IEORE4407: Game Theory Models of Operation

Lecture Notes

Arya Gadage
Columbia University
Instructor: Prof. Jay Sethuraman

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Static Games with Complete Information

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- n players.
- Player i has action set A_i.
- Payoff: $u_i : A_1 \times A_2 \times \ldots \times A_n \to \mathbb{R}$
- $u_i(a_1, a_2, \ldots, a_i, \ldots, a_n)$ is the payoff to player i for choosing a_i.
- Actions are common knowledge.
- One-shot game.
- Each player picks an action "simultaneously" (without knowing others' choices).

	L	R
U	$9, 9$	$0, 10$
D	$10, 0$	$4, 1$

- Row player can choose U or D.
- Column player can choose L or R.

Dominant Strategies

The "best" choice never depends on what the other person does.

- Row player should pick D anyways.

- Column player should pick R anyways.
- We assume players are rational.
- Expected outcome: $(9, 4)$
- This is the "Prisoner's Dilemma."
- If they interact: Row $\to U$, Column $\to L$.

For strategies:
$(s_1, s_2, \ldots, s_i, \ldots, s_n)$, s_i is the strategy for player i.
s_i^* is a dominant strategy for player i if:

$$u_i(s_1, s_2, \ldots, s_i^*, \ldots, s_n) > u_i(s_1, s_2, \ldots, s_i, \ldots, s_n)$$

for all $s_1 \in A_1, s_2 \in A_2, \ldots$.

Dominant Strategy Equilibrium

If a game is such that each player i has a dominant strategy s_i^*, then the game has a dominant strategy equilibrium $(s_1^*, s_2^*, \ldots, s_n^*)$.

Dominance Solvable

$\tilde{s}$ is a "dominated" strategy for player i if there is another strategy $\tilde{s_i}$ such that:

$$u_i(s_1, s_2, \ldots, \tilde{s_i}, \ldots, s_n) < u_i(s_1, s_2, \ldots, s_i, \ldots, s_n)$$

Static games

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Read: Game theory in supply chain analysis (survey paper). (Cachon & Netessine)

Definitions

Game

- A game is defined with n players ($n \geq 2$).
- Set of actions available to i: $A_1, A_2, \ldots, A_n$.
- Strategy space of i: $S_1, S_2, \ldots, S_n$.
- Payoff function: $u_1, u_2, \ldots, u_n$.

$$u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

$$u_i(s_1, s_2, \ldots, s_i, \ldots, s_n) = \text{Payoff to } i \text{ when players choose } s_1, s_2, \ldots$$

- Depends on $(s_1, s_2, \ldots, s_n)$, not just on s_i.
- Could be different for different players.

Dominant Strategy

s_i is a dominant strategy if:

$$u_i(t_i, \ldots, \underline{s_i}, \ldots, t_n) > u_i(t_i, \ldots, \underline{s_i'}, \ldots, t_n)$$

$$\forall t_1 \in S_1, t_2 \in S_2, \ldots, t_n \in S_n \quad \text{where} \quad s_i \neq s_i'$$

Strictly Dominated Strategy

Let $t_i \in S_i, s_i \in S_i$.

t_i is strictly dominated by s_i if:

$$u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

- A strictly dominated strategy may not exist.
- A rational player will never play a strictly dominated strategy.

	C	M	H
L	6/6	2/8	0/5
M	8/2	4/1	1/3
H	4/10	3/1	2/2

For row players, L is dominated.
For column players, C is dominated.

$\therefore$ Game is effectively

	M	H
M	4/1	1/3
H	3/1	2/1

Rationalizability and Beliefs

Belief of player $i \equiv$ some random $B_{-i} \subseteq S_{-i}$.

S_i : Player's best response to B_{-i}.

$$(u_i(s_i, B_{-i}) \geq u_i(s_i', B_{-i}) \quad \forall s_i' \in S_i)$$

$(s_1, s_2, \ldots, s_n)$ is rationalizable if s_i is a best response to some B_{-i} for each player i.

- Belief of i about j can be different from belief of k about j.
- Rationalizability allows for that.

Examples

Example 1:

	D	C
D	$-1/-1$	$9/0$
C	$0/9$	$-1/-2$

- C is dominant strategy for each player.
- D is dominated.
- (C, C) can be rationalized as N.E.

