Note Template

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Abstract

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Contents

Chapter 1

Introduction

1.1 Definitions

Definition 1.1.1 (Sequence). A *sequence* is an ordered list of numbers which follows a specific pattern. Each number in this pattern is called a *term*. Sequences can be finite or infinite (i.e. they may or may not have a last term).

Example. Some examples of sequences include:

- $2, 4, 6, 8, 10, \ldots$ (even numbers)
- $1, 4, 9, 16, 25, \ldots$ (perfect squares)
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (reciprocals of natural numbers)

For the rest of these definitions, we'll use the first of these examples. Note that when a sequence is written with ellipses $(\ldots)$, it is assumed to be infinite. The first term of this sequence is 2, the second is 4, the third is 6, and so on. But what about the 100th term? Clearly we can't just keep counting! Instead, we define a *general formula* for the nth term, which allows us to quickly calculate the value of any term.

Notation. The nth term of a sequence is denoted by u_n. In this case, the first term is $u_1 = 2$, the second is $u_2 = 4$, and so on.

The formula for the nth term (for the sequence of even numbers) can then be written as:

$$u_n = 2n \tag{1.1}$$

To find the value of any term in the sequence, we simply substitute the term number, n, into the general formula. Now, we can easily find the 100th term:

$$u_{100} = 2 \times 100 = 200$$

Clearly, this is much more efficient than counting all the way up to the 100th term!

Definition 1.1.2 (Series). A *series* is the sum of the terms of a sequence. If we take the first n terms of a sequence and add them together, we get a finite series. If we add up all the terms of an infinite sequence, we get an infinite series.

For the sequence we defined earlier $(2, 4, 6, 8, \ldots)$, the series formed by adding the first n terms is denoted by S_n. In this case:

$$\begin{aligned}
S_1 &= 2 \\
S_2 &= 2 + 4 = 6 \\
S_3 &= 2 + 4 + 6 = 12 \\
&\vdots \\
S_n &= 2 + 4 + 6 + \ldots + 2n \\
&\vdots \\
S_\infty &= 2 + 4 + 6 + 8 + \ldots
\end{aligned}$$

Where S_∞ represents the infinite series formed by adding all the terms of the sequence. Note that this infinite series 'diverges,' meaning that the terms of the sum keep growing, so the sum's value is undefined. We will discuss this in more detail later on.

1.2 General Formulae

Finding general formulae is a key part of working with sequences and series. Often, the challenge lies in identifying the pattern. In this section, I'll discuss some common types of sequences, and hopefully demonstate what to look out for.

Definition 1.2.1 (Arithmetic Sequence). An *arithmetic sequence* is a sequence in which the difference between consecutive terms is a constant, called the common difference, d.

Exercise. Find the general formula for the following arithmetic sequence:

$$5, 11, 17, 23, 29, \ldots$$

Answer. You can quickly identify an arithmetic sequence by checking whether the difference between consecutive terms remains constant. In this case, the common difference is $d = 6$. The general formula for the nth term can be expressed as:

$$u_n = u_1 + (n-1)d = 5 + (n-1) \times 6 = 6n - 1$$

We'll discuss where this formula comes from in the next section. ⊛

Definition 1.2.2 (Geometric Sequence). A *geometric sequence* is a sequence in which each term is found by multiplying the previous term by a constant, called the common ratio, r.

Exercise. Find the general formula for the following geometric sequence:

$$3, 6, 12, 24, 48, \ldots$$

Answer. Recognizing a geometric sequence is similarly straightforward; check whether the ratio (division) between consecutive terms remains constant. In this case, the common ratio is $r = 2$. The general formula for the nth term can be expressed as:

$$u_n = u_1 \times r^{n-1} = 3 \times 2^{n-1}$$

We'll discuss where this formula comes from in the next section. ⊛

There are a few other common sequences which you should recognize on sight, such as:

- Perfect squares: $1, 4, 9, 16, 25, \ldots$ with general formula $u_n = n^2$
- Perfect cubes: $1, 8, 27, 64, 125, \ldots$ with general formula $u_n = n^3$

It is also possible that you may be given a sequence of fractions, where either the numerator, denominator, or both follow a specific pattern. In such cases, we can consider the patterns separately.

