COMPACT

TOPOLOGICAL SPACES

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CERTIFICATE-I

This is to certify that the thesis entitles "TOPOLOGICAL SPACES" submitted by Aryabrata dev to the Orissa University of Agriculture and Technology, Bhubaneswar in partial fulfilment of the requirements for the award of degree is faithful record of bonafide and original research work carried out by ARYABRATA DEV under my guidance and supervision. No part of this thesishas been submitted for any other degree or diploma.

It is further certified that the assistance and help received by him from various sources during the course of investigation has been duly acknowledged.

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CERTIFICATE-II

This is to certify that the thesis entitles "TOPOLOGICAL SPACES" submitted by Aryabrata dev to the Orissa University of Agriculture and Technology, Bhubaneswar in partial fulfilment of the requirements for the award of degree of has been approved/disapproved by chairman,advisory committee and the external examiner.

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DECLARATION

I hereby declare that the major thesis entitled "TOPOLOGICAL SPACES" submitted to Department of Mathematics, CBSH, OUAT, Bhubaneswar for the partial fulfilment of the requirement for the degree of bachelor science (B.Sc) is an authentic record of the work carried out by me under the guidance of Mrs.MAMATA KULIA, prof in mathematics, college of basic science and humanities, OUAT, Bhubaneswar, Odisha, and that result or any part of the result have not be submitted anywhere for any other degree or diploma or any other qualification.

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Compact topological spaces

1.1 Introduction

The theory of topological spaces or, as it is called, point set or general topology, has become one of the elementary building blocks underlying divers branches of mathemat-ics. Its concepts and methods have enriched numerous other elds of mathematics and given considerable impulse to their further development. For these reasons, topology counts among those few basic structures, which alone give access to modern mathe-matical research. In this work, we have tried to accumulate several types of concepts of separation axioms on topological space, obtained from di erent texts. Numerous proofs, examples and solved problems are included in this work. Up to now, no work ever complete. We only try to discuss the clear concepts of compactness on topological spaces, its properties and applications, so that one can see these materials concerning compactness on topological spaces at one place.

OBJECTIVE:

The advantage in compact spaces is that one may study the whole space by studying a nite number of open subsets. We shall see this when we prove that a continuous function f : X ! Y from a compact metric space X to a metric space Y is uniformly continuous. In conclu-sion we shall examine some compact surfaces that may be formed by identifying edges of a rectangle.

Procedure:

In the proposed study, rst, we shall give some necessary de nitions and states some necessary theorems in order to present the paper in a self contained manner. Afterward, we shall prove the most important and popular theorem namely Heine-Borel Theorem. Then we shall study locally compact spaces of a metric space and in general of a topological space. Next, we shall discuss compact and nally compact spaces and some of their important properties. Finally, we shall study completely compact spaces, anti-compact spaces and completely dense subsets in a space and some of their important properties.

Conclusion:

In this study we show that compactness, limit point compactness and sequentially compactness are equivalent in metrizable spaces. We intro-duce it here as an interesting application of the Tychono theorem. We show that every compact space is locally compact but not conversely. We also show that the product of nitely many compact spaces is a lo-cally compact space. We show that Stone-Cech compacti cation in a space is unique. We also show that every regular nally compact space is normal. This research work would give some remarkable results which can be used to study the whole topological space by studying a nite number of open sets.

1.2 Compact Spaces and Their Properties

De nition 1.2.1. A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.

De nition 1.2.2. A space X is said to be compact if every open covering A of X contains a nite subcollection that also covers X.

Example 1.2.1. The real line R is not compact, for the covering of R by open intervals

A = f(n; n + 2) j n 2 Zg

contains no nite subcollection that covers R.

Example 1.2.2. The following subspace of R is compact:

$$X = \{0\} \cup \{1/n \mid n \in Z_+\}$$

Given an open covering A of X, there is an element U of A containing 0. The set U contains all but nitely many of the points $1/n$; choose, for each point of X not in U, an element of A containing it. The collection consisting of these elements of A, along with the element U, is a nite subcollection of A that covers X.

Example 1.2.3. Any space X containing only nitely many points is necessarily compact, because in this case every open covering of X is nite.

