IEORE4407: Game Theory Models of Operation

Lecture Notes

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October 6, 2025

Static Games with Complete Information

10 September 2025

- n players.
- Player i has action set A_i.
- Payoff: $u_i : A_1 \times A_2 \times \ldots \times A_n \to \mathbb{R}$
- $u_i(a_1, a_2, \ldots, a_i, \ldots, a_n)$ is the payoff to player i for choosing a_i.
- Actions are common knowledge.
- One-shot game.
- Each player picks an action "simultaneously" (without knowing others' choices).

	L	R
U	$9, 9$	$0, 10$
D	$10, 0$	$4, 1$

- Row player can choose U or D.
- Column player can choose L or R.

Dominant Strategies

The "best" choice never depends on what the other person does.

- Row player should pick D anyways.

- Column player should pick R anyways.
- We assume players are rational.
- Expected outcome: $(9, 4)$
- This is the "Prisoner's Dilemma."
- If they interact: Row $\to U$, Column $\to L$.

For strategies:
$(s_1, s_2, \ldots, s_i, \ldots, s_n)$, s_i is the strategy for player i.
s_i^* is a dominant strategy for player i if:

$$u_i(s_1, s_2, \ldots, s_i^*, \ldots, s_n) > u_i(s_1, s_2, \ldots, s_i, \ldots, s_n)$$

for all $s_1 \in A_1, s_2 \in A_2, \ldots$.

Dominant Strategy Equilibrium

If a game is such that each player i has a dominant strategy s_i^*, then the game has a dominant strategy equilibrium $(s_1^*, s_2^*, \ldots, s_n^*)$.

Dominance Solvable

$\tilde{s}$ is a "dominated" strategy for player i if there is another strategy $\tilde{s_i}$ such that:

$$u_i(s_1, s_2, \ldots, \tilde{s_i}, \ldots, s_n) < u_i(s_1, s_2, \ldots, s_i, \ldots, s_n)$$

Static games

15 September 2025
Read: Game theory in supply chain analysis (survey paper). (Cachon & Netessine)

Definitions

Game

- A game is defined with n players ($n \geq 2$).
- Set of actions available to i: $A_1, A_2, \ldots, A_n$.
- Strategy space of i: $S_1, S_2, \ldots, S_n$.
- Payoff function: $u_1, u_2, \ldots, u_n$.

$$u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

$$u_i(s_1, s_2, \ldots, s_i, \ldots, s_n) = \text{Payoff to } i \text{ when players choose } s_1, s_2, \ldots$$

- Depends on $(s_1, s_2, \ldots, s_n)$, not just on s_i.
- Could be different for different players.

Dominant Strategy

s_i is a dominant strategy if:

$$u_i(t_i, \ldots, \underline{s_i}, \ldots, t_n) > u_i(t_i, \ldots, \underline{s_i'}, \ldots, t_n)$$

$$\forall t_1 \in S_1, t_2 \in S_2, \ldots, t_n \in S_n \quad \text{where} \quad s_i \neq s_i'$$

Strictly Dominated Strategy

Let $t_i \in S_i, s_i \in S_i$.

t_i is strictly dominated by s_i if:

$$u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

- A strictly dominated strategy may not exist.
- A rational player will never play a strictly dominated strategy.

	C	M	H
L	6/6	2/8	0/5
M	8/2	4/1	1/3
H	4/10	3/1	2/2

For row players, L is dominated.
For column players, C is dominated.

$\therefore$ Game is effectively

	M	H
M	4/1	1/3
H	3/1	2/1

Rationalizability and Beliefs

Belief of player $i \equiv$ some random $B_{-i} \subseteq S_{-i}$.

S_i : Player's best response to B_{-i}.

$$(u_i(s_i, B_{-i}) \geq u_i(s_i', B_{-i}) \quad \forall s_i' \in S_i)$$

$(s_1, s_2, \ldots, s_n)$ is rationalizable if s_i is a best response to some B_{-i} for each player i.

- Belief of i about j can be different from belief of k about j.
- Rationalizability allows for that.