Exercise. Find the general formula for the following sequence:

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \ldots$$

Answer. In this case, the numerators follow an arithmetic sequence with a common difference of 2, while the denominators follow a geometric sequence with a common ratio of 2. Thus, we can express the general term as:

$$u_n = \frac{1 + 2(n-1)}{2 \times 2^{n-1}}$$

⊛

1.3 Sigma Notation

When dealing with series, especially those with many terms, it is often convenient to use sigma notation to represent the sum concisely. Here is an example of a sum expressed in sigma notation:

Example. The sum of the first n even numbers can be expressed as:

$$S_n = \sum_{i=1}^{n} 2i$$

Explanation. In the above example, the statement beneath the sigma symbol ($\sum$) indicates that we are initializing a variable $i = 1$. We then increment the variable until it reaches n. This maximum value is indicated by the number (or variable) above the sigma symbol. The terms of the sum are given by the expression inside the sigma symbol, telling us to sum $2i$ for each value of i from 1 to n. Hence, the above example tell us:

$$S_n = 2(1) + 2(2) + 2(3) + \ldots + 2(n)$$

⊛

As we said earlier, this sum diverges, so it doesn't have a definite value. If we instead consider the sum of the reciprocals of the even numbers, we find that it actually does have a value:

$$S_\infty = \sum_{i=1}^{\infty} \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots = 1$$

It's quite strange that the sum of an infinite number of terms can have a finite value. We'll explore this more with a geometric proof later on.

Exercises 1.1 (Introduction to Sequences and Series).

a) Find the general formula for the following sequences:
   i) $1, 3, 5, 7, 9, \ldots$
   ii) $2, 6, 18, 54, 162, \ldots$
   iii) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$
   iv) $4, 9, 16, 25, 36, \ldots$

b) Task B
   a) Item b1)
   b) Item b2)
   c) Item b3)
   d) Item b4)

c) Task C
   a) Item b1)
   b) Item b2)
   c) Item b3)
   d) Item b4)

Chapter 2

Arithmetic Sequences and Series

2.1 General Formula

Proposition 2.1.1. The general formula for an arithmetic sequence is given by:

$$u_n = u_1 + (n-1)d$$

where u_1 is the first term, d is the common difference, and n is the term number.

Proof. We know that the first term is simply u_1. We obtain the second term by simply adding d to the first term:

$$u_2 = u_1 + d$$

We employ a similar approach for the remaining terms:

$$\begin{aligned} u_3 &= u_2 + d = u_1 + 2d \\ u_4 &= u_3 + d = u_1 + 3d \\ &\vdots \\ u_n &= u_1 + (n-1)d \end{aligned}$$

■

Determining the general formula of any arithmetic sequence is simply an exercise in determining the first term and the common difference. This is not always straightforward.

Exercise. An arithmetic sequence has third term 11 and seventh term 23. Find the general formula for this sequence.

Answer. In this question, we are not given the first term of the common difference. We are told:

$$\begin{aligned} u_3 &= 11 \\ u_7 &= 23 \end{aligned}$$

Using the general formula for arithmetic sequences, we can express these terms in terms of u_1 and d:

$$\begin{aligned} u_3 &= u_1 + 2d = 11 \quad (1) \\ u_7 &= u_1 + 6d = 23 \quad (2) \end{aligned}$$

Now, we must isolate and solve for u_1 and d. We can do this by subtracting equation (1) from equation (2):

$$
\begin{aligned}
(u_1 + 6d) - (u_1 + 2d) &= 23 - 11 \\
4d &= 12 \\
d &= 3
\end{aligned}
$$

We can now substitute this value of d back into equation (1) to find u_1:

$$
\begin{aligned}
u_1 + 2(3) &= 11 \\
u_1 + 6 &= 11 \\
u_1 &= 5
\end{aligned}
$$

We find the general formula:

$$u_n = 5 + (n-1)3 = 3n + 2$$

⊛

Finding the general formula for an arithmetic sequence is almost always a matter of identifying the first term and the common difference, and then applying the general formula. The above exercise shows you how to achieve this for more general questions.