Example 1.2.4. The interval $(0; 1]$ is not compact; the open covering

$$A = \{(1/n; 1] \mid n \in Z_+\}$$

contains no nite subcollection covering $(0; 1]$, Nor is the interval $(0; 1)$ compact; the same argument applies. On the other hand, the interval $[0; 1]$ is compact.

In general, it takes some e ort to decide whether a given space is compact or not. First we shall prove some general theorems that show us how to construct new compact spaces out of existing ones. Then in the next section we shall show certain speci c spaces are compact. These spaces include all closed intervals in the real line, and all closed and bounded subsets of R^n.

Let us rst prove some facts about subspaces. If Y is a subspace of X, a collection A of subsets of X is said to cover Y if the union of its elements contains Y.

Theorem 1.2.1. Let Y be a subspace of X. Then Y is compact if and only if covering of Y by sets open in X contains a nite subcollection covering Y.

Proof. Suppose that Y is compact and A = fA g 2J is a covering of Y by sets open in X. Then the collection

fA \ Y j 2 Jg

is a covering of Y by sets open in Y; hence a nite subcollection

fA 1 \ Y; A 2 \ Y; :::; A n \ Yg

covers Y. Then fA 1 ; A 2 ; :::; A n g is a subcollection of A that covers Y.

Conversely, suppose that given condition holds; we wish to prove Y compact. Let A^0 = fA^0 g be a covering of Y by sets open in Y. For each , choose a set A open in X such that

A^0 =A [Y

The collection A = fA g is a covering of Y by sets open in X. By hypothesis, some nite subcollection fA 1 ; A 2 ; :::; A n g covers Y. Then fA^0 1 ; A^0 2 ; :::A^0 n g is a subcolletion of A^0 that covers Y. □

Theorem 1.2.2. Every closed subspace of a compact space is compact.

Proof. Let Y be a closed subspace of the compact space X. Given a covering A of Y by sets open in X, let us form an open covering B of X by adjoining to A the single open set X-Y, that is,

B = A [fX-Yg:

Some nite subcollection of B covers X. If this subcollection contains the set X-Y, discard X-Y; otherwise, leave the subcollection alone. The resulting collection is a nite subcollection of A that covers Y. □

De nition 1.2.3. A topological space X is called a Hausdor space if for each pair x_1; x_2 of disjoint points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 1.2.3. Every compact subspace of a Hausdor space is closed.

Proof. Let Y be a compact subspace of the Hausdor space X. We shall prove that X-Y is open, so that Y is closed. Let x_0 be a point of X-Y.

We show that there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively(using the Hausdor condition). The collection fV_y j y 2 Y g is a covering of Y by sets open in X; therefore, nitely many of them V_{y_1} ; V_{y_2} ; :::V_{y_n} cover Y. The open set

$$V = V_{y_1} [V_{y_2} [::: V_{y_n}$$

contains Y, and it is disjoint from the open set

$$U = U_{y_1} \backslash U_{y_2} \backslash ::: \backslash U_{y_n}$$

formed by taking the intersection of the corresponding neighborhood of x_0. For if z is a point of V , then z 2 V_{y_i} for some i, hence z 2= U_{y_i} and so z 2= U. See gure

Then U is a neighborhood of x_0 disjoint from y. □

Theorem 1.2.4. If Y is a compact subspace of the Hausdor space X and x_0 in not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Proof. Let Y be compact subspace of the Hausdor space X. Let x_0 2= Y, For each y Y we get a disjoint open neighborhood U and V_y containing x and y respectively. The collection fV_y j y 2 Y g is cover of Y, So nite many of them fV_{y_1} ; V_{y_2} ; :::V_{y_n} g cover Y. Let

$$V = \overset{n}{\underset{i=1}{[}} V_{y_i} :$$

Now U $\overset{T}{}$ V_i = ;; for all i then U \ V = ; Again Y $S^{n}_{i=1}$ V_{y_i} = V

Thus, for x_0 and Y we get a disjoint open neighborhood U and V . □

Theorem 1.2.5. The image of a compact space under a continues map is compact.

