(a) $\sqrt{\frac{s(s-a)}{bc}}$ (b) $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(c) $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ (d) $\sqrt{\frac{(s-b)(s-c)}{bc}}$

2. *In any triangle ABC, a, b, c are sides of the ΔABC and s is semi perimeter of the triangle. Which one of the following is equal to $\sin\frac{A}{2}$?*

(a) $\sqrt{\frac{s(s-a)}{bc}}$ (b) $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(c) $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ (d) $\sqrt{\frac{(s-b)(s-c)}{bc}}$

3. *In any ΔABC, $b\cos C + c\cos B$ is equal to*

(a) a (b) b (c) c (d) R

4. *In any ΔABC, $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C$ is equal to*
   (a) s (b) 2s (c) 3s (d) 4s

5. *In any ΔABC, if $\sin A + \sin B + \sin C$ is equal to*
   (a) $\frac{R}{s}$ (b) $\frac{s}{2R}$ (c) $\frac{s}{R}$ (d) $\frac{R}{2s}$

6. *In any ΔABC, if $\sin^2 A + \sin^2 B = \sin^2 C$ then the triangle is*
   (a) equilateral (b) isosceles
   (c) right angled (d) scalene

7. *In any ΔABC, if $2\cos A \sin C = \sin B$. Which one is the type of ΔABC?* **[2081 Supp. Set**
   (a) scalene (b) isosceles
   (c) equilateral (d) right angled

8. *In any ΔABC, if $\frac{a}{b} = \frac{\cos A}{\cos B}$, then the triangle is*
   (a) equilateral (b) isosceles
   (c) right angled (d) right angled isosceles triangle

9. *In any ΔABC, if $\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C}$, then the triangle is*
   (a) equilateral (b) isosceles
   (c) right angled (d) scalene

10. *If the sides a, b, c of the ΔABC are such that $a = k^2 - k + 1$, $b = k^2 + 1$ and $c = k^2 - 1$, then the great*
    angle is
    (a) A (b) B (c) C (d) None.

Answer Key

1. a	2. d	3. a	4. b	5. c	6. c	7. b	8. b	9. a	10. b

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Prove that: $1 - \tan\frac{A}{2}.\tan\frac{B}{2} = \frac{2c}{a+b+c}$*
   (b) *If $\sin C.\sin(A-B) = \sin A.\sin(B-C)$, prove that a^2, b^2, c^2 are in A.P.* **[3] [2081 Se**

Soln: (a) Here, L.H.S. $= 1 - \tan\frac{A}{2}.\tan\frac{B}{2}$

$$= 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$= 1 - \sqrt{\frac{(s-b)(s-c)(s-c)(s-a)}{s^2(s-a)(s-b)}}$$

$$= 1 - \frac{s-c}{s}$$

$$= \frac{s-s+c}{s}$$

$$= \frac{c}{s} = \frac{c}{\frac{a+b+c}{2}} = \frac{2c}{a+b+c} = \text{R.H.S. Proved.}$$

(b) Given, $\sin C \sin(A-B) = \sin A \sin(B-C)$

or, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

or, $\frac{\sin(B+C)}{\sin(A+B)} = \frac{\sin(A-B)}{\sin(B-C)}$ [$\because \sin(B+C) = \sin A$ and $\sin(A+B) = \sin C$]

or, $\sin(B+C).\sin(B-C) = \sin(A+B).\sin(A-B)$

or, $\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$ [$\because \sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B$]

or, $\left(\frac{b}{2R}\right)^2 - \left(\frac{c}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 - \left(\frac{b}{2R}\right)^2$

or, $\frac{b^2-c^2}{4R^2} = \frac{a^2-b^2}{4R^2}$

or, $b^2 - c^2 = a^2 - b^2$

or, $2b^2 = a^2 + c^2$

$\therefore$ a^2, b^2, c^2 are in A.P.

2. (a) *In any ΔABC, Prove that $a^2 + b^2 + c^2 - 2(bc\cos A + ca\cos B + ab\cos C) = 0$.* **[2]**
   (b) *If $\frac{1}{p+r} = \frac{3}{p+q+r} - \frac{1}{q+r}$ in a triangle PQR, prove that $\angle R = 60°$.* **[3] [2081 Set V]**

Soln: (a) Here, L.H.S $= a^2 + b^2 + c^2 - 2(bc\cos A + ca\cos B + ab\cos C)$

$$= a^2 + b^2 + c^2 - 2\left[bc.\frac{b^2+c^2-a^2}{2bc} + ca.\frac{c^2+a^2-b^2}{2ca} + ab.\frac{a^2+b^2-c^2}{2ab}\right]$$ [$\because$ By using cosine law]

$$= a^2 + b^2 + c^2 - \frac{2}{2}(b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2)$$

$$= a^2 + b^2 + c^2 - (a^2 + b^2 + c^2)$$

$$= 0$$

= **R.H.S. Proved.**

(b) Given, $\frac{1}{p+r} = \frac{3}{p+q+r} - \frac{1}{q+r}$

or, $\frac{1}{p+r} + \frac{1}{q+r} = \frac{3}{p+q+r}$

or, $\frac{q+r+p+r}{(p+r)(q+r)} = \frac{3}{p+q+r}$

or, $(p+q+2r)(p+q+r) = 3(p+r)(q+r)$

or, $p^2 + pq + pr + pq + q^2 + qr + 2pr + 2qr + 2r^2 = 3(pq + pr + qr + r^2)$

or, $p^2 + q^2 + 2r^2 + 2pq + 3pr + 3qr = 3pq + 3pr + 3qr + 3r^2$

or, $p^2 + q^2 + 2r^2 - 3r^2 = 3pq - 2pq$

or, $p^2 + q^2 - r^2 = pq$

Dividing both side by 2pq we get

$$\frac{p^2+q^2-r^2}{2pq} = \frac{pq}{2pq}$$

or, $\cos R = \frac{1}{2}$

or, $\cos R = \cos 60°$

$\therefore$ $\angle R = 60°$

3. (a) *Prove that $a(b\cos C - c\cos B) = b^2 - c^2$*

(b) *Show that $2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right)$ is less than the perimeter of ΔABC.* [3] [2081 Optio...]

Soln: (a) Here, L.H.S $= a(b\cos C - c\cos B)$

$$= a\left(b.\frac{a^2+b^2-c^2}{2ab} - c.\frac{c^2+a^2-b^2}{2ca}\right)$$

$$= \frac{a^2+b^2-c^2}{2} - \frac{c^2+a^2-b^2}{2}$$

$$= \frac{a^2+b^2-c^2-c^2-a^2+b^2}{2}$$

$$= \frac{2b^2-2c^2}{2} = b^2 - c^2.$$

= **R.H.S. Proved.**

(b) Here, $2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = a.\,2\sin^2\frac{C}{2} + c.2\sin^2\frac{A}{2}$

$$= a(1-\cos C) + c(1-\cos A)$$

$$= a - a\cos C + c - c\cos A$$

$$= a + c - (a\cos C + c\cos A)$$

$$= a + c - b$$

Since, perimeter $(2s) = a + b + c$

$\therefore\ 2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) <$ perimeter. **Hence Proved.**

4. (a) *In a ΔABC, prove that $\frac{c - b\cos A}{b - c\cos A} = \frac{\cos B}{\cos C}$*

(b) *If $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$, prove that $C = 45°$ or $135°$.*

Soln: (a) By projection law, $c = a\cos B + b\cos A$

$b = c\cos A + a\cos C$

Now, L.H.S. $= \frac{c - b\cos A}{b - c\cos A} = \frac{a\cos B + b\cos A - b\cos A}{c\cos A + a\cos C - c\cos A} = \frac{a\cos B}{a\cos C} = \frac{\cos B}{\cos C}$ = **R.H.S. Proved.**

(b) Given, $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$

or, $(a^2)^2 + (b^2)^2 - 2c^2(a^2+b^2) + c^4 = 0$

or, $(a^2+b^2)^2 - 2a^2b^2 - 2c^2(a^2+b^2) + c^4 = 0$

or, $(a^2+b^2)^2 - 2(a^2+b^2)c^2 + (c^2)^2 = 2a^2b^2$

or, $(a^2+b^2-c^2)^2 = 2a^2b^2$

$\therefore\ a^2 + b^2 - c^2 = \pm\sqrt{2}ab$

Dividing both side by 2ab we get

$$\frac{a^2+b^2-c^2}{2ab} = \pm\frac{\sqrt{2}ab}{2ab}$$

or, $\cos C = \pm\frac{1}{\sqrt{2}}$

Taking +ve sign,

$\cos C = \frac{1}{\sqrt{2}} = \cos 45°$

$\therefore\ C = 45°$

taking –ve sing

$\cos C = \frac{-1}{\sqrt{2}} = \cos 135°$

$C = 135°$

$\therefore\ C = 45°$ or $135°$

5. (a) *In a ΔABC, proved that $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$* [2]

(b) *Prove that: $\cos A + \cos B = \frac{2(a+b)}{c}\sin^2\frac{C}{2}$* [3]

Soln: (a) Here, L.H.S. $= (b + c)\cos A + (c + a)\cos B + (a + b)\cos C$

$= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$

$= c + b + a$ **(∵ By projection law)**

$= a + b + c$

= **R.H.S. Proved.**

(b) Here, R.H.S. $= \frac{2(a+b)}{c}\sin^2\frac{C}{2}$

$$= \frac{2(2R\sin A + 2R\sin B)}{2R\sin C} \times \sin^2\frac{C}{2} \quad \text{(∵ By sine law)}$$

$$= \frac{2(\sin A + \sin B)}{\sin C} \times \sin^2\frac{C}{2}$$

$$= 2 \times \frac{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{2\sin\frac{C}{2}.\cos\frac{C}{2}} \times \sin^2\frac{C}{2}$$

$$= 2 \times \frac{\cos\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}} \times \sin\frac{C}{2} \quad \left[∵ \sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}\right]$$

$$= 2\cos\left(\frac{A-B}{2}\right).\cos\left(\frac{A+B}{2}\right) \quad \left[∵ \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}\right]$$

$$= \cos\left(\frac{A+B+A-B}{2}\right) + \cos\left(\frac{A+B-A+B}{2}\right)$$

$= \cos A + \cos B.$

= **L.H.S. Proved.**

6. (a) *Prove that $\frac{\cos A + \cos B}{a + b} + \frac{\cos C}{c} = \frac{1}{c}$* [2]

(b) *Prove that $(b + c - a)\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = 2a\cot\frac{A}{2}$* [3]

Soln: (a) Here, L.H.S. $= \frac{\cos A + \cos B}{a + b} + \frac{\cos C}{c}$

$$= \frac{c\cos A + c\cos B + a\cos C + b\cos C}{(a+b)c}$$

$$= \frac{(b\cos C + c\cos B) + (c\cos A + a\cos C)}{(a+b)c}$$

$$= \frac{a+b}{(a+b).c} \quad \text{(∵ By using projection law)}$$

$$= \frac{1}{c}$$

= **R.H.S. Proved.**

(b) Here, L.H.S. $= (b+c-a)\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right)$

$$= (2s-a-a)\left[\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}\right]$$

$$= (2s-2a)\sqrt{\frac{s}{s-a}}\left[\sqrt{\frac{s-b}{s-c}} + \sqrt{\frac{s-c}{s-b}}\right]$$

$$= 2(s-a) \times \frac{\sqrt{s}}{\sqrt{s-a}} \times \frac{s-b+s-c}{\sqrt{(s-c)(s-b)}}$$

$$= 2\sqrt{(s-a)^2} \times \frac{\sqrt{s}}{\sqrt{s-a}} \times \frac{2s-b-c}{\sqrt{(s-b)(s-c)}}$$

$$= 2\sqrt{(s-a)} \times \sqrt{s} \times \frac{a+b+c-b-c}{\sqrt{(s-b)(s-c)}}$$

$$= 2\sqrt{(s-a)} \times \sqrt{s} \times \frac{a}{\sqrt{(s-b)(s-c)}}$$

$$= 2a \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= 2a\cot\frac{A}{2}$$

= **R.H.S. Proved.**

7. *(a)* ***In any ΔABC, prove that: $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$***

(b) ***Prove that: $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$.***

Soln: (a) Here, L.H.S. $= a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= 2R\sin A.\sin(B-C) + 2R\sin B.\sin(C-A) + 2R\sin C.\sin(A-B)$ (∵ By sine la

$= 2R[\sin A.\sin(B-C) + \sin B.\sin(C-A) + \sin C.\sin(A-B)]$

$= 2R[\sin(B+C).\sin(B-C) + \sin(C+A).\sin(C-A) + \sin(A+B).\sin(A-B)]$

(∵ $\sin(A+B) = \sin C$)

$= 2R(\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$

$= 2R \times 0$

$= 0$

= **R.H.S. Proved.**

(b) Here, L.H.S. $= a\cos A + b\cos B + c\cos C$

$= 2R\sin A.\cos A + 2R\sin B.\cos B + 2R\sin C.\cos C$ [∵ By sine law]

$= R[\sin 2A + \sin 2B + \sin 2C]$

$= R\left[2\sin\left(\frac{2A+2B}{2}\right).\cos\left(\frac{2A-2B}{2}\right) + \sin 2C\right]$

$= R[2\sin(A+B).\cos(A-B) + \sin 2C]$

$= R.[2\sin C.\cos(A-B) + 2\sin C.\cos C]$ [∵ $\sin(A+B) = \sin C$]

$= R.2\sin C[\cos(A-B) + \cos C]$

$= R.2\sin C[\cos(A-B) - \cos(A+B)]$ [∵ $\cos(A+B) = -\cos C$]

$= R.2\sin C.2\sin A.\sin B$

$= 4R\sin A.\sin B.\sin C.$

= **R.H.S. Proved.**

8. *(a)* ***If $2\cos A = \sin B : \sin C$, show that the triangle is isosceles.*** [2]

(b) ***In any ΔABC, prove that: $\frac{\cos B - \cos C}{1+\cos A} = \frac{c-b}{a}$.*** [3]

Soln: (a) Given, $2\cos A = \frac{\sin B}{\sin C}$

or, $2.\frac{b^2+c^2-a^2}{2bc} = \frac{\frac{b}{2R}}{\frac{c}{2R}}$ (∵ By using sine and cosine law.)

or, $\frac{b^2+c^2-a^2}{bc} = \frac{b}{c}$

or, $b^2+c^2-a^2 = b^2$

or, $c^2 - a^2 = 0$

or, $c^2 = a^2$

∴ $c = a$

Since, two sides are equal, then Δ is an isosceles.

(b) Here, L.H.S. $= \frac{\cos B - \cos C}{1+\cos A}$

$= \frac{a\cos B - a\cos C}{a(1+\cos A)}$ [∵ Multiplying both numerator and denominator by a]

$= \frac{(c - b\cos A) - (b - c\cos A)}{a(1+\cos A)}$ [∵ By using projection law]

$= \frac{c - b\cos A - b + c\cos A}{a(1+\cos A)}$

$= \frac{(c-b) + c\cos A - b\cos A}{a(1+\cos A)}$

$= \frac{1(c-b) + \cos A(c-b)}{a(1+\cos A)}$

$= \frac{(c-b)(1+\cos A)}{a(1+\cos A)} = \frac{c-b}{a}$

= **R.H.S. Proved.**

9. *(a)* ***In any ΔABC, prove that*** [2]

$bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2} + ab\cos^2\frac{C}{2} = s^2$, where s is semi-perimeter of the triangle

(b) ***Prove that: $\frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(A-B)}{\sin C} = 0$*** [3]

Soln: (a) Here, L.H.S. $= ab\cos^2\frac{C}{2} + bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2}$

$= ab.\frac{s(s-c)}{ab} + bc.\frac{s(s-a)}{bc} + ca.\frac{s(s-b)}{ca}$ [∵ Using half angle formula]

$= s(s-c) + s(s-a) + s(s-b)$

$= s[s-c+s-a+s-b]$

$= s[3s-(a+b+c)]$

$= s(3s-2s)$

$= s \times s = s^2$

= **R.H.S. Proved.**

(b) Here, L.H.S. $= \frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C}$

$= \frac{(2R\sin A)^2 . \sin(B-C)}{\sin A} + \frac{(2R\sin B)^2 \sin(C-A)}{\sin B} + \frac{(2R\sin C)^2 \sin(A-B)}{\sin C}$

$= \frac{4R^2 \sin^2 A \sin(B-C)}{\sin A} + \frac{4R^2 \sin^2 B \sin(C-A)}{\sin B} + \frac{4R^2 \sin^2 C \sin(A-B)}{\sin C}$

$= 4R^2 [\sin A . \sin(B-C) + \sin B . \sin(C-A) + \sin C . \sin(A-B)]$

$= 4R^2 [\sin(B+C). \sin(B-C) + \sin(C+A). \sin(C-A) + \sin(A+B) . \sin(A-B)]$

[$\because$ **sin (B + C) = sin A and so on**]

$= 4R^2 [\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B]$

[$\because$ **$\sin(A+B). \sin(A-B) = \sin^2 A - \sin^2 B$**]

$= 4R^2 \times 0$

$= 0$

= **R.H.S. Proved.**

10. (a) *If a, b, c are in A.P., show that $a \sin^2\frac{C}{2} + c \sin^2\frac{A}{2} = \frac{b}{2}$.*

(b) *Prove that in any ΔABC, $\frac{b^2-c^2}{a^2} \sin 2A + \frac{c^2-a^2}{b^2} \sin 2B + \frac{a^2-b^2}{c^2} \sin 2C = 0$*

Soln: (a) Given a, b, c are in A.P. So, $2b = a + c$

L.H.S. $= a \sin^2\frac{C}{2} + c \sin^2\frac{A}{2}$

$= a\left(\frac{1-\cos C}{2}\right) + c\left(\frac{1-\cos A}{2}\right)$ [$\because \sin^2\frac{C}{2} = \frac{1-\cos C}{2}$]

$= \frac{a - a\cos C + c - c\cos A}{2}$

$= \frac{1}{2}[a + c - (a\cos C + c\cos A)]$

$= \frac{1}{2}[a+c-b]$ [$\because$ **$b = c\cos A + a\cos C$**]

$= \frac{1}{2}(2b-b) = \frac{b}{2}$.

= **R.H.S. Proved.**

(b) Here, L.H.S. $= \frac{b^2-c^2}{a^2} \sin 2A + \frac{c^2-a^2}{b^2} \sin 2B + \frac{a^2-b^2}{c^2} \sin 2C$

$= \frac{b^2-c^2}{a^2} . 2\sin A . \cos A + \frac{c^2-a^2}{b^2} . 2\sin B . \cos B + \frac{a^2-b^2}{c^2} 2\sin C . \cos C$

$= \frac{b^2-c^2}{a^2} 2 \cdot \frac{a}{2R} \cdot \frac{b^2+c^2-a^2}{2bc} + \frac{c^2-a^2}{b^2} 2 \cdot \frac{b}{2R} \cdot \frac{c^2+a^2-b^2}{2ca} + \frac{a^2-b^2}{c^2} \cdot 2 \cdot \frac{c}{2R} \cdot \frac{a^2+b^2-c^2}{2ab}$

$= \frac{(b^2-c^2)(b^2+c^2-a^2)}{2Rabc} + \frac{(c^2-a^2)(c^2+a^2-b^2)}{2Rabc} + \frac{(a^2-b^2)(a^2+b^2-c^2)}{2Rabc}$

$= \frac{1}{2Rabc}[b^4 - c^4 - a^2b^2 + c^2a^2 + c^4 - a^4 - b^2c^2 + a^2b^2 + a^4 - b^4 - c^2a^2 + b^2c^2]$

$= \frac{1}{2Rabc} \times 0 = 0$. = **R.H.S. Proved.**

11. (a) *In ΔABC, If $\angle B = 60°$, then show that : $(a+b+c)(a-b+c) = 3ca$* [2]

(b) *In a ΔABC, If $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then prove that $\angle C = 60°$.* [3]

Soln: (a) Here, L.H.S. $= (a+b+c)(a-b+c)$

$= 2s(a+c-b)$ [$\because$ **$a+b+c = 2s$**]

$= 2s(2s-b-b)$

$= 2s(2s-2b)$

$= 4s(s-b)$

$= 4\frac{s(s-b)}{ca} . ca$

$= 4 . \left(\cos^2\frac{B}{2}\right) . ca$ [$\because \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$]

$= 4ca . \cos^2\frac{60°}{2}$ [$\because \angle B = 60°$]

$= 4ca . \cos^2 30°$

$= 4ca . \frac{3}{4}$ [$\because \cos 30° = \frac{\sqrt{3}}{2}$]

$= 3ca$ = **R.H.S. Proved.**

(b) Given, $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$

or, $\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)\left(\frac{a}{2R} + \frac{b}{2R} - \frac{c}{2R}\right) = 3 . \frac{a}{2R} \cdot \frac{b}{2R}$ ($\because$ **By sine law**)

or, $\frac{(a+b+c)}{2R} \times \frac{(a+b-c)}{2R} = \frac{3ab}{4R^2}$

or, $\frac{(a+b)^2 - c^2}{4R^2} = \frac{3ab}{4R^2}$

or, $a^2 + 2ab + b^2 - c^2 = 3ab$

or, $a^2 + b^2 - c^2 = ab$

Dividing both sides by 2ab we get

$\frac{a^2+b^2-c^2}{2ab} = \frac{ab}{2ab}$

or, $\cos C = \frac{1}{2}$

or, $\cos C = \cos 60°$

$\therefore$ $\angle C = 60°$ **Proved.**

12. (a) *In ΔABC, prove that $b^2 \sin 2C + c^2 \sin 2B = 2ab \sin C$* [2]

(b) *If $b - a = kc$, prove that $\cot\left(\frac{B-A}{2}\right) = \frac{1 + k\cos B}{k \sin B}$* [3]

Soln: (a) Here, L.H.S. $= b^2 \sin 2C + c^2 \sin 2B$

$= b^2 . 2\sin C . \cos C + c^2 2\sin B . \cos B$

$= b^2 . 2 . \frac{c}{2R} \cdot \frac{a^2+b^2-c^2}{2ab} + c^2 . 2 \cdot \frac{b}{2R} \cdot \frac{c^2+a^2-b^2}{2ca}$ ($\because$ **By using sine and cosine law**)

$= \frac{bc}{2aR}(a^2+b^2-c^2) + \frac{bc}{2aR}(c^2+a^2-b^2)$

$= \frac{bc}{2aR}(a^2+b^2-c^2+c^2+a^2-b^2) = \frac{bc}{2aR} \times 2a^2 = \frac{abc}{R} = 2ab \times \frac{c}{2R} = 2ab\sin C$

= **R.H.S. Proved.**

(b) Given, $b - a = kc \Rightarrow k = \frac{b-a}{c}$.

$$\text{R.H.S.} = \frac{1 + k\cos B}{k \sin B}$$

$$= \frac{1 + \left(\frac{b-a}{c}\right)\cos B}{\left(\frac{b-a}{c}\right)\sin B}$$

$$= \frac{c + (b-a)\cos B}{(b-a)\sin B}$$

$$= \frac{a\cos B + b\cos A + b\cos B - a\cos B}{(b-a).\sin B} \quad [\because c = a\cos B + b\cos A]$$

$$= \frac{b(\cos A + \cos B)}{(b-a).\sin B}$$

$$= \frac{2R\sin B(\cos A + \cos B)}{(2R\sin B - 2R\sin A)\sin B}$$

$$= \frac{\cos A + \cos B}{\sin B - \sin A}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{B+A}{2}\right).\sin\left(\frac{B-A}{2}\right)}$$

$$= \frac{\cos\left(\frac{B-A}{2}\right)}{\sin\left(\frac{B-A}{2}\right)} \quad [\because \cos(A-B) = \cos(B-A)]$$

$$= \cot\left(\frac{B-A}{2}\right) = \text{L.H.S. Proved.}$$

13. ***In any ΔABC, if $8R^2 = a^2 + b^2 + c^2$, prove that the triangle is right angled.***

Soln: Given, $a^2 + b^2 + c^2 = 8R^2$

or, $(2R\sin A)^2 + (2R\sin B)^2 + (2R\sin C)^2 = 8R^2$

or, $4R^2\sin^2 A + 4R^2\sin^2 B + 4R^2\sin^2 C = 8R^2$

or, $2R^2(2\sin^2 A + 2\sin^2 B + 2\sin^2 C) = 8R^2$

or, $2\sin^2 A + 2\sin^2 B + 2\sin^2 C = 4$

or, $1 - \cos 2A + 1 - \cos 2B + 2\sin^2 C = 4$

or, $2 - (\cos 2A + \cos 2B) + 2\sin^2 C = 4$

or, $2 - 2\cos\frac{2A+2B}{2}.\cos\frac{2A-2B}{2} + 2\sin^2 C = 4$

or, $2 - 2\cos(A+B).\cos(A-B) + 2\sin^2 C = 4$

or, $-2\cos(A+B).\cos(A-B) + 2\sin^2 C = 2$

or, $-\cos(A+B).\cos(A-B) + \sin^2 C = 1$

or, $\cos C.\cos(A-B) + 1 - \cos^2 C = 1 \quad (\because \cos(A+B) = -\cos C)$

or, $\cos C[\cos(A-B) - \cos C] = 0$

or, $\cos C.[\cos(A-B) + \cos(A+B)] = 0$

or, $\cos C.2\cos A.\cos B = 0$

or, $\cos A.\cos B.\cos C = 0$

Either $\cos A = 0 \Rightarrow A = 90°$

or, $\cos B = 0 \Rightarrow B = 90°$

or, $\cos C = 0 \Rightarrow C = 90°$

$\therefore$ ABC must be right angled triangle.

14. ***In any ABC, If $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$, then prove that the triangle is either isosceles or right angled.*** *[4]*

Soln: Given, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$

or, $\cos A.\sin C + 2\sin C.\cos C = \cos A.\sin B + 2\sin B.\cos B$

or, $\cos A.\sin C + \sin 2C = \cos A.\sin B + \sin 2B$

or, $\sin 2B - \sin 2C + \cos A(\sin B - \sin C) = 0$

or, $2\cos\left(\frac{2B+2C}{2}\right).\sin\left(\frac{2B-2C}{2}\right) + \cos A.(\sin B - \sin C) = 0$

or, $2\cos(B+C).\sin(B-C) + \cos A.(\sin B - \sin C) = 0$

or, $-2\cos A.\sin(B-C) + \cos A.(\sin B - \sin C) = 0$

or, $-\cos A[2\sin(B-C) - (\sin B - \sin C)] = 0$

or, $\cos A.\left[2.2\sin\left(\frac{B-C}{2}\right).\cos\left(\frac{B-C}{2}\right) - 2\cos\left(\frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)\right] = 0$

or, $\cos A.2\sin\left(\frac{B-C}{2}\right)\left[2\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right)\right] = 0.$

or, $\cos A.\sin\left(\frac{B-C}{2}\right).\left[2\cos\frac{B-C}{2} - \cos\left(\frac{B+C}{2}\right)\right] = 0$

Since, $\cos\left(\frac{B-C}{2}\right) > \cos\left(\frac{B+C}{2}\right)$. So, $2\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right) \neq 0$

$\therefore$ $\cos A.\sin\left(\frac{B-C}{2}\right) = 0$

Either, $\cos A = 0 \Rightarrow A = 90°$

or, $\sin\left(\frac{B-C}{2}\right) = 0 \Rightarrow \frac{B-C}{2} = 0$

i.e. $B - C = 0$

i.e. $B = C$

$\therefore$ ΔABC is either right angle or isosceles.

15. ***State cosine law. using cosine law, prove that***

(a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ ***(b) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$*** *[1+2+2]*

Soln: Cosine law: In any ΔABC,

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ i.e. $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ i.e. $b^2 = c^2 + a^2 - 2ca\cos B$

(iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ i.e. $c^2 = a^2 + b^2 - 2ab\cos C.$

(a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

We have, $\cos A = 1 - 2\sin^2\frac{A}{2}$

or, $\frac{b^2+c^2-a^2}{2bc} = 1 - 2\sin^2\frac{A}{2}$

or, $2\sin^2\frac{A}{2} = 1 - \frac{b^2+c^2-a^2}{2bc}$

or, $2\sin^2\frac{A}{2} = \frac{2bc-b^2-c^2+a^2}{2bc}$

or, $\sin^2\frac{A}{2} = \frac{a^2-b^2+2bc-c^2}{4bc}$

$= \frac{a^2-(b-c)^2}{4bc}$

$= \frac{(a+b-c)(a-b+c)}{4bc}$

$= \frac{(a+b-c)(a+c-b)}{4bc}$

$= \frac{(2s-c-c)(2s-b-b)}{4ab}$ $\quad (\because 2s = a+b+c)$

$= \frac{(2s-2c)(2s-2b)}{4bc}$

$= \frac{2(s-c).2(s-b)}{4bc}$

$= \frac{(s-b)(s-c)}{bc}$

Since, $0 < \frac{A}{2} < 90°$, so, $\sin\frac{A}{2}$ is positive.

$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

(b) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

We have, $\cos A = 2\cos^2\frac{A}{2} - 1$

or, $\frac{b^2+c^2-a^2}{2bc} + 1 = 2\cos^2\frac{A}{2}$

or, $\frac{b^2+c^2-a^2+2bc}{2bc} = 2\cos^2\frac{A}{2}$

or, $\cos^2\frac{A}{2} = \frac{b^2+2bc+c^2-a^2}{4bc}$

or, $\cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc}$

$= \frac{(b+c+a)(b+c-a)}{4bc}$

$= \frac{2s(2s-a-a)}{4bc}$ $\quad (\because 2s = a+b+c)$

$= \frac{2s.(2s-2a)}{4bc} = \frac{2s.2(s-a)}{4bc} = \frac{s(s-a)}{bc}$

since $0 < \frac{A}{2} < 90°$ so, $\cos\frac{A}{2}$ is positive.

$\therefore \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

16. ***State the sine law. Using sine Law, prove that***
$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$. **[1+3]**

Soln: In any ΔABC, we have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circum-radius of ABC.

Next part:

By the sine law we have, $a = 2R\sin A$ and $b = 2R\sin B$, then

$\frac{a-b}{a+b} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}$

$= \frac{\sin A - \sin B}{\sin A + \sin B}$

$= \frac{2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}$

$= \cot\left(\frac{A+B}{2}\right).\tan\left(\frac{A-B}{2}\right)$

$= \cot\left(\frac{\pi-C}{2}\right).\tan\left(\frac{A-B}{2}\right)$

$= \cot\left(\frac{\pi}{2}-\frac{C}{2}\right).\tan\left(\frac{A-B}{2}\right)$

$= \tan\left(\frac{A-B}{2}\right).\tan\frac{C}{2}$

Now, $\frac{a-b}{a+b}\cot\frac{C}{2} = \tan\left(\frac{A-B}{2}\right).\tan\frac{C}{2}.\cot\frac{C}{2}$

$= \tan\frac{A-B}{2}$

$\therefore \quad \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$.

17. (a) ***Use sine law to prove that the projection law.*** **[2]**

(b) ***In ΔABC, $A = 45°$, $B = 75°$ prove that $a + c\sqrt{2} - 2b = 0$.*** **[3] [2081 Supp. Set A]**

Soln: (a) Projection Law: In any ΔABC,

(i) $a = b\cos C + c\cos B$

(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

Proof:

By the sine law we have, $a = 2R \sin A$

$= 2R \sin (B + C)$ $\quad$ [$\because A = \pi - (B + C)$]

$= 2R (\sin B . \cos C + \cos B. \sin C)$

$= 2R \sin B. \cos C + 2R \sin C . \cos B$

$= b \cos C + c \cos B$

So, similarly we can show that other two (b and c) relations.

(b) In ΔABC, $A = 45°$, $B = 75°$

$\therefore \quad C = 180° - (A + B)$

$= 180° - (45° + 75°) = 60°$

By sine law, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{a}{\sin 45°} = \frac{b}{\sin 75°} = \frac{c}{\sin 60°}$

or, $\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}}{2}} = k$ (say)

$\therefore \quad a = \frac{k}{\sqrt{2}}, b = \frac{(\sqrt{3}+1)k}{2\sqrt{2}}$ and $c = \frac{\sqrt{3}}{2}k$

Now, L.H.S. $= a + c\sqrt{2} - 2b$

$= \frac{k}{\sqrt{2}} + \frac{\sqrt{3}}{2} . k\sqrt{2} - 2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k$

$= k\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{3}+1}{\sqrt{2}}\right)$

$= k\left(\frac{1+\sqrt{3}-\sqrt{3}-1}{\sqrt{2}}\right)$

$= k \times \frac{0}{2}$

$= 0.$

= R.H.S. **Proved.**

18. ***(a)*** ***State and prove sine law.*** ***[3] [2081 Supp. Set B]***

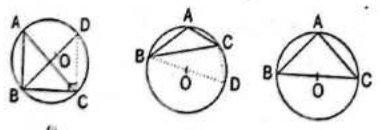
(b) ***In any ΔPQR, prove that: $p (\sin Q - \sin R) + q (\sin R - \sin P) + r (\sin P - \sin Q) = 0$.*** ***[2] [2081 Supp. Set B]***

Soln: (a) **Sine Law :** In any ΔABC, we have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Where, R is the circum-radius of ΔABC

Proof:

Let, O be the circum-centre and R the circum-radius of the ΔABC. Then we have three possible cases.



(i) $\quad$ (ii) $\quad$ (iii)

Case (I) when $\angle A$ is an acute angle in fig (i), draw BOD = 2R and join CD. Then we have $\angle A = \angle D$ and $\angle BCD = 90°$

Now, $\sin D = \frac{BC}{BD}$

or, $\sin A = \frac{a}{2R}$

$\Rightarrow \quad \frac{a}{\sin A} = 2R$

Case (II) when $\angle A$ is an obtuse in fig (ii). Draw BOD = 2R and join CD. Then, $\angle A + \angle D = 180°$ and $\angle BCD = 90°$

Now, $\sin D = \frac{BC}{BD}$

or, $\sin (180° - A) = \frac{BC}{BD}$

or, $\sin A = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R$

Case (III) when $\angle A$ is right angle in fig (iii). then $\frac{BC}{BC} = 1$

or, $\frac{a}{2R} = \sin 90°$

or, $\frac{a}{2R} = \sin A \Rightarrow \frac{a}{\sin A} = 2R$

Thus, in all cases, $\frac{a}{\sin A} = 2R.$

Similarly, we can show that $\frac{b}{\sin B} = 2R, \frac{c}{\sin C} = 2R.$

Combining these we get, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$

(b) L.H.S. $= p (\sin Q - \sin R) + q (\sin R - \sin P) + r (\sin P - \sin Q)$

$= 2R \sin P (\sin Q - \sin R) + 2R \sin Q (\sin R - \sin P) + 2R \sin R (\sin P - \sin Q)$

$= 2R (\sin P \sin Q - \sin P \sin R + \sin Q \sin R - \sin Q \sin P + \sin R \sin P - \sin R \sin Q)$

$= 2R \times 0$

$= 0.$

= R.H.S. **Proved.**

19. ***(a)*** ***State and prove cosine law of trigonometry.*** ***[3]***

(b) ***In a ΔABC, prove that $(b. \sin B - c \cos A. \sin B) = (a \sin A - c \sin A \cos B)$*** ***[2] [2081 Supp. Set A]***

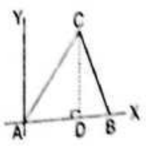
Soln: (a) Cosine law: In any ΔABC,

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ $\quad$ i.e. $a^2 = b^2 + c^2 - 2bc \cos A$

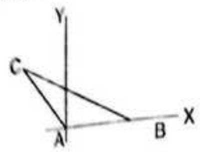
(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ $\quad$ i.e. $b^2 = c^2 + a^2 - 2ca \cos B$

(iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ $\quad$ i.e. $c^2 = a^2 + b^2 - 2ab \cos C$

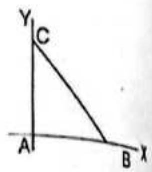
Proof: Let, ABC be any $\triangle$ place in the standard positions so that the vertex A is at the origin and AB lies along the positive x-axis. Then we have three possible cases.



(i)



(ii)



(iii)

Draw CD ⊥ AB in fig (i), then we have, $\cos A = \frac{AD}{AC} = \frac{AD}{b}$

$\Rightarrow$ $AD = b\cos A$

Again, $\sin A = \frac{CD}{AC} = \frac{CD}{b}$

$\Rightarrow$ $CD = b \sin A$

Co-ordinate of C is (AD, CD)

i.e. (b cos A, b sin A)

Then for all figures the co-ordinates of the vertices are A(0,0), B (c, 0) and C (b cos A, b sin A).

Now, using distance formula, we have

$BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

or, $a^2 = (b\cos A - c)^2 + (b \sin A - 0)^2$

or, $a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2 \sin^2 A$

or, $a^2 = b^2(\cos^2 A + \sin^2 A) - 2bc\cos A + c^2$

or, $a^2 = b^2 + c^2 - 2bc\cos A$

$\therefore$ $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Similarly placing the vertices B and C in the standard position and proceeding with similar calculations as above we can obtain the remaining relations.

(b) L.H.S. = b sin B – C cos A sin B

= sin B (b – c cos A)

= sin B (c cos A + a cos C – c cos A)

= sin B . a cos C

= sin B .2R sin A . cos C

= sin A (2R sin B. cos C)

= sin A (b cos C)

= sin A (a – c cos B)

= a sin A – c sin A . cos B **[$\because$ a = c cos B + b cos C]**

= **R.H.S. Proved.**

□□□

Chapter 7 Solution of Triangle

7.1 Solution of Triangle

Basic Formulae and Key Points

1. If three angles of a triangle are given, we use sine law to find the ratio of three sides.

   $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ i.e. a: b: c = sin A: sin B: sin C

2. If three sides or the ratio of three sides of a triangle are given, we use cosine law to find three angles.

   $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

3. If two sides and the included angle are given, then first we use cosine law and then sine law to find the remaining parts.
4. If two angles and one side of a triangle are given then we use sine law to find the remaining parts.
5. If two sides and opposite angle are given, then we use sine law to find the remaining parts.
6. If two sides and the included angle are given, then first we use cosine law and then sine law to find the remaining parts.

Notes:
(a) A + B + C = 180°.
(b) In a triangle the angle opposite to the smallest side is smallest.
(c) In a triangle the angle opposite to the greatest side is greatest.
(d) In a ambiguous case we get two solutions.

sine or cosine value of 15°, 75° and 105°

(a) $\sin 15° = \sin(45° - 30°)$

$= \sin 45° . \cos 30° - \cos 45° . \sin 30°$

$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

Similarly, we can find $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(b) $\sin 75° = \sin(45° + 30°)$

$= \sin 45° . \cos 30° + \cos 45° . \sin 30°$

$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

Similarly, we can find $\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(c) $\sin 105° = \sin(60° + 45°)$

$= \sin 60° . \cos 45° + \cos 60° . \sin 45°$

$= \frac{\sqrt{3}}{2} . \frac{1}{\sqrt{2}} + \frac{1}{2} . \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

Similarly, we can find $\cos 105° = \frac{1-\sqrt{3}}{2\sqrt{2}}$

Group 'A' (Multiple Choice Questions and Answers)

1. *In a triangle ABC, $a = 1$, $b = \sqrt{3}$ and $\angle C = 30°$. Which one of the following is that type of triangle?* [2081 Set W]

(a) isosceles and obtuse angled (b) equilateral
(c) right angled (d) isosceles triangle

2. *In ΔABC, $\angle A = 30°$, $\angle B = 45°$, which one of the following is a : c?* [2081 Set V]

(a) $\frac{\sqrt{2}}{\sqrt{3}+1}$ (b) $\frac{\sqrt{3}+1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\frac{2\sqrt{2}}{\sqrt{3}+1}$

3. *In a triangle ABC, if $\angle A = 60°$, $\angle B = 30°$ then a : b : c is*

(a) $1 : \sqrt{3} : 2$ (b) $\sqrt{3} : 1 : 2$
(c) $\sqrt{3} : 1 : \sqrt{2}$ (d) $3 : 1 : 2$

4. *In a triangle ABC, if $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$, then sin A =*

(a) $\sqrt{\frac{5}{3}}$ (b) $\sqrt{\frac{3}{5}}$ (c) $\sqrt{\frac{5}{3}}$ (d) $\sqrt{\frac{2}{5}}$

5. *In ΔABC, if a = 3 cm, b = 5 cm and c = 7 cm. Which one is the greatest angle of ΔABC?*

(a) 150° (b) 135° (c) 120° (d) 75°

6. *In ΔABC, $\angle A = 75°$, $\angle B = 60°$, which one of the following is b:c?*

(a) $2 : \sqrt{6}$ (b) $\sqrt{6} : 2$ (c) $\sqrt{3} : 1$ (d) $2 : \sqrt{3}$

7. *In ΔABC, if $a = 2$, $b = 1 + \sqrt{3}$ and $\angle C = 60°$, then c is equal to*

(a) $2\sqrt{3}$ (b) $3\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\sqrt{6}$

8. *In any ΔABC, if $a = 2$, $c = \sqrt{6}$ and $\angle C = 60°$ then $\angle A$ is equal to*

(a) 30° (b) 45° (c) 60° (d) 90°

9. *In ΔABC, If $a = \frac{1}{\sqrt{2}+1}$, $b = \frac{1}{\sqrt{2}-1}$ and $\angle C = 60°$, then c =*

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $\sqrt{6}$

10. *In a ΔABC, sinA: sinB: sinC:: 5:12:13, what type of triangle is ABC?* [Supp. 2081 Set-A]

(a) Acute angled (b) Obtuse angled
(c) Right angled (d) Right angled isosceles

Answer Key

1. a	2. a	3. b	4. d	5. c	6. b	7. d	8. b	9. d	10. c

Group 'B' or 'C' (Subjective Questions and Answers)

1. *If three sides of a triangle are proportional to $2 : \sqrt{6} : \sqrt{3}+1$, find the angles.* [2] [2081 Set W]

Soln: Let, in ΔABC, $a = 2k$, $b = \sqrt{6}\,k$ and $c = (\sqrt{3} + 1)k$

By cosine law,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(\sqrt{6}k)^2 + \{(\sqrt{3}+1)k\}^2 - (2k)^2}{2.\sqrt{6}.k.\,(\sqrt{3}+1)\,k}$$

$$= \frac{6k^2 + (3 + 2\sqrt{3} + 1)k^2 - 4k^2}{2\sqrt{6}\,(\sqrt{3}+1)k^2}$$

$$= \frac{(6 + 4 + 2\sqrt{3} - 4)k^2}{2\sqrt{6}\,(\sqrt{3}+1)k^2}$$

$$= \frac{6 + 2\sqrt{3}}{2\sqrt{6}\,(\sqrt{3}+1)} = \frac{2\sqrt{3}\,(\sqrt{3}+1)}{2\sqrt{6}\,(\sqrt{3}+1)} = \sqrt{\frac{3}{6}} = \frac{1}{\sqrt{2}} = \cos 45°.$$

$\therefore$ A = 45°

Again, $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

$$= \frac{\{(\sqrt{3}+1)k\}^2 + (2k)^2 - (\sqrt{6}\,k)^2}{2\,(\sqrt{3}+1)\,k.\,2K}$$

$$= \frac{(3 + 2\sqrt{3} + 1)\,k^2 + 4k^2 - 6k^2}{4\,(\sqrt{3}+1)\,k^2}$$

$$= \frac{(4 + 2\sqrt{3} + 4 - 6)k^2}{4\,(\sqrt{3}+1)k^2} = \frac{2 + 2\sqrt{3}}{4(\sqrt{3}+1)} = \frac{2\,(1+\sqrt{3})}{4\,(\sqrt{3}+1)} = \frac{1}{2} = \cos 60°$$

$\therefore$ B = 60°

Now, C = 180° – (A + B)

= 180° – (45° + 60°) = 75°

$\therefore$ A = 45°, B = 60°, C = 75°.

2. *If $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3}+1$, solve the triangle.* [2]

Soln: In ΔABC, $a = 2$, $b = \sqrt{2}$ and $c = \sqrt{3} + 1$

By cosine law, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{(\sqrt{2})^2 + (\sqrt{3}+1)^2 - 2^2}{2.\sqrt{2}.\,(\sqrt{3}+1)}$$

$$= \frac{2 + 3 + 2\sqrt{3} + 1 - 4}{2\sqrt{2}\,(\sqrt{3}+1)}$$

$$= \frac{2\sqrt{3}+2}{2\sqrt{2}\,(\sqrt{3}+1)} = \frac{2\,(\sqrt{3}+1)}{2\sqrt{2}\,(\sqrt{3}+1)} = \frac{1}{\sqrt{2}} = \cos 45°$$

$\therefore$ A = 45°

Again, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

$= \dfrac{(\sqrt{3}+1)^2 + 2^2 - (\sqrt{2})^2}{2(\sqrt{3}+1).2}$

$= \dfrac{3 + 2\sqrt{3} + 1 + 4 - 2}{4(\sqrt{3}+1)}$

$= \dfrac{6 + 2\sqrt{3}}{4(\sqrt{3}+1)} = \dfrac{2\sqrt{3}(\sqrt{3}+1)}{4(\sqrt{3}+1)} = \dfrac{\sqrt{3}}{2} = \cos 30°$

$\therefore$ $B = 30°$

Now, $C = 180° - (A + B)$

$= 180° - (45° + 30°) = 105°$

$\therefore$ $A = 45°, B = 30°$ and $C = 105°$

3. ***In any ΔABC, $a = 1$, $b = \sqrt{3}$ and $C = 30°$, solve the triangle.*** [2]

Soln: In ΔABC, $a = 1$, $b = \sqrt{3}$ and $C = 30°$

By cosine law,

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

or, $\cos 30° = \dfrac{1 + 3 - c^2}{2.1.\sqrt{3}}$

or, $\dfrac{\sqrt{3}}{2} = \dfrac{4 - c^2}{2\sqrt{3}}$

or, $3 = 4 - c^2$

or, $c^2 = 1$

$\therefore$ $c = 1$

Since, $a = c = 1$

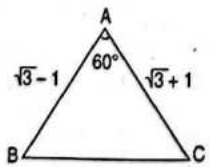
$\therefore$ $A = C = 30°$

Now, $B = 180° - (A + C) = 180° - (30° + 30°) = 120°$

$c = 1$, $A = 30°$ and $B = 120°$

4. ***Two sides of a triangle are $\sqrt{3} + 1$ and $\sqrt{3} - 1$ and the include angle is 60°, solve the triangle.*** [3]

Soln: Let, in ΔABC, $b = \sqrt{3} + 1$, $c = \sqrt{3} - 1$ and $\angle A = 60°$.



By cosine law,

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

or, $\cos 60° = \dfrac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 - a^2}{2(\sqrt{3}+1).(\sqrt{3}-1)}$

or, $\dfrac{1}{2} = \dfrac{3 + 2\sqrt{3} + 1 + 3 - 2\sqrt{3} + 1 - a^2}{2(3-1)}$

or, $\dfrac{1}{2} = \dfrac{8 - a^2}{2.2}$

or, $2 = 8 - a^2$

or, $a^2 = 6$

$\therefore$ $a = \sqrt{6}$

Again, by sine law, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

or, $\dfrac{\sqrt{6}}{\sin 60°} = \dfrac{\sqrt{3} - 1}{\sin C}$

or, $\dfrac{\sqrt{6}}{\frac{\sqrt{3}}{2}} = \dfrac{\sqrt{3} - 1}{\sin C}$

or, $\sin C = \dfrac{\sqrt{3} - 1}{\sqrt{6}} \times \dfrac{\sqrt{3}}{2}$

$= \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \sin 15°$ $\left(\because \sin 15° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}\right)$

$\therefore$ $C = 15°$

Now, $B = 180° - (A + B)$

$= 180° - (60° + 15°) = 105°$

$\therefore$ Required solution is $\sqrt{6}$, 15° and 105°.

i.e. $a = \sqrt{6}$, $C = 15°$ and $B = 105°$

5. ***If $C = 30°$, $B = 45°$ and $c = 6\sqrt{2}$, solve the triangle.*** [2]

Soln: In ΔABC, $C = 30°$, $B = 45°$ and $c = 6\sqrt{2}$

Since, $A + B + C = 180°$

$A = 180° - (B + C) = 180° - (45° + 30°) = 105°$

By using sine law, $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or, $\dfrac{b}{\sin 45°} = \dfrac{6\sqrt{2}}{\sin 30°}$

or, $\dfrac{b}{\frac{1}{\sqrt{2}}} = \dfrac{6\sqrt{2}}{\frac{1}{2}}$

or, $\sqrt{2}\,b = 12\sqrt{2}$

$\therefore$ $b = 12$

Again, by sine law,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

or, $\dfrac{a}{\sin 105°} = \dfrac{12}{\sin 45°}$

or, $a = \dfrac{12}{\sin 45°} \times \sin 105°$

or, $a = \dfrac{12}{\frac{1}{\sqrt{2}}} \times \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

$= 6(\sqrt{3} + 1)$

$\therefore$ $A = 105°$, $b = 12$ and $a = 6(\sqrt{3} + 1)$

6. ***If $A = 60°$, $B = 75°$ and $a = 2\sqrt{3}$, solve the triangle.***

Soln: In ΔABC, $A = 60°$, $B = 75°$ and $a = 2\sqrt{3}$

Since, $A + B + C = 180°$

$\therefore\ C = 180° - (A + B)$

$= 180° - (60° + 75°) = 45°$

By sine law, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2\sqrt{3}}{\sin 60°} = \frac{c}{\sin 45°}$

or, $\frac{2\sqrt{3}}{\sin 60°} \times \sin 45° = c$

or, $\frac{2\sqrt{3}}{\frac{\sqrt{3}}{2}} \times \frac{1}{\sqrt{2}} = c$

or, $c = 2\sqrt{3} \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$

Again, by sine law,

$\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{2\sqrt{3}}{\sin 60} = \frac{b}{\sin 75}$

or, $b = \frac{2\sqrt{3}}{\sin 60} \times \sin 75$

$= \frac{2\sqrt{3}}{\frac{\sqrt{3}}{2}} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$ $\quad [\because \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}]$

$= 2\sqrt{3} \times \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$

$= \sqrt{2}(\sqrt{3}+1) = \sqrt{6} + \sqrt{2}$

$\therefore\ C = 45°$, $b = \sqrt{6} + \sqrt{2}$ and $c = 2\sqrt{2}$

7. ***If $A = 15°$, $b = \sqrt{6}$ and $c = \sqrt{3} + 1$. Solve the triangle.***

Soln: In ΔABC, $b = \sqrt{6}$, $c = \sqrt{3} + 1$ and $A = 15°$

By cosine law, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

or, $\cos 15° = \frac{(\sqrt{6})^2 + (\sqrt{3}+1)^2 - a^2}{2\sqrt{6}(\sqrt{3}+1)}$

or, $\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{6 + 3 + 2\sqrt{3} + 1 - a^2}{2\sqrt{6}(\sqrt{3}+1)}$

or, $\sqrt{3} + 1 = \frac{10 + 2\sqrt{3} - a^2}{\sqrt{3}(\sqrt{3}+1)}$

or, $\sqrt{3}(\sqrt{3}+1)^2 = 10 + 2\sqrt{3} - a^2$

or, $\sqrt{3}(3 + 2\sqrt{3} + 1) = 10 + 2\sqrt{3} - a^2$

or, $\sqrt{3}(4 + 2\sqrt{3}) = 10 + 2\sqrt{3} - a^2$

or, $a^2 = 10 + 2\sqrt{3} - 4\sqrt{3} - 6$

$= 4 - 2\sqrt{3}$

$= \sqrt{3}^2 - 2\sqrt{3} + 1 = (\sqrt{3} - 1)^2$

$\therefore\ a = \sqrt{3} - 1$

Again, by sine law, $\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{\sqrt{3}-1}{\sin 15°} = \frac{\sqrt{6}}{\sin B}$

or, $\sin B = \frac{\sqrt{6}}{\sqrt{3}-1} \times \sin 15°$

$= \frac{\sqrt{6}}{\sqrt{3}-1} \times \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} = \sin 60°.$

$\therefore\ B = 60°$

Now, $C = 180° - (A + B) = 180° - (15° + 60°) = 105°.$

$\therefore\ a = \sqrt{3} - 1$, $B = 60°$ and $C = 105°$

8. ***In a triangle ABC, $a = 2$, $b = \sqrt{6}$ and $\angle A = 45°$ solve the triangle.*** *[2] [2081 Set - V]*

Soln: In ΔABC, $a = 2$, $b = \sqrt{6}$ and $\angle A = 45°$

By sine law,

$\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{2}{\sin 45°} = \frac{\sqrt{6}}{\sin B}$

or, $\sin B = \frac{\sqrt{6}}{2} \times \sin 45°$

$= \frac{\sqrt{6}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2}$

$\Rightarrow\ B = 60°$ or $120°$

Case (I) when $B = 60°$

$\therefore\ C = 180° - (A + B)$

$= 180° - (45° + 60°)$

$= 75°$

Again, by sine law, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2}{\sin 45°} = \frac{c}{\sin 75°}$

or, $c = \frac{2}{\sin 45°} \times \sin 75°$

$= \frac{2}{\frac{1}{\sqrt{2}}} \times \frac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{3}+1$

$\therefore$ $B = 60°$, $C = 75°$ and $c = \sqrt{3}+1$.

Case (II) when $B = 120°$

$\therefore$ $C = 180° - (A + B)$

$= 180° - (45° + 120°)$

$= 15°$

Again, by sine law $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2}{\sin 45°} = \frac{c}{\sin 15°}$

or, $\frac{2}{\frac{1}{\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

or, $2\sqrt{2} = c \times \frac{2\sqrt{2}}{\sqrt{3}-1}$

$\therefore$ $c = \sqrt{3}-1$

$\therefore$ $B = 120°$, $C = 15°$ and $c = \sqrt{3}-1$

9. ***If $b = 2$, $c = \sqrt{3}-1$ and $C = 15°$. Solve the triangle***

Soln: In ABC, $b = 2$, $c = \sqrt{3}-1$ and $C = 15°$

By using sine law, $\frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{2}{\sin B} = \frac{\sqrt{3}-1}{\sin 15°}$

or, $\sin B = \frac{2}{\sqrt{3}-1} \times \sin 15°$

$= \frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$

$\Rightarrow$ $B = 45°$ or $135°$

Case (I) when $B = 45°$

$\therefore$ $A = 180° - (B + C)$

$= 180° - (45° + 15°) = 120°$

Again, by sine law $\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{a}{\sin 120°} = \frac{2}{\sin 45°}$

or, $a = \frac{2}{\sin 45°} \times \sin 120°$

$= \frac{2}{\frac{1}{\sqrt{2}}} \times \frac{\sqrt{3}}{2} = \sqrt{6}$

$\therefore$ $A = 120°$, $B = 45°$ and $a = \sqrt{6}$.

Case (II) when $B = 135°$

$\therefore$ $A = 180° - (B + C)$

$= 180° - (135° + 15°) = 30°$

By sine law $\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{a}{\sin 30°} = \frac{2}{\sin 135°}$

or, $a = \frac{2}{\sin 135°} \times \sin 30°$

$= \frac{2}{\frac{1}{\sqrt{2}}} \times \frac{1}{2} = \sqrt{2}$.

$\therefore$ $A = 30°$, $B = 135°$ and $a = \sqrt{2}$.

10. ***If the angles of a triangle are to one another as 1 : 2 : 3. Prove that the corresponding sides are $1 : \sqrt{3} : 2$.*** [2]

Soln: Let, In ΔABC, $A = k$, $B = 2k$ and $C = 3k$.

Since, $A + B + C = 180°$

or, $k + 2k + 3k = 180°$

or, $6k = 180°$

$k = 30°$

$\therefore$ $A = 30°$, $B = 2 \times 30° = 60°$ and $C = 3 \times 30° = 90°$

Now by sine law

$a : b : c = \sin A : \sin B : \sin C$

$= \sin 30° : \sin 60° : \sin 90°$

$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$= 1 : \sqrt{3} : 2$ ($\because$ **Multiplying by 2**)

□□□

Again, by sine law, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2}{\sin 45^\circ} = \frac{c}{\sin 75^\circ}$

or, $c = \frac{2}{\sin 45^\circ} \times \sin 75^\circ$

$= \frac{2}{\frac{1}{\sqrt{2}}} \times \frac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{3}+1$

$\therefore$ $B = 60^\circ$, $C = 75^\circ$ and $c = \sqrt{3}+1$.

Case (III) when $B = 120^\circ$

$\therefore$ $C = 180^\circ - (A + B)$

$= 180^\circ - (45^\circ + 120^\circ)$

$= 15^\circ$

Again, by sine law $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2}{\sin 45^\circ} = \frac{c}{\sin 15^\circ}$

or, $\frac{2}{\frac{1}{\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

or, $2\sqrt{2} = c \times \frac{2\sqrt{2}}{\sqrt{3}-1}$

$\therefore$ $c = \sqrt{3}-1$

$\therefore$ $B = 120^\circ$, $C = 15^\circ$ and $c = \sqrt{3}-1$

9. ***If $b = 2$, $c = \sqrt{3}-1$ and $C = 15^\circ$. Solve the triangle***

Soln: In ABC, $b = 2$, $c = \sqrt{3}-1$ and $C = 15^\circ$

By using sine law, $\frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{2}{\sin B} = \frac{\sqrt{3}-1}{\sin 15^\circ}$

or, $\sin B = \frac{2}{\sqrt{3}-1} \times \sin 15^\circ$

$= \frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$

$\Rightarrow$ $B = 45^\circ$ or 135°

Case (I) when $B = 45^\circ$

$\therefore$ $A = 180^\circ - (B + C)$

$= 180^\circ - (45^\circ + 15^\circ) = 120^\circ$

Again, by sine law $\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{a}{\sin 120^\circ} = \frac{2}{\sin 45^\circ}$

or, $a = \frac{2}{\sin 45^\circ} \times \sin 120^\circ$

$= \frac{2}{\frac{1}{\sqrt{2}}} \times \frac{\sqrt{3}}{2} = \sqrt{6}$

$\therefore$ $A = 120^\circ$, $B = 45^\circ$ and $a = \sqrt{6}$.

Case (II) when $B = 135^\circ$

$\therefore$ $A = 180^\circ - (B + C)$

$= 180^\circ - (135^\circ + 15^\circ) = 30^\circ$

By sine law $\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{a}{\sin 30^\circ} = \frac{2}{\sin 135^\circ}$

or, $a = \frac{2}{\sin 135^\circ} \times \sin 30^\circ$

$= \frac{2}{\frac{1}{\sqrt{2}}} \times \frac{1}{2} = \sqrt{2}$.

$\therefore$ $A = 30^\circ$, $B = 135^\circ$ and $a = \sqrt{2}$.

10. ***If the angles of a triangle are to one another as 1 : 2 : 3. Prove that the corresponding sides are 1: $\sqrt{3}$: 2.*** [2]

Soln: Let, In ΔABC, $A = k$, $B = 2k$ and $C = 3k$.

Since, $A + B + C = 180^\circ$

or, $k + 2k + 3k = 180^\circ$

or, $6k = 180^\circ$

$k = 30^\circ$

$\therefore$ $A = 30^\circ$, $B = 2 \times 30^\circ = 60^\circ$ and $C = 3 \times 30^\circ = 90^\circ$

Now by sine law

$a : b : c = \sin A : \sin B : \sin C$

$= \sin 30^\circ : \sin 60^\circ : \sin 90^\circ$

$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$= 1 : \sqrt{3} : 2$ ($\because$ **Multiplying by 2**)

□□□

3. Analytic Geometry

Chapter 8 Conic Section

8.1 Circle

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Basic Formulae and Key Points

1. **Equation of Circle in Different Forms**
   (i) Centre at the origin: $x^2 + y^2 = r^2$
   (ii) Central form: $(x-h)^2 + (y-k)^2 = r^2$
   (iii) General equation of circle: $x^2 + y^2 + 2gx + 2fy + c = 0$.
   Where, Centre of the circle $= (-g, -f)$
   Radius of the circle $= \sqrt{g^2 + f^2 - c}$.
2. **Recap:**

   Equation of the circle touching the x-axis: $(x-h)^2 + (y-k)^2 = k^2$
   Equation of the circle touching the y-axis: $(x-h)^2 + (y-k)^2 = h^2$
   Equation of the circle touching both axes: $(x-h)^2 + (y-h)^2 = h^2$ or $(x-k)^2 + (y-k)^2 = k^2$
   Equation of the circle in diameter form: $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

   Tangents and Normals of Circle:
   (i) Equation of tangent to the circle $x^2 + y^2 = a^2$ at a point (x_1, y_1) is $xx_1 + yy_1 = a^2$.
   (ii) Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at point (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
   (iii) Equation of normal to the circle $x^2 + y^2 = a^2$ at a point (x_1, y_1) is $xy_1 = yx_1$.
   (iv) Equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at point (x_1, y_1) is $xy_1 + yx_1 + g(y - y_1) - f(x - x_1) = 0$.
   (v) The condition that a straight line $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$ is $c = \pm a\sqrt{1+m^2}$.
3. **Length of Tangent From an External Point:**
   (i) The length of tangent to the circle $x^2 + y^2 = a^2$ from an external point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 - a^2}$.
   (ii) The length of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from an external point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Group 'A' (Multiple Choice Questions and Answers)

1. *The condition that the line $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$ if*
   (a) $c = \pm\sqrt{1 + a^2 m^2}$ (b) $c = \pm\sqrt{1+m^2}$
   (c) $c = \pm a\sqrt{1+m^2}$ (d) $c = \pm\sqrt{a^2 + m^2}$

2. *If a be the radius of a circle which touches x-axis, then its equation is*
   (a) $x^2 + y^2 - 2hx - 2ay + h^2 = 0$ (b) $x^2 + y^2 - 2ax - 2ky + k^2 = 0$
   (c) $x^2 + y^2 = a^2$ (d) $x^2 - 2ay + y^2 = a^2$
3. *If the circle touches both axes, then the equation of the circle is*
   (a) $(x-a)^2 + y^2 = a^2$ (b) $x^2 + (y-a)^2 = a^2$
   (c) $(x-a)^2 + (y-a)^2 = r^2$ (d) $(x-a)^2 + (y-a)^2 = a^2$
4. *The equation of tangent to the circle $x^2 + y^2 = a^2$ at a point (x_1, y_1) is*
   (a) $xx_1 + yy_1 = 0$ (b) $xx_1 + yy_1 + a^2 = 0$
   (c) $xx_1 + yy_1 = a^2$ (d) $xy_1 = x_1y$
5. *The equation of the tangent to the circle $x^2 + y^2 = 5$ at the point (1, 2) is*
   (a) $x + 2y = 0$ (b) $x - 2y = 5$
   (c) $x + 2y = 5$ (d) $2x + y = 5$
6. *The equation of normal to the circle $x^2 + y^2 = 25$ at (4,3) is*
   (a) $3x - 4y = 0$ (b) $4x - 3y = 0$
   (c) $3x + 4y = 0$ (d) $4x + 3y = 0$
7. *The length of the tangent to the circle $x^2 + y^2 - 4x + 6y - 1 = 0$ from (3,4) is*
   (a) 5 (b) 6 (c) 7 (d) 8
8. *Find the value of k so, that the length of the tangent from (5,4) to the circle $x^2 + y^2 + 2ky = 0$ is 1.*
   (a) 3 (b) 4 (c) −3 (d) −5
9. *Find the value of k if the line $y = 2x + k$ is tangent to the circle $x^2 + y^2 = 5$.*
   (a) $\pm\sqrt{5}$ (b) ± 5 (c) ± 2 (d) ± 4
10. *Find the condition that the line $\ell x + my + n = 0$ should be a normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$*
    (a) $n = g\ell - fm$ (b) $n = g\ell + fm$ (c) $n = fm - g\ell$ (d) None

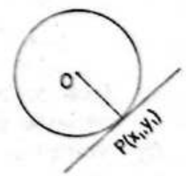
Answer Key

1. c	2. a	3. d	4. c	5. c	6. a	7. b	8. d	9. b	10. b

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Find the equation of tangent to the circle $x^2 + y^2 = a^2$ at a point (x_1, y_1) on the circle.* [3]
   (b) *Find the equation of the tangent to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at (2, 3).* [2]

Soln: (a) Let, $P(x_1, y_1)$ be any point on the circle $x^2 + y^2 = a^2$ where centre = (0, 0) and radius = a. Then, $x_1^2 + y_1^2 = a^2$ ......(i)



Slope of radius OP $= \dfrac{y_1 - 0}{x_1 - 0} = \dfrac{y_1}{x_1}$

$\therefore$ Slope of tangent at P $= -\dfrac{x_1}{y_1}$

[∵ Tangent at P is perpendicular to the radius.]

Now, the equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

or, $yy_1 - y_1^2 = -x_1x + x_1^2$

or, $xx_1 + yy_1 = x_1^2 + y_1^2$

$\therefore$ $xx_1 + yy_1 = a^2$ **[∵ $x_1^2 + y_1^2 = a^2$]**

Which is the required equation of tangent to the circle $x^2 + y^2 = a^2$.

(b) Here, given equation of circle is

$$x^2 + y^2 - 2x - 4y + 3 = 0 \text{ ......(i)}$$

Comparing (i) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$2g = -2,\ 2f = -4,\ c = 3$$

$\Rightarrow$ $g = -1$ and $f = -2$

Now, the equation of tangent to the circle (i) at (2, 3) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

or, $x.2 + y.3 + (-1)(x + 2) + (-2)(y + 3) + 3 = 0$

or, $2x + 3y - x - 2 - 2y - 6 + 3 = 0$

$\therefore$ $x + y = 5$ **Ans.**

2. (a) ***Find the equation of the normal to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1).***

(b) ***Find the condition that the line $\ell x + my + n = 0$ should be a normal to the ci[rcle] $x^2 + y^2 + 2gy + 2fy + c = 0$.***

Soln: (a) Here,

The equation of circle is $x^2 + y^2 = a^2$

Where centre = (0, 0) and radius = a.