Examples

Example 1:

	D	C
D	$-1/-1$	$9/0$
C	$0/9$	$-1/-2$

- C is dominant strategy for each player.
- D is dominated.
- (C, C) can be rationalized as N.E.

Example 2:

	O	F
O	$2/1$	$0/0$
F	$0/0$	$1/2$

- No dominant/dominated strategy.
- N.E. $= (0, 0)$ (pure strategies)
- Rationalizable strategies: all rationalizable (O, F), (F, O), etc.

Example 3:

	L	R
U	$8/8$	$5/9$
D	$9/5$	$6/6$

- D is dominant.
- U is dominated.
- R is dominant.
- L is dominated.

Example 4:

	L	M
V	1/0	1/2
H	0/3	2/0

- R is dominated.
- Only N.E. rationalizable.
- V is dominating, M is dominating.

More Games

17 September 2025

1 Cournot Competition

17 September 2025

1.1 Setup

- 2 firms: i and j (quantity competition)
- Marginal production cost $= c$ (cost to produce 1 item)
- Firms choose quantities q_i and q_j simultaneously
- Unit price $= a - (q_i + q_j)$

Question: What are equilibrium quantities chosen by the firms?
(In game theory context)

- Number of players = 2 (i and j)
- Strategy space:

$$\text{Firm } i : [0, \infty) \tag{1}$$
$$\text{Firm } j : [0, \infty) \tag{2}$$

Payoff for firm i: (Revenue - cost = Profit)

$$U_i(q_i, q_j) = q_i \cdot (\text{unit price} - \text{marginal cost}) \tag{3}$$
$$= q_i(a - q_i - q_j - c) \tag{4}$$
$$\therefore U_i(q_i, q_j) = q_i(a - q_i - q_j - c) \tag{5}$$

1.2 Finding Nash Equilibrium

(q_i^*, q_j^*) are Nash Equilibrium if:

$$U_i(q_i^*, q_j^*) \geq U_i(q_i, q_j^*) \quad \forall q_i^* \neq q_i \tag{6}$$
$$U_i(q_i^*, q_j^*) \geq U_i(q_i^*, q_j) \quad \forall q_j^* = q_j \tag{7}$$

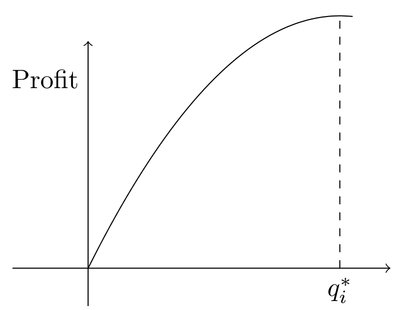
Suppose firm j produces $\hat{q}_j$. Then q_i^* is the optimal reaction of firm i:

$$\max_{q_i} \left(q_i(a - q_i - \hat{q}_j - c)\right) \tag{8}$$

where $\hat{q}_j$ is a known fixed number (essentially a constant).

$$\pi(q_i) = q_i(a - q_i - \hat{q}_j - c) \tag{9}$$
$$\pi'(q_i) = (a - q_i - \hat{q}_j - c) + q_i(-1) = 0 \tag{10}$$
$$a - q_i - \hat{q}_j - c - q_i = 0 \tag{11}$$
$$\therefore q_i^* = \frac{a - \hat{q}_j - c}{2} \tag{12}$$

Since $\pi''(q_i) = -2 < 0 \Rightarrow q_i^*$ is a max.
If you plot a profit function:



Similarly, $q_j^* = \frac{a - q_i^* - c}{2}$
For both firms to get equilibrium:

$$q_i^* = \frac{a - q_j^* - c}{2} \tag{13}$$
$$q_j^* = \frac{a - q_i^* - c}{2} \tag{14}$$

Solving:

$$\therefore q_i^* = q_j^* = \frac{a - c}{3} \tag{15}$$

If both firms were owned by the same entity:

$$\max_{q_i, q_j} \{q_i(a - q_i - q_j - c) + q_j(a - q_i - q_j - c)\} \tag{16}$$

$$= \max_{q_i, q_j} \{(q_i + q_j)(a - c - (q_i + q_j))\} \tag{17}$$

Consider $q_i + q_j = q$:

$$\max_{q} \{q(a - c - q)\} \tag{18}$$

In that case:

$$q_i^{opt} + q_j^{opt} = \frac{a - c}{2} \tag{19}$$

$$\therefore q_i^{opt} = q_j^{opt} = \frac{1}{4}(a - c) \tag{20}$$

(Like prisoner's dilemma)

