Proofs

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Abstract

These notes correspond to the *Proofs* portion of the *Number and Algebra* section of the IB AA HL Mathematics syllabus. These notes are not mathematically rigorous, but I believe them to be exhaustive of the syallabus content. These notes are part of a series of notes on various topics in the syllabus. The complete repository of notes can be found at:

https://github.com/aryakakodkar/ibmathnotes/

Contents

Chapter 1

Introduction

1.1 Proofs

In Mathematics, it is usually not sufficient to simply state that some is true; We must also provide a *proof* that the statement is indeed true. A proof is a logical argument that demonstrates the truth of a mathematical statement, based on previously established statements. A number of different kinds of proofs exist, each with its own nuances. We will begin with the simplest kind:

> **Definition 1.1.1** (Direct Proof). A *direct proof* is a method of proving a mathematical statement by assuming the premises are true and using logical reasoning to arrive at the conclusion.

When I say *premises*, I mean the initial assumptions or conditions of the statement. For those confused, an example may clear things up:

> **Example.** Prove that if n is an even integer, then n^2 is also even.

Let's break down the statement in the above example to determine what the premises are, and what the conclusion of our proof must be. The statement begins with "if n is an even integer", which is our premise. We assume that n is indeed an even integer, because we don't need to consider the alternative case (that n is odd), since the statement in the example doesn't concern it.

Our conclusion will be that n^2 is even, if n is even. For a direct proof, we must show this to be true using logical reasoning. Enough talk, let's get to the proof itself:

> **Proposition 1.1.1.** If n is an even integer, then n^2 is also even.

Proof. Let n be an even integer. By definition, an even integer is a multiple of 2. Therefore, we can express n as:

$$n = 2k$$

for any integer k (e.g. if $n = 8$, then $k = 4$ — this can be done for any even n). Now, we will compute n^2:

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

We clearly see that n^2 is a multiple of 2 (since $2k^2$ is an integer). Therefore, by definition, n^2 is even. ■

In the above proof, we have mathematically re-written our premise (n even $\to n = 2k$), and used it to logically arrive at our conclusion. It is common to be uncomfortable with how I came up with the first line of this proof (rewriting n as $2k$). This is a skill that comes with practice, and simply being exposed to many proofs. Let's look at a few more examples of direct proofs, and the most common examples of premises.

Exercise (Easy). Prove that the sum of any 3 consecutive integers is divisible by 3.

Answer. Let us begin by defining our premise. It is possible to rewrite this statement as: if 3 integers are consecutive, prove that their sum is divisible by 3. Therefore, our premise is that we have 3 consecutive integers. We can write this by considering the first integer to be some integer n. Then, the next two consecutive integers are $n+1$ and $n+2$. The sum of these integers is:

$$\begin{aligned} \text{Sum} &= n + (n+1) + (n+2) \\ &= 3n + 3 \\ &= 3(n+1) \end{aligned}$$

Since $n+1$ is an integer, we see that the sum of the 3 consecutive integers is a multiple of 3. Therefore, the sum is divisible by 3. ⊛

Exercise (Easy). Prove that $x^2 - 3x + 3$ is always positive for all real values of x.

Answer. Again, it may help to re-write the statement in our 'if-then' format. This one becomes: if x is a real number, prove that $x^2 - 3x + 3 > 0$. Therefore, our premise is that x is a real number.

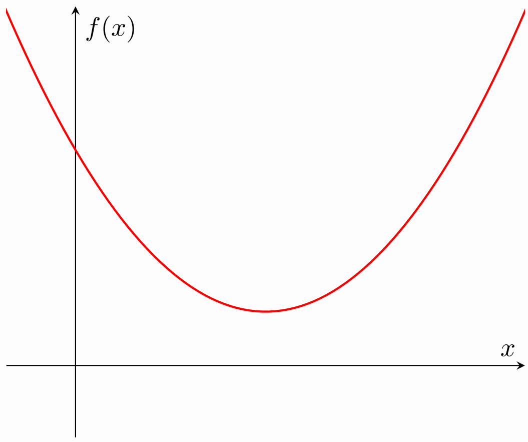

Figure 1.1: The graph of the quadratic function $f(x) = x^2 - 3x + 3$.

For those familiar with quadratics (if you aren't, see my notes on functions), we note that this function is concave up. For it to be positive at all real x, it cannot cross the x-axis (if it does, the value of $f(x)$ reaches 0, which is not positive). Equivalently, the function must have no real roots. Its discriminant must therefore be less than 0:

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(3) \\ &= 9 - 12 \\ &= -3 \end{aligned}$$

Since $D < 0$, the quadratic has no real roots, and is therefore always positive for all real values of x. ⊛

Chapter 2

Proofs by Counterexample and Contradiction

2.1 Proof by Counterexample

Definition 2.1.1 (Proof by Counterexample). A *proof by counterexample* is a method of disproving a mathematical statement by providing a specific example that contradicts the statement.