The equation of tangent at point P (x_1, y_1) is

$$xx_1 + yy_1 = a^2$$

$\therefore$ Slope of tangent $= -\dfrac{x_1}{y_1}$

So, slope of normal $= \dfrac{y_1}{x_1}$ $\left[\because \text{slope of normal} = -\dfrac{1}{\text{slope of tangent}}\right]$

Hence, the equation of normal is

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

or, $x_1y - x_1y_1 = xy_1 - x_1y_1$

$\therefore$ $x_1y = y_1x$ which is the required equation of normal to the circle $x^2 + y^2 = a^2$ at a point (x_1, y_1).

(b) Here, given equation of line is $\ell x + my + n = 0$ .....(i)

Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ......(ii)

Where, centre = $(-g, -f)$ and radius = $\sqrt{g^2 + f^2 - c^2}$

The line (i) is normal to the circle (ii), if the line (i) passes through the centre $(-g, -f)$ of the circ[le]

So, we have

$$-\ell g - mf + n = 0$$

$\therefore$ $n = \ell g + mf$ which is the required condition. **Ans.**

3. (a) ***Show that the tangents to the circle $x^2 + y^2 = 100$ at the points (6, 8) and (8, -6) [are] perpendicular to each other.***

(b) ***What is the condition for the line $y = mx + c$ to be tangent to the circle $x^2 + y^2 = a^2$.***

(c) ***What is the formula to find the equation of the normal to the circle $x^2 + y^2 = a^2$?***

Soln: (a) The equation of the circle is $x^2 + y^2 = 100$ .... (i)

Where, centre is (0, 0) and radius is 10.

The equation of tangent to the circle at point (6, 8) is

$x.6 + y.8 = 100$ $[\because xx_1 + yy_1 = a^2]$

or, $6x + 8y = 100$

$\therefore$ $3x + 4y = 50$ ....(ii)

Similarly, the equation of tangent to the circle at (8, -6) is

$$xx_1 + yy_1 = a^2$$

or, $8x - 6y = 100$

$\therefore$ $4x - 3y = 50$ ....(iii)

The slope of tangent (ii), $m_1 = -\dfrac{x\text{-coeff.}}{y\text{-coeff.}} = \dfrac{-3}{4}$

The slope of tangent (iii), $m_2 = -\dfrac{x\text{-coeff.}}{y\text{-coeff.}} = \dfrac{-4}{-3} = \dfrac{4}{3}$

Now, $m_1 \times m_2 = \dfrac{-3}{4} \times \dfrac{4}{3} = -1$

Hence, the tangents (ii) and (iii) are perpendicular to each other. **Proved**

(b) The condition for the line $y = mx + c$ to be tangent to the circle $x^2 + y^2 = a^2$ is

$$c = \pm a\sqrt{1 + m^2}$$

(c) The formula to find the equation of the normal to the circle $x^2 + y^2 = a^2$ is $xy_1 = yx_1$.

4. (a) ***Find the condition that the line $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$.*** [3]

(b) ***Find the equation to tangent.*** [1]

Soln: (a) The equation of circle is $x^2 + y^2 = a^2$ ......(i) where centre = (0, 0) and radius = a.

Also, equation of line is $y = mx + c$ ....(ii)

Now, solving equations (i) and (ii), we get

$$x^2 + (mx + c)^2 = a^2$$

or, $x^2 + m^2x^2 + 2\,mxc + c^2 = a^2$

or, $(1 + m^2)x^2 + 2\,mxc + (c^2 - a^2) = 0$ ....(iii)

The equation (iii) is quadratic in x.

The line (ii) will be tangent to the circle (i) if the discriminant of (iii) is zero i.e. $b^2 - 4ac = 0$. So

$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

or, $4m^2c^2 - 4[c^2 - a^2 + m^2c^2 - a^2m^2] = 0$

or, $m^2c^2 - c^2 + a^2 - m^2c^2 + a^2m^2 = 0$

or, $a^2 + a^2m^2 = c^2$

or, $c^2 = a^2(1 + m^2)$

$\therefore$ $c = \pm a\sqrt{1 + m^2}$, which is the required condition. **Ans.**

(b) The equation of the tangent is

$$y = mx \pm a\sqrt{1 + m^2} \textbf{ Ans.}$$

5. (a) ***Define tangent and normal to the circle.*** [2]

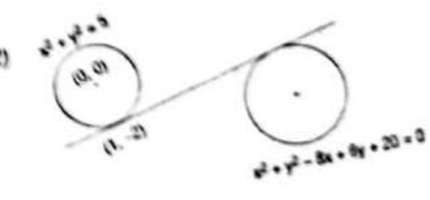
(b) ***Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point (1, -2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find the point of contact.*** [3]

Soln: (a) **Tangent to a circle:** A straight line is said to be a tangent to a circle if it touches the circumference of the circle at a single point only.

Normal to a circle: A straight line which is perpendicular to the tangent to the circle at a point is called normal to a circle at a point.

(b) Given equation of the circle is $x^2 + y^2 = 5$ ......(i)

The equation of the tangent to the circle (i) at (1, -2) is



$x.1 + y.(-2) = 5$

or, $x - 2y = 5$

i.e. $x = 2y + 5$ ....(iv)

The equation of another circle is

$x^2 + y^2 - 8x + 6y + 20 = 0$ ....(ii)

Now, solving equation (•) and (ii) we get

$(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$

or, $5y^2 + 10y + 5 = 0$

or, $5(y^2 + 2y + 1) = 0$

or, $y^2 + 2y + 1 = 0$

or, $(y + 1)^2 = 0$

$\therefore$ $y = -1, -1$

When, $y = -1$, $x = 2(-1) + 5 = 3$

As the line $x - 2y = 5$ touches the circle (i) at $(1, -2)$ meets the another circle (ii) at coincident point $(3, -1)$. Hence, the tangent to the circle (i) also touches the circle (ii) and the point of contact is $(3, -1)$. **Ans.**

6. (a) ***Find the equations of the tangents and normal to the circle $x^2 + y^2 - 3x + 10y - 15 = 0$ at the point $(4, -11)$.*** [3]

(b) ***Prove that the line $5x + 12y + 78 = 0$ is tangent to the circle $x^2 + y^2 = 36$.*** [2]

Soln: (a) Given equation of circle is $x^2 + y^2 - 3x + 10y - 15 = 0$ ......(i)

Comparing eq^n. (i) with the equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -3,\ 2f = 10\ \&\ c = -15$

$\Rightarrow\ g = \frac{-3}{2},\ f = 5\ \&\ c = -15$

The equation of tangent at point $(4, -11)$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

or, $x.4 + y(-11) + \left(\frac{-3}{2}\right)(x + 4) + 5(y - 11) - 15 = 0$

or, $4x - 11y - \frac{3x}{2} - 6 + 5y - 55 - 15 = 0$

or, $\frac{8x - 22y - 3x - 12 + 10y - 110 - 30}{2} = 0$

or, $5x - 12y - 152 = 0$

$\therefore$ $5x - 12y - 152 = 0$ which is the required equation of tangent to the circle (i)

Again, let the equation of the normal to the line $5x - 12y - 152 = 0$ be

$12x + 5y + k = 0$ ......(i)

Since, (i) passes through $(4, -11)$, so

$12 \times 4 + 5.(-11) + k = 0$

or, $48 - 55 + k = 0$

$\therefore$ $k = 7$

$\therefore$ The equation of normal is $12x + 5y + 7 = 0$. **Ans.**

(b) Given equation of line is $5x + 12y + 78 = 0$ ......(i) and

Given equation of circle is $x^2 + y^2 = 36$ ......(ii)

Where, centre = (0, 0) and radius = 6.

Now, the line (i) will be a tangent to a circle (ii), if the length of perpendicular from the centre of the circle to the line is equal to the radius of the circle.

$= \pm\frac{5.0 + 12.0 + 78}{\sqrt{5^2 + 12^2}} = \pm\frac{78}{13} = 6$ which is the radius of the given circle.

Hence, the length of perpendicular from origin to the line $5x + 12y + 78 = 0$ is equal to radius of circle (ii).

$\therefore$ Line (i) is tangent to the circle (ii). **Proved.**

7. (a) ***Find the condition that the line $px + qy = r$ is tangent to the circle $x^2 + y^2 = a^2$.*** [2]

(b) ***Find the value of k so that the line $4x + 3y + k = 0$ may touch the circle $x^2 + y^2 - 4x + 10y + 4 = 0$.*** [3]

Soln: (a) Given circle is $x^2 + y^2 = a^2$ ....(i)

Where, centre = (0, 0) and radius = a.

Also, given line is $px + qy = r$ ....(ii)

The line (i) will be tangent to the circle (ii) if the length of perpendicular from (0, 0) to line (i) is equal to the radius of the circle. so,

$\pm\frac{p.0 + q.0 - r}{\sqrt{p^2 + q^2}} = a$

or, $\frac{r^2}{p^2 + q^2} = a^2$

$\therefore$ $a^2(p^2 + q^2) = r^2$, which is the required condition.

(b) Here, given circle is $x^2 + y^2 - 4x + 10y + 4 = 0$ ....(i)

Comparing (i) with $x^2 + y^2 + 2gx + 2fy + c = 0$.

Where, $2g = -4$, $2f = 10\ \&\ c = 4$

$\therefore$ $g = -2,\ f = 5\ \&\ c = 4$

$\therefore$ Centre $(-g, -f) = (2, -5)$

Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + 5^2 - 4} = 5$

Also given that the equation of tangent is $4x + 3y + k = 0$ ......(ii)

Now, length of perpendicular from $(2, -5)$ to the line (ii) is equal to radius of the circle (i). So,

$\pm\frac{4.2 + 3.(-5) + k}{\sqrt{4^2 + 3^2}} = 5$

or, $\pm\frac{k - 7}{5} = 5$

or, $\pm(k - 7) = 25$

Taking (+ve) sign: $k - 7 = 25\ \therefore k = 32$

Taking (–ve) sign: $-(k - 7) = 25$

or, $k - 7 = -25$

$\therefore$ $k = -18$

Hence, $k = -18$ or 32 **Ans.**

8. ***Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.*** [3]

Soln: The equations of the circles are

$x^2 + y^2 + 2ax + c^2 = 0$ ......(i)

$x^2 + y^2 + 2by + c^2 = 0$ ......(ii)

Let, C_1, C_2 be the centres and R_1, R_2 be the radii of the circles (i) and (ii) respectively. Then

From (i), centre $(C_1) = (-g, -f) = (-a, 0)$

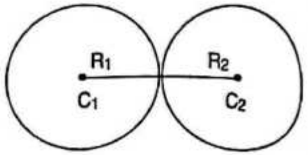
Radius $(R_1) = \sqrt{g^2 + f^2 - c} = \sqrt{a^2 + 0 - c^2}$

$= \sqrt{a^2 - c^2}$

Also, from (ii), centre $(C_2) = (-g, -f) = (0, -b)$

Radius $(R_2) = \sqrt{g^2 + f^2 - c} = \sqrt{0^2 + (b)^2 - c^2}$

$= \sqrt{b^2 - c^2}$



Now, two circles touch each other if the distance between their centres is equal to sum or difference of their radii.

i.e. if $C_1C_2 = R_1 \pm R_2$

or, $\sqrt{(-a-0)^2 + (0+b)^2} = \sqrt{a^2-c^2} \pm \sqrt{b^2-c^2}$

or, $\sqrt{a^2+b^2} = \sqrt{a^2-c^2} \pm \sqrt{b^2-c^2}$

or, $a^2+b^2 = a^2-c^2 \pm 2\sqrt{a^2-c^2}\,.\sqrt{b^2-c^2} + b^2-c^2$

or, $0 = -2c^2 \pm 2\sqrt{a^2-c^2}\,.\sqrt{b^2-c^2}$

or, $c^2 = \pm\sqrt{a^2-c^2}\sqrt{b^2-c^2}$

Squaring both sides, we get

$c^4 = (a^2-c^2)(b^2-c^2)$

or, $c^4 = a^2b^2 - a^2c^2 - b^2c^2 + c^4$

or, $a^2c^2 + b^2c^2 = a^2b^2$

Dividing both sides by $a^2b^2c^2$, we get

$$\frac{a^2c^2}{a^2b^2c^2} + \frac{b^2c^2}{a^2b^2c^2} = \frac{a^2b^2}{a^2b^2c^2}$$

$\therefore\ \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ is the conditions for given circles.

9. ***Find the equation of tangent and normal to the circle $x^2 + y^2 = 13$ at the point (2, 3).*** *[3] [2081 Set -*

Soln: Here, given circle is

$x^2 + y^2 = 13$ ....(i)

The equation of tangent to the circle (i) at (2, 3) is

$xx_1 + yy_1 = a^2$

or, $x.2 + y.3 = 13$

$\therefore$ $2x + 3y = 13$ **Ans.**

The slope of the tangent (m) $= \frac{-\text{x - coeff.}}{\text{y - coeff.}} = \frac{-2}{3}$

Slope of normal $(m_1) = \frac{-1}{\text{slope of tangent}} = \frac{-1}{\frac{-2}{3}} = \frac{3}{2}$

Now, the equation of the normal at (2, 3) is

$y - 3 = \frac{3}{2}(x-2)$

or, $2y - 6 = 3x - 6$

$\therefore$ $3x - 2y = 0$ **Ans.**

10. ***Find the equation of tangents to the circle $x^2 + y^2 = 25$ drawn through the point (13, 0).*** *[3] [2081 Set -*

Soln: The equation of the line through a point (13, 0) is

$y - 0 = m(x - 13)$

or, $y - mx + 13m = 0$ ....(i)

The equation of circle is $x^2 + y^2 = 25$.........(ii)

Where, centre = (0, 0) and radius = 5

Now, the line (i) will be tangent to the circle (ii) if

Radius = length of the perpendicular from the centre to the line (i). So,

$$5 = \pm\frac{0 - 0 + 13m}{\sqrt{(-m)^2 + 1^2}}$$

$$5 = \frac{13m}{\sqrt{1+m^2}}$$

Squaring both sides, we get

$25 + 25m^2 = 169m^2$

or, $144m^2 - 25 = 0$

or, $(12m - 5)(12m + 5) = 0$

$\therefore$ $m = \frac{5}{12}, \frac{-5}{12}$

When, $m = \frac{5}{12}$, the eq^n. of tangent at (13, 0) is

$y - 0 = \frac{5}{12}(x - 13)$

or, $12y = 5x - 65$

$\therefore$ $5x - 12y - 65 = 0$

When, $m = \frac{-5}{12}$, the equation of tangent at (13, 0) is

$y - 0 = \frac{-5}{12}(x - 13)$

$12y = -5x + 65$

$\therefore$ $5x + 12y - 65 = 0$ **Ans.**

11. ***The condition for a straight line $y = mx + c$ to be a tangent to a circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$. Justify it with an example.*** *[2] [2081 'Optional']*

Soln: Let, $x^2 + y^2 = 25$ ..... (i)

Which is in the form of $x^2 + y^2 = a^2$. Let's take a point (3, 4) which satisfies the circle (i). So, the equation of tangent at point (3, 4) is

$xx_1 + yy_1 = a^2$

or, $x \times 3 + y \times 4 = 25$

or, $3x + 4y = 25$

or, $4y = -3x + 25$

$\therefore$ $y = \frac{-3}{4}x + \frac{25}{4}$ which is in the form of $y = mx + c$.

Where, $m = \frac{-3}{4}$ and $c = \frac{25}{4}$

Also, we have $a^2 = 25$

Now, for the condition for tangency

$c^2 = a^2(1 + m^2)$

or, $\left(\frac{25}{4}\right)^2 = 25\left(1 + \left(\frac{-3}{4}\right)^2\right)$

or, $\frac{625}{16} = 25\left(1 + \frac{9}{16}\right)$

or, $\frac{625}{16} = 25 \times \frac{25}{16}$

$\therefore$ $\frac{625}{16} = \frac{625}{16}$ (Which is true)

Hence, the line $3x + 4y = 25$ is tangent to the circle (i). **Ans.**

8.2 Parabola

Basic Formulae and Key Points

1. Some of the Important Terms:
   (i) **Focal Distance:** The distance of any point on the parabola from the focus is called the focal distance or focal radius of the point.
   (ii) **Focal Length:** The distance of the focus from the vertex is called the focal length of a parabola.
   (iii) **Focal Chord:** Any chord of the parabola passing through the focus is called the focal chord of a parabola.
   (iv) **Latus Rectum:** The focal chord which is perpendicular to the axis of the parabola is called the latus rectum or focal width of the parabola.
   (v) **Eccentricity:** The eccentricity of the parabola is the ratio of the distance of a point on the parabola from the focus to its perpendicular distance from the directrix. It is denoted by e.

Note: If e = 1, then the conic is a parabola.

2. Formula Table

Equation of parabola	Vertex	Focus	Equation of Directrix	Length of Latus Rectum	Equation of Axis
$y^2 = 4ax$	(0, 0)	(a, 0)	$x + a = 0$	4a	$y = 0$
$x^2 = 4ay$	(0, 0)	(0, a)	$y + a = 0$	4a	$x = 0$
$(y - k)^2 = 4a(x - h)$	(h, k)	(h + a, k)	$x + a = h$	4a	$y = k$
$(x - h)^2 = 4a(y - K)$	(h, k)	(h, k + a)	$y + a = k$	4a	$x = h$

3. Tangent and Normal:
   (i) Equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$.
   (ii) Equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.
   (iii) Equation of tangent to the parabola in slope (m) form is $y = mx + \frac{a}{m}$.
   (iv) Equation of normal to the parabola in slope (m) form is $y = mx - 2am - am^3$.
   (v) The line $y = mx + c$ will be the tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

Group 'A' (Multiple Choice Questions and Answers)

1. *The focus of the parabola $y^2 = 16x$ is*
   (a) (– 4, 0) (b) (16, 0) (c) (8, 0) (d) (4, 0)
2. *The length of the latus rectum of the parabola $y^2 = 12x$ is*
   (a) 12 (b) 6 (c) 24 (d) 18
3. *The vertex of the parabola $y^2 = 8x$ is*
   (a) (2, 0) (b) (0, 2) (c) (0, –2) (d) (0, 0)
4. *The equation of the directrix of the parabola $y^2 = 16x$ is*
   (a) $x = 4$ (b) $x = -4$ (c) $y = 4$ (d) $y = -4$
5. *The equation of the tangent to the parabola $y^2 = 8x$ at (2, –4) is*
   (a) $x + y - 2 = 0$ (b) $x - y + 2 = 0$ (c) $x + y + 2 = 0$ (d) $x - y - 2 = 0$
6. *The equation of the tangent to the parabola in the slope form is*
   (a) $y = mx$ (b) $y = mx + c$ (c) $y = mx + \frac{a}{m}$ (d) $y = mx - \frac{a}{m}$
7. *The chord of a conic that passes through the focus and perpendicular to the axis is*
   (a) Latus rectum (b) Directrix
   (c) Line of axis (d) Y-axis
8. *The parametric equation of $y^2 = 6x$ are*
   (a) $x = 3t^2, y = 4t$ (b) $x = 2t^2, y = \frac{3}{2}t$
   (c) $x = \frac{3}{2}t^2, y = 3t$ (d) $x = t, y = t$
9. *The vertex of the parabola $y^2 - 2x + 6y + 3 = 0$ is*
   (a) (– 3, 3) (b) (3, 3) (c) (– 3, 2) (d) (– 3, – 3)
10. *The equation of the parabola with focus at (3, 0) and directrix $x = -3$ is*
   (a) $y^2 = 6x$ (b) $y^2 = 12x$ (c) $y^2 = 8x$ (d) $y^2 = -6x$

Answer Key

1. d	2. a	3. d	4. b	5. c	6. c	7. a	8. c	9. d	10. b

Group 'B' or 'C' (Subjective Questions and Answers)

1. *Does a conic, $y^2 = 12x$ have two tangents from the point (6, 9)? Justify it with calculation.* [3] [2081 Set E/W]

Soln: Here, given equation of the parabola is

$y^2 = 12x$

Comparing this equation with $y^2 = 4ax$, we get

$4a = 12$

$\therefore\ a = 3$

The equation of tangent to the parabola $y^2 = 4ax$ with slope m is $y = mx + \frac{a}{m}$ ---- (i)

Since the tangent passes through the point (6, 9)

Then,

$9 = 6m + \frac{3}{m}$

or, $9 = \frac{6m^2 + 3}{m}$

or, $6m^2 + 3 = 9m$

or, $6m^2 - 9m + 3 = 0$

or, $2m^2 - 3m + 1 = 0$

or, $2m^2 - 2m - m + 1 = 0$

or, $2m(m - 1) - 1(m - 1) = 0$

or, $(m - 1)(2m - 1) = 0$

$\therefore\ m = 1$ or $m = \frac{1}{2}$

When, m = 1, from (i)

$y = x + \frac{3}{1}$

or, $y = x + 3$

$\therefore$ $x - y + 3 = 0$

When, $m = \frac{1}{2}$, from (i)

$y = \frac{1}{2}x + \frac{3}{\frac{1}{2}}$

or, $y = \frac{1}{2}x + 6$

or, $y = \frac{x + 12}{2}$

or, $x + 12 = 2y$

$\therefore$ $x - 2y + 12 = 0$

$\therefore$ The required equations of tangents are $x - y + 3 = 0$ and $x - 2y + 12 = 0$.

Hence, the given conic has two tangents from the point (6, 9). **Ans.**

2. (a) ***Define a parabola in conic section.***
(b) ***When does a conic section become a parabola in terms of its eccentricity?***
(c) ***Write the standard equation of a parabola.***
(d) ***Find the focus and directrix of the parabola $y^2 - 4y - 8x - 20 = 0$.***

Soln: (a) **Parabola:** A parabola is the locus of a point which moves in a plane, such that its distance from fixed point is equal to its distance from a fixed straight line.

(b) In terms of eccentricity (e), a conic section becomes a parabola when the eccentricity (e) is equal to 1 that is e = 1. **Ans.**

(c) The standard equation of a parabola is given by $y^2 = 4ax$.

(d) Here,

Given parabola is

$y^2 - 4y - 8x - 20 = 0$

or, $y^2 - 4y + 4 - 4 - 8x - 20 = 0$

or, $(y - 2)^2 - 8x - 24 = 0$

or, $(y - 2)^2 = 8x + 24$

or, $(y - 2)^2 = 8(x + 3)$

Comparing this equation with

$(y - k)^2 = 4a(x - h)$

We have, $h = -3$, $k = 2$ and $4a = 8$

$\Rightarrow a = 2$

Now, focus = $(h + a, k) = (-3 + 2, 2) = (-1, 2)$

Also, equation of directrix is

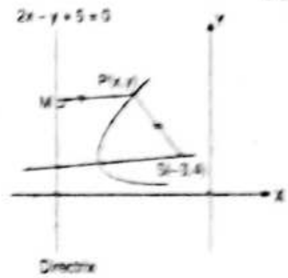
$x = h - a$

or, $x = -3 - 2$

$\therefore$ $x + 5 = 0$ **Ans.**

3. (a) ***Find the equation of the parabola whose focus is at the point (-3, 4) and the directrix is $2x + 5 = y$.*** [3] [2081 NEB Set V]
(b) ***Write the focus of the parabola having the equation $x^2 = 4ay$.*** [2]

Soln: (a) Here, given



Focus (S) = (-3, 4)

Equation of the directrix is $2x + 5 = y$ i.e. $2x - y + 5 = 0$

Let, P(x, y) be any point on the parabola such that P(x, y) is equidistant from focus (- 3, 4) and directrix $2x - y + 5 = 0$

Then by definition,

$PS = PM$

or, $PS^2 = PM^2$

or, $(x + 3)^2 + (y - 4)^2 = \left\{\frac{2x - y + 5}{\sqrt{2^2 + (-1)^2}}\right\}^2$

or, $x^2 + 6x + 9 + y^2 - 8y + 16 = \frac{(2x - y + 5)^2}{5}$

or, $5\{x^2 + 6x + 9 + y^2 - 8y + 16\} = (2x - y + 5)^2$

or, $5x^2 + 30x + 125 + 5y^2 - 40y = 4x^2 + y^2 + 25 - 4xy - 10y + 20x$

$\therefore$ $x^2 + 4y^2 + 4xy + 10x - 30y + 100 = 0$ Which is the required equation of the parabola. **Ans.**

(b) The focus of the parabola having the equation $x^2 = 4ay$ is (0, a). **Ans.**

4. (a) ***What is the latus rectum of a parabola?*** [1]
(b) ***Write the coordinates of the ends of the latus rectum and the length of the latus rectum.*** [2]
(c) ***Find the equation of the tangent to the parabola $y^2 = 16x$ at the point (4, 8).*** [2]

Soln: (a) **Latus rectum:** Latus rectum is the chord of the parabola passing through the focus and perpendicular to the axis.

(b) The coordinates of the ends of the latus rectum are (a, 2a) and (a, −2a) respectively and the length of the latus rectum = 4a.

(c) Here, given equation of parabola is $y^2 = 16x$.

Comparing it with $y^2 = 4ax$, then

$4a = 16$

$\Rightarrow a = 4$

Now, the equation of tangent to the parabola at (4, 8) is

$yy_1 = 2a(x + x_1)$

or, $y.8 = 2 \times 4(x + 4)$

$\therefore$ $y = x + 4$ **Ans.**

5. ***Given the equation of the parabola, $y^2 = 6y - 12x + 45$. Find***
(a) ***The coordinates of the focus.*** [1]
(b) ***The coordinates of the vertex.*** [1]
(c) ***The equation of the directrix.*** [1]
(d) ***The length of the latus rectum.*** [1]
(e) ***The equation of the axis.*** [1]

Soln: Here, the equation of the parabola is

$y^2 = 6y - 12x + 45$

or, $y^2 - 6y + 9 = -12x + 45 + 9$

or, $(y - 3)^2 = -12\left(x - \frac{9}{2}\right)$ ....(i)

Comparing (i) with the equation of parabola

$(y-k)^2 = 4a(x-h)$ we get

$h = \frac{9}{2}$, $k = 3$, $4a = -12$

$\therefore$ $a = -3$

Then,

(a) Focus $(h+a, k) = \left(\frac{9}{2} + (-3), 3\right) = \left(\frac{3}{2}, 3\right)$

(b) Vertex $(h, k) = \left(\frac{9}{2}, 3\right)$

(c) Equation of the directrix is: $x = h - a$

$\Rightarrow x = \frac{9}{2} + 3$

$\Rightarrow x = \frac{15}{2}$

$\therefore$ $2x - 15 = 0$ **Ans.**

(d) Length of latus rectum $= |4a|$

$= |4 \times (-3)|$

$= 12$

(e) The equation of the axis is $y = k$

i.e. $y = 3$ **Ans.**

6. (a) ***Prove that the line $\ell x + my + n = 0$ touches the parabola $y^2 = 4ax$ if $\ell n = am^2$.***

(b) ***Find the equation of the tangent to the parabola $y^2 = 9x$ at $(4, -6)$.***

Soln: (a) Here,

Equation of parabola,

$y^2 = 4ax$ .... (i)

Also, given line is

$\ell x + my + n = 0$

or, $my = -\ell x - n$

$\therefore$ $y = \frac{-\ell}{m}x - \frac{n}{m}$

Comparing this with $y = mx + c$, where slope $(m) = -\frac{\ell}{m}$ and $c = -\frac{n}{m}$

We know that the line $y = mx + c$ will be tangent to the parabola (i) if $c = \frac{a}{m}$

So, $\frac{-n}{m} = \frac{a}{\frac{-\ell}{m}}$

or, $\frac{\ell n}{m^2} = a$

$\therefore$ $\ell n = am^2$ **Hence Proved.**

(b) Here, given parabola is

$y^2 = 9x$

Comparing it with $y^2 = 4ax$, we get

$4a = 9$

$\therefore$ $a = \frac{9}{4}$

Now, the equation of tangent to the given parabola at $(4, -6)$ is

$yy_1 = 2a(x + x_1)$

or, $y.(-6) = 2.\frac{9}{4}(x+4)$

or, $-6y = \frac{9}{2}(x+4)$

or, $-2y = \frac{3}{2}(x+4)$

or, $-4y = 3x + 12$

$\therefore$ $3x + 4y + 12 = 0$ **Ans.**

7. ***Find the condition that a line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$. Also, Find the equation of the tangent in the slope form.*** ***[3+1=4]***

Soln: Let, the equation of the st. line be

$y = mx + c$ ....(i) and

The equation of the parabola be

$y^2 = 4ax$ ....(ii)

Solving (i) and (ii), we get

$(mx + c)^2 = 4ax$

or, $m^2x^2 + 2mxc + c^2 - 4ax = 0$

or, $m^2x^2 + 2x(mc - 2a) + c^2 = 0$

or, $m^2x^2 + 2(mc - 2a)x + c^2 = 0$ ....(iii)

Which is quadratic in x. The line (i) will be tangent to the parabola (ii) if the discriminant of (iii) is equal to 0.

i.e. $b^2 - 4ac = 0$

or, $\{2(mc - 2a)\}^2 - 4.m^2.c^2 = 0$

or, $4\{m^2c^2 - 4amc + 4a^2\} - 4m^2c^2 = 0$

or, $m^2c^2 - 4amc + 4a^2 - m^2c^2 = 0$

or, $-4amc + 4a^2 = 0$

or, $4amc = 4a^2$

or, $c = \frac{a}{m}$.

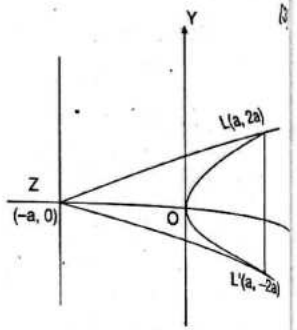
$\therefore$ The line $y = mx + c$ will be tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

Also, the equation of tangent in slope form is $y = mx + \frac{a}{m}$. **Ans.**

8. ***Show that the lines joining the ends of the latus rectum of the parabola $y^2 = 4ax$ to the point of intersection of the directrix and the axis are at right angles.***

Soln: Let, L(a, 2a) and L' (a, −2a) be the ends of the latus rectum of the parabola $y^2 = 4ax$.

Also, Z(−a, 0) be the point of intersection of the directrix and the axis of parabola.



Now, slope of LZ $= \frac{2a-0}{a+a} = 1$

Slope of L'Z $= \frac{-2a-0}{a+a} = -1$

Here, slope of LZ × slope of L'Z = 1 (−1) = −1

Hence, LZ is perpendicular to L'Z.

$\therefore$ The two lines LZ and L'Z are at right angle. **Hence Proved.**

9. ***Find the equation of the normal to the parabola $y^2 = 5x$ perpendicular to the line $x + 2y = 7$.***

Soln: Here, given parabola is $y^2 = 5x$.

Comparing it with $y^2 = 4ax$, we get

$4a = 5$

$\therefore \quad a = \frac{5}{4}$

We know, the equation of normal is

$y = mx - 2am - am^3$

$y = mx - 2 \times \frac{5}{4} m - \frac{5}{4} m^3$

$\therefore \quad y = mx - \frac{5}{2} m - \frac{5}{4} m^3$ ....(i)

Where, m is the slope of normal.

Also, the slope of the given line $x + 2y = 7$, is $\frac{-1}{2}$ $\left[\because \textbf{Slope of the line (m)} = \frac{\textbf{-x- coeff.}}{\textbf{y- coeff.}}\right]$

Since, the normal (i) is perpendicular to the line $x + 2y = 7$ so,

$m \times \left(\frac{-1}{2}\right) = -1$

$\therefore \quad m = 2$

Putting the value of m in (i), we get

$y = 2x - \frac{5}{2} \times 2 - \frac{5}{4} \times 2^3$

or, $y = 2x - 5 - 10$

$\therefore$ $y = 2x - 15$, is the required equation of the normal. **Ans.**

10. ***Find the equation of the tangent and normal to the parabola $y^2 = 16x$ at each end of the latus rectum.***

Soln: Given equation of parabola is

$y^2 = 16x$

Comparing it with $y^2 = 4ax$

Where, $4a = 16$

$\therefore \quad a = 4$

Hence, the ends of the latus rectum are (a, 2a) and (a, − 2a) i.e. (4, 8) and (4, − 8) respectively.

Now, the equation of tangent to $y^2 = 16x$ at (4, 8) is

$yy_1 = 2a\,(x + x_1)$

or, $y \times 8 = 2 \times 4\,(x + 4)$

or, $8y = 8\,(x + 4)$

$\therefore \quad x - y + 4 = 0$

Slope of the tangent = 1

Slope of the normal = −1

Again, the equation of normal at (4, 8) is

$y - 8 = -1\,(x - 4)$

or, $y - 8 = -x + 4$

$\therefore \quad x + y - 12 = 0$

Also, the equation of tangent to $y^2 = 16x$ at (4, −8) is

$yy_1 = 2a\,(x + x_1)$

or, $y.(-8) = 2 \times 4\,(x + 4)$

or, $-8y = 8\,(x + 4)$

or, $-y = x + 4$

$\therefore \quad x + y + 4 = 0$

The slope of tangent = −1

The slope of normal = 1

$\therefore$ The equation of normal at (4, −8) is

$y + 8 = 1\,(x - 4)$

or, $y + 8 = x - 4$

$\therefore \quad x - y - 12 = 0$. **Ans.**

11. ***Prove that the line $3x + 4y + 6 = 0$ is tangent to the parabola $2y^2 = 9x$. Find its point of contact.*** [3]

Soln: Here, given line is

$3x + 4y + 6 = 0$

or, $y = \frac{-3}{4} x - \frac{6}{4}$ ......(i)

Also, given parabola is

$2y^2 = 9x$

or, $y^2 = \frac{9}{2} x$ ....(ii)

Solving (i) & (ii), we get

$\left(-\frac{3}{4} x - \frac{6}{4}\right)^2 = \frac{9}{2} x$

or, $\left\{\frac{-3}{4}(x + 2)\right\}^2 = \frac{9}{2} x$

or, $\frac{9}{16}(x^2 + 4x + 4) = \frac{9}{2} x$

or, $x^2 + 4x + 4 = 8x$

or, $x^2 - 4x + 4 = 0$

or, $(x - 2)^2 = 0$

or, $(x - 2)(x - 2) = 0$

$\therefore \quad x = 2, 2$

Using x = 2, 2 in (i) y = − 3, − 3

$\therefore$ The line (i) and parabola (ii) intersect at two coincident point (2, −3).
Hence the given line is tangent to the given parabola.

$\therefore$ The point of contact is (2, −3) **Ans.**

Group 'A' (Multiple Choice Questions and Answers)

1. *In a school ground, the area is the set of all points in a plane. The sum of whose distances from two fixed points is constant. The enclosed area represented gives a geometrical form* *[2079 Set - J]*
   (a) circle (b) parabola (c) ellipse (d) hyperbola

2. *The eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is* *[2079 Set - K]*
   (a) $\frac{3}{5}$. (b) $\frac{5}{3}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

3. *Which of the following curve is an ellipse?* *[2079 Optional]*
   (a) $\frac{4x^2}{25}-\frac{9y^2}{16}=1$ (b) $\frac{3x^2}{4}-\frac{y^2}{9}=1$
   (c) $\frac{x^2}{49}-\frac{y^2}{36}=-1$ (d) $9x^2+4y^2+8y-40=0$

4. *$y^2 = 16 - 4x^2$ is a conic section. The eccentricity of the conic section (e) is .....* *[2079 GIF]*
   (a) 0 (b) 1 (c) less than 1 (d) greater than 1

5. *Which of the following conic is represented by the equation $4x^2 - 16x + y^2 = 48$?* *[2080 Set - G]*
   (a) Hyperbola (b) Ellipse (c) Parabola (d) Circle

6. *In the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has foci ($\pm$c, 0). when will be the ellipse become a circle?*
   (a) $\sqrt{a^2+b^2}=0$ (b) $\sqrt{b^2-a^2}=0$ (c) $\sqrt{a^2-b^2}=0$ (d) $c = \grave{a} = b$

7. *If a conic section has eccentricity (e) = $\frac{\sqrt{a^2-b^2}}{a}$, what is the equation of that conic section?*
   (a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (c) $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ (d) $\frac{x^2}{b^2}+\frac{y^2}{b^2}=1$

8. *Which of the following has an eccentricity less than one?*
   (a) Circle (b) Parabola (c) Hyperbola (d) Ellipse

9. *The vertices of the ellipse $9x^2 + 4y^2 = 36$ are*
   (a) $(0, \pm\sqrt{5})$ (b) $(\pm\sqrt{5}, 0)$ (c) $(0, \pm 3)$ (d) $(\pm 3, 0)$

10. *The foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are*
   (a) $(\pm 3, 0)$ (b) $(0, \pm\sqrt{5})$ (c) $(0, \pm 2)$ (d) $(\pm 2, 0)$

11. *The equation of the directrices of the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ are*
   (a) $x = \pm 2$ (b) $x = \pm 8$ (c) $y = \pm 2$ (d) $y = \pm 8$

12. *If the distance from the focus is 10 units and the distance from the directrix is 30 units, then what is the name of the conic?*
   (a) Circle (b) Parabola (c) Hyperbola (d) Ellipse

Hints: $e = \frac{\text{distance of the point from the focus}}{\text{distance of the point from the directrix}}$

$= \frac{10}{30}$

$= 0.33 < 1$ [$\because$ The eccentricity is less than 1 so the conic is an ellipse.]

Answer Key

1. c	2. a	3. d	4. c	5. b	6. d	7. b	8. d	9. c	10. d
11. b	12. d								

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Define an ellipse.* [1]

   (b) *Find the eccentricity and foci of the ellipse $9x^2 + 4y^2 - 18x - 16y - 11 = 0$* [2] [2080 Set - V]

Soln: (a) **Ellipse:** An ellipse is the locus of a point in a plane such that the sum of the distances of the point from two fixed points is constant.

(b) Given equation of ellipse is

$9x^2 + 4y^2 - 18x - 16y - 11 = 0$

or, $9x^2 - 18x + 9 - 9 + 4y^2 - 16y + 16 - 16 - 11 = 0$

or, $9(x^2 - 2x + 1) - 9 + 4(y^2 - 4y + 4) - 27 = 0$

or, $9(x-1)^2 + 4(y-2)^2 - 36 = 0$

or, $9(x-1)^2 + 4(y-2)^2 = 36$

or, $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$ ....(i)

Comparing (i) with the eq^n. of ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$h = 1, \quad k = 2$

$a^2 = 4, \quad b^2 = 9$

$\Rightarrow a = 2, \quad b = 3$

Since, $b > a$, so the major axis is along the y-axis.

Now, eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

Foci $= (h, k \pm be) = (1, 2 \pm 3 \times \frac{\sqrt{5}}{3}) = (1, 2 \pm \sqrt{5})$ **Ans.**

2. (a) *Find the eccentricity and the coordinates of the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$* [2]

   (b) *Find the equation of ellipse whose major axis is twice the minor axis and passes through the point (1, 0).* [2] [2081 Set - V]

Soln: (a) Given equation of ellipse is

$\frac{x^2}{9} + \frac{y^2}{16} = 1$ ....(i)

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 9, \quad b^2 = 16$

$a = 3, \quad b = 4$

Since, $b > a$, so the major axis is along y-axis.

Now, eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

Foci $= (0, \pm be) = (0, \pm 4\frac{\sqrt{7}}{4}) = (0, \pm\sqrt{7})$

(b) Let, the equation of ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(i) where $a > b$.

By question,

Major axis = 2 times of minor axis

or, $2a = 2 \times 2b$

or, $a = 2b$ ....(ii)

Also, if (i) passes through the point (1, 0), then

$\frac{1}{a^2} + \frac{0}{b^2} = 1$

or, $a^2 = 1$

$\therefore a = 1$

From (ii), $a = 2b$

or, $1 = 2b$

$\therefore b = \frac{1}{2}$

Putting the values of a and b in (i), we get

$\frac{x^2}{1} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$

$\therefore x^2 + 4y^2 = 1$, which is the required equation of ellipse. **Ans.**

3. *Find the equation of the ellipse whose foci are (± 2, 0) and the length of the latus rectum is 6 units.* [2] [2080 Optional]

Soln: Let, the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(i) where $a > b$.

Also, we have

Focus $(\pm ae, 0) = (\pm 2, 0)$

$\therefore ae = 2$ ....(ii)

And the length of latus rectum $= \frac{2b^2}{a} = 6$

$\Rightarrow b^2 = 3a$ ......(iii)

We have,

$\Rightarrow a^2 = b^2 + c^2$ [$\because c = ae$]

$\Rightarrow a^2 = 3a + 4$

$\Rightarrow a^2 - 3a - 4 = 0$

$\Rightarrow a^2 - 4a + a - 4 = 0$

$\Rightarrow a(a - 4) + 1(a - 4)$

$\Rightarrow (a - 4)(a + 1) = 0$

$\therefore a = 4, -1$ [$\because$ 'a' must be greater than b]

The possible value of a = 4

Then, $b^2 = 3a = 3 \times 4 = 12$

The equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{x^2}{16} + \frac{y^2}{12} = 1$, which is the required equation of ellipse. **Ans.**

4. *Find the coordinates of the foci and the length of the latus rectum of the conic* $\frac{x^2}{25}+\frac{y^2}{100}=1$. [3] [2081 - Optional]

Soln: Given equation of conic is

$$\frac{x^2}{25}+\frac{y^2}{100}=1 \text{ ....(i)}$$

Comparing it with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 25$, and $b^2 = 100$

$\therefore$ $a = 5$, $\therefore$ $b = 10$

Since a < b, so the major axis is along the y-axis.

Now, eccentricity (e) $= \sqrt{1-\frac{a^2}{b^2}}$

$$= \sqrt{1-\frac{25}{100}} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Foci $= (0, \pm be) = \left(0, \pm 10 \times \frac{\sqrt{3}}{2}\right) = (0, \pm 5\sqrt{3})$

Length of latus rectum $= \frac{2a^2}{b} = \frac{2\times 25}{10} = 5$ **Ans.**

5. *Find the equation of the ellipse in standard form with its length of the major axis 12 and eccentricity* $\frac{2}{3}$. [2] [2080 GIE Set A]

Soln: Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ .... (i) where } a > b$$

Given that, length of major axis = 12

i.e. $2a = 12$

$\therefore$ $a = 6$

Also, we have

$$e = \frac{2}{3}$$

or, $\sqrt{1-\frac{b^2}{a^2}} = \frac{2}{3}$

or, $\sqrt{1-\frac{b^2}{36}} = \frac{2}{3}$

or, $1-\frac{b^2}{36} = \frac{4}{9}$

or, $\frac{36-b^2}{36} = \frac{4}{9}$

or, $36 - b^2 = 16$

or, $b^2 = 20$

Substituting the value of a^2 and b^2 in (i), we get

$\frac{x^2}{36}+\frac{y^2}{20}=1$ which is the required equation of the ellipse. **Ans.**

6. *Find the equation of ellipse passing through the points (–2, 2) and (–3, 1).* [2] [2080 GIE Set B]

Soln: Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ .... (i) where } a > b$$

If (i) passes through the points (–2, 2) and (–3, 1), then we have,

$$\frac{(-2)^2}{a^2}+\frac{2^2}{b^2}=1$$

or, $\frac{4}{a^2}+\frac{4}{b^2}=1$ .... (ii)

Also,

$$\frac{(-3)^2}{a^2}+\frac{1^2}{b^2}=1$$

or, $\frac{9}{a^2}+\frac{1}{b^2}=1$ .... (iii)

Multiplying equation (iii) by 4 and subtracting (ii) from (iii) we get

$$\frac{36}{a^2}+\frac{4}{b^2}=4$$

$$\frac{4}{a^2}+\frac{4}{b^2}=1$$

(–) (–) (–)

$$\frac{32}{a^2}=3$$

$\therefore$ $a^2 = \frac{32}{3}$

From (ii),

$$\frac{4}{\frac{32}{3}}+\frac{4}{b^2}=1$$

or, $\frac{4}{b^2} = 1-\frac{12}{32}$

or, $\frac{4}{b^2} = \frac{20}{32}$

$\therefore$ $b^2 = \frac{32\times 4}{20} = \frac{32}{5}$

Substituting the values of a^2 and b^2 in (i), we get

$$\frac{x^2}{\frac{32}{3}}+\frac{y^2}{\frac{32}{5}}=1$$

or, $\frac{3x^2}{32}+\frac{5y^2}{32}=1$

$\therefore$ $3x^2 + 5y^2 = 32$ which is the required equation. **Ans.**

7. ***Find the eccentricity and length of latus rectum of the curve*** $\frac{x^2}{64} + \frac{y^2}{81} = 1$. **[2] [2079 Optio...]**

Soln: Here, given equation of ellipse is

$$\frac{x^2}{64}+\frac{y^2}{81}=1 \text{ ....(i)}$$

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 64, \quad b^2 = 81$

$\Rightarrow \quad a = 8, \quad b = 9$

Since, $b > a$, so the major axis is along the y-axis.

Now, eccentricity $(e) = \sqrt{1-\frac{a^2}{b^2}}$

$= \sqrt{1-\frac{64}{81}}$

$= \sqrt{\frac{17}{81}}$

$= \frac{\sqrt{17}}{9}$

$\therefore$ The length of the latus rectum $= \frac{2a^2}{b} = \frac{2 \times 64}{9}$

$= \frac{128}{9}$ **Ans.**

8. ***Given the equation of an ellipse,*** $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$, ***find***

(a) *The eccentricity.*

(b) *The coordinates of the vertices.*

(c) *The coordinates of the centre.*

(d) *The foci.*

(e) *The length of the latus rectum.*

Soln: Here, given equation of an ellipse is $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$ ....(i)

Comparing (i), with the equation of ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$h = -2, \quad k = 5$

$a^2 = 16, \quad b^2 = 9$

$\Rightarrow \quad a = 4, \quad b = 3$

Since, $a > b$ so, the major axis is along the x-axis.

Now,

(a) Eccentricity $(e) = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$

(b) The coordinates of the vertices $= (h \pm a, k)$

$= (-2 \pm 4, 5)$

$= (-6, 5)$ and $(2, 5)$

(c) The coordinates of the centre $= (h, k) = (-2, 5)$

(d) The coordinates of the foci $= (h \pm ae, k)$

$= (-2 \pm 4 \times \frac{\sqrt{7}}{4}, 5)$

$= (-2 \pm \sqrt{7}, 5)$

(e) The length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$

9. **(a)** ***What is the major axis in the ellipse?*** ***[1]***

(b) ***Find the equation of the ellipse whose latus rectum is half the major axis and focus is at (3, 0).*** ***[2]***

Soln: (a) **Major axis:** The straight line passing through the foci is called the major axis.

(b) Let, the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ......(i), where } a > b.$$

Given that,

Focus $(\pm ae, 0) = (3, 0)$

$\therefore \quad ae = 3$

The length of latus rectum $= \frac{1}{2} \times 2a$

or, $\frac{2b^2}{a} = a$

or, $2a^2(1-e^2) = a^2$ **$[\because e^2 = 1 - \frac{b^2}{a^2} \Rightarrow b^2 = a^2(1-e^2)]$**

or, $1 - e^2 = \frac{1}{2}$

or, $e^2 = 1 - \frac{1}{2}$

$\therefore \quad e = \frac{1}{\sqrt{2}}$

Now, $ae = 3$

or, $a \times \frac{1}{\sqrt{2}} = 3$

$\therefore \quad a = 3\sqrt{2}$

Also, $b^2 = a^2(1-e^2) = 9 \times 2 \times \frac{1}{2} = 9$

Now, from equation (i),

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{18} + \frac{y^2}{9} = 1$

$\therefore \quad x^2 + 2y^2 = 18$ which is the required equation of ellipse. **Ans.**

10. (a) *Find the equation of the ellipse in the standard form whose focus is at (−2, 0) and ve (5, 0).*

(b) *What shape would the ellipse be if $a < b$?*

Soln: (a) Here, given

Vertex $(\pm a, 0) = (5, 0)$

$\therefore \quad a = 5$

Also, foci $(\pm ae, 0) = (-2, 0)$

i.e. $\quad -ae = -2$

$\therefore \quad ae = 2$

or, $\quad 5e = 2 \qquad [\because a = 5]$

$\therefore \quad e = \dfrac{2}{5}$

We know,

$$b^2 = a^2(1 - e^2) = 5^2\left(1 - \frac{4}{25}\right)$$

$$= 21$$

Now, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{x^2}{25} + \frac{y^2}{21} = 1 \textbf{ Ans.}$$

(b) If $a < b$, then the major axis of the ellipse is along the y-axis and hence the shape of the ellipse vertically elongated.

11. *Analyze the ellipse represented by the equation $x^2 + 4y^2 - 4x + 24y + 24 = 0$.*

(a) *Show that the given equation represents an ellipse.*

(b) *Find the eccentricity of the ellipse.*

(c) *Calculate the length of the minor axis.*

(d) *Determine the length of the latus rectum.*

Soln: (a) Given equation is,

$$x^2 + 4y^2 - 4x + 24y + 24 = 0$$

or, $\quad x^2 - 4x + 4 - 4 + 4y^2 + 24y + 36 - 36 + 24 = 0$

or, $\quad (x-2)^2 + 4(y^2 + 6y + 9) - 36 + 24 - 4 = 0$

or, $\quad (x-2)^2 + 4(y+3)^2 = 16$

or, $\quad \dfrac{(x-2)^2}{16} + \dfrac{(y+3)^2}{4} = 1$ which is in the form of $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

Hence the given equation represents an equation of an ellipse. **Proved.**

(b) We have, $h = 2, \quad k = -3$

$a^2 = 16, \quad b^2 = 4$

$\Rightarrow a = 4, \quad b = 2$

Now,

$\therefore$ Eccentricity $(e) = \sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{1 - \dfrac{4}{16}} = \sqrt{\dfrac{12}{16}} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$

(c) Length of the minor axis $= 2b = 2 \times 2 = 4$

(d) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{2 \times 4}{4} = 2$ **Ans.**

8.4 Hyperbola

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Basic Formulae and Key Points

1. **Important Terms Relating to the Different Parts of Hyperbola**

Foci: The two fixed points in hyperbola are called foci.

Transverse Axis: The straight line passing through foci is called transverse axis.

Centre: The midpoint of the line joining the two foci is called the centre of the hyperbola.

Conjugate Axis: The straight line passing through the centre and perpendicular to the transverse axis is called conjugate axis.

Vertices: The points where the hyperbola intersects the axis are called the vertices.

Latus Rectum: The straight line segment passing through focus and perpendicular to transverse axis is called latus rectum. There are two latus rectum in a hyperbola.

2. **Formula Table:**

Equation of Hyperbola	Vertex	Focus	Equation of Directrix	length of Latus Rectum	Length of Transverse Axis	Length of Conjugate Axis	Eccentricity
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$(\pm a, 0)$	$(\pm ae, 0)$	$x = \pm\frac{a}{e}$	$\frac{2b^2}{a}$	2a	2b	$e = \sqrt{1 + \frac{b^2}{a^2}}$
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$	$(0, \pm b)$	$(0, \pm be)$	$y = \pm\frac{b}{e}$	$\frac{2a^2}{b}$	2b	2a	$e = \sqrt{1 + \frac{a^2}{b^2}}$
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$(h \pm a, k)$	$(h \pm ae, k)$	$x = h \pm \frac{a}{e}$	$\frac{2b^2}{a}$	2a	2b	$e = \sqrt{1 + \frac{b^2}{a^2}}$
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$	$(h, k \pm b)$	$(h, k \pm be)$	$y = k \pm \frac{b}{e}$	$\frac{2a^2}{b}$	2b	2a	$e = \sqrt{1 + \frac{a^2}{b^2}}$

Note: If $e > 1$, the conic section is a hyperbola.

Group 'A' (Multiple Choice Questions and Answers)

1. *Which one of the following has transverse axis and conjugate axis?* *[2081 Set V]*

(a) $y^2 - 4y - 4x + 4 = 0$ (b) $2y^2 - 3x^2 - 6 = 0$

(c) $2y^2 + 3x^2 - 6 = 0$ (d) $2x^2 + 2y^2 = 72$

2. *If a conic section has eccentricity $(e) = \dfrac{\sqrt{a^2 + b^2}}{a}$, what is the equation of that conic section?* *[2081 Set E/W]*

(a) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(c) $x^2 + y^2 = a^2$ (d) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$

3. *Which one of the following conic section has eccentricity greater than one?* *[2080 Optional]*

(a) $25x^2 + 16y^2 = 1$ (b) $3(x+2)^2 + 4(y-1)^2 = 192$

(c) $9x^2 - 144 = 16y^2$ (d) $y^2 = 20 - x^2$

4. *$x^2 = 5y^2 + 20$ represents the equation of a conic section. The eccentricity(e) of the conic section is always ....* *[2079 - GIE]*

(a) $e = 0$ (b) $e = 1$ (c) $e > 1$ (d) $e < 1$

5. *The path traced by a moving point such that the difference of its point from two fixed point is*

(a) parabola (b) circle (c) hyperbola (d) ellipse

6. *Hyperbola has .... vertices and .... foci.*

(a) two, one (b) one, one (c) one, two (d) two, two

7. *The hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ has the transverse axis is along*

(a) x-axis (b) y -axis
(c) not through the centre but parallel to x-axis. (d) not through the centre but parallel to y-axis.

8. *What is the conjugate axis length for the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$?*

(a) 5 units (b) 4 units (c) 8 units (d) 10 units

9. *The coordinates of foci of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ is*

(a) $(\pm 5, 0)$ (b) $(\pm 4, 0)$ (c) $(\pm 3, 0)$ (d) none

10. *Find the length of the latus rectum for the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is*

(a) $\frac{3}{8}$ (b) $\frac{8}{3}$ (c) 9 (d) none

11. *Distance between foci of hyperbola is*

(a) $\frac{a}{e}$ (b) $\frac{2a}{e}$ (c) ae (d) 2ae

Answer Key

1. b	2. a	3. c	4. c	5. d	6. d	7. a	8. c	9. a	10. b
11. d									

Group 'B' or 'C' (Subjective Questions and Answers)

1. *Find the equation of the hyperbola with vertices (0, – 4) and (0, 4) and foci (0, 6) and (0, – 6).* *[3] [2081 Optional*

Soln: Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad ......(i)$$

Now, we have

Vertices $(0, \pm b) = (0, \pm 4)$

$\therefore \quad b = 4 \Rightarrow b^2 = 16$

Foci $(0, \pm be) = (0, \pm 6)$

i.e. $\quad be = 6$

or, $\quad 4e = 6$

$\therefore \quad e = \frac{6}{4} = \frac{3}{2}$

Then,

$$e^2 = 1 + \frac{a^2}{b^2}$$

or, $\left(\frac{3}{2}\right)^2 = \frac{b^2+a^2}{b^2}$

or, $\frac{9}{4} = \frac{4^2+a^2}{4^2}$

or, $\frac{9}{4} = \frac{16+a^2}{16}$

or, $a^2 = 36 - 16$

$\therefore \quad a^2 = 20$

Putting the values of a^2 and b^2 in (i), we get

$\frac{x^2}{20}-\frac{y^2}{16}=-1$ which is the required equation of the hyperbola. **Ans.**

2. *Find the foci and centre of the hyperbola with equation $9x^2 - 16y^2 - 144 = 0$.* *[2] [2080 Set G]*

Soln: Here, given equation of hyperbola is

$9x^2 - 16y^2 - 144 = 0$

or, $9x^2 - 16y^2 = 144$

or, $\frac{9x^2}{144}-\frac{16y^2}{144}=1$ [$\because$ Dividing both sides by 144]

or, $\frac{x^2}{16}-\frac{y^2}{9}=1$

or, $\frac{(x-0)^2}{16}-\frac{(y-0)^2}{9}=1 \quad .... (i)$

Comparing (i) with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get

$h = 0, k = 0$

$a^2 = 16 \Rightarrow a = 4,$

$b^2 = 9 \Rightarrow b = 3$

$\therefore \quad$ Centre $(h, k) = (0, 0)$

We have,

Eccentricity $(e) = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$\therefore \quad$ Foci $= (\pm ae, 0) = \left(\pm 4 \times \frac{5}{4}, 0\right) = (\pm 5, 0)$ **Ans.**

3. *Given the equation of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.*

Answer the following question. *[1]*

(a) *Write the vertices of the hyperbola.* *[1]*

(b) *Find the coordinates of the foci.* *[1]*

(c) *Write the formula to find the eccentricity.* *[1]*

(d) *How do you calculate the length of the latus rectum of a hyperbola?* *[1]*

(e) *Define transverse axis in a hyperbola.*

Soln: (a) Vertices of the hyperbola is $(\pm a, 0)$

(b) Foci are at $(\pm ae, 0)$

(c) Eccentricity $(e) = \sqrt{1+\frac{b^2}{a^2}}$

(d) The length of the latus rectum $= \frac{2b^2}{a}$

(e) The straight line passing through foci is called transverse axis. The transverse axis is along the x-axis, and its length is 2a.

4. *(a) What is a hyperbola in conic section?*
(b) Determine the equation of the hyperbola with a focus at (5, 0) and vertex (3, 0.)

Soln: (a) **Hyperbola:** A hyperbola is the locus of of a point in a plane such that the difference of the distance from two fixed points is constant.

(b) Let, the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .... (i).

Now, we have

Vertex (a, 0) = (3, 0)

$\therefore$ a = 3 then $a^2 = 9$

Also, focus (ae, 0) = (5, 0)

or, ae = 5

or, 3e = 5

$\therefore$ $e = \frac{5}{3}$

Now, we have,

$e^2 = 1 + \frac{b^2}{a^2}$

or, $\left(\frac{5}{3}\right)^2 = 1 + \frac{b^2}{3^2}$

or, $\frac{25}{9} = \frac{9 + b^2}{9}$

or, $b^2 + 9 = 25$

$\therefore$ $b^2 = 16$

Now, substituting the values of a^2 and b^2 in equation (i), we get

$\frac{x^2}{9} - \frac{y^2}{16} = 1$ **Ans.**

5. *Given the equation $9x^2 - 16y^2 + 18x + 32y - 151 = 0$*
(a) Show that it represents a hyperbola.
(b) Find the coordinates of the vertices.
(c) Determine the eccentricity.
(d) Find coordinates of the foci.

Soln: (a) Given equation is,

$9x^2 - 16y^2 + 18x + 32y - 151 = 0$

or, $9x^2 + 18x - 16y^2 + 32y - 151 = 0$

or, $9(x^2 + 2x) - 16(y^2 - 2y) - 151 = 0$

or, $9(x^2 + 2x + 1 - 1) - 16(y^2 - 2y + 1 - 1) - 151 = 0$

or, $9(x + 1)^2 - 9 - 16(y - 1)^2 + 16 - 151 = 0$

or, $9(x + 1)^2 - 16(y - 1)^2 - 144 = 0$

or, $9(x + 1)^2 - 16(y - 1)^2 = 144$

or, $\frac{9(x+1)^2}{144} - \frac{16(y-1)^2}{144} = 1$

or, $\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1$ which is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$

Hence, the given equation represents an equation of hyperbola. **Proved.**

(b) We have, h = –1, k = 1

$a^2 = 16$, $b^2 = 9$

$\therefore$ a = 4, b = 3

Now, the coordinates of the vertices = (h ± a, k)

= (– 1 ± 4, 1)

= (– 5, 1) and (3, 1) **Ans.**

(c) Eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

(d) Foci = (h ± ae, k) = $(-1 \pm 4 \times \frac{5}{4}, 1)$

= (–1 ± 5, 1)

= (– 6, 1) and (4,1) **Ans.**

6. *Given the equation of a hyperbola: $9x^2 - 16y^2 = 144$. Find the following.*
(a) The length of the transverse axis. *[1]*
(b) The length of the conjugate axis. *[1]*
(c) The coordinates of the vertices. *[1]*
(d) The eccentricity. *[1]*
(e) The coordinates of the foci. *[1]*

[2080 GIE Set B]

Soln: Here,

The given equation of the hyperbola is

$9x^2 - 16y^2 = 144$

or, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ......(i)

Comparing (i) with the equation of hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$a^2 = 16$, $b^2 = 9$

$\therefore$ a = 4, b = 3

Now,

(a) The length of transverse axis = 2a = 2 × 4 = 8

(b) The length of conjugate axis = 2b = 2 × 3 = 6

(c) The coordinates of the vertices = (± a, 0) = (± 4, 0)

(d) The eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}}$

$= \sqrt{1 + \frac{9}{16}}$

$= \frac{5}{4}$

(e) Foci (± ae, 0) = $(\pm 4 \times \frac{5}{4}, 0)$

= (± 5, 0) **Ans.**

7. *Find the equation of a hyperbola with a focus at (–7, 0) and eccentricity $\frac{7}{4}$.*

Soln: Here, given that

Focus $(-ae, 0) = (-7, 0)$

$\therefore \quad ae = 7$ .....(i)

Also, eccentricity $(e) = \frac{7}{4}$

Then from (i),

$ae = 7$

$a \times \frac{7}{4} = 7$

$\therefore \quad a = 4$

Again, we have

$e = \sqrt{1 + \frac{b^2}{a^2}}$

or, $\frac{7}{4} = \sqrt{1 + \frac{b^2}{16}}$

Squaring both sides, we get

$\frac{49}{16} = 1 + \frac{b^2}{16}$

or, $\frac{49}{16} = \frac{16 + b^2}{16}$

or, $b^2 + 16 = 49$

$\therefore \quad b^2 = 33$

Now, the equation of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{x^2}{16} - \frac{y^2}{33} = 1$

Which is the required equation of hyperbola. **Ans.**

8. *Find the equation of the hyperbola with vertex (8, 0) and passing through the point $(8\sqrt{2}, 4)$.*

Soln: Let, the equation of hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ......(i)

Where, vertex $(a, 0) = (8, 0)$

$\therefore \quad a = 8$

Now, equation (i) becomes,

$\frac{x^2}{64} - \frac{y^2}{b^2} = 1$ ....(ii)

Also, eq^n. passes through the point $(8\sqrt{2}, 4)$.

So, $\frac{(8\sqrt{2})^2}{64} - \frac{4^2}{b^2} = 1$

or, $2 - \frac{16}{b^2} = 1$

or, $\frac{16}{b^2} = 1$

$\therefore \quad b^2 = 16$

Putting the value of b^2 in (ii), we get

$\frac{x^2}{64} - \frac{y^2}{16} = 1$ which is the required equation of hyperbola. **Ans.**

9. *Find the eccentricity, coordinates of the vertices and the foci of the hyperbola $5x^2 - 20y^2 - 20x = 0$.* [3]

Soln: Here, the equation of hyperbola is

$5x^2 - 20y^2 - 20x = 0$

or, $5x^2 - 20x - 20y^2 = 0$

or, $5(x^2 - 4x + 4 - 4) - 20y^2 = 0$

or, $5(x-2)^2 - 20 - 20y^2 = 0$

or, $5(x-2)^2 - 20y^2 = 20$

or, $\frac{5(x-2)^2}{20} - \frac{20y^2}{20} = 1$

or, $\frac{(x-2)^2}{4} - y^2 = 1$ ....(i)

Comparing (i) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$,

We have, $h = 2$, $\quad k = 0$

$a^2 = 4$, $\quad b^2 = 1$

$\Rightarrow \quad a = 2$, $\quad b = 1$

Now, vertices $= (h \pm a, k) = (2 \pm 2, 0) = (0, 0)$ & $(4, 0)$

Eccentricity $(e) = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

Foci $= (h \pm ae, k) = (2 \pm 2 . \frac{\sqrt{5}}{2}, 0) = (2 \pm \sqrt{5}, 0)$ **Ans.**

10. *Find the latus rectum of the conic section $9y^2 - 4x^2 = 36$.* [2] [2080 Set I]

Soln: Here, given equation is

$9y^2 - 4x^2 = 36$

Dividing both sides by 36, we get

$\frac{y^2}{4} - \frac{x^2}{9} = 1$ ..... (i)

Comparing (i) with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get

$\therefore \quad a^2 = 9$, $\quad b^2 = 4$

$a = 3 \quad b = 2$

$\therefore \quad$ Length of latus rectum $= \frac{2a^2}{b} = \frac{2 \times 9}{2} = 9$ **Ans.**

❑❑❑

4. Vectors

Chapter 9 Product of Vectors

9.1 Scalar (dot) Product of Two Vectors

Basic Formulae and Key Points

1. The scalar product of two plane vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ denoted by $\vec{a}$. $\vec{b}$ is defined by $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$.

   Similarly, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be two space vectors then

   $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

2. **Angular Form:** If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then scalar product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \cdot \vec{b}$ is defined by $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = ab \cos\theta$, $0 \le \theta \le \pi$.

3. Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

   Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

4. **Condition of Perpendicularity:** Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular (orthogonal) if $\vec{a} \cdot \vec{b} = 0$.