Even though $\frac{1}{4}(a - c)$ is better, if q_i plays that, j will play $q_j^* = \left(\frac{a-c}{2}\right) - \frac{1}{2}\left(\frac{a-c}{4}\right)$

$\therefore$ they will both end up playing sub-optimally, i.e., $\frac{(a-c)}{3}$

1.3 Why will q_j produce more?

q_j produces more, even though price will go down; the volume still gives q_j more profit than q_i.

So when we say:

$$q_i^* = \frac{a - q_j^* - c}{2} \tag{21}$$

So whatever i thinks j will do:

$$q_i^* = \left(\frac{a - c}{2}\right) - \frac{1}{2}(q_j^*) \tag{22}$$

Since q_j cannot be negative ($q_j \geq 0$):

$$\Rightarrow q_i^* \leq \left(\frac{a - c}{2}\right) \tag{23}$$

$\therefore$ i should never produce more than $\left(\frac{a-c}{2}\right)$

Similarly, $q_j^* \leq \left(\frac{a-c}{2}\right)$

We know $q_i^{opt} = q_j^{opt} = \frac{1}{4}(a - c)$

$\therefore$ min quantity i & j can produce: $\frac{1}{4}(a - c)$

(See full working in Tirole's textbook)

2 Bertrand Duopoly

("Price competition")

2.1 Setup

- 2 firms: i and j
- Prices: P_i and P_j where $a > 0$, $b > 0$, and c is the same

Demand:

$$\text{Firm } i = a - P_i + bP_j \tag{24}$$
$$\text{Firm } j = a - P_j + bP_i \tag{25}$$

Market Size: "a"
Production cost $= c$
Equilibrium prices: (P_i^*, P_j^*)?
Payoff for firm i: (price - cost)

$$\pi_i(P_i) = (a - P_i + bP_j)(P_i - c) \tag{26}$$

what i thinks choice j makes

$$\therefore P_i^* = \arg\max_{P_i \geq c} [(a - P_i + bP_j)(P_i - c)] \tag{27}$$

$$P_i^* = \frac{1}{2}(a + bP_j^* + c) \tag{28}$$
$$P_j^* = \frac{1}{2}(a + bP_i^* + c) \tag{29}$$

2.2 Solving the 2 equations

$$P_i^* = P_j^* = \frac{a+c}{2-b} \quad (b < 2) \tag{30}$$

This is the equilibrium strategy.

3 Commons Problem

Public good: K agents $(k_1, k_2, \ldots, k_n)$ units
Agent i can claim any amount k_i
Utility for $i = \ln(k_i) + \ln\left(k - \frac{\sum k_j}{3}\right)$
log is concave $\Rightarrow$ these are good models of utility function (e.g., utility of money)
Let's try for 3 agents: (k_1^*, k_2^*, k_3^*)
Suppose agent j uses k_2^*, agent 3 uses k_3^*

Then agent 1's optimal solution:

$$\max_{k_1} \left[\ln(k_1) + \ln\left(k - k_1 - k_2^* - k_3^*\right)\right] \tag{31}$$

Taking derivatives:

$$\frac{1}{k_1} - \frac{1}{k - k_1 - k_2^* - k_3^*} = 0 \tag{32}$$

$$\therefore k - k_1 - k_2^* - k_3^* = k_1 \tag{33}$$

$$\Rightarrow \therefore k_1 = \frac{k - k_2^* - k_3^*}{2} \tag{34}$$

Assume all k_i are identical:

$$\therefore k^* = \frac{k - 2k^*}{2} \tag{35}$$

$$\Rightarrow \therefore k^* = \frac{k}{4} \tag{36}$$

In general: $k_i^* = \frac{k}{(n+1)}$ if we maximize utility for each agent
Optimal consumption for 3 agents (as a whole) :