2.2 Arithmetic Series

Definition 2.2.1 (Arithmetic Series). An *arithmetic series* is the sum of the terms of an arithmetic sequence. If we take the first n terms of an arithmetic sequence and add them together, we get a finite arithmetic series. Infinite arithmetic series are not considered, as they always diverge.

Remark. An arithmetic series can be represented using sigma notation, as the sum of the terms of an arithmetic sequence. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ can be expressed as:

$$S_n = \sum_{k=1}^{n} u_k$$

where S_n is the sum of the first n terms, and u_k is the kth term of the arithmetic sequence.

We will use the above to prove a useful formula for finding the sum of an arithmetic series, so it is important that it makes sense. If it doesn't, try looking back over our discussion of sigma notation.

Proposition 2.2.1. The sum of the first n terms of an arithmetic series is given by:

$$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

where u_1 is the first term, u_n is the nth term, and n is the number of terms.

Proof. First, we write our sum in sigma notation:

$$S_n = \sum_{k=1}^{n} u_k \tag{2.1}$$

We know that the kth term of an arithmetic sequence can be expressed as:

$$u_k = u_1 + (k-1)d$$

Substituting this into equation (2.1), we get:

$$S_n = \sum_{k=1}^{n} \left[u_1 + (k-1)d\right] \tag{2.2}$$

Remember, this is just a sum of terms, so we can split into two separate sums:

$$S_n = \sum_{k=1}^{n} u_1 + \sum_{k=1}^{n} (k-1)d \tag{2.3}$$

We can factor out the d from the second sum, since it is constant with respect to k:

$$S_n = \sum_{k=1}^{n} u_1 + d\sum_{k=1}^{n} (k-1) \tag{2.4}$$

Notice that the second sum is now just d times the sum of the first $n-1$ natural numbers. The sum of the first m natural numbers is given by:

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

(We won't prove this in these notes, but a proof can be found in the notes on 'Proofs'). Using this formula, we can express the second sum in equation (2.4) as:

$$\sum_{k=1}^{n} (k-1) = \sum_{j=0}^{n-1} j = \frac{(n-1)n}{2} \tag{2.5}$$

The first term in equation (2.4) is simply n copies of u_1, since it is constant with respect to k, so we can express it as u_1 multiplied by n:

$$\sum_{k=1}^{n} u_1 = nu_1 \tag{2.6}$$

Substituting equations (2.5) and (2.6) back into equation (2.4), we get:

$$\begin{aligned} S_n &= nu_1 + d \cdot \frac{(n-1)n}{2} \\ &= nu_1 + \frac{d(n-1)n}{2} \\ &= \frac{2nu_1}{2} + \frac{d(n-1)n}{2} \\ &= \frac{n}{2}\left[2u_1 + (n-1)d\right] \end{aligned}$$

■

Remark. Notice that in the above proof, we could also have expressed the sum as:

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(u_1 + u_n) \end{aligned}$$

This is because $u_n = u_1 + (n-1)d$ by the general formula for arithmetic sequences.

If the proofs above don't make immediate sense to you, they don't need to. In general, you won't need to be able to understand the proof to use the formula, but it's often quite helpful to see where it comes from. Nonetheless, you are likely better served by a few worked examples.

Exercise (Easy). Bob arranges his books on a shelf such that the number of books on successive shelves forms an arithmetic sequence. If the first shelf has 20 books, and the fifth shelf has 44 books:

(a) How many books are there on the

(i) 8th shelf?

(ii) nth shelf?

(b) Given that there are 12 shelves in total, how many books does Bob have?