5. **Angle Between two Vectors**

   If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

6. **Properties:**

   (a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

   (b) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

   (c) $\vec{a} \cdot \vec{a} = a^2$

   (d) $k(\vec{a} \cdot \vec{b}) = (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$

7. **If $\vec{i} = (1, 0, 0)$, $\vec{j} = (1, 0, 0)$ and $\vec{k} = (0, 0, 1)$ are three mutually perpendicular unit vectors in space then,**

   $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$

   $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = \vec{k} \cdot \vec{i} = 0$

Aid to memory

·	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

8. **Recap:**

   (i) **Magnitude of a Vector:**

   The magnitude or modulus of a vector $\overrightarrow{OP} = x\vec{i} + y\vec{j}$ is denoted by $|\overrightarrow{OP}|$ and given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

   In case of space vector, $\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$ then

   $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

   (ii) **Unit Vector:** The unit vector in the direction of vector $\vec{a}$ denoted by $\hat{a}$ is given by

   $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$, where $|\vec{a}| \neq 0$

Group 'A' (Multiple Choice Questions and Answers)

1. *Which one of the following is the angle between two vectors $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$?* [2081 Set – W]

   (a) 0° (b) 60° (c) 90° (d) 180°

2. *If α be the angle between two vectors $\vec{a}$ and $\vec{b}$ then what is the $\vec{a} \cdot \vec{b}$ equal to?*

   (a) $|\vec{a}||\vec{b}|$ (b) $|\vec{a}||\vec{b}| \sin\alpha$ (c) $|\vec{a}||\vec{b}| \cos\alpha$ (d) $\sqrt{a^2 + b^2}$

3. *Which of the following statement is correct?*

   (a) $\vec{a} \cdot \vec{b} = \vec{b} \times \vec{a}$ (b) $\vec{a} \cdot (\vec{b} \cdot \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$

   (c) $\vec{i} \cdot \vec{i} = 0$ (d) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

4. *The projection of $\vec{a}$ on $\vec{b}$ is equal to*

   (a) $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ (b) $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

   (c) $\dfrac{|\vec{a}|}{\vec{a} \cdot \vec{b}}$ (d) $\dfrac{|\vec{b}|}{\vec{a} \cdot \vec{b}}$

5. *What is $\vec{k} \cdot \vec{i}$ equal to?*

   (a) 0 (b) 1 (c) −1 (d) ∞

6. *The value of 'm' when $2\vec{i} + m\vec{j} - 3\vec{k}$ and $2\vec{i} + 3\vec{j} + 4\vec{k}$ are perpendicular is*

(a) $\frac{-8}{3}$ (b) $\frac{8}{3}$ (c) $\frac{3}{8}$ (d) $\frac{-3}{8}$

7. *Dot product of two non zero vectors is zero ($\vec{a}..\vec{b} = 0$), when angle between them is*

(a) 30° (b) 60° (c) 45° (d) 90°

8. *Compute the dot product of $\vec{a} = 5\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = -2\vec{i} - 3\vec{j} + 6\vec{k}$*

(a) 2 (b) 3 (c) – 2 (d) – 3

9. *The scalar product is also known as:*

(a) additional product (b) negative product
(c) dot product (d) cross product

10. *The two vectors have magnitudes 3 and 5. If the angle between them is 60°, then dot product of two vector will be*

(a) 6.5 (b) 7.5 (c) 8.4 (d) 7.9

11. *When the product of two vectors results into a scalar quantity, then the product is called*

(a) cross product (b) dot product
(c) multiplication (d) all of the above

12. *If the scalar projection of $\vec{a} = \lambda\vec{i} + \vec{j} + 4\vec{k}$ on $\vec{b} = 2\vec{i} + 6\vec{j} + 3\vec{k}$ is 4 units then λ is equal to*

(a) 5 (b) 3 (c) 1 (d) 5

13. *If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then*

(a) $\vec{a} \perp \vec{b}$ (b) $\vec{a} \parallel \vec{b}$ (c) $\vec{a} = 0$ (d) $\vec{b} = 0$

Answer Key

1. b	2. c	3. d	4. b	5. a	6. b	7. d	8. a	9. c	10. b
11. b	12. d	13. a							

Group 'B' or 'C' (Subjective Questions and Answers)

1. *The dot product of two non-zero vectors given a positive real numbers. Justify it with example.* *[2] [2081 - Set E & ...*

Soln: The question is incorrect because the dot product of two non-zero vectors can be any real number, not necessarily positive.

Here, we may take an example that gives the product of two non-zero vectors in a positive real number.

Let, $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$. Then

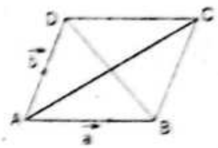
$$\vec{a}.\vec{b} = (2\vec{i} + 3\vec{j} + \vec{k}).(\vec{i} + 2\vec{j} + 3\vec{k})$$
$$= 2.1 + 3.2 + 1.3$$
$$= 2 + 6 + 3$$

= 11 which is a positive real number. **Ans.**

2. *In a rhombus, two of the diagonals are perpendicular to each other. Verify it by taking vector dot product of two vectors.* *[2] [2080 – Set V]*

Soln: Let, ABCD be rhombus with diagonals AC and BD.

Let, $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$.



Since, $\vec{AB}$ and $\vec{AD}$ are vectors of equal length in a rhombus

So, $|\vec{AB}| = |\vec{AD}|$

$\Rightarrow$ a = b

Now,

$$\vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$$
$$\vec{BD} = \vec{BA} + \vec{AD} = -\vec{AB} + \vec{AD} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}$$
$$\vec{AC}.\vec{BD} = (\vec{a} + \vec{b}).(\vec{b} - \vec{a})$$
$$= \vec{b}^2 - \vec{a}^2$$
$$= b^2 - a^2$$
$$= b^2 - a^2 \quad [\because a = b]$$
$$= 0$$

This verifies that the diagonals of rhombus are perpendicular to each other. **Ans.**

3. *Find the dot product of the vectors $2\vec{i} - \vec{j} + 2\vec{k}$ and $3\vec{i} - 2\vec{j} + 6\vec{k}$.* *[2] [2081 – Optional]*

Soln: Let, $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$

$\vec{b} = 3\vec{i} - 2\vec{j} + 6\vec{k}$

Now, $\vec{a}.\vec{b} = (2\vec{i} - \vec{j} + 2\vec{k}).(3\vec{i} - 2\vec{j} + 6\vec{k})$
$$= 2.3 + (-1).(-2) + 2.6$$
$$= 6 + 2 + 12$$

= 20 **Ans.**

4. *If $\vec{a} = (1, 1, -2)$ and $\vec{b} = (2, 1, -1)$,*

(a) *Find $\vec{a}.\vec{b}$* *[1]*

(b) *Find the projection of $\vec{a}$ on $\vec{b}$* *[1]*

(c) *Find the cosine of the angle between the two vectors $\vec{a}$ and $\vec{b}$* *[2]*

(d) *Write one property of dot product.* *[1]*

Soln: Here, $\vec{a} = (1, 1, -2)$ and $\vec{b} = (2, 1, -1)$

(a) Now, $\vec{a}.\vec{b} = (1, 1, -2).(2, 1, -1)$
$$= 1.2 + 1.1 + (-2).(-1)$$
$$= 2 + 1 + 2$$
$$= 5$$

(b) We have,

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$

Now, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \dfrac{5}{\sqrt{6}}$ **Ans.**

(c) Here, $\vec{a} = (1, 1, -2)$ & $\vec{b} = (2, 1, -1)$, then

$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

If θ be the angle between $\vec{a}$ and $\vec{b}$ then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{5}{6}\right)$ **Ans.**

(d) The scalar product of two vectors is commutative i.e. if $\vec{a}$ and $\vec{b}$ be two vectors, then

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

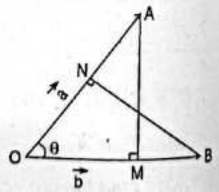
5. (a) *Define scalar product of two vectors.* [2]

(b) *Give the geometrical interpretation of the scalar product of two vectors $\vec{a}$ and $\vec{b}$.* [2]

(c) *What is the relationship between two vectors if their dot product is zero?* [1]

Soln: (a) **Scalar product of two vectors:** Scalar (dot) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\cdot\vec{b}$, is defined by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where a and b are the magnitude of the vectors $\vec{a}$ and $\vec{b}$. θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) **Geometrical Interpretation:**



Let, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ and θ be the angle between the two vectors $\vec{a}$ and $\vec{b}$.

AM ⊥ OB and BN ⊥ OA are drawn.

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= ab\cos\theta$

$= (OA)(OB\cos\theta)$

$= OA.\,ON$

$=$ (Magnitude of $\vec{a}$). (Projection of $\vec{b}$ on $\vec{a}$)

Similarly $\vec{a}\cdot\vec{b}$ = (Magnitude of $\vec{b}$). (projection of $\vec{a}$ on $\vec{b}$)

Hence the scalar product of two vectors is the product of the magnitude of one of the vectors and the projection of the second vector on the first.

(c) If the dot product of two vectors is zero, the relationship between the vectors is that they are perpendicular or orthogonal to each other.

6. *If $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ are any two vectors,*

(a) *Write the formula to find angle between two vectors $\vec{a}$ and $\vec{b}$.* [1]

(b) *Find the projection of $\vec{b}$ on $\vec{a}$.* [1]

(c) *Find the cosine of the angle between two vectors $\vec{a}$ and $\vec{b}$.* [1]

(d) *If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that $\vec{a}$ is perpendicular to $\vec{b}$.* [2]

Soln: (a) If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

(b) Here, $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + \vec{k}$

Now, $\vec{a}\cdot\vec{b} = (\vec{i} + 2\vec{j} - \vec{k}).(\vec{i} - \vec{j} + \vec{k})$

$= 1.1 + 2.(-1) + (-1).1$

$= 1 - 2 - 1$

$= -2$

$|\vec{a}| = \sqrt{1^2 + 2^2 (-1)^2} = \sqrt{6}$

$\therefore$ Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{-2}{\sqrt{6}}$

(c) Here, from (b), $\vec{a}\cdot\vec{b} = -2$, $|\vec{a}| = \sqrt{6}$

$|\vec{b}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

Now, if θ be the angle between them, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-2}{\sqrt{6}\sqrt{3}} = \frac{-2}{3\sqrt{2}} = \frac{-\sqrt{2}}{3}$$

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{-\sqrt{2}}{3}\right)$ **Ans.**

(d) Here, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

Squaring both sides, we get

or, $|\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

or, $(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$

or, $a^2 + 2\vec{a}\cdot\vec{b} + b^2 = a^2 - 2\vec{a}\cdot\vec{b} + b^2$

or, $2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{b} = 0$

or, $4\vec{a}\cdot\vec{b} = 0$

or, $\vec{a}\cdot\vec{b} = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{b}$. **Hence Proved.**

7. (a) *If $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$, prove that $|\vec{a}|=|\vec{b}|$.*

(b) *If θ is the angle between two unit vectors $\vec{a}$ and $\vec{b}$, show that $\frac{1}{2}|\vec{a}-\vec{b}|=\sin\frac{\theta}{2}$*

(c) *Show that the scalar product of two vectors is commutative.*

Soln: (a) Here,

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$

or, $\vec{a}.\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}.\vec{b}=0$

or, $\vec{a}^2-\vec{a}.\vec{b}+\vec{a}.\vec{b}-\vec{b}^2=0$ $[\because \vec{a}.\vec{b}=\vec{b}.\vec{a}]$

or, $|\vec{a}|^2-|\vec{b}|^2=0$

or, $|\vec{a}|^2=|\vec{b}|^2$

$\therefore$ $|\vec{a}|=|\vec{b}|$ **Hence roved.**

(b) Here, given that

$|\vec{a}|=a=1$ and $|\vec{b}|=b=1$

Now, we have

$|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})^2$ $[\because |\vec{a}|^2=\vec{a}^2]$

$=\vec{a}^2-2\vec{a}.\vec{b}+\vec{b}^2$

$=a^2-2\vec{a}.\vec{b}+b^2$

$=1-2|\vec{a}||\vec{b}|\cos\theta+1$

$=2-2.1.1\cos\theta$

$=2(1-\cos\theta)$

$=2.(2\sin^2\frac{\theta}{2})$

or, $|\vec{a}-\vec{b}|^2=(2\sin\frac{\theta}{2})^2$

or, $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

$\therefore$ $\frac{1}{2}|\vec{a}-\vec{b}|=\sin\frac{\theta}{2}$ **Hence Proved.**

(c) Let, $\vec{a}=(x_1, y_1, z_1)$ and $\vec{b}=(x_2, y_2, z_2)$ be any two space vectors then

$\vec{a}.\vec{b}=(x_1, y_1, z_1).(x_2, y_2, z_2)$

$=x_1x_2+y_1y_2+z_1z_2$

$=x_2x_1+y_2y_1+z_2z_1$

$=(x_2, y_2, z_2).(x_1, y_1, z_1)$

$=\vec{b}.\vec{a}$

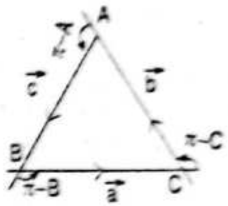
i.e. $\vec{a}.\vec{b}=\vec{b}.\vec{a}$

Hence, scalar product of two vectors is commutative. **Proved.**

8. (a) *In any triangle ABC, Prove vectorially that c = a cosB + b cosA* [3]

(b) *Show that the vectors $2\vec{i}+3\vec{j}-8\vec{k}$ and $2\vec{i}+4\vec{j}-2\vec{k}$ are orthogonal.* [2]

Soln: (a) In ABC, let $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$ and $\overrightarrow{AB}=\vec{c}$



Now, by using triangle law of vector addition

$\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{CB}$

or, $\vec{c}=-\overrightarrow{CA}-\overrightarrow{BC}$

or, $\vec{c}=-\vec{b}-\vec{a}$

Taking scalar product on both sides by $\vec{c}$, we get

$\vec{c}.\vec{c}=-\vec{b}.\vec{c}-\vec{a}.\vec{c}$

or, $\vec{c}^2=-bc\cos(\pi-A)-ac\cos(\pi-B)$

or, $c^2=bc\cos A+ac\cos B$

or, $c^2=c(b\cos A+a\cos B)$

$\therefore$ $c=a\cos B+b\cos A$ **Hence Proved.**

(b) Let, $\vec{a}=2\vec{i}+3\vec{j}-8\vec{k}$ and

$\vec{b}=2\vec{i}+4\vec{j}+2\vec{k}$

Now, $\vec{a}.\vec{b}=(2\vec{i}+3\vec{j}-8\vec{k}).(2\vec{i}+4\vec{j}+2\vec{k})=4+12-16=0$

Hence, $\vec{a}.\vec{b}=0$ which implies that $\vec{a}$ and $\vec{b}$ are orthogonal. **Proved.**

9. (a) *Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.* [2]

(b) *In any triangle, prove vectorially that*
$a^2=b^2+c^2-2bc\cos A$ [3]

Soln: (a) Here, given that

Magnitude of two vectors $\vec{a}$ and $\vec{b}$ is same

So, $|\vec{a}|=|\vec{b}|$

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$\theta=60°$ and $\vec{a}.\vec{b}=\frac{1}{2}$

Now, we have,

$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

or, $\frac{1}{2}=|\vec{a}||\vec{a}|\cos 60°$ $[\because |\vec{a}|=|\vec{b}|]$

or, $\frac{1}{2}=|\vec{a}|^2.\frac{1}{2}$

or, $|\vec{a}|^2=1$

$\therefore$ $|\vec{a}|=\pm 1$

7. (a) If $(\vec a+\vec b)\cdot(\vec a-\vec b)=0$, prove that [illegible]

(b) If θ is the angle between two unit vectors [illegible]

(c) Show that the scalar product of two vectors is commutative.

Soln: (a) Here,

$(\vec a+\vec b)\cdot(\vec a-\vec b)=0$

or, $\vec a\cdot\vec a-\vec a\cdot\vec b+\vec b\cdot\vec a-\vec b\cdot\vec b=0$ $[\because \vec a\cdot\vec b=\vec b\cdot\vec a]$

or, $\vec a^2-\vec a\cdot\vec b+\vec a\cdot\vec b-\vec b^2=0$

or, $|\vec a|^2-|\vec b|^2=0$

or, $|\vec a|^2=|\vec b|^2$

∴ $|\vec a|=|\vec b|$ Hence proved.

(b) Here, given that

$|\vec a|=a=1$ and $|\vec b|=b=1$

Now, we have

$|\vec a-\vec b|^2=(\vec a-\vec b)^2$ $[\because |\vec a|^2=\vec a^2]$

$=\vec a^2-2\vec a\cdot\vec b+\vec b^2$

$=a^2-2\vec a\cdot\vec b+b^2$

$=1-2|\vec a||\vec b|\cos\theta+1$

$=2-2\cdot 1\cos\theta$

$=2(1-\cos\theta)$

$=2\cdot 2\sin^2\frac{\theta}{2}$

or, $|\vec a-\vec b|^2=(2\sin\frac{\theta}{2})^2$

or, $|\vec a-\vec b|=2\sin\frac{\theta}{2}$

∴ $\frac{1}{2}|\vec a-\vec b|=\sin\frac{\theta}{2}$ Hence proved.

(c) Let $\vec a=(x_1,y_1,z_1)$ and $\vec b=(x_2,y_2,z_2)$ be any two vectors. Then

$\vec a\cdot\vec b=(x_1,y_1,z_1)\cdot(x_2,y_2,z_2)$

$=x_1x_2+y_1y_2+z_1z_2$

$=x_2x_1+y_2y_1+z_2z_1$

$=(x_2,y_2,z_2)\cdot(x_1,y_1,z_1)$

$=\vec b\cdot\vec a$

i.e. $\vec a\cdot\vec b=\vec b\cdot\vec a$

Hence, scalar product of two vectors [illegible] proved.

7. (a) If $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$, prove that $|\vec{a}|=|\vec{b}|$.

(b) If θ is the angle between two unit vectors $\vec{a}$ and $\vec{b}$, show that $\frac{1}{2}|\vec{a}-\vec{b}|=\sin\frac{\theta}{2}$

(c) Show that the scalar product of two vectors is commutative.

Soln: (a) Here,

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$$

or, $\vec{a}.\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}.\vec{b}=0$ $\quad [\because \vec{a}.\vec{b}=\vec{b}.\vec{a}]$

or, $\vec{a}^2-\vec{a}.\vec{b}+\vec{a}.\vec{b}-\vec{b}^2=0$

or, $|\vec{a}|^2-|\vec{b}|^2=0$

or, $|\vec{a}|^2=|\vec{b}|^2$

$\therefore$ $|\vec{a}|=|\vec{b}|$ **Hence roved.**

(b) Here, given that

$|\vec{a}|=a=1$ and $|\vec{b}|=b=1$

Now, we have $\quad [\because |\vec{a}|^2=\vec{a}^2]$

$$|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})^2$$
$$=\vec{a}^2-2\vec{a}.\vec{b}+\vec{b}^2$$
$$=a^2-2\vec{a}.\vec{b}+b^2$$
$$=1-2|\vec{a}||\vec{b}|\cos\theta+1$$
$$=2-2.1.1\cos\theta$$
$$=2(1-\cos\theta)$$
$$=2.(2\sin^2\frac{\theta}{2})$$

or, $|\vec{a}-\vec{b}|^2=(2\sin\frac{\theta}{2})^2$

or, $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

$\therefore$ $\frac{1}{2}|\vec{a}-\vec{b}|=\sin\frac{\theta}{2}$ **Hence Proved.**

(c) Let, $\vec{a}=(x_1, y_1, z_1)$ and $\vec{b}=(x_2, y_2, z_2)$ be any two space vectors then

$$\vec{a}.\vec{b}=(x_1, y_1, z_1).(x_2, y_2, z_2)$$
$$=x_1x_2+y_1y_2+z_1z_2$$
$$=x_2x_1+y_2y_1+z_2z_1$$
$$=(x_2, y_2, z_2).(x_1, y_1, z_1)$$
$$=\vec{b}.\vec{a}$$

i.e. $\vec{a}.\vec{b}=\vec{b}.\vec{a}$

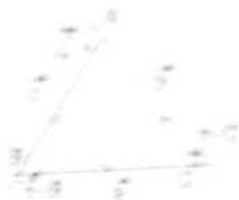
Hence, scalar product of two vectors is commutative. **Proved.**

8. (a) In any triangle ABC, Prove vectorially that $c=a\cos B+b\cos A$ [3]

(b) Show that the vectors $2\vec{i}+3\vec{j}-8\vec{k}$ and $2\vec{i}+4\vec{j}+2\vec{k}$ are orthogonal. [2]

Soln: (a) In ABC, let $\vec{BC}=\vec{a}$, $\vec{CA}=\vec{b}$ and $\vec{AB}=\vec{c}$

Now, by using triangle law of vector addition



$$\vec{AB}=\vec{AC}+\vec{CB}$$

or, $\vec{c}=-\vec{CA}-\vec{BC}$

or, $\vec{c}=-\vec{b}-\vec{a}$

Taking scalar product on both sides by $\vec{c}$, we get

$$\vec{c}.\vec{c}=-\vec{b}.\vec{c}-\vec{a}.\vec{c}$$

or, $\vec{c}^2=-bc\cos(\pi-A)-ac\cos(\pi-B)$

or, $c^2=bc\cos A+ac\cos B$

or, $c^2=c(b\cos A+a\cos B)$

$\therefore$ $c=a\cos B+b\cos A$ **Hence Proved.**

(b) Let, $\vec{a}=2\vec{i}+3\vec{j}-8\vec{k}$ and

$\vec{b}=2\vec{i}+4\vec{j}+2\vec{k}$

Now, $\vec{a}.\vec{b}=(2\vec{i}+3\vec{j}-8\vec{k}).(2\vec{i}+4\vec{j}+2\vec{k})=4+12-16=0$

Hence, $\vec{a}.\vec{b}=0$ which implies that $\vec{a}$ and $\vec{b}$ are orthogonal. **Proved.**

9. (a) Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$. [2]

(b) In any triangle, prove vectorially that $a^2=b^2+c^2-2bc\cos A$ [3]

Soln: (a) Here, given that

Magnitude of two vectors $\vec{a}$ and $\vec{b}$ is same

So, $|\vec{a}|=|\vec{b}|$

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\theta=60° \text{ and } \vec{a}.\vec{b}=\frac{1}{2}$$

Now, we have,

$$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

or, $\frac{1}{2}=|\vec{a}||\vec{a}|\cos 60°$ $\quad [\because |\vec{a}|=|\vec{b}|]$

or, $\frac{1}{2}=|\vec{a}|^2.\frac{1}{2}$

or, $|\vec{a}|^2=1$

$\therefore$ $|\vec{a}|=\pm 1$

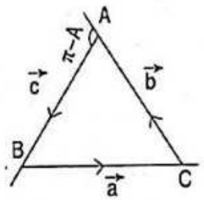
Since, magnitude of vectors is not negative,

So, $|\vec{a}| = 1$

$\therefore$ $|\vec{a}| = |\vec{b}| = 1$ **Ans.**

(b) In triangle ABC, let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

Now, by using triangle law of vector addition



$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

or, $\overrightarrow{BC} = -\overrightarrow{AB} - \overrightarrow{CA}$

or, $\vec{a} = -\vec{c} - \vec{b}$

or, $\vec{a} = -(\vec{b} + \vec{c})$

Squaring both sides, we get

$(\vec{a})^2 = \{-(\vec{b} + \vec{c})\}^2$

or, $a^2 = (\vec{b} + \vec{c})^2$

or, $a^2 = \vec{b}^2 + 2\vec{b}.\vec{c} + \vec{c}^2$

or, $a^2 = b^2 + 2bc\cos(\pi - A) + c^2$

or, $a^2 = b^2 - 2bc\cos A + c^2$ $[\because \cos(\pi - A) = -\cos A]$

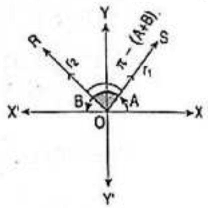
$\therefore$ $a^2 = b^2 + c^2 - 2bc\cos A$. **Hence Proved.**

10. ***Define scalar product of two vectors.***

Prove by vector method that, cos (A + B) = cos A cos B − sinA sinB [3]

Soln: Scalar product of two vectors: Scalar (dot) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined by $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$. Where, a and b are the magnitude of the vectors $\vec{a}$ and $\vec{b}$, θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Second part:



Let, XOX' and YOY' be two mutually perpendicular straight lines representing X-axis and Y-axis respectively.

Let, $\angle XOS = A$ and $\angle ROX' = B$ so that $\angle SOR = \pi - (A + B)$.

Also let $|\overrightarrow{OS}| = OS = r_1$ and $|\overrightarrow{OR}| = OR = r_2$, then the coordinates of S and R are $(r_1\cos A, r_1\sin A)$ and $(-r_2\cos B, r_2\sin B)$.

$\therefore$ $\overrightarrow{OS} = (r_1\cos A, r_1\sin A)$ and

$\overrightarrow{OR} = (-r_2\cos B, r_2\sin B)$

Now, $\overrightarrow{OS}.\overrightarrow{OR} = (r_1\cos A, r_1\sin A).(-r_2\cos B, r_2\sin B)$

$= -r_1r_2\cos A\cos B + r_1r_2\sin A\sin B$

$= -r_1r_2(\cos A\cos B - \sin A\sin B)$

Since, $\{\pi - (A + B)\}$ is the angle between $\overrightarrow{OS}$ and $\overrightarrow{OR}$

So,

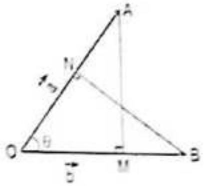
$$\cos\{\pi - (A + B)\} = \frac{\overrightarrow{OS}.\overrightarrow{OR}}{|\overrightarrow{OS}||\overrightarrow{OR}|}$$

or, $$-\cos(A + B) = \frac{-r_1r_2(\cos A.\cos B - \sin A\sin B)}{r_1r_2}$$

$\therefore$ $\cos(A + B) = \cos A.\cos B - \sin A.\sin B$. **Hence Proved.**

11. ***Give geometrical interpretation of scaler product of two vectors. Prove by vector method that cos (A − B) = cosA cosB + sinA sinB.*** [2+3=5]

Soln: 1st part: Geometrical Interpretation:



Let, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let, θ be the angle between the two vectors $\vec{a}$ and $\vec{b}$.

$AM \perp OB$ and $BN \perp OA$ are drawn.

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= ab\cos\theta$

$= (OA)(OB\cos\theta)$

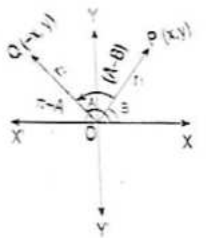
$= OA.\ ON$

$=$ (Magnitude of $\vec{a}$) . (Projection of $\vec{b}$ on $\vec{a}$)

Similarly $\vec{a}.\vec{b}$ = (Magnitude of $\vec{b}$). (projection of $\vec{a}$ on $\vec{b}$)

Hence the scalar product of two vectors is the product of the magnitude of one of the vectors and the projection of the second vector on the first.

2nd part:



Let, XOX' and YOY' be two mutually perpendicular straight lines representing X-axis and Y-axis respectively.

Let, $\angle XOP = B$ and $\angle XOQ = A$

So, that $\angle POQ = A - B$. Also, let $|\overrightarrow{OP}| = OP = r_1$ and $|\overrightarrow{OQ}| = r_2$, then the coordinates of P and Q are $(r_1\cos B, r_1\sin B)$ and $(r_2\cos(\pi - A), r_2\sin(\pi - A)) = (-r_2\cos A, r_2\sin A)$, since the point Q lies in second quadrant so, coordinates of Q will be $(r_2\cos A, r_2\sin A)$.

$\therefore$ $\overrightarrow{OP}.\overrightarrow{OQ} = (r_1\cos B, r_1\sin B).(r_2\cos A, r_2\sin A)$

$= r_1r_2\cos A\cos B + r_1r_2\sin A\sin B$

$= r_1r_2(\cos A\cos B + \sin A\sin B)$

Since, $(A - B)$ is the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, so

$$\cos(A - B) = \frac{\overrightarrow{OP}\cdot\overrightarrow{OQ}}{|\overrightarrow{OP}||\overrightarrow{OQ}|}$$

$$= \frac{r_1r_2(\cos A\cos B + \sin A\sin B)}{r_1r_2}$$

$\therefore$ $\cos(A - B) = \cos A\cos B + \sin A\sin B$. **Hence Proved.**

9.2 Vector (Cross) Product of Two Vectors

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Basic Formulae and Key Points

1. If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note: The elements of the vector product $\vec{a} \times \vec{b}$ of two space vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ can be obtained by the following rule

$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

$$a_2b_3 - a_3b_2 \quad a_3b_1 - a_1b_3 \quad a_1b_2 - a_2b_1$$

$\therefore \quad \vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

2. $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \vec{n} = ab \sin\theta\, \vec{n}$

Where, $\vec{n}$ is the unit vector normal to $\vec{a}$ and $\vec{b}$.

3. If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

4. Properties:

(a) $\vec{a} \times \vec{0} = \vec{0}$

(b) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(c) $(\vec{a} \times \vec{b}) . \vec{a} = 0$ and $(\vec{a} \times \vec{b}) . \vec{b} = 0$

(d) $\vec{a} \times \vec{b} = 0$, if and only if $\vec{a} \parallel \vec{b}$

(e) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

5. Unit Vectors:

(a) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$

(b) $\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$

(c) $\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{k} = -\vec{j}$

6. Area of Geometrical Figure Using Vectors

(i) The area of a parallelogram whose adjacent sides are represented by $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$

(ii) The area of the tringle OAB in which $\overrightarrow{OA} = \vec{a}$ and $= \overrightarrow{OB} = \vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$

(iii) The area of the parallelogram with diagonals $\vec{c}$ and $\vec{d}$ is $\frac{1}{2}|\vec{c} \times \vec{d}|$.

Note: (i) Vector perpendicular to $\vec{a}$ and $\vec{b} = \vec{a} \times \vec{b}$

(ii) Unit vector perpendicular to $\vec{a}$ and $\vec{b} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Group 'A' (Multiple Choice Questions and Answers)

1. *It is given that $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} \times \vec{b}| = |\vec{a} . \vec{b}|$. Which is the angle between $\vec{a}$ and $\vec{b}$?* [2081 Set-V]

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

2. *What is the value of $\vec{i}.(\vec{j} \times \vec{k}) + \vec{j}.(\vec{k} \times \vec{i}) - \vec{k}.(\vec{i} \times \vec{j})$? Where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along x-axis, y-axis and z-axis respectively.*

(a) –3 (b) –1 (c) 1 (d) 3

3. *Given $\vec{a} . \vec{b} = 48, |\vec{a}| = 15$ and $|\vec{b}| = 4$, what is the value of $|\vec{a} \times \vec{b}|$?*

(a) 12 (b) 36 (c) 48 (d) 60

4. *Consider $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ Which one of the following is parallel to $(\vec{b} - \vec{c})$?*

(a) $\vec{a} - \vec{b}$ (b) $\vec{a} - \vec{c}$ (c) $\vec{a} - \vec{d}$ (d) $\vec{c} \times \vec{d}$

5. *Cross product is also known as ....*

(a) scalar product (b) vector product

(c) dot product (d) Multiplication

6. *what will be the cross product of the vectors $2\vec{i} + 3\vec{j} + \vec{k}$ and $3\vec{i} + 2\vec{j} + \vec{k}$?*

(a) $\vec{i} + 2\vec{j} + \vec{k}$ (b) $2\vec{i} + 3\vec{j} + \vec{k}$

(c) $\vec{i} + \vec{j} - 5\vec{k}$ (d) $2\vec{i} - \vec{j} + 5\vec{k}$

7. *The angle between the vector $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ is*

(a) 0° (b) 90° (c) 180° (d) 270°

8. *If two vectors $\vec{a}$ and $\vec{b}$ are parallel then*

(a) $\vec{a} . \vec{b} = 0$ (b) $\vec{a} \times \vec{b} = 0$ (c) ab = 0 (d) $\vec{a} \times \vec{b} = 1$

9. *(0, 0, 1) × (1, 0, 0) is equal to*

(a) (0, 0, 0) (b) (1, 0, 0) (c) (0, 1, 0) (d) (0, 0, 1)

10. *Area of the parallelogram whose diagonals are represented by $\vec{a}$ and $\vec{b}$ is*

(a) $\vec{a} \times \vec{b}$ (b) $|\vec{a} \times \vec{b}|$ (c) $\frac{1}{2}|\vec{a} \times \vec{b}|$ (d) $\vec{a} . \vec{b}$

11. *ABCD is a parallelogram. Which one of the following represents area of the parallelogram?*
(a) Magnitude of vector product of two vectors along AB and BD.
(b) Magnitude of vector product of two vectors along AB and DC.
(c) Magnitude of vector product of two vectors along AC and BD.
(d) Magnitude of vector product of two vectors along AB and AD.

12. *If $\vec{a} = 2\vec{i}$ and $\vec{b} = 3\vec{j}$ where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along X, Y and Z-axes respectively, then the value of $\vec{b} \times \vec{a}$ is equal to*
(a) $-6\vec{k}$ (b) $6\vec{k}$ (c) $6\vec{i}$ (d) $6\vec{j}$

13. *Which one of the following is the angle between two vectors $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$?*
(a) 0° (b) 60° (c) 90° (d) 180°

14. *The value of $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$ is*
(a) $(a^2 - b^2)$ (b) $2(\vec{a} \times \vec{b})$ (c) 2ab (d) None

15. *The value of $\vec{a} \times \vec{a}$ is*
(a) 1 (b) -1 (c) 0 (d) $|\vec{a}|^2$

16. *What is $(2\vec{a} - 3\vec{b}) \times (2\vec{a} + 3\vec{b})$ equal to?*
(a) $\vec{0}$ (b) $\vec{a} \times \vec{b}$ (c) $12(\vec{a} \times \vec{b})$ (d) $4a^2 - 9b^2$

Answer Key

1. c	2. c	3. b	4. c	5. b	6. c	7. c	8. b	9. c	10. c
11. d	12. a	13. b	14. b	15. c	16. c				

Group 'B' or 'C' (Subjective Questions and Answers)

1. *Find the area of parallelogram whose diagonals are represented by the vectors $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $3\vec{i} - 5\vec{j} + 2\vec{k}$* [3] [2081 Set - V]

Soln: Let, $\vec{a} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\vec{c} = 3\vec{i} - 5\vec{j} + 2\vec{k}$

Now, $\vec{a} \times \vec{c} = \begin{vmatrix} 3 & -4 \\ -5 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -4 \\ 3 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix}\vec{k}$

$= (6 - 20)\vec{i} - (4 + 12)\vec{j} + (-10 - 9)\vec{k}$

$= -14\vec{i} - 16\vec{j} - 19\vec{k}$

$|\vec{a} \times \vec{c}| = \sqrt{(-14)^2 + (-16)^2 + (-19)^2}$

$= \sqrt{196 + 256 + 361} = \sqrt{813}$

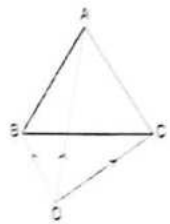
$\therefore$ The area of the parallelogram determined by the diagonal $\vec{a}$ and $\vec{c} = \frac{1}{2}|\vec{a} \times \vec{c}|$

$= \frac{1}{2} \times \sqrt{813}$

$= \frac{\sqrt{813}}{2}$ sq. units. **Ans.**

2. *Find the area of a triangle formed by the points whose position vectors are $2\vec{i} - \vec{j} + 3\vec{k}$, $\vec{i} - \vec{j} - 2\vec{k}$ and $\vec{i} + 2\vec{j} + 3\vec{k}$.* [3] [2081 Set - W]

Soln: Let, O be the origin and A, B, C be three points with given position vectors respectively. Then



$\vec{OA} = 2\vec{i} - \vec{j} + 3\vec{k}$

$\vec{OB} = \vec{i} - \vec{j} - 2\vec{k}$

$\vec{OC} = \vec{i} + 2\vec{j} + 3\vec{k}$

Now, $\vec{BC} = \vec{OC} - \vec{OB}$

$= \vec{i} + 2\vec{j} + 3\vec{k} - \vec{i} + \vec{j} + 2\vec{k}$

$= 3\vec{j} + 5\vec{k}$

$\vec{BA} = \vec{OA} - \vec{OB}$

$= 2\vec{i} - \vec{j} + 3\vec{k} - \vec{i} + \vec{j} + 2\vec{j} = \vec{i} + 5\vec{k}$

$\vec{BC} \times \vec{BA} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 5 \\ 1 & 0 & 5 \end{vmatrix}$

$= \begin{vmatrix} 3 & 5 \\ 0 & 5 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 5 \\ 1 & 5 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}\vec{k}$

$= (15 - 0)\vec{i} - (0 - 5)\vec{j} + (0 - 3)\vec{k} = 15\vec{i} + 5\vec{j} - 3\vec{k}$

Also, $|\vec{BC} \times \vec{BA}| = \sqrt{15^2 + 5^2 + (-3)^2}$

$= \sqrt{225 + 25 + 9} = \sqrt{259}$

$\therefore$ The area of a triangle $= \frac{1}{2}|\vec{BC} \times \vec{BA}| = \frac{1}{2}\sqrt{259}$ sq. units. **Ans.**

3. *If $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k}$*

(a) *Find a unit vector perpendicular to $\vec{a}$ and $\vec{b}$.* [2]

(b) *Find the sine of the angle between $\vec{a}$ and $\vec{b}$.* [1]

(c) *Prove that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.* [2]

Soln: (a) Here, given $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}\vec{k}$

$= 8\vec{i} - 8\vec{j} - 8\vec{k}$

Also, $|\vec{a} \times \vec{b}| = \sqrt{8^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$

Now,

$$\text{Unit vector perpendicular to } \vec{a} \text{ and } \vec{b} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$= \frac{8\vec{i} - 8\vec{j} - 8\vec{k}}{8\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$$

(b) Here, $|\vec{a}| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{2^2+(-2)^2+4^2} = 2\sqrt{6}$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14}\times 2\sqrt{6}} = \frac{4}{2\sqrt{7}} = \frac{2}{\sqrt{7}} \text{ Ans.}$$

(c) We know, $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, if $(\vec{a}\times\vec{b}).\vec{a} = 0$ and $(\vec{a}\times\vec{b}).\vec{b} = 0$.

Now, $(\vec{a}\times\vec{b}).\vec{a} = (8\vec{i} - 8\vec{j} - 8\vec{k}).(3\vec{i} + \vec{j} + 2\vec{k})$

$= 24 - 8 - 16$

$= 0$

Again,

$(\vec{a}\times\vec{b}).\vec{b} = (8\vec{i} - 8\vec{j} - 8\vec{k}).(2\vec{i} - 2\vec{j} + 4\vec{k})$

$= 16 + 16 - 32$

$= 0$ **Hence Proved.**

4. *If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, prove that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.* [3]

Soln: We have,

$\vec{a} + \vec{b} + \vec{c} = 0$

or, $\vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = 0$ [$\because$ Taking cross product on both sides by $\vec{a}$]

or, $0 + \vec{a}\times\vec{b} - \vec{c}\times\vec{a}$ [$\because \vec{a}\times\vec{a} = 0$]

$\therefore$ $\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ .... (i)

Again, $\vec{a} + \vec{b} + \vec{c} = 0$

or, $\vec{a}\times\vec{b} + \vec{b}\times\vec{b} + \vec{c}\times\vec{b} = 0$ [$\because$ Taking cross product on both sides by $\vec{b}$]

or, $\vec{a}\times\vec{b} + 0 - \vec{b}\times\vec{c} = 0$

or, $\vec{a}\times\vec{b} - \vec{b}\times\vec{c} = 0$

$\therefore$ $\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ .....(ii)

Now, form (i) & (ii), $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$. **Hence Proved.**

5. *Find the area of parallelogram determined by the vectors $-\vec{i} - \vec{j} + \vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$.* [2]

Soln: Let, $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = -3\vec{i} - 2\vec{j} + \vec{k}$

$$\text{Now, } \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 1 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix}\vec{k}$$

$$= (-1+2)\vec{i} - (-1+3)\vec{j} + (-2+3)\vec{k}$$

$$= \vec{i} - 2\vec{j} + \vec{k}$$

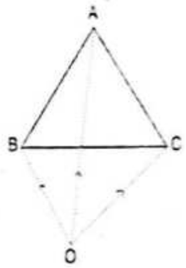
Now, Area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ is

$|\vec{a}\times\vec{b}| = \sqrt{1^2 + (-2)^2 + (1)^2} = \sqrt{6}$ sq. units **Ans.**

6. *Prove that area of triangle ABC $= \frac{1}{2}|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c} + \vec{a}|$.* [3]

Soln: Let, O be the origin, then

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$



Now,

By triangle law of vector addition,

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \vec{c} - \vec{b}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= \vec{a} - \vec{b}$

Now,

$$\overrightarrow{BC}\times\overrightarrow{BA} = (\vec{c} - \vec{b})\times(\vec{a} - \vec{b})$$

$$= \vec{c}\times\vec{a} - \vec{c}\times\vec{b} - \vec{b}\times\vec{a} + \vec{b}\times\vec{b}$$

$$= \vec{c}\times\vec{a} - \vec{c}\times\vec{b} - \vec{b}\times\vec{a} + \vec{b}\times\vec{b}$$

$$= \vec{c}\times\vec{a} + \vec{b}\times\vec{c} + \vec{a}\times\vec{b} + 0$$

$$= \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$$

$\therefore$ Vector area of $\triangle ABC = \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$

$= \frac{1}{2}|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|$ **Hence Proved.**

7. (a) *Define cross product of two vectors.* [2]

(b) *Prove that* $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$ [3]

Soln: (a) **Cross product of two vectors:** The vector product is also known as cross product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \times \vec{b}$ is defined by $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n} = ab \sin\theta\, \hat{n}$ where, a and b are magnitudes of $\vec{a}$ and $\vec{b}$ respectively, θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

(b) If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$, then

$\vec{a} \cdot \vec{b} = ab \cos\theta$ ....(i) and

$\vec{a} \times \vec{b} = ab \sin\theta\, \hat{n}$ ......(ii)

Squaring and adding (i) & (ii) we get

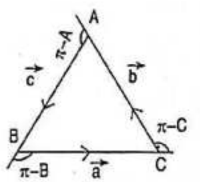
$$(\vec{a} \cdot \vec{b})^2 + (\vec{a} \times \vec{b})^2 = (ab \cos\theta)^2 + (ab \sin\theta\, \hat{n})^2$$
$$= a^2b^2 \cos^2\theta + a^2b^2 \sin^2\theta\, (\hat{n})^2$$
$$= a^2b^2 \cos^2\theta + a^2b^2 \sin^2\theta \qquad [\because \hat{n}^2 = 1]$$
$$= a^2b^2 (\sin^2\theta + \cos^2\theta)$$
$$= a^2b^2$$

$\therefore$ $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$. **Hence Proved.**

8. *Prove, in any triangle, by vector method that* $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ [3]

Soln: In triangle ABC, suppose that $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$

By triangle law of vector addition,



$$\vec{BC} = \vec{BA} + \vec{AC}$$

or, $\vec{a} = -\vec{AB} - \vec{CA}$

or, $\vec{a} = -\vec{c} - \vec{b}$

$\therefore$ $\vec{a} + \vec{b} + \vec{c} = 0$ ....(i)

Taking cross product with $\vec{a}$ in (i), we get

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

or, $0 + \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = 0$ $\qquad [\because \vec{a} \times \vec{a} = 0]$

or, $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ....(ii)

Also, taking cross product with $\vec{b}$ in (i), we get

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{b} + \vec{c} \times \vec{b} = 0$$

or, $\vec{a} \times \vec{b} + 0 - \vec{b} \times \vec{c} = 0$

$\therefore$ $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ....(iii)

From (ii) and (iii), we get

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ .... (iv)

Taking modulus on each side on (iv)

$$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

or, $ab \sin(\pi - C) = bc \sin(\pi - A) = ca \sin(\pi - B)$

or, $ab \sin C = bc \sin A = ca \sin B$

or, $\dfrac{ab \sin C}{abc} = \dfrac{bc \sin A}{abc} = \dfrac{ca \sin B}{abc}$

$\therefore$ $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. **Hence Proved.**

9. *Prove that the area of triangle ABC whose vertices are A (1, 2, 3), B (2, 5, –1) and C (–1, 1, 2) is* $\dfrac{\sqrt{155}}{2}$ *sq. units.* [3]

Soln: Let, O be the origin, then

$\vec{OA} = (1, 2, 3)$

$\vec{OB} = (2, 5, -1)$

$\vec{OC} = (-1, 1, 2)$

Now, $\vec{AB} = \vec{OB} - \vec{OA} = (2, 5, -1) - (1, 2, 3) = (2-1, 5-2, -1-3) = (1, 3, -4)$

$\vec{AC} = \vec{OC} - \vec{OA} = (-1, 1, 2) - (1, 2, 3) = (-1-1, 1-2, 2-3) = (-2, -1, -1)$

For, $\vec{AB} \times \vec{AC}$

$$\begin{array}{cccccc} 1 & 3 & -4 & 1 & 3 \\ -2 & -1 & -1 & -2 & -1 \end{array}$$

$= (-3-4,\ 8+1,\ -1+6) = (-7, 9, 5)$

$\therefore$ $\vec{AB} \times \vec{AC} = (-7, 9, 5)$

$|\vec{AB} \times \vec{AC}| = \sqrt{(-7)^2 + 9^2 + 5^2} = \sqrt{49 + 81 + 25} = \sqrt{155}$

$\therefore$ Area of ΔABC determined by $\vec{AB}$ and $\vec{AC}$ is

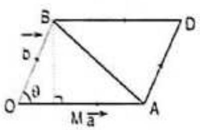
$\dfrac{1}{2}|\vec{AB} \times \vec{AC}| = \dfrac{1}{2} \times \sqrt{155} = \dfrac{\sqrt{155}}{2}$ sq. units. **Ans.**

10. (a) *Give the geometrical interpretation of the vector(cross) product of two vectors.* [2]

(b) *If* $\vec{a}, \vec{b}, \vec{c}$ *are three non-zero vectors, prove that:* $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. [2]

Soln: (a) **Geometrical Interpretation:**

Let, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and ∠AOB = θ. Complete the parallelogram OADB.



Let, us draw BM ⊥ OA.

Now,

Area of Parallelogram OADB = (OA) (BM) $\qquad$ [∵ Area of ▱ = base x height]

$= (OA)(OB \sin\theta)$

$= ab \sin\theta$

$= |\vec{a} \times \vec{b}|$

So, $|\vec{a} \times \vec{b}|$ is the area of the parallelogram having adjacent sides represented by $\vec{a}$ and $\vec{b}$.

(b) Let, $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$

Then,

$$\vec{b} + \vec{c} = (b_1, b_2, b_3) + (c_1, c_2, c_3)$$
$$= (b_1 + c_1, b_2 + c_2, b_3 + c_3)$$

Now,

$$\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

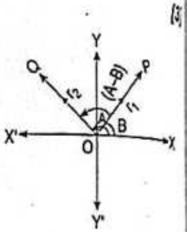
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad [\because \text{By the property of determinant}]$$

$\therefore \quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. **Hence Proved.**

11. *Prove by vector method that sin (A – B) = sin A cos B – cos A sin B.*

Soln: Let, XOX' and YOY' be two mutually perpendicular st. lines representing X-axis and Y-axis respectively. Let, $\angle XOQ = A$ and $\angle XOP = B$, so that $\angle POQ = A - B$. If $OP = r_1$ and $OQ = r_2$, then the coordinates of P and Q are $(r_1 \cos B, r_1 \sin B)$ and $(r_2 \cos A, r_2 \sin A)$.



$\therefore \quad \overrightarrow{OP} = (r_1 \cos B, r_1 \sin B) = (r_1 \cos B, r_1 \sin B, 0)$

$\therefore \quad \overrightarrow{OQ} = (r_2 \cos A, r_2 \sin A) = (r_2 \cos A, r_2 \sin A, 0)$.

Now, for $\overrightarrow{OP} \times \overrightarrow{OQ}$

$$\begin{array}{cccccc} r_1 \cos B & r_1 \sin B & 0 & r_1 \cos B & r_1 \sin B \\ r_2 \cos A & r_2 \sin A & 0 & r_2 \cos A & r_2 \sin A \end{array}$$

$$\overrightarrow{OP} \times \overrightarrow{OQ} = (0, 0, r_1 r_2 \sin A \cos B - r_1 r_2 \cos A \sin B)$$

$$|\overrightarrow{OP} \times \overrightarrow{OQ}| = \sqrt{0^2 + 0^2 + \{r_1 r_2 (\sin A \cos B - \cos A \sin B)\}^2}$$

$\therefore \quad |\overrightarrow{OP} \times \overrightarrow{OQ}| = r_1 r_2 (\sin A \cos B - \cos A \sin B)$

Since, (A – B) is the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, so

$$\sin (A - B) = \frac{|\overrightarrow{OP} \times \overrightarrow{OQ}|}{|\overrightarrow{OP}|\ |\overrightarrow{OQ}|}$$

$$= \frac{r_1 r_2 (\sin A \cos B - \cos A \sin B)}{r_1 r_2}$$

$\therefore \quad \sin (A - B) = \sin A \cos B - \cos A \sin A$ **Hence Proved.**

Unit-wise Model Questions For Trigonometry, Analytic Geometry and Vectors

Set 1

Group 'A'

3. In $\triangle ABC$, $\angle A = 30°$, $\angle B = 45°$, which one of the following is a:c?

(a) $\frac{\sqrt{2}}{\sqrt{3}+1}$ (b) $\frac{3+1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\frac{2\sqrt{2}}{\sqrt{3}-1}$ [Ans: a]

4. Which one of the following has transverse axis and conjugate axis?

(a) $y^2 - 4y - 4x + 4 = 0$ (b) $2y^2 - 3x^2 - 6 = 0$
(c) $2y^2 + 3x^2 - 6 = 0$ (d) $2x^2 + 2y^2 = 72$ [Ans: b]

5. It is given that $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} \times \vec{b}| = |\vec{a} . \vec{b}|$. What is the angle between $\vec{a}$ and $\vec{b}$?

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$ [Ans: c]

Group 'B'

14. (a) If $\frac{1}{p+r} = \frac{3}{p+q+r} - \frac{1}{q+r}$ in a triangle PQR, prove that $\angle R = 60°$. [3]

(b) Find the eccentricity and foci of the ellipse

$9x^2 + 4y^2 - 18x - 16y - 11 = 0$ [2] [Ans: $\frac{\sqrt{5}}{3}$, $(1, 2 \pm \sqrt{5})$]

15. (a) Find the equations of tangent and normal to the circle $x^2 + y^2 = 13$ at the point (2, 3). [3]

[Ans: $2x+3y = 13$, $3x-2y = 0$]

(b) In a rhombus, two of the diagonals are perpendicular to each other. Verify it by taking vector dot product of two vectors. [2]

Group 'C'

21. (a) Find the equation of the parabola whose focus is at the point (–3, 4) and the directrix is $2x + 5 = y$.

[3] [Ans: $x^2 + 4y^2 + 4xy + 10x - 30y + 100 = 0$]

(b) Find the area of parallelogram whose diagonals are represented by the vectors $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $3\vec{i} - 5\vec{j} + 2\vec{k}$.

[3] [Ans: $\frac{\sqrt{813}}{2}$ sq.units]

(c) In a triangle ABC, $a = 2$, $b = \sqrt{6}$ and $\angle A = 45°$. Solve the triangle.

[2] [Ans: $B = 60°$, $C = 75°$ & $C = \sqrt{3}+1$ Or, $B = 120°$, $C = 15°$, C & $\sqrt{3} - 1$]

Set 2

Group 'A'

3. In a triangle ABC, $a = 1$, $b = \sqrt{3}$ and $\angle C = 30°$. Which one of the following is the type of triangle?

(a) isosceles and obtuse-angled (b) equilateral
(c) right-angled (d) isosceles triangle [Ans: a]

4. If a conic section has eccentricity (e) = $\frac{\sqrt{a^2+b^2}}{a}$, what is the equation of that conic section?

(a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(c) $x^2+y^2=a^2$ (d) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ [Ans: a]

5. Which one of the following is the angle between two vectors $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$?

(a) 0° (b) 60° (c) 90° (d) 180° [Ans: b]

Group 'B'

14. (a) If sin C. sin (A – B) = sin A sin (B – C), prove that a^2, b^2, c^2 are in A.P. [3]

(b) Find the equation of ellipse whose major axis is twice the minor axis and passes through the point (1, 0). [2] [Ans: $x^2+4y^2=1$]

15. (a) Does a conic, $y^2=12x$ have two tangents from the point (6, 9)? Justify it with calculation. [3] [Ans: Yes]

(b) The dot product of two non-zero vectors gives a positive real number. Justify it with example. [2]

Group 'C'

21. (a) Find the equation of tangents to the circle $x^2+y^2=25$ drawn through the point (13, 0). [3] [Ans: $12y=\pm 5(x-13)$]

(b) If three sides of a triangle are proportional to $2:\sqrt{6}:\sqrt{3}+1$, find the angles. [2] [Ans: A = 45°, B = 60°, C = 75°]

(c) Find the area of a triangle formed by the points whose position vectors are $2\vec{i}-\vec{j}+3\vec{k}$, $\vec{i}-\vec{j}-2\vec{k}$ and $i+2\vec{j}+3\vec{k}$. [3] [Ans: $\frac{1}{2}\sqrt{259}$ sq. units]

Set 3

Group 'A'

3. In a ΔABC, sinA : sinB : sinC : : 5 : 12 : 13, what type of triangle is ABC?

(a) Acute angled (b) Obtuse angled

(c) Right angled (d) Right angled isosceles [Ans: c]

4. Which one of the following is equation of tangent to the circle $3x^2+3y^2=48$ at a point (a, b)?

(a) $3a^2+3b^2=48$ (b) $ax+by=16$

(c) $ax+by=48$ (d) $ay-bx=0$ [Ans: b]

5. It is given that $\vec{a}$ and $\vec{b}$ are non-zero vectors and $\frac{\theta}{2}$ be the angle between $\vec{a}$ and $\vec{b}$. Which one of the following represents $\vec{a}.\vec{b}$?

(a) ab cosθ (b) ab sinθ

(c) ab cos$\frac{\theta}{2}$ (d) ab sin$\frac{\theta}{2}$ [Ans: c]

Group 'B'

14. (a) In ΔABC, A = 45°, B = 75°, prove that $a+c\sqrt{2}-2b=0$. [3]

(b) Vector product of two vectors are parallel if their vector product gives null vector. Justify with example. [2]

15. (a) Find the vertex and focus of the parabola $x^2+12y=0$. [2] [Ans: (0, 0), (0, –3)]

(b) For the ellipse $9x^2+4y^2=36$, find the eccentricity and foci. [3] [Ans: $\frac{\sqrt{5}}{3}$, $(0,\pm\sqrt{5})$]

Group 'C'

21. (a) In a ΔABC, prove that (b sinB – c cosA.sinB) = (a sinA – c sinA.cosB). [2]

(b) Find the area of parallelogram whose diagonals are represented by the vectors $\vec{i}+\vec{j}-3\vec{k}$ and $3\vec{i}-3\vec{j}+\vec{k}$. [3] [Ans: $\sqrt{14}$ Sq. unit]

(c) For the hyperbola given by $9x^2-16y^2=144$, find the latus rectum and equations of directrices. [3] [Ans: $\frac{9}{2}$, $5x=\pm16$]

Set 4

Group 'A'

3. In any ΔABC, 2cosA.sinC = sinB. Which one is the type of ABC?

(a) Scalene (b) Isosceles (c) Equilateral (d) Right angled [Ans: b]

4. The centre of the circle $x^2+y^2+4x-6y+5=0$ is

(a) (2, 3) (b) (–2, 3) (c) (2, –3) (d) (–2, –3) [Ans: b]

5. If $\vec{p}$ and $\vec{q}$ are like and parallel to each other then the angle between two vectors $\vec{p}$ and $\vec{q}$ is

(a) 270° (b) 90° (c) 45° (d) 0° [Ans: d]

Group 'B'

14. (a) State and prove Sine Law. [3]

(b) In any ΔPQR, prove that, [2]
p(sinQ – sinR) + q(sinR – sinP) + r(sinP – sinQ) = 0

15. (a) Is the angle between the lines represented by $2\vec{i}-\vec{j}+\vec{k}$ and $\vec{i}-\vec{j}$ acute? Explain it. [2] [Ans: Yes, 30°]

(b) The dot product of two vector and cross product of two vectors are interrelated. Justify the statement. [2]

Group 'C'

21. (a) Find the equation of the parabola whose focus is at the (4, 5) and directrix is $3x+4y-5=0$. [4] [Ans: $16x^2+9y^2-170x-210y-24xy+1000=0$]

(b) Find the eccentricity and foci of the hyperbola $\frac{x^2}{36}-\frac{y^2}{25}=1$. [2] [Ans: $\frac{\sqrt{61}}{6}$, $(\pm\sqrt{61},0)$]

(c) Show that $2x^2+3y^2-12=0$ represent the locus of an ellipse. [2]

□□□

5. Statistics and Probability

Chapter 10

Correlation and Regression

10.1 Correlation

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Basic Formulae and Key Points

Karl Pearson's Correlation Coefficient:

Karl Pearson's correlation coefficient is denoted by r_{xy} or simply r and can be calculated using the following formulas depending on the available data:

1. $r = \dfrac{\text{Cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}}$

   Where, $\text{Cov}(X, Y) = \dfrac{1}{n}\Sigma(X-\overline{X})(Y-\overline{Y})$

2. $r = \dfrac{n\Sigma XY - \Sigma X\Sigma Y}{\sqrt{n\Sigma X^2-(\Sigma X)^2}\sqrt{n\Sigma Y^2-(\Sigma Y)^2}}$

3. $r = \dfrac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}$, Where, $X = X - \overline{X}$, $Y = Y - \overline{Y}$

4. $r = \dfrac{\Sigma xy}{n\sigma_x\sigma_y}$, Where, σ_x and σ_y are the standard deviations of X and Y series respectively.

5. $r = \dfrac{n\Sigma uv - \Sigma u\Sigma v}{\sqrt{n\Sigma u^2-(\Sigma u)^2}\sqrt{n\Sigma v^2-(\Sigma v)^2}}$

Notes:

- The coefficient of correlation lies between – 1 and 1.
- Correlation is symmetric for both variables x and y.
- For a positive correlation: the values increase together.
- For a negative correlation: one value decreases as the other increases.
- The coefficient of correlation is the geometric mean of the two regression coefficients i.e. $r = \pm\sqrt{b_{xy}\,.\,b_{yx}}$.
- If r = 1, there is a perfect positive correlation between the two variables.
- If r = – 1, there is a perfect negative correlation between the two variables.
- If r = 0, then the two variables are uncorrelated.

Rank Correlation:

When, we come across spearman's rank correlation, we may find two types of problems,

(i) When ranks are given

(ii) When ranks are not given

Rank Correlation $(R) = 1 - \dfrac{6\Sigma d^2}{n(n^2-1)}$

Repeated Rank:

Rank Correlation $(R) = 1 - \dfrac{6\left\{\Sigma d^2 + \dfrac{m_1(m_1^2-1)}{12} + \dfrac{m_2(m_2^2-1)}{12} + \ldots\right\}}{n(n^2-1)}$

Where, m_1 = Number of repetitions of first variate values

m_2 = Number of repetitions of second variate vales

Note: Rank correlation coefficient (R) also lies between –1 and +1 i.e. $-1 \le R \le 1$.

Group 'A' (Multiple Choice Questions and Answers)

1. *The correlation for the values of two variables moving in the same direction is*
   (a) Perfect Positive (b) Positive
   (c) Negative (d) No correlation
2. *The correlation for the values of two variables moving in the opposite direction is*
   (a) Positive (b) Negative (c) Linear (d) Non-linear
3. *Choose the correct example for positive correlation.*
   (a) Weight and income (b) Price and demand
   (c) The repayment period and EMI (d) Income and expenditure
4. *Which of the following would be considered a very strong negative correlation?*
   (a) 0.89 (b) – 0.89 (c) – 0.9 (d) 0.09
5. *The correlation coefficient is*
   (a) $r = \dfrac{\sigma_x\sigma_y}{\text{Cov}(x,y)}$ (b) $r = \dfrac{\text{Cov}(x,y)}{\sigma_x\sigma_y}$
   (c) $r = \dfrac{\text{Cov}(x,y)}{\sigma_y}$ (d) $r = \dfrac{\text{Cov}(x,y)}{\sigma_x}$
6. *Two variables x and y are not correlated if the correlation coefficient (r) between them is*
   (a) 0 (b) 1 (c) – 1 (d) 0.5
7. *Which of the following is not a possible value of the correlation coefficient?*
   (a) – 0.9 (b) 0 (c) 0.15 (d) 1.5
8. *The correlation coefficient is used to determine:*
   (a) a specific value of the y-variable given a specific value of the x-variable
   (b) a specific value of the x-variable given a specific value of the y-variable
   (c) the strength of the relationship between the x and y variables
   (d) none of the above
9. *Calculate 'r' if $\Sigma x^2 = 114$, $\Sigma y^2 = 422$, $\Sigma xy = 174$.*
   (a) 0.201 (b) – 0.512 (c) 0.793 (d) 1
10. *From the following data calculate the correlation coefficient: $\Sigma xy = 120$, $\Sigma x^2 = 90$, $\Sigma y^2 = 640$.*
    (a) – 0.5 (b) 0.5 (c) 0.25 (d) 0.005
11. *If the covariance between x and y variables is 12.5 and variance of x and y are respectively 16.4 and 13.8. Find the coefficient of correlation between them.*
    (a) 0.38 (b) – 0.3808 (c) 0.8308 (d) 0.0038

12. *Find the correlation coefficient from the data:*

$\Sigma(X-\overline{X})^2 = 40$, $\Sigma(Y-\overline{Y})^2 = 63$ and $\Sigma(X-\overline{X})(Y-\overline{Y}) = 35$.

(a) 0.679 (b) 0.79 (c) 0.697 (d) 0.89

13. *For 10 pair of observations, $\Sigma d^2 = 120$. Find the value of the rank correlation coefficient.*

(a) 0.5 (b) 0.25 (c) 0.26 (d) 0.27

14. *State whether you would expect a positive, negative or no correlation in the weight of the load of trucks and their petrol consumption.*

(a) No correlation (b) Negative correlation
(c) Positive correlation (d) None

15. *State whether you would expect a positive, negative or no correlation in the age of husbands and wives*

(a) No correlation (b) Negative correlation
(c) Positive correlation (d) None

16. *The degree of relationship between two variables in terms of their performance is determined by*

(a) Spearman's rank correlation coefficient (b) Karl Pearson's correlation coefficient
(c) Scatter diagram (d) None of them

17. *If there is perfect disagreement between the marks in Geography and Statistics, then what would be the value of rank correlation coefficient?*

(a) Any value (b) Only 1 (c) Only −1 (d) 0

Explanation: Correlatión is 1 if the agreement between the two rankings is perfect. −1 if the disagreement between the two ranking is perfect. Thus, one ranking is the reverse of the other.

Answer Key

1. b	2. b	3. d	4. c	5. b	6. a	7. d	8. c	9. c	10. b
11. c	12. c	13. d	14. b	15. c	16. a	17. c			

Group 'B' (Subjective Questions and Answers)

1. *Calculating the coefficient of rank correlation between age (in yrs) and weight (in kg) of the following observations.* [3] [2080 GIE Set A]

Age in yrs (X)	12	14	16	18	20
Weight in kg (Y)	25	32	40	50	56

Soln:

Calculation of coefficient of rank correlation

Age in yrs. (X)	Weight in Kg (Y)	Rank of X (R_1)	Rank of Y (R_2)	$d = R_1 - R_2$	d^2
12	25	5	5	0	0
14	32	4	4	0	0
16	40	3	3	0	0
18	50	2	2	0	0
20	56	1	1	0	0
				$\Sigma d = 0$	$\Sigma d^2 = 0$

Here, $n = 5$, $\Sigma d^2 = 0$

Now, by formula,

$$\text{Coefficient of rank correlation (R)} = 1 - \frac{6\Sigma d^2}{n(n^2-1)} = 1 - \frac{6\times 0}{5(5^2-1)} = 1 - 0 = 1 \text{ Ans.}$$

2. *You are given the following data for two variables x and y:*

The number of data points, n = 15

The standard deviation of x, $\sigma_x = 3.2$

The standard deviation of y, $\sigma_y = 3.4$

The sum of the product of the deviations from the mean for x and y = $\Sigma(x-\bar{x})(y-\bar{y}) = 122$.

(a) *Write the formula to find the correlation coefficient r when the covariance between x and y is given.* [1]

(b) *Calculate the correlation coefficient r between two variables x and y using the given data.* [2]

(c) *What does the calculated correlation coefficient indicate about the relationship between x and y?* [1]

(d) *Where does r lie? Is your answer within the range?* [1]

Soln: (a) The formula to calculate correlation coefficient r when the covariance between two variables x and y is $r = \dfrac{\text{Cov}(x,y)}{\sqrt{\text{var}(x)}\sqrt{\text{var}(y)}}$

(b) $$\text{Correlation coefficient (r)} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\,\sigma_x\sigma_y} = \frac{122}{15\times 3.2\times 3.4} = 0.75$$

(c) Since, r = 0.75 which indicates a strong positive relationship between the variable x and y.

(d) The correlation coefficient r always lies between −1 and +1. Since, r = 0.75 which is within the range.

3. *If n = 10, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$.*

(a) *Under what condition are two variables x and y considered uncorrelated?* [1]

(b) *What does it indicate if two variables are uncorrelated?* [1]

(c) *Find the correlation coefficient between the two variables from the data given.* [2]

(d) *Interpret the result.* [1]

Soln: (a) Two variables x and y are considered uncorrelated if the correlation between them is zero. i.e. $(r_{xy}) = 0$.

(b) If two variables x and y are uncorrelated then it indicates that there is no linear relationship between the two variables x and y.

(c) Given, n = 10, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$,

$\Sigma Y^2 = 580$, $\Sigma XY = 415$

Correlation coefficient (r) = ?

Now,

$$\text{Correlation coefficient (r)} = \frac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{(n\Sigma Y)^2 - (\Sigma Y)^2}}$$

$$= \frac{10 \times 415 - 60 \times 60}{\sqrt{10 \times 400 - 60^2}\sqrt{10 \times 580 - 60^2}}$$

$$= \frac{4150 - 3600}{\sqrt{400}\sqrt{2200}}$$

$$= \frac{500}{20\sqrt{2200}}$$

$= 0.59$ **Ans.**

(d) Since, $r = 0.59$ which is positive. So, the correlation between two variables are positively correlated. This means that as one variable increases, the other variable tends to increase as well.