Example 2:

	O	F
O	$2/1$	$0/0$
F	$0/0$	$1/2$

- No dominant/dominated strategy.
- N.E. $= (0, 0)$ (pure strategies)
- Rationalizable strategies: all rationalizable (O, F), (F, O), etc.

Example 3:

	L	R
U	$8/8$	$5/9$
D	$9/5$	$6/6$

- D is dominant.
- U is dominated.
- R is dominant.
- L is dominated.

Example 4:

	L	M
V	1/0	1/2
H	0/3	2/0

- R is dominated.
- Only N.E. rationalizable.
- V is dominating, M is dominating.

More Games

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1 Cournot Competition

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1.1 Setup

- 2 firms: i and j (quantity competition)
- Marginal production cost $= c$ (cost to produce 1 item)
- Firms choose quantities q_i and q_j simultaneously
- Unit price $= a - (q_i + q_j)$

Question: What are equilibrium quantities chosen by the firms?
(In game theory context)

- Number of players $= 2$ (i and j)
- Strategy space:

$$\text{Firm } i : [0, \infty) \tag{1}$$
$$\text{Firm } j : [0, \infty) \tag{2}$$

Payoff for firm i: (Revenue - cost = Profit)

$$U_i(q_i, q_j) = q_i \cdot (\text{unit price} - \text{marginal cost}) \tag{3}$$
$$= q_i(a - q_i - q_j - c) \tag{4}$$
$$\therefore U_i(q_i, q_j) = q_i(a - q_i - q_j - c) \tag{5}$$

1.2 Finding Nash Equilibrium

(q_i^*, q_j^*) are Nash Equilibrium if:

$$U_i(q_i^*, q_j^*) \geq U_i(q_i, q_j^*) \quad \forall q_i^* \neq q_i \tag{6}$$
$$U_i(q_i^*, q_j^*) \geq U_i(q_i^*, q_j) \quad \forall q_j^* = q_j \tag{7}$$

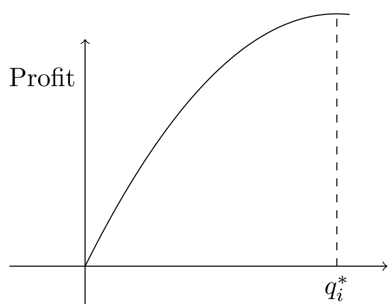
Suppose firm j produces $\hat{q}_j$. Then q_i^* is the optimal reaction of firm i:

$$\max_{q_i} \left(q_i(a - q_i - \hat{q}_j - c)\right) \tag{8}$$

where $\hat{q}_j$ is a known fixed number (essentially a constant).

$$\pi(q_i) = q_i(a - q_i - \hat{q}_j - c) \tag{9}$$
$$\pi'(q_i) = (a - q_i - \hat{q}_j - c) + q_i(-1) = 0 \tag{10}$$
$$a - q_i - \hat{q}_j - c - q_i = 0 \tag{11}$$
$$\therefore q_i^* = \frac{a - \hat{q}_j - c}{2} \tag{12}$$

Since $\pi''(q_i) = -2 < 0 \Rightarrow q_i^*$ is a max.
If you plot a profit function:



Similarly, $q_j^* = \frac{a - q_i^* - c}{2}$
For both firms to get equilibrium:

$$q_i^* = \frac{a - q_j^* - c}{2} \tag{13}$$
$$q_j^* = \frac{a - q_i^* - c}{2} \tag{14}$$

Solving:

$$\therefore q_i^* = q_j^* = \frac{a - c}{3} \tag{15}$$

If both firms were owned by the same entity:

$$\max_{q_i, q_j} \{q_i(a - q_i - q_j - c) + q_j(a - q_i - q_j - c)\} \tag{16}$$

$$= \max_{q_i, q_j} \{(q_i + q_j)(a - c - (q_i + q_j))\} \tag{17}$$

Consider $q_i + q_j = q$:

$$\max_{q} \{q(a - c - q)\} \tag{18}$$

In that case:

$$q_i^{opt} + q_j^{opt} = \frac{a - c}{2} \tag{19}$$

$$\therefore q_i^{opt} = q_j^{opt} = \frac{1}{4}(a - c) \tag{20}$$

(Like prisoner's dilemma)

Even though $\frac{1}{4}(a-c)$ is better, if q_i plays that, j will play $q_j^* = \left(\frac{a-c}{2}\right) - \frac{1}{2}\left(\frac{a-c}{4}\right)$

$\therefore$ they will both end up playing sub-optimally, i.e., $\frac{(a-c)}{3}$

1.3 Why will q_j produce more?

q_j produces more, even though price will go down; the volume still gives q_j more profit than q_i.