$\therefore k_i^* = \frac{k}{2n}$ maximizes overall utility for everybody

3.1 Notes

- Each agent's optimal problem means they give less weighting to overall consumption, because they only care about their own utility.
- If society wants to optimize overall welfare, there needs to be more weight on overall consumption and less on purely private benefits.
- agents consume more than they should compared to the optimal social solution. This is because of externalities — if part of the consumption cost is incurred by society as a whole, an individual tends to ignore it. As a result, we over-utilize the commons, since everyone is acting in their own self-interest without accounting for the shared damage.
- To regulate that, we can introduce tolls, penalties, or corrective taxes (Pigouvian taxes). These policies work by making private agents "internalize" the externality, i.e. aligning private incentives with the social optimum.

4 Final Offer Arbitration

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4.1 Setup

- Union $\sim$ arbitrator
- Firms
- Firm arguing over wages; wise arbitrator

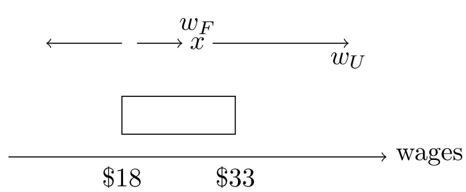
Game:

- Firm proposes a wage offer w_F simultaneously
- Union chooses a wage after w_U

Arbitrator has an ideal settlement:

- Required to choose w_F or w_U

Arbitrator commits to choosing after doesn't in x.
From the union, believe x is a randomly distributed with distribution $F(x)$.
Example: $F(x)$



Then w_F is chosen (x is closer to w_F)

4.2 Equilibrium Analysis

What do the firm & union do in equilibrium?
Analysis: Suppose union picks w_U, firm picks w_F

$$P(w_U \text{ is chosen}) = P\left(x > \frac{w_U + w_F}{2}\right) = 1 - F\left(\frac{w_U + w_F}{2}\right) \tag{37}$$

$$P(w_F \text{ is chosen}) = P\left(x < \frac{w_U + w_F}{2}\right) = F\left(\frac{w_U + w_F}{2}\right) \tag{38}$$

$\therefore$ Expected wage agreement is: $G(w_U, w_F)$

$$= w_F \cdot F\left(\frac{w_U + w_F}{2}\right) + w_U \cdot \left(1 - F\left(\frac{w_U + w_F}{2}\right)\right) \tag{39}$$

If (w_U^*, w_F^*) is NE (Nash equilibrium), then:
w_F^* should minimize $G(w_U^*, w_F)$ over all w_F
w_U^* should maximize $G(w_U, w_F^*)$ over all w_U
$\therefore w_F$ should solve:

$$\min_{w_F} \left\{ w_F \cdot F\left(\frac{w_F + w_U^*}{2}\right) + w_U \cdot \left(1 - F\left(\frac{w_F + w_U^*}{2}\right)\right)\right\} \tag{40}$$

Similarly:

$$\max_{w_U} \left\{ w_F^* \cdot F\left(\frac{w_F^* + w_U}{2}\right) + w_U \cdot \left(1 - F\left(\frac{w_F^* + w_U}{2}\right)\right)\right\} \tag{41}$$

First order conditions:
At $w_F = w_F^*$:

$$0 = \frac{1}{2} \cdot w_F \cdot F'\left(\frac{w_F + w_U^*}{2}\right) + F\left(\frac{w_F + w_U^*}{2}\right) - w_U \cdot \frac{1}{2} \cdot F'\left(\frac{w_F + w_U^*}{2}\right) \tag{42}$$

At $w_U = w_U^*$:

$$0 = w_F \cdot F'\left(\frac{w_F + w_U^*}{2}\right) \cdot \frac{1}{2} - 1 + \left(1 - F\left(\frac{w_F + w_U^*}{2}\right) - w_U \cdot \frac{f(w_F^* + w_U)}{2}\right) \tag{43}$$

Equating both:

$$\frac{1}{2}(w_U^* - w_F^*) \cdot F'\left(\frac{w_U^* + w_F^*}{2}\right) = F\left(\frac{w_F^* + w_U^*}{2}\right) \tag{44}$$

$$\Rightarrow F\left(\frac{w_F^* + w_U^*}{2}\right) = \frac{1}{2} \tag{45}$$

At equilibrium, the avg of their choices would be at the median of the observations.