4. *From the data given in the table below:*

X	*10*	*12*	*20*	*?*	*16*	*14*
Y	*9*	*12*	*15*	*18*	*14*	*16*

(a) *Find the missing item of X series, If arithmetic mean of X (AM) = 15.* [2]

(b) *Calculate the correlation coefficient by Karl Pearson's method.* [3]

Soln: (a) Let, a be the missing item in x series. Then

$\Sigma X = 10 + 12 + 20 + a + 16 + 14 = 72 + a$

$n = 6$

Given that,

AM of X $(\overline{X}) = 15$

We know,

$$\text{Mean } (\overline{X}) = \frac{\Sigma X}{n}$$

or, $15 = \dfrac{72 + a}{6}$

or, $90 = 72 + a$

or, $a = 90 - 72$

$\therefore$ $a = 18$

(b) Calculation of correlation coefficient:

X	Y	X^2	Y^2	XY
10	9	100	81	90
12	12	144	144	144
20	15	400	225	300
18	18	324	324	324
16	14	256	196	224
14	16	196	256	224
$\Sigma X = 90$	$\Sigma Y = 84$	$\Sigma X^2 = 1420$	$\Sigma Y^2 = 1226$	$\Sigma XY = 1306$

Here, n = 6

$$\text{We have, correlation coefficient (r)} = \frac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$$

$$= \frac{6 \times 1306 - 90 \times 84}{\sqrt{6 \times 1420 - (90)^2}\sqrt{6 \times 1226 - (84)^2}}$$

$$= \frac{276}{\sqrt{420}\sqrt{300}}$$

$= 0.78$ **Ans.**

5. *The rank of 10 students of same batch in two subjects A and B are given below.*

Rank of A	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Rank of B	*6*	*7*	*5*	*10*	*3*	*9*	*4*	*1*	*8*	*2*

(a) *What will be the value of the sum of the rank differences (Σd) if the ranks are perfectly matched?* [1]

(b) *Calculate the Spearman's rank correlation coefficient from the data given.* [3]

(c) *What is the range of the rank correlation coefficient?* [1]

Soln: (a) If the ranks are perfectly matched then the sum of the rank differences (Σd) will be equal to 0.

(b) Calculation of Spearman's rank correlation coefficient:

Rank of A (R_1)	Rank of B (R_2)	$d = R_1 - R_2$	d^2
1	6	−5	25
2	7	−5	25
3	5	−2	4
4	10	−6	36
5	3	2	4
6	9	−3	9
7	4	3	9
8	1	7	49
9	8	1	1
10	2	8	64
		$\Sigma d = 0$	$\Sigma d^2 = 226$

Here, $n = 10$, $\Sigma d^2 = 226$

Rank correlation coefficient (R) =?

Now,

$$R = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6 \times 226}{10\,(10^2 - 1)}$$

$$= 1 - \frac{1356}{990}$$

$= 1 - 1.37$

$= -0.37$

(c) The range of the rank correlation coefficient is from − 1 to + 1.

6. *From the information given in the table below,*

Age of sisters (yrs)	35	45	60	80	90
Age of brothers (yrs)	30	25	55	70	85

(a) *Find the average ages of sisters and brothers.* [3]

(b) *Determine the degree of relationship between the ages of sisters and their brother from the data.* [2] [2080 Optional]

Soln: Calculation of correlation coefficient:

Sisters (X)	Brothers (Y)	$x = X - \overline{X}$	$y = Y - \overline{Y}$	x^2	y^2	xy
35	30	−27	−23	729	529	621
45	25	−17	−28	289	784	476
60	55	−2	2	4	4	−4
80	70	18	17	324	289	306
90	85	28	32	784	1024	896
$\Sigma X = 310$	$\Sigma Y = 265$			$\Sigma x^2 = 2130$	$\Sigma y^2 = 2630$	$\Sigma xy = 2295$

(a) Here, $n = 5$, $\Sigma X = 310$ & $\Sigma Y = 265$

Now, average age of sisters $(\overline{X}) = \frac{\Sigma X}{n} = \frac{310}{5} = 62$

Average age of brothers $(\overline{Y}) = \frac{\Sigma Y}{n} = \frac{265}{5} = 53$

(b) Now, we have $(r) = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}} = \frac{2295}{\sqrt{2130}\sqrt{2630}} = \frac{2295}{46.15 \times 51.28} = 0.97$ **Ans.**

> **Note:** Finding the degree of relationship between the two variables is equivalent to calculating the correlation between them.

7. *The following table shows the ages of husbands and their wives.*

Age of husbands (yrs)	*23*	*22*	*20*	*24*	*23*	*26*	*27*	*28*	*30*	*20*
Age of wives (yrs)	*20*	*18*	*23*	*20*	*21*	*21*	*22*	*24*	*25*	*26*

(a) *Determine how the ages of the husband and their wives are correlated.* [2]

(b) *Interpret the relationship between ages of husband and wives based on the calculated correlation value.* [3]

Soln: (a) Computation of correlation coefficient:

Age of husband (X)	Age of wife (Y)	X^2	Y^2	XY
23	20	529	400	460
22	18	484	324	396
20	23	400	529	460
24	20	576	400	480
23	21	529	441	483
26	21	676	441	546
27	22	729	484	594
28	24	784	576	672
30	25	900	625	750
20	26	400	676	520
$\Sigma X = 243$	$\Sigma Y = 220$	$\Sigma X^2 = 6007$	$\Sigma Y^2 = 4896$	$\Sigma XY = 5361$

Here,

The number of items (n) = 10

We know,

$$\text{Correlation coefficient } (r) = \frac{n\Sigma XY - \Sigma X \Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$$

$$= \frac{10 \times 5361 - 243 \times 220}{\sqrt{10 \times 6007 - (243)^2}\sqrt{10 \times 4896 - (220)^2}} = \frac{150}{\sqrt{1021}\sqrt{560}} = 0.198$$

(b) **Interpretation:** Since, $r = 0.198$, so there is low degree of relationship between the ages of husbands and their wives. **Ans.**

8. *Calculate Spearman's rank correlation from the following data:* [5]

X	*10*	*12*	*8*	*15*	*20*	*25*	*40*
Y	*15*	*10*	*6*	*25*	*16*	*12*	*8*

Soln: Calculation of Rank correlation coefficient

X	Y	Rank of X (R_1)	Rank of Y (R_2)	$d = R_1 - R_2$	d^2
10	15	6	3	3	9
12	10	5	5	0	0
8	6	7	7	0	0
15	25	4	1	3	9
20	16	3	2	1	1
25	12	2	4	−2	4
40	8	1	6	−5	25
				$\Sigma d = 0$	$\Sigma d^2 = 48$

Here, $n = 7$

Now, we have

$$\text{Spearman's Rank Correlation coefficient } (R) = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6 \times 48}{7(7^2 - 1)} = 1 - \frac{288}{336} = 1 - 0.857 = 0.143 \text{ Ans.}$$

9. *The marks obtained by 8 students in Mathematics and Physics are given below:* [5]

S. No.	1	2	3	4	5	6	7	8
Marks in Math	40	60	35	68	70	96	70	84
Marks in Physics	48	62	28	52	85	90	52	73

Find the rank correlation coefficient between the marks in Mathematics and Physics.

Soln:

Calculation of rank correlation coefficient

S.No.	Marks in Math (X)	Marks in Physics (Y)	R_1	R_2	$d = R_1 - R_2$	d^2
1	40	48	7	7	0	0
2	60	62	6	4	2	4
3	35	28	8	8	0	0
4	68	52	5	5.5	−0.5	0.25
5	70	85	3.5	2	1.5	2.25
6	96	90	1	1	0	0
7	70	52	3.5	5.5	−2	4
8	84	73	2	3	−1	1
					$\Sigma d = 0$	$\Sigma d^2 = 11.5$

Here, no. of items(n) = 8

Marks 70 in Mathematics is repeated 2 times,

So, $m_1 = 2$

Also, marks 52 in Physics is repeated 2 times,

So, $m_2 = 2$

Now, we have

$$R = 1 - \frac{6\left\{\Sigma d^2 + \frac{m_1(m_1^2-1)}{12} + \frac{m_2(m_2^2-1)}{12}\right\}}{n(n^2-1)}$$

$$= 1 - \frac{6\left\{11.5 + \frac{2(2^2-1)}{12} + \frac{2(2^2-1)}{12}\right\}}{8(8^2-1)}$$

$$= 1 - \frac{6\left\{11.5 + \frac{1}{12}\times 6 + \frac{1}{12}\times 6\right\}}{504}$$

$$= 1 - \frac{6\left\{11.5 + \frac{1}{2} + \frac{1}{2}\right\}}{504} = 1 - \frac{6\times 12.5}{504} = 1 - 0.148 = 0.852 \text{ Ans.}$$

10. (a) *Define correlation with an example.* [1]

(b) *Calculate the coefficient of correlation between X and Y series from the following data.* [2]

	x	y
Number of pairs of observation	15	15
Arithmetic mean	25	18
Standard deviation	3.01	3.03
$\Sigma(x-\overline{x})(y-\overline{y}) = 122$		

Soln: (a) Correlation is a statistical tool that helps to measure the degree of relationship between two variables. Two variables are said to be correlated when the value of one variable changes with the change in the value of the other variable.

For example: Relationship between amount of rainfall and yield of rice.

(b) Here,

No. of pair of observation (n) = 15

Standard deviation of x-series(σ_x) = 3.01

Standard deviation of y-series(σ_y) = 3.03

Coefficient of correlation (r) =?

We have,

$$\text{Correlation coefficient (r)} = \frac{\Sigma(x-\overline{x})(y-\overline{y})}{n\,\sigma_x\sigma_y}$$

$$= \frac{122}{15\times 3.01\times 3.03}$$

$$= \frac{122}{136.8}$$

$$= 0.89 \text{ Ans.}$$

10.2 Regression

Basic Formulae and Key Points

1. Regression Equation of y on x (Regression Line of x on y):

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

Where, $b_{yx} = \dfrac{n\Sigma xy - \Sigma x\,\Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$

2. Regression Equation of x on y (Regression Line of x on y):

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

Where, $b_{xy} = \dfrac{n\Sigma xy - \Sigma x\,\Sigma y}{n\Sigma y^2 - (\Sigma y)^2}$

Remember: $\overline{x}$ and $\overline{y}$ are the means of x-series and y-series respectively. Also, b_{yx} & b_{xy} are respectively regression coefficients of y on x and x on y.

3. The regression coefficients can be expressed in terms of correlation coefficient and standard deviation as follow:

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} \text{ and } b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

Some Important Notes:

(i) The range of regression coefficient is $-\infty$ to ∞.

(ii) The correlation coefficient r is the geometric mean of two regression coefficients b_{yx} and b_{xy}.

i.e. $r = \pm\sqrt{b_{yx}\times b_{xy}}$

(iii) The algebraic sign of two regression coefficients b_{yx} and b_{xy} and correlation coefficient are always same.

(iv) If the value of one of two regression coefficients is less than unity, then the value of other regression coefficient is greater than unity.

i.e. If $b_{yx} < 1$ then $b_{xy} > 1$.

(v) The product of two regression coefficients must be less or equal to 1. i.e. $b_{yx}.\,b_{xy} \le 1$.

(vi) If the correlation coefficient r = 0, then the two lines of regression are perpendicular to each other.

Group 'A' (Multiple Choice Questions and Answers)

1. *The degree of relationship between the two variables is measured by ....*

(a) correlation analysis (b) skewness
(c) regression analysis (d) dispersion

2. *To determine the height of a person when his weight is given is:*

(a) correlation problem (b) regression problem
(c) both (a) and (b) (d) none

3. *A process by which we estimate the value of dependent variable on the basis of one or more independent variables is called:*

(a) correlation (b) regression (c) residual (d) slope

4. *When b_{xy} is positive, then b_{yx} will be:*

(a) negative (b) zero (c) one (d) positive

5. *b_{yx} is called regression coefficient of*

(a) X on Y (b) Y on X (c) both (d) none

6. r, b_{yx} and b_{xy} all have .... sign.
   (a) different (b) same (c) both (d) none
7. *In the line $y = \frac{19}{12} - \frac{1}{4}x$, b_{yx} is equal to*
   (a) $\frac{19}{12}$ (b) $\frac{1}{4}$ (c) $-\frac{1}{4}$ (d) none
8. *In the equation $x = \frac{46}{9} - \frac{y}{3}$, b_{yx} is equal to*
   (a) $\frac{-y}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{3}$ (d) $\frac{46}{9}$
9. *The product of two regression coefficient is*
   (a) 0 (b) less than or equal to 1
   (c) 1 (d) greater than 1
10. *Two regression lines coincide when ....*
   (a) $r = 0$ (b) $r = 2$ (c) $r = +1$ (d) $r = -1$
11. *The correlation coefficient r is the .... of the two regressions coefficient b_{yx} and b_{xy}:*
   (a) A.M. (b) G.M. (c) H.M. (d) None
12. *Two lines of regression are $x + 2y = 5$ and $2x + 3y = 8$. Then mean of x and y are*
   (a) 1 and 2 (b) 3 and 4 (c) 2 and 3 (d) 1 and 4
13. *If $r = 0$ then the angle between the two lines of regression is:*
   (a) 0° (b) 30° (c) 60° (d) 90°

Answer Key

1. c	2. b	3. b	4. d	5. b	6. b	7. c	8. b	9. b	10. c
11. b	12. a	13. d							

Group 'B' (Subjective Questions and Answers)

1. *The equations of two regression lines are $3X + 4Y = 65$, $3X + Y = 32$. Find,*
   (a) *the mean of X and the mean of Y.* [2]
   (b) *the regression coefficients.* [2]
   (c) *the correlation coefficient of X and Y.* [1]

Soln: (a) Let, $\overline{X}$ and $\overline{Y}$ be the means of X and Y series respectively. Then,

$$3\overline{X} + 4\overline{Y} = 65 \text{ ....(i)} \qquad [\because \text{Two lines of regression intersect at } (\overline{X}, \overline{Y})]$$

$$3\overline{X} + \overline{Y} = 32 \text{ ....(ii)}$$

Solving (i) & (ii), we get

$$\begin{array}{r} 3\overline{X} + 4\overline{Y} = 65 \\ 3\overline{X} + \overline{Y} = 32 \\ (-) \quad (-) \quad (-) \\ \hline 3\overline{Y} = 33 \end{array}$$

or, $\overline{Y} = 11$

Substituting the value of $\overline{Y}$ in (i), we get

$$3\overline{X} + 4 \times 11 = 65$$

or, $3\overline{X} = 21$

or, $\overline{X} = 7$

$\therefore$ $\overline{X} = 7, \overline{Y} = 11$

(b) Let, the regression line of Y on X be $3X + 4Y = 65$,

Then,

$$4Y = -3X + 65$$

or, $Y = -\frac{3}{4}X + \frac{65}{4}$

$\therefore$ $b_{yx} = -\frac{3}{4}$

Again, the regression line of X on Y be $3X + Y = 32$,

Then,

$$3X = -Y + 32$$

or, $X = \frac{-1}{3}Y + \frac{32}{3}$

$\therefore$ $b_{xy} = -\frac{1}{3}$

Since, the product of two regression coefficients must be less than or equal to 1. So, we have

$$b_{xy}.\, b_{yx} = \left(\frac{-1}{3}\right)\left(\frac{-3}{4}\right)$$

$$= \frac{1}{4} < 1$$

Hence, our assumption is correct.

(c) We have,

$$b_{yx} = -\frac{3}{4}$$

$$b_{xy} = -\frac{1}{3}$$

Now, $r = \pm\sqrt{b_{yx} \times b_{xy}}$

$$= -\sqrt{\left(\frac{-3}{4}\right)\left(\frac{-1}{3}\right)}$$

$$= -\frac{1}{2} \text{ Ans.}$$

Answer Key

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6. b	7. c	8. b

… series respectively. Then,

Two lines of regression intersect at ($\bar{X}$, $\bar{Y}$)

2. *From the given data,*

Variable	x	y
Mean	6	9
Standard deviation	4	6
Coefficient of correlation = $\frac{2}{3}$		

(a) *Write two regression coefficients in terms of correlation coefficient and standard deviation and calculate them from the information given in the table.* [2]

(b) *Find the regression equation of x on y.* [1]

(c) *Find the regression equation of y on x.* [1]

(d) *Verify whether the product of b_{yx} and b_{xy} is less or equal to 1.* [1]

Soln: (a) The two regression coefficients can be expressed in terms of correlation coefficient and standard deviation as follows:

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} \text{ and } b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

Where, r = correlation coefficient

σ_x = Standard deviation of x-series

σ_y = Standard deviation of y-series

Also, we have

$$\overline{x} = 6, \overline{y} = 9, \sigma_x = 4, \sigma_y = 6 \text{ \& } r = \frac{2}{3}$$

Now,

$$b_{xy} = r \times \frac{\sigma_x}{\sigma_y} = \frac{2}{3} \times \frac{4}{6} = \frac{4}{9}$$

$$b_{yx} = r \times \frac{\sigma_y}{\sigma_x} = \frac{2}{3} \times \frac{6}{4} = 1$$

(b) The regression equation of x on y is,

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

or, $x - 6 = \frac{4}{9}(y - 9)$

or, $9x - 54 = 4y - 36$

or, $9x - 4y = -36 + 54$

or, $9x - 4y = 18$

or, $x = \frac{4}{9}y + \frac{18}{9}$

$\therefore$ $x = \frac{4}{9}y + 2$

(c) The regression equation of y on x is

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

or, $y - 9 = 1(x - 6)$

or, $y - 9 = x - 6$

or, $y = x - 6 + 9$

$\therefore$ $y = x + 3$ **Ans.**

(d) Here, we have

$$b_{yx} = 1 \text{ and } b_{xy} = \frac{4}{9}$$

Now, $b_{yx} \times b_{xy} = 1 \times \frac{4}{9} = \frac{4}{9} = 0.44 < 1$. **Hence verified.**

3. *The regression coefficients, $b_{xy} = 1.5$, $b_{yx} = 0.65$ and arithmetic means $\overline{X} = 36$, $\overline{Y} = 52$.*

(a) *Find the regression equation of Y on X.*

(b) *Find the regression equation of X on Y.*

(c) *Find the estimated value of Y when X = 60.*

(d) *What do we call the variable whose value is to be predicted?*

(e) *Find the correlation coefficient between the two variables.*

Soln: (a) Here, $b_{xy} = 1.5$ $\quad b_{yx} = 0.65$

$$\overline{X} = 36 \qquad \overline{Y} = 52$$

Now, the regression equation of Y on X is

$$Y - \overline{Y} = b_{yx}(X - \overline{X})$$

or, $Y - 52 = 0.65(X - 36)$

or, $Y = 0.65X - 23.4 + 52$

$\therefore$ $Y = 0.65X + 28.6$

(b) The regression equation of X on Y is

$$X - \overline{X} = b_{xy}(Y - \overline{Y})$$

or, $X - 36 = 1.5(Y - 52)$

or, $X = 1.5Y - 78 + 36$

$\therefore$ $X = 1.5Y - 42$

(c) The estimated value of Y when X = 60 should be calculated from the line Y on X.

When, X = 60; from (a) We have,

$Y = 0.65 \times 60 + 28.6 = 67.6$ **Ans.**

(d) The variable whose value is to be predicted in a regression analysis is called the dependent variable.

(e) We have,

$b_{xy} = 1.5$ and $b_{yx} = 0.65$

Now, the correlation coefficient between two variable is given by

$$r = \sqrt{b_{xy} \times b_{yx}}$$

$$= \sqrt{1.5 \times 0.65}$$

$$= \sqrt{0.975}$$

$= 0.98$ **Ans.**

4. (a) ***Define the regression.*** [1]

(b) ***Find the regression equation of X on Y from the following data.*** [3]

X	5	9	13	17	21
Y	3	8	13	18	23

(c) ***Estimate the value of X when Y = 18.*** [1]

Soln: (a) Regression is the study of the relationship between variables, which is used to predict the value of one variable when the value of another is known.

(b) Computation of regression equation of X on Y.

X	Y	Y^2	XY
5	3	9	15
9	8	64	72
13	13	169	169
17	18	324	306
21	23	529	483
$\Sigma X = 65$	$\Sigma Y = 65$	$\Sigma Y^2 = 1095$	$\Sigma XY = 1045$

Here, n = 5

$$\overline{X} = \frac{\Sigma X}{n} = \frac{65}{5} = 13$$

$$\overline{Y} = \frac{\Sigma Y}{n} = \frac{65}{5} = 13$$

$$b_{xy} = \frac{n\Sigma XY - \Sigma X\Sigma Y}{n\Sigma Y^2 - (\Sigma X)^2}$$

$$= \frac{5 \times 1045 - 65 \times 65}{5 \times 1095 - (65)^2}$$

$$= \frac{5225 - 4225}{5475 - 4225}$$

$$= \frac{1000}{1250}$$

$$= 0.8$$

(c) The regression equation of X on Y is

$$X - \overline{X} = b_{xy}(Y - \overline{Y})$$

or, $X - 13 = 0.8(Y - 13)$

or, $X = 0.8Y + 2.6$

When, $Y = 18$, $X = 0.8 \times 18 + 2.6$

$= 14.4 + 2.6 = 17.$

5. ***The following results were obtained with respect to two variables x and y:***
$\Sigma x = 30, \Sigma y = 42, \Sigma xy = 199, \Sigma x^2 = 184, \Sigma y^2 = 318$ & $n = 6$.
Find the following:
- ***(a) The regression coefficients.*** [2]
- ***(b) Correlation coefficient between x and y.*** [1]
- ***(c) Regression equation of y on x.*** [1]
- ***(d) Estimate the value of y when x = 10.*** [1]

Soln: (a) The regression coefficient of y on x is

$$b_{yx} = \frac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{6 \times 199 - 30 \times 42}{6 \times 184 - (30)^2}$$

$$= -\frac{66}{204} = -0.32$$

Also, the regression coefficient of x on y is

$$b_{xy} = \frac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma y^2 - (\Sigma y)^2}$$

$$= \frac{6 \times 199 - 30 \times 42}{6 \times 318 - (42)^2}$$

$$= \frac{1194 - 1260}{1908 - 1764}$$

$$= \frac{-66}{144}$$

$$= \frac{-11}{24}$$

$$= -0.46$$

(b) Correlation coefficient $(r) = \sqrt{b_{xy} \times b_{yx}}$

$$= \sqrt{(-0.46) \times (-0.32)}$$

$$= -\sqrt{0.144}$$

$$= -0.38$$

(c) We have, $\overline{x} = \frac{\Sigma x}{n} = \frac{30}{6} = 5$

$$\overline{y} = \frac{\Sigma y}{n} = \frac{42}{6} = 7$$

Now, the regression equation of y on x is :

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

or, $y - 7 = -0.32(x - 5)$

or, $y - 7 = -0.32x + 1.6$

or, $y = -0.32x + 1.6 + 7$

$\therefore$ $y = -032x + 8.6$

(d) When, $x = 10$, $y = -0.32 \times 10 + 8.6$

$= -3.2 + 8.6 = 5.4$ **Ans.**

6. ***From the following data, compute the line of regression for estimating age on weight and estimate the most probable age on a weight of 37 kg.*** [5]

Age (X)	5	15	30	45	50	60
Weight (Y)	10	35	50	65	55	45

Soln:

Computation of line of regression of X on Y:

Age (X)	Weight (Y)	XY	Y^2
5	10	50	100
15	35	525	1225
30	50	1500	2500
45	65	2925	4225
50	55	2750	3025
60	45	2700	2025
$\Sigma X = 205$	$\Sigma Y = 260$	$\Sigma XY = 10450$	$\Sigma Y^2 = 13100$

Here,

No. of items (n) = 6

$$\overline{X} = \frac{\Sigma X}{n} = \frac{205}{6} = 34.17$$

$$\overline{Y} = \frac{\Sigma Y}{n} = \frac{260}{6} = 43.33$$

$$b_{xy} = \frac{n\Sigma XY - \Sigma X\Sigma Y}{n\Sigma Y^2 - (\Sigma Y)^2}$$

$$= \frac{6 \times 10450 - 205 \times 260}{6 \times 13100 - (260)^2} = \frac{9400}{11000} = 0.85$$

The regression equation of X on Y is

$X - \overline{X} = b_{xy}(Y - \overline{Y})$

or, $X - 34.17 = 0.85(Y - 43.33)$

or, $X - 34.17 = 0.85Y - 36.83$

or, $X = 0.85Y + 34.17 - 36.83$

$\therefore$ $X = 0.85Y - 2.66$

When, $Y = 37$, $X = 0.85 \times 37 - 2.66 = 28.79$

$\therefore$ Most probable age = 28.79 **Ans.**

7. *The following table gives the age and weight of school children in a locality.* *[2081 Set E/W]*

Age in year	4	5	7	9	10	11
Weight in kg.	20	25	28	30	32	33

(a) *Find the co-efficient of correlation between age and the weight.* *[2]*

(b) *Estimate the weight when the age is 12 years.* *[3]*

Soln:

Calculation of correlation coefficient

Age in year (X)	Weight in kg (Y)	X^2	Y^2	XY
4	20	16	400	80
5	25	25	625	125
7	28	49	784	196
9	30	81	900	270
10	32	100	1024	320
11	33	121	1089	363
$\Sigma X = 46$	$\Sigma Y = 168$	$\Sigma X^2 = 392$	$\Sigma Y^2 = 4822$	$\Sigma XY = 1354$

Here, n = 6

(a) Correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$

$$= \frac{6 \times 1354 - 46 \times 168}{\sqrt{6 \times 392 - (46)^2}\ \sqrt{6 \times 4822 - (168)^2}}$$

$$= \frac{396}{\sqrt{236}\sqrt{708}}$$

$= 0.97$ **Ans.**

(b) To estimate the weight (Y) for a given age (X), we need to determine the regression equation of Y on X.

Here, we have

$$\overline{X} = \frac{\Sigma X}{n} = \frac{46}{6} = 7.67$$

$$\overline{Y} = \frac{\Sigma Y}{n} = \frac{168}{6} = 28$$

We know,

$$b_{YX} = \frac{n\Sigma XY - \Sigma X\Sigma Y}{n\Sigma X^2 - (\Sigma X)^2}$$

$$= \frac{6 \times 1354 - 46 \times 168}{6 \times 392 - (46)^2}$$

$= 1.68$

Now,

The regression equation of Y on X is

$Y - \overline{Y} = b_{YX}(X - \overline{X})$

or, $Y - 28 = 1.68(X - 7.67)$

or, $Y = 1.68X - 12.89 + 28$

$\therefore$ $Y = 1.68X + 15.11$

When, X = 12

$Y = 1.68 \times 12 + 15.11$

$= 35.27$

$\therefore$ The estimated weight is 35.27 kg. **Ans.**

8. *The supply and price of a commodity for the last six year is given below:* *[2081 Set V]*

Price in Rs. per kg	100	110	112	115	120	140
Supply in kg	30	40	45	20	55	55

(a) *Find the co-efficient of correlation between price and supply.* *[2]*

(b) *Estimate supply in kg on which rate of price is Rs. 150.* *[3]*

Soln:

Calculation of correlation coefficient

X	Y	X^2	Y^2	XY
100	30	10000	900	3000
110	40	12100	1600	4400
112	45	12544	2025	5040
115	20	13225	400	2300
120	55	14400	3025	6600
140	55	19600	3025	7700
$\Sigma X = 697$	$\Sigma Y = 245$	$\Sigma X^2 = 81869$	$\Sigma Y^2 = 10975$	$\Sigma XY = 29040$

Here, n = 6

(a) Correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X\, \Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$

$= \dfrac{6 \times 29040 - 697 \times 245}{\sqrt{6 \times 81869 - (697)^2}\sqrt{6 \times 10975 - (245)^2}}$

$= \dfrac{3475}{\sqrt{5405}\sqrt{5825}}$

= 0.62 **Ans.**

(b) To estimate the supply (Y) in kg for a given rate of price (X), we need to determine the regression equation of Y on X.

Here, we have

$\overline{X} = \dfrac{\Sigma X}{n} = \dfrac{697}{6} = 116.17$

$\overline{Y} = \dfrac{\Sigma Y}{n} = \dfrac{245}{6} = 40.83$

We know,

$b_{YX} = \dfrac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2}$

$= \dfrac{6 \times 29040 - 697 \times 245}{6 \times 81869 - (697)^2}$

$= \dfrac{3475}{5405}$

$= 0.64$

Now,

The regression equation of Y on X is

$Y - \overline{Y} = b_{YX}(X - \overline{X})$

or, $Y - 40.83 = 0.64(X - 116.17)$

or, $Y = 0.64X - 74.35 + 40.83$

$\therefore$ $Y = 0.64X - 33.52$

When, X = 150

$Y = 0.64 \times 150 - 33.52$

$= 62.48$

$\therefore$ The estimated supply is 62.48 kg. **Ans.**

□□□

Chapter 11

Probability

11.1 Probability

Basic Formulae and Key Points

1. **The probability of the happening of an event E is denoted by P (E) and is given by**

$P(E) = \dfrac{\text{No. of favourable cases}}{\text{Total no. of possible cases}} = \dfrac{m}{n}$

2. **P (E) Satisfies the Condition: $0 \le P(E) \le 1$.**

Note: i) If E is an impossible event then P(E) = 0

ii) If E is a sure event then P(E) = 1

iii) The sum of the probabilities of the occurrence and non-occurrence of an event is unity.

i.e. $P(E) + P(\overline{E}) = 1$

3. **Conditional Probability:**

Let, A and B be two dependent events, then the probability of occurrence of an event A when it is given that the event B has already occurred is known as the conditional probability of the event A. It is denoted by P(A/B) and is given by

(i) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$, $P(A) \neq 0$

Where, $P(A \cap B)$ is the probability of the simultaneous occurrence of the events A and B.

4. **Addition Rule of Probability:**

(i) $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Where, $P(A \cap B)$ is the probability of the simultaneous occurrence of the events A and B.

(ii) If A and B are mutually exchange events, then $P(A \cap B) = 0$ so that

$P(A \cup B) = p(A \cup B) = P(A) + P(B)$.

(iii) $P(A \cup B \cup C) = P(A) + p(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

If A, B and C are mutually exclusive events, then $P(A \cap B) = P(B \cap C) = P(C \cap A) = P(A \cap B \cap C) = 0$, so that $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

5. **Multiplicative Law of Probability of Dependent Events:**

(i) $P(A \cap B) = P(A).P(B/A)$

Where, P(B/A) is the conditional probability of B given that the event A has already occurred.

(ii) $P(A \cap B) = P(B).P(A/B)$

Note:

(i) If A and B are independent events then,

$P(A \cap B) = P(A) \times P(B)$.

(ii) The probabilities of an event that does not affect one another with replacement are independent. The probabilities of an event that affects one another without replacement are dependent.

6. **Use of Permutation & Combination in Probability:**

(i) The number of permutation of n distinct objects taken r at a time is given by

$$P(n,r) = \frac{n!}{(n-r)!},\ n \geq r.$$

(ii) The number of permutation of n objects taken all of them at a time when p objects are of one kind, q objects are of second kind and r objects are of third kind and so on is $\frac{n!}{p!\,q!\,r!}$.

(iii) The total number of selections of a set of n different objects taken r at a time is given by

$$C(n,r) = \frac{n!}{(n-r)!\,r!},\ n \geq r.$$

Group 'A' (Multiple Choice Questions and Answers)

1. *If A and B are two not-independent events, then the probability that both A and B will happen together is:*
   (a) $P(A \cap B) = P(A) \times P(B)$
   (b) $P(A \cap B) = P(A) \times P(B/A)$
   (c) $P(A \cap B) = P(A) + P(B)$
   (d) $P(A \cap B) = P(A)$

2. *Given that E and F are events such that $P(E) = 0.6$, $P(F) = 0.3$ and $P(E \cap F) = 0.2$, then the value of $P(E/F)$ is:*
   (a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) 0 (d) none

3. *If E_1 and E_2 are two independent events such that $P(E_1) = 0.3$ and $P(E_2) = 0.4$ then the value of $P(E_1 \cap E_2)$ is*
   (a) 0.7 (b) 0.75 (c) 0.12 (d) 0.21

4. *Two events are said to be independent if*
   (a) each outcome has equal chance of occurrence.
   (b) there is a common event in between them.
   (c) one does not affect the occurrence of the other.
   (d) both events have only one point.

5. *When a coin and a die are thrown, the number of all possible cases is*
   (a) 7 (b) 8 (c) 12 (d) 0

6. *The conditional probability of B given A is*
   (a) $\frac{P(A \cap B)}{P(B)}$ (b) $\frac{P(A \cap B)}{P(A)}$ (c) $\frac{P(A \cup B)}{p(B)}$ (d) $\frac{P(A \cup B)}{P(A)}$

7. *The probability of not getting 2, when a die is thrown is*
   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{6}$ (d) $\frac{5}{6}$

8. *The probability of sure event is*
   (a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

9. *Consider A and B be two dependent events. If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{13}$ and $P(A \cap B) = \frac{1}{26}$, what is $P(B/A)$?*
   (a) $\frac{1}{52}$ (b) $\frac{1}{26}$ (c) $\frac{1}{13}$ (d) $\frac{1}{2}$

10. *What is the probability of getting 53 Friday or Saturday in a leap year?*
    (a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{1}{122}$

11. *Let, A and B be two dependent events. If $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$ and $P(A \cap B) = \frac{2}{5}$, what is the value of $P(A/B)$?*
    (a) equal to P(B/A)
    (b) equal to P(A)
    (c) Less then $P(A \cap B)$
    (d) less than P(B/A)

Answer Key

1. b	2. a	3. c	4. c	5. c	6. b	7. d	8. b	9. c	10. c
11. d									

Group 'B' (Subjective Questions and Answers)

1. (a) *Define conditional probability.* [1]
   (b) *Consider two events A and B where $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.18$. Find $P(B/A)$.* [1]
   (c) *If the probability of a student passing both mathematics and physics is 0.3 and the probability of passing mathematics is 0.5, what is the conditional probability that a student passes physics given that he/she has passed in mathematics?* [2]
   (d) *Given two events A and B such that $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$. Determine whether the events A and B are independent or not.* [1]

Soln: (a) Let, A and B be two dependent events, then the probability of occurrence of an even A when it is given that the event B has already occurred is known as the conditional probability of the event A. It is denoted by P(A/B).

(b) Here, $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.18$, $P(B/A) = ?$

Now, we have

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{0.18}{0.3}$$

$$= 0.6 \text{ Ans.}$$

(c) Here, given that

P(M) = Probability of passing a student in Mathematics = 0.5

$P(M \cap P)$ = Probability of passing in Mathematics and Physics = 0.3

P(P/M) = Probability of passing in Physics given that he/she passed in Mathematics = ?

Now, we have

$$P(P/M) = \frac{(M \cap P)}{P(M)} = \frac{0.3}{0.5} = 0.6 \text{ Ans.}$$

(d) Here, $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$.

Now, $P(A) \times P(B) = 0.4 \times 0.5 = 0.2$

Since, $P(A \cap B) = 0.2$ and $P(A) \times P(B) = 0.2$

i.e. $P(A \cap B) = P(A) \times P(B)$

Hence, the events A and B are independent.

2. *Consider the following contingency table:*

Person	Right-handed	Left-handed	Total
Male	0.41	0.08	0.49
Female	0.45	0.06	0.51
Total	0.86	0.14	1

Find the probability that a random selected person is:

(a) *A male given that he is right-handed.*
(b) *Right-handed given that he is a male.* [1]
(c) *A female given that she is left-handed.* [1]
(d) *What is the probability that a randomly selected person is either male or left-handed?* [1]
(e) *Determine whether the events "being a female" and being left-handed are independent. Justify* [1] [1]

Soln: Here, from the information given in the table.

Let, M = male, F = Female, R = Right-handed, L = Left-handed

P(M) = Probability of male = 0.49, P(F) = Probability of female = 0.51

$P(M\cap R)$ = Probability of a selected person is a male given that he is right-handed = 0.41

$P(F\cap L)$ = Probability of a selected person is a female given that she is left-handed = 0.06

P(L) = Probability of left-handed = 0.14

P(R) = Probability of right-handed = 0.86

(a) Now,

$$P(M/R) = \frac{p(M\cap R)}{P(R)} = \frac{0.41}{0.86} = 0.476$$

(b) $P(R/M) = \frac{P(R\cap M)}{P(M)} = \frac{0.41}{0.49} = 0.84$

(c) $P(F/L) = \frac{P(F\cap L)}{P(L)} = \frac{0.06}{0.14} = 0.43$

(d) P (Either male or left-handed) = $P(M\cup L)$ =?

Now, $P(M\cup L) = P(M) + P(L) - P(M\cap L)$
$= 0.49 + 0.14 - 0.08$
$= 0.55$ **Ans.**

(e) Here, $P(F\cap L) = 0.06$

Also, $P(F)\times P(L) = 0.51\times 0.14 = 0.071$

Since, $P(F\cap L) \neq P(F)\times P(L)$

So, the events F and L are not independent.

Hence, they are dependent to each other.

3. *In a certain college 30% students fail in physics, 25% fails in mathematics and 10% fails in both. If a student is chosen at random, find the probability that*

(a) *She fails in physics given that she has failed mathematics.* [2]
(b) *She fails in mathematics given that she has failed in physics.* [2]
(c) *Define dependent events.* [1]

Soln: Let, A and B denote the events that a students fail in physics and mathematics respectively.

Then, P(A) = 30% = 0.3, P(3) = 25% = 0.25 and $P(A\cap B)$ = 10% = 0.10

(a) The probability that she fails in physics given that she has failed in mathematics is

$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{0.10}{0.25} = \frac{2}{5}$$

(b) The probability that she fails in mathematics given that she has failed in physics is

$$P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{0.10}{0.30} = \frac{1}{3}$$

(c) **Dependent events:** Events are said to be dependent if the occurrence of one event affects the occurrence of the other.

4. *A class consists of 60 boys and 40 girls. If two students are chosen at random, what will be the probability that (a) both are boys (b) both are girls (c) one boy and one girl?* [1+1+1]

Soln: Total no. of students = 60 + 40 = 100

Total no. of possible cases (n) = 2 students out of 100 can be selected in $^{100}C_2$ ways.

(a) **Both are boys**

No. of favourable cases (m) = 2 boys out of 60 boys can be selected in $^{60}C_2$ ways.

$$\therefore \quad P(\text{both boys}) = \frac{m}{n} = \frac{^{60}C_2}{^{100}C_2} = \frac{59}{165}$$

(b) **Both are girls**

No. of favourable cases (m) = 2 girls out of 40 girls can be selected in $^{40}C_2$ ways.

$$\therefore \quad P(\text{both girls}) = \frac{m}{n} = \frac{^{40}C_2}{^{100}C_2} = \frac{26}{165}$$

(c) **One boy and one girl**

No. of favourable cases (m) = one boy from 60 boys and one girl from 40 girls can be selected in $^{60}C_1\times{}^{40}C_1$ ways.

$$\therefore \quad P(\text{one boy and one girl}) = \frac{m}{n} = \frac{^{60}C_1\times{}^{40}C_1}{^{100}C_2} = \frac{60\times 40\times 2}{100\times 99} = \frac{16}{33} \text{ Ans.}$$

5. *If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A\cap B) = \frac{1}{4}$, find*

(a) *P (A/B) and (B/A)* [1]
(b) *$P(\bar{A})$ and $P(\bar{B})$* [1]
(c) *$P(\bar{A}/\bar{B})$ and $(\bar{B}/\bar{A})$* [2]
(d) *If two events are mutually exclusive, what is the probability that both occur at the same time?* [1]

Soln: (a) $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

Again, $P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$

(b) $P(A') = 1 - P(A) = 1 - \frac{3}{8} = \frac{5}{8}$ and

$P(B') = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$

(c) By De Morgon's law

$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - \left[\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right]$

$= 1 - \frac{5}{8}$

$= \frac{3}{8}$

$\therefore \quad P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$ and $P(B'/A') = \frac{P(A' \cap B')}{P(A')} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$ Ans.

(d) If two events are mutually exclusive, then the probability that both occur at the same time is zero.

6. *A family has two children. what is the probability that the second child is a boy given that their first child is girl?* [3]

Soln: Let, B and G represent boy and Girl respectively.

Sample space (s) = {BB, BG, GB, GG}

$\therefore$ n(s) = 4

Let, G: event of first child is a girl

So, G = {GB, GG}

$\therefore$ (G) = 2

Also, B: event of second child is a boy

So, B = {BB, GB}

$\therefore$ n (B) = 2 and

$n(B \cap G) = 1$

Now, $P(B) = \frac{n(B)}{n(s)} = \frac{2}{4} = \frac{1}{2}$

$n(G) = \frac{n(G)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

$P(B \cap G) = \frac{n(B \cap G)}{n(s)} = \frac{1}{4}$

$\therefore$ P(B/G) = Probability that the second child is a boy given that first child is a girl

$= \frac{p(B \cap G)}{p(G)}$

$= \frac{\frac{1}{4}}{\frac{1}{2}}$

$= \frac{1}{2}$

7. *A lot contains 10 items of which 3 are defective. Three items are chosen from the lot at a random one after another without replacement. Find the probability that:*

(a) All three are defective

(b) Only first one is defective [2+2]

Soln: Here, total no. of items = 10

No. of defective item = 3

No. of non–defective item = 7

(a) All there are defective

Since, the items are not replaced, so

P(First defective item) $= \frac{m}{n} = \frac{3}{10}$

P(Second defective item) $= \frac{m}{n} = \frac{2}{9}$

P(Third defective item) $= \frac{m}{n} = \frac{1}{8}$

$\therefore$ The probability of getting all three defective items $= \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{1}{120}$ Ans.

(b) Only first one is defective

Now, P(First defective item) $= \frac{m}{n} = \frac{3}{10}$

P(Second non defective item) $= \frac{m}{n} = \frac{7}{9}$

P(Third non defective item) $= \frac{m}{n} = \frac{6}{8}$

$\therefore$ The probability of first defective and 2^{nd} and 3^{rd} are non–defective $= \frac{3}{10} \times \frac{7}{9} \times \frac{6}{8} = \frac{7}{40}$ Ans.

Unit-wise Model Questions For Statistics and Probability

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Set 1

Group 'A'

6. In a school, there were 100 students, 35% students failed in mathematics, 20% students failed in science and 15% failed in both of the subjects. A student selected at random, the probability of the student failed in mathematics given that failed in science already is

(a) $\frac{3}{7}$ (b) $\frac{4}{7}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$ **[Ans: c]**

Group 'B'

17. The supply and price of a commodity for the last six year is given below.

Price in Rs. per kg	100	110	112	115	120	140
Supply in kg	30	40	45	20	55	55

(a) Find the coefficient of correlation between price and supply. [2] **[Ans: 0.62]**

(b) Estimate supply in kg on which rate of price is Rs. 150. [3] **[Ans: Rs. 62.5]**

Set 2

Group 'A'

6. Let A and B be two dependent events. If $P(A) = 0.5$, $P(B) = 0.75$ and $P(A \cap B) = 0.4$. What is the value of $P(A/B)$?

(a) equal to P(B) (b) less then $P(A \cap B)$
(c) less than $P(B/A)$ (d) equal to $P(B/A)$ [Ans: c]

Group 'B'

17. The following table gives the age and weight of school children in a locality.

age in year	4	5	7	9	10	11
weight in kg.	20	25	28	30	32	33

(a) Find the co-efficient of correlation between age and the weight. [2] [Ans: 0.968]
(b) Estimate the weight when the age is 12 years. [3] [Ans: 35.27]

Set 3

Group 'A'

6. Two dependent events A and B are given in such a way $P(A \cap B) = 0.78$ and $P(A) = 0.82$. What is the value of $P(B/A)$?

(a) 0.49 (b) 0.78 (c) 0.82 (d) 0.95 [Ans: d]

Group 'B'

16. The following table represents the yearly profit and yearly expenditure in crores rupees of a company.

Yearly Profit (in crore rupees)	50	60	100	150	200	240
Yearly expenditure (in crore rupees)	10	11	14	15	18	22

(a) Find the Pearson's correlation coefficient between yearly profit and yearly expenditure. [3] [Ans: 0.98]
(b) Find the regression coefficient of yearly profit on yearly expenditure. [2] [Ans: X = 16.9Y – 120.17]

Set 4

Group 'A'

6. In a family there are 2 male and 3 female, if two person are chosen at random, what will the probability that both are female?

(a) $\frac{9}{20}$ (b) $\frac{4}{25}$ (c) $\frac{9}{25}$ (d) $\frac{3}{10}$ [Ans: d]

Group 'B'

16. Find the correlation coefficient between X and Y and regression equation of Y on X from given data. [2+3]

Marks obtained by boys (X)	53	56	59	70	50
Marks obtained by girls (Y)	62	65	47	52	46

[Ans: r = – 0.105, Y = –0.1189X + 61.248]

□□□

6. Calculus

Chapter 12

Derivatives

12.1 Derivative and its Application

Basic Formulae and Key Points

1. **Hyperbolic Functions**

(a) $\sinh x = \frac{e^x - e^{-x}}{2}$ (b) $\cosh x = \frac{e^x + e^{-x}}{2}$

(c) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (d) $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}, x \neq 0$

(e) $\text{cosech } x = \frac{2}{e^x - e^{-x}}, x \neq 0$ (f) $\text{sech} x = \frac{2}{e^x + e^{-x}}$

2. **Some Identities**

(i) (a) $\cosh^2 x - \sinh^2 x = 1$
(b) $\text{sech}^2 x + \tanh^2 x = 1$
(c) $\coth^2 x - \text{cosech}^2 x = 1$

(ii) (a) $\cosh^2 x + \sinh^2 x = \cosh 2x$
(b) $\sinh 2x = 2 \sinh x. \cosh x$

3. **Derivatives of Hyperbolic Functions**

(a) $\frac{d}{dx}(\sinh x) = \cosh x$ (d) $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$

(b) $\frac{d}{dx}(\cosh x) = \sinh x$ (e) $\frac{d}{dx}(\text{sech } x) = -\text{sech } x . \tanh x$

(c) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$ (f) $\frac{d}{dx}(\text{cosech } x) = -\text{cosech } x . \coth x$

4. **Derivatives of Inverse Hyperbolic Functions:**

(a) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ (d) $\frac{d}{dx}(\coth^{-1} x) = \frac{-1}{x^2-1}, |x| > 1$

(b) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, (|x| > 1)$ (e) $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, (|x| < 1)$

(c) $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} (|x| < 1)$ (f) $\frac{d}{dx}(\text{cosech}^{-1} x) = \frac{-1}{x\sqrt{1+x^2}}$

Multiple Choice Questions and Answers)

-ivative of $\sinh x$ w. r. t. 'x' is

(a) cosh x (b) $-\cosh x$ (c) $-h\cosh x$ (d) $\frac{1}{h}\cosh x$

derivative of $\cosh x$ w. r. t. 'x' is

(a) $-\sinh x$ (b) $\sinh x$ (c) $-h\sinh x$ (d) $\frac{1}{h}\sinh x$

3. *The derivative of $\coth x$ w. r. t. 'x' is*

(a) cosech x. coth x (b) – cosech x. coth x
(c) $\text{cosech}^2 x$ (d) $-\text{cosech}^2 x$

4. *The derivative of $\tanh x$ w. r. t. 'x' is*

(a) sech x (b) sech x. tanh x (c) $\text{sech}^2 x$ (d) $-\text{sech}^2 x$

5. *The derivative of sech x w. r. t. 'x' is*

(a) tanh x (b) – tanh x (c) – sech x. tanh x (d) sech x tanh x

6. *The derivative of cosech x w. r. t. 'x' is*

(a) coth x (b) – coth x
(c) coesch x. coth x (d) – coesech x. coth x

7. *If $y = \sinh^{-1} x$ then $\frac{dy}{dx}$=*

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{x^2-1}}$ (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{1}{x\sqrt{1-x^2}}$

8. *If $y = \cosh^{-1}x$ then $\frac{dy}{dx}$ =*

(a) $\frac{1}{\sqrt{1-x^2}}$, (|x| < 1) (b) $\frac{1}{\sqrt{x^2-1}}$, (|x| < 1)
(c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{1}{\sqrt{x^2-1}}$, (|x| > 1)

9. *Which one of the following is the derivative of $\tanh^{-1}x$?* [2081 Set – W]

(a) $\frac{1}{1-x^2}$, |x| < 1 (b) $\frac{1}{\sqrt{1-x^2}}$, |x| < 1
(c) $\frac{-1}{1+x^2}$, |x| < 1 (d) $\frac{-1}{1-x^2}$, |x| < 1

10. *Which is a derivative of $\coth^{-1}x$?* [2081 Supp. Set – B]

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$
(c) $\frac{1}{x^2-1}$, (|x| > 1) (d) $\frac{-1}{x^2-1}$, (|x| > 1)

11. *If $y = \text{sech}^{-1}x$ then $\frac{dy}{dx}$ =*

(a) $\frac{1}{x\sqrt{1-x^2}}$, (|x| < 1) (b) $\frac{-1}{x\sqrt{1-x^2}}$, (|x| < 1)
(c) $\frac{-1}{x\sqrt{x^2-1}}$, (|x| > 1) (d) $\frac{1}{x\sqrt{x^2-1}}$, (|x| > 1)

12. *Which one of the following is the derivative of $\text{cosech}^{-1}x$?* [2081 Set V]

(a) $\frac{1}{x\sqrt{x^2+1}}$ (b) $\frac{-1}{x\sqrt{x^2+1}}$
(c) $\frac{1}{x\sqrt{1-x^2}}$, (|x| < 1) (d) $\frac{-1}{x\sqrt{1-x^2}}$, (|x| < 1)

13. *What is the derivative of $y = 2\sinh\frac{x}{2}$ with respect to x?* [2080 Set – G]

(a) $\cosh\frac{x}{2}$ (b) $2\cosh\frac{x}{2}$ (c) $\cosh\frac{x}{4}$ (d) $2\cosh\frac{x}{4}$

14. *The derivative of $\ln\left(\cosh\frac{x}{a}\right)$ is*

(a) $\frac{1}{a}\coth\frac{x}{a}$ (b) $\frac{-1}{a}\coth\frac{x}{a}$ (c) $\frac{1}{a}\tanh\frac{x}{a}$ (d) $\frac{-1}{a}\tan h\frac{x}{a}$

15. *The derivative of $e^{\sinh x}$ is*

(a) $e^{\sinh x}$ (b) $e^{\cosh x}$
(c) $\sinh x.e^{\sinh x-1}$ (d) $\cosh x\ e^{\sinh x}$

16. *Which one of the following is derivative of $\tanh^{-1}2x$ with respect to x (|x| < 1)?* [2081 Supp. Set – A]

(a) $\frac{1}{1+4x^2}$ (b) $\frac{2}{1+4x^2}$ (c) $\frac{1}{1-4x^2}$ (d) $\frac{2}{1-4x^2}$

Answer Key

1. a	2. b	3. d	4. c	5. c	6. d	7. c	8. d	9. a	10. d
11. b	12. b	13. a	14. c	15. d	16. d				

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *What is the derivative of cosh x?* [1]

(b) *Find the derivative of $x^{\sinh x}$.* [2]

Soln: (a) The derivative of cosh x is sinh x.

i.e. $\frac{d}{dx}(\cosh x) = \sinh x.$

(b) Let, $y = x^{\sinh x}$

Taking ln on both sides we get,

$$\ln y = \ln x^{\sinh x}$$
$$= \sinh x.\ \ln x$$

Diff. both sides w. r. t. 'x' we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sinh x.\ \ln x)$$

or, $\frac{d(\ln y)}{dy}\cdot\frac{dy}{dx} = \sinh x.\frac{d}{dx}(\ln x) + \ln x.\frac{d}{dx}(\sinh x)$

or, $\frac{1}{y}\cdot\frac{dy}{dx} = \sinh x.\frac{1}{x} + \ln x.\ \cosh x$

$\therefore$ $\frac{dy}{dx} = y\left(\frac{1}{x}\sinh x + \cosh x.\ \ln x\right)$

$$= x^{\sinh x}\left(\frac{1}{x}\sinh x + \cosh x.\ \ln x\right).$$

2. (a) *Write the derivatives of $y = \sinh^{-1} x$ and $y = \tanh x$.* [1+1] [2080 Set I]

(b) *Find the derivative of $x^{\cosh^2\frac{x}{a}}$.* [2]

Soln: (a) If $y = \sinh^{-1} x$ then, $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

and $y = \tanh x$ then, $\frac{dy}{dx} = \text{sech}^2 x$.

(b) Let, $y = x^{\cosh^2\frac{x}{a}}$

Taking ln on both sides we get

$\ln y = \ln x^{\cosh^2\frac{x}{a}} = \cosh^2\frac{x}{a}.\ln x$

Diff. both sides w. r. t. 'x' we get,

$$\frac{d(\ln y)}{dx} = \frac{d}{dx}\left(\cosh^2\frac{x}{a}.\ln x\right)$$

or, $\frac{d(\ln y)}{dy}.\frac{dy}{dx} = \cosh^2\frac{x}{a}.\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}\left(\cosh^2\frac{x}{a}\right)$

or, $\frac{1}{y}.\frac{dy}{dx} = \cosh^2\frac{x}{a}.\frac{d}{dx}(\ln x) + \ln x\frac{d\left(\cosh^2\frac{x}{a}\right)}{d\left(\cosh\frac{x}{a}\right)}.\frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}.\frac{d\left(\frac{x}{a}\right)}{dx}$

$$\frac{1}{y}.\frac{dy}{dx} = \cosh^2\frac{x}{a}.\frac{1}{x} + \ln x.2\cosh\frac{x}{a}.\sinh\frac{x}{a}.\frac{1}{a}$$

$\therefore \frac{dy}{dx} = y\left(\frac{1}{x}\cosh^2\frac{x}{a} + \frac{1}{a}.\ln x.\sinh\frac{2x}{a}\right)$

$$= x^{\cosh^2\frac{x}{a}}\left(\frac{1}{x}\cosh^2\frac{x}{a} + \frac{1}{a}\ln x.\sinh\frac{2x}{a}\right).$$

3. (a) *What is the derivative of ln sinh x?* [1]

(b) *Find the derivative of $\left(\cosh\frac{x}{a}\right)^{\ln x}$.* [2]

Soln: (a) Let, $y = \ln \sinh x$

Diff. both sides w. r. t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}(\ln \sinh x)$$

$$= \frac{d(\ln \sinh x)}{d(\sinh x)}.\frac{d}{dx}(\sinh x)$$

$$= \frac{1}{\sinh x}.\cosh x = \coth x.$$

(b) Let, $y = \left(\cosh\frac{x}{a}\right)^{\ln x}$

Taking ln on both sides we get,

$\ln y = \ln\left(\cosh\frac{x}{a}\right)^{\ln x}$

$= \ln x.\ln\left(\cosh\frac{x}{a}\right)$

Diff. both sides w. r. t. 'x' we get

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left\{\ln x.\ln\left(\cosh\frac{x}{a}\right)\right\}$$

or, $\frac{d(\ln y)}{dy}.\frac{dy}{dx} = \ln x\frac{d}{dx}\left\{\ln\left(\cosh\frac{x}{a}\right)\right\} + \ln\left(\cosh\frac{x}{a}\right).\frac{d}{dx}(\ln x)$

or, $\frac{1}{y}.\frac{dy}{dx} = \ln x.\frac{d\left\{\ln\left(\cosh\frac{x}{a}\right)\right\}}{d\left(\cosh\frac{x}{a}\right)}.\frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}.\frac{d\left(\frac{x}{a}\right)}{dx} + \ln\left(\cosh\frac{x}{a}\right).\frac{1}{x}$

or, $\frac{1}{y}.\frac{dy}{dx} = \ln x.\frac{1}{\cosh\frac{x}{a}}.\sinh\frac{x}{a}.\frac{1}{a} + \frac{1}{x}.\ln\left(\cosh\frac{x}{a}\right)$

$\therefore \frac{dy}{dx} = y\left(\frac{1}{a}\ln x.\tan\frac{x}{a} + \frac{1}{x}\ln\cosh\frac{x}{a}\right)$

$$= \left(\cosh\frac{x}{a}\right)^{\ln x}\left(\frac{1}{a}\ln x.\tan\frac{x}{a} + \frac{1}{x}\ln\cosh\frac{x}{a}\right)$$

4. *Find the derivatives of*

(a) *$\sinh^{-1}(\cosh x)$* (b) *$(\ln x)^{\sinh x}$* [1+2]

Soln: (a) Let, $y = \sinh^{-1}(\cosh x)$

Diff. both sides w.r.t. 'x' We get

$$\frac{dy}{dx} = \frac{d}{dx}\{\sinh^{-1}(\cosh x)\}$$

$$= \frac{d\{\sinh^{-1}(\cosh x)\}}{d(\cosh x)}.\frac{d}{dx}(\cosh x)$$

$$= \frac{1}{\sqrt{1+\cosh^2 x}}.\sinh x = \frac{\sinh x}{\sqrt{1+\cosh^2 x}}$$

(b) Let, $y = (\ln x)^{\sinh x}$

Taking ln on both sides we get,

$\ln y = \ln(\ln x)^{\sinh x}$

$= \sinh x.\ln(\ln x)$

Diff. both sides w. r. t. 'x' we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\{\sinh x.\ln(\ln x)\}$$

$$\frac{d(\ln y)}{dy}.\frac{dy}{dx} = \sinh x.\frac{d}{dx}\{\ln(\ln x)\} + \ln(\ln x).\frac{d}{dx}(\sinh x)$$

or, $\frac{1}{y}\frac{dy}{dx} = \sinh x.\frac{d\{\ln(\ln x)\}}{d(\ln x)}.\frac{d(\ln x)}{dx} + \ln(\ln x).\cosh x$

or, $\frac{1}{y}.\frac{dy}{dx} = \sinh x.\frac{1}{\ln x}.\frac{1}{x} + \cosh x.\ln(\ln x)$

$\therefore \frac{dy}{dx} = y\left(\frac{1}{x\ln x}.\sinh x + \cosh x.\ln(\ln x)\right)$

$$= (\ln x)^{\sinh x}\left(\frac{1}{x\ln x}.\sinh x + \cosh x.\ln(\ln x)\right)$$

5. *Find the derivatives of*

(a) $\text{sech}(\tan^{-1}x)$ (b) $\left(\sinh\frac{x}{a}\right)^{x^2}$ [1+2]

Soln: (a) Let, $y = \text{sech}(\tan^{-1}x)$

Diff. both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}[\text{sech}(\tan^{-1}x)]$$

$$= \frac{d[\text{sech}(\tan^{-1}x)]}{d(\tan^{-1}x)} \cdot \frac{d}{dx}(\tan^{-1}x)$$

$$= -\text{sech}(\tan^{-1}x).\tanh(\tan^{-1}x).\frac{1}{1+x^2}$$

(b) Let, $y = \left(\sinh\frac{x}{a}\right)^{x^2}$

Taking ln on both sides we get

$$\ln y = \ln\left(\sinh\frac{x}{a}\right)^{x^2} = x^2\ln\left(\sinh\frac{x}{a}\right)$$

Diff. both sides w.r.t. 'x' we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left\{x^2\ln\left(\sinh\frac{x}{a}\right)\right\}.$$

or, $$\frac{d(\ln y)}{dy} \cdot \frac{dy}{dx} = x^2\frac{d}{dx}\left(\ln\sinh\frac{x}{a}\right) + \ln\sinh\frac{x}{a} \cdot \frac{d}{dx}(x^2)$$

or, $$\frac{1}{y}\frac{dy}{dx} = x^2\frac{d\left(\ln\sinh\frac{x}{a}\right)}{d\left(\sinh\frac{x}{a}\right)} \cdot \frac{d\left(\sinh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)} \cdot \frac{d}{dx}\left(\frac{x}{a}\right) + \ln\sinh\frac{x}{a}.2x$$

$$= x^2.\frac{1}{\sinh\frac{x}{a}}.\cosh\frac{x}{a}.\frac{1}{a} + 2x\ln\sinh\frac{x}{a} = \frac{x^2}{a}\coth\frac{x}{a} + 2x\ln\sinh\frac{x}{a}$$

$$\therefore \frac{dy}{dx} = y\left(\frac{x^2}{a}\coth\frac{x}{a} + 2x\ln\sinh\frac{x}{a}\right)$$

$$= \left(\sinh\frac{x}{a}\right)^{x^2}\left(\frac{x^2}{a}\coth\frac{x}{a} + 2x\ln\sinh\frac{x}{a}\right)$$

6. ***Find the derivatives of***

(a) $\ln\left(\cosh\frac{x}{a}\right)$ (b) $(\cosh x)^{\sinh^{-1}x}$ [1+2]

Soln: (a) Let, $y = \ln\left(\cosh\frac{x}{a}\right)$

Diff. both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\ln\left(\cosh\frac{x}{a}\right)\right\}$$

$$= \frac{d\left\{\ln\left(\cosh\frac{x}{a}\right)\right\}}{d\left(\cosh\frac{x}{a}\right)} \cdot \frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{1}{\cosh\frac{x}{a}}.\sinh\frac{x}{a}.\frac{1}{a} = \frac{1}{a}\tanh\frac{x}{a}$$

(b) Let, $y = (\cosh x)^{\sinh^{-1}x}$

Taking ln on both sides we get

$$\ln y = \ln(\cosh x)^{\sinh^{-1}x}$$

$$= \sinh^{-1}x.\ln(\cosh x)$$

Diff. both sides w.r.t. 'x' we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\sinh^{-1}x.\ln(\cosh x)]$$

or, $$\frac{d(\ln y)}{dy} \cdot \frac{dy}{dx} = \sinh^{-1}x.\frac{d}{dx}(\ln\cosh x) + \ln\cosh x.\frac{d}{dx}(\sinh^{-1}x)$$

or, $$\frac{1}{y} \cdot \frac{dy}{dx} = \sinh^{-1}x.\frac{d(\ln\cosh x)}{d(\cosh x)} \cdot \frac{d}{dx}(\cosh x) + \ln\cosh x.\frac{1}{\sqrt{1+x^2}}$$

$$= \sinh^{-1}x.\frac{1}{\cosh x}.\sinh x + \frac{1}{\sqrt{1+x^2}}.\ln\cosh x$$

$$= \sinh^{-1}x.\tanh x + \frac{\ln\cosh x}{\sqrt{1+x^2}}$$

$$\therefore \frac{dy}{dx} = y\left(\sinh^{-1}x.\tanh x + \frac{\ln\cosh x}{\sqrt{1+x^2}}\right)$$

$$= (\cosh x)^{\sinh^{-1}x}\left(\sinh^{-1}x.\tanh x + \frac{\ln\cosh x}{\sqrt{1+x^2}}\right)$$

7. *Find the derivatives of*

(a) $2\tanh^{-1}\left(\tan\frac{x}{2}\right)$ (b) $x^{\cosh\frac{x}{a}}$ [2+2]

Soln: (a) Let, $y = 2\tanh^{-1}\left(\tan\frac{x}{2}\right)$

Diff. both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{2\tanh^{-1}\left(\tan\frac{x}{2}\right)\right\}$$

$$= 2\frac{d\left\{\tanh^{-1}\left(\tan\frac{x}{2}\right)\right\}}{d\left(\tan\frac{x}{2}\right)} \cdot \frac{d\left(\tan\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= 2.\frac{1}{1-\tan^2\frac{x}{2}}.\sec^2\frac{x}{2}.\frac{1}{2}$$

$$= \frac{\frac{1}{\cos^2\frac{x}{2}}}{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$$

$$= \frac{1}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{1}{\cos x} = \sec x.$$

(b) Let, $y = x^{\cosh\frac{x}{a}}$

Taking ln on both sides we get

$$\ln y = \ln x^{\cosh\frac{x}{a}}$$
$$= \cosh\frac{x}{a}.\ \ln x$$

Diff. both sides w.r.t. 'x' we get,

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\cosh\frac{x}{a}\ln x\right)$$

or, $$\frac{d(\ln y)}{dy}\cdot\frac{dy}{dx} = \cosh\frac{x}{a}\cdot\frac{d}{dx}(\ln x) + \ln x.\frac{d}{dx}\left(\cosh\frac{x}{a}\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cosh\frac{x}{a}\cdot\frac{1}{x} + \ln x.\frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{1}{x}\cosh\frac{x}{a} + \ln x.\sinh\frac{x}{a}\cdot\frac{1}{a}$$

$$\therefore \frac{dy}{dx} = y\left(\frac{1}{x}\cosh\frac{x}{a} + \frac{\ln x}{a}\sinh\frac{x}{a}\right)$$

$$= x^{\cosh\frac{x}{a}}\left(\frac{1}{x}\cosh\frac{x}{a} + \frac{\ln x}{a}.\sinh\frac{x}{a}\right).$$

8. *Find the derivatives of*

(a) Arc tan sinh x *(b)* $e^{\cosh^{-1}x}$ *(c)* $\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}$ [1 + 1 + 3]

Soln: (a) Let, $y = \text{Arc}\tan\sinh x$

$$= \tan^{-1}(\sinh x)$$

Diff. both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sinh x)$$
$$= \frac{d(\tan^{-1}\sinh x)}{d(\sinh x)}\cdot\frac{d(\sinh x)}{dx}$$
$$= \frac{1}{1+\sinh^2 x}\cdot\cosh x = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$$

(b) Let, $y = e^{\cosh^{-1}x}$

Diff. both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\cosh^{-1}x}\right)$$
$$= \frac{d\left(e^{\cosh^{-1}x}\right)}{d(\cosh^{-1}x)}\cdot\frac{d}{dx}(\cosh^{-1}x)$$
$$= e^{\cosh^{-1}x}\cdot\frac{1}{\sqrt{x^2-1}} = \frac{e^{\cosh^{-1}x}}{\sqrt{x^2-1}}.$$

(c) Let, $y = \left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}$

Taking ln on both sides we get,

$$\ln y = \ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}$$
$$= nx.\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$$

Diff. both sides w.r.t. 'x' we get,

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left\{nx.\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}$$

$$\frac{d(\ln y)}{dy}\cdot\frac{dy}{dx} = nx\frac{d}{dx}\left\{\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\} + \ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\cdot\frac{d}{dx}(nx)$$

or, $$\frac{1}{y}\frac{dy}{dx} = nx\frac{d\left\{\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}}{d\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}\cdot\frac{d\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}\times\frac{d}{dx}\left(\frac{x}{a}\right) + \ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right).n$$

or, $$\frac{1}{y}\frac{dy}{dx} = nx.\frac{1}{\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}\cdot\left(\cosh\frac{x}{a} + \sinh\frac{x}{a}\right).\frac{1}{a} + n.\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$$

$$= \frac{nx}{a} + n.\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$$

$$\therefore \frac{dy}{dx} = ny\left\{\frac{x}{a} + \ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}$$

$$= n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left\{\frac{x}{a} + \ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}.$$

9. *Find the derivatives of*

(a) $\coth x - \frac{1}{3}\coth^3 x$ *(b)* $2\tan^{-1}\left(\tanh\frac{x}{2}\right)$ [2+2]

Soln: (a) Let, $y = \coth x - \frac{1}{3}\coth^3 x$

Diff. both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\coth x - \frac{1}{3}\coth^3 x\right)$$
$$= \frac{d}{dx}(\coth x) - \frac{1}{3}\frac{d(\coth^3 x)}{d(\coth x)}\cdot\frac{d(\coth x)}{dx}$$
$$= -\operatorname{cosech}^2 x - \frac{1}{3}.3\coth^2 x.(-\operatorname{cosech}^2 x)$$
$$= \operatorname{cosech}^2 x.\coth^2 x - \operatorname{cosech}^2 x$$
$$= \operatorname{cosech}^2 x(\coth^2 x - 1)$$
$$= \operatorname{cosech}^2 x.\operatorname{cosech}^2 x$$
$$= \operatorname{cosech}^4 x.$$

(b) Let, $y = 2\tan^{-1}\left(\tanh\frac{x}{2}\right)$

Diff. both sides w.r.t. 'x' we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{2\tan^{-1}\left(\tanh\frac{x}{2}\right)\right\}$$

$$= 2\frac{d\left\{\tan^{-1}\left(\tanh\frac{x}{2}\right)\right\}}{d\left(\tanh\frac{x}{2}\right)} \cdot \frac{d\left(\tanh\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= 2 \cdot \frac{1}{1+\tanh^2\frac{x}{2}} \cdot \operatorname{sech}^2\frac{x}{2} \cdot \frac{1}{2}$$

$$= \frac{\frac{1}{\cosh^2\frac{x}{2}}}{1+\frac{\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}}}$$

$$= \frac{1}{\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}}$$

$$= \frac{1}{\cosh x}$$

$= \operatorname{sech} x.$

□□□

Chapter 13 Applications of Derivatives

13.1 Applications of Derivatives

A. L Hospital's Rule

Basic Formulae and Key Points

If f(x) and g(x) and their derivatives f'(x) and g'(x) are continuous at x = a and if f(a) = g(a) = 0 then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$, $g'(a) \neq 0$.