Answer. We are given:

$$\begin{aligned} u_1 &= 20 \\ u_5 &= 44 \end{aligned}$$

Using the general formula for arithmetic sequences, we can express u_5 in terms of u_1 and d:

$$\begin{aligned} u_5 = u_1 + 4d &= 44 \\ 20 + 4d &= 44 \\ 4d &= 24 \\ d &= 6 \end{aligned}$$

We can now find the required terms:

(a) The number of books on:

(i) The 8th shelf:

$$\begin{aligned} u_8 &= u_1 + 7d \\ &= 20 + 7(6) \\ &= 62 \end{aligned}$$

(ii) The nth shelf:

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 20 + (n-1)(6) \\ &= 6n + 14 \end{aligned}$$

(b) The total number of books is the sum of the number of books on each shelf. Using the formula for the sum of an arithmetic series, we get:

$$\begin{aligned} S_{12} &= \frac{12}{2}[2(20) + (12-1)(6)] \\ &= 6[40 + 66] \\ &= 6(106) \\ &= 636 \end{aligned}$$

⊛

Exercise (Medium). Alice is a student who loves reading books. By the end of her first week at school, she reads 1 book. By the end of her fifth week, she reads 4 books. Assuming that the number of books she reads each week forms an arithmetic sequence, find:

(a) The general formula for the number of books she reads in the nth week.

(b) The week in which Alice will read her 1000th book.

Answer. We are given:

$$u_1 = 1$$
$$u_5 = 4$$

Using the general formula for arithmetic sequences, we can express u_5 in terms of u_1 and d:

$$\begin{aligned} u_5 = u_1 + 4d &= 4 \\ 1 + 4d &= 4 \\ 4d &= 3 \\ d &= \frac{3}{4} \end{aligned}$$

We can now find the required terms:

(a) The general formula for the number of books she reads in the nth week:

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 1 + (n-1)\left(\frac{3}{4}\right) \\ &= \frac{3n - 3 + 4}{4} \\ &= \frac{3n+1}{4} \end{aligned}$$

(b) To find the week in which Alice will read her 1000th book, we need to find the number of weeks that will have passed such that the total number of books read (i.e. the sum of the number of books read in each week up to that week) is equal to 1000. Hence, we set $S_n = 1000$ and solve for n using the formula for the sum of an arithmetic series:

$$S_n = \frac{n}{2}[2u_1 + (n-1)d] = 1000$$

Now, it is possible to solve this equation analytically (i.e. by hand), but it turns out to be much easier to solve this on your calculator. Simply set one equation equal to $y = 1000$ and the other equal to $y = \frac{x}{2}[2(1) + (x-1)(\frac{3}{4})]$, where, in this case, x represents n:

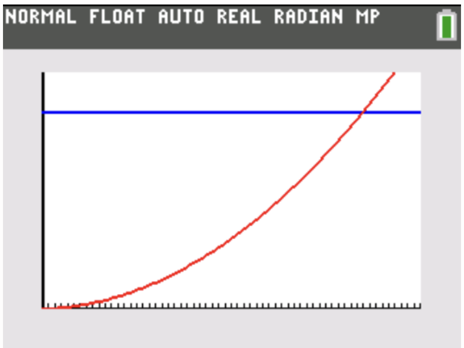

Figure 2.1: Graphical solution to find the week in which Alice reads her 1000th book.

We can then use the intersect function on the calculator to find the value of x (i.e. n) where the two lines meet. We find that $n \approx 57.4$. Since $n < 57$, Alice will not have read her 1000th book by the end of week 57, but she will have by the end of week 58. Therefore, Alice will read her 1000th book in week 58. ⊛

Exercise (Hard). Charlie sells math notes on the web. Each customer pays \$16.50 per month for access to all his notes. By the end of the first month, he has 12 customers. By the end of the fourth month, he has 27 customers. Assuming that the number of customers he has each month forms an arithmetic sequence, how much revenue will Charlie have generated by the end of the first year?