Proof. Let f : X ! Y be continues; Let X be compact. Let A be covering of the set f(X) by sets open in Y. The collection

$$ff\ ^{1}(A)\ j\ A\ 2\ Ag$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence nitely many of then , say

$$f\ ^{1}(A_1);\ f\ ^{1}(A_2);\ :::f\ ^{1}(A_n)$$

cover X. Then the set A_1; A_2; :::A_n cover f(X). □

Theorem 1.2.6. Let f : X ! Y be a bijective continuous function. If X is compact and Y is Hausdor then f is a homeomorphism.

Proof. We know image of closed set of X under continuous function f are closed in Y; this will prove continuity of the map f ^{1}. If A is closed in X, then A is compact, by theorem 1.2.2. Therefore, by the theorem 1.2.5. f(A) is compact. Since Y is Hausdor , f(A) is closed in Y, by theorem 1.2.3. □

Theorem 1.2.7. Let a < b. Then [a; b] is a compact space, where a; b 2 R

Proof. Suppose that A = fU_i g_{i2I} is an open cover of [a; b]. We prove that there exist a nite subcover. Let C = fc 2 [a; b] : [a; c] can be covered by a nite subcollection of Ag. We aim to show that b 2 C. Notice that C is bounded from above by b, as C [a; b]. Furthermore , as we can pick a single open set U_j containing a, We have [a; a +] U_j for some > 0. Hence a + 2 C for some . Thus C is a non-empty bounded subset of real numbers. Let x denote the least upper bound of C. Then x 2 [a; b] as ^{0}a^{0} is bounded by x and x is not greater than upper bond of C. Note that x 6= a as a + x for the previously constructed. We claim that x = b. Otherwise take an open set U_j which contains x. It must also contain an interval (x "; x + ") [a; b]. But the property of the supreme we can take an element c 2 C s.t x " < c < x. We can take a nite open sub-cover of [a; c], say with element U_{i_1} ; U_{i_2} ; :::; U_{i_n} . But then adding the set U_j we obtain a nite open sub-cover of [a; x+ $\frac{"}{2}$].

This implies that $x + \frac{\epsilon}{2} \in C$, contradicting $x = \sup C$. Thus we must have $x = b$. The same trick we used to show that $x = b$ is used to show that $b \in C$. Take an open set U_j containing b. It must contain same half-open interval $(b - \epsilon; b] \subset U_j$. Then the property of the supreme we can take $c \in C$ s.t $b - \epsilon < c < b$. But we can take a nite open sub-cover $U_{i1}; U_{i2}; \dots U_{in}$ of $[a; c]$. Then $U_j; U_{i_1}; U_{i_2}; \dots; U_{i_n}$ is a nite open subcover of $[a; b]$. Thus $b \in C$. So $[a; b]$ can be covered with a nite open sub-cover, and $[a; b]$ is compact. □

Remark 1.2.1. On the above theorem we prove that interval $[a; b]$ of R is compact. Then by theorem 1.2.2 any closed subset of $[a; b]$ be compact. On the other hand, it follows from theorem 1.2.3 that the interval $(a; b]$ and $(a; b)$ in R can't be compact because they are not closed in Hausdor space R.

1.3 product spaces and Cantors intersection Theo-rem

Product Spaces

Just as the product of connected spaces is connected, the product of compact spaces is compact.

Theorem

If X and Y are compact spaces, then $X \times Y$ is compact. In order to prove this, it helps to distinguish a valuable lemma.

Lemma 6. (The Tube Lemma)

Let X and Y be spaces, with Y compact. If $N \subset X \times Y$ is open, and a slice $\{x_0\} \times Y \subset N$ for some $x_0 \in X$, then there is a nbhd W of x_0 such that $W \times Y \subset N$

Proof. For each y, consider a basis element $U_y \times V_y$ of $X \times Y$ suchthat $x_0 \times y \in U_y \times V_y \subset N$. (We can do this as N contains all the points of the slice x_0 Y and is an open set). Note that y $V_y$andx$_0$ U_y for all y. Then the open sets fV_y g$_{y2Y}$ form an open cover of the fac-tor space Y , because if y Y , then fx$_0$g y 2 U_y V_y . As Y is compact, we can select a nite subcover V_{y_1} ; :::; V_{y_n} such that Y = [$^n{}_{i=1}V_{y_i}$

. Let W = \$^n{}_{i=1}U_{y_i}$, a nbhd of x_0. Then W is the desired nbhd of x_0. Suppose that (z; y) 2 W Y . Then there is an j such that y 2 V_{y_j} , as the collection fV_{y_i} g$^n{}_{i=1}$ covers Y . As z 2 W = \$^n{}_{i=1}U_{y_i}$, we have (z; y) 2 U_j V_j N.