Note: If $f'(a) = g'(a) = 0$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}$ and so on.

Group 'A' (Multiple Choice Questions and Answers)

1. $\lim_{x\to 0}\frac{\sin x}{x}$ is equal to [2079 GIE Set A]

   (a) $-\infty$ (b) 0 (c) 1 (d) ∞

2. $\lim_{x\to\infty}\frac{\ln x}{x}$ is equal to [2079 GIE Set B]

   (a) -1 (b) 0 (c) 1 (d) ∞

3. Which one of the following is equal to $\lim_{x\to 0}\frac{3x-\sin x}{2x}$?

   (a) 3 (b) 2 (c) 1.5 (d) 1

4. Which one of the following is equal to $\lim_{x\to 0}\frac{e^{3x}-1}{2x}$? [2081 Set V]

   (a) 0 (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) 3

5. Which one of the following is equal to $\lim_{x\to\frac{1}{2}}\frac{\tan\pi x}{\sec^2\pi x}$? [2081 Set W]

   (a) $\frac{-1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1

6. According to L Hospital's rule, the value of $\lim_{x\to 0}\frac{x^3}{4\sin x}$ is equal to

   (a) $\frac{3}{4}$ (b) 0 (c) $\frac{1}{4}$ (d) ∞

7. The value of $\lim_{x\to 2}\frac{x^4-16}{x-2}$ is

   (a) 0 (b) 16 (c) 32 (d) 64

8. $\lim_{x\to a} \frac{x-a}{x^n - a^n}$ is equal to

(a) $\frac{0}{0}$ (b) $n.x^{n-1}$ (c) $n.a^{n-1}$ (d) $\frac{1}{n} a^{1-n}$

9. $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

(a) 0 (b) 1 (c) ∞ (d) n

10. $\lim_{x\to 0} \frac{\sin px}{\sin qx}$

(a) pq (b) $\frac{p}{q}$ (c) $\frac{q}{p}$ (d) 1

11. $\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$

(a) 1 (b) 0 (c) −1 (d) Does not exist

12. $\lim_{x\to 0} \frac{e^{2x}-1}{\ln(x+1)}$ is equal to

(a) 1 (b) −1 (c) 2 (d) 0

13. $\lim_{x\to e} \frac{\ln x - 1}{x-e}$ is equal to

(a) e (b) $\frac{1}{e}$ (c) e^2 (d) $\frac{1}{e^2}$

14. $\lim_{x\to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

15. $\lim_{x\to \infty} \frac{x^3}{e^x}$

(a) 0 (b) 1 (c) 2 (d) −2

16. The value of $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$ [2081 Supp. Set - B]

(a) −1 (b) 0 (c) 1 (d) 2

Answer Key

1. c	2. b	3. d	4. c	5. c	6. b	7. c	8. d	9. d	10. b
11. c	12. c	13. b	14. d	15. a	16. b				

Group 'B' or 'C' (Subjective Questions and Answers)

1. *Define L Hospital's rule. Using L Hospital's rule evaluate* $\lim_{x\to 2} \frac{x^4-16}{x^3-8}$. [2]

Soln: **L Hospital's rule:** If f(x) and g(x) and their derivatives f'(x) and g'(x) are continuous at x = a and if f(a) = g(a) = 0 then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}, \text{ Provided that } g'(a) \neq 0.$$

"Next part"

$$\lim_{x\to 2} \frac{x^4-16}{x^3-8} \quad (\text{form } \tfrac{0}{0})$$

$$= \lim_{x\to 2} \frac{4x^3}{3x^2} = \frac{4\times 2^3}{3\times 2^2} = \frac{4}{3}\times 2 = \frac{8}{3}$$

2. *Apply L Hospital's rule to evaluate:* $\lim_{x\to 1} \frac{x^3-2x^2+1}{3x^3-3x}$ [3] [2081 Supp. Set - A]

Soln: Here, $\lim_{x\to 1} \frac{x^3-2x^2+1}{3x^3-3x}$ (form $\frac{0}{0}$)

$$= \lim_{x\to 1} \frac{3x^2-4x}{9x^2-3} = \frac{3.1-4.1}{9.1-3} = \frac{3-4}{9-3} = -\frac{1}{6} \text{ Ans.}$$

3. *L Hospital's rule is useful to evaluate the limit while we get indeterminate form. Justify the statement with suitable example.* [2] [2080 Set I]

Soln: For the given statement we take an example $\lim_{x\to 0} \frac{\tan ax}{\tan bx}$

It takes the form $\frac{0}{0}$ when x = 0.

i.e. $\lim_{x\to 0} \frac{\tan ax}{\tan bx}$ (form $\frac{0}{0}$)

$$= \lim_{x\to 0} \frac{a \sec^2 ax}{b \sec^2 bx} \quad (\because \text{ by L Hospital's rule})$$

$$= \frac{a.1}{b.1} = \frac{a}{b}$$

From above example we say that L Hospital's rule is useful to evaluate the limit while we get indeterminate form.

4. *Using L Hospital's rule, evaluate* $\lim_{x\to\infty} \frac{\ln(2x^2+5)}{\ln(3x^3-4)}$. [3] [2080 Set G]

Soln: $\lim_{x\to\infty} \frac{\ln(2x^2+5)}{\ln(3x^3-4)}$ (form $\frac{\infty}{\infty}$)

$$= \lim_{x\to\infty} \frac{\frac{1}{2x^2+5}\times 4x}{\frac{1}{3x^3-4}\times 9x^2}$$

$$= \lim_{x\to\infty} \frac{4x(3x^3-4)}{9x^2(2x^2+5)}$$

$$= \lim_{x\to\infty} \frac{4(3x^3-4)}{9x(2x^2+5)}$$

$= \lim_{x\to\infty} \frac{12x^3-16}{18x^3+45x}$ (form $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{36x^2}{54x^2+45}$ (form $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{72x}{108x}$ (form $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{72}{108}$

$= \frac{2}{3}$.

5. *Using L Hospital's rule, evaluate* $\lim_{x\to 0} \frac{e^x-x-1}{x^2}$. [2]

Soln: Here, $\lim_{x\to 0} \frac{e^x-x-1}{x^2}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{e^x-1}{2x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{e^x}{2}$

$= \frac{1}{2}$

6. *Using L Hospital's rule, evaluate* $\lim_{x\to 0} \frac{x-\sin x}{x^3}$. [2]

Soln: $\lim_{x\to 0} \frac{x-\sin x}{x^3}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{\sin x}{6x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{\cos x}{6}$

$= \frac{1}{6}$

7. *Using L Hospital's rule, evaluate* $\lim_{x\to 0} \frac{\tan x-x}{x-\sin x}$. [2]

Soln: $\lim_{x\to 0} \frac{\tan x-x}{x-\sin x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{\sec^2x-1}{1-\cos x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{2\sec x.\sec x.\tan x}{\sin x}$

$= \lim_{x\to 0} \frac{2\sec^2x.\tan x}{\sin x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{2\sec^2x.\sec^2x+\tan x.4\sec x.\sec x.\tan x}{\cos x}$

$= \lim_{x\to 0} \frac{2\sec^4x+4\sec^2x.\tan^2x}{\cos x} = \frac{2.1+4.1.0}{1} = \frac{2}{1} = 2$

8. *Using L Hospital's rules evaluate:* $\lim_{x\to 0} \frac{x^2-\sin^2x}{x^2}$. [2]

Soln: $\lim_{x\to 0} \frac{x^2-\sin^2x}{x^2}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{2x-2\sin x.\cos x}{2x}$

$= \lim_{x\to 0} \frac{2x-\sin 2x}{2x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{2-2\cos 2x}{2}$

$= \frac{2-2.1}{2}$

$= \frac{0}{2} = 0.$

9. *Using L Hospital's rule, evaluate:* $\lim_{x\to 0} \frac{e^x+e^{-x}-2\cos x}{\sin^2x}$. [2]

Soln: $\lim_{x\to 0} \frac{e^x+e^{-x}-2\cos x}{\sin^2x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{e^x-e^{-x}+2\sin x}{2\sin x.\cos x}$

$= \lim_{x\to 0} \frac{e^x-e^{-x}+2\sin x}{\sin 2x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{e^x+e^{-x}+2\cos x}{2\cos 2x}$

$= \frac{1+1+2.1}{2.1}$

$= \frac{4}{2}$

$= 2.$

10. *Using L Hospital's rule, evaluate:* $\lim_{x\to 0} \frac{x.e^x-\ln(1+x)}{x^2}$. [2]

Soln: $\lim_{x\to 0} \frac{x.e^x-\ln(x+1)}{x^2}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{x.e^x+e^x.1-\frac{1}{x+1}}{2x}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{x.e^x+e^x.1+e^x+\frac{1}{(x+1)^2}}{2}$

$= \frac{0+1+1+1}{2}$

$= \frac{3}{2}$.

11. Using L Hospital's rule, evaluate: $\lim\limits_{x\to0}\frac{x-\sin x.\cos x}{x^3}$. [2]

Soln: $\lim\limits_{x\to0}\frac{x-\sin x.\cos x}{x^3}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{2x-2\sin x.\text{Cos}x}{2x^3}$ (Multiplying both numerator and denominator by 2)

$=\lim\limits_{x\to0}\frac{2x-\sin 2x}{2x^3}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{2-2\cos 2x}{6x^2}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{-2\times-2\sin 2x}{12x}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{\sin 2x}{3x}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{2\cos 2x}{3}$

$=\frac{2\times1}{3}=\frac{2}{3}$.

12. Using L Hospital's rule, evaluate: $\lim\limits_{x\to0}\frac{(e^x-1)\tan x}{x^2}$. [2]

Soln: $\lim\limits_{x\to0}\frac{(e^x-1)\tan x}{x^2}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{(e^x-1).\sec^2x+\tan x.e^x}{2x}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{(e^x-1).2\sec x.\text{Sec}x.\tan x+\sec^2x.e^x+\tan x.e^x+e^x.\sec^2x}{2}$

$=\frac{0+1+0+1}{2}$

$=\frac{2}{2}=1.$

13. Using L Hospital's rule, evaluate: $\lim\limits_{x\to0}\frac{\ln(\tan x)}{\ln x}$. [2]

Soln: $\lim\limits_{x\to0}\frac{\ln(\tan x)}{\ln x}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{\frac{1}{\tan x}.\sec^2x}{\frac{1}{x}}$

$=\lim\limits_{x\to0}\frac{x\sec^2x}{\tan x}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to0}\frac{x.2\sec x.\sec x.\tan x+\sec^2x.1}{\sec^2x}$

$=\lim\limits_{x\to0}\frac{2x.\tan x.\sec^2x+\sec^2x}{\sec^2x}$

$=\frac{2\times0+1}{1}=1$

14. Using L Hospital's rule, evaluate $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 5x}$. [2]

Soln: $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 5x}$ (form $\frac{\infty}{\infty}$)

$=\lim\limits_{x\to\frac{\pi}{2}}\frac{\cot 5x}{\cot x}$ (form $\frac{0}{0}$)

$=\lim\limits_{x\to\frac{\pi}{2}}\frac{-5.\text{cosec}^25x}{-\text{cosec}^2x}$

$=\lim\limits_{x\to\frac{\pi}{2}}\frac{5\,\text{cosec}^25x}{\text{cosec}^2x}$

$=5\times\frac{1}{1}$

$=5.$

15. Using L Hospital's rule, evaluate: $\lim\limits_{x\to0}\tan x.\ln\sin x$. [2]

Soln: $\lim\limits_{x\to0}\tan x.\ \ln\sin x$ (form $0\times\infty$)

$=\lim\limits_{x\to0}\frac{\ln\sin x}{\cot x}$ (form $\frac{\infty}{\infty}$)

$=\lim\limits_{x\to0}\frac{\frac{1}{\sin x}\times\cos x}{-\text{cosec}^2x}$

$=-\lim\limits_{x\to0}\frac{\cos x}{\sin x}\times\sin^2x$

$=-\lim\limits_{x\to0}(\sin x.\cos x)=-(0\times1)=0.$

16. Using L Hospital's rule, evaluate : $\lim\limits_{x\to\infty}\frac{x^n}{e^x}$. [2]

Soln: $\lim\limits_{x\to\infty}\frac{x^n}{e^x}$ (form $\frac{\infty}{\infty}$)

$=\lim\limits_{x\to\infty}\frac{n.x^{n-1}}{e^x}$ (form $\frac{\infty}{\infty}$)

$=\lim\limits_{x\to\infty}\frac{n.(n-1).x^{n-2}}{e^x}$ (form $\frac{\infty}{\infty}$)

$=\lim\limits_{x\to\infty}\frac{n(n-1).(n-2)\ldots3.2.1}{e^x}$ [$\because$ Continuing the process of up to n^{th} term]

$=n!\lim\limits_{x\to\infty}\frac{1}{e^x}=n!\times0=0.$

B. Differentials, Tangent and Normal :

Basic Formulae and Key Points

1. Let, $y = f(x)$ be a function of x, then
   (i) Differential of x is $dx = \Delta x$.
   (ii) Differential of y is $dy = f'(x)\,dx$.
2. Approximate increase in y is $dy = f'(x)\,dx$.
3. Actual increase in y is $\Delta y = f(x + \Delta x) - f(x)$.
4. Error in the estimate $= |\Delta y - dy|$
5. % Error $= \frac{\text{Error}}{y} \times 100\%$

Note: $dx = \Delta x$ but $dy \neq \Delta y$.

6. If $\frac{dy}{dx}$ at $(x_1, y_1) = m$ = Slope of tangent to the curve at point (x_1, y_1), then
   (i) Equation of tangent at (x_1, y_1) is $y - y_1 = m(x - x_1)$
   (ii) Equation of Normal at (x_1, y_1) is $y - y_1 = \frac{-1}{m}(x - x_1)$.
   (iii) If the tangent is parallel to the x–axis then $\frac{dy}{dx} = 0$.
   (iv) If the tangent is parallel to the y–axis then $\frac{dx}{dy} = 0$.
7. If m_1 and m_2 be the slopes of tangents to the two curves at each point of intersection and θ be the angle between them, then $\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1.m_2}$
   (i) If $m_1 \times m_2 = -1$, then two curves intersects orthogonally i.e. perpendicularly.
   (ii) If $m_1 = m_2$, then two curves touches each other i.e. they have a common tangent.

Group 'A' (Multiple Choice Questions and Answers)

1. *If $y = x^2 + 4$, find dy when $x = 2$ and $\Delta x = 0.01$.*
   (a) 0.02 (b) 0.2 (c) 0.04 (d) 0.4
2. *If $y = f(x) = x^3 + 3$, calculate Δy when $x = 1$ and $\Delta x = 0.1$.*
   (a) 0.0331 (b) 0.331 (c) 3.31 (d) – 0.331
3. *If $y = x^3$, then the approximate change in y when x changes form 5 to 5.01 is*
   (a) 0.075 (b) 0.75 (c) – 0.75 (d) 0.575
4. *If $y = \sqrt{x}$, then the actual change in y when x changes from 4 to 4.41 is*
   (a) 0.41 (b) 0.041 (c) 0.1 (d) 0.01
5. *The edge of a cube increases from 10 cm to 10.025 cm. What would be the approximate increment in volume?*
   (a) 10^3 cm³ (b) 10.025 cm³ (c) 7.5187 cm³ (d) 7.5 cm³
6. *What is the angle between the tangents to the curve $x^2 = 6y - 15$ at the points (1, 9) and (–9, 0)?* [2079 Optional Set A]
   (a) 45° (b) 60° (c) 90° (d) 120°

7. *Tangent to the curve $y = x^3$ at $x = 2$ and $x = -2$ are* [2079 Set - K]
   (a) Parallel (b) Perpendicular
   (c) intersecting (d) does not exist
8. *What is the point on the curve $y = 2x^2 - 4x - 3$ at which the tangent of the curve is parallel to the line $4x - y + 2 = 0$?* [2079 Set - J]
   (a) (2, –3) (b) (–1 ,3) (c) (0, –3) (d) (0,2)
9. *Which one of the following is equation of tangent to the curve $y = x^3 - 2x^2 + 4$ at the point (2,4)?* [2080 Optional]
   (a) $4x - y - 4 = 0$ (b) $4x + y - 4 = 0$
   (c) $x - 4y - 4 = 0$ (d) $4x - y + 4 = 0$
10. *Which one of the following represents the equation of tangent to the curve $y^2 = 4x$ at the point (1, 2)?*
    (a) $x + y + 1 = 0$ (b) $x - y + 1 = 0$
    (c) $x + y - 1 = 0$ (d) $x - y - 1 = 0$
11. *If $\frac{dy}{dx} = \frac{-2}{5}$, what is the slope of normal at (1,1) for $y = f(x)$?* [2080 Set I]
    (a) $\frac{-2}{5}$ (b) $\frac{-5}{2}$ (c) $\frac{2}{5}$ (d) $\frac{5}{2}$
12. *What is the slope of tangent of the curve $y = x^2$ at $(\frac{1}{2}, \frac{1}{4})$?* [2080 GIE Set B]
    (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2
13. *Which one of the following represents the equation of normal to the curve $x^2 = 2y$ at the point (–2, 2)?* [2081 Set - V]
    (a) $2x + y + 6 = 0$ (b) $2x - 2y + 6 = 0$
    (c) $2x - y + 6 = 0$ (d) $x - 2y + 6 = 0$
14. *Which one of the following is the angle made by the tangent to the curve $2y = 2 - x^2$ at $x = 1$?* [2081 Set W]
    (a) 0° (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$
15. *If the slope of the tangent to the curve $y = x^2 - 4x + 4$ at a point is 2, then the point is*
    (a) $(2, \frac{1}{4})$ (b) (2, – 2) (c) (4, 2) (d) (3,1)
16. *What is the slope of normal to the curve $y = 2x^2 + 3x + 5$ at (–2, 7)?* [2081 Supp. Set – A]
    (a) –5 (b) $-\frac{1}{5}$ (c) $\frac{1}{5}$ (d) 5

Answer Key

1. c	2. b	3. b	4. c	5. d	6. c	7. a	8. a	9. a	10. b
11. d	12. c	13. d	14. d	15. d	16. c				

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Define differential.* [1]

(b) *If $y = \frac{x^2}{2} + 3x$, $x = 2$ and $dx = 0.5$, then*

(i) *What is the value of Δy?* [1]

(ii) *What is the value of dy?* [1]

(iii) *Find error and percentage error in the approximation.* [1+1]

Soln: (a) Differential: Let, y = f(x) be a function of x. Then

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)\,dx$.

(b) Given, $y = f(x) = \frac{x^2}{2} + 3x$, $x = 2$, $dx = \Delta x = 0.5$

(i) We have, $\Delta y = f(x + \Delta x) - f(x)$

$$= f(2 + 0.5) - f(2)$$
$$= f(2.5) - f(2)$$
$$= \left[\frac{(2.5)^2}{2} + 3 \times 2.5\right] - \left(\frac{2^2}{2} + 3 \times 2\right)$$
$$= 10.625 - 8$$
$$= 2.625$$

(ii) $\frac{dy}{dx} = \frac{2x}{2} + 3$

$\because$ $dy = (x + 3)\,dx = (2 + 3) \times 0.5 = 2.5$

(iii) Error $= \Delta y - dy$

$$= 2.625 - 2.5 = 0.125$$

Again, Percentage error $= \frac{\text{Error}}{y} \times 100\%$

$$= \frac{0.125}{8} \times 100\% \quad [\because y = f(2)]$$
$$= 1.5625\ \%$$

2. (a) *If the radius of a sphere changes from 3 cm to 3.01 cm. Find the approximate increments in the volume and the surface area of the sphere.* [1+1]

(b) *Use differential to approximate $\sqrt{36.6}$.* [2]

Soln: (a) Let, x be the radius, V be the volume and s be the surface area of the sphere.

Then, $V = \frac{4}{3}\pi x^3$, $S = 4\pi x^2$.

$x = 3$ cm and $dx = 3.01 - 3 = 0.01$ cm.

Now, $V = \frac{4}{3}\pi x^3$ then $\frac{dV}{dx} = \frac{4}{3}\pi . 3x^2$

$\therefore$ $dV = 4\pi x^2\,dx$

$$= 4\pi \times 3^2 \times 0.01 = 0.36\,\pi\ \text{cm}^3.$$

$\therefore$ Approximate change in the volume $= 0.36\pi$ cm^3.

Again, $s = 4\pi x^2$ then

$\frac{ds}{dx} = 4\pi . 2x$

$\therefore$ $ds = 8\pi x\,dx = 8\pi \times 3 \times 0.1 = 2.4\pi$ cm^2

$\therefore$ Approximate change in the surface area $= 2.4\pi$ cm^2.

(b) Let, $y = f(x) = \sqrt{x}$.

Let, $x = 36$ and $\Delta x = dx = 0.6$

Then, we have $\Delta y = f(x + \Delta x) - f(x)$

or, $\Delta y = f(36.6) - f(36)$

or, $\Delta y = \sqrt{36.6} - \sqrt{36}$

$\therefore$ $\sqrt{36.6} = 6 + \Delta y$ ....(i)

Again, $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore$ $dy = \frac{dx}{2\sqrt{x}} = \frac{0.6}{2\sqrt{36}} = \frac{0.6}{2 \times 6} = 0.05.$

Since, dy is approximately equal to Δy.

Now, by (i), $\sqrt{36.6} \approx 6 + dy$

$$= 6 + 0.05 = 6.05$$

$\therefore$ The approximate value of $\sqrt{36.6}$ is 6.05.

3. (a) *Find the approximate increase in the volume of the cube if the edge increases from 10 cm to 10.025 cm. Also calculate the percentage error in the use of differential approximation.* [3]

(b) *Use a differential to approximate $\frac{1}{10.1}$.* [2]

Soln: (a) Let, x be the edge and V be the volume of the cube, then $v = x^3$.

$x = 10$ cm, $dx = \Delta x = 10.025 - 10 = 0.025$ cm.

Since, $v = x^3$ then $\frac{dv}{dx} = 3x^2$

$\therefore$ $dv = 3x^2\,dx$

$$= 3 \times 10^2 \times 0.025 = 7.5\ \text{cm}^3$$

Again, $\Delta V = f(x + \Delta x) - f(x)$ $\quad [\because v = f(x)]$

$$= f(10.025) - f(10)$$
$$= (10.025)^3 - (10)^3$$
$$= 1007.5187 - 1000$$
$$= 7.5187\ \text{cm}^3$$

Now, Error $= \Delta v - dv$

$$= 7.5187 - 7.5 = 0.0187\ \text{cm}^3$$

Again, % Error $= \frac{\text{Error}}{v} \times 100\%$

$$= \frac{0.0187}{(10)^3} \times 100\% = 0.00187\ \%$$

(b) Let, $y = f(x) = \frac{1}{x}$.

Let, $x = 10$ and $\Delta x = dx = 0.1$

Then we have $\Delta y = f(x + \Delta x) - f(x)$

or, $\Delta y = f(10.1) - f(10)$

or, $\Delta y = \frac{1}{10.1} - \frac{1}{10}$

$\therefore$ $\frac{1}{10.1} = \Delta y + 0.1$ ....(i).

Again, $\frac{dy}{dx} = \frac{-1}{x^2}$

$\therefore \quad dy = \frac{-dx}{x^2} = -\frac{0.1}{10^2} = -0.001.$

Since, dy is approximately equal to Δy.

Now, by (i). $\frac{1}{10.1} \approx dy + 0.1$

$= -0.001 + 0.1 = 0.099.$

$\therefore$ The approximate value of $\frac{1}{10.1}$ is 0.099.

4. ***A circular copper plate is heated so that its radius increases from 5 cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.*** [2]

Soln: Let, x be the radius and A be the area of the circular plate.

Then $A = \pi x^2$, $x = 5$ and $dx = \Delta x = 0.06$

We have, $A = \pi x^2$ then $\frac{dA}{dx} = 2\pi x$

$\therefore \quad dA = 2\pi x dx$

$= 2\pi \times 5 \times 0.06$

$= 0.6\pi \text{ cm}^2$

$\therefore$ The approximate increase in area (dA) = $0.6\pi \text{ cm}^2$.

Again, actual increase in area $(\Delta A) = f(x + \Delta x) - f(x)$

$= f(5.06) - f(5)$

$= \pi(5.06)^2 - \pi \times 5^2$

$= \pi[(5.06)^2 - 5^2]$

$= \pi \times 0.6036$

$= 0.6036\pi \text{ cm}^2.$

5. ***The edge of a cube was found to be 10 cm with a possible error 0.01 cm. Use a differential to estimate the approximate***
   (a) ***Volume of the cube.*** [2]
   (b) ***Surface area of the cube.*** [2]

Soln: Let, x be the edge, v be the volume and A be the surface area of the cube, then $v = x^3$, $A = 6x^2$.

Also, x = 10 cm and $dx = \Delta x = 0.01$ cm.

(a) We have, $v = x^3$ then $dv = 3x^2dx$

$= 3 \times 10^2 \times 0.01 = 3 \text{ cm}^3$

Volume (v) $= x^3 = 10^3 = 1000 \text{ cm}^3$

$\therefore$ Approximate volume of cube = v + dv

$= (1000 + 3) \text{ cm}^3 = 1003 \text{ cm}^3.$

(b) We have, $A = 6x^2$ then $dA = 12xdx$

$= 12 \times 10 \times 0.01 = 1.2 \text{ cm}^2$

Surface area (A) $= 6x^2$

$= 6 \times 10^2 = 600 \text{ cm}^2$

$\therefore$ Approximate surface area of the cube = A + dA

$= 600 + 1.2$

$= 601.2 \text{ cm}^2.$

6. ***Suman and Binita are studying about application of derivative in a class. They ask each the quiz questions as given below. On the basis of these questions answer the following.***
   (a) ***Give the geometrical meaning of derivative.*** [1]
   (b) ***Write the slope of tangent and normal to the curve $y = f(x)$ at (x_1, y_1).*** [1+1]
   (c) ***Write the approximate increase and actual increase in y to the curve $y = f(x)$.*** [1+1]

Soln: (a) Let, y = f(x) be a function, then the derivative of y with respect to 'x' i.e. $\frac{dy}{dx}$ at any given point $P(x_1, y_1)$ always represents the slope of the tangent to the curve y = f(x) at that point.

(b) If y = f(x) then $\frac{dy}{dx} = f'(x)$.

Now, slope of tangent at $(x_1, y_1) = f'(x_1)$

and slope of normal at $(x_1, y_1) = \frac{-1}{f'(x_1)}$.

(c) If y = f(x) be a function of x, then approximate increase in y is $dy = f'(x)dx$ and actual Increase in y is $\Delta y = f(x + \Delta x) - f(x)$.

7. (a) ***Write a condition when tangents are parallel to the y-axis?*** [1]
   (b) ***Find the slope and the inclination with the x-axis of the tangent of $2x^2 + 3y^2 + 8x + 3 = 0$ at (–1, 1)*** [2]

Soln: (a) If the tangents are parallel to the y-axis then $\frac{dx}{dy} = 0$.

(b) Given curve is $2x^2 + 3y^2 + 8x + 3 = 0$

Diff. both sides with respect to 'x' we get

$4x + 6y\frac{dy}{dx} + 8 = 0$

or, $6y\frac{dy}{dx} = -4x - 8$

$\therefore \quad \frac{dy}{dx} = \frac{-4(x+2)}{6y} = \frac{-2(x+2)}{3y}$

At (–1, 1), $\frac{dy}{dx} = \frac{-2(-1+2)}{3 \times 1} = \frac{-2}{3}$

$\therefore$ Slope of tangent $= \frac{-2}{3}$.

If θ be the inclination of tangent with the x-axis then $\tan\theta = \frac{-2}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{-2}{3}\right)$ Ans.

8. (a) ***Write a condition when tangents are parallel to the x-axis?*** [1]
   (b) ***At what angle does the curve $y(1 + x) = x$ cut the x-axis?*** [2]

Soln: (a) If the tangents are parallel to the x-axis then $\frac{dy}{dx} = 0$.

(b) Given curve is y(1 + x) = x ....(i)

Since, (i) cuts the x-axis at y = 0.

$\therefore \quad 0 \times (1 + x) = x$

or, x = 0

$\therefore$ (i) meets the x-axis at (0, 0).

Now, Diff. both sides of (i) with respect to 'x' we get

$$y.1 + (1 + x)\frac{dy}{dx} = 1$$

or, $(x+1)\frac{dy}{dx} = 1 - y$

$\therefore \frac{dy}{dx} = \frac{1-y}{x+1}$

At (0, 0), $\frac{dy}{dx} = \frac{1-0}{0+1} = 1$

If θ be the angle made by the curve on x-axis then $\tan\theta = 1$

or, $\tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

9. (a) *Find the point on the curve $4y = x^2$, where the tangent drawn makes an angle 45° with the x-axis.* [2]

(b) *Show that the equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ at the point (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$.* [2]

Soln: (a) Given curve is $4y = x^2$ ....(i)

Diff. both sides with respect to 'x' we get,

$$4\frac{dy}{dx} = 2x$$

$\therefore \frac{dy}{dx} = \frac{x}{2}$

$\therefore$ Slope of tangent $= \frac{x}{2}$

Since, the tangent makes an angle 45° with the x-axis. Then slope of tangent = tan45°

or, $\frac{x}{2} = 1$

$\therefore x = 2$

From (i), $4y = 2^2 \Rightarrow y = 1$

$\therefore$ required point is (2, 1).

(b) Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

Diff. both sides with respect to 'x' we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$

$= \frac{-b^2x}{a^2y}$

At (a, b), $\frac{dy}{dx} = \frac{-b^2.a}{a^2.b} = \frac{-b}{a}$

$\therefore$ Slope of tangent (m) $= \frac{-b}{a}$

Now, equation of tangent at (a, b) is $y - y_1 = m(x - x_1)$

or, $y - b = \frac{-b}{a}(x - a)$

or, $ay - ab = -bx + ab$

or, $bx + ay = 2ab$

Dividing both sides by ab we get,

$\frac{x}{a} + \frac{y}{b} = 2$ **Hence Proved.**

10. (a) *Find the points on the curve $x^2 + y^2 = 36$ at which the tangents are parallel to*
(i) *x-axis* (ii) *y-axis* [3]

(b) *Find the equation of the normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1,1).* [2]

Soln: (a) Given curve is $x^2 + y^2 = 36$ ....(i).

Diff. both sides with respect to 'x' we get

$$2x + 2y\frac{dy}{dx} = 0$$

or, $2y\frac{dy}{dx} = -2x$

$\therefore \frac{dy}{dx} = \frac{-x}{y}$

(i) If the tangents are parallel to x-axis, then $\frac{dy}{dx} = 0$

or, $\frac{-x}{y} = 0$

$\therefore x = 0$

Form (i), When, $x = 0$, $y^2 = 36$

$\therefore y = \pm 6$

Required points are (0, 6) and (0, − 6).

(ii) If the tangents are parallel to y-axis, then $\frac{dx}{dy} = 0$

or, $\frac{-y}{x} = 0$

$y = 0$

From (i), When, $y = 0$, $x^2 = 36$

$\therefore x = \pm 6$

$\therefore$ Required points are (6, 0) and (−6, 0).

(b) Given curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$

Diff. both sides w.r.t. 'x' we get,

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or, $\frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = \frac{-2}{3}x^{\frac{-1}{3}}$

$\therefore \frac{dy}{dx} = -\frac{x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

Now, Diff. both sides of (i) with respect to 'x' we get

$$y.1 + (1 + x)\frac{dy}{dx} = 1$$

or, $(x+1)\frac{dy}{dx} = 1 - y$

$\therefore \frac{dy}{dx} = \frac{1-y}{x+1}$

At (0, 0), $\frac{dy}{dx} = \frac{1-0}{0+1} = 1$

If θ be the angle made by the curve on x-axis then $\tan\theta = 1$

or, $\tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

9. (a) *Find the point on the curve $4y = x^2$, where the tangent drawn makes an angle 45° with the x-axis.* [2]

(b) *Show that the equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ at the point (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$.* [2]

Soln: (a) Given curve is $4y = x^2$ ....(i)

Diff. both sides with respect to 'x' we get,

$$4\frac{dy}{dx} = 2x$$

$\therefore \frac{dy}{dx} = \frac{x}{2}$

$\therefore$ Slope of tangent $= \frac{x}{2}$

Since, the tangent makes an angle 45° with the x-axis. Then slope of tangent = tan45°

or, $\frac{x}{2} = 1$

$\therefore x = 2$

From (i), $4y = 2^2 \Rightarrow y = 1$

$\therefore$ required point is (2, 1).

(b) Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

Diff. both sides with respect to 'x' we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$

$= \frac{-b^2x}{a^2y}$

At (a, b), $\frac{dy}{dx} = \frac{-b^2.a}{a^2.b} = \frac{-b}{a}$

$\therefore$ Slope of tangent (m) $= \frac{-b}{a}$

Now, equation of tangent at (a, b) is $y - y_1 = m(x - x_1)$

or, $y - b = \frac{-b}{a}(x - a)$

or, $ay - ab = -bx + ab$

or, $bx + ay = 2ab$

Dividing both sides by ab we get,

$\frac{x}{a} + \frac{y}{b} = 2$ **Hence Proved.**

10. (a) *Find the points on the curve $x^2 + y^2 = 36$ at which the tangents are parallel to*

(i) *x-axis* (ii) *y-axis* [3]

(b) *Find the equation of the normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1,1).* [2]

Soln: (a) Given curve is $x^2 + y^2 = 36$ ....(i).

Diff. both sides with respect to 'x' we get

$$2x + 2y\frac{dy}{dx} = 0$$

or, $2y\frac{dy}{dx} = -2x$

$\therefore \frac{dy}{dx} = \frac{-x}{y}$

(i) If the tangents are parallel to x-axis, then $\frac{dy}{dx} = 0$

or, $\frac{-x}{y} = 0$

$\therefore x = 0$

Form (i), When, $x = 0$, $y^2 = 36$

$\therefore y = \pm 6$

Required points are (0, 6) and (0, – 6).

(ii) If the tangents are parallel to y-axis, then $\frac{dx}{dy} = 0$

or, $\frac{-y}{x} = 0$

$y = 0$

From (i), When, $y = 0$, $x^2 = 36$

$\therefore x = \pm 6$

$\therefore$ Required points are (6, 0) and (–6, 0).

(b) Given curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$

Diff. both sides w.r.t. 'x' we get,

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or, $\frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = \frac{-2}{3}x^{\frac{-1}{3}}$

$\therefore \frac{dy}{dx} = -\frac{x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

At (1, 1), $\frac{dy}{dx} = -\frac{1}{1} = -1$

$\therefore$ Slope of tangent (m) = – 1

Now, eq^n. of normal at (1, 1) is

$y - y_1 = \frac{-1}{m}(x - x_1)$

or, $y - 1 = \frac{-1}{-1}(x - 1)$

or, $y - 1 = x - 1$

or, $x = y$

$\therefore$ $x - y = 0$, which is the required equation.

11. (a) *Find the equation of the tangent and normal to the curve $y = x^3 - 2x^2 + 4$ at (2,4).* [2]

(b) *Find the points on the curve $4y = x^4 - 8x^2$ where the tangents are parallel to the x-axis.* [2]

Soln: (a) Given curve is $y = x^3 - 2x^2 + 4$.

Then $\frac{dy}{dx} = 3x^2 - 4x$

At (2,4), $\frac{dy}{dx} = 3 \times 2^2 - 4 \times 2 = 12 - 8 = 4$

$\therefore$ Slope of tangent (m) = 4

Equation of tangent at (2, 4) is $y - y_1 = m(x - x_1)$

or, $y - 4 = 4(x - 2)$

or, $4x - 8 = y - 4$

$\therefore$ $4x - y = 4$

Again, equation of normal at (2,4) is $y - y_1 = \frac{-1}{m}(x - x_1)$

or, $y - 4 = \frac{-1}{4}(x - 2)$

or, $4y - 16 = -x + 2$

$\therefore$ $x + 4y = 18$

(b) Given curve is $4y = x^4 - 8x^2$ ....(i)

Diff. both sides with respect to 'x' we get,

$4\frac{dy}{dx} = 4x^3 - 16x$

$\therefore$ $\frac{dy}{dx} = x^3 - 4x$

Since, the tangent is parallel to the x-axis, then $\frac{dy}{dx} = 0$

or, $x^3 - 4x = 0$

or, $x(x^2 - 4) = 0$

$\therefore$ $x = 0, \pm 2$

putting the values of the x in (i) we get,

When, $x = 0$, $4y = 0 - 0 \Rightarrow y = 0$

When, $x = \pm 2$, $4y = (\pm 2)^4 - 8(\pm 2)^2$

or, $4y = 16 - 32$

or, $4y = -16$

$\therefore$ $y = -4$

$\therefore$ The required points are (0, 0) and ($\pm$ 2, – 4).

12. (a) *Find the point on the curve $y = 3x^2 + x$ where the tangent is parallel to the line $y = 7x + 3$.* [2]

(b) *Find the angle of intersection of the curves $y^2 = x^3$ and $y = 2x$ at (0, 0).* [2]

Soln: (a) Given curve is $y = 3x^2 + x$ ....(i)

Then $\frac{dy}{dx} = 6x + 1$

$\therefore$ Slope of tangent $(m_1) = 6x + 1$

Again, slope of given line $y = 7x + 3$ is $m_2 = 7$.

Since, the tangent is // to the given line, then $m_1 = m_2$

or, $6x + 1 = 7$

or, $6x = 6$

$\therefore$ $x = 1$

From (i), $y = 3.1^2 + 1 = 4$

$\therefore$ Required point is (1, 4).

(b) Given curves are $y^2 = x^3 \Rightarrow y = x^{\frac{3}{2}}$ ....(i)

and $y = 2x$ ....(ii)

Diff. (i) with respect to 'x' we get $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

At (0, 0), $\frac{dy}{dx} = \frac{3}{2} \times 0 = 0$

i.e. $m_1 = 0$

Again, diff. (ii) with respect to 'x' we get $\frac{dy}{dx} = 2$

At (0, 0), $\frac{dy}{dx} = 2$

i.e. $m_2 = 2$

If θ be the angle of intersection then $\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1.m_2}$

$= \pm \frac{0 - 2}{1 + 0 \times 2} = \pm 2$

$\therefore$ $\theta = \tan^{-1}(\pm 2)$

Hence, the required angle $(\theta) = \tan^{-1}(\pm 2)$.

13. *Show that the curves $y = x^3 + 2$ and $y = 2x^2 + 2$ have a common tangent at the point (0, 2).* [3]

Soln: Given curve is $y = x^3 + 2$ ....(i)

Diff. both sides with respect to 'x' we get $\frac{dy}{dx} = 3x^2$

At (0, 2), $\frac{dy}{dx} = 3 \times 0 = 0$

Slope of tangent (m) = 0

Equation of tangent at (0, 2) is $y - 2 = 0(x - 0)$

$\Rightarrow y = 2$

Again, another curve is $y = 2x^2 + 2$ ....(i)

At (1, 1), $\frac{dy}{dx} = -\frac{1}{1} = -1$

$\therefore$ Slope of tangent (m) = − 1

Now, eq^n. of normal at (1, 1) is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

or, $y - 1 = \frac{-1}{-1}(x - 1)$

or, $y - 1 = x - 1$

or, $x = y$

$\therefore$ $x - y = 0$, which is the required equation.

11. (a) *Find the equation of the tangent and normal to the curve $y = x^3 - 2x^2 + 4$ at (2,4).* [2]

(b) *Find the points on the curve $4y = x^4 - 8x^2$ where the tangents are parallel to the x-axis.* [2]

Soln: (a) Given curve is $y = x^3 - 2x^2 + 4$.

Then $\frac{dy}{dx} = 3x^2 - 4x$

At (2,4), $\frac{dy}{dx} = 3 \times 2^2 - 4 \times 2 = 12 - 8 = 4$

$\therefore$ Slope of tangent (m) = 4

Equation of tangent at (2, 4) is $y - y_1 = m(x - x_1)$

or, $y - 4 = 4(x - 2)$

or, $4x - 8 = y - 4$

$\therefore$ $4x - y = 4$

Again, equation of normal at (2,4) is $y - y_1 = \frac{-1}{m}(x - x_1)$

or, $y - 4 = \frac{-1}{4}(x - 2)$

or, $4y - 16 = -x + 2$

$\therefore$ $x + 4y = 18$

(b) Given curve is $4y = x^4 - 8x^2$ ....(i)

Diff. both sides with respect to 'x' we get,

$$4\frac{dy}{dx} = 4x^3 - 16x$$

$\therefore$ $\frac{dy}{dx} = x^3 - 4x$

Since, the tangent is parallel to the x-axis, then $\frac{dy}{dx} = 0$

or, $x^3 - 4x = 0$

or, $x(x^2 - 4) = 0$

$\therefore$ $x = 0, \pm 2$

putting the values of the x in (i) we get,

When, $x = 0$, $4y = 0 - 0 \Rightarrow y = 0$

When, $x = \pm 2$, $4y = (\pm 2)^4 - 8(\pm 2)^2$

or, $4y = 16 - 32$

or, $4y = -16$

$\therefore$ $y = -4$

$\therefore$ The required points are (0, 0) and (± 2, − 4).

12. (a) *Find the point on the curve $y = 3x^2 + x$ where the tangent is parallel to the line $y = 7x + 3$.* [2]

(b) *Find the angle of intersection of the curves $y^2 = x^3$ and $y = 2x$ at (0, 0).* [2]

Soln: (a) Given curve is $y = 3x^2 + x$ ....(i)

Then $\frac{dy}{dx} = 6x + 1$

$\therefore$ Slope of tangent $(m_1) = 6x + 1$

Again, slope of given line $y = 7x + 3$ is $m_2 = 7$.

Since, the tangent is // to the given line, then $m_1 = m_2$

or, $6x + 1 = 7$

or, $6x = 6$

$\therefore$ $x = 1$

From (i), $y = 3.1^2 + 1 = 4$

$\therefore$ Required point is (1, 4).

(b) Given curves are $y^2 = x^3 \Rightarrow y = x^{\frac{3}{2}}$ ....(i)

and $y = 2x$ ....(ii)

Diff. (i) with respect to 'x' we get $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

At (0, 0), $\frac{dy}{dx} = \frac{3}{2} \times 0 = 0$

i.e. $m_1 = 0$

Again, diff. (ii) with respect to 'x' we get $\frac{dy}{dx} = 2$

At (0, 0), $\frac{dy}{dx} = 2$

i.e. $m_2 = 2$

If θ be the angle of intersection then $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1.m_2}$

$$= \pm\frac{0 - 2}{1 + 0 \times 2} = \pm 2$$

$\therefore$ $\theta = \tan^{-1}(\pm 2)$

Hence, the required angle (θ) = $\tan^{-1}(\pm 2)$.

13. *Show that the curves $y = x^3 + 2$ and $y = 2x^2 + 2$ have a common tangent at the point (0, 2).* [3]

Soln: Given curve is $y = x^3 + 2$ ....(i)

Diff. both sides with respect to 'x' we get $\frac{dy}{dx} = 3x^2$

At (0, 2), $\frac{dy}{dx} = 3 \times 0 = 0$

Slope of tangent (m) = 0

Equation of tangent at (0, 2) is $y - 2 = 0(x - 0)$

$\Rightarrow y = 2$

Again, another curve is $y = 2x^2 + 2$ ....(i)

Diff. both sides with respect to 'x' we get $\frac{dy}{dx} = 4x$

At $(0, 2) = \frac{dy}{dx} = 4 \times 0 = 0$

Slope of tangent (m) = 0

Equation of tangent at (0, 2) is $y - 2 = 0\,(x - 0)$

$$\Rightarrow y = 2$$

$\therefore$ Both the equation (i) and (ii) have a common tangent at (0, 2). **Hence Proved.**

14. (a) *Find the point of intersection of the curves $y = 6 - x^2$ and $x^2 = 2y$.* [1]

(b) *Find the angle of intersection of the curves.* [3]

Soln: (a) Given curves are $y = 6 - x^2$ ....(i)

and $x^2 = 2y$ ....(ii)

Eliminating 'x' between (i) and (ii) we get $y = 6 - 2y$

or, $3y = 6$

$\therefore$ $y = 2$

Putting the value of y in equation (ii) we get $x^2 = 2 \times 2$

or, $x^2 = 4$

$\therefore$ $x = \pm 2$

$\therefore$ The point of intersection of the curves (i) and (ii) are (2, 2) and (–2, 2).

(b) Diff. (i) with respect to 'x' we get $\frac{dy}{dx} = -2x = m_1$ (say)

Again, diff. (ii) with respect to 'x' we get $2x = 2\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = x = m_2 \text{ (say)}$$

At (2, 2)

$m_1 = -2 \times 2 = -4$

$m_2 = 2$

If θ be the angle of intersection between (i) and (ii) then, $\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 . m_2}$

$$= \pm \frac{-4-2}{1+(-4).2}$$

$$= \pm \frac{6}{7}$$

$$\therefore \theta = \tan^{-1}\left(\pm \frac{6}{7}\right) \text{ at } (2, 2).$$

Again, at (– 2, 2)

$m_1 = -2 \times (-2) = 4$

$m_2 = -2$

If θ be the angle of intersection of (i) and (ii) then, $\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 . m_2}$

$$= \pm \frac{4+2}{1+4(-2)}$$

$$= \pm \frac{6}{7}$$

$$\theta = \tan^{-1}\left(\pm \frac{6}{7}\right) \text{ at } (-2, 2).$$

15. *The equation of a curve is $y^3 + 4xy = 16$.*

(a) *Show that $\frac{dy}{dx} = -\frac{4y}{3y^2 + 4x}$.* [1]

(b) *Show that the curve has no stationary points.* [1]

(c) *Find the co-ordinate of the point on the curve where the tangent is parallel to the y-axis.* [2]

Soln: (a) Given curve is $y^3 + 4xy = 16$ ....(i)

Diff. both sides with respect to 'x' we get

$$3y^2\frac{dy}{dx} + 4\left(x\frac{dy}{dx} + y.1\right) = 0$$

or, $3y^2\frac{dy}{dx} + 4x\frac{dy}{dx} + 4y = 0$

or, $(3y^2 + 4x)\frac{dy}{dx} = -4y$

$\therefore$ $\frac{dy}{dx} = -\frac{4y}{3y^2 + 4x}$

(b) For stationary point, we have $\frac{dy}{dx} = 0$

or, $\frac{-4y}{3y^2 + 4x} = 0$

or, $-4y = 0$

$\therefore$ $y = 0$

When, y = 0, from (i), $0^3 + 4x.0 = 16$

or, 0 = 16 (false)

$\therefore$ The point y = 0 does not exist.

i.e. the curve has no stationary point.

(c) If the tangent is parallel to the y-axis then $\frac{dx}{dy} = 0$

or, $-\frac{3y^2 + 4x}{4y} = 0$

or, $3y^2 + 4x = 0$

or, $x = -\frac{3y^2}{4}$ ....(ii)

Solving (i) and (ii) we get

$$y^3 + 4.\left(\frac{-3y^2}{4}\right).y = 16$$

or, $y^3 - 3y^3 = 16$

or, $-2y^3 = 16$

or, $y^3 = -8$

$\therefore$ $y = -2$

Putting the value of y in equation (ii) we get $x = \frac{-3\,(-2)^2}{4} = -3$

$\therefore$ Required point is (– 3, – 2).

13.2 Derivative as Rate Measure

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Basic Formulae and Key Points

If s denote the distance covered by a particle in a straight line in time t, then the rate of change of displacement 's' with respect to time is the velocity of particle i.e. velocity

$(V) = \frac{ds}{dt}$

Similarly, acceleration $(a) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Group 'A' (Multiple Choice Questions and Answers)

1. *The side of a square is increasing at the rate of 0.2 cm/sec. At what rate is the perimeter increasing when the side of the square is 12 cm?*
   (a) 0.8 cm/sec (b) 4.8 cm/sec (c) 48 cm/sec. (d) 144 cm/ sec.
2. *The side of a square sheet is increasing at the rate of 5 cm/ min. At what rate is the area increasing when the side is 12 cm long?*
   (a) 20 cm²/min. (b) 25 cm² /min. (c) 120 cm²/min. (d) 144 cm²/min.
3. *If a radius of circular plate on heating is increasing at the rate of 25 cm/sec, then the rate of increase of its circumference is*
   (a) 25π cm / sec (b) 50π cm/sec (c) 2.5π cm/ sec (d) 5π cm/sec
4. *If a stone is thrown into a pond, then it produces a circular ripple that increase at the rate of 0.25 cm/sec. How fast is the area growing when the radius is 7 cm?*
   (a) 7π cm²/sec (b) 14π cm²/sec (c) 2.8π cm²/cm (d) 3.5π cm²/ sec
5. *The rate of change of the volume of a cylinder of radius r and height h with respect to a change in the radius is*
   (a) $\pi r^2 h$ (b) πr^2 (c) $2\pi rh$ (d) πrh
6. *The volume of a sphere is increasing at the rate of 25 cm³/sec. At what rate the radius is increasing at the instant when the total surface area of a sphere is 10π cm²?*
   (a) $\frac{\pi}{4}$ cm/sec (b) $\frac{4}{\pi}$ cm/sec (c) $\frac{5}{2\pi}$ cm/sec (d) 100π cm/sec
7. *If the radius of a circle increases at a constant rate then the area of the circle*
   (a) increases at a constant rate.
   (b) increases at a rate proportional to the radius of the circle.
   (c) increases at a rate inversely proportional to the radius.
   (d) none of them
8. *If the area of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ sq. unit/sec, then the rate of increase of it's side when the side is 2 unit is*
   (a) 1 unit/sec (b) 2 unit/sec (c) $\sqrt{3}$ unit/sec (d) 3 unit/sec
9. *The rate of change of the diagonal of a square having length ℓ with respect to it's area A is*
   (a) $\frac{1}{A}$ (b) A^2 (c) ℓ^2 (d) $\frac{1}{\ell\sqrt{2}}$
10. The rate of change of volume of a sphere with respect to it's surface area, when the radius is 3 cm is
   (a) 1 cm (b) 1.5 cm (c) 2 cm (d) 3 cm

Answer Key

1. a	2. c	3. b	4. d	5. c	6. c	7. b	8. b	9. d	10. b

Group 'B' or 'C' (Subjective Questions and Answers)

1. *A spherical ball of salt dissolving in water decreases it's volume at the rate of 0.75 cm³/ min. Find the rate at which the radius of the salt is decreasing when it's radius is 6 cm.* [2]

Soln: Let, r be the radius and V be the volume of the spherical ball of salt in time 't'

Given, $\frac{dV}{dt} = 0.75$ cm³/min.

At r = 6cm, $\frac{dr}{dt} = ?$

We have, $V = \frac{4}{3}\pi r^3$

Diff. both sides with respect to 't' we get,

$$\frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 \frac{dr}{dt}$$

or, $0.75 = 4\pi r^2 \frac{dr}{dt}$

or, $\frac{75}{100} = 4\pi \times 6^2 \times \frac{dr}{dt}$

or, $\frac{3}{4} = 4\pi \times 36 \times \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{3}{4 \times 4\pi \times 36}$

$= \frac{1}{192\pi}$ cm/min.

2. *A spherical ball of camphor is evaporating is such a way that the rate of decrease in volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate. Explain.* [3] [2081 Supp. Set B]

Soln: Let, r be the radius, V be the volume and s be the surface area of the spherical ball of camphor in time 't'.

Since, $s = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

By given $\frac{dV}{dt} \propto s$.

or, $\frac{dV}{dt} = -ks$, where $-k$ is a constant.

Now, $V = \frac{4}{3}\pi r^3$

Diff. both sides with respect to 't' we get

$$\frac{dV}{dt} = \frac{4}{3}\pi .3r^2 \frac{dr}{dt}$$

or, $-ks = 4\pi r^2 \frac{dr}{dt}$

or, $-ks = s\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = -k$

This shows that the radius is decreasing at a constant rate.

3. ***If the volume of an expanding cube is increasing at the rate of 4 ft³/min, how fast is its surface area increasing when the surface area is 24 sq. ft.?*** [3]

Soln: Let, x be the length of side, V be the volume and s be the surface area of cube in time 't'.

Given, $\frac{dV}{dt} = 4$ ft³/min.

At $s = 24$ ft², $\frac{ds}{dt} = ?$

We have, $s = 6x^2$

or, $24 = 6x^2$

or, $x^2 = 4$

$\therefore$ $x = 2$ ft.

Now, $V = x^3$

Diff. both side with respect to 't' we get,

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

or, $4 = 3 \times 2^2 \times \frac{dx}{dt}$

or, $\frac{dx}{dt} = \frac{4}{3 \times 4} = \frac{1}{3}$ ft/min

Again, $s = 6x^2$

Diff. both sides with respect to 't' we get $\frac{ds}{dt} = 12x\frac{dx}{dt} = 12 \times 2 \times \frac{1}{3}$ ft²/min $= 8$ ft²/min.

4. ***The volume of a spherical ballon is increasing at the rate of 25 cm³/sec. Find the rate of change of its surface at the instant when it's radius is 5 cm.*** [3]

Soln: Let, r be the radius, V be the volume and s be the surface of a spherical ballon in time 't'.

Given that, $\frac{dV}{dt} = 25$ cm³/sec

At $r = 5$ cm, $\frac{ds}{dt} = ?$

We have, $V = \frac{4}{3}\pi r^3$

Diff. both sides with respect to 't' we get,

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt}$$

or, $25 = 4\pi \times 5^2\frac{dr}{dt}$

$\therefore$ $\frac{dr}{dt} = \frac{25}{4\pi \times 25} = \frac{1}{4\pi}$ cm/sec

Again, $s = 4\pi r^2$

Diff. both sides with respect to 't' we get, $\frac{ds}{dt} = 8\pi r\frac{dr}{dt}$

$= 8\pi \times 5 \text{ cm} \times \frac{1}{4\pi}$ cm/sec

$= 10$ cm²/sec

5. ***If the area of a circle increases at a uniform rate, show that the rate of increase of the perimeter varies inversely as the radius.*** [3]

Soln: Let, r be the radius, A be the area and P be the perimeter of the circle in time 't'.

Since, the area of a circle increases at a uniform rate, so let $\frac{dA}{dt} = k$, where k is constant.

We have, $A = \pi r^2$

Diff. both sides with respect to 't' we get

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

or, $k = 2\pi r\frac{dr}{dt}$

$\therefore$ $\frac{dr}{dt} = \frac{k}{2\pi r}$

Again, $P = 2\pi r$

Diff. both sides with respect to 't' we get,

$$\frac{dP}{dt} = 2\pi\frac{dr}{dt}$$

$$= 2\pi \cdot \frac{k}{2\pi r}$$

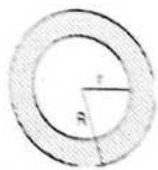
$$= \frac{k}{r}$$

$\therefore$ $\frac{dP}{dt} \propto \frac{1}{r}$

This shows that the rate of increase of the perimeter varies inversely as the radius.

6. ***Two concentric circles are expanding in such a way that the radius of the inner circle is increasing at the rate of 6 cm /sec and that of the outer circle at the rate of 2.5 cm/sec. At a certain time the radius of the inner and the outer circle are respectively 20 cm and 32 cm. At that time, how fast is the area between the circles increasing or decreasing?*** [3]

Soln: Let, r be the radius of inner circle, R be the radius of outer circle and s be the area between two circles in time 't'.



Given that, $\frac{dr}{dt} = 6$ cm/sec, $\frac{dR}{dt} = 2.5$ cm/sec

At, $r = 20$ cm, $R = 32$ cm, $\frac{ds}{dt} = ?$

We have, $s = \pi R^2 - \pi r^2$

Diff. both sides with respect to 't' we get, $\frac{ds}{dt} = 2\pi R\frac{dR}{dt} - 2\pi r\frac{dr}{dt}$

$$= 2\pi\left(R\frac{dR}{dt} - r\frac{dr}{dt}\right)$$

$$= 2\pi(32 \times 2.5 - 20 \times 6)$$

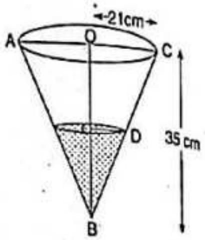
$$= 2\pi \times (-40)$$

$$= -80\pi \text{ cm}^2\text{/sec}$$

Hence the area between the circles is decreasing at the rate of 80π cm²/sec.

7. ***Water flows into a inverted conical tank at the rate of 36 cm³/min. When the depth of water is 12 cm, how fast is level rising, if the radius of base and height of the tank is 21 cm and 35 cm respectively.*** ***[3] [2081 Set – V]***

Soln: Let, ABC be the conical tank with height (BO) = 35 cm and radius (OC) = 21 cm. Let, r be the radius, h be the height and V be the volume of water in conical tank in time 't'.



Given that, $\frac{dV}{dt} = 36$ cm³/min.

At h = 12 cm, $\frac{dh}{dt} = ?$

Since, ΔBED ~ ΔBOC.

$\therefore \frac{BE}{BO} = \frac{ED}{OC}$

or, $\frac{h}{35\text{ cm}} = \frac{r}{21\text{ cm}}$

or, $\frac{h}{5} = \frac{r}{3}$

$\therefore r = \frac{3h}{5}$

Now, $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times \frac{9h^2}{25} \times h = \frac{3\pi}{25} h^3$

Diff. both sides with respect to 't'. we get,

$\frac{dV}{dt} = \frac{3\pi}{25} \times 3h^2 \frac{dh}{dt}$

or, $36 = \frac{3\pi}{25} \times 3 \times 144 \times \frac{dh}{dt}$

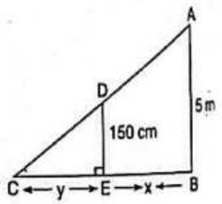
or, $36 \times 25 = 9\pi \times 144 \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{36 \times 25}{9\pi \times 144} = \frac{25}{36\pi}$ cm/min.

8. ***A man 150 cm tall, walks away from a source of light situated at the top of a pole 5m high at the rate of 0.7m/s. Find the rate at which***
 (a) His shadow is lengthening.
 (b) The tip of his shadow is moving, when he is 2 m away from the pole. ***[3+1]***

Soln: (a) Let, AB = 5 m be the pole.

Let, DE be the position of the man, BE = x and CE = y in time 't'.



Given that, $\frac{dx}{dt} = 0.7$ m/s.

At x = 2m, $\frac{dy}{dt} = ?$

Since, ΔABC ~ ΔDEC.

$\therefore \frac{AB}{DE} = \frac{BC}{CE}$

or, $\frac{5m}{1.5m} = \frac{x+y}{y}$ [∵ DE = 150 cm = 1.5 m]

or, $\frac{50}{15} = \frac{x+y}{y}$

or, $15x + 15y = 50y$

or, $15x = 35y$

or, $3x = 7y$

Diff. both sides with respect to 't' we get, $3\frac{dx}{dt} = 7\frac{dy}{dt}$

or, $3 \times 0.7\text{m/s} = 7 \times \frac{dy}{dt}$

$\therefore \frac{dy}{dt} = \frac{3 \times 0.7}{.7} = 0.3$ m/s

∴ His shadow is lengthening at the rate of 0.3 m/s

(b) $\frac{d}{dt}(x + y) = ?$

We have, $\frac{d}{dt}(x + y) = \frac{dx}{dt} + \frac{dy}{dt}$

$= 0.3$ m/s $+ 0.7$ m/s

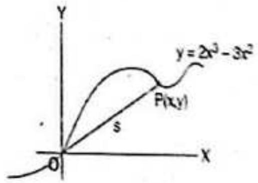
$= 1$m/s

∴ Tip of the shadow is moving at 1m/s.

9. ***A point is moving along the curve $y = 2x^3 - 3x^2$ in such a way that its x-coordinate is increasing at the rate of 4 ft/sec. Find the rate at which the distance of the point from the origin is increasing when the point at (2, 4).*** ***[3]***

Soln: Let, P(x, y) be the moving point along the curve $y = 2x^3 - 3x^2$ ....(i)

Let, OP = s in time 't'.



Given that, $\frac{dx}{dt} = 4$ ft/sec.

At (2,4), $\frac{ds}{dt} = ?$

Diff. both sides of (i) with respect to 't' we get,

$\frac{dy}{dt} = (6x^2 - 6x)\frac{dx}{dt}$

$= (6 \times 2^2 - 6 \times 2) \times 4$ ft/sec

$= 48$ ft/sec

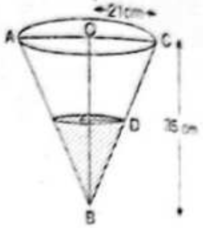
Now, using distance formula,

$s^2 = (x - 0)^2 + (y - 0)^2$

or, $s^2 = x^2 + y^2$

7. ***Water flows into a inverted conical tank at the rate of 36 cm³/min. When the depth of water is 12 cm, how fast is level rising, if the radius of base and height of the tank is 21 cm and 35 cm respectively.*** **[3] [2081 Set – V]**

Soln: Let, ABC be the conical tank with height (BO) = 35 cm and radius (OC) = 21 cm. Let, r be the radius, h be the height and V be the volume of water in conical tank in time 't'.



Given that, $\frac{dV}{dt} = 36\ cm^3/min$

At h = 12 cm, $\frac{dh}{dt} = ?$

Since, $\Delta BED \sim \Delta BOC$

$\therefore \frac{BE}{BO} = \frac{ED}{OC}$

or, $\frac{h}{35\ cm} = \frac{r}{21\ cm}$

or, $\frac{h}{5} = \frac{r}{3}$

$\therefore r = \frac{3h}{5}$

Now, $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times \frac{9h^2}{25} \times h = \frac{3\pi}{25} h^3$

Diff. both sides with respect to 't', we get,

$\frac{dV}{dt} = \frac{3\pi}{25} \times 3h^2 \frac{dh}{dt}$

or, $36 = \frac{3\pi}{25} \times 3 \times 144 \times \frac{dh}{dt}$

or, $36 \times 25 = 9\pi \times 144 \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{36 \times 25}{9\pi \times 144} = \frac{25}{36\pi}$ cm/min.

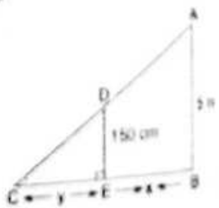
8. ***A man 150 cm tall, walks away from a source of light situated at the top of a pole 5m high at the rate of 0.7m/s. Find the rate at which*** **[3+]**

(a) ***His shadow is lengthening.***

(b) ***The tip of his shadow is moving, when he is 2 m away from the pole.***

Soln: (a) Let, AB = 5 m be the pole.

Let, DE be the position of the man, BE = x and CE = y in time 't'.



Given that, $\frac{dx}{dt} = 0.7$ m/s

At x = 2m, $\frac{dy}{dt} = ?$

Since, $\Delta ABC \sim \Delta DEC$

$\frac{AB}{DE} = \frac{BC}{CE}$

or, $\frac{5m}{1.5m} = \frac{x+y}{y}$

or, $\frac{50}{15} = \frac{x+y}{y}$

or, $15x + 15y = 50y$

or, $15x = 35y$

or, $3x = 7y$

Diff. both sides with respect to 't' we get, $3\frac{dx}{dt} = 7\frac{dy}{dt}$

or, $3 \times 0.7 m/s = 7 \times \frac{dy}{dt}$

$\therefore \frac{dy}{dt} = \frac{3 \times 0.7}{7} = 0.3$ m/s

$\therefore$ His shadow is lengthening at the rate of 0.3 m/s

(b) $\frac{d}{dt}(x+y) = ?$

We have, $\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$

$= 0.7 m/s + 0.3 m/s$

$= 1 m/s$

$\therefore$ Tip of the shadow is moving at 1 m/s.

