CS271: Data Structures

Name: Aryah Rao and John Yoon

Instructor: Dr. Stacey Truex

Project 0: Sets - Proofs

1. Prove inductively that a set S with cardinality $n \geq 1$ has exactly 2^n unique subsets

   (a) Hypothesis: for all $n \geq 1$, $P(n)$: a set S of cardinality n has exactly 2^n unique subsets.

   (b) Base case: When $n = 1$, the set S only has one element. Therefore the subsets of set S would be the empty set and the set containing the single element. We have $2^n = 2^1 = 2$ which is true. Therefore, our P(n) holds for our base case.

   (c) Inductive Hypothesis: We assume that $P(k')$ is true for all sets S with cardinality of values of $k' \in [1, k]$.

   (d) Inductive Step: Let us show that $P(k + 1)$ is true. Suppose there exists a set S with cardinality $k + 1$. We pick an element x from the set S and say that it is the $k + 1$st element. By taking all the subsets of set S that does not include the $k + 1$st element of the set S, these subsets are subsets of set of size k. In other words, these subsets are set S without the element x. Using our induction hypothesis, there are 2^k unique subsets. Now we look at the subsets of set S that contains the element x. This means that there exists equal amount of subsets from the set S with the size k with addition of element x. Thus, there are 2^k subsets of S that includes the element x. We can now add all the subsets we have got from both of the cases. Then, set S with cardinality of $k + 1$, we have that $2^k + 2^k = 2^{k+1}$. Therefore, $P(k + 1)$ is true.

   (e) Conclusion: We know that $P(1)$ is true from the base case. Since we know that $P(1)$ is true and we showed that $P(1)$ implies $P(2)$ in the inductive step, this means that $P(2)$ is true. Since we know that $P(2)$ is true and we showed that $P(2)$ implies $P(3)$ in the inductive step, this means that $P(3)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 1$.

2. Prove inductively that a set S with cardinality $n \geq 2$ has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2.

   (a) Hypothesis: for all $n \geq 2$, $P(n)$: a set S with cardinality n has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2.

   (b) Base case: When $n = 2$, the set S has two elements. This means the only subset of cardinality 2 is the set itself. So we have 1 subset of cardinality 2 and $\frac{n(n-1)}{2} = \frac{2(2-1)}{2} = 1$. Since $1 = 1$, our $P(n)$ holds for our base case.

(c) Inductive Hypothesis: We assume that $P(k')$ is true for all sets S with cardinality of values of $k' \in [2, k]$.

(d) Inductive Step: We will show that $P(k+1)$ is true. Consider a set S of cardinality $k+1$. We consider another set $S-x$ of cardinality k and call it T(x being a single element in set S. Therefore, we have another set T that resembles set S without the single random element x. From the inductive hypothesis, we know that set T has $\frac{k(k-1)}{2}$ unique subsets of cardinality 2. We know that set S contains just 1 extra element, x than set T, so to get the subsets of S, we can simply group x to the subsets of set T. Therefore, set S will have $\frac{k(k-1)}{2} + k$ unique subsets of cardinality 2.
$\frac{k(k-1)}{2} + k = \frac{(k^2-k)}{2} + \frac{2k}{2}$
$= \frac{k^2-k+2k}{2}$
$= \frac{k^2+k}{2}$
$= \frac{k(k+1)}{2}$
Therefore, set S has exactly $= \frac{k(k+1)}{2}$ unique subsets of cardinality 2, so $P(k+1)$ is true.

(e) Conclusion: We know that $P(2)$ is true from the base case. Since we know that $P(2)$ is true and we showed that $P(2)$ implies $P(3)$ in the inductive step, this means that $P(3)$ is true. Since we know that $P(3)$ is true and we showed that $P(3)$ implies $P(4)$ in the inductive step, this means that $P(4)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 1$.

3. Prove inductively that the complement of the union of any n sets $S_1, S_2, ..., S_n$ is equivalent to the intersection of each of their individual complements (i.e., that $\overline{S_1 \cup S_2 ... \cup S_n} = \overline{S_1} \cap \overline{S_2} \cap \overline{S_n})$ for all $n \geq 1$. Hint: it may be helpful to remember De Morgan's Law: $\overline{S \cup T} = \overline{S} \cap \overline{T}$

(a) Hypothesis: for all $n \geq 1$, $P(n)$: the complement of the union of any n sets $S_1, S_2, ..., S_n$ is equivalent to the intersection of each of their individual complements (i.e., that $\overline{S_1 \cup S_2 ... \cup S_n} = \overline{S_1} \cap \overline{S_2} \cap \overline{S_n})$

(b) Base case: When $n = 1$, the complement of the union of n sets would simply be $\overline{S_1}$ and the intersection of each of their individual complements would also just be $\overline{S_1}$. Since $\overline{S_1} = \overline{S_1}$, our P(1) holds for our base case.

(c) Inductive Hypothesis: We assume that $P(k')$ is true for all k' sets $S_1, S_2, ..., S_{k'}$, $k' \in [1, k]$.

(d) Inductive Step: We will show that $P(k+1)$ is true. We have the complement of the union of $k+1$ sets: $\overline{(S_1 \cup S_2 \cup \ldots \cup S_k) \cup S_{k+1}}$. Substituting into the De Morgan's Law, we get $\overline{(S_1 \cup S_2 \cup \ldots \cup S_k) \cup S_{k+1}} = \overline{(S_1 \cup S_2 \cup \ldots \cup S_k)} \cap \overline{S_{k+1}}$. By the inductive hypothesis we get, $\overline{S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1}} = \overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_k} \cap \overline{S_{k+1}}$. Therefore, $P(k+1)$ is true.

(e) Conclusion: We know that $P(1)$ is true from the base case. Since we know that $P(1)$ is true and we showed that $P(1)$ implies $P(2)$ in the inductive step, this means that $P(2)$ is true. Since we know that $P(2)$ is true and we showed that $P(2)$ implies $P(3)$ in the inductive step, this means that $P(3)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 1$.

4. Prove by contradiction that the intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set (i.e., $S_1 \cap (S_2 \setminus S_1) = \emptyset$)

   (a) Assume for a contradiction that the intersection of any set S_1 with the difference of any set S_2 and S_1 is not an empty set (i.e., $S_1 \cap (S_2 \setminus S_1) \neq \emptyset$). This means that S_1 and $(S_2 \setminus S_1)$ contains at least one element that is the same. By definition, $(S_2 \setminus S_1)$ is a set of elements in S_2 that is not contained in S_1. This means that $(S_2 \setminus S_1)$ cannot contain any elements that are in S_1 since we are getting a set of elements that are only in S_2. This contradicts our assumption that S_1 and $(S_2 \setminus S_1)$ share at least one common element. Therefore, the intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set (i.e., $S_1 \cap (S_2 \setminus S_1) = \emptyset$).