Proof. (Theorem). Let fP_i g$_{i2I}$ be a collection of open subsets of X Y covering X Y . For each x 2 X, we can cover the space fxg Y with nitely many open sets, say $P_k{}^X$ for k = 1; :::; n_X(the x indicates that these open sets are intended to cover the slice fxg Y . By the tube lemma we can take an open set W ^{X}such that x 2 W ^{X} and W ^{X} Y Q^X = [$^n{}_{k=1^x}P_X{}^k$. Then the open sets fW ^{X}g cover the compact space X. Thus we can take a nite subcover W $^X{}_1$; ::::W $^X{}_m$. Claim: The collection of all $P_k{}^{X_j}$ for k = 1; :::; n_{X_j} and j = 1,...,m, form a subcover of

. For if x y $\sum$ n xj X Y , we can pick W $^X{}_j$ such that x 2 W $^X{}_j$. Then
w $^x{}_j$ Y j [$_{k=1}$, so that there exists k between 1 and n such that x xj

x y 2 P_k . Thus we have constructed the nite subcover of X Y , so that the latter set is compact.

Corollary. Any nite product of compact spaces is compact. Proof. If X_1; :::; X_n is a collection of spaces, then (X_1 ::: Xn1) Xn is home-omorphic to X_1 ::: Xn via the map ((x_1; :::; $x_{n\ 1}$; x_n) ! (x_1; :::; x_n) .

Corollary. Any product of closed intervals in R, say $Q^n{}_{i=1}[a_i; b_i]$,

is compact in the product topology.

1.4 Metric Spaces and Finite intersection property

Metric Spaces

As already mentioned, a metric space is just a set X equipped with a function $d : X X ! R$ which measures the distance $d(x, y)$ between points $x; y$ 2 X. For the theory to work, we need the function d to have properties similar to the distance functions we are familiar with. So what properties do we expect from a measure of distance?

First of all, the distance $d(x; y)$ should be a non-negative number, and it should only be equal to zero if $x = y$. Second, the distance $d(x; y)$ from x to y should equal the distance $d(y; x)$ from y to x. Note that this is not always a reasonable assumption if we , e.g., measure the distance from x to y by the time it takes to walk from x to y, $d(x; y)$ and $d(y; x)$ may be di erent but we shall restrict ourselves to situations where the condition is satis ed. The third condition we shall need, says the distance obtained by going directly from x to y, should always be less than or equal to the distance we get when we stop at a third point z along the way, i.e.

$$d(x; y) \quad d(x; z) + d(z; x):$$

It turns out that these conditions are the only ones we need, and we sum them up in a formal de nition.

De nition 1.4.1. A metric space (X,d) consist of a non-empty set X and a function $d : X X ! [0; 1)$ such that:

[Positivity]: For all x,y 2 X, $d(x,y) \quad 0$ with equality if and only if $x = y$.

[Symmetry]: For all x,y 2 X, $d(x; y) = d(x; y)$.

[Triangle inequality]: For all $x; y; z$ 2 X

$$d(x; y) \quad d(x; z) + d(y; z):$$

A function d satisfying above three condition, is called a metric on X.

comment: When it is clear or irrelevent which metric d we have in mind, we shall often refer to "the metric space X" rather than "the metric space (X,d)". Let us take some example of the metric space.