$$P(w_F) = P(w_U) = \frac{1}{2} \tag{46}$$

$$\therefore w_U^* - w_F^* = \frac{1}{F'\left(\frac{w_F^* + w_U^*}{2}\right)} \tag{47}$$

(density fn.)
Example: $F(x)$ is normally distributed $\sim N(m, \sigma^2)$

$$\therefore \frac{w_U^* - w_F^*}{2} = \frac{1}{f(m)} = \sqrt{2\pi}\sigma \tag{48}$$

$$\frac{w_U^* + w_F^*}{2} = m \quad \text{(for normal distribution, mean = median = } m\text{)} \tag{49}$$

Thus:

$$w_F^* = m - \left(\sqrt{\frac{\pi}{2}}\right)\sigma \tag{50}$$

$$w_U^* = m + \left(\sqrt{\frac{\pi}{2}}\right)\sigma \tag{51}$$

$\Rightarrow$ "gap" increased with uncertainty.

$\Rightarrow$ If they don't know what the arbitrator picks, firm & union tend to choose close to end points

5 Bertrand Competition (Variation)

2 firms: i & j, unit production cost $= c$

Prices: P_i, P_j

$$\pi(P_i, P_j) = \begin{cases} (a - P_i) \cdot (P_i - c) & P_i < P_j \\ 0 & (a - P_i) \cdot (P_i - c) \\ P_i > P_j & \\ \frac{(a-P_i)}{2} \cdot (P_i - c) & P_i = P_j \end{cases} \tag{52}$$

Note: $P_i > c$ can never be supported as a NE

$\Rightarrow P_i^* = P_j^* = c$ is the unique NE

5.1 Discrete Model

Suppose Ps are required to be in the set $\{c, c + \varepsilon, c + 2\varepsilon, c + 3\varepsilon, \ldots\}$

"ε" = increment

What are the Nash equilibria in this model?

(c, c) is a NE

$(c + \varepsilon, c + \varepsilon)$ is also a NE

(Though one firm can undercut to c, they reduce their profit to 0.)

6 Mixed Strategies (Nash Equilibria)

Exercise: Find a mixed strategy equilibrium

	U	D
Alice	0,0	0,-1
Bob	0,-10	-90,-6

Suppose Alice plays U with prob. (p), D with prob. $(1-p)$
Bob plays L with prob. q, R with prob. $(1-q)$
What is Bob's best response?
If Bob picks L, expected payoff:

$$p(0) + (1-p)(-10) = 10p - 10 \tag{53}$$

Bob picks R, expected payoff:

$$p(-1) + (1-p)(-6) = 5p - 6 \tag{54}$$

$\Rightarrow$ Bob's best response

$$10p - 10 > 5p - 6 \tag{55}$$

$$5p > 4 \tag{56}$$

$$p > \frac{4}{5} \tag{57}$$

Best response $= \begin{cases} L & \text{if } p > 4/5 \\ R & \text{if } p < 4/5 \\ \{L, R\} & \text{if } p = 4/5 \end{cases}$
Suppose Bob plays L with prob. q, R with prob. $(1-q)$
Alice's best response:
If Alice plays U expected payoff:

$$q(0) + (1-q)(0) = 0 \tag{58}$$

If Alice plays D expected payoff:

$$q(10) + (1-q)(-90) = 100q + 90 \tag{59}$$

Alice's best response:

$$\begin{cases} U & \text{if } q < 9/10 \\ D & \text{if } q > 9/10 \\ \{U, D\} & \text{if } q = 9/10 \end{cases} \tag{60}$$

	O	F
O	2,1	0,0
F	0,0	1,2

$(0,0)$ (F, F) are pure strategy NE
Suppose Alice plays O with prob. (p), F with prob. $(1-p)$
Bob's best response:
If bob plays $O = p(1) + (1-p)(0) = p$

$$F = p(0) + (1-p)2 = 2 - 2p \tag{61}$$

$$p > 2/3 \tag{62}$$
$$p < 2/3 \tag{63}$$

$$\text{Bob's best response} = \begin{cases} O & \text{if } p > 2/3 \\ F & \text{if } p < 2/3 \\ \{O, F\} & \text{if } p = 2/3 \end{cases}$$