9. ***A point is moving along the curve*** [illegible] ***the rate of 4 ... when the point at (2, 4).***

Soln: Let P(x, y) be the moving point along the curve ...

Let, OP = s in time 't'.

Given that, $\frac{dx}{dt} = 4$ ...

At (2, 4), $\frac{ds}{dt} = ?$

Diff. both sides of (i) with respect to 't' we get,

$\frac{dy}{dt} = (6x^2 - 6x)\frac{dx}{dt}$

$= (6 \times 2^2 - 6 \times 2) \times 4$ ...

$= 48$ ...

Now, using distance formula,

$s^2 = (x-0)^2 + (y-0)^2$

or, $s^2 = x^2 + y^2$

Diff. both sides of (i) with respect to 't' we get,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

or, $s\frac{ds}{dt} = x\frac{dx}{dt}$

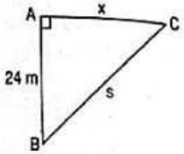
$\therefore$ $\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$

$$= \frac{\frac{1}{2}}{\frac{5}{6}} \times 15 = \frac{1}{2} \times \frac{6}{5} \times 15 \text{ mph} = 9 \text{ mph}.$$

13. ***A kite is 24 m high and there are 25 meters of cord out. If the kite moves horizontally at the rate of 36 km/hr directly away from the person who is flying it, how fast is the cord out?*** [3]

Soln: Let, C be the position of the kite and BC be the cord out.

Let, AC = x and BC = s in time 't'.



Given, $\frac{dx}{dt} = 36$ km/hr.

At $s = 25$ m, $\frac{ds}{dt} = ?$

From figure, $24^2 + x^2 = s^2$ ....(i)

When, $s = 25$ m, $24^2 + x^2 = 25^2$

or, $x^2 = 625 - 576$

or, $x^2 = 49$

$\therefore$ $x = 7$m

Now, diff. both side of (i) with respect to 't' we get,

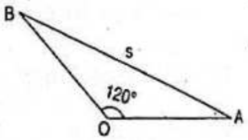
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

or, $\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$

$$= \frac{7m}{25m} \times 36 \text{ km/hr} = 10.08 \text{ km/hr}.$$

14. ***Two cars start at the same time from the junction of two roads one on each road, with uniform speed v mph. If the roads are inclined at 120°, show that the distance between them increases at the rate of $\sqrt{3}$ v mph.*** [3]

Soln: Let, A and B are position of two cars in time 't' with uniform speed v mph. Let, AB = s in time 't'.



Let, $\angle AOB = 120°$.

The distance travelled by first car in t hours i.e. OA = v × t miles.

The distance travelled by second car in t hours i.e. OB = v × t = vt miles.

Now by cosine law, $AB^2 = OA^2 + OB^2 - 2OA.OB \cos\theta$

or, $s^2 = (vt)^2 + (vt)^2 - 2.vt.vt. \cos 120°$

or, $s^2 = v^2t^2 + v^2t^2 - 2v^2t^2 \times \left(\frac{-1}{2}\right)$

$= 3v^2t^2$

$\therefore$ $s = \sqrt{3}\,vt$

Diff. both sides with respect to 't' we get

$\frac{ds}{dt} = \sqrt{3}v$ mph. **Hence Proved.**

15. ***A curve $y^2 = 8x$ changes its abscissa and ordinate at the same rate. The ordinate is double of its abscissa. Justify it with calculation.*** [2] [2081 Optional]

Soln: Given curve is $y^2 = 8x$ ......(i).

Since, abscissa and ordinate change at the same rate. So, let $\frac{dy}{dt} = \frac{dx}{dt} = k$.

Diff. both sides of (i) with respect 't' we get $2y\frac{dy}{dt} = 8\frac{dx}{dt}$

or, $2y.k = 8.k$

or, $y = 4$

Now, putting the value of y in (i) we get,

$4^2 = 8x$

or, $16 = 8x$

$\therefore$ $x = 2$

Clearly $y = 2x$.

$\therefore$ The ordinate is double of its abscissa. Hence Justified.

16. ***Illustrate derivative as a rate measure with suitable example.*** [2]

Soln: Let, us consider the example of car travelling along a straight road.

The position of the car at time t is given by the function s(t) where s represents the distance travelled and t represents time.

Then $\frac{ds}{dt}$ represents the rate at which the position of the car changes with respect to time.

For example, if $s(t) = 2t^2 + 3t$, where s is in meters and t is in seconds, then the derivative $\frac{ds}{dt} = 4t + 3$ represents the velocity of the car at time t.

At $t = 2$, $\frac{ds}{dt} = 4 \times 2 + 3 = 11$

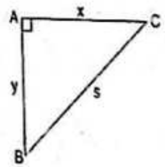
i.e. at t = 2 seconds the car is moving at a velocity 11 m/sec.

Note: Students can use any correct example for this type of question.

17. ***Two cars start from certain places at the same instant. One goes east 60 km/hr and other goes south at 80 km/hr. How fast is the distance between them increasing? Express in symbolic form.*** [2] [2081 Set - W]

Soln: Let, B and C be position of two cars after t hours.

Let, AC = x, AB = y and BC = s in time t.



Then $\frac{dx}{dt} = 60$ km/hr, $\frac{dy}{dt} = 80$ km/hr.

$\frac{ds}{dt} = ?$

From figure, $s^2 = x^2 + y^2$

Diff.t both sides with respect to 't' we get,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or, $s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

$\therefore$ $\frac{ds}{dt} = \frac{1}{s}(x \times 60 + y \times 80)$ km/hr

$$= \frac{60x + 80y}{\sqrt{x^2 + y^2}} \text{ km/hr}.$$

□□□

Chapter 14

Antiderivative

14.1 Antiderivative

Basic Formulae and Key Points

1. **Standard Integrals (I)**

(a) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ or $-\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + c$

(b) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(c) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(d) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$ or $-\cos^{-1}\left(\frac{x}{a}\right) + c$

(e) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c$ or $\sinh^{-1}\left(\frac{x}{a}\right) + c$

(f) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$ or $\cosh^{-1}\left(\frac{x}{a}\right) + c$

Notes: (i) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

(ii) $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

2. **Standard Integrals (II)**

(a) $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

(b) $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

(c) $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Note: $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c, (n \neq -1)$

3. **Standard Integrals (III)**

(a) $\int \operatorname{cosec} x\,dx = \ln\left|\tan\frac{x}{2}\right| + c$

(b) $\int \sec x\,dx = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

4. **Integration of Hyperbolic Functions**

(a) $\int \sinh x\,dx = \cosh x + c$

(b) $\int \cosh x\,dx = \sinh x + c$

(c) $\int \tanh x\,dx = \ln|\cosh x| + c$

(d) $\int \coth x\,dx = \ln|\sinh x| + c$

(e) $\int \operatorname{cosech} x\,dx = \ln\left|\tanh\frac{x}{2}\right| + c$

(f) $\int \operatorname{sech} x\,dx = 2\tan^{-1}\left(\tanh\frac{x}{2}\right) + c$

(g) $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$

(h) $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

(i) $\int \operatorname{sech} x . \tanh x\,dx = -\operatorname{sech} x + c$

(j) $\int \operatorname{cosech} x . \coth x\,dx = -\operatorname{cosech} x + c$

5. **Integration of Rational Fractions**

(a) **Proper Rational Fraction:** A rational fraction $\frac{f(x)}{g(x)}$ is said to be proper if the degree of the polynomial g(x) in the denominator is greater than that of the polynomial f(x) in the numerator.

e.g.: $\frac{3x-4}{(x+2)(5x+7)}$

(b) **Improper Rational Fraction:** A rational fraction $\frac{f(x)}{g(x)}$ is said to be improper if the degree of the polynomial f(x) in the numerator is greater than or equal to that of the polynomial g(x) in the denominator.

e.g.: $\frac{x^2+2}{x^2-7}, \frac{3x^4+8x+2}{x^2-3x+1}$

Notes:

Proper Rational Fractions	Corresponding Partial Fraction
i. $\frac{f(x)}{(x+p)(x+q)}$	$\frac{A}{x+p}+\frac{B}{x+q}$
ii. $\frac{f(x)}{(x+p)(x+q)^2}$	$\frac{A}{x+p}+\frac{B}{x+q}+\frac{C}{(x+q)^2}$
iii. $\frac{f(x)}{(x+p)(qx^2+rx+s)}$	$\frac{A}{x+p}+\frac{Bx+c}{qx^2+rx+s}$

Group 'A' (Multiple Choice Questions and Answers)

1. Which one of the following is the result of $\int \frac{dx}{x^2+4}$? [2081 Optional]

(a) $\frac{1}{2}\tan^{-1}\left(\frac{2}{x}\right)+c$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$

(c) $2\tan^{-1}\left(\frac{x}{2}\right)+c$ (d) $\tan^{-1}\left(\frac{x}{2}\right)+c$

2. What is the integral of $\int \frac{dx}{x^2-36}$? [2080 GIE Set A]

(a) $\frac{1}{12}\ln\left(\frac{x-6}{x+6}\right)+c$ (b) $\frac{1}{2}\ln\left(\frac{x+6}{x-6}\right)+c$

(c) $\frac{1}{6}\ln\left(\frac{x-6}{x+6}\right)+c$ (d) $\frac{1}{6}\ln\left(\frac{x+6}{x-6}\right)+c$

3. What is the integral of $\int \frac{dx}{x^2-1}$? [2080 Set – G]

(a) $2\log\left(\frac{x+1}{x-1}\right)+c$ (b) $\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)+c$

(c) $\log\left(\frac{x-1}{x+1}\right)+c$ (d) $\frac{1}{2}\log\left(\frac{x+1}{x-1}\right)+c$

4. Which one of the following is the integral of $\int \frac{dx}{\sqrt{4x^2-9}}$? [2080 Set – I]

(a) $\frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right|+c$ (b) $2\log\left|x+\sqrt{x^2-\frac{9}{4}}\right|+c$

(c) $\frac{1}{2}\log\left|x-\sqrt{x^2-\frac{9}{4}}\right|+c$ (d) $2\log\left|x-\sqrt{x^2-\frac{9}{4}}\right|+c$

5. Which of the following is the integral value of $\int \frac{dx}{4x^2+1}$? [2080 Optional, 2081 Supp. Set B]

(a) $\tan^{-1}(2x)+c$ (b) $\frac{1}{2}\tan^{-1}(2x)+c$

(c) $2\tan^{-1}\left(\frac{x}{2}\right)+c$ (d) $2\tan^{-1}(2x)+c$

6. $\int \frac{dx}{\sqrt{1-x^2}}$ is equal to .... [2079 GIE Set A]

(a) $\tan^{-1}(x)+k$ (b) $\cos^{-1}(x)+k$

(c) $\sec^{-1}(x)+k$ (d) $\sin^{-1}(x)+k$

7. $\int \frac{dx}{a^2-x^2}$ is equal to ..... [2079 GIE Set B]

(a) $\ln\left|\frac{a+x}{a-x}\right|+c$ (b) $\ln\left|\frac{x-a}{x+a}\right|+c$

(c) $\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$ (d) $\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$

8. If $\int \frac{dx}{4+9x^2} = \frac{1}{6}\tan^{-1}(ax)+c$, then the value of a is

(a) 3 (b) 2 (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

9. $\int \frac{(6x+1)\,dx}{9x^2+3x+10} =$

(a) $\ln|9x^2+3x+10|+c$ (b) $3\ln|9x^2+3x+10|+c$

(c) $\frac{1}{3}\ln|9x^2+3x+10|+c$ (d) $\frac{1}{9}\ln|9x^2+3x+10|+c$

10. $\int \frac{(x-2)\,dx}{\sqrt{2x^2-8x+7}} =$

(a) $\ln\left|\sqrt{4x^2-8x+7}\right|+c$ (b) $4\ln|2x^2-8x+7|+c$

(c) $\frac{1}{2}\sqrt{2x^2-8x+7}+c$ (d) $2\sqrt{2x^2-8x+7}+c$

11. $\int \sqrt{4-x^2}\,dx =$

(a) $\frac{x\sqrt{4-x^2}}{2}-4\sin^{-1}\frac{x}{2}+c$ (b) $\frac{x\sqrt{4-x^2}}{2}+4\sin^{-1}\left(\frac{x}{2}\right)+c$

(c) $\frac{x\sqrt{4-x^2}}{2}-2\sin^{-1}\left(\frac{x}{2}\right)+c$ (d) $\frac{x\sqrt{4-x^2}}{2}+2\sin^{-1}\left(\frac{x}{2}\right)+c$

12. $\int \sqrt{x^2-36}\,dx =$

(a) $\frac{x\sqrt{x^2-36}}{2}-18\ln\left|x+\sqrt{x^2-36}\right|+c$ (b) $\frac{x\sqrt{x^2-36}}{2}+18\ln\left|x+\sqrt{x^2-36}\right|+c$

(c) $\frac{x\sqrt{x^2-36}}{2}-18\ln\left|x-\sqrt{x^2-36}\right|+c$ (d) $\frac{x\sqrt{x^2-36}}{2}-\frac{1}{2}\ln\left|x+\sqrt{x^2-36}\right|+c$

13. If $\int \frac{dx}{(x+1)(x+2)} = \log(x+a)-\log(x+b)+c$, then the values of a and b are respectively

(a) 2, 3 (b) 1, 3 (c) 1, 2 (d) –2, –3

14. If $\int \frac{dx}{5x^2-4} = \frac{1}{4a}\log\left|\frac{ax-2}{ax+2}\right|+c$, then the value of a is

(a) $\frac{2}{5}$ (b) $\frac{\sqrt{2}}{\sqrt{5}}$ (c) $\sqrt{2}$ (d) $\sqrt{5}$

15. If $\int \frac{dx}{e^x+e^{-x}} = \tan^{-1}(f(x))+c$, then the f(x) =

(a) e^{-x} (b) e^x (c) e^{2x} (d) e^x+e^{-x}

Answer Key

1. b	2. a	3. b	4. a	5. b	6. d	7. c	8. c	9. c	10. c
11. d	12. a	13. c	14. d	15. b					

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Write the integral of* $\int \frac{dx}{a^2-x^2}$. [1] [2081 Set W]

(b) *Integrate* $\int \frac{dx}{x^2+9}$. [2] [2081 Set W]

(c) *Integrate* $\int \frac{dx}{1+x-x^2}$. [2]

Soln: (a) The integral of $\int \frac{dx}{a^2-x^2}$ is $\frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$.

(b) $\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+3^2}$

$$= \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + c.$$

(c) $\int \frac{dx}{1+x-x^2} = -\int \frac{dx}{x^2-x-1}$

$$= -\int \frac{dx}{x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} - 1}$$

$$= -\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$$

$$= \int \frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2.\frac{\sqrt{5}}{2}} \ln \left| \frac{\frac{\sqrt{5}}{2} + \left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2} - \left(x-\frac{1}{2}\right)} \right| + c$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1} \right| + c$$

2. (a) *Write the meaning of integral of f(x).* [1] [2080 Set I]

(b) *Write the integral of* $\int \frac{dx}{a^2+x^2}$. [1] [2080 GIE Set A]

(c) *Prove that* $\int \frac{dx}{1-e^x} = \log|e^x| - \log|1-e^x| + c$ [3] [2080 Set G]

Soln: (a) Let, f(x) and g(x) be two functions such that $\frac{d}{dx}(g(x)) = f(x)$ then we say that g(x) is an integral of f(x).

We write $\int f(x)\,dx = g(x) + c$

(b) The integral of $\int \frac{dx}{a^2+x^2}$ is $\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$.

(c) Let, $I = \int \frac{dx}{1-e^x}$

Put, $1 - e^x = y \Rightarrow -e^x\,dx = dy$

$\Rightarrow \quad dx = -\frac{dy}{e^x} = -\frac{dy}{1-y}$

Now, $I = -\int \frac{dy}{(1-y).y}$

$$= -\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy$$

$$= -\left[\log|y| + \frac{\log(1-y)}{-1}\right] + c$$

$$= -\log|y| + \log|1-y| + c$$

$$= -\log|1-e^x| + \log|e^x| + c$$

$$= \log|e^x| - \log|1-e^x| + c$$

3. (a) *Write the integral of* $\int \frac{dx}{x^2-a^2}$. [1] [2081 Set V]

(b) *Integrate* $\int \frac{dx}{e^x+e^{-x}}$. [2]

(c) *Integrate* $\int \frac{dx}{\sqrt{2ax-x^2}}$. [2]

Soln: (a) The integral of $\int \frac{dx}{x^2-a^2}$ is $\frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$.

(b) Let, $I = \int \frac{dx}{e^x+e^{-x}} = \int \frac{dx}{e^x+e^{-x}} \times \frac{e^x}{e^x} = \int \frac{e^x\,dx}{(e^x)^2+1}$

Put, $y = e^x \Rightarrow dy = e^x\,dx$

Now, $I = \int \frac{dy}{y^2+1^2}$

$$= \frac{1}{1} \tan^{-1}\left(\frac{y}{1}\right) + c = \tan^{-1}(e^x) + c$$

(c) Let, $I = \int \frac{dx}{\sqrt{2ax-x^2}}$

$$= \int \frac{dx}{\sqrt{a^2-a^2+2ax-x^2}}$$

$$= \int \frac{dx}{\sqrt{a^2-(a^2-2ax+x^2)}}$$

$$= \int \frac{dx}{\sqrt{a^2-(x-a)^2}}$$

$$= \sin^{-1}\left(\frac{x-a}{a}\right) + c.$$

4. (a) *Write the integral of* $\int \frac{dx}{\sqrt{x^2-a^2}}$. [1] [2081 Optional]

(b) *Integrate* $\int \frac{dx}{3-2x-x^2}$ [2]

(c) *Integrate* $\int \sqrt{\frac{1+x}{1-x}}\, dx$ [2]

Soln: (a) The integral of $\int \frac{dx}{\sqrt{x^2-a^2}}$ is $\ln|x+\sqrt{x^2-a^2}|+c$

(b) Let, $I = \int \frac{dx}{3-2x-x^2}$

$= -\int \frac{dx}{x^2+2x-3}$

$= -\int \frac{dx}{x^2+2x+1-4}$

$= -\int \frac{dx}{(x+1)^2-4}$

$= \int \frac{dx}{2^2-(x+1)^2}$

$= \frac{1}{2.2}\ln\left|\frac{2+(x+1)}{2-(x+1)}\right| + c = \frac{1}{4}\ln\left|\frac{2+x+1}{2-x-1}\right| + c = \frac{1}{4}\ln\left|\frac{3+x}{1-x}\right| + c$

(c) Let, $I = \int \sqrt{\frac{1+x}{1-x}}\, dx$

$= \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}}\, dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}\, dx$

$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\, dx}{\sqrt{1-x^2}}$

$= \int \frac{dx}{\sqrt{1^2-x^2}} - \frac{1}{2}\int \frac{-2x\, dx}{\sqrt{1-x^2}}$

$= \sin^{-1}\left(\frac{x}{1}\right) - \frac{1}{2}\cdot 2\sqrt{1-x^2} + c = \sin^{-1}x - \sqrt{1-x^2} + c.$

5. *Evaluate*

(a) $\int \frac{dx}{(2x-1)\sqrt{4x-3}}$ [2] [2079 Optional Set A]

(b) $\int \frac{dx}{4x^2-9}$ [3] [2079 Optional Set A]

Soln: (a) Let, $I = \int \frac{dx}{(2x-1)\sqrt{4x-3}}$

Put, $4x-3 = y^2 \Rightarrow 4\,dx = 2y\,dy$

i.e. $dx = \frac{y\,dy}{2}$

and $4x = y^2+3$

i.e. $x = \frac{y^2+3}{4}$

Now, $I = \frac{1}{2}\int \frac{y\,dy}{\left\{2\left(\frac{y^2+3}{4}\right)-1\right\}.\sqrt{y^2}}$

$= \frac{1}{2}\int \frac{y\,dy}{\left(\frac{y^2+3}{2}-1\right).y}$

$= \frac{1}{2}\int \frac{dy}{\frac{y^2+3-2}{2}}$

$= \int \frac{dy}{y^2+1^2} = \frac{1}{1}\tan^{-1}\left(\frac{y}{1}\right) + c = \tan^{-1}\left(\sqrt{4x-3}\right) + c.$

(b) Let, $I = \int \frac{dx}{4x^2-9}$

$= \frac{1}{4}\int \frac{dx}{x^2-\frac{9}{4}}$

$= \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{3}{2}\right)^2} = \frac{1}{4}\cdot\frac{1}{2.\frac{3}{2}}\ln\left|\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right| + c = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + c$

6. (a) *Write the integral of* $\int \frac{dx}{\sqrt{a^2-x^2}}$ [1]

(b) *Integrate* $\int \frac{2x+3}{4x^2+1}\, dx$ [2]

(c) *Integrate* $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $(\beta > \alpha)$ [2]

Soln: (a) The integral of $\int \frac{dx}{\sqrt{a^2-x^2}}$ is $\sin^{-1}\left(\frac{x}{a}\right) + c.$

(b) Let, $I = \int \frac{2x+3}{4x^2+1}\, dx$

$= \int \frac{2x\,dx}{4x^2+1} + \int \frac{3\,dx}{4x^2+1}$

$= \frac{1}{4}\int \frac{8x\,dx}{4x^2+1} + \frac{3}{4}\int \frac{dx}{x^2+\frac{1}{4}}$

$= \frac{1}{4}\ln(4x^2+1) + \frac{3}{4}\int \frac{dx}{x^2+\left(\frac{1}{2}\right)^2}$

$= \frac{1}{4}\ln(4x^2+1) + \frac{3}{4}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) + c$

$= \frac{1}{4}\ln(4x^2+1) + \frac{3}{2}\tan^{-1}(2x) + c$

Let. $I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $(\beta > \alpha)$

Put, $x - \beta = y^2$ then $x = y^2 + \beta$ and $dx = 2y\,dy$

Now,

$$I = \int \frac{2y\,dy}{\sqrt{(y^2+\beta-\alpha)\cdot y^2}}$$

$$= \int \frac{2y\,dy}{y\sqrt{y^2+\beta-\alpha}}$$

$$= 2\int \frac{dy}{\sqrt{y^2+\left(\sqrt{\beta-\alpha}\right)^2}}$$

$$= 2\ln\left(y+\sqrt{y^2+\left(\sqrt{\beta-\alpha}\right)^2}\right)+c$$

$$= 2\ln\left(\sqrt{x-\beta}+\sqrt{x-\beta+\beta-\alpha}\right)+c$$

$$= 2\ln\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)+c$$

7. *Evaluate:*

(a) $\int \frac{x\,dx}{\sqrt{3x^2+4}}$ [1]

(b) $\frac{(6x+1)\,dx}{x^2+9}$ [2]

(c) $\int \frac{dx}{\sqrt{1+e^{-2x}}}$ [2]

Soln: (a) $\int \frac{x\,dx}{\sqrt{3x^2+4}} = \frac{1}{6}\int \frac{6x\,dx}{\sqrt{3x^2+4}}$

$$= \frac{1}{6}\cdot 2\sqrt{3x^2+4}+c$$

$$= \frac{1}{3}\sqrt{3x^2+4}+c$$

(b) $\int \frac{(6x+1)\,dx}{x^2+9} = \int \frac{6x\,dx}{x^2+9} + \int \frac{dx}{x^2+9}$

$$= 3\int \frac{2x\,dx}{x^2+9} + \int \frac{dx}{x^2+3^2}$$

$$= 3\ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

(c) Let, $I = \int \frac{dx}{\sqrt{1+e^{-2x}}}$

$$= \int \frac{dx}{\sqrt{1+\frac{1}{e^{2x}}}}$$

$$= \int \frac{dx}{\frac{\sqrt{e^{2x}+1}}{e^x}}$$

$$= \int \frac{e^x\,dx}{\sqrt{(e^x)^2+1}}$$

Put, $y = e^x \Rightarrow dy = e^x\,dx$

Now, $I = \int \frac{dy}{\sqrt{y^2+1^2}}$

$$= \ln\left|y+\sqrt{y^2+1^2}\right|+c$$

$$= \ln\left|e^x+\sqrt{e^{2x}+1}\right|+c$$

8. *Evaluate: (a)* $\int (2x-5)\sqrt{x^2-5x+1}\,dx$ [1]

(b) $\int \frac{dx}{x+\sqrt{x^2-1}}$ [2]

(c) $\int \sqrt{4x^2+4x+5}\,+dx$ [2]

Soln: (a) $\int (2x-5)\sqrt{x^2-5x+1}\,dx = \int (x^2-5x+1)^{\frac{1}{2}}(2x-5)\,dx$

$$= \frac{(x^2-5x+1)^{\frac{3}{2}}}{\frac{3}{2}}+c$$

$$= \frac{2}{3}(x^2-5x+1)^{\frac{3}{2}}+c$$

(b) $\int \frac{dx}{x+\sqrt{x^2-1}} = \int \frac{dx}{x+\sqrt{x^2-1}} \times \frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$

$$= \int \frac{x-\sqrt{x^2-1}}{x^2-x^2+1}\,dx$$

$$= \int x\,dx - \int \sqrt{x^2-1^2}\,dx$$

$$= \frac{x^2}{2} - \left[\frac{x\sqrt{x^2-1^2}}{2} - \frac{1^2}{2}\ln\left|x+\sqrt{x^2-1^2}\right|\right]+c$$

$$= \frac{x^2}{2} - \frac{x}{2}\sqrt{x^2-1} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+c.$$

(c) Let, $I = \int \sqrt{4x^2+4x+5}\,dx$

$$= \int \sqrt{(2x^2)+2.2x.1+1^2+4}\,dx$$

$$= \int \sqrt{(2x+1)^2+2^2}\,dx$$

Put, $y = 2x+1 \Rightarrow dy = 2dx$

i.e. $dx = \frac{dy}{2}$

Now, $I = \frac{1}{2}\int \sqrt{y^2+2^2}\,dy$

$$= \frac{1}{2}\left[\frac{y\sqrt{y^2+2^2}}{2} + \frac{2^2}{2}\ln\left|y+\sqrt{y^2+2^2}\right|\right]+c$$

$$= \frac{1}{4}(2x+1)\sqrt{4x^2+4x+5} + \ln\left|(2x+1)+\sqrt{4x^2+4x+5}\right|+c$$

valuate

(a) $\int\sqrt{2ax-x^2}\,dx$ [2]

(b) $\int(2-x)\sqrt{16-6x-x^2}\,dx$ [3]

Soln: (a) $\int\sqrt{2ax-x^2}\,dx$

$$=\int\sqrt{a^2-a^2+2ax-x^2}\,dx$$
$$=\int\sqrt{a^2-(a^2-2ax+x^2)}\,dx$$
$$=\int\sqrt{a^2-(x-a)^2}\,dx$$
$$=\frac{1}{2}(x-a)\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+c$$
$$=\frac{1}{2}(x-a)\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+c$$

(b) $\int(2-x)\sqrt{16-6x-x^2}\,dx$

$$=\frac{1}{2}\int(4-2x)\sqrt{16-6x-x^2}\,dx$$
$$=\frac{1}{2}\int(-6-2x+10)\sqrt{16-6x-x^2}\,dx$$
$$=\frac{1}{2}\int(-6-2x)\sqrt{16-6x-x^2}\,dx+\frac{10}{2}\int\sqrt{16-6x-x^2}\,dx$$
$$=\frac{1}{2}\int(16-6x-x^2)^{\frac{1}{2}}(-6-2x)\,dx+5\int\sqrt{5^2-(x+3)^2}\,dx$$
$$=\frac{1}{2}\frac{(16-6x-x^2)^{\frac{3}{2}}}{\frac{3}{2}}+5\left[\frac{1}{2}(x+3)\sqrt{5^2-(x+3)^2}+\frac{5^2}{2}\sin^{-1}\left(\frac{x+3}{5}\right)\right]+c$$
$$=\frac{1}{3}(16-6x-x^2)^{\frac{3}{2}}+\frac{5}{2}(x+3)\sqrt{16-6x-x^2}+\frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right)+c$$

10. (a) *Evaluate* $\int\frac{dx}{1+3\cos^2x}$ [2]

(b) *Are there two integrals obtained by* $\int\frac{dx}{a+b\cos x}$*? Justify, obtain the result when a = 3 and b = 2.* [3] [2080 Set I]

Soln: (a) Let, $I=\int\frac{dx}{1+3\cos^2x}$

$$=\int\frac{\sec^2x\,dx}{\sec^2x+3}\quad(\because \text{Dividing both numerator and denominator by } \cos^2x.)$$
$$=\int\frac{\sec^2x\,dx}{1+\tan^2x+3}=\int\frac{\sec^2x\,dx}{\tan^2x+4}$$

Put, $y=\tan x\Rightarrow dy=\sec^2x\,dx$

$$\therefore\quad I=\int\frac{dy}{y^2+2^2}=\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)+c=\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right)+c.$$

(b) Let, $I=\int\frac{dx}{a+b\cos x}$

$$=\int\frac{dx}{a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$
$$=\int\frac{(1+\tan^2\frac{x}{2})\,dx}{a+a\tan^2\frac{x}{2}+b-b\tan^2\frac{x}{2}}$$
$$=\int\frac{\sec^2\frac{x}{2}\,dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

Put, $y=\tan\frac{x}{2}\Rightarrow dy=\sec^2\frac{x}{2}\cdot\frac{1}{2}dx$

$$\Rightarrow 2dy=\sec^2\frac{x}{2}\,dx$$

Case (I): If $a>b$, then $I=\int\frac{2dy}{(a+b)+(a-b)\,y^2}$

$$=\frac{2}{a-b}\int\frac{dy}{\frac{(a+b)}{(a-b)}+y^2}$$
$$=\frac{2}{a-b}\int\frac{dy}{y^2+\left(\sqrt{\frac{a+b}{a-b}}\right)^2}$$
$$=\frac{2}{a-b}\times\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan^{-1}\left(\frac{y}{\sqrt{\frac{a+b}{a-b}}}\right)+c$$
$$=\frac{2}{(\sqrt{a-b})^2}\times\frac{\sqrt{a-b}}{\sqrt{a+b}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\,y\right)+c$$
$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)+c.$$

Case (II): When $a<b$, then $I=\int\frac{2dy}{(a+b)+(a-b)\,y^2}$

$$=\int\frac{2dy}{(a+b)-(b-a)\,y^2}$$
$$=\frac{2}{b-a}\int\frac{dy}{\frac{(a+b)}{(b-a)}-y^2}$$
$$=\frac{2}{b-a}\int\frac{dy}{\left(\sqrt{\frac{b+a}{b-a}}\right)^2-y^2}$$

$$= \frac{2}{b-a} \times \frac{1}{2\sqrt{\frac{b+a}{b-a}}} \ln\left|\frac{\sqrt{\frac{b+a}{b-a}}+y}{\sqrt{\frac{b+a}{b-a}}-y}\right| + c$$

$$= \frac{1}{(\sqrt{b-a})^2} \times \frac{\sqrt{b-a}}{\sqrt{b+a}} \ln\left|\frac{\sqrt{b+a}+\sqrt{b-a}.y}{\sqrt{b+a}-\sqrt{b-a}.y}\right| + c$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{\sqrt{b+a}+\sqrt{b-a}.\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}.\tan\frac{x}{2}}\right| + c$$

From above two cases, we found two integrals of $\int \frac{dx}{a+b\cos x}$ when $a > b$ and $a < b$ respectively.

Hence Justified.

"Last part"

When, $a = 3$ and $b = 2$. i.e. $a > b$.

$$I = \frac{2}{\sqrt{3^2-2^2}} \tan^{-1}\left(\sqrt{\frac{3-2}{3+2}} \tan\frac{x}{2}\right) + c$$

$$= \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{1}{\sqrt{5}} \tan\frac{x}{2}\right) + c$$

11. (a) *What is the integral of $\int \sinh x\, dx$?* [1]

(b) *The integral of $\int \frac{dx}{a+b\cos x}$ depends upon the values of a and b explain.* [4] [2081 Optional]

Soln: (a) The integral of $\int \sinh x\, dx$ is $\cosh x + c$.

(b) See Question no. 10 (b)

12. (a) *Evaluate $\int \frac{dx}{1-3\sin x}$* [3] [2079 GIE Set A]

(b) *Integrate $\int \frac{dx}{3\sin x - 4\cos x}$* [2] [2081 Set V]

Soln: (a) Let, $I = \int \frac{dx}{1-3\sin x}$

$$= \int \frac{dx}{1-3.\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \int \frac{(1+\tan^2\frac{x}{2})\,dx}{1+\tan^2\frac{x}{2}-6\tan\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}-6\tan\frac{x}{2}+1}$$

put, $y = \tan\frac{x}{2} \Rightarrow dy = \sec^2\frac{x}{2}\cdot\frac{1}{2}dx$

$$\Rightarrow 2dy = \sec^2\frac{x}{2}\, dx$$

Now, $I = \int \frac{2dy}{y^2-6y+1}$

$$= \int \frac{2dy}{y^2-2.y.3+3^2+1-9}$$

$$= \int \frac{2dy}{(y-3)^2-8}$$

$$= 2\int \frac{dy}{(y-3)^2-(2\sqrt{2})^2}$$

$$= 2 \times \frac{1}{2.2\sqrt{2}} \ln\left|\frac{y-3-2\sqrt{2}}{y-3+2\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{\tan\frac{x}{2}-3-2\sqrt{2}}{\tan\frac{x}{2}-3+2\sqrt{2}}\right| + c$$

(b) Let, $I = \int \frac{dx}{3\sin x - 4\cos x}$

Put, $3 = r\cos\theta$ and $-4 = r\sin\theta$

$\therefore\ 3^2 + (-4)^2 = r^2(\cos^2\theta + \sin^2\theta)$

or, $25 = r^2 \Rightarrow r = 5$

Again, $\frac{r\sin\theta}{r\cos\theta} = \frac{-4}{3}$

or, $\tan\theta = \frac{-4}{3}$

$\therefore\ \theta = \tan^{-1}\left(\frac{-4}{3}\right)$

Now,

$$I = \int \frac{dx}{r\cos\theta\ \sin x + r\sin\theta\cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\theta)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\ln\left|\tan\left(\frac{x+\theta}{2}\right)\right| + c$$

$$= \frac{1}{5}\ln\left|\tan\left\{\frac{x}{2}+\frac{1}{2}\tan^{-1}\left(\frac{-4}{3}\right)\right\}\right| + c.$$

13. (a) Evaluate $\int \frac{\sin 2x}{(\sin x + \cos x)^2} dx$ [2]

(b) Evaluate $\int \frac{dx}{2 + 3\cos x}$ [3]

Soln: (a) Let, $I = \int \frac{\sin 2x}{(\sin x + \cos x)^2} dx$

$$= \int \frac{1 + \sin 2x - 1}{(\sin x + \cos x)^2} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x + 2\sin x \cos x - 1}{(\sin x + \cos x)^2} dx$$

$$= \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)^2} dx$$

$$= \int dx - \int \frac{dx}{(\sin x + \cos x)^2}$$

$$= x - \int \frac{\sec^2 x\, dx}{(\tan x + 1)^2}$$ [$\because$ Dividing both numerator and denominator by $\cos^2 x$.]

Put, $y = \tan x + 1 \Rightarrow dy = \sec^2 x\, dx$

Now, $I = x - \int \frac{dy}{y^2}$

$$= x - \frac{y^{-1}}{-1} + c = x + \frac{1}{y} + c = x + \frac{1}{1 + \tan x} + c.$$

(b) Let, $I = \int \frac{dx}{2 + 3\cos x}$

$$= \int \frac{dx}{2 + 3\left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)}$$

$$= \int \frac{\left(1 + \tan^2 \frac{x}{2}\right) dx}{2 + 2\tan^2 \frac{x}{2} + 3 - 3\tan^2 \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2}\, dx}{5 - \tan^2 \frac{x}{2}}$$

Put, $y = \tan \frac{x}{2}$ $\Rightarrow dy = \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx$

$\Rightarrow 2dy = \sec^2 \frac{x}{2}\, dx$

Now,

$$I = \int \frac{2dy}{(\sqrt{5})^2 - y^2} = 2\int \frac{dy}{(\sqrt{5})^2 - y^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{5}} \ln \left| \frac{\sqrt{5} + y}{\sqrt{5} - y} \right| + c$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5} + \tan \frac{x}{2}}{\sqrt{5} - \tan \frac{x}{2}} \right| + c$$

14. (a) Evaluate $\int \frac{\coth x\, dx}{\sinh x - 9\,\mathrm{cosech}\, x}$

(b) Evaluate $\int \frac{dx}{1 + \sin x + \cos x}$ [2]

Soln: (a) Let, $I = \int \frac{\coth x\, dx}{\sinh x - 9\,\mathrm{cosech}\, x}$ [3]

$$= \int \frac{\frac{\cosh x}{\sinh x} dx}{\sinh x - \frac{9}{\sinh x}}$$

$$= \int \frac{\cosh x\, dx}{\sinh^2 x - 9}$$

Put, $\sinh x = y \Rightarrow \cosh x\, dx = dy$.

Now, $I = \int \frac{dy}{y^2 - 3^2}$

$$= \frac{1}{2.3} \ln \left| \frac{y - 3}{y + 3} \right| + c$$

$$= \frac{1}{6} \ln \left| \frac{\sinh x - 3}{\sinh x + 3} \right| + c$$

(b) Let, $I = \int \frac{dx}{1 + \sin x + \cos x}$

$$= \int \frac{dx}{1 + \frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$$

$$= \int \frac{\left(1 + \tan^2 \frac{x}{2}\right) dx}{1 + \tan^2 \frac{x}{2} + 2\tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}$$

$$= \int \frac{\sec^2 \frac{x}{2}\, dx}{2 + 2\tan \frac{x}{2}} = \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}\, dx}{1 + \tan \frac{x}{2}}$$

Put, $y = 1 + \tan \frac{x}{2}$ $\Rightarrow dy = \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx$

$\Rightarrow 2dy = \sec^2 \frac{x}{2} dx$

Now,

$$I = \frac{1}{2} \int \frac{2dy}{y}$$

$$= \int \frac{dy}{y}$$

$$= \ln |y| + c = \ln \left| 1 + \tan \frac{x}{2} \right| + c$$

15 (a) *Evaluate* $\int \frac{(\cos x - \sin x)\,dx}{\sqrt{\sin 2x}}$ [2]

(b) *Evaluate* $\int \frac{dx}{3 + 5\cosh x}$ [3]

Soln: (a) Let, $I = \int \frac{(\cos x - \sin x)\,dx}{\sqrt{\sin 2x}}$

$$= \int \frac{(\cos x - \sin x)\,dx}{\sqrt{1 + \sin 2x - 1}}$$

$$= \int \frac{(\cos x - \sin x)\,dx}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x - 1}}$$

$$= \int \frac{(\cos x - \sin x)\,dx}{\sqrt{(\sin x + \cos x)^2 - 1}}$$

Put, $y = \sin x + \cos x \Rightarrow dy = (\cos x - \sin x)\,dx$

Now, $I = \int \frac{dy}{\sqrt{y^2 - 1^2}}$

$$= \ln\left|y + \sqrt{y^2 - 1^2}\right| + c$$

$$= \ln\left|(\sin x + \cos x) + \sqrt{\sin 2x}\right| + c$$

(b) Let, $I = \int \frac{dx}{3 + 5\cosh x}$

$$= \int \frac{dx}{3\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + 5\left(\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{3\cosh^2\frac{x}{2} - 3\sinh^2\frac{x}{2} + 5\cosh^2\frac{x}{2} + 5\sinh^2\frac{x}{2}}$$

$$= \int \frac{dx}{2\sinh^2\frac{x}{2} + 8\cosh^2\frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2\frac{x}{2}\,dx}{2\tanh^2\frac{x}{2} + 8}$$ [∵ Dividing both numerator and denominator by $\cosh^2\frac{x}{2}$]

Put, $y = \tanh\frac{x}{2} \Rightarrow dy = \operatorname{sech}^2\frac{x}{2}\cdot\frac{1}{2}\,dx$

$$\Rightarrow 2dy = \operatorname{sech}^2\frac{x}{2}\,dx$$

Now,

$$I = \int \frac{2dy}{2y^2 + 8} = \int \frac{dy}{y^2 + 4} = \int \frac{dy}{y^2 + 2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tanh\frac{x}{2}\right) + c.$$

16. (a) *Evaluate* $\int \frac{dx}{\sin x + \cos x}$ [2]

(b) *Evaluate* $\int \frac{(1 - \cos x)\,dx}{\sin x + \tan x}$ [3]

Soln: (a) Let, $I = \int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{1.\sin x + 1.\cos x}$

Put, $1 = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow\ 1^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow\ 2 = r^2$

$\therefore\ r = \sqrt{2}$

Again,

$$\frac{r\sin\theta}{r\cos\theta} = \frac{1}{1}$$

$\Rightarrow\ \tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore\ \theta = \frac{\pi}{4}$

$$I = \int \frac{dx}{r\cos\theta\,\sin x + r\sin\theta\,\cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x + \theta)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x + \theta)\,dx$$

$$= \frac{1}{r}\ln\left|\tan\left(\frac{x + \theta}{2}\right)\right| + c$$

$$= \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x}{2} + \frac{1}{2}\cdot\frac{\pi}{4}\right)\right| + c$$

$$= \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + c$$

(b) Let, $I = \int \frac{1 - \cos x}{\sin x + \tan x}\,dx$

$$= \int \frac{1 - \cos x}{\sin x + \frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x\,(1 - \cos x)}{\sin x\cos x + \sin x}\,dx$$

$$= \int \frac{\cos x\,(1 - \cos x)}{\sin x\,(1 + \cos x)} \times \frac{1 + \cos x}{1 + \cos x}\,dx$$

$$= \int \frac{\cos x\,(1 - \cos^2 x)}{\sin x\,(1 + \cos x)^2}\,dx$$

$$= \int \frac{\cos x\,.\sin^2 x}{\sin x\,(1 + \cos x)^2}\,dx = \int \frac{\sin x\cos x\,dx}{(1 + \cos x)^2}$$

Put, $y = \cos x \Rightarrow dy = -\sin x\, dx$

$\Rightarrow -dy = \sin x\, dx$

Now,

$$I = -\int \frac{y\, dy}{(1+y)^2}$$

$$= -\int \frac{(1+y-1)\, dy}{(1+y)^2}$$

$$= -\int \frac{dy}{1+y} + \int \frac{dy}{(1+y)^2}$$

$$= -\ln|1+y| + \left(\frac{1+y}{-1}\right)^{-1} + c$$

$$= -\ln|1+\cos x| - \frac{1}{1+\cos x} + c.$$

17. (a) Evaluate $\int \frac{x}{(x-a)(x-b)} dx$ [2]

(b) $\int \frac{x}{(x-1)(x^2+1)} dx$ [3] [2081 Set V]

Soln: (a) Let, $\frac{x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

$\therefore$ $x = A(x-b) + B(x-a)$ ....(i)

Put, $x = b$ in (i) we get, $b = B(b-a)$

$\Rightarrow$ $B = \frac{b}{b-a}$

Again, put $x = a$ in (i) we get, $a = A(a-b)$

$\Rightarrow$ $A = \frac{a}{a-b}$

Now,

$$\int \frac{x\, dx}{(x-a)(x-b)} = \frac{a}{a-b}\int \frac{dx}{x-a} + \frac{b}{b-a}\int \frac{dx}{x-b}$$

$$= \frac{a}{a-b} \ln|x-a| - \frac{b}{a-b} \ln|x-b| + c$$

$$= \frac{1}{a-b}\left[a \ln|x-a| - b \ln|x-b|\right] + c$$

(b) Let, $\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\therefore$ $x = A(x^2+1) + (Bx+C)(x-1)$ ....(i)

Put, $x = 1$, in (i) we get $1 = 2A$

$\Rightarrow$ $A = \frac{1}{2}$

Again, put $x = 0$ in (i) we get $0 = A - C$

$\Rightarrow$ $0 = \frac{1}{2} - C$

$\therefore$ $C = \frac{1}{2}$

Again, put $x = -1$ in (i) we get,

$-1 = 2A + (-B + C)(-2)$

or, $-1 = 2A + 2B - 2C$

or, $-1 = 2 \cdot \frac{1}{2} + 2B - 2 \cdot \frac{1}{2}$

or, $-1 = 1 + 2B - 1$

or, $-1 = 2B$

$\therefore$ $B = \frac{-1}{2}$

$$\text{Now, } \int \frac{x\, dx}{(x-1)(x^2+1)} = \frac{1}{2}\int \frac{dx}{x-1} + \int \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} dx$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{2}\int \frac{x\, dx}{x^2+1} + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{4}\int \frac{2x\, dx}{x^2+1} + \frac{1}{2}\int \frac{dx}{x^2+1^2}$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \cdot \frac{1}{1} \tan^{-1}\left(\frac{x}{1}\right) + c$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2}\tan^{-1}x + c$$

18. (a) Evaluate $\int \frac{dx}{(4+x^2)(9+x^2)}$. [2] [2079 GIE Set – A]

(b) Using partial fraction integrate $\int \frac{8}{(2x-1)(2x+1)} dx$. [3] [2080 Set – I]

Soln: (a) Let, $I = \int \frac{dx}{(4+x^2)(9+x^2)}$

Since, $(9+x^2) - (4+x^2) = 9 + x^2 - 4 - x^2 = 5$.

$$\therefore \quad I = \frac{1}{5}\int \left(\frac{1}{4+x^2} - \frac{1}{9+x^2}\right) dx$$

$$= \frac{1}{5}\int \left(\frac{1}{x^2+2^2} - \frac{1}{x^2+3^2}\right) dx$$

$$= \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right] + c$$

(b) Let, $\frac{8}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1}$

$\therefore$ $8 = A(2x+1) + B(2x-1)$ ....(i).

Put, $x = \frac{1}{2}$ in (i) we get, $8 = A(1+1)$

$\Rightarrow$ $2A = 8$

$\therefore$ $A = 4$

Again, put $x = \frac{-1}{2}$ in (i) we get, $8 = B(-1-1)$

$\Rightarrow$ $-2B = 8$

$\therefore$ $B = -4$

Now, $\int\frac{8}{(2x-1)(2x+1)}dx = \int\frac{4}{2x-1}dx + \int\frac{-4}{2x+1}dx$

$= 2\int\frac{2\,dx}{2x-1} - 2\int\frac{2\,dx}{2x+1}$

$= 2\ln|2x-1| - 2\ln|2x+1| + c$

$= 2\ln\left|\frac{2x-1}{2x+1}\right| + c$

19. ***Define proper rational fraction with an example.***

Integrate $\int\frac{2x^2+3}{x^3+3x^2+2x}dx$ using concept of partial fraction. [1+4] [2079 Set – K]

Soln: Proper rational fraction: A rational fraction $\frac{f(x)}{g(x)}$ is said to be proper if the degree of the polynomial g(x) in the denominator is greater than that of the polynomial f(x) in the numerator.

Example: $\frac{x}{(x+1)(3x-1)}, \frac{2x^2+5}{(x+2)(x-3)^2}$.

"Next Part"

Let, $I = \int\frac{(2x^2+3)\,dx}{x^3+3x^2+2x} = \int\frac{(2x^2+3)\,dx}{x(x^2+3x+2)} = \int\frac{(2x^2+3)\,dx}{x(x+2)(x+1)}$

Let, $\frac{2x^2+3}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

$\therefore$ $2x^2+3 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$ ....(i).

Put, $x = 0$ in (i) we get, $3 = 2A \Rightarrow A = \frac{3}{2}$.

Put, $x = -1$ in (i) we get, $5 = -B \Rightarrow B = -5$

Again, put $x = -2$ in (i) we get, $11 = 2C \Rightarrow C = \frac{11}{2}$

Now,

$I = \frac{3}{2}\int\frac{dx}{x} - 5\int\frac{dx}{x+1} + \frac{11}{2}\int\frac{dx}{x+2}$

$= \frac{3}{2}\ln|x| - 5\ln|x+1| + \frac{11}{2}\ln|x+2| + c$

20. **(a)** ***Integrate $\int\frac{x^2\,dx}{(x+2)^2(x+3)}$***

What concept is used to integrate the above integral? [2] [2080 GIE Set – A]

(b) ***Integrate $\int\frac{5dx}{(x+5)(2x^2+5)}$*** [3]

Soln: (a) Let, $\frac{x^2}{(x+2)^2(x+3)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$\therefore$ $x^2 = A(x+2)^2 + B(x+3)(x+2) + C(x+3)$ ...... (i)

Put, $x = -2$ in (i) we get, $4 = C \Rightarrow C = 4$

Put, $x = -3$ in (i) we get, $9 = A \Rightarrow A = 9$

Again, put $x = 0$ in (i) we get, $0 = 4A + 6B + 3C$

or, $4.9 + 6B + 3.4 = 0$

or, $6B = -48$

$\therefore$ $B = -8$

Now, $\int\frac{x^2\,dx}{(x+2)^2(x+3)} = 9\int\frac{dx}{x+3} - 8\int\frac{dx}{x+2} + 4\int\frac{dx}{(x+2)^2}$

$= 9\ln|x+3| - 8\ln|x+2| + 4.\frac{(x+2)^{-1}}{-1} + c$

$= 9\ln|x+3| - 8\ln|x+2| - \frac{4}{x+2} + c$

Concept of partial fraction is used to integrate the above integral.

(b) Let, $\frac{5}{(x+5)(2x^2+5)} = \frac{A}{x+5} + \frac{Bx+C}{2x^2+5}$

$\therefore$ $5 = A(2x^2+5) + (Bx+C)(x+5)$ ....(i).

Put, $x = -5$ in (i) we get, $5 = 55A \Rightarrow A = \frac{1}{11}$

Again, put $x = 0$ in (i) we get, $5 = 5A + 5C$

or, $1 = A + C$

or, $1 = \frac{1}{11} + C$

$\therefore$ $C = 1 - \frac{1}{11} = \frac{10}{11}$

Now, put $x = 1$ in (i) we get, $5 = 7A + 6B + 6C$

or, $5 = \frac{7}{11} + 6B + 6.\frac{10}{11}$

or, $5 - \frac{7}{11} - \frac{60}{11} = 6B$

or, $6B = \frac{-12}{11}$

$\therefore$ $B = \frac{-2}{11}$

Now,

$$\int\frac{5dx}{(x+5)(2x^2+5)} = \int\frac{\frac{1}{11}}{x+5}dx + \int\frac{\frac{-2}{11}x+\frac{10}{11}}{2x^2+5}dx$$

$$= \frac{1}{11}\int\frac{dx}{x+5} - \frac{2}{11}\int\frac{x\,dx}{2x^2+5} + \frac{10}{11}\int\frac{dx}{2x^2+5}$$

$$= \frac{1}{11}\ln|x+5| - \frac{2}{11}\times\frac{1}{4}\int\frac{4x\,dx}{2x^2+5} + \frac{10}{11.2}\int\frac{dx}{x^2+\left(\sqrt{\frac{5}{2}}\right)^2} + c$$

$$= \frac{1}{11}\ln|x+5| - \frac{1}{22}\ln|2x^2+5| + \frac{5}{11}\cdot\frac{1}{\sqrt{\frac{5}{2}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{5}{2}}}\right) + c$$

$$= \frac{1}{11}\ln|x+5| - \frac{1}{22}\ln|2x^2+5| + \frac{\sqrt{10}}{11}\tan^{-1}\left(\sqrt{\frac{2}{5}}\,x\right) + c$$

21. (a) Integrate $\int \frac{dx}{x(1-x)}$ [2] [2079 Set – J]

(b) Integrate $\int \frac{dx}{x^4+x^2+1}$ [3] [2079 GIE Set B]

Soln: (a) Let, $I = \int \frac{dx}{x(1-x)}$

Since, $x + (1-x) = x + 1 - x = 1$

$$\therefore \quad I = \int\left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$

$$= \ln|x| + \frac{\ln|1-x|}{-1} + c$$

$$= \ln|x| - \ln|1-x| + c = \ln\left|\frac{x}{1-x}\right| + c.$$

(b) $I = \int \frac{dx}{x^4+x^2+1}$

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+x^2+1} dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)\,dx}{x^4+x^2+1} - \frac{1}{2}\int \frac{(x^2-1)\,dx}{x^4+x^2+1}$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+1+\frac{1}{x^2}} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+1+\frac{1}{x^2}}$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2.x.\frac{1}{x}+1} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2.x.\frac{1}{x}+1}$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$$

Put, $x - \frac{1}{x} = y \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dy$ and put, $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

Now, $I = \frac{1}{2}\int \frac{dy}{y^2+(\sqrt{3})^2} - \frac{1}{2}\int \frac{dt}{t^2-1^2}$

$$= \frac{1}{2}\times\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right) - \frac{1}{2}\cdot\frac{1}{2.1}\ln\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right) - \frac{1}{4}\ln\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) - \frac{1}{4}\ln\left|\frac{x^2+1-x}{x^2+1+x}\right| + c$$

22. (a) Evaluate $\int \frac{(2x-11)dx}{x^2+x-2}$ [2]

(b) Evaluate $\int \frac{x^3\,dx}{2x^4-3x^2-5}$ [3]

Soln: (a) Let, $I = \int \frac{(2x-11)dx}{x^2+x-2} = \int \frac{(2x-11)dx}{(x+2)(x-1)}$

Let, $\frac{(2x-11)}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$\therefore \quad 2x - 11 = A(x-1) + B(x+2)$ ....(i)

Put, $x = -2$ in (i) we get, $-15 = -3A \Rightarrow A = 5$

Put, $x = 1$ in (i) we get, $-9 = 3B \Rightarrow B = -3$

Now, $I = \int \frac{5}{x+2}dx + \int \frac{-3\,dx}{x-1}$

$$= 5\int\frac{dx}{x+2} - 3\int\frac{dx}{x-1}$$

$$= 5\ln|x+2| - 3\ln|x-1| + c.$$

(b) Put, $x^2 = y$ then $2x\,dx = dy$

i.e. $x\,dx = \frac{dy}{2}$

Now, $\int \frac{x^3\,dx}{2x^4-3x^2-5} = \int \frac{x^2.x\,dx}{2(x^2)^2-3x^2-5}$

$$= \frac{1}{2}\int\frac{y\,dy}{2y^2-3y-5}$$

$$= \frac{1}{2}\int\frac{y\,dy}{2y^2-5y+2y-5}$$

$$= \frac{1}{2}\int\frac{y\,dy}{y(2y-5)+1(2y-5)}$$

$$= \frac{1}{2}\int\frac{y\,dy}{(y+1)(2y-5)}$$

Let, $\frac{y}{(y+1)(2y-5)} = \frac{A}{y+1} + \frac{B}{2y-5}$

$\therefore \quad y = A(2y-5) + B(y+1)$ ....(i)

Put, $y = -1$ in (i) we get, $-1 = -7A \Rightarrow A = \frac{1}{7}$

Again, put $y = 0$ in (i) we get, $0 = -5A + B$

or, $B = 5A = \frac{5}{7}$

Now, $\int \frac{x^3dx}{2x^4-3x^2-5} = \frac{1}{2}\int \frac{y\,dy}{(y+1)(2y-5)}$

$$= \frac{1}{2}\left[\int \frac{\frac{1}{7}}{y+1}dy + \int \frac{\frac{5}{7}}{2y-5}dy\right]$$

$$= \frac{1}{2}\left[\frac{1}{7}\int \frac{dy}{y+1} + \frac{5}{7}\int \frac{dy}{2y-5}\right]$$

$$= \frac{1}{2}\left[\frac{1}{7}\ln|y+1| + \frac{5}{7}\,\frac{\ln|2y-5|}{2}\right] + c$$

$$= \frac{1}{14}\ln|x^2+1| + \frac{5}{28}\ln|2x^2-5| + c$$

23. *Give an example of proper rational fraction and improper rational fraction.*

Integrate $\int \frac{x^2}{x^4-2x^2-15}dx$ [2 + 3]

Soln: Example of proper rational fraction: $\frac{x-5}{x^2+x-20}$

Example of improper rational fraction: $\frac{x^3}{(x+3)(x+2)}$

"Next part"

Put, $x^2 = y$, then $\frac{x^2}{x^4-2x^2-15} = \frac{y}{y^2-2y-15} = \frac{y}{(y+3)(y-5)}$

Let, $\frac{y}{(y+3)(y-5)} = \frac{A}{y+3} + \frac{B}{y-5}$

$\therefore\quad y = A(y-5) + B(y+3)$ ....(i)

Put, $y = 5$ in (i) we get, $5 = 8B \Rightarrow B = \frac{5}{8}$

Put, $y = -3$ in (i) we get, $-3 = -8A \Rightarrow A = \frac{3}{8}$

Now, $\int \frac{x^2dx}{x^4-2x^2-15} = \int\left(\frac{\frac{3}{8}}{y+3} + \frac{\frac{5}{8}}{y-5}\right)dx$

$$= \frac{3}{8}\int \frac{dx}{x^2+3} + \frac{5}{8}\int \frac{dx}{x^2-5}$$

$$= \frac{3}{8}\int \frac{dx}{x^2+(\sqrt{3})^2} + \frac{5}{8}\int \frac{dx}{x^2-(\sqrt{5})^2}$$

$$= \frac{3}{8}\times\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{5}{8}\times\frac{1}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$$

$$= \frac{\sqrt{3}}{8}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{\sqrt{5}}{16}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c.$$

24. *Evaluate* $\int \frac{dx}{(x-2)^2(x-3)^3}$ [4]

Soln: Let, $I = \int \frac{dx}{(x-2)^2(x-3)^3} = \int \frac{dx}{\left(\frac{x-2}{x-3}\right)^2(x-3)^5}$

Put, $\frac{x-2}{x-3} = y$

or, $x-2 = xy-3y$

or, $3y-2 = xy-x$

$\therefore\quad x = \frac{3y-2}{y-1}$

Also $\frac{dx}{dy} = \frac{(y-1).3-(3y-2).1}{(y-1)^2} = \frac{3y-3-3y+2}{(y-1)^2} = \frac{-1}{(y-1)^2}$

$\therefore\quad dx = \frac{-dy}{(y-1)^2}$

and $x-3 = \frac{3y-2}{y-1} - 3 = \frac{3y-2-3y+3}{y-1} = \frac{1}{y-1}$

Now, $I = \int \frac{1}{y^2\left(\frac{1}{y-1}\right)^5} \times \frac{-dy}{(y-1)^2}$

$$= -\int \frac{(y-1)^3\,dy}{y^2}$$

$$= -\int\left(\frac{y^3-3y^2+3y-1}{y^2}\right)dy$$

$$= -\int\left(y-3+\frac{3}{y}-y^{-2}\right)dy$$

$$= -\left(\frac{y^2}{2} - 3y + 3\ln(y) - \frac{y^{-1}}{-1}\right) + c$$

$$= -\left(\frac{y^2}{2} - 3y + 3\ln(y) + \frac{1}{y}\right) + c$$

$$= -\frac{1}{2}\left(\frac{x-2}{x-3}\right)^2 + 3\left(\frac{x-2}{x-3}\right) - 3\ln\left|\frac{x-2}{x-3}\right| - \left(\frac{x-3}{x-2}\right) + c.$$

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Chapter 15

Differential Equations

15.1 Differential Equations

Basic Formulae and Key Points

1. **Differential Equation:** Any equation which involves the differential coefficients with or without the independent variable or dependent variable or both is called a differential equation.

   e.g. $\frac{dy}{dx} = x$

   e.g. $\frac{d^2y}{dx^2} - y = 0$

2. **Order of a Differential Equation:** The order of a differential equation is the order of the highest derivative involving in the equation.

3. **Degree of a Differential Equation:** The degree of the differential equation is the degree (power) of derivative of highest order when differential equation should be polynomial in derivatives.

   Note: Before finding the degree of a differential equation, it is noted that the coefficient should be free from radical sign.

4. **Equation of First Order and First Degree**

   A differential equation of the form $\frac{dy}{dx} = f(x, y)$ and $f(x, y)\,dx + g(x, y)\,dy = 0$ are called a first order and first degree differential equation.

   (i) **Variable Separable Form:** An equation of the form X dx = Y dy where X is the function of x alone and Y is the function of y alone is called variable separate form.

   e.g. $(x^2 - 1)\,dx = (y^2 + 1)\,dy$

   (ii) **Homogeneous Differential Equation:** A differential equation of first order and first degree is said to be homogeneous if it can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

   e.g. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

   This type of equation can be solved by putting $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

   (iii) **Exact Differential Equation:** A differential equation of the form M dx + N dy = 0 is called exact if there exist a function f(x, y) such that M dx + N dy = d(f(x, y))

   e.g. $x\,dy + y\,dx = 0$

   Since, $x\,dy + y\,dx = d(xy)$

   (iv) **Linear Differential Equation:** A first order and first degree differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x or constants, is called a linear differential equation.

   e.g. $\frac{dy}{dx} + xy = 1$

   Note: To solve such equation, we multiply both sides by $e^{\int P dx}$.
   i.e. Integrating factor (I.F.) $= e^{\int P dx}$

Group 'A' (Multiple Choice Questions and Answers)

1. What is the order of the differential equation $\left(\frac{d^4y}{dx^4}\right)^2 + \left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{d^2y}{dx^2}\right)^2 + y = 0$? [2079 Optional Set A]

   (a) 5 (b) 4 (c) 3 (d) 2

2. If the degree of the differential equation $\frac{d^4y}{dx^4} + 2\left(\frac{d^3y}{dx^3}\right)^2 + 3\left(\frac{d^2y}{dx^2}\right)^3 + 4\left(\frac{dy}{dx}\right)^4 + 5 = 0$ is 1, what is its order? [2079 Set-J]

   (a) 1 (b) 2 (c) 3 (d) 4

3. What is the degree of ordinary differential equation $\frac{d^3s}{dt^3} = \left[4 + \left(\frac{d^2s}{dt^2}\right)^2\right]^{\frac{1}{2}}$? [2081 Optional]

   (a) 1 (b) 2 (c) 3 (d) 4

4. If p and q are the order and degree of differential equation $4\frac{\left(\frac{dy}{dx}\right)^3}{\frac{d^2y}{dx^2}} + \frac{d^2y}{dx^2} = x$, then pq =

   (a) 2 (b) 4 (c) 6 (d) 8

5. The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{dy}{dx}\right)$ is

   (a) 1 (b) 2 (c) 3 (d) Not defined

6. The differential equation with general solution $x^2 + y^2 = c^2$ is

   (a) $x + y\frac{dy}{dx} = 0$ (b) $x - y\frac{dy}{dx} = 0$

   (c) $y + x\frac{dy}{dx} = 0$ (d) $y - x\frac{dy}{dx} = 0$

7. Which one of the following is the solution of differential equation $x\,dy - y\,dx = 0$? [2081 Set-V]

   (a) $x = cy$ (b) $y = cx$ (c) $xy = c$ (d) $x - y = c$

8. Which one of the following is an example of homogeneous differential equation of first order? [2081 Optional]

   (a) $3x\,dy + 2y\,dx = 4$ (b) $x\,dy - y\,dx + 1 = 0$

   (c) $x\,dx + y\,dy = 2$ (d) $(x^2 + xy)\,dy - (xy - y^2)\,dx = 0$

9. Which one of the following is not a homogenous differential equation?

   (a) $xdy + ydx = 0$ (b) $\frac{dy}{dx} = \frac{y + 1}{x + y + 1}$

   (c) $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$ (d) $(x^2 + y^2)\,dx = 2xy\,dy$

10. Which one is the exact differential equation?

    (a) $2xy\,dy - y^2\,dx = 0$ (b) $2xy\,dx - x^2\,dy = 0$

    (c) $xdy - ydx = 0$ (d) $2xy\,dy + y^2dx = 0$.

11. The integrating factor, by which $2xydy - y^2\,dx = 0$ must be multiplied to make it exact, is

    (a) $\frac{1}{x}$ (b) $\frac{1}{y}$ (c) $\frac{1}{x^2}$ (d) $\frac{1}{y^2}$

12. *The total differential of $\frac{ydx - xdy}{x^2 + y^2}$ is*

(a) $d\left(\tan^{-1}\frac{y}{x}\right)$ (b) $d\left(\tan^{-1}\frac{x}{y}\right)$

(c) $d\{\ln(x^2 + y^2)\}$ (d) $d\{\frac{1}{2}\ln(x^2 + y^2)\}$

13. *Which one of the following differential equations gives integrating factor?* [2081 Set- W]

(a) $xdy - y^2dx = 0$ (b) $x^2dy - xy^2dx = 0$

(c) $\sin^2x\frac{dy}{dx} + y = 2$ (d) $3xy\,dy - y^2dx = 0$

14. *The integrating factor of differential equation $\sin x\frac{dy}{dx} + y\cos x = x\sin x$ is* [2079 Set- K]

(a) sinx (b) cosx (c) tanx (d) cotx

15. *What is the integration factor of the differential equation $\cos^2x\frac{dy}{dx} + y = 1$?*

(a) tanx (b) $e^{\tan x}$ (c) $\sec^2x$ (d) $e^{\sec x}$

16. *Which one of the following is integrating factor of $\frac{dy}{dx} + \frac{1}{x}y = x^2$?* [Supp. 2081 Set- A]

(a) $\frac{1}{x}$ (b) x (c) e^x (d) logx

Answer Key

1. a	2. d	3. b	4. d	5. d	6. a	7. a or b	8. d	9. b	10. d
11. c	12. b	13. c	14. a	15. b	16. b				

Group 'B' or 'C' (Subjective Questions and Answers)

1. (a) *Write the degree of $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 5y + 6 = 0$.* [1] [2080 Optional]

(b) *Find the differential equation of $y = 2x + cx^2$.* [2]

(c) *Solve the differential equation $y(1 + x)\,dx - xdy = 0$.* [2] [2080 Optional]

Soln: (a) Here, highest derivative in the given equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 5y + 6 = 0$ is 2 and its power is 1.

$\therefore$ Degree = 1.