Example 1.4.1. If we let $d(x; y)=j\ x\ y\ j$,(R; d) is a metric space. The rst two conditions are obviously satis ed, and the third follows from the ordinary triangle inequality for real numbers:

$d(x,y)=|(x;y)|=|(x-z)+(z-y)|\le|x-z|+|z-y|=d(x;z)+d(z;y):$

Example 1.4.2. If we let $d(x;y)=|x-y|$, (R^n,d) is a metric space. The first two condition are obviously satisfied, and the third follows from the triangle inequality for vectors the same way as above :

$$d(x;y)=|(x;y)|=|(x-z)+(z-y)|\le|x-z|+|z-y|=d(x;z)+d(z;y):$$

Example 1.4.3. Assume that we want to move from one point $x=(x_1;x_2)$ in the plane to another $y=(y_1;Y_2)$, but that we are only allows to move horizontally and vertically. If we first move horizentally from $(x_1;x_2)$ to $(y_1;x_2)$ and then from $(y_1;x_2)$ to $(y_1;y_2)$, the total distance is

$$d(x;y)=|y_1-x_1|+|y_2-x_2|:$$

This gives us a metric on R^2 which is different from the usual metric in Example 2. It is ofte referred to as the Manhattan metric or the taxi cab metric

Also in this case the first two conditions of a metric space are obviously satisfied. To prove the triangle inequality, observe that for any third point $z=(z_1;z_2)$, we have

$$d(x;y)=|y_1-x_1|+|y_2-x_1|$$

$$\begin{aligned}
&=|(y_1-z_1)+(z_1-x_1)|+|(z_1-x_1)|+|(y_2-z_2)+(z_2-x_2)| \\
&\le|(y_1-z_1|+|z_1-x_1|+|y_2-z_2|+|z_2-x_2| \\
&=|z_1-x_1|+|z_2-x_2|+|y_1-z_1|+|y_2-z_2|= \\
&= d(x;z)+d(z;y);
\end{aligned}$$

where we used the ordinary triangle inequality for real numbers to get from the second to the third line.

Example 1.4.4. We shall now take a look at an example of different kind. Assume that we want to send messages in a language of N sym-bols (letters,numbers,punctuation marks, space,etc.) We assume that all messages have the same length K(if they too short or too long, we either ll them out or break them into pieces). We let X be the set of all messages, i.e. all sequences of symbols from the language of length K. If $x = (x_1; x_2; :::; x_k)$ and $y = (y_1; y_2; :::; y_k)$ are two message, we de ne, $d(x; y)$ = the number of indices n such that x_n not equal to y_n. It is not hard to check that d is a metric. It is usually referred to as the Hamming-metric, and is much used in coding theory where it serves as a measure of how much a message gets distorted during transmission.

Example 1.4.5. The matrices in this example may seem rather strange. Although they are not very useful in applications, they are handy to know about as they are totally di erent from the metrics we are used to form R^n and may help sharpen our intuition of how a metric can be. Let X be any non-empty set, and de ne:

$$d(x; y) = \begin{cases} 0 & \text{If} \quad x = y \\ 1 & \text{otherwise;} \end{cases}$$

It is not hard to check that d is a metric on X, usually referred to as the discrete metric.

Example 1.4.6. There are many ways to make new metric spaces from old. The simplest is the subspace metric: If (X,d) is a metric space and A is a non-empty subset of X, we can make a metric d_A on A bu putting $d_A(x; y) = d(x; y)$ for all $x; y$ 2 A we simply restrict the metric on A. It is trivial to check that d_A is a metric on A.

A collection A of subsets of a set X is said to have the nite intersection properties(FIP) if the intersection over any nite subcollection of A is non-empty. It has the strong nite intersection property(SFIP) if the intersection over any nite subcollection of A is in nite.
A centered system of sets is the collection of sets with the nite intersection property.

De nition 1.4.2. A collection C of subsets of X is said to have the nite intersetion property if for every nite subcollection

$$fC_1; C_2; :::; C_ng$$

of C, the intersection $C_1 \setminus C_2 \setminus ::: \setminus C_n$ is nonempty.

Theorem 1.4.1. Let X be a topological space. Then X is compact if and only if for every collection C of closed sets in X having the nite intersection property, the intersection $_{c2C}$ C of all the elements of C is nonempty.

Proof. Given a collection A of subsets of X, Let

$$C = fX\text{-}A \ j \ A \ 2 \ Ag$$

be the collection of their complements. Then the following statements holds.