Suppose Bob plays O with prob. q, F with prob. $(1-q)$
If Alice picks $O = q(2) + (1-q)(0) = 2q$

$$F = q(0) + (1-q)1 = 1 - q \tag{64}$$

$$2q > 1 - q \tag{65}$$
$$q > \frac{1}{3} \tag{66}$$

$$\text{Alice's best response} = \begin{cases} O & \text{if } q > 1/3 \\ F & \text{if } q < 1/3 \\ \{O, F\} & \text{if } q = 1/3 \end{cases}$$

$\therefore p = \frac{2}{3}, q = \frac{1}{3}$ is N.E $\leftarrow$ mixed strategy Nash equilibria
$p = 1, q = 1$ is N.E
$p = 0, q = 0$ is N.E

7 Two-Person Zero Sum Games

$$a_{ij} = \begin{cases} \text{Payoff to Alice (from Bob) if} \\ \text{Alice plays strategy } i \text{ \& Bob} = j \end{cases} \tag{67}$$

		Bob		
		$y_1, y_2 \dots$	$\dots$	y_n
4*Alice	x_1			
	x_2		a_{ij}	
	$\vdots$			
	x_m			

$(a_{ij} - a_{ij})$ zero sum
Alice's $x_i^* = (x_1^*, x_2^*, \ldots, x_m^*)$ are NE if & only if:
Bob's $y_j^* = (y_1^*, y_2^*, \ldots, y_n^*)$ are NE if & only if

$$U(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_j^* \tag{68}$$

$\therefore x^*$ & y^* are NE if & only if:
[box=$\boxed{\,}$]align U(x, y*) $\leq U(x^*, y^*) \leq U(x^*, y)$

7.1 Alternative Approach

Fix Bob: $(y_1^*, y_2^*, \ldots, y_n^*)$
Alice's expected payoff for playing strategy i (Alice plays i^{th} row):

$$= a_i y_1^* + a_2 y_2^* + \cdots + a_{in} y_n^* \tag{69}$$

Alice's best response $= \max_{1 \leq i \leq m} \left\{ \sum_{j=1}^{n} a_{ij} y_j^* \right\}$

7.2 Extended

Let $V^* = \max \left\{ \sum_{j=1}^{n} a_{ij} y_j^* \right\}$
If for some k, $\sum_{j=1}^{n} a_{kj} y_j^* < V^*$ then $x_k^* = 0$
So: $\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_j^* \right) x_i^* = \sum_{i=1}^{m} V^* x_i^* = V^*$
Also, $V^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_j^* \right) x_i^* \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_j^* \right) x_i^*$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} x_i^* \right) y_j^* \tag{70}$$

Bob's problem: (Bob plays j^{th} column)
For any $(x_1, \ldots, x_m)$ that Alice chooses
Bob's payoff $= \sum_{i=1}^{m} a_{ij} x_i^*$
$\therefore$ Bob's response $w = \min_{1 \leq j \leq n} \{ \sum_{i=1}^{m} a_{ij} x_i^* \}$
(Remember - zero sum game so if Alice = 9, Bob = -9)

7.3 Completing the Proof

$$\therefore W^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} x_i^* \right) y_j^* \tag{71}$$

$$\geq \sum_{j=1}^{m} \left(\sum_{i=1}^{m} a_{ij} x_i^* \right) y_j^* \tag{72}$$

$\therefore (x^*, y^*)$ is a N-E if $\Rightarrow$

[box=$\boxed{\;}$]align $\mathrm{U}_1(x, y^*) \leq U_1(x^*, y^*)$
$\forall (x_1, x_2, \ldots, x_m)$
s.t. $\sum x_i = 1$
$x_i \geq 0$