Answer. We are given:

$$\begin{aligned} u_1 &= 12 \\ u_4 &= 27 \end{aligned}$$

Using the general formula for arithmetic sequences, we can express u_4 in terms of u_1 and d:

$$\begin{aligned} u_4 = u_1 + 3d &= 27 \\ 12 + 3d &= 27 \\ 3d &= 15 \\ d &= 5 \end{aligned}$$

We now have a general formula for the number of customers paying Charlie in the nth month:

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 12 + (n-1)(5) \\ &= 5n + 7 \end{aligned}$$

Thus, the revenue generated in the nth month is given by the product of the number of customers and the price:

$$R_n = 16.50 \times u_n = 16.50(5n + 7) = 82.5n + 115.5$$

The monthly revenue also forms an arithmetic sequence, with first term $R_1 = 198$ and common difference $d_R = 82.5$. We can now use the formula for the sum of an arithmetic series to find the total revenue generated by the end of the first year (i.e. after 12 months):

$$\begin{aligned} S_{12} &= \frac{12}{2}[2(198) + (12-1)(82.5)] \\ &= 6[396 + 907.5] \\ &= 6(1303.5) \\ &= 7821 \end{aligned}$$

Thus, Charlie will have generated a total revenue of \$7821 by the end of the first year. ⊛

Exercises 2.1 (Arithmetic Sequences and Series).

a) Find the general formula for the following sequences:
   - i) $1, 3, 5, 7, 9, \ldots$
   - ii) $2, 6, 18, 54, 162, \ldots$
   - iii) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$
   - iv) $4, 9, 16, 25, 36, \ldots$

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   - c) Item b3)
   - d) Item b4)

Chapter 3

Geometric Sequences and Series

3.1 General Formula

Proposition 3.1.1. The general formula for the nth term of a geometric sequence is given by:

$$u_n = u_1 \times r^{n-1} \tag{3.1}$$

where u_1 is the first term and r is the common ratio.

Proof. We know that the first term is simply u_1. We obtain the second term by multiplying the first term by r:

$$u_2 = u_1 \times r \tag{3.2}$$

We employ a similar approach for the remaining terms:

$$u_3 = u_2 \times r = u_1 \times r^2 \tag{3.3}$$
$$u_4 = u_3 \times r = u_1 \times r^3 \tag{3.4}$$
$$\vdots \tag{3.5}$$
$$u_n = u_1 \times r^{n-1} \tag{3.6}$$

This proof is quite similar to that of the arithmetic sequence's general formula. ■

As with arithmetic sequences, solving problems relating to geometric sequences requires you to identify the first term and the common ratio. From there, you can plug values into the general formula to find the desired term.

3.2 Geometric Series

Definition 3.2.1 (Geometric Series)**.** A *geometric series* is the sum of the terms of a geometric sequence. If we take the first n terms of a geometric sequence and add them together, we get a finite geometric series.

With geometric sequences, there is an additional consideration. The sums of infinite geometric series are not necessarily undefined, as with arithmetic series.

Definition 3.2.2 (Convergence)**.** A series is said to *converge* if the sum approaches a finite value as more terms are added. If the sum does not approach a finite value, the series is said to *diverge*. To determine whether a geometric series converges or diverges, we look at the common ratio, r. If $|r| < 1$, the series converges; otherwise, it diverges.

Intuition. To understand why a geometric series converges when $|r| < 1$, consider the behavior of the terms in the series. When the common ratio r is between -1 and 1, each successive term in the geometric sequence becomes smaller in magnitude. As more terms are added to the series, the contributions of these smaller terms become negligible, leading the sum to approach a finite limit. On the other hand, if $|r| \geq 1$, the terms do not decrease in magnitude, and the sum continues to grow without bound, resulting in divergence.

We will consider the above in more detail when we consider the infinite sum of a geometric series. First, let's start with finite sums:

Proposition 3.2.1. The sum of the first n terms of a geometric series can be calculated using the formula:

$$S_n = u_1 \frac{1 - r^n}{1 - r} \quad \text{for } r \neq 1 \tag{3.7}$$

where u_1 is the first term and r is the common ratio.