(b) Given, $y = 2x + cx^2$ ....(i)

Differentiating both sides with respect to 'x' we get

$\frac{dy}{dx} = 2 + 2cx$

or, $\frac{dy}{dx} = 2 + 2x\left(\frac{y - 2x}{x^2}\right)$ [$\because$ Using (i)]

or, $\frac{dy}{dx} = 2 + 2\left(\frac{y - 2x}{x}\right)$

or, $x\frac{dy}{dx} = 2x + 2y - 4x$

or, $x\frac{dy}{dx} + 2x - 2y = 0$.

Which is the required equation.

(c) Given differential equation is $y(1 + x)\,dx - xdy = 0$.

Dividing both sides by xy, we get

$\frac{y(1 + x)dx}{xy} - \frac{xdy}{xy} = 0$

or, $\left(\frac{1 + x}{x}\right)dx - \frac{dy}{y} = 0$

or, $\left(\frac{1}{x} + 1\right)dx - \frac{dy}{y} = 0$.

Integrating on both sides, we get

$\ln x + x - \ln y = c$

$\therefore$ $\ln\left(\frac{x}{y}\right) + x = c$, which is the required solution.

2. (a) *Write the degree and order of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^3y}{dx^3}\right)^2$.* [1+1] [2080 Set- I]

(b) *Solve: $\frac{dy}{dx} = \frac{1 - y}{2x + 1}$.* [3] [2081 Set- W]

Soln: (a) Here, the highest derivative in the given equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^3y}{dx^3}\right)^2$ is 3 and its power is 2.

$\therefore$ Order = 3 and degree = 2.

(b) Given differential equation is $\frac{dy}{dx} = \frac{1 - y}{2x + 1}$.

or, $\frac{dx}{2x + 1} = \frac{dy}{1 - y}$

or, $\frac{dx}{2x + 1} + \frac{dy}{y - 1} = 0$

Integrating on both sides, we get

$\frac{\ln(2x + 1)}{2} + \ln(y - 1) = \ln c$

or, $\frac{1}{2}\ln(2x + 1) + \ln(y - 1) = \ln c$

or, $\ln(2x + 1)^{\frac{1}{2}} + \ln(y - 1) = \ln c$

or, $\ln\{\sqrt{2x + 1}\,.\,(y - 1)\} = \ln c$

$\therefore$ $(y - 1)\sqrt{2x + 1} = c$, which is the required solution.

3. (a) *What is the degree and order of differential equation $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 5y = 0$.* [1+1] [2080 GIE Set B].

(b) *Find the differential equation of $y = ae^x$* [1]

(c) *Solve: $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$* [2] [2080 Set- G]

Soln: (a) Here, the highest derivative in the given equation $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 5y = 0$ is 2 and its power is 1.

$\therefore$ Degree = 1 and order = 2.

(b) Given, $y = ae^x$ ....(i)

Differentiating both sides with respect to 'x', we get $\frac{dy}{dx} = ae^x$

or, $\frac{dy}{dx} = y$ [∵ Using (i)]

∴ $\frac{dy}{dx} - y = 0.$

Which is the required equation.

(c) Given differential equation is $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

or, $\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$

or, $\frac{dx}{x^2+1} - \frac{dy}{y^2+1} = 0$

Integrating both sides, we get

$\tan^{-1} x - \tan^{-1} y = \tan^{-1} c$

or, $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1} c$

∴ $\frac{x-y}{1+xy} = c$

i.e. $x - y = c(1 + xy)$.

4. (a) *Write the order of differential equation $\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{dy}{dx}\right)^2 + 5 = 0.$* [1] [2080 Set-V]

(b) *Solve the differential equation $xy(y+1)\,dy - (x^2+1)dx = 0.$* [2] [2080 Set-I]

(c) *Show that $y = a\cos 3x$ is a solution of $\frac{d^2y}{dx^2} + 9y = 0.$* [2]

Soln: (a) Here, the highest derivatives in the given equation $\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{dy}{dx}\right)^2 + 5 = 0$ is 3.

∴ Order = 3

(b) Given differential equation is $xy(y+1)\,dy - (x^2+1)\,dx = 0.$

Dividing both sides by x, we get

$\frac{xy(y+1)\,dy}{x} - \frac{(x^2+1)\,dx}{x} = 0$

or, $(y^2+y)\,dy - (x + \frac{1}{x})\,dx = 0$

Integrating on both sides, we get

$\frac{y^3}{3} + \frac{y^2}{2} - \left(\frac{x^2}{2} + \ln x\right) = c$

or, $\frac{y^3}{3} + \frac{y^2}{2} - \frac{x^2}{2} - \ln x = c$

or, $2y^3 + 3y^2 - 3x^2 - 6\ln x = 6c$

∴ $2y^3 + 3y^2 - 3x^2 - 6\ln x = K$, (Where, $K = 6c$).

Which is the required solution.

(c) Given, $y = a\cos 3x$ ....(i)

Diff. both sides with respect to 'x', we get

$\frac{dy}{dx} = -3a\sin 3x$

Again, differentiating both sides with respect to 'x', we get

$\frac{d^2y}{dx^2} = -3a \times 3\cos 3x$

$= -9 \times a\cos 3x$

$= -9y$ [∵ Using (i)]

∴ $\frac{d^2y}{dx^2} + 9y = 0.$

This shows that $y = a\cos 3x$ is solution of given differential equation.

5. (a) *Solve: $\frac{dy}{dx} = \frac{e^x+1}{y}$.* [1]

(b) *Solve: $(xy^2 + x)\,dx + (x^2y + y)\,dy = 0$* [2]

(c) *Solve: $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$* [2]

Soln: (a) Given equation is, $\frac{dy}{dx} = \frac{e^x+1}{y}$

or, $y\,dy = (e^x + 1)\,dx$

Integrating on both sides, we get $\frac{y^2}{2} = e^x + x + \frac{c}{2}$

∴ $y^2 = 2e^x + 2x + c$

(b) Given differential equation is $(xy^2 + x)\,dx + (x^2y + y)\,dy = 0$

or, $x(y^2+1)\,dx + y(x^2+1)\,dy = 0$

Dividing both sides by $(x^2+1)(y^2+1)$, we get

$\frac{x(y^2+1)dx}{(x^2+1)(y^2+1)} + \frac{y(x^2+1)dy}{(x^2+1)(y^2+1)} = 0$

or, $\frac{x\,dx}{x^2+1} + \frac{y\,dy}{y^2+1} = 0$

or, $\frac{2x\,dx}{x^2+1} + \frac{2y\,dy}{y^2+1} = 0$

Integrating on both sides, we get

$\ln(x^2+1) + \ln(y^2+1) = \ln c$

or, $\ln\{(x^2+1)(y^2+1)\} = \ln c$

∴ $(x^2+1)(y^2+1) = c$

(c) Given differential equation is $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

or, $\frac{dy}{dx} + \frac{2\cos^2 y}{2\sin^2 x} = 0$

or, $\frac{dy}{dx} + \frac{\cos^2 y}{\sin^2 x} = 0$

or, $\sin^2 x\,dy + \cos^2 y\,dx = 0$

Dividing both sides by $\sin^2x \cos^2y$, we get

$$\frac{\sin^2x dy}{\sin^2x.\cos^2y} + \frac{\cos^2y dx}{\sin^2x \cos^2y} = 0$$

or, $\sec^2y\, dy + \operatorname{cosec}^2x\, dx = 0$

Integrating on both sides, we get

$\tan y - \cot x = c.$

6. (a) *Write the degree of* $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = x^2 \sin\left(\frac{dy}{dx}\right)$. [1]

(b) *Solve:* $\frac{dy}{dx} = \frac{2x+1}{5y^4+1}$ [2]

(c) *Solve:* $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ [2]

Soln: (a) Since, the differential equation is not a polynomial in all differential coefficients. So, its degree is not defined.

(b) Given differential equation is $\frac{dy}{dx} = \frac{2x+1}{5y^4+1}$.

or, $(5y^4 + 1)\, dy = (2x + 1)\, dx$

Integrating both sides, we get,

$$\frac{5y^5}{5} + y = \frac{2x^2}{2} + x + c$$

$\therefore$ $y^5 + y = x^2 + x + c.$

(c) Given differential equation is $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

or, $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$

or, $\sqrt{1-x^2}\, dy + \sqrt{1-y^2}\, dx = 0$

Dividing both sides by $\sqrt{1-x^2}\sqrt{1-y^2}$, we get

$$\frac{\sqrt{1-x^2}\, dy}{\sqrt{1-x^2}.\sqrt{1-y^2}} + \frac{\sqrt{1-y^2}\, dx}{\sqrt{1-x^2}.\sqrt{1-y^2}} = 0$$

or, $\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$

Integrating on both sides, we get

$\sin^{-1}y + \sin^{-1}x = \sin^{-1}c$

or, $\sin^{-1}\left(y\sqrt{1-x^2} + x\sqrt{1-y^2}\right) = \sin^{-1}c$

$\therefore$ $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c.$

7. (a) *A differential equation is in the form Xdx = Ydy where X is the function of x alone and Y is the function of y alone. Name the differential equation.* [1]

(b) *Solve:* $x\frac{dy}{dx} + y - 1 = 0$ [2]

(c) *Solve:* $\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^3}{e^y}$ [2]

Soln: (a) Differential equation with variable separable form.

(b) Given differential equation is $x\frac{dy}{dx} + (y-1) = 0$

or, $x dy + (y - 1)\, dx = 0$

Dividing both sides by $x(y - 1)$, we get

$$\frac{x dy}{x(y-1)} + \frac{(y-1)\, dx}{x(y-1)} = 0$$

or, $\frac{dy}{y-1} + \frac{dx}{x} = 0$

Integrating on both sides, we get

$\ln(y-1) + \ln x = \ln c$

or, $\ln\{x(y-1)\} = \ln c$

$\therefore$ $x(y-1) = c.$

(c) Given differential equation is $\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^3}{e^y}$

or, $e^y dy = (e^x + x^3)dx$

Integrating on both sides, we get

$e^y = e^x + \frac{x^4}{4} + c.$

8. (a) *Solve:* $(1+x^2)\frac{dy}{dx} = 1$ [2]

(b) *A differential equation of the first degree and first order is homogeneous, if it satisfies the condition* $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. *Justify the statement with an example and solve it.* [3] [2081 Set V]

Soln: (a) Given differential equation is $(1+x^2)\frac{dy}{dx} = 1$.

or, $dy = \frac{dx}{1+x^2}$

Integrating both sides we get, $y = \tan^{-1}x + c$.

(b) For differential equation of first degree and first order, let's take on an example $\frac{dy}{dx} = \frac{y+x}{x}$ ....(i)

Now, above equation can be written as $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{x} = \frac{y}{x} + 1 = \frac{y}{x} + \left(\frac{y}{x}\right)^0$. Which is in the form $f\left(\frac{y}{x}\right)$.

$\therefore$ It is homogeneous differential equation.

Now, put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(ii)

From (i) and (ii) we get, $v + x\frac{dv}{dx} = \frac{vx+x}{x}$

or, $v + x\frac{dv}{dx} = v + 1$

or, $x\frac{dv}{dx} = 1$

or, $dv = \frac{dx}{x}$

Integrating both sides we get,

$v = \ln x + \ln c$

or, $\frac{y}{x} = \ln cx$

$\therefore$ $y = x \ln cx$

Which is the required solution.

Note: Students can use any correct example for this type of question.

9. (a) *Solve:* $\frac{dy}{dx} + 4x = 2e^{2x}$. [2]

(b) *Write an example of homogeneous differential equation of first order and solve it.* [3]

Soln: (a) Given differential equation is $\frac{dy}{dx} + 4x = 2e^{2x}$.

or, $\frac{dy}{dx} = 2e^{2x} - 4x$

or, $dy = (2e^{2x} - 4x)\,dx$

Integrating on both sides, we get

$$y = \frac{2e^{2x}}{2} - \frac{4x^2}{2} + c$$

$\therefore$ $y = e^{2x} - 2x^2 + c$.

(b) See Solution of Q. No. 8 (b).

10. (a) *Solve:* $x^2\,dy - y^2\,dx = 0$. [2]

(b) *Identify the homogeneous differential equation and solve it.*

$(2xy + y^2)\,dy + (y^2 + x)\,dx = 0$, $(1 - x^2)\frac{dy}{dx} - xy = 1$ and $\frac{dy}{dx} + \frac{x^2 - y^2}{3xy} = 0$. [3] [2080 Optional]

Soln: (a) Given differential equation is $x^2dy - y^2\,dx = 0$

Dividing both sides by x^2y^2, we get $\frac{x^2dy}{x^2y^2} - \frac{y^2dx}{x^2y^2} = 0$

or, $y^{-2}\,dy - x^{-2}\,dx = 0$

Integrating both sides, we get $\frac{y^{-1}}{-1} - \frac{x^{-1}}{-1} = c$

or, $\frac{1}{x} - \frac{1}{y} = c$

or, $y - x = c\,xy$.

(b) From first equation, $\frac{dy}{dx} = -\frac{y^2 + x}{2xy + y^2}$

From second equation, $\frac{dy}{dx} = \frac{1 + xy}{1 - x^2}$.

Since, R.H.S. of both equations cannot be in the form $\frac{y}{x}$ or $\frac{x}{y}$.

Now, from third equation $\frac{dy}{dx} = \frac{y^2 - x^2}{3xy}$ ....(i)

Which is in the form of $\frac{y}{x}$, so this equation is homogeneous.

So, put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(ii)

From (i) and (ii), we get $v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{3x\,.\,vx}$

or, $x\frac{dv}{dx} = \frac{v^2 - 1}{3v} - v$

or, $x\frac{dv}{dx} = \frac{v^2 - 1 - 3v^2}{3v}$

or, $x\frac{dv}{dx} = \frac{-2v^2 - 1}{3v}$

or, $\frac{3v}{2v^2 + 1}\,dv = -\frac{dx}{x}$

or, $\frac{3}{4}\left(\frac{4v}{2v^2 + 1}\right)dv + \frac{dx}{x} = 0$

Integrating both sides, we get

$$\frac{3}{4}\ln(2v^2 + 1) + \ln x = \ln c$$

or, $\ln(2v^2 + 1)^{\frac{3}{4}} + \ln x = \ln c$

or, $\ln\left\{x(2v^2 + 1)^{\frac{3}{4}}\right\} = \ln c$

or, $x\left(2\frac{y^2}{x^2} + 1\right)^{\frac{3}{4}} = c$

or, $x\left(\frac{2y^2 + x^2}{x^2}\right)^{\frac{3}{4}} = c$

or, $x(x^2 + 2y^2)^{\frac{3}{4}} = x\sqrt{x}\,c$

$\therefore$ $(x^2 + 2y^2)^{\frac{3}{4}} = c\sqrt{x}$

11. *Solve:* $2xy\frac{dy}{dx} = x^2 + y^2$. [3] [2080 GIE Set B]

Soln: Given differential equation is $2xy\frac{dy}{dx} = x^2 + y^2$

or, $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ ....(i), which is homogeneous.

So, put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ......(ii)

From (i) and (ii) we get, $v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x\,.vx}$

or, $v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$

or, $x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v$

or, $x\frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v}$

or, $x\frac{dv}{dx} = \frac{1 - v^2}{2v}$

or, $\frac{2v}{1 - v^2}\,dv = \frac{dx}{x}$

or, $\frac{2v}{v^2 - 1}\,dv + \frac{dx}{x} = 0$

Integrating on both sides we get,

$\ln(v^2 - 1) + \ln x = \ln c$

or, $\ln\{(v^2 - 1)\,x\} = \ln c$

$\therefore$ $\left(\frac{y^2}{x^2} - 1\right)x = c$

or, $\frac{(y^2 - x^2)\,x}{x^2} = c$

$\therefore$ $y^2 - x^2 = cx$

12. Solve: $x^2\frac{dy}{dx}+y^2=xy$ [3]

Soln: Given differential equation is $x^2\frac{dy}{dx}+y^2=xy$

or, $\frac{dy}{dx}=\frac{xy-y^2}{x^2}$ ....(i) Which is homogeneous.

So put $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$ ....(ii)

Form (i) and (ii) we get, $v+x\frac{dv}{dx}=\frac{x.vx-v^2x^2}{x^2}$

or, $v+x\frac{dv}{dx}=v-v^2$

or, $x\frac{dv}{dx}=-v^2$

or, $-v^{-2}dv=\frac{dx}{x}$

Integrating, $\frac{-v^{-1}}{-1}=\ln x+c$

or, $\frac{1}{v}=\ln x+c$

or, $\frac{1}{\frac{y}{x}}=\ln x+c$

or, $\frac{x}{y}=\ln x+c$

$\therefore$ $x=y(\ln x+c)$.

13. Solve: $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$. [3]

Soln: Given differential equation is $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$ ....(i). Which is homogeneous.

So, put $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$ ....(ii)

From (i) and (ii) we get, $v+x\frac{dv}{dx}=\frac{vx}{x}+\tan\left(\frac{vx}{x}\right)$

or, $v+x\frac{dv}{dx}=v+\tan v$

or, $x\frac{dv}{dx}=\frac{\sin v}{\cos v}$

or, $\frac{\cos v}{\sin v}dv=\frac{dx}{x}$

Integrating, $\ln(\sin v)=\ln x+\ln c$

or, $\ln(\sin v)=\ln(cx)$

or, $\sin v=cx$

$\therefore$ $\sin\left(\frac{y}{x}\right)=cx$.

14. Solve: $(x^2-y^2)\frac{dy}{dx}=xy$. [3]

Soln: Given differential equation is $(x^2-y^2)\frac{dy}{dx}=xy$

or, $\frac{dy}{dx}=\frac{xy}{x^2-y^2}$ ....(i). Which is homogeneous.

So, put $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$ ....(ii)

From (i) and (ii) we get, $v+x\frac{dv}{dx}=\frac{x.vx}{x^2-v^2x^2}$

or, $x\frac{dv}{dx}=\frac{v}{1-v^2}-v$

or, $x\frac{dv}{dx}=\frac{v-v+v^3}{1-v^2}$

or, $x\frac{dv}{dx}=\frac{v^3}{1-v^2}$

or, $\left(\frac{1-v^2}{v^3}\right)dv=\frac{dx}{x}$

or, $\left(v^{-3}-\frac{1}{v}\right)dv=\frac{dx}{x}$

Integrating, $\frac{v^{-2}}{-2}-\ln v=\ln x+\ln c$

or, $\frac{-1}{2v^2}=\ln v+\ln x+\ln c$

or, $\frac{-1}{\frac{2y^2}{x^2}}=\ln(vxc)$

or, $\frac{-x^2}{2y^2}=\ln(yc)$

or, $-x^2=2y^2\ln(cy)$

$\therefore$ $x^2+2y^2\ln(cy)=0$.

15. Solve: $x^2ydx=(x^3+y^3)dy$. [3]

Soln: Given differential equation is $x^2ydx=(x^3+y^3)dy$

or, $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$ ....(i). Which is homogeneous.

So, put $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$ ....(ii)

From (i) and (ii), we get $v+x\frac{dv}{dx}=\frac{x^2.vx}{x^3+v^3x^3}$

or, $x\frac{dv}{dx}=\frac{v}{1+v^3}-v$

or, $x\frac{dv}{dx}=\frac{v-v-v^4}{1+v^3}$

or, $x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$

... dv ...

Integrati... $\ln v + \ln x$ ...

or, $\frac{-1}{3v^3} + \ln(vxc) = 0$

or, $\frac{-1}{3.\frac{y^3}{x^3}} + \ln\left(\frac{y}{x}.xc\right) = 0$

or, $\frac{-x^3}{3y^3} + \ln(cy) = 0$

or, $-x^3 + 3y^3 \ln(cy) = 0$

or, $x^3 = 3y^3 \ln(cy)$

16. *Solve:* $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ ....(i) [3]

Soln: Given differential equation is $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ ....(i). Which is homogeneous.

So, put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ .... (ii)

From (i) and (ii) we get, $2\left(v + x\frac{dv}{dx}\right) = \frac{vx}{x} + \frac{v^2x^2}{x^2}$

or, $2v + 2x\frac{dv}{dx} = v + v^2$

or, $2x\frac{dv}{dx} = v + v^2 - 2v$

or, $2x\frac{dv}{dx} = v^2 - v$

or, $\frac{2dv}{v(v-1)} = \frac{dx}{x}$

or, $2\left(\frac{1}{v-1} - \frac{1}{v}\right) dv = \frac{dx}{x}$

Integrating, $2\{\ln(v-1) - \ln v\} = \ln x + \ln c$

or, $2\ln\left(\frac{v-1}{v}\right) = \ln cx$

or, $\ln\left(\frac{v-1}{v}\right)^2 = \ln cx$

or, $\left(\frac{v-1}{v}\right)^2 = cx$

or, $\left(\frac{\frac{y}{x}-1}{\frac{y}{x}}\right)^2 = cx$

or, $\left(\frac{y-x}{y}\right)^2 = cx$

$\therefore$ $(y-x)^2 = cxy^2$

17. *Does* $x\frac{dy}{dx} = \sqrt{x^2+y^2} + y$ ***represent a homogeneous differential equation? Give reason. Also solve the equation.*** [3]

Soln: Given differential equation is $x\frac{dy}{dx} = \sqrt{x^2+y^2} + y$

or, $\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}$ ....(i) Which is in the form $\frac{y}{x}$. So, it is homogeneous.

Now, put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(ii)

From (i) and (ii), we get $v + x\frac{dv}{dx} = \frac{\sqrt{x^2+x^2v^2}+vx}{x}$

or, $v + x\frac{dv}{dx} = \frac{x\sqrt{1+v^2}}{x} + \frac{vx}{x}$

or, $v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$

or, $x\frac{dv}{dx} = \sqrt{1+v^2}$

or, $\frac{dv}{\sqrt{v^2+1}} = \frac{dx}{x}$

Integrating, $\ln(v + \sqrt{v^2+1}) = \ln x + \ln c$

or, $\ln(v + \sqrt{v^2+1}) = \ln cx$

$\therefore$ $\frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = cx$

or, $\frac{y}{x} + \frac{\sqrt{y^2+x^2}}{x} = cx$

$\therefore$ $y + \sqrt{x^2+y^2} = cx^2$.

18. *Solve:* $\frac{dy}{dx} = \frac{y+1}{x+y+1}$ [3]

Soln: Given differential equation is $\frac{dy}{dx} = \frac{y+1}{x+y+1}$, Which is not homogeneous.

So, put $z = y + 1 \Rightarrow \frac{dz}{dx} = \frac{dy}{dx}$.

Now, above equation becomes, $\frac{dz}{dx} = \frac{z}{x+z}$ ....(i) Which is homogeneous.

So, put $z = vx$, then $\frac{dz}{dx} = v + x\frac{dv}{dx}$ ......(ii)

From (i) and (ii) we get, $v + x\frac{dv}{dx} = \frac{vx}{x+vx}$

or, $x\frac{dv}{dx} = \frac{v}{1+v} - v$

or, $x\frac{dv}{dx} = \frac{v-v-v^2}{1+v}$

or, $x\frac{dv}{dx} = \frac{-v^2}{1+v}$

or, $\frac{1+v}{v^2}dv = -\frac{dx}{x}$

or, $\left(v^{-2}+\frac{1}{v}\right)dv + \frac{dx}{x} = 0$

Integrating, $\frac{v^{-1}}{-1} + \ln v + \ln x + \ln c = 0$

or, $-\frac{1}{v} + \ln(vxc) = 0$

or, $-\frac{1}{\frac{z}{x}} + \ln(z.c) = 0$

or, $-\frac{x}{z} + \ln(cz) = 0$

or, $-\frac{x}{y+1} + \ln\{c(y+1)\} = 0$

or, $\ln\{c(y+1)\} = \frac{x}{y+1}$

$\therefore$ $c(y+1) = e^{\frac{x}{y+1}}$.

19. (a) ***Give an example of exact differential equation, homogeneous differential equation and a standard integral.*** *[3] [2079 GIE Set-A]*

(b) ***Solve the differential equation $(x + 2y - 3)\,dy - (2x - y + 1)\,dx = 0$*** *[2]*

Soln: (a) An example of exact differential equation is $xdy + ydx = 0$ because it can be written as $d(xy) = 0$.

An example of homogeneous differential is $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$ because it can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

An example of standard integral is $\int\frac{dx}{x^2+9}$ because it is in the form of $\int\frac{dx}{x^2+a^2}$.

> ***Note:*** Students can use any correct example for this type of question.

(b) Given equation is, $(x + 2y - 3)\,dy - (2x - y + 1)\,dx = 0$

or, $xdy + 2y\,dy - 3dy - 2x\,dx + y\,dx - dx = 0$

or, $(xdy + ydx) + 2y\,dy - 2x\,dx - 3dy - dx = 0$

or, $d(xy) + d(y^2) - d(x^2) - d(3y) - dx = 0$

or, $d(xy + y^2 - x^2 - 3y - x) = 0$

Integrating, $xy + y^2 - x^2 - 3y - x = c$.

20. (a) ***Write an example of exact differential equation in x and y.*** *[1] [2080 Set-V]*

(b) ***Solve: $2xy\,dx - x^2\,dy = 0$*** *[2]*

(c) ***Solve: $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$*** *[2]*

Soln: (a) An example of exact differential equation is $xdy + ydx = 0$ because it can be written as $d(xy) = 0$.

(b) Given differential equation is $2xy\,dx - x^2dy = 0$

or, $y(2x\,dx) - x^2\,dy = 0$

or, $y\,d(x^2) - x^2\,dy = 0$

Dividing both sides by y^2 we get,

$$\frac{y\,d(x^2) - x^2\,dy}{y^2} = 0$$

or, $d\left(\frac{x^2}{y}\right) = 0$

Integrating, $\frac{x^2}{y} = c$

$\therefore$ $x^2 = cy$

(c) Given differential equation $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$

or, $x^2\,dx - aydx - axdy + y^2dy = 0$

or, $d\left(\frac{x^3}{3}\right) - a(ydx + xdy) + d\left(\frac{y^3}{3}\right) = 0$

or, $d\left(\frac{x^3}{3}\right) - ad(xy) + d\left(\frac{y^3}{3}\right) = 0$

or, $d\left(\frac{x^3}{3} - axy + \frac{y^3}{3}\right) = 0$

Integrating, $\frac{x^3}{3} - axy + \frac{y^3}{3} = \frac{c}{3}$

or, $x^3 - 3axy + y^3 = c$.

21. (a) ***An exact differential equation have always the mathematical form like $M(x, y)\,dx + N(x, y)\,dy = 0$. Take a simple example to justify the statement with characteristics of $M(x, y)$ and $N(x, y)$.*** *[3] [2081 Optional]*

(b) ***Solve: $(x + y^2)\,dx = 2xy\,dy$.*** *[2]*

Soln: (a) **An exact differential equation always has the form $M(x, y)\,dx + N(x, y)\,dy = 0$.**

Consider the differential equation $2xy\,dx + x^2\,dy = 0$ ....(i).

Where, $M(x, y) = 2xy$ and $N(x, y) = x^2$.

Here, M and N are functions of x or y or both.

Now, equation (i) can be written as $y(2x\,dx) + x^2\,dy = 0$

or, $y\,d(x^2) + x^2\,dy = 0$

or, $d(x^2y) = 0$, Which is an exact differential equation.

> ***Note:*** Students can use any correct example for this type of question.

(b) Given differential equation is $(x + y^2)\,dx = 2xy\,dy$

or, $2xy\,dy - (x + y^2)\,dx = 0$

or, $x(2y\,dy) - xdx - y^2\,dx = 0$

or, $xd(y^2) - y^2dx - x\,dx = 0$

Dividing both sides by x^2 we get,

$$\frac{xd(y^2) - y^2\,dx}{x^2} - \frac{xdx}{x^2} = 0$$

or, $d\left(\frac{y^2}{x}\right) - \frac{dx}{x} = 0$

or, $d\left(\frac{y^2}{x}\right) - d(\ln x) = 0$

or, $d\left(\frac{y^2}{x} - \ln x\right) = 0$

Integrating, $\frac{y^2}{x} - \ln x = c$

or, $\frac{y^2}{x} = c + \ln x$

$\therefore$ $y^2 = x(\ln x + c)$.

22. (a) ***Solve:*** $xdy + (x + y)\,dx = 0$. [2]

(b) ***Solve:*** $dy\,(x + y + 1) = dx\,(x - y + 1)$. [3] [2080 GIE Set-B]

Soln: (a) Given differential equation is $xdy + (x + y)\,dx = 0$

or, $xdy + xdx + ydx = 0$

or, $x\,dx + (xdy + ydx) = 0$

or, $d\left(\frac{x^2}{2}\right) + d\,(xy) = 0$

or, $d\left(\frac{x^2}{2} + xy\right) = 0$

Integrating, $\frac{x^2}{2} + xy = c$.

(b) Given differential equation is $dy\,(x + y + 1) = dx\,(x - y + 1)$.

or, $(x + y + 1)\,dy - (x - y + 1)\,dx = 0$

or, $x\,dy + y\,dy + dy - x\,dx + y\,dx - dx = 0$

or, $(x\,dy + y\,dx) + y\,dy - x\,dx + dy - dx = 0$

or, $d\,(xy) + d\left(\frac{y^2}{2}\right) - d\left(\frac{x^2}{2}\right) + dy - dx = 0$

or, $d\left(xy + \frac{y^2}{2} - \frac{x^2}{2} + y - x\right) = 0$

Integrating, $xy + \frac{y^2}{2} - \frac{x^2}{2} + y - x = \frac{c}{2}$

$\therefore$ $2xy + y^2 - x^2 + 2y - 2x = c$

23. (a) ***A differential equation is in the form*** $\frac{dy}{dx} + Py = Q$, ***where P and Q are functions of x.***

(i) ***Name the differential equation.*** [1] [2080 Set-W]

(ii) ***Write the integrating factor I.F. of above differential equation.*** [1] [2081 Optional]

(b) ***Solve:*** $\frac{dy}{dx} + 2y\tan x = \sin x$. [3]

Soln: (a) (i) Linear differential equation.

(ii) Integrating factor (I.F.) $= e^{\int P dx}$

(b) Given differential equation is $\frac{dy}{dx} + 2y\tan x = \sin x$ ....(i), Which is linear.

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get $P = 2\tan x$, $Q = \sin x$.

I.F. $= e^{\int P dx}$

$= e^{2\int \tan x\,dx} = e^{2\ln \sec x} = e^{\ln \sec^2 x} = \sec^2 x$

Multiplying both sides of (i) by $\sec^2 x$ we get

$\sec^2 x\frac{dy}{dx} + \sec^2 x.\,2y\tan x = \sec^2 x.\sin x$

or, $\frac{d}{dx}(\sec^2 x\,.y) = \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$

$d\,(y.\sec^2 x) = \sec x\,.\,\tan x\,dx$

Integrating we get, $y\sec^2 x = \sec x + c$.

24. ***Which type of differential equation*** $\sin x\frac{dy}{dx} + y\cos x = x\sin x$ ***represents? Also solve it*** [3] [2081 Set-W]

Soln: Given differential equation is $\sin x\frac{dy}{dx} + y\cos x = x\sin x$

$\therefore$ $\frac{dy}{dx} + \frac{\cos x}{\sin x}y = x$ ....(i)

Which is in the form $\frac{dy}{dx} + Py = Q$. So, it is linear differential equation.

Since, $P = \frac{\cos x}{\sin x}$ and $Q = x$

I.F. $= e^{\int P dx}$

$= e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln(\sin x)} = \sin x$

Multiplying both sides of (i) by $\sin x$ we get,

$\sin x\frac{dy}{dx} + \cos x.\,y = x\sin x$

or, $\frac{d}{dx}(y\sin x) = x\sin x$

or, $d\,(y\sin x) = x\sin x\,dx$

Integrating,

$y\sin x = \int x.\sin x\,dx$

or, $y\sin x = x\int \sin x\,dx - \int\left[\frac{dx}{dx}\int \sin x\,dx\right]dx$

or, $y\sin x = x(-\cos x) - \int 1.\,(-\cos x)\,dx$

or, $y\sin x = -x\cos x + \int \cos x dx$

or, $y\sin x = -x\cos x + \sin x + c$

$\therefore$ $y\sin x = \sin x - x\cos x + c$.

25. (a) ***Write a differential equation in linear form.*** [1] [2080 GIE Set A]

(b) ***Solve:*** $(x + 1)\frac{dy}{dx} + 2y = \frac{e^x}{x + 1}$ [3]

Soln: (a) A differential equation in linear form is $\frac{dy}{dx} + Py = Q$. Where P and Q are function of x.

e.g. $\frac{dy}{dx} + 2xy = \sin x$.

(b) Given differential equation is $(x+1)\frac{dy}{dx} + 2y = \frac{e^x}{x+1}$

or, $\frac{dy}{dx} + \frac{2}{x+1}y = \frac{e^x}{(x+1)^2}$ ....(i). Which is linear.

Comparing (i) with $\frac{dy}{dx} + Py = Q$. We get, $P = \frac{2}{x+1}$, $Q = \frac{e^x}{(x+1)^2}$

I.F. $= e^{\int Pdx}$

$= e^{\int \frac{2}{x+1}dx} = e^{2\ln(x+1)} = e^{\ln(x+1)^2} = (x+1)^2$

Multiplying both sides of (i) by $(x+1)^2$ we get

$(x+1)^2\frac{dy}{dx} + 2(x+1)y = e^x$

or, $\frac{d}{dx}\{y(x+1)^2\} = e^x$

or, $d\{y(x+1)^2\} = e^x dx$

Integrating we get, $y(x+1)^2 = e^x + c$.

26. *Find the solution of differential equation $x\frac{dy}{dx} + y - x\ln x = 0$.* [3] [2080 Set-I]

Soln: Given differential equation is $x\frac{dy}{dx} + y - x\ln x = 0$

or, $x\frac{dy}{dx} + y = x\ln x$

$\therefore$ $\frac{dy}{dx} + \frac{1}{x}y = \ln x$ ....(i). Which is linear.

Comparing (i) with $\frac{dy}{dx} + Py = Q$. We get, $P = \frac{1}{x}$, $Q = \ln x$.

I.F. $= e^{\int Pdx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiplying both sides of (i) by x we get,

$x\frac{dy}{dx} + y = x\ln x$

or, $\frac{d}{dx}(yx) = x\ln x$

or, $d(xy) = x\ln x\, dx$

Integrating we get, $xy = \int \ln x . x\, dx$

or, $xy = \ln x\int x\,dx - \int\left[\frac{d}{dx}(\ln x)\int x\,dx\right]dx$

or, $xy = \ln x . \frac{x^2}{2} - \int\left(\frac{1}{x} . \frac{x^2}{2}\right)dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2} . \frac{x^2}{2} + c$

$\therefore$ $xy = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$.

27. *Solve: $(1+x)\,dy = dx\,(1 + xy - x)$* [3] [2079 Set- K]

Soln: Given differential equation is $(1+x)\,dy = dx\,(1+xy-x)$

or, $(1+x)\frac{dy}{dx} = 1 + xy - x$

or, $(1+x)\frac{dy}{dx} - xy = 1 - x$

$\therefore$ $\frac{dy}{dx} - \frac{x}{1+x}y = \frac{1-x}{1+x}$ ....(i).Which is linear.

Comparing (i) with $\frac{dy}{dx} + Py = Q$. We get, $P = \frac{-x}{x+1}$, $Q = \frac{1-x}{1+x}$

I.F. $= e^{\int Pdx}$

$= e^{-\int \frac{x}{x+1}dx}$

$= e^{-\int \frac{x+1-1}{x+1}dx}$

$= e^{-\int\left(1 - \frac{1}{x+1}\right)dx}$

$= e^{\int\left(\frac{1}{x+1} - 1\right)dx}$

$= e^{\ln(x+1) - x}$

$= e^{\ln(x+1)} . e^{-x}$

$= (x+1)e^{-x}$

Multiplying both sides of (i) by $(x+1)e^{-x}$, we get

$(x+1)e^{-x}\frac{dy}{dx} - x . e^{-x}y = e^{-x}(1-x)$

or, $\frac{d}{dx}\{(x+1)e^{-x} . y\} = e^{-x}(1-x)$

or, $d\{y(x+1)e^{-x}\} = (1-x)e^{-x}dx$

Integrating we get,

$y.(x+1)e^{-x} = \int(1-x)e^{-x}dx$

$= (1-x)\int e^{-x}dx - \int\left[\frac{d}{dx}(1-x)\int e^{-x}dx\right]dx$

$= (1-x)\frac{e^{-x}}{-1} - \int(-1)\frac{e^{-x}}{-1}dx$

$= -(1-x)e^{-x} - \int e^{-x}dx$

$= (x-1)e^{-x} - \frac{e^{-x}}{-1} + c$

$= (x-1+1)e^{-x} + c$

$\therefore$ $y(x+1)e^{-x} = xe^{-x} + c$

$\therefore$ $y(x+1) = x + ce^x$

28. (a) *The concept of antiderivative is necessary for solving a differential equation. Justify the statement with an example.* [2] [2081 Set – V]

(b) *Solve:* $\frac{dt}{dx} = \frac{e^{\tan^{-1}x} - t}{1+x^2}$ [3] [2081 Set – V]

Soln: (a) The concept of an antiderivative is necessary for solving a differential equation. This is because many differential equations involve finding a function whose derivative matches a given expression. The antiderivative provides a way to reverse the process of differentiation to recover the original function.

Example: $\frac{dy}{dx} = 2x$.

This equation tells us that the derivative of some unknown function f(x) with respect to x is 2x.

To solve this differential equation, we need to find a function f(x) such that its derivative is 2x.

To do this, we use the concept of the antiderivative. The antiderivative of 2x with respect to x is $f(x) = \int 2x\,dx$.

Calculating the antiderivative $f(x) = x^2 + C$, where C is the constant of integration.

Thus, the general solution to the differential equation $\frac{dy}{dx} = 2x$ is $f(x) = x^2 + C$.

Note: Students can use any correct example for this type of question.

(b) Given differential equation is $\frac{dt}{dx} = \frac{e^{\tan^{-1}x} - t}{1+x^2}$

or, $\frac{dt}{dx} = \frac{e^{\tan^{-1}x}}{1+x^2} - \frac{t}{1+x^2}$

or, $\frac{dt}{dx} + \frac{1}{1+x^2}t = \frac{e^{\tan^{-1}x}}{1+x^2}$ ....(i). Which is linear.

Comparing (i) with $\frac{dt}{dx} + Pt = Q$. We get, $P = \frac{1}{1+x^2}$, $Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1}x}$$

Multiplying both sides of (i) by $e^{\tan^{-1}x}$ we get,

$$e^{\tan^{-1}x}\frac{dt}{dx} + \frac{e^{\tan^{-1}x}}{1+x^2}t = \frac{\left(e^{\tan^{-1}x}\right)^2}{1+x^2}$$

or, $\frac{d}{dx}\left(e^{\tan^{-1}x}.t\right) = \frac{\left(e^{\tan^{-1}x}\right)^2}{1+x^2}$

$\therefore$ $d\left(e^{\tan^{-1}x}.t\right) = \frac{\left(e^{\tan^{-1}x}\right)^2}{1+x^2}dx$

Integrating, $e^{\tan^{-1}x}.t = \int \frac{\left(e^{\tan^{-1}x}\right)^2}{1+x^2}dx$

or, $e^{\tan^{-1}x}.t = \int (e^z)^2 dz$ [$\because$ Put, $z = \tan^{-1}x \Rightarrow dz = \frac{dx}{1+x^2}$]

or, $e^{\tan^{-1}x}.t = \int e^{2z}dz$

or, $e^{\tan^{-1}x}.t = \frac{e^{2z}}{2} + c$

or, $e^{\tan^{-1}x}.t = \frac{e^{2\tan^{-1}x}}{2} + c$

$\therefore$ $t = \frac{e^{\tan^{-1}x}}{2} + c.e^{-\tan^{-1}x}$

29. *Solve:* $x\ln x\frac{dy}{dx} + y = 2\ln x$. [3]

Soln: Given equation is, $x\ln x\frac{dy}{dx} + y = 2\ln x$

or, $\frac{dy}{dx} + \frac{1}{x\ln x}.y = \frac{2}{x}$ ....(i). Which is linear.

So, comparing (i) with $\frac{dy}{dx} + Py = Q$. We get, $P = \frac{1}{x\ln x}$, $Q = \frac{2}{x}$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int \frac{1}{x\ln x}dx} = e^{\int \frac{\frac{1}{x}}{\ln x}dx} = e^{\ln(\ln x)} = \ln x$$

Multiplying both sides of (i) by lnx we get,

$$\ln x\frac{dy}{dx} + \frac{1}{x}y = \frac{2\ln x}{x}$$

or, $\frac{d}{dx}(\ln x.\, y) = \frac{2\ln x}{x}$

or, $d(y\ln x) = \frac{2\ln x}{x}dx$

Integrating both sides we get,

$$y\ln x = \int \frac{2\ln x}{x}dx$$

or, $y\ln x = 2\int \ln x.\frac{dx}{x}$

or, $y\ln x = 2\int t\,dt$ [$\because t = \ln x \Rightarrow dt = \frac{dx}{x}$]

or, $y\ln x = 2.\frac{t^2}{2} + c$

or, $y\ln x = t^2 + c$

$\therefore$ $y\ln x = (\ln x)^2 + c$.

[illegible] $x = \ln c$ [illegible]

$\therefore \frac{x^2+y^2}{x} = c$

$\therefore x^2 + y^2 = cx$ ....(iii)

When $x = 1$, $y = 1$ then (iii) becomes $1 + 1 = c$

$\therefore c = 2$

$\therefore$ Required solution is $x^2 + y^2 = 2x$.

(b) Given equation is, $\frac{dy}{dx} - y \tan x = e^x \sec x$

or, $\frac{dy}{dx} - \tan x . y = e^x \sec x$......(i). Which is linear.

So, comparing (i) with $\frac{dy}{dx} + Py = Q$. We get $P = -\tan x$, $Q = e^x \sec x$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int -\tan x \, dx}$$

$$= e^{\int \frac{-\sin x \, dx}{\cos x}}$$

$$= e^{\ln \cos x}$$

$$= \cos x$$

Multiplying both sides of (i) by cosx we get,

$$\cos x \frac{dy}{dx} - \cos x . \tan x . y = \cos x . e^x \sec x$$

or, $\frac{d}{dx}(y \cos x) = e^x$

or, $d(y \cos x) = e^x dx$

Integrating we get, $y \cos x = e^x + c$ ....(ii)

When, $x = 0$, $y = 1$,

$\therefore$ Equation (ii) becomes $1.1 = 1 + c$

$\therefore c = 0$

$\therefore$ Required solution is $y \cos x = e^x$

(c) Given equation is, $\frac{dy}{dx} - y \tan x = -y \sec^2 x$

or, $\frac{dy}{dx} + (\sec^2 x - \tan x) y = 0$ ....(i). Which is linear.

So, comparing (i) with $\frac{dy}{dx} + Py = Q$. We get, $P = \sec^2 x - \tan x$, $Q = 0$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int (\sec^2 x - \tan x) dx}$$

$$= e^{\int \left(\sec^2 x - \frac{\sin x}{\cos x}\right) dx}$$

$$= e^{\tan x + \ln \cos x}$$

$$= e^{\tan x} . e^{\ln \cos x} = e^{\tan x} . \cos x$$

Multiplying both sides of (i) by $\cos x . e^{\tan x}$ we get

$$\cos x . e^{\tan x} \frac{dy}{dx} + \cos x . e^{\tan x} (\sec^2 x - \tan x) y = 0$$

or, $\frac{d}{dx}(y . \cos x . e^{\tan x}) = 0$

or, $d(y . \cos x . e^{\tan x}) = 0$

Integrating, $y \cos x \, e^{\tan x} = c$

(d) Given equation is, $y^2 dx + 2xy \, dy = 0$.

or, $2xy \, dy = -y^2 dx$

or, $\frac{dy}{dx} = -\frac{y^2}{2xy}$

or, $\frac{dy}{dx} = -\frac{y}{2x}$ ....(i).

Which is homogeneous as it is in the form of $f\left(\frac{y}{x}\right)$.

So, put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Now, (i) becomes, $v + x\frac{dv}{dx} = -\frac{vx}{2x}$

or, $x\frac{dv}{dx} = -\frac{v}{2} - v$

or, $x\frac{dv}{dx} = -\frac{3v}{2}$

or, $2\frac{dv}{v} = -3\frac{dx}{x}$

or, $2\frac{dv}{v} + 3\frac{dx}{x} = 0$

Integrating, $2\ln v + 3\ln x = \ln c$

or, $\ln v^2 + \ln x^3 = \ln c$

or, $\ln(v^2 x^3) = \ln c$

$\therefore v^2 x^3 = c$

or, $\frac{y^2}{x^2} . x^3 = c$

$\therefore xy^2 = c$.

Integrating, $\ln(v^2+1) = -\ln x + \ln c$

or, $\ln\left(\frac{y^2}{x^2}+1\right) + \ln x = \ln c$

or, $\ln\left(\frac{y^2+x^2}{x^2}\right).x = \ln c$

$\therefore \frac{x^2+y^2}{x} = c$

$\therefore x^2+y^2 = cx$ ....(iii)

When $x = 1, y = 1$ then (iii) becomes $1 + 1 = c$

$\therefore$ $c = 2$

$\therefore$ Required solution is $x^2+y^2 = 2x$.

(b) Given equation is, $\frac{dy}{dx} - y\tan x = e^x \sec x$

or, $\frac{dy}{dx} - \tan x.y = e^x \sec x$......(i). Which is linear.

So, comparing (i) with $\frac{dy}{dx} + Py = Q$. We get $P = -\tan x$, $Q = e^x \sec x$

I.F. $= e^{\int Pdx}$

$= e^{\int -\tan x\, dx}$

$= e^{\int \frac{-\sin x\, dx}{\cos x}}$

$= e^{\ln \cos x}$

$= \cos x$

Multiplying both sides of (i) by cosx we get,

$\cos x \frac{dy}{dx} - \cos x.\tan x.y = \cos x.\, e^x \sec x$

or, $\frac{d}{dx}(y\cos x) = e^x$

or, $d(y\cos x) = e^x dx$

Integrating we get, $y\cos x = e^x + c$ ....(ii)

When, $x = 0, y = 1$,

$\therefore$ Equation (ii) becomes $1.1 = 1 + c$

$\therefore$ $c = 0$

$\therefore$ Required solution is $y\cos x = e^x$

(c) Given equation is, $\frac{dy}{dx} - y\tan x = -y\sec^2 x$

or, $\frac{dy}{dx} + (\sec^2 x - \tan x)y = 0$ ....(i). Which is linear.

So, comparing (i) with $\frac{dy}{dx} + Py = Q$. We get, $P = \sec^2 x - \tan x$, $Q = 0$

I.F. $= e^{\int Pdx}$

$= e^{\int(\sec^2 x - \tan x)dx}$

$= e^{\int\left(\sec^2 x - \frac{\sin x}{\cos x}\right)dx}$

$= e^{\tan x + \ln\cos x}$

$= e^{\tan x}.\, e^{\ln\cos x} = e^{\tan x}.\cos x$

Multiplying both sides of (i) by $\cos x.\, e^{\tan x}$ we get

$\cos x.\, e^{\tan x}\frac{dy}{dx} + \cos x.\, e^{\tan x}(\sec^2 x - \tan x)y = 0$

or, $\frac{d}{dx}(y.\cos x.e^{\tan x}) = 0$

or, $d(y.\cos x.e^{\tan x}) = 0$

Integrating, $y\cos x\, e^{\tan x} = c$

(d) Given equation is, $y^2 dx + 2xy\, dy = 0$.

or, $2xy\, dy = -y^2 dx$

or, $\frac{dy}{dx} = -\frac{y^2}{2xy}$

or, $\frac{dy}{dx} = -\frac{y}{2x}$ ....(i).

Which is homogeneous as it is in the form of $f\left(\frac{y}{x}\right)$.

So, put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Now, (i) becomes, $v + x\frac{dv}{dx} = -\frac{vx}{2x}$

or, $x\frac{dv}{dx} = -\frac{v}{2} - v$

or, $x\frac{dv}{dx} = -\frac{3v}{2}$

or, $2\frac{dv}{v} = -3\frac{dx}{x}$

or, $2\frac{dv}{v} + 3\frac{dx}{x} = 0$

Integrating, $2\ln v + 3\ln x = \ln c$

or, $\ln v^2 + \ln x^3 = \ln c$

or, $\ln(v^2 x^3) = \ln c$

$\therefore$ $v^2 x^3 = c$

or, $\frac{y^2}{x^2}.x^3 = c$

$\therefore$ $xy^2 = c$.

15.2 Application of Differential Equation

1. *A college hostel accommodating 1000 students, one of them came from abroad with infection of corona virus then the hostel was isolated. If the rate at which the virus spreads is assumed to be proportional to the product of the number 'N' of infected students and number of non-infected students and the number of infected students in 50 after 4 days.*
   - (a) *Express the above information in the form of differential equation.* [2]
   - (b) *Solve the differential equation.* [2]
   - (c) *Show that more than 95% students will be infected after 10 days.* [4]

Soln: (a) Here, total number of students = 1000

Number of infected students = N

$\therefore$ Number of non-infected students = 1000 – N

By the question, $\frac{dN}{dt} \propto N(1000-N)$

$\Rightarrow \frac{dN}{dt} = kN(1000-N)$ ....(i)

Where, k is proportional constant.

(b) We have, $\frac{dN}{dt} = kN(1000-N)$

or, $\frac{dN}{N(1000-N)} = k\,dt$

Integrating both sides we get,

$\frac{1}{1000}\int \frac{1000\,dN}{N(1000-N)} = \int k\,dt$

or, $\int \frac{(1000-N+N)dN}{N(1000-N)} = 1000\int k\,dt$

or, $\int\left[\frac{1000-N}{N(1000-N)} + \frac{N}{N(1000-N)}\right]dN = 1000\int k\,dt$

or, $\int\left(\frac{1}{N} + \frac{1}{1000-N}\right)dN = 1000\int k\,dt$

or, $\ln N - \ln(1000-N) = 1000\,kt + c$

or, $\ln\left(\frac{N}{1000-N}\right) = 1000\,kt + c$ ....(ii)

Initially, one student was infected.

i.e. t = 0, N = 1

Then, $\ln\left(\frac{1}{1000-1}\right) = 1000\,k \times 0 + c$

or, $\ln\left(\frac{1}{999}\right) = c$

Now, (ii) becomes, $\ln\left(\frac{N}{1000-N}\right) = 1000\,kt + \ln\left(\frac{1}{999}\right)$

or, $\ln\left(\frac{N}{1000-N}\right) - \ln\left(\frac{1}{999}\right) = 1000\,kt$

$\therefore \ln\left(\frac{999\,N}{1000-N}\right) = 1000\,kt$ ....(iii).

Again, when t = 4, N = 50, then

$\ln\left(\frac{999 \times 50}{1000-50}\right) = 1000\,k \times 4$

or, $\ln\left(\frac{49950}{950}\right) = 4000\,k$

or, $4000\,k = \ln(52.58)$

$\therefore k = 0.0009906$

Now, (iii) becomes, $\ln\left(\frac{999\,N}{1000-N}\right) = 1000\,t \times 0.0009906 = 0.9906\,t$

$\therefore \frac{999N}{1000-N} = e^{0.9906t}$

or, $999N = 1000\,e^{0.9906t} - N\,e^{0.9906t}$

or, $999N + N\,e^{0.9906t} = 1000\,e^{0.9906t}$

$\therefore N = \frac{1000\,e^{0.9906t}}{999 + e^{0.9906t}} = \frac{1000}{999\,e^{-0.9906t} + 1}$

(c) When, t = 10, $N = \frac{1000}{1 + 999 \times e^{-0.9906 \times 10}} = 953$

Now, Infected % = $\frac{\text{Infected students}}{\text{Total students}} \times 100 = \frac{953}{1000} \times 100 = 95.3\%$

This shows that more than 95% students will be infected after 10 days.

2. *A water tank is filled in such a way that the rate at which the depth of water increases is proportional to the square roots of the depth. Initially the depth is 'n' meters.*
   - (a) *Write down a differential equation for h.* [2]
   - (b) *Show that $\sqrt{h} = \frac{kt}{2} + \sqrt{n}$ where k is a constant.* [4]
   - (c) *When, h = 16 and t = 6 hours, prove that $n = (4-3k)^2$* [2] [2079 Set – K]

Soln: (a) Let, h be the depth of water.

Then by question, $\frac{dh}{dt} \propto \sqrt{h}$

$\therefore \frac{dh}{dt} = k\sqrt{h}$, where k is proportional constant.

or, $\frac{dh}{\sqrt{h}} = k\,dt$ ......(i)

(b) We have, $\frac{dh}{\sqrt{h}} = k\,dt$

Integrating, both sides we get $\frac{h^{\frac{1}{2}}}{\frac{1}{2}} = kt + c$

or, $2\sqrt{h} = kt + c$ ....(ii)

When, t = 0, h = n.

$\therefore 2\sqrt{n} = 0 + c$

$\therefore c = 2\sqrt{n}$

Now, (ii) becomes, $2\sqrt{h} = kt + 2\sqrt{n}$

or, $\sqrt{h} = \frac{kt}{2} + \sqrt{n}$ ....(iii)

(c) When, h = 16, t = 6

Now, (iii) becomes, $\sqrt{16} = \frac{k.6}{2} + \sqrt{n}$

or, $4 = 3k + \sqrt{n}$

or, $4 - 3k = \sqrt{n}$

$\therefore$ $n = (4-3k)^2$. Hence Proved.

3. *An equation relating to stability of an aeroplane is $\frac{dv}{dt} = g\cos\alpha - kv$, where v is the velocity and g, α, k are constant.*

(a) *Under which condition the solution of the equation does not exist? Give reason.* [3] [2080 Set – G]

(b) *Find an expression for velocity, if v = 0 what t = 0.* [2]

Soln: (a) Given differential equation is $\frac{dv}{dt} = g\cos\alpha - kv$

or, $\frac{dv}{dt} + kv = g\cos\alpha$ ....(i). Which is linear.

Comparing (i) with $\frac{dv}{dt} + Pv = Q$. We get, P = k, Q = g cosα

$\text{I.F.} = e^{\int P dt} = e^{\int k dt} = e^{kt}$

Multiplying both sides of (i) by e^{kt} we get

$e^{kt}\frac{dv}{dt} + e^{kt}kv = g\cos\alpha\, e^{kt}$

or, $\frac{d}{dt}(e^{kt}v) = g\cos\alpha\, e^{kt}$

or, $d(ve^{kt}) = g\cos\alpha\, e^{kt}dt$

Integrating both sides we get,

$ve^{kt} = g\cos\alpha\frac{e^{kt}}{k} + c$

$\therefore$ $v = \frac{g\cos\alpha}{k} + c.\,e^{-kt}$ ....(ii)

Which is the solution of the given differential equation,

Here, solution does not exist for k = 0 because when k = 0, v is undefined.

(b) When, v = 0 and t = 0.

Equation (ii) becomes, $0 = \frac{g\cos\alpha}{k} + c.\,e^{-k\times 0}$

or, $\frac{-g\cos\alpha}{k} = c$

Putting the value of c in (ii) we get

$v = \frac{g\cos\alpha}{k} - \frac{g\cos\alpha}{k}e^{-kt}$

$\therefore$ $v = \frac{g\cos\alpha}{k}\left(1 - e^{-kt}\right)$.

4. *The volume of an expanding cube is increasing w[illegible]y time 't' it is found that the rate of increase of the volume is directly proportional to the s[illegible] of the cube. Show that the length of the side of the cube 'ℓ' is a linear function of time 't', if ℓ [illegible]hen t = 0 and ℓ = 3 when t = 1, find the value of 'ℓ' when t = 8. (length ℓ and time t being measured in cm and seconds respectively).* [5] [2079 Optional]

Soln: Let, v be the volume and s be the surface area of cube, then $v = \ell^3$ and $s = 6\ell^2$.

By the question, $\frac{dv}{dt} \propto s$

or, $\frac{dv}{dt} = ks$, where k is proportional constant

or, $\frac{d\ell^3}{dt} = k\,6\ell^2$

or, $3\ell^2\frac{d\ell}{dt} = 6\,\ell^2 k$

or, $\frac{d\ell}{dt} = 2k$

or, $d\ell = 2k\,dt$

Integrating, we get, $\ell = 2kt + c$ ....(i).

$\therefore$ Length (ℓ) is linear in time 't'.

Again, if $\ell = 1$, t = 0.

$\therefore$ $1 = 0 + c$

$\therefore$ $c = 1$

From (i) $\ell = 2kt + 1$ ....(ii)

Again, if $\ell = 3$, t = 1.

$\therefore$ $3 = 2k + 1$

or, $2k = 2$

$\therefore$ $k = 1$

So, equation (ii) becomes, $\ell = 2.1.t + 1$

$\therefore$ $\ell = 2t + 1$ ....(ii)

Now, when t = 8, $\ell = 2 \times 8 + 1 = 17$ cm

$\therefore$ The value of ℓ when t = 8 seconds is 17 cm.

5. *If the population of city increases at a rate proportional to the number of its inhabitants at any time. If the population is doubled in 40 yrs, in how many years will be tripled?* [3]

Soln: Let, p be the population of a city at any time 't' years.

By the question, $\frac{dp}{dt} \propto p$

or, $\frac{dp}{dt} = kp$, where k is proportional constant.

or, $\frac{dp}{p} = kdt$

Integrating we get, $\ln p = kt + c$ ....(i)

Let, initial population (p) = N, when t = 0, then

$\ln N = 0 + c$

$\therefore$ $c = \ln N$

Now, (i) become,

$\ln p = kt + \ln N$ ....(ii)

When, t = 40 yrs, p = 2N

$\therefore$ $\ln 2N = k \times 40 + \ln N$

or, $\ln 2N - \ln N = 40k$

or, $\ln\left(\frac{2N}{N}\right) = 40k$

or, $\ln 2 = 40k$

$k = \frac{\ln 2}{40} = 0.0173$

$\therefore$ (ii) becomes, $\ln p = 0.0173t + \ln N$ ....(iii)

When, P = 3N, then $\ln 3N = 0.0173t + \ln N$

or, $\ln\left(\frac{3N}{N}\right) = 0.0173t$

or, $t = \frac{\ln 3}{0.0173} = 63.5036$

$\therefore$ Population will be tripled after more than 63 years.

6. ***The growth rate of bacteria population is proportional to its size. Initially the population is 10000 and after 10 days it is 25,000. What is the population size after 20 days?*** [3]

Soln: Let, p be the size of the population.

Then by the equation, $\frac{dp}{dt} \propto p$.

or, $\frac{dp}{dt} = kp$, where k is proportional constant.

or, $\frac{dp}{p} = k\,dt$

Integrating we get, $\ln p = kt + c$ ....(i).

When, t = 0, p = 10000

$\therefore$ $\ln 10000 = 0 + c$ $\therefore c = \ln 10000$

Now, (i) becomes, $\ln p = kt + \ln 10000$ ....(ii)

When, t = 10 days, p = 25,000.

$\therefore$ $\ln 25000 = k \times 10 + \ln 10000$

or, $\ln 25000 - \ln 10000 = 10k$

or, $10k = \ln\left(\frac{25000}{10000}\right)$

or, $10k = \ln(2.5)$

$\therefore$ $k = \frac{\ln(2.5)}{10} = 0.0916$

$\therefore$ (ii) becomes, $\ln p = 0.0916t + \ln 10000$ ....(iii).

Now, when t = 20 days,

$\ln p = 0.0916 \times 20 + \ln 10000$

or, $\ln p - \ln 10000 = 1.8326$

or, $\ln\left(\frac{p}{10000}\right) = 1.8326$

or, $\frac{p}{10000} = e^{1.8326}$

or, $p = 10000 \times e^{1.8326}$

$= 62501$

$\therefore$ *The size of the population after 20 days is 62501.*

Unit-wise Model Qu... Calculus

Set 1

Group 'A'

7. Which one of the following is the derivative of cosec h^{-1} ...

(a) $\frac{1}{x\sqrt{x^2+1}}$ (b) $\frac{-1}{x\sqrt{x^2+1}}$ (c) $\frac{1}{x\sqrt{1-x^2}}$ (|x|<1) (d) $\frac{-1}{x\sqrt{1-x^2}}$ (|x|<1) **[Ans: b]**

8. Which one of the following is equal to $\lim_{x \to 0} \frac{e^{3x}-1}{2x}$?

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) 3 **[Ans: c]**

9. Which one of the following represents the equation of normal to the curve $x^2 = 2y$ at the point (−2, 2)?

(a) $2x + y + 6 = 0$ (b) $2x - 2y + 6 = 0$

(c) $2x - y + 6 = 0$ (d) $x - 2y + 6 = 0$ **[Ans: d]**

10. Which one of the following is the solution of differential equation $x\,dy - y\,dx = 0$?

(a) $x = cy$ (b) $y = cx$ (c) $xy = c$ (d) $x - y = c$ **[Ans: a & b]**

Group 'B'

16. (a) Write the order of differential equation $\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{dy}{dx}\right)^2 + 5 = 0$. **[1] [Ans: 3]**

(b) Write the derivative of sinh x with respect to x. **[1] [Ans: coshx]**

(c) Write an example of exact differential equation in x and y. [1]

(d) Write the integral of $\int\frac{1}{x^2 - a^2}dx$. **[1] [Ans: $\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$]**

(e) State L – Hospital's rule. [1]

18. (a) Integrate: $\int\frac{dx}{3\sin x - 4\cos x}$ **[2] [Ans: $\frac{1}{5}\ln\left|\text{Tan}\left(\frac{x}{2} - \frac{1}{2}\text{Tan}^{-1}\frac{4}{3}\right)\right| + C$]**

(b) Solve: $\frac{dt}{dx} = \frac{e^{\tan^{-1}x} - t}{1 + x^2}$. **[3] [Ans: $t = \frac{1}{2}e^{\text{Tan}^{-1}x} + (e^{-\text{Tan}^{-1}x})$]**

Group 'C'

22. (a) Water flows into an inverted conical tank at the rate of 36 cm³/min. When the depth of water is 12 cm, how fast is level rising, if the radius of base and height of the tank are 21 cm and 35 cm respectively? **[3] [Ans: $\frac{25}{36\pi}$ cm/min]**

(b) The concept of anti-derivative is necessary for solving a differential equation. Justify the statement with an example. [2]

(c) A differential equation of the first degree and first order is homogenous if it satisfies the condition $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. Justify the statement with an example and solve it. [3]

Set 2

Group 'A'

7. Which one of the following is the derivative of $\tanh^{-1}x$?

(a) $\frac{1}{1-x^2}$, $|x|<1$
(b) $\frac{1}{\sqrt{1-x^2}}$, $|x|<1$
(c) $\frac{-1}{1+x^2}$, $|x|<1$
(d) $\frac{-1}{1-x^2}$, $|x|<1$

[Ans: a]

8. Which one of the following is equal to $\lim\limits_{x\to\frac{1}{2}} \frac{\tan \pi x}{\sec^2 \pi x}$?

(a) $-\frac{1}{2}$
(b) $\frac{1}{2}$
(c) 0
(d) 1

[Ans: c]

9. Which one of the following is the angle made by the tangent to the curve $2y = 2 - x^2$ at $x = 1$?

(a) $0°$
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$
(d) $\frac{3\pi}{4}$

[Ans: d]

10. Which one of the following differential equations gives integrating factor?

(a) $x dy - y^2 dx = 0$
(b) $x^2 dy - xy^2 dx = 0$
(c) $\sin^2 x \frac{dy}{dx} + y = 2$
(d) $3xy\, dy - y^2 dx = 0$

[Ans: c]

Group 'B'

16. (a) Write the slope of tangent to the curve $y = f(x)$ at (x_1, y_1). [1] [Ans: slope $\left(\frac{dy}{dx}\right) = f'(x_1)$]

(b) Write the derivative of cosechx with respect to x. [1] [Ans: – cosechx.cothx]

(c) A differential equation is in the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only. Name the differential equation. [1] [Ans: Linear]

(d) Write the integral of $\int \frac{1}{a^2 - x^2} dx$. [1] [Ans: $\frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right| + C$]

(e) Write a characteristic of L-Hospital's rule. [1]

18. (a) Integrate: $\int \frac{dx}{x^2+9}$. [2] [Ans: $\frac{1}{3} \mathrm{Tan}^{-1}\left(\frac{x}{3}\right) + C$]

(b) Solve: $\frac{dy}{dx} = \frac{1-y}{2x+1}$. [3] [Ans: $(y-1)\sqrt{2x+1} = C$]

Group 'C'

22. (a) Which type of differential equation does $\sin x \frac{dy}{dx} + y \cos x = x \sin x$ represent? Also solve it. [3] [Ans: Linear, $y \sin x = -\cos x + \sin x + C$]

(b) Evaluate: $\int \frac{x}{(x-1)(x^2+1)} dx$. [3] [Ans: $\frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \mathrm{Tan}^{-1}x + C$]

(c) Two cars start from certain places at the same instant. One goes east at 60km/hr and the other goes south at 80km/hr. How fast is the distance between them increasing? Express in symbolic form. [2] [Ans: $\frac{60x + 80y}{\sqrt{x^2+y^2}}$ km/hr]

Set 3

Group 'A'

7. Which one of the following is derivative of $\tanh^{-1} 2x$ ...

(a) $\frac{1}{1+4x^2}$
(b) $\frac{2}{1+4x^2}$
(c) ...

8. What is the slope of normal to the curve $y = 2x^2 + 3x$ ...

(a) –5
(b) $\frac{1}{-5}$
(c) $\frac{1}{5}$

9. In a differential equation $\frac{dy}{dx} = x^2y$ which one of the following ...

(a) $x \neq 0$
(b) $y \neq 0$
(c) ...

10. Which one of the following is integrating factor of $\frac{dy}{dx} + \frac{1}{x} y$ ...

(a) $\frac{1}{x}$
(b) x
(c) e^x

Group 'B'

17. Answer the following

(a) Write the equation of normal to the curve $y = f(x)$ at ...

