Proofs

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Abstract

These notes correspond to the *Proofs* portion of the *Number and Algebra* section of the IB AA HL Mathematics syllabus. These notes are not mathematically rigorous, but I believe them to be exhaustive of the syallabus content. These notes are part of a series of notes on various topics in the syllabus. The complete repository of notes can be found at:

https://github.com/aryakakodkar/ibmathnotes/

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Chapter 1

Introduction

1.1 Proofs

In Mathematics, it is usually not sufficient to simply state that some is true; We must also provide a *proof* that the statement is indeed true. A proof is a logical argument that demonstrates the truth of a mathematical statement, based on previously established statements. A number of different kinds of proofs exist, each with its own nuances. We will begin with the simplest kind:

Definition 1.1.1 (Direct Proof)**.** A *direct proof* is a method of proving a mathematical statement by assuming the premises are true and using logical reasoning to arrive at the conclusion.

When I say *premises*, I mean the initial assumptions or conditions of the statement. For those confused, an example may clear things up:

Example. Prove that if n is an even integer, then n^2 is also even.

Let's break down the statement in the above example to determine what the premises are, and what the conclusion of our proof must be. The statement begins with "if n is an even integer", which is our premise. We assume that n is indeed an even integer, because we don't need to consider the alternative case (that n is odd), since the statement in the example doesn't concern it.

Our conclusion will be that n^2 is even, if n is even. For a direct proof, we must show this to be true using logical reasoning. Enough talk, let's get to the proof itself:

Proposition 1.1.1. If n is an even integer, then n^2 is also even.

Proof. Let n be an even integer. By definition, an even integer is a multiple of 2. Therefore, we can express n as:

$$n = 2k$$

for any integer k (e.g. if $n = 8$, then $k = 4$ — this can be done for any even n). Now, we will compute n^2:

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

We clearly see that n^2 is a multiple of 2 (since $2k^2$ is an integer). Therefore, by definition, n^2 is even. ■

In the above proof, we have mathematically re-written our premise (n even $\rightarrow n = 2k$), and used it to logically arrive at our conclusion. It is common to be uncomfortable with how I came up with the first line of this proof (rewriting n as $2k$). This is a skill that comes with practice, and simply being exposed to many proofs. Let's look at a few more examples of direct proofs, and the most common examples of premises.

Exercise (Easy). Prove that the sum of any 3 consecutive integers is divisible by 3.

Answer. Let us begin by defining our premise. It is possible to rewrite this statement as: if 3 integers are consecutive, prove that their sum is divisible by 3. Therefore, our premise is that we have 3 consecutive integers. We can write this by considering the first integer to be some integer n. Then, the next two consecutive integers are $n+1$ and $n+2$. The sum of these integers is:

$$\begin{aligned} \text{Sum} &= n + (n+1) + (n+2) \\ &= 3n + 3 \\ &= 3(n+1) \end{aligned}$$

Since $n+1$ is an integer, we see that the sum of the 3 consecutive integers is a multiple of 3. Therefore, the sum is divisible by 3. ⊛

Exercise (Easy). Prove that $x^2 - 3x + 3$ is always positive for all real values of x.

Answer. Again, it may help to re-write the statement in our 'if-then' format. This one becomes: if x is a real number, prove that $x^2 - 3x + 3 > 0$. Therefore, our premise is that x is a real number.

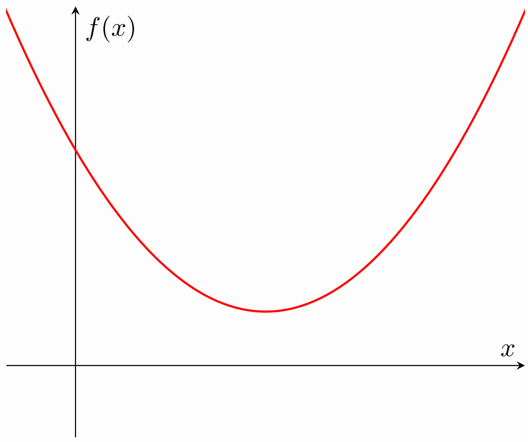

Figure 1.1: The graph of the quadratic function $f(x) = x^2 - 3x + 3$.

For those familiar with quadratics (if you aren't, see my notes on functions), we note that this function is concave up. For it to be positive at all real x, it cannot cross the x-axis (if it does, the value of $f(x)$ reaches 0, which is not positive). Equivalently, the function must have no real roots. Its discriminant must therefore be less than 0:

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(3) \\ &= 9 - 12 \\ &= -3 \end{aligned}$$

Since $D < 0$, the quadratic has no real roots, and is therefore always positive for all real values of x. ⊛