Proof. Let the first n terms of a geometric series be denoted by S_n:

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n \tag{3.8}$$
$$= u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} \tag{3.9}$$

Multiplying both sides by r gives:

$$rS_n = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^n \tag{3.10}$$

Subtracting equation (3.10) from equation (3.9), we get:

$$\begin{aligned} S_n - rS_n &= u_1 - u_1 r^n \\ S_n(1 - r) &= u_1(1 - r^n) \\ S_n &= u_1 \frac{1 - r^n}{1 - r} \end{aligned}$$

■

The formula is valid for all geometric series. We employ a similar proof for the sum of an infinite geometric series, which only applies when the series converges.

Proposition 3.2.2. The sum of an infinite geometric series converges to:

$$S_\infty = \frac{u_1}{1 - r} \quad \text{for } |r| < 1 \tag{3.11}$$

where u_1 is the first term and r is the common ratio.

Proof. Let's start at the same place as we did for the finite sum, by writing out th complete sum:

$$S_\infty = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \ldots \tag{3.12}$$

Multiplying both sides by r gives:

$$rS_\infty = u_1 r + u_1 r^2 + u_1 r^3 + \ldots \tag{3.13}$$

Subtracting equation (3.13) from equation (3.12), we get:

$$\begin{aligned} S_\infty - rS_\infty &= u_1 \\ S_\infty(1 - r) &= u_1 \\ S_\infty &= \frac{u_1}{1 - r} \end{aligned}$$

However, this only holds true if the series converges, which occurs when $|r| < 1$. ■

Chapter 4

Applications of Sequences and Series

4.1 Simple Interest

Definition 4.1.1 (Interest). *Interest* is the cost of borrowing money, usually expressed as a percentage of the principal amount borrowed, over a specific period of time.

Definition 4.1.2 (Simple Interest). *Simple interest* is a way of calculating interest on a principal amount where the interest is calculated only on the original principal, not on the accumulated interest.

If you're not familiar with either of the above defitions (or both), I hope that the following example will help clarify things.

Example. Let's say you borrow \$1000 from a bank at an annual simple interest rate of 5% for 3 years. The \$1000 your borrowed is called the *principal* amount. The interest rate is 5% per year, which means that each year, you will owe 5% of the principal amount as interest. That means that your annual interest (payment) is:

$$\text{Annual Interest} = \text{Principal} \times \text{Interest Rate} = 1000 \times 0.05 = \$50$$

So, each year, you owe \$50 in interest. Over 3 years, the total interest you owe is:

$$\text{Total Interest} = \text{Annual Interest} \times \text{Number of Years} = 50 \times 3 = \$150$$

If, instead of being asked to pay the interest annually, you are asked to pay the interest (and the principal) back all at once at the end of the 3 years, you would owe:

$$\text{Principal} + (\text{Annual Interest} \times \text{Number of Years}) = 1000 + (50 \times 3) = \$1150$$

Note that it's fairly straightforward to extend this example to any number of years. For n years worth of interst, we find:

$$\begin{aligned} \text{Total Interest} &= \text{Annual Interest} \times \text{Number of Years} = 50 \times n \\ \text{Total Amount Owed} &= \text{Principal} + (\text{Annual Interest} \times \text{Number of Years}) = 1000 + (50 \times n) \end{aligned}$$

Hopefully, you can see that the total amount owed forms an arithmetic sequence, with the nth term corresponding to the total amount owed after n years. Each year, the amount increases by a constant amount (\$50), which is the common difference of the arithmetic sequence.

It's important to note that in the above case of simple interest, the interest payment remains constant each year because it's calculated only based on the original principal. This is in contrast to compound interest.

4.2 Compound Interest

Definition 4.2.1. *Compound interest* is a way of calculating interest where the interest is calculated on both the original principal and the accumulated interest from previous periods.

Essentially, with compound interest, you generate interest, which is added back onto the principal. Consequently, you earn interest on the new, larger principal amount in the next period. This leads to an exponential growth of the total amount owed, as opposed to linear growth.