So when we say:

$$q_i^* = \frac{a - q_j^* - c}{2} \tag{21}$$

So whatever i thinks j will do:

$$q_i^* = \left(\frac{a - c}{2}\right) - \frac{1}{2}(q_j^*) \tag{22}$$

Since q_j cannot be negative ($q_j \geq 0$):

$$\Rightarrow q_i^* \leq \left(\frac{a - c}{2}\right) \tag{23}$$

$\therefore$ i should never produce more than $\left(\frac{a-c}{2}\right)$

Similarly, $q_j^* \leq \left(\frac{a-c}{2}\right)$

We know $q_i^{opt} = q_j^{opt} = \frac{1}{4}(a-c)$

$\therefore$ min quantity i & j can produce: $\frac{1}{4}(a-c)$

(See full working in Tirole's textbook)

2 Bertrand Duopoly

("Price competition")

2.1 Setup

- 2 firms: i and j
- Prices: P_i and P_j where $a > 0$, $b > 0$, and c is the same

Demand:

$$\text{Firm } i = a - P_i + bP_j \tag{24}$$
$$\text{Firm } j = a - P_j + bP_i \tag{25}$$

Market Size: "a"
Production cost $= c$
Equilibrium prices: (P_i^*, P_j^*)?
Payoff for firm i**:** (price - cost)

$$\pi_i(P_i) = (a - P_i + bP_j)(P_i - c) \tag{26}$$

what i thinks choice j makes

$$\therefore P_i^* = \arg\max_{P_i \geq c} [(a - P_i + bP_j)(P_i - c)] \tag{27}$$

$$P_i^* = \frac{1}{2}(a + bP_j^* + c) \tag{28}$$
$$P_j^* = \frac{1}{2}(a + bP_i^* + c) \tag{29}$$

2.2 Solving the 2 equations

$$P_i^* = P_j^* = \frac{a + c}{2 - b} \quad (b < 2) \tag{30}$$

This is the equilibrium strategy.

3 Commons Problem

Public good: K agents $(k_1, k_2, \ldots, k_n)$ units
Agent i can claim any amount k_i
Utility for $i = \ln(k_i) + \ln\left(k - \frac{\sum k_j}{3}\right)$
log is concave $\Rightarrow$ these are good models of utility function (e.g., utility of money)
Let's try for 3 agents: (k_1^*, k_2^*, k_3^*)
Suppose agent j uses k_2^*, agent 3 uses k_3^*

Then agent 1's optimal solution:

$$\max_{k_1} [\ln(k_1) + \ln(k - k_1 - k_2^* - k_3^*)] \tag{31}$$

Taking derivatives:

$$\frac{1}{k_1} - \frac{1}{k - k_1 - k_2^* - k_3^*} = 0 \tag{32}$$

$$\therefore k - k_1 - k_2^* - k_3^* = k_1 \tag{33}$$

$$\Rightarrow \therefore k_1 = \frac{k - k_2^* - k_3^*}{2} \tag{34}$$

Assume all k_i are identical:

$$\therefore k^* = \frac{k - 2k^*}{2} \tag{35}$$

$$\Rightarrow \therefore k^* = \frac{k}{4} \tag{36}$$

In general: $k_i^* = \frac{k}{(n+1)}$ if we maximize utility for each agent
Optimal consumption for 3 agents (as a whole) :

$\therefore k_i^* = \frac{k}{2n}$ maximizes overall utility for everybody

3.1 Notes

- Each agent's optimal problem means they give less weighting to overall consumption, because they only care about their own utility.
- If society wants to optimize overall welfare, there needs to be more weight on overall consumption and less on purely private benefits.
- agents consume more than they should compared to the optimal social solution. This is because of externalities — if part of the consumption cost is incurred by society as a whole, an individual tends to ignore it. As a result, we over-utilize the commons, since everyone is acting in their own self-interest without accounting for the shared damage.
- To regulate that, we can introduce tolls, penalties, or corrective taxes (Pigouvian taxes). These policies work by making private agents "internalize" the externality, i.e. aligning private incentives with the social optimum.