(b) It is given that $y = f(x)$ be a function and $f'(x)$ is first order ... (x) and dx.

(c) Write the integral of $\sqrt{a^2 - x^2}$ with respect to x.

(d) Which term is same as $\int \mathrm{cosech}\, x\, dx$? Write it.

(e) Write the general form of first order exact differential equation ...

18. (a) Apply L-Hospital's rule to evaluate.

$$\lim_{x\to 1} \frac{x^3 - 2x^2 + 1}{3x^3 - 3x}.$$

(b) Evaluate: $\int \frac{dx}{4x^2+9}$

Group 'C'

22. (a) The rate of change of volume of sphere is proportional to square of its diameter ...

(b) The integration at $\frac{x^3+x}{x-1}$ with respect to x requires concept of ... fraction. Justify it.

(c) $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ is general form of a first degree linear equation ... Q(x) and solve it.

Set 2

Group 'A'

7. Which one of the following is the derivative of $\tanh^{-1}x$?

(a) $\frac{1}{1-x^2}$, $|x| < 1$ (b) $\frac{1}{\sqrt{1-x^2}}$, $|x| < 1$

(c) $\frac{-1}{1+x^2}$, $|x| < 1$ (d) $\frac{-1}{1-x^2}$, $|x| < 1$ [Ans: a]

8. Which one of the following is equal to $\lim\limits_{x \to \frac{1}{2}} \frac{\tan \pi x}{\sec^2 \pi x}$?

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1 [Ans: c]

9. Which one of the following is the angle made by the tangent to the curve $2y = 2 - x^2$ at $x = 1$?

(a) 0° (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$ [Ans: d]

10. Which one of the following differential equations gives integrating factor?

(a) $xdy - y^2dx = 0$ (b) $x^2dy - xy^2dx = 0$

(c) $\sin^2 x \frac{dy}{dx} + y = 2$ (d) $3xy\,dy - y^2dx = 0$ [Ans: c]

Group 'B'

16. (a) Write the slope of tangent to the curve $y = f(x)$ at (x_1, y_1). [1] [Ans: slope $\left(\frac{dy}{dx}\right) = f'(x_1)$]

(b) Write the derivative of cosechx with respect to x. [1] [Ans: – cosechx.cothx]

(c) A differential equation is in the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only. Name the differential equation. [1] [Ans: Linear]

(d) Write the integral of $\int \frac{1}{a^2 - x^2} dx$. [1] [Ans: $\frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right| + C$]

(e) Write a characteristic of L-Hospital's rule. [1]

18. (a) Integrate: $\int \frac{dx}{x^2+9}$. [2] [Ans: $\frac{1}{3} \operatorname{Tan}^{-1}\left(\frac{x}{3}\right) + C$]

(b) Solve: $\frac{dy}{dx} = \frac{1-y}{2x+1}$. [3] [Ans: $(y-1)\sqrt{2x+1} = C$]

Group 'C'

22. (a) Which type of differential equation does $\sin x \frac{dy}{dx} + y \cos x = x \sin x$ represent? Also solve it.

[3] [Ans: Linear, $y \sin x = -\cos x + \sin x + C$]

(b) Evaluate: $\int \frac{x}{(x-1)(x^2+1)} dx$. [3] [Ans: $\frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \operatorname{Tan}^{-1}x + C$]

(c) Two cars start from certain places at the same instant. One goes east at 60km/hr and the other goes south at 80km/hr. How fast is the distance between them increasing? Express in symbolic form.

[2] [Ans: $\frac{60x + 80y}{\sqrt{x^2+y^2}}$ km/hr]

Set 3

Group 'A'

7. Which one of the following is derivative of $\tanh^{-1}2x$ with respect to $x(|x| < 1)$?

(a) $\frac{1}{1+4x^2}$ (b) $\frac{2}{1+4x^2}$ (c) $\frac{1}{1-4x^2}$ (d) $\frac{2}{1-4x^2}$ [Ans: d]

8. What is the slope of normal to the curve $y = 2x^2 + 3x + 5$ at $(-2, 7)$?

(a) –5 (b) $\frac{1}{-5}$ (c) $\frac{1}{5}$ (d) 5 [Ans: c]

9. In a differential equation $\frac{dy}{dx} = x^2y$ which one of the following is necessary condition?

(a) $x \neq 0$ (b) $y \neq 0$ (c) $x^2 > 0$ (d) $y = 0$ [Ans: b]

10. Which one of the following is integrating factor of $\frac{dy}{dx} + \frac{1}{x}y = x^2$?

(a) $\frac{1}{x}$ (b) x (c) e^x (d) logx [Ans: b]

Group 'B'

17. Answer the following

(a) Write the equation of normal to the curve $y = f(x)$ at (x_1, y_1) if $f'(x)$ exist. [1] $\left[\text{Ans: } y - y_1 = \frac{-1}{f'(x)}(x - x_1)\right]$

(b) It is given that $y = f(x)$ be a function and $f'(x)$ is first order derivative in terms of x. Write dy in term of f'(x) and dx. [1] [Ans: $dy = f'(x)dx$]

(c) Write the integral of $\sqrt{a^2 - x^2}$ with respect to x. [1] $\left[\text{Ans: } \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$

(d) Which term is same as $\int$cosech x dx ? Write it. [1] $\left[\text{Ans: } \ln\left|\tanh\frac{x}{2}\right| + c\right]$

(e) Write the general form of first order exact differential equation. [1] [Ans: $M(x, y)dx + N(x, y)dy = 0$]

18. (a) Apply L-Hospital's rule to evaluate.

$\lim\limits_{x \to 1} \frac{x^3 - 2x^2 + 1}{3x^3 - 3x}$. [3] $\left[\text{Ans: } \frac{-1}{6}\right]$

(b) Evaluate: $\int \frac{dx}{4x^2 + 9}$ [2] $\left[\text{Ans: } \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c\right]$

Group 'C'

22. (a) The rate of change of volume of sphere is proportional to square of it's diameter. Explain. [3]

(b) The integration at $\frac{x^3 + x}{x - 1}$ with respect to x requires concept of rational functions by partial fraction. Justify it. [2]

(c) $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ is general form of a first degree linear equation take suitable function P(x) and Q(x) and solve it. [3]

Set 4

Group 'A'

7. Which is a derivative of $\coth^{-1} x$?

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$ (c) $\frac{1}{x^2-1}$ (x > 1) (d) $-\frac{1}{x^2-1}$ (x > 1) [Ans: d]

8. The value of $\lim_{x \to 0} \frac{1-\cos x}{\tan x}$

(a) – 1 (b) 0 (c) 1 (d) 2 [Ans: b]

9. Which of the following is an integral value of $\int \frac{dx}{4x^2+1}$?

(a) $\tan^{-1}(2x) + C$ (b) $\frac{1}{2}\tan^{-1}(2x) + C$

(c) $2\tan^{-1}\left(\frac{x}{2}\right) + C$ (d) $2\tan^{-1}(2x) + C$ [Ans: b]

10. A metallic circular plate, the rate of change of radius for particular time is 1 cm/s, is heated then the rate of increasing area of that plate in cm²/sec is ......

(a) $2\pi r$ (b) 2π (c) πr (d) πr^2 [Ans: a]

Group 'B'

17. (a) Find the derivatives of $\cosh^{-1}x$ with respect to x. [2] $\left[\text{Ans: } \frac{1}{\sqrt{x^2-1}} (x > 1)\right]$

(b) Using simple partial fraction, find the integral of $\int \frac{10}{(4y-1)(4y+1)} dy$. [3] $\left[\text{Ans: } \frac{5}{4}\ln\left|\frac{4y-1}{4y+1}\right| + c\right]$

18. Study the given ordinary differential equation and answer the following questions.

$(1 + x^2)\frac{dy}{dx} + ky = e^{\tan^{-1}x}$

(a) Write the order of the differential equation. [1] [Ans: Order = 1]

(b) Write the degree of the differential equation. [1] [Ans: Degree = 1]

(c) Write the type of given ordinary differential equation. [1] [Ans: Linear]

(d) If k is not a constant number, then whose function may be the k? [1] [Ans: Function of x]

(e) Write any one use of differential equation. [1]

Group 'C'

22. (a) A spherical ball of camphor is evaporating in such a way that the rate of decrease in volume at any instant is proportional to the surface Area. Prove that the radius is decreasing at a constant rate. Explain. [5]

(b) Are you agree that derivative and anti-derivative are not same? Justify your answer. [3]

□□□

7. Computational Methods

Chapter 16 System of Linear Equations

16.1 System of Linear Equations

Basic Formulae and Key Points

1. **Gauss Elimination Method:**
To solve the system of linear equations by Gaussian Elimination method, we proceed the following two steps.
Step-I: Forward Elimination
Step-II: Backward Substitution

2. **Solving the System of Linear Equations**
   (i) **Consistent and Independent System:** If the system has exactly one solution, it is called a consistent and independent system.
   (ii) **Consistent and Dependent System:** If the system has infinite number of solution, it is called a consistent and dependent system.
   (iii) **Inconsistent and Independent System:** If the system has no solution, it is called an inconsistent and independent system.

3. **Gaussian Elimination with Partial Pivoting Method:** If the system of linear equations be
$a_{11}x + a_{12}y + a_{13}z = b_1$
$a_{21}x + a_{22}y + a_{23}z = b_2$
$a_{31}x + a_{32}y + a_{33}z = b_3$ and
If $|a_{21}| > |a_{11}|$ we interchange first and second rows.
If $|a_{31}| > |a_{11}|$ we interchange first and third rows.

Note: If the system of equations has a zero pivot element, then the following method should be adopted.

4. **If the System of Linear Equations be in the Form**
$a_{12}y + a_{13}z = b_1$
$a_{21}x + a_{22}y + a_{23}z = b_2$
$a_{31}x + a_{32}y + a_{33}z = b_3$ and
   (i) If $|a_{21}| > |a_{31}|$ then we interchange first and second rows.
   (ii) If $|a_{31}| > |a_{21}|$ then we interchange first and third rows.

Group 'A' (Multiple Choice Questions and Answers)

1. *What is the number of solutions of the system of linear equations x + y = 5 and x + y = 7?* *[Model Set- 2079]*

(a) One solution (b) No solution
(c) Infinite solutions (d) More than one solution

Set 4

Group 'A'

7. Which is a derivative of $\coth^{-1} x$?

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$ (c) $\frac{1}{x^2-1}$ (x > 1)

8. The value of $\lim_{x \to 0} \frac{1-\cos x}{\tan x}$

(a) -1 (b) 0 (c) 1

9. Which of the following is an integral value of $\int \frac{dx}{4x^2+1}$?

(a) $\tan^{-1}(2x) + C$ (b) $\frac{1}{2}\tan^{-1}(2x) + C$

(c) $2\tan^{-1}\left(\frac{x}{2}\right) + C$ (d) $2\tan^{-1}(2x) + C$

10. A metallic circular plate, the rate of change of radius for ... increasing area of that plate in cm²/sec is ......

(a) $2\pi r$ (b) 2π (c) πr

Group 'B'

17. (a) Find the derivatives of $\cosh^{-1}x$ with respect to x

(b) Using simple partial fraction, find the integral of $\int \frac{}{(4y - \ldots)}$

18. Study the given ordinary differential equation and answer the ...

$(1+x^2)\frac{dy}{dx} + ky = e^{\tan^{-1}x}$

(a) Write the order of the differential equation.

(b) Write the degree of the differential equation.

(c) Write the type of given ordinary differential equation.

(d) If k is not a constant number, then whose function may be t...

(e) Write any one use of differential equation.

Group 'C'

22. (a) A spherical ball of camphor is evaporating in such a way that th... instant is proportional to the surface Area. Prove that the radiu... Explain.

(b) Are you agree that derivative and anti-derivative are not same? Jus...

□□□

Group 'B' (Subjective Questions and Answers)

(a) *What are the two major steps to solve a system of linear equation using the Gauss Elimination method?* [1]

(b) *Using Gauss-elimination method, solve the equations $x + 2y = 5$, $5x - 3y = -1$* [2]

(a) The two major steps to solve a system of linear equation using Gauss Elimination method are:

(i) Forward elimination

(ii) Backward substitution

Given system of equations are

$x + 2y = 5$ ....(i)

$5x - 3y = -1$ ....(ii)

Multiplying equation (i) by 5 and subtracting (ii) from (i), we get

$$\begin{array}{r} 5x + 10y = 25 \\ 5x - 3y = -1 \\ (-)\ (+)\quad (+) \\ \hline 13y = 26 \ldots\text{(iii)} \end{array}$$

Now, we have the following equations

$x + 2y = 5$ ....(i)

$13y = 26$ ....(iii)

From (iii), $13y = 26$

$\therefore\ y = 2$

Substituting the value of y in (i), we get

$x + 2 \times 2 = 5$

or, $x = 5 - 4$

$x = 1$

$x = 1, y = 2$ Ans.

...ne *the consistency of the following system by the Gauss elimination method:* [3]

... $+ x_3 = -3$, $3x_1 + x_2 - 2x_3 = -2$, $2x_1 + 4x_2 + 7x_3 = 7$

... equations are

$x_1 + x_2 + x_3 = -3$ ....(i)

$3x_1 + x_2 - 2x_3 = -2$ ....(ii)

$2x_1 + 4x_2 + 7x_3 = 7$ ....(iii)

...ying equation (i) by 3 and subtracting from (ii), we get

$$\begin{array}{r} 3x_1 + x_2 - 2x_3 = -2 \\ 3x_1 + 3x_2 + 3x_3 = -9 \\ (-)\ (-)\quad (-)\qquad (+) \\ \hline -2x_2 - 5x_3 = 7 \ldots\text{(iv)} \end{array}$$

...ying equation (i) by 2 and subtracting it from (iii), we get

$$\begin{array}{r} 2x_1 + 4x_2 + 7x_3 = 7 \\ 2x_1 + 2x_2 + 2x_3 = -6 \\ (-)\ (-)\quad (-)\qquad (+) \\ \hline 2x_2 + 5x_3 = 13 \ldots\text{(v)} \end{array}$$

2. ***While solving a system of three linear equations in the three variables x, y, z by using Gauss elimination method, what happens to the system if the third variable z is found to be a free variable?*** [2079 Set J]
   (a) The system becomes consistent with infinite solutions.
   (b) The system becomes consistent with an unique solution.
   (c) The system becomes inconsistent with no solution.
   (d) The system cannot be solved by Gauss elimination method.
3. ***When Gauss forward elimination method is used for solving the equations $3x + 4y = 18$ ....(i) and $3y - x = 7$ ....(ii), we apply the operation like..........***
   (a) eq^n. (i) + 4 eq^n. (ii) (b) eq^n. (i) + 3 eq^n. (ii)
   (c) eq^n. (i) + eq^n. (ii) (d) eq^n. (ii) + 3eq^n. (i)
4. ***During the process of solving system of linear equation is x, y, z if we get $0.z = 0$, then the system is*** [2079 Set K]
   (a) inconsistent and has no solution. (b) consistent and has unique solution.
   (c) inconsistent and but has no solution. (d) consistent and has infinitely solution.
5. ***Two simultaneous linear equations are given as $2x + y = 5$ and $x + 4y = 6$. What is the equation obtained after eliminating x?*** [2080]
   (a) $7y = 5$ (b) $7y = 6$ (c) $7y = 7$ (d) $7y = 12$
6. ***What is the nature of the system of equations $3x + y = 5$ and $3x + 3y = 12$?*** [2080 Optional]
   (a) Consistent and independent. (b) Inconsistent and dependent.
   (c) Inconsistent and independent. (d) Consistent and dependent.
7. ***The system of linear equation $2x - y = 0$ and $2x + y = 3$ has ....*** [2079 GIE – A]
   (a) No solution (b) Infinitely many solution
   (c) One solution (d) More than one solutions, but finite
8. ***The system of linear equations $x - y = 5$ and $4x - 4y = 20$ has ....*** [2079 GIE – B]
   (a) No solution (b) Infinitely many solutions
   (c) One solution (d) More than one solutions but finite
9. ***Given below are the system of linear equations in two variables x and y.*** [2079 Optional – A]
   (a) $x - y = 15$, $x - 2y = 10$ (b) $x - y = 5$, $2x - 2y = 9$
   (c) $4x - 4y = 32$, $x - y = 8$ (d) $x + 2y = 3$, $4x - 5y = 6$
   Which of the above system of linear equations has infinite solution?
10. ***To solve $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ by Gauss elimination method, the coefficient of the first variable in the first equation must be***
    (a) unity (b) zero (c) non–zero (d) negative
11. ***In Gauss elimination the given system of simultaneous equation is transformed into***
    (a) lower triangular matrix (b) unit matrix
    (c) transpose matrix (d) upper triangular matrix
12. ***The elimination process in Gauss elimination method is also known as.............***
    (a) Forward elimination (b) Backward elimination
    (c) Sideways elimination (d) Crossways elimination

Answer Key

1. b	2. a	3. b	4. d	5. c	6. a	7. c	8. d	9. c	10. c
11. d	12. a								

Group 'B' (Subjective Questions and Answers)

1. (a) ***What are the two major steps to solve a system of linear equation using the Gauss Elimination method?*** [1]
   (b) ***Using Gauss-elimination method, solve the equations $x + 2y = 5$, $5x - 3y = -1$*** [2]

Soln: (a) The two major steps to solve a system of linear equation using Gauss Elimination method are:
(i) Forward elimination
(ii) Backward substitution

(b) Given system of equations are

$x + 2y = 5$ ....(i)

$5x - 3y = -1$ ....(ii)

Multiplying equation (i) by 5 and subtracting (ii) from (i), we get

$$\begin{aligned} 5x + 10y &= 25 \\ 5x - 3y &= -1 \\ (-)\quad (+)\quad &\quad (+) \\ \hline 13y &= 26 \text{ ...(iii)} \end{aligned}$$

Now, we have the following equations

$x + 2y = 5$ ....(i)

$13y = 26$ ....(iii)

From (iii), $13y = 26$

$\therefore$ $y = 2$

Substituting the value of y in (i), we get

$x + 2 \times 2 = 5$

or, $x = 5 - 4$

$\therefore$ $x = 1$

$\therefore$ $x = 1, y = 2$ Ans.

2. ***Examine the consistency of the following system by the Gauss elimination method:*** [3]
   $x_1 + x_2 + x_3 = -3$, $3x_1 + x_2 - 2x_3 = -2$, $2x_1 + 4x_2 + 7x_3 = 7$

Soln: Given equations are

$x_1 + x_2 + x_3 = -3$ ....(i)

$3x_1 + x_2 - 2x_3 = -2$ ....(ii)

$2x_1 + 4x_2 + 7x_3 = 7$ ....(iii)

Multiplying equation (i) by 3 and subtracting from (ii), we get

$$\begin{aligned} 3x_1 + x_2 - 2x_3 &= -2 \\ 3x_1 + 3x_2 + 3x_3 &= -9 \\ (-)\quad (-)\quad (-)\quad &\quad (+) \\ \hline -2x_2 - 5x_3 &= 7 \text{ ....(iv)} \end{aligned}$$

Multiplying equation (i) by 2 and subtracting it from (iii), we get

$$\begin{aligned} 2x_1 + 4x_2 + 7x_3 &= 7 \\ 2x_1 + 2x_2 + 2x_3 &= -6 \\ (-)\quad (-)\quad (-)\quad &\quad (+) \\ \hline 2x_2 + 5x_3 &= 13 \text{...(v)} \end{aligned}$$

Adding equation (iv), and (v), we get,

$$\begin{array}{r} -2x_2 - 5x_3 = 7 \\ +2x_2 + 5x_3 = 13 \\ \hline 0.x_3 = 20 \text{ ....(vi)} \end{array}$$

Now, we have the following equations

$x_1 + x_2 + x_3 = -3$ ....(i)

$-2x_2 - 5x_3 = 7$ ....(iv)

$0.x_3 = 20$ ....(vi)

Since, no value of x_3 that satisfies the equation (vi). So, the system has no solution.

Hence, the system of equation is inconsistent. Ans.

3. (a) *When solving a system of three linear equation in x, y and z using the Gauss elimination method, under what condition does the system have infinitely many solutions?* [1]

(b) *Solve the following equation by Gauss elimination method: $x_1 - x_2 + x_3 = 1$, $3x_1 + x_2 + 5x_3 = 11$, $4x_1 + 2x_2 + 7x_3 = 16$* [2]

Soln: (a) If forward elimination gives $0.z = 0$. Here, z is a free variable and can take any value. So, the system has infinitely many solutions.

(b) Given equations are,

$x_1 - x_2 + x_3 = 1$ ....(i)

$3x_1 + x_2 + 5x_3 = 11$ ....(ii)

$4x_1 + 2x_2 + 7x_3 = 16$ ....(iii)

Multiplying equation (i) by 3 and subtracting it from (ii), we get

$$\begin{array}{r} 3x_1 + x_2 + 5x_3 = 11 \\ 3x_1 - 3x_2 + 3x_3 = 3 \\ (-) \quad (+) \quad (-) \quad (-) \\ \hline 4x_2 + 2x_3 = 8 \text{....(iv)} \end{array}$$

Multiplying equation (i) by 4 and subtracting it from equation (iii), we get

$$\begin{array}{r} 4x_1 + 2x_2 + 7x_3 = 16 \\ 4x_1 - 4x_2 + 4x_3 = 4 \\ (-) \quad (+) \quad (-) \quad (-) \\ \hline 6x_2 + 3x_3 = 12 \text{.....(v)} \end{array}$$

Again, multiplying equation (iv) by 3 and (v) by 2 and subtracting, we get

$$\begin{array}{r} 12x_2 + 6x_3 = 24 \\ 12x_2 + 6x_3 = 24 \\ (-) \quad (-) \quad (-) \\ \hline 0.x_3 = 0 \text{......(vi)} \end{array}$$

Now, we have the following three equations

$x_1 - x_2 + x_3 = 1$ ....(i)

$4x_2 + 2x_3 = 8$ ....(iv)

$0.x_3 = 0$:..............(vi)

Here, equation (vi) is true for all values of x_3. In this case x_3 is a free variable, so we can assign any value for x_3. So, the system has infinitely many solutions.

If $x_3 = k$, then from (iv), $x_2 = \frac{8-2k}{4} = \frac{4-k}{2}$

Again, putting the values of x_2 and x_3 in (i), we get

$$x_1 = 1 + x_2 - x_3$$

$$= 1 + \frac{4-k}{2} - k$$

$$= \frac{2+4-k-2k}{2} = \frac{6-3k}{2}$$

$\therefore x_1 = \frac{6-3k}{2}, x_2 = \frac{4-k}{2}, x_3 = k$ Ans.

4. *Using Gauss-elimination method, solve the following system of equation: $x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$.* [3]

Soln: Given equations are:

$x + 3y - 2z = 5$ ....(i)

$3x + 5y + 6z = 7$ ....(ii)

$2x + 4y + 3z = 8$ ....(iii)

Multiplying equation (i) by 3 and subtracting it from equation (ii), we get

$$\begin{array}{r} 3x + 5y + 6z = 7 \\ 3x + 9y - 6z = 15 \\ (-) \quad (-) \quad (+) \quad (-) \\ \hline -4y + 12z = -8 \text{..............(iv)} \end{array}$$

Multiplying equation (i) by 2 and subtracting from equation (iii), we get

$$\begin{array}{r} 2x + 4y + 3z = 8 \\ 2x + 6y - 4z = 10 \\ (-) \quad (-) \quad (+) \quad (-) \\ \hline -2y + 7z = -2 \text{..............(v)} \end{array}$$

Again, multiplying equation (v) by 2 and subtracting equation (iv) from (v), we get

$$\begin{array}{r} -4y + 14z = -4 \\ -4y + 12z = -8 \\ (+) \quad (-) \quad (+) \\ \hline 2z = 4 \text{..............(vi)} \end{array}$$

Now, we have the following three equations

$x + 3y - 2z = 5$ ....(i)

$-4y + 12z = -8$ ....(iv)

$2z = 4$ ....(vi)

From equation. (vi), we have, $2z = 4$

$\therefore z = 2$

From equation (iv), $-4y + 12 \times 2 = -8$

or, $-4y = -8 - 24$

or, $-4y = -32$

$\therefore y = 8$

Now, substituting the values of y and z in eq^n.(i), we get

$x + 3 \times 8 - 2 \times 2 = 5$

or, $x = 5 - 24 + 4$

$\therefore x = -15$

$\therefore x = -15, y = 8, z = 2$ Ans.

16.2 Gauss Seidel Method

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Basic Formulae and Key Points

1. **Diagonally Dominant:** A system of linear equations is said to diagonally dominant if the absolute value of the diagonal coefficient is greater than the sum of the absolute values of the other coefficients in that row. Thus, a system

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ is diagonally dominant

If $|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$

2. **Well-conditioned and Ill-conditioned System**

Well-conditioned: A system of equation is said to be well-conditioned if a small change in the coefficient of the variable in the system of equation shows a small deviation in the solution.

Ill-conditioned: A system of equation is said to be ill-conditioned if a small change in the coefficient of the variable in the system of equation shows a larger deviation in the solution.

Group 'A' (Multiple Choice Questions and Answers)

1. *The system of equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ can be solved by Gauss seidel method if*
   (a) $a_1 = b_1$
   (b) $|a_1| < |b_1|$
   (c) $|a_1| > |b_1|$
   (d) $|a_1| > |b_1|$ and $|b_2| > |a_2|$
2. *A simple way for initial guess in Gauss seidel method is to assume that*
   (a) they are all ones
   (b) they are all negatives
   (c) they are all positive
   (d) they are all zeros
3. *While solving by Gauss seidel method, which of the following is the first iterative solution system: $x - 2y = 1$ and $x + 4y = 4$?*
   (a) (1, 0.75) (b) (0.25, 1) (c) (0,0) (d) (1, 0.65)
4. *The system of equations $3x_1, + x_2 = 5$ and $x_1 - 3x_2 = 5$ is diagonally dominant because................*
   (a) $|3| + |1| < 5$
   (b) $|1| + |-3| < 5$
   (c) $|3| + |1| = |1| + |-3|$
   (d) $|3| > 1$ and $|-3| > 1$
5. *The system of equations $3x - y = -4$ and $2x + 5y = 2$ is diagonally dominant so the system is*
   (a) oscillatory (b) divergent (c) convergent (d) none
6. *The system of equations*
   $x_1 + 5x_2 - x_3 = 10$ ....(i)
   $x_1 + x_2 + 8x_3 = 20$ ....(ii)
   $4x_1 + 2x_2 + x_3 = 14$ ......(iii)
   Will be diagonally dominant if the order of equations be
   (a) (iii), (ii) and (i)
   (b) (iii), (i) and (ii)
   (c) (i), (ii) and (iii)
   (d) (i), (iii) and (ii)
7. *If a small change in the coefficient of the variable in the system of equations shows a small deviation in the solution then the system is*
   (a) ill-conditioned
   (b) well-conditioned
   (c) both ill and well-conditioned
   (d) cannot be determined

8. *The solution of the system of equations $3x_1 + 2x_2 = -9$ and $2x_1 - 3x_2 = -6$ is second iteration using Gauss seidel method is*
   (a) $x_1 = 0; x_2 = 0$
   (b) $x_1 = 0, x_2 = 3$
   (c) $x_1 = -3, x_2 = 0$
   (d) $x_1 = 1, x_2 = 2$
9. *The system of equations are as follows:*
   $x_1 + 2x_2 + 3x_3 = 5$
   $2x_1 + 3x_2 + x_3 = 1$
   $3x_1 + 2x_2 + x_3 = 3$
   Assuming initial guess as $x_1 = x_2 = x_3 = 0$, the value of x_3 after the first iteration using Gauss seidel method as per the given order of equations is ....
   (a) -3 (b) 5 (c) 6 (d) -6
10. *Which of the following system of linear equations has a strictly diagonally dominant?*
   (a) $2x + y = 1$, $x - y = 4$
   (b) $3x + 5y = 2$, $x + y = -3$
   (c) $4x = 2y - z - 1$, $x + z = -4$, $3x - 5y + z = 3$
   (d) $3x - y = 4$, $2x + 5y = 2$

Answer Key

1. d	2. d	3. a	4. d	5. c	6. b	7. b	8. c	9. d	10. d

Group 'B' (Subjective Questions and Answers)

1. (a) *What does it mean for a system of linear equations to be diagonally dominant?* [2]
   (b) *Test whether the system of equation $12x_1 + 3x_2 - 5x_3 = 1$, $x_1 + 5x_2 + 3x_3 = 28$, $3x_1 + 7x_2 + 13x_3 = 1$.* [2]

Soln: (a) **Diagonally Dominant:** A system of linear equations is said to diagonally dominant if the absolute value of the diagonal coefficient is greater than the sum of the absolute values of the other coefficients in that row. Thus, a system

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ is diagonally dominant, if

$|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$

(b) Here, given equations are

$12x_1 + 3x_2 - 5x_3 = 1$

$x_1 + 5x_2 + 3x_3 = 28$

$3x_1 + 7x_2 + 13x_3 = 1$

Since, $|12| > |3| + |-5|$

$|5| > |1| + |3|$

$|13| > |3| + |7|$

So, the given system of equations is diagonally dominant. **Ans.**

2. (a) ***During the process of solving a system of linear equations using the Gauss seidel method, what should be the coefficients of the leading diagonal?*** [1]

(b) ***Given the system,***

$x + 2y = 5$ ***and*** $3x + y = 5$.

Is the system diagonally dominant? If not how can we make the system diagonally dominant? [2]

Soln: (a) During the process of solving a system of linear equations using the Gauss seidel method, the coefficient of the leading diagonal must be non-zero.

(b) **1st part:** Here, given system of equations are

$x + 2y = 5$ and

$3x + y = 5$

Since, $|1| \not> |2|$ and

$|1| \not> |3|$, so the given system is not diagonally dominant.

2nd part: We can make the system diagonally dominant if we interchange the equations as follows:

$3x + y = 5$ and

$x + 2y = 5$

Now, the system of equations is diagonally dominant.

3. ***Using Gauss seidel method, solve the equations $3x + 2y = -9$, $2x - 3y = -6$.*** [2]

Soln: Given equations are

$3x + 2y = -9$

$2x - 3y = -6$

Since, $|3| > |2|$ and $|-3| > |2|$. The given system of equation is diagonally dominant. Hence, we can apply Gauss Seidel method.

The given system of equations can be written as, $x = \frac{-9-2y}{3}$ ....(i)

$y = \frac{2x+6}{3}$ ....(ii)

Suppose initial approximation is $x = 0$, $y = 0$

Iteration - I: $x = \frac{-9-2\times 0}{3} = -3$

$y = \frac{2\times(-3)+6}{3} = 0$

Iteration - II: $x = \frac{-9-2\times 0}{3} = -3$

$y = \frac{2\times(-3)+6}{3} = 0$

Here, in first and second iterations, the values of x and y are the same. So, $x = -3$, $y = 0$

4. ***Solve the following equations using Gauss–seidel method. $3x_1 + x_2 = 5$, $x_1 - 3x_2 = 5$*** [3]

Soln: Here, $|3| > |1|$ and $|-3| > |1|$

$\therefore$ Given system is diagonally dominant.

Now, the above system can be written as

$x_1 = \frac{5-x_2}{3}$ .... (i)

$x_2 = \frac{x_1-5}{3}$ .... (ii)

Suppose $(x_1, x_2) = (0, 0)$

Iteration - I: $x_1 = \frac{5-x_2}{3} = \frac{5-0}{3} = 1.667$

$x_2 = \frac{x_1-5}{3} = \frac{1.667-5}{3} = -1.111$

Iteration - II: $x_1 = \frac{5-x_2}{3} = \frac{5-(-1.111)}{3} = 2.037$

$x_2 = \frac{x_1-5}{3} = \frac{2.037-5}{3} = -0.987$

Iteration - III: $x_1 = \frac{5-x_2}{3} = \frac{5-(-0.987)}{3} = 1.996$

$x_2 = \frac{x_1-5}{3} = \frac{1.996-5}{3} = -1.001$

Iteration - IV: $x_1 = \frac{5-x_2}{3} = \frac{5-(-1.001)}{3} = 2.000$

$x_2 = \frac{x_1-5}{3} = \frac{2.000-5}{3} = -1.000$

Iteration - V: $x_1 = \frac{5-x_2}{3} = \frac{5-(-1)}{3} = 2$

$x_2 = \frac{x_1-5}{3} = \frac{2-5}{3} = -1$

Here, in the iterations IV and V, the values of x_1 and x_2 are the same. So, $x_1 = 2$ and $x_2 = -1$ **Ans.**

5. ***Solve the following equations using Gauss Seidel method:***

$3x + 4y + 8z = 7$, $x + 20y + z = -18$ and $25x + y - 5z = 19$ [3]

Soln: Given system of equations are

$3x + 4y + 8z = 7$

$x + 20y + z = -18$

$25x + y - 5z = 19$

Here, from the first equation, $|3| \not> |4| + |8|$ and from the third equation, $|-5| \not> |25| + |1|$.

So, the system is not diagonally dominant. We can rewrite the system so that it would be diagonally dominant.

Now, interchanging first and third equations, so as to make them diagonally dominant.

Now, we have

$25x + y - 5z = 19$

$x + 20y + z = -18$

$3x + 4y + 8z = 7$

These equations can be written as

$x = \frac{19-y+5z}{25}$ ....(i)

$y = \frac{-18-x-z}{20}$ ....(ii)

$z = \frac{7-3x-4y}{8}$ ....(iii)

2. (a) ***During the process of solving a system of linear equations using the Gauss ... what should be the coefficients of the leading diagonal?***

(b) ***Given the system,***

$x + 2y = 5$ and $3x + y = 5$.

Is the system diagonally dominant? If not how can we make the system diagonally ...

Soln: (a) During the process of solving a system of linear equations using the Gauss seidel ... coefficient of the leading diagonal must be non-zero.

(b) **1st part:** Here, given system of equations are

$x + 2y = 5$ and

$3x + y = 5$

Since, $|1| \ngtr |2|$ and

$|1| \ngtr |3|$, so the given system is not diagonally dominant.

2nd part: We can make the system diagonally dominant if we interchange the equations as follows:

$3x + y = 5$ and

$x + 2y = 5$

Now, the system of equations is diagonally dominant.

3. ***Using Gauss seidel method, solve the equations $3x + 2y = -9$, $2x - 3y = -6$.*** [2]

Soln: Given equations are

$3x + 2y = -9$

$2x - 3y = -6$

Since, $|3| > |2|$ and $|-3| > |2|$. The given system of equation is diagonally dominant. Hence, we can apply Gauss Seidel method.

The given system of equations can be written as, $x = \dfrac{-9 - 2y}{3}$ ....(i)

$$y = \frac{2x + 6}{3} \quad \text{..............(ii)}$$

Suppose initial approximation is $x = 0$, $y = 0$

Iteration - I: $x = \dfrac{-9 - 2 \times 0}{3} = -3$

$$y = \frac{2 \times (-3) + 6}{3} = 0$$

Iteration - II: $x = \dfrac{-9 - 2 \times 0}{3} = -3$

$$y = \frac{2 \times (-3) + 6}{3} = 0$$

Here, in first and second iterations, the values of x and y are the same. So, $x = -3$, $y = 0$

4. ***Solve the following equations using Gauss–seidel method. $3x_1 + x_2 = 5$, $x_1 - 3x_2 = 5$*** [3]

Soln: Here, $|3| > |1|$ and $|-3| > |1|$

$\therefore$ Given system is diagonally dominant.

Now, the above system can be written as

$x_1 = \dfrac{5 - x_2}{3}$ .... (i)

$x_2 = \dfrac{x_1 - 5}{3}$ .... (ii)

Suppose initial guess be x = 0, y = 0 and z = 0

Iteration -I: $x = \frac{19 - y + 5z}{25} = \frac{19 - 0 - 5 \times 0}{25} = 0.76$

$y = \frac{-18 - x - z}{20} = \frac{-18 - 0.76 - 0}{20} = -0.938$

$z = \frac{7 - 3x - 4y}{8} = \frac{7 - 3 \times (0.76) - 4 \times (-0.938)}{8} = 1.059$

Iteration -II: $x = \frac{19 - y + 5z}{25} = \frac{19 - (-0.938) + 5 \times 1.059}{25} = 1.009$

$y = \frac{-18 - x - z}{20} = \frac{-18 - 1.009 - 1.059}{20} = -1.003$

$z = \frac{7 - 3x - 4y}{8} = \frac{7 - 3 \times 1.009 - 4 \times (-1.003)}{8} = 0.998$

Iteration -III: $x = \frac{19 - y + 5z}{25} = \frac{19 - (-1.003) + 5 \times 0.998}{25} = 1.000$

$y = \frac{-18 - x - z}{20} = \frac{-18 - 1 - 0.998}{20} = -1.000$

$z = \frac{7 - 3x - 4y}{8} = \frac{7 - 3 \times 1 - 4 \times (-1)}{8} = 1$

Iteration -IV: $x = \frac{19 - y + 5z}{25} = \frac{19 - (-1) + 5 \times 1}{25} = 1$

$y = \frac{-18 - x - z}{20} = \frac{-18 - 1 - 1}{20} = -1$

$z = \frac{7 - 3x - 4y}{8} = \frac{7 - 3 \times 1 - 4 \times (-1)}{8} = 1$

Here, in iterations (III) and (IV), the values of x, y and z are equal. So, x = 1, y = −1 and z = 1 **Ans.**

6. ***Examine whether the following system of equations are ill conditioned.*** [2]

$2x_1 + x_2 = 25$

$2.001x_1 + x_2 = 25.01$

Soln: Let, A = the coefficient matrix

$= \begin{pmatrix} 2 & 1 \\ 2.001 & 1 \end{pmatrix}$

Now,

$|A| = \begin{vmatrix} 2 & 1 \\ 2.001 & 1 \end{vmatrix}$

$= 2 - 2.001$

$= -0.001$ whose absolute value is too small.

Hence, the equations are ill-conditioned.

> **Note:** If two lines appear nearly parallel, use the determinant method to identify an ill-conditioned system. Otherwise, convert the given system into y = mx + c form and then apply the determinant technique accordingly.

7. (a) ***Write the conditions for the system of equations***

$a_{11}x + a_{12}y = b_1$

$a_{21}x + a_{22}y = b_2$ ***to be ill-conditioned.***

(b) ***Examine whether the following system of equations are ill-conditioned.*** [1]

$21x - 16y = 4$

$3x + 2y = 2$

Soln: (a) Given system of equations can be written as [2]

$\begin{pmatrix} a_{21} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

If the given system is ill-conditioned, then the determinant of the coefficient matrix should be nearly equal to zero. So, the required condition is

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$

i.e. $a_{11}a_{22} - a_{21}a_{12} = 0$ **Ans.**

(b) Here, given system of equations are

$21x - 16y = 4$

$3x + 2y = 2$

Let, us change the system into y = mx + c form as

$1.3125x - y = 0.25$ ....(i) $[\because y = \frac{21}{16}x - \frac{4}{16}]$

$1.5x + y = 1$ ....(ii) $[\because y = \frac{-3}{2}x + 1]$

Let, A = the coefficient matrix formed by (i) and (ii)

$= \begin{pmatrix} 1.3125 & -1 \\ 1.5 & 1 \end{pmatrix}$

Now, $|A| = \begin{vmatrix} 1.3125 & -1 \\ 1.5 & 1 \end{vmatrix}$

$= 1.3125 + 1.5$

$= 2.8125$

Which is not small. Hence the equations are well conditioned.

8. (a) ***Define well-conditioned and ill-conditioned of a system of equation.*** [2]

(b) ***Is the system*** $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ ***well-conditioned? Justify your answers.*** [1]

Soln: (a) **Well-conditioned:** A system of equation AX = B is said to be well-conditioned if a small change in the coefficient of the variable in the system of equations shows a small deviation in the solution.

Ill-conditioned: A system of equation AX = B is said to be ill-conditioned if a small change in the coefficient of the variable in the system of equations shows a large deviation in the solution.

(b) Let, $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$ whose absolute value is not very small or not close to zero.

Hence, the system is well-conditioned.

□□□

Chapter 17

Linear Programming

17.1 Linear Programming

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Basic Formulae and Key Points

1. **Basic Solution (BS):** This solution is obtained by setting any n variables (among m + n variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these variables is non-zero. Such m variables are called basic variables and remaining n zero valued variables are called non basic variable
2. **Basic Feasible Solution (BFS):** It is a basic solution which also satisfies the non-negativity restrictions.
3. **Objective Function:**
   A linear function which is to be optimized (maximized or minimized) is said to be the objective function.
4. **Decision Variable:**
   The independent variables involved in the linear programming problems are called decision variables.
5. **Slack Variable:**
   If the inequalities are of the type '$\le$' then we add a new non-negative variables S_i (where, i = 1,2,3 .... k) in left side to change the inequalities into equalities then the new non-negative variables S_i are called the slack variables.
6. **Surplus Variable:**
   If the inequalities are of the type '$\ge$' then we subtract a new non-negative variables S_i (Where, i = 1,2,3...k) in the left side to change the given inequalities into equalities then the new non-negative variables S_i are called the surplus variable.
7. **Optimal Solution:**
   Any feasible solution which minimizes or maximizes the objective function of a general LPP is called optimal solution to the LPP.
8. **Simplex Method:**
   For solving LPP consisting two or more decision variables and a large number of constraints, the graphical method will not be convenient. In such cases another finite iterative method called "simplex method" is used to find optimal solution of the LPP.
9. **Steps of Solving the Simplex Method:**

(i) Convert all constraints into equations by **adding slack variables.**
(ii) Create the **initial simplex tableau.**
(iii) Locate the most negative entry in the bottom row to determine the pivot column.
(iv) Find the smallest ratio of **"R.H.S. – column"** with their corresponding pivot column in pivot row.
(v) Use elementary row operations so that the **pivot values is 1,** and all other entries in the **entering column are 0.** This process is called **pivoting.**
(vi) If all the entries in the last row (or bottom row) are non-negative, this is the final table. If not go back to step no. (iii) to determine the pivot column and repeat steps (iv) & (v).
(vii) From the final table, the LPP has a maximum solution (optimal solution), which is given by the entry in the **lower right corner** of the table.

Note: (i) If there are two equal most negative entries in the last row, we may choose either column (any one column) as the pivot column.
(ii) In order to find the pivot element, if a smallest ratio is negative, it should be ignored. instead, only the positive ratios should be considered when selecting the pivot element.

Group 'A' (Multiple Choice Questions and Answers)

1. ***Which one of the following is true in case of simplex method of linear programming?***
(a) The constant of constraints equation may be positive or negative.
(b) Inequalities are not converted into equations.
(c) It cannot be used for two variable problems.
(d) The simplex algorithm is an iterative procedure.

2. ***The non-negative variable that has to be subtracted from a constraint inequality of the form $\ge$ to change it to an equation is called:***
(a) Slack variable (b) Surplus variable
(c) Artificial slack variable (d) None of these

3. ***The pivot element in the simplex tableau is***
(a) any element in pivot column.
(b) any element in pivot row.
(c) an element common to pivot column and pivot row.
(d) any non–zero element.

4. ***The optimal value of the objective function is obtained when all entries in the last row of the simplex tableau is***
(a) positive (b) negative (c) non-positive (d) non-negative

5. ***In the simplex tableau, the pivot element is***

x	y	r	s	z	RHS
3	2	1	0	0	6
1	–3	0	1	0	4
–5	–3	0	0	1	0

(a) 1 (b) 3 (c) 6 (d) 4

6. ***Which row is the pivot row in the simplex tableau given below?***

x	y	r	s	z	RHS
6	8	1	1	0	100
4	3	– 2	0	1	90
–2	– 5	– 6	0	0	0

(a) Row 1 (b) Row 2 (c) Row 3 (d) None

7. ***What would you conclude to find the optimum solutions from the simplex tableau given below?***

x	y	r	s	z	RHS
0	1	1	0	0	24
–2	2	0	1	0	60
– 40	– 35	0	0	1	0

(a) 0 (b) No solution (c) 24 (d) 60

8. *If the given feasible solution is optimal, then basic feasible solution is also:*
(a) optimal (b) not optimal (c) may be optimal or not (d) none of these

9. *The number at the intersection of key row and key column is known as*
(a) Pivot column. (b) Pivot row.
(c) Pivot element. (d) None.

10. *In simplex method if the last row of the simplex tableau has more than one negative value then we choose .... to identify the pivot column.*
(a) two negative values (b) any one negative value
(c) one most negative value (d) all negative value

11. *The presentation of the objective function max. $z = 3x + 4y$ after introducing non-negative slack variables r and s is*
(a) $-3x - 4y + z = 0$ (b) $-3x - 4y + 0.r + z = 0$
(c) $-3x - 4y + 0.s + z = 0$ (d) $-3x - 4y + 0.r + 0.s + 1.z = 0$

12. *According to algebra of simplex method slack variables are assigned zero coefficients because*
(a) no contribution in objective function (b) high contribution in objective function
(c) divisor contribution in objective function (d) bare contribution in objective function.

Answer Key

1. d	2. b	3. c	4. d	5. b	6. a	7. b	8. a	9. c	10. c
11. d	12. a								

Group 'B' (Subjective Questions and Answers)

1. *Use the simplex method and maximize $z = 15x + 12y$ subject to $2x + 3y \le 21$, $3x + 2y \le 24$, $x, y \ge 0$.* *[5] [2081 Set E/W]*

Soln: Let, r and s be the non-negative slack variables. Adding the slack variables, then the given LPP can be written as

$2x + 3y + r = 21$
$3x + 2y + s = 24$
$15x + 12y = z$ such that
$2x + 3y + r + 0.s + 0.z = 21$
$3x + 2y + 0.r + s + 0.z = 24$
$-15x - 12y + 0.r + 0.s + z = 0$
$x, y, r, s \ge 0$

Initial simplex tableau is given as below:

Basic variable	x	y	r	s	z	R.H.S.
r	2	3	1	0	0	21
s	$\boxed{3}$	2	0	1	0	24
	−15	−12	0	0	1	0
	↑					

Here, the most negative value in the last row is −15. So, first column is the pivot column. Since, $\frac{21}{2} = 10.5$ and $\frac{24}{3} = 8$ and $8 < 10.5$. The minimum ratio is 8. So, $\boxed{3}$ in first column is the pivot element.

Now, applying $R_2 \to \frac{1}{3} R_2$

Basic variable	x	y	r	s	z	R.H.S.
r	2	3	1	0	0	21
x	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	8
	−15	−12	0	0	1	0

Applying $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 15R_2$

Basic variable	x	y	r	s	z	R.H.S.
r	0	$\frac{5}{3}$	1	$\frac{-2}{3}$	0	5
x	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	8
	0	−2	0	5	1	120
		↑				

This is not the optimal solution as last row contains negative entry. So, second column is the pivot column.

Since, $\frac{5}{\frac{5}{3}} = \frac{15}{5} = 3$ and $\frac{8}{\frac{2}{3}} = \frac{24}{2} = 12$ and $3 < 12$. So, $\boxed{\frac{5}{3}}$ is the pivot element.

Again, applying $R_1 \to \frac{3}{5} R_1$

Basic variable	x	y	r	s	z	R.H.S.
y	0	1	$\frac{3}{5}$	$\frac{-2}{5}$	0	3
x	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	8
	0	−2	0	5	1	120

Applying $R_2 \to R_2 - \frac{2}{3} R_1$ and $R_3 \to R_3 + 2R_1$

Basic variable	x	y	r	s	z	R.H.S.
y	0	1	$\frac{3}{5}$	$\frac{-2}{5}$	0	3
x	1	0	$\frac{-2}{5}$	$\frac{3}{5}$	0	6
	0	0	$\frac{6}{5}$	$\frac{21}{5}$	1	126

Since, all the entries in the last row are non-negative, so the solution is optimal.

$\therefore$ Max. $z = 126$ where $x = 6$, $y = 3$

Also, max. $z = 15x + 12y = 15 \times 6 + 12 \times 3 = 126$ which is verified.

2. *Using simplex method maximize $P(x, y) = 15x + 10y$ subject to $2x + y \le 10$, $x + 3y \le 12$, $x, y \ge 0$.* [5] [2081 – Set V]

Soln: Let, r and s be the non-negative slack variables. Adding the slack variables, then the given LPP can be written as

$2x + y + r = 10$
$x + 3y + s = 12$
$15x + 10y = P$ such that
$2x + y + r + 0.s + 0.P = 10$
$x + 3y + 0.r + s + 0.P = 12$
$-15x - 10y + 0.r + 0.s + P = 0$

Initial simplex tableau is given as below:

Basic variable	x	y	r	s	P	R.H.S.
r	$\boxed{2}$	1	1	0	0	10
s	1	3	0	1	0	12
	–15	–10	0	0	1	0
	↑					

Here, –15 is the most negative entry, so first column is the pivot column. Since, $\frac{10}{2} = 5$, $\frac{12}{1} = 12$ and $5 < 12$. The minimum ratio is 5. So, $\boxed{2}$ is the pivot element.

Applying $R_1 \to \frac{1}{2}R_1$

Basic variable	x	y	r	s	P	R.H.S.
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5
s	1	3	0	1	0	12
	–15	–10	0	0	1	0

Again, Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 15R_1$

Basic variable	x	y	r	s	P	R.H.S.
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5
s	0	$\frac{5}{2}$	$\frac{-1}{2}$	1	0	7
	0	$\frac{-5}{2}$	$\frac{15}{2}$	0	1	75
		↑				

As the last row contains negative entry. So, this is not the optimal solution.
Here, the second column is the pivot column.

Since, $\frac{5}{\frac{1}{2}} = 10$, $\frac{7}{\frac{5}{2}} = \frac{14}{5} = 2.8$ and $2.8 < 10$,

So, $\boxed{\frac{5}{2}}$ is the pivot element.

Now, applying $R_2 \to \frac{2}{5}R_2$

Basic variable	x	y	r	s	P	R.H.S.
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5
y	0	1	$\frac{-1}{5}$	$\frac{2}{5}$	0	$\frac{14}{5}$
	0	$\frac{-5}{2}$	$\frac{15}{2}$	0	1	75

Again, applying $R_1 \to R_1 - \frac{1}{2}R_2$ and $R_3 \to R_3 + \frac{5}{2}R_2$

Basic variable	x	y	r	s	P	R.H.S.
x	1	0	$\frac{3}{5}$	$\frac{-1}{5}$	0	$\frac{18}{5}$
y	0	1	$\frac{-1}{5}$	$\frac{2}{5}$	0	$\frac{14}{5}$
	0	0	7	1	1	82

Since, all the entries in the last row are non-negative. So, the optimal solution is obtained.

$\therefore$ Max. $P(x, y) = 82$ at $x = \frac{18}{5}$ and $y = \frac{14}{5}$ **Ans.**

Also, max. $P(x, y) = 15x + 10y = 15 \times \frac{18}{5} + 10 \times \frac{14}{5}$

$= 54 + 28 = 82$ which is verified.

1. *Apply simplex method to maximize $z = 7x_1 + 5x_2$ subject to $x_1 + 2x_2 \le 6$, $4x_1 + 3x_2 \le 12$, $x_1, x_2 \ge 0$.* [3]

Soln: Let, x_3 and x_4 be non-negative slack variables. Adding the slack variables, then

We have,

$x_1 + 2x_2 + x_3 = 6$
$4x_1 + 3x_2 + x_4 = 12$
$7x_1 + 5x_2 = z$ such that
$x_1 + 2x_2 + x_3 + 0.x_4 + 0.z = 6$
$4x_1 + 3x_2 + 0.x_3 + x_4 + 0.z = 12$
$-7x_1 - 5x_2 + 0.x_3 + 0.x_4 + z = 0$
$x_1, x_2, x_3, x_4 \ge 0$

The initial simplex tableau is given below:

Basic variable	x_1	x_2	x_3	x_4	z	R.H.S.
x_3	1	2	1	0	0	6
x_4	$\boxed{4}$	3	0	1	0	12
	–7	–5	0	0	1	0
	↑					

Here, the most negative value in the last row is – 7. So, first column is the pivot column. Since, $\frac{6}{1} = 6$, $\frac{12}{4} = 3$ and $3 < 6$. The minimum ratio is 3. So, $\boxed{4}$ is the pivot element.

Applying $R_2 \rightarrow \frac{1}{4} R_2$

Basic variable	x_1	x_2	x_3	x_4	z	R.H.S.
x_3	1	2	1	0	0	6
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	3
	–7	–5	0	0	1	0

Applying $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 + 7R_2$

Basic variable	x_1	x_2	x_3	x_4	z	R.H.S.
x_3	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	0	3
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	3
	0	$\frac{1}{4}$	0	$\frac{7}{4}$	1	21

Since, all the entries in the last row are non-negative, so the solution is optimal.

$\therefore$ Max. $z = 21$ at $x_1 = 3$ and $x_2 = 0$.

Also, maximum $z = 7x_1 + 5x_2$

$= 7 \times 3 + 5 \times 0$

$= 21$ which is verified.

4. ***Use the simplex method to maximize $P = x + y$ subject to constraints $x + 2y \leq 6$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.*** [4]

Soln: Let, r and s be the non-negative slack variables. Adding the slack variables. Then we have,

$x + 2y + r = 6$

$3x + 2y + s = 12$

$x + y = P$ such that

$x + 2y + r + 0.s + 0.P = 6$

$3x + 2y + 0.r + s + 0.P = 12$

$-x - y + 0.r + 0.s + P = 0$

The initial simplex tableau is given below:

Basic variable	x	y	r	s	P	R.H.S.
r	1	$\boxed{2}$	1	0	0	6
s	3	2	0	1	0	12
	–1	–1 ↑	0	0	1	0

Since, the last row has two equal negative entries, we may choose any one of the 1^{st} or 2^{nd} column to determine the pivot element. Let, us choose 2^{nd} column as pivot column. since, $\frac{6}{2} = 3$, $\frac{12}{2} = 6$ and $3 < 6$. The minimum ratio is 3. So, $\boxed{2}$ is the pivot element.

Applying $R_1 \rightarrow \frac{1}{2} R_1$

Basic variable	x	y	r	s	P	R.H.S.
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	3
s	3	2	0	1	0	12
	–1	–1	0	0	1	0

Again, applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 + R_1$

Basic variable	x	y	r	s	P	R.H.S.
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	3
s	$\boxed{2}$	0	–1	1	0	6
	$\frac{-1}{2}$ ↑	0	$\frac{1}{2}$	0	1	3

As the last row contains negative entry so, this is not the optimal solutions. Here, the first column is the pivot column. Since, $\frac{3}{\frac{1}{2}} = 6$, $\frac{6}{2} = 3$ and $3 < 6$. Since, 3 is minimum ratio, so $\boxed{2}$ is the pivot element.

Now, applying $R_2 \rightarrow \frac{1}{2} R_2$

Basic variable	x	y	r	s	p	R.H.S.
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	3
x	1	0	$\frac{-1}{2}$	$\frac{1}{2}$	0	3
	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	1	3

Again, applying $R_1 \rightarrow R_1 - \frac{1}{2} R_2$ and $R_3 \rightarrow R_3 + \frac{1}{2} R_2$

Basic variable	x	y	r	s	P	R.H.S.
y	0	1	$\frac{3}{4}$	$\frac{-1}{4}$	0	1.5
x	1	0	$\frac{-1}{2}$	$\frac{1}{2}$	0	3
	0	0	$\frac{1}{4}$	$\frac{1}{4}$	1	4.5

Since, all the entries in last row are non-negative so the solution in optimal.

$\therefore$ Max. $P = 4.5$ at $x = 3$ and $y = 1.5$

Also, max. $P = x + y = 3 + 1.5 = 4.5$ which is verified.

5. *Form the dual problem corresponding to each of the following LPP.*

(a) *Maximize $P = 25x + 45y$ subject to $x + 3y \le 21$, $2x + 3y \le 24$, $x, y \ge 0$.* [2]

(b) *Minimize $W = 12x + 18y$ s.t.* [2]

$x + 2y \ge 8$

$4x + 4y \ge 24$

$x, y \ge 0$

Soln: (a) Let, A = Augmented matrix formed from the given constraints and objective function.

$$= \left(\begin{array}{cc|c} 1 & 3 & 21 \\ 2 & 3 & 24 \\ \hline 25 & 45 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 2 & 25 \\ 3 & 3 & 45 \\ \hline 21 & 24 & 0 \end{array}\right)$$

Now, the corresponding dual problem of given LPP is

Minimize $P^* = 21u + 24v$

Subject to: $u + 2v \ge 25$

$3u + 3v \ge 45$

$u, v \ge 0$ **Ans.**

(b) Let, A = Augmented matrix formed from the constraints and objective function.

$$= \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 4 & 4 & 24 \\ \hline 12 & 18 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 4 & 12 \\ 2 & 4 & 18 \\ \hline 8 & 24 & 0 \end{array}\right)$$

Now,

The corresponding dual problem of given LPP is

Maximize $W^* = 8u + 24v$ s.t.

$u + 4v \le 12$

$2u + 4v \le 18$

$u, v \ge 0$ **Ans.**

6. ***Using the simplex method, minimize $W = 3x + 2y$ subject to $2x + y \ge 4$, $x + 2y \ge 4$, $x, y \ge 0$.*** [5]

Soln: Let, A = Augmented matrix formed from the constraints and objective function.

$$= \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 2 & 4 \\ \hline 3 & 2 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 2 \\ \hline 4 & 4 & 0 \end{array}\right)$$

Now,

The corresponding dual problem of the given LPP is

Maximize $W^* = 4u + 4v$

Subject to: $2u + v \le 3$

$u + 2v \le 2$

$u, v \ge 0$

Let, x and y be the non-negative slack variables. Adding the slack variables. Then the above LPP can be written as $2u + v + x = 3$

$u + 2v + y = 2$

$4u + 4v = W^*$ such that

$2u + v + x + 0.y + 0.W^* = 3$

$u + 2v + 0.x + y + 0.W^* = 2$

$-4u - 4v + 0.x + 0.y + W^* = 0$

The initial simplex tableau is given below:

Basic variable	u	v	x	y	W*	R.H.S.
x	$\boxed{2}$	1	1	0	0	3
y	1	2	0	1	0	2
	–4	–4	0	0	1	0
	↑					

Since, the last row has two equal negative entries, we may choose any one of the 1st or 2nd column to determine the pivot element. Let, us choose 1st column as pivot column. Since, $\frac{3}{2} = 1.5$, $\frac{2}{1} = 2$ and $1.5 < 2$. The minimum ratio is 1.5. So, $\boxed{2}$ is the pivot element.

Applying, $R_1 \to \frac{1}{2}R_1$

Basic variable	u	v	x	y	W*	R.H.S.
u	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y	1	2	0	1	0	2
	–4	–4	0	0	1	0

Again, applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 4R_1$

Basic variable	u	v	x	y	w*	R.H.S.
u	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y	0	$\boxed{\frac{3}{2}}$	$\frac{-1}{2}$	1	0	$\frac{1}{2}$
	0	–2	2	0	1	6
		↑				

It is not the optimal solution because last row has still one negative entry. Second column is the pivot column. since $\frac{\frac{3}{2}}{\frac{1}{2}} = 3$, $\frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$ and $\frac{1}{3} < 3$. The minimum ratio is $\frac{1}{3}$. So, $\boxed{\frac{3}{2}}$ is the pivot element.

5. *Form the dual problem corresponding to each of the following LPP.*

(a) *Maximize $P = 25x + 45y$ subject to $x + 3y \le 21$, $2x + 3y \le 24$, $x, y \ge 0$.* [2]

(b) *Minimize $W = 12x + 18y$ s.t.* [2]

$$x + 2y \ge 8$$
$$4x + 4y \ge 24$$
$$x, y \ge 0$$

Soln: (a) Let, A = Augmented matrix formed from the given constraints and objective function.

$$= \left(\begin{array}{cc|c} 1 & 3 & 21 \\ 2 & 3 & 24 \\ \hline 25 & 45 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 2 & 25 \\ 3 & 3 & 45 \\ \hline 21 & 24 & 0 \end{array}\right)$$

Now, the corresponding dual problem of given LPP is

Minimize $P^* = 21u + 24v$

Subject to: $u + 2v \ge 25$

$3u + 3v \ge 45$

$u, v \ge 0$ **Ans.**

(b) Let, A = Augmented matrix formed from the constraints and objective function.

$$= \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 4 & 4 & 24 \\ \hline 12 & 18 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 4 & 12 \\ 2 & 4 & 18 \\ \hline 8 & 24 & 0 \end{array}\right)$$

Now,

The corresponding dual problem of given LPP is

Maximize $W^* = 8u + 24v$ s.t.

$u + 4v \le 12$

$2u + 4v \le 18$

$u, v \ge 0$ **Ans.**

6. *Using the simplex method, minimize $W = 3x + 2y$ subject to $2x + y \ge 4$, $x + 2y \ge 4$, $x, y \ge 0$.* [5]

Soln: Let, A = Augmented matrix formed from the constraints and objective function.

$$= \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 2 & 4 \\ \hline 3 & 2 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 2 \\ \hline 4 & 4 & 0 \end{array}\right)$$

Now,

The corresponding dual problem of the given LPP is

Maximize $W^* = 4u + 4v$

Subject to: $2u + v \le 3$

$u + 2v \le 2$

$u, v \ge 0$

Let, x and y be the non-negative slack variables. Adding the slack variables. Then the above LPP can be written as $2u + v + x = 3$

$u + 2v + y = 2$

$4u + 4v = W^*$ such that

$2u + v + x + 0.y + 0.W^* = 3$

$u + 2v + 0.x + y + 0.W^* = 2$

$-4u - 4v + 0.x + 0.y + W^* = 0$

The initial simplex tableau is given below:

Basic variable	u	v	x	y	W*	R.H.S.
x	$\boxed{2}$	1	1	0	0	3
y	1	2	0	1	0	2
	–4	–4	0	0	1	0
	↑					

Since, the last row has two equal negative entries, we may choose any one of the 1st or 2nd column to determine the pivot element. Let, us choose 1st column as pivot column. Since, $\frac{3}{2} = 1.5$, $\frac{2}{1} = 2$ and $1.5 < 2$. The minimum ratio is 1.5. So, $\boxed{2}$ is the pivot element.

Applying, $R_1 \to \frac{1}{2}R_1$

Basic variable	u	v	x	y	W*	R.H.S.
u	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y	1	2	0	1	0	2
	–4	–4	0	0	1	0

Again, applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 4R_1$

Basic variable	u	v	x	y	W*	R.H.S.
u	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y	0	$\boxed{\frac{3}{2}}$	$\frac{-1}{2}$	1	0	$\frac{1}{2}$
	0	–2	2	0	1	6
		↑				

It is not the optimal solution because last row has still one negative entry. Second column is the pivot column. since $\frac{\frac{3}{2}}{\frac{1}{2}} = 3$, $\frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$ and $\frac{1}{3} < 3$. The minimum ratio is $\frac{1}{3}$. So, $\boxed{\frac{3}{2}}$ is the pivot element.

Now, applying $R_2 \to \frac{2}{3} R_2$

Basic variable	u	v	x	y	W*	R.H.S.
u	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
v	0	1	$\frac{-1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
	0	−2	2	0	1	6

Applying $R_1 \to R_1 - \frac{1}{2} R_2$ and $R_3 \to R_3 + 2R_2$

Basic variable	u	v	x	y	w*	R.H.S.
u	1	0	$\frac{2}{3}$	$\frac{-1}{3}$	0	$\frac{4}{3}$
v	0	1	$\frac{-1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
	0	0	$\frac{4}{3}$	$\frac{4}{3}$	1	$\frac{20}{3}$

Since, all entries in the last row are non-negative, so the solution in optimal.

$\therefore$ Max. $W^* = \frac{20}{3}$ at $u = \frac{4}{3}$ and $v = \frac{1}{3}$

i.e. Minimum $W = \frac{20}{3}$ at $x = \frac{4}{3}$ and $y = \frac{4}{3}$ **Ans.**

□□□

Or

8. Mechanics

Chapter 18

Statics

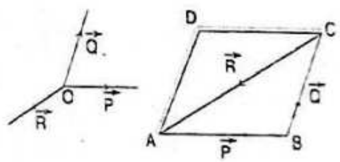
18.1 Statics

Basic Formulae and Key Points

1. **Equilibrium of Forces:** A number of forces acting upon a particle are said to be equilibrium if and only if their resultant is zero.
2. **Concurrent Forces:** The number of forces are said to be the concurrent forces if the lines of action of all the forces intersect at a point.
3. **Triangle of Forces:** If three forces acting at a point be represented in magnitude and direction by the three sides of a triangle taken in order then the forces are in equilibrium.

Proof:

Let, the forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ acting at the point O be represented in magnitude and direction by the sides AB, BC and CA of the triangle ABC. Let, us complete the parallelogram ABCD. Since, BC and AD are equal and parallel, so $\overrightarrow{BC} = \overrightarrow{AD} = \vec{Q}$.



Now, using parallelogram of forces we have

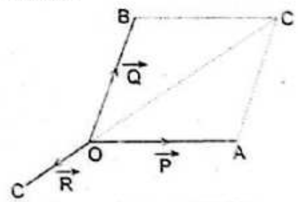
$$\vec{P} + \vec{Q} + \vec{R} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$$
$$= \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CA}$$
$$= \overrightarrow{AC} + \overrightarrow{CA} = -\overrightarrow{CA} + \overrightarrow{CA} = 0$$

Hence the given forces are in equilibrium.

4. **Converse of Triangle of Forces:** If three forces acting at a point are in equilibrium, then they can be represented in magnitude and direction by the sides of a triangle taken in order.

Proof:

Let, forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ acting at a point O be in equilibrium. Then



$$\vec{P} + \vec{Q} + \vec{R} = 0$$

or, $\vec{P} + \vec{Q} = -\vec{R}$

Taking two of the three forces, say $\overrightarrow{OA} = \vec{P}$, $\overrightarrow{OB} = \vec{Q}$. Let, us complete the parallelogram OACB.

Then $\overrightarrow{AC} = \overrightarrow{OB} = \vec{Q}$.