1. A is a collection of open sets if and only if C is a collection of closed sets.
2. The collection A covers X if and only if the intersection $_{c2C}$ C of

   all the elements of C is empty. T
3. The rst sub-collection fA_1; A_2; :::A_ng of A covers X if and only if the intersection of the corresponding elements C_i = X-A_i of C is empty.

The rst statement is trivial, while the second and third follows from De-Morgan's law:

$$X \ [_{2J} (A) = \ \backslash_{2J} (X \quad A):$$

The proof of the theorem now proceeds in two easy steps: taking the contraceptive (of the theorem), and then the complement (of the sets)!

The statement that X is compact is equivalent to saying:" Given any collection A of open set of X, if A covers X, then some nite subcollection of A covers X". This statement is equivalent to its contraceptive, which is following: "Given any collection A of open sets, if no nite subcollection of A covers X, then A does not cover X.". Letting C be, as earlier, the collection fX A j A 2 Ag and applying (1),(2)&(3), we see that this statement is in turn equivalent to the following:"Given any collection C of closed sets, if every nite intersection of elements of C is non-empty, then the intersection of all the elements of C is non-empty". This is just the condition of our theorem. □

A special case of the theorem occurs when we have a nested sequence $C_1 \supset C_2 \supset \dots \supset C_n \supset C_{n+1} \supset \dots$ of closed sets in a compact space X. If each of the C_n is non-empty, then the collection $\mathcal{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ automatically has the nite intersection property. Then the intersection

$$\bigcap_{n \in \mathbb{Z}_+} C_n$$

is non-empty.

1.5 Generalized Tube lemma and Examples

Theorem 1.5.1. Consider the product space $X \times Y$, Where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is neighborhood of x_0 in X.

1.5.1 Examples and properties

1. This theorem is not any more hold if non- of the set X and Y are compact. For example, Let Y be the Y-axis in $\mathbb{R}^2$, and let

$$N = \left\{ x \times y \mid |x| < \frac{1}{y^2 + 1} \right\}$$

Then N is an open set containing the set $0 \times \mathbb{R}$, But it contain no tube about $0 \times \mathbb{R}$. It is illustrated in this gure,

2. The tube lemma can be used to prove that if X and Y are compact topological spaces, the $X \times Y$ is compact as follows:

   Let $\{G_a\}$ be an open cover of $X \times Y$, for each x belonging to X, cover the slice $\{x\} \times Y$ by nitely many elements of $\{G_a\}$ (this is possible science $\{x\} \times Y$ is compact being homomorphic to Y). Call the union of these nitely many elements N_x. By the tube lemma, there is an open set of the form $W_x \times Y$ containing $\{x_0\} \times Y$ and containing N_x. The collection of all W_x for 'x' belonging to X is an open cover of X and hence has a nite sub-cover $W_{x1}; W_{x2}; \dots W_{xn}$.

Then for each x_j, W_{x_j} Y is contained in N_{x_j} . Using the fact that each N_{x_j} is the nite union of elements of G_a and that the nite collection (W_{x_1} Y) [(W_{x_2} Y) [::: [(W_{x_n} Y) covers X Y, the collection N_{x_1} [N_{x_2} [:::N_{x_n} is a nite sub-cover of X Y.

1.5.2 Generalized tube lemma

Let X and Y be topological spaces and consider the product space X Y. Let A be a compact subspace of X and B be a compact subset of Y. If N is an open set containing A B, then there exist U open in X and V open in Y s.t A B U V N.

The Tube Lemma is a useful tool in working with Cartesian products of nitely many compact spaces. A general discussion is followed by three applications of the lemma.

Tubes are one type of open subsets of the Cartesian product X Y . The Tube Lemma is applicable when one of the factors is compact. Let Y be the factor that is compact. A good way of thinking about the lemma is that when you consider the slices fxg Y as points, the tubes G Y , where x 2 G , behave like a base. The following is a statement of the lemma.

The Tube Lemma can be used in proving that the product of two com-pact spaces is compact. By induction, it follows that the product of nitely many compact spaces is compact. However, the lemma cannot be used in proving the compactness of product space with in nitely many compact factors (the Tychono Theorem).