Bob:

$$U_2(x^*, y) \leq U_2(x^*, y^*) \tag{73}$$
$$\forall (y_1, \ldots, y_n) \tag{74}$$
$$\text{s.t. } \sum y_j^* = 1 \tag{75}$$
$$y_j \geq 0 \tag{76}$$

$$-U_1(x^*, y) \leq -U_1(x^*, y^*) \tag{77}$$
$$\Rightarrow U_1(x^*, y) \geq U_1(x^*, y^*) \tag{78}$$

[box=$\boxed{\;}$]align $\therefore U_1(x, y^*) \leq U_1(x^*, y^*) \leq U_1(x^*, y)$

7.4 Bob's Perspective

$$U_1(x^*, y^*) \leq U_1(x^*, y) \tag{79}$$
$$(\Rightarrow) \quad \leq \min_{y \in S_2} \{U_1(x^*, y^*)\} \tag{80}$$
$$\leq \max_{x \in S_1} \left\{ \min_{y \in S_2} \{U_1(x, y)\} \right\} \tag{81}$$

7.5 Alice's Perspective

$$U_1(x^*, y^*) \geq U_1(x, y^*) \tag{82}$$
$$(\Rightarrow) \quad \geq \max_{x \in S_1} \{U_1(x, y^*)\} \tag{83}$$
$$\geq \min_{y \in S_2} \left\{ \max_{x \in S_1} \{U_1(x, y)\} \right\} \tag{84}$$

$\therefore$ N-E for 2 person zero sum games:
$U_1(x^*, y^*) = \max_{x \in S_1} \min_{y \in S_2} \{U_1(x, y)\}$

7.6 Example with Payoff Matrix

		Bob		
		q_1	q_2	q_3
3*Alice	P_1	5	1	1
	P_2	3	0	8
	P_3	4	4	0

Alice's payoff:

$$U : 5q_1 + q_2 + q_3 \tag{85}$$
$$M : 3q_1 + 0 + 3q_3 \tag{86}$$
$$D : 4q_1 + 4q_2 + 0 \tag{87}$$

Min W:

$$W \geq 5q_1 + q_2 + q_3 \tag{88}$$
$$W \geq 3q_1 + 3q_3 \tag{89}$$
$$W \geq 4q_1 + 4q_2 \tag{90}$$

$$q_1 + q_2 + q_3 = 1 \tag{91}$$
$$q_1, q_2, q_3 \geq 0 \tag{92}$$

Bob's payoff/loss:

$$L : 5p_1 + 3p_2 + 4p_3 \tag{93}$$
$$C : p_1 + 4p_3 \tag{94}$$
$$R : p_1 + 3p_2 \tag{95}$$

Bob wants to min:

$$\min_{q_1,q_2,q_3} \{\max \{5q_1 + q_2 + q_3, 3q_1 + 3q_3, 4q_1 + 4q_2\}\} \tag{96}$$
$$q_i \geq 0 \tag{97}$$
$$\sum q_i = 1 \tag{98}$$

This is equivalent to: W
Bob's payoff/loss (zero sum):

$$L : 5p_1 + 3p_2 + 4p_3 \tag{99}$$
$$C : p_1 + 4p_3 \tag{100}$$
$$R : p_1 + 3p_2 \tag{101}$$

Bob's optimal loss:

$$\max_{p_1,p_2,p_3} \{\min \{5p_1 + 3p_2 + 4p_3, p_1 + 4p_3, p_1 + 3p_2\}\} \tag{102}$$
$$\sum p_i = 1 \tag{103}$$
$$p_i \geq 0 \tag{104}$$

$\therefore$ LP problem in this case:

Max V

$$V \leq 5p_1 + 3p_2 + 4p_3 \tag{105}$$
$$V \leq p_1 + 4p_3 \tag{106}$$
$$V \leq p_1 + 3p_2 \tag{107}$$

8 Sequential Games / Extensive Form Games

6 October 2025

8.1 Games that unfold over time

8.1.1 Ingredients:

1. Set of players N
2. Payoff function for each players - payoff depends on outcomes & outcomes depend on actions of all players
3. Sequence in which players move
4. Available actions when it is a players turn to move
5. Knowledge of a player when it's their turn to move
6. Moves by Nature (= Prob. over exogenous events)

GAME TREES (analogous to decision trees for single player problems)