When proving a statement to be true, we must show that it holds for all possible cases. When proving it to be false, we need only to find a single case where the statement does not hold. For example, the statement "All prime numbers are odd" can be disproven by providing a single counterexample: the number 2, which is prime but not odd. Proofs by counterexample are most useful when you are asked to disprove a statement (they are not particularly useful for proving statements to be true).

2.2 Proof by Contradiction

Definition 2.2.1 (Proof by Contradiction). A *proof by contradiction* is a method of proving a mathematical statement by assuming the negation of the statement is true, and showing that this assumption leads to a logical contradiction.

The definition of a proof by contradiction is rather abstract. In the case of a direct proof, we assume the premises are true and use logical reasoning to arrive at the conclusion. In a proof by contradition, we still assume that the premises are true, but that the conclusion is false. We then use logical reasoning to arrive at a contradiction (we will arrive at a statement which contradicts the premises which we assumed to be true).

Proposition 2.2.1. $\sqrt{2}$ is irrational.

Proof. We first remind ourselves of the defintion of a rational number (and indirectly, an irrational number):

Definition 2.2.2 (Rational Number). A *rational number* is a number that can be expressed as the quotient or fraction $\frac{p}{q}$ of two integers, where p and q are integers and $q \neq 0$. An *irrational number* is a number that cannot be expressed as such a quotient.

An irrational number is a number that cannot be expressed as such a quotient. To prove that $\sqrt{2}$ is irrational, we will use a proof by contradiction. We will assume that $\sqrt{2}$ is rational, and show that this assumption leads to a contradiction.

If $\sqrt{2}$ is rational, then we can express it as:

$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers with no common factors (i.e., the fraction is in simplest form), and $q \neq 0$. Squaring both sides, we get:

$$2 = \frac{p^2}{q^2}$$

Multiplying both sides by q^2, we have:

$$2q^2 = p^2$$

This implies that p^2 is even (since it is equal to $2q^2$, which is a multiple of 2). Therefore, p must also be even (because the square of an odd number is odd, as you can hopefully prove yourself). We can express p as:

$$p = 2k$$

for some integer k. Substituting this back into the equation $2q^2 = p^2$, we get:

$$2q^2 = (2k)^2$$

Simplifying, we have:

$$2q^2 = 4k^2$$

Dividing both sides by 2, we get:

$$q^2 = 2k^2$$

This implies that q^2 is even, and therefore q must also be even. However, this contradicts our initial assumption that p and q have no common factors (since both are even, they share a factor of 2). Therefore, we conclude that the assumptions we made at the start of this proof do not apply: $\sqrt{2}$ is not rational (i.e. it is irrational). ■

Since proofs by contradiction are often less intuitive than direct proofs, I will consider a few more examples of proofs by contradiction. Note that it is somewhat rare to be asked to prove something by contradiction in the IB.

Exercise (Medium). Prove by contradiction that if the integer n is odd, then n^2 is also odd.

Answer. Again, to perform this proof, we must first assume the opposite of the conclusion provided in the statement. In this case, the conclusion is: n^2 is odd. Therefore, we will assume that n^2 is even.

We assume that n is odd, and that n^2 is even. Since n^2 is even, we can express it as:

$$n^2 = 2k \Rightarrow n \times n = 2k$$

for some integer k. But we know that the product of two odd numbers is necessarily odd (we will skip the rigorous proof, but you are encouraged to try to produce it yourself). Hence, we arrive at a contradiction. Therefore, we must conclude that our assumption that n^2 is even is false. Thus, if n is odd, then n^2 is also odd.

We could equivalently conclude that our assumption that n is odd is the incorrect one, and in doing so, we automatically prove that if n^2 is even, then n is also even. This is called the *contrapositive* of the original statement. It has the equivalent logical implication as the original statement. This fact is non-examinable. ⊛

Exercise (Hard). Prove that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$.

Answer. A quick reminder: $\mathbb{R}$ is the set of all real numbers. To prove this statement by contradiction, we will assume that there exists some $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$. We will then show that this assumption leads to a contradiction.

Starting from our assumption, we have:

$$\frac{1}{x-2} = 1 - x$$

Multiplying both sides by $x-2$, we get:

$$1 = (1-x)(x-2)$$

Expanding the right-hand side, we have:

$$1 = x - 2 - x^2 + 2x$$

Simplifying, we get:

$$1 = -x^2 + 3x - 2$$

Rearranging, we have:

$$x^2 - 3x + 3 = 0$$

To determine whether this quadratic equation has any real solutions, we can calculate the discriminant:

$$D = b^2 - 4ac = (-3)^2 - 4(1)(3) = 9 - 12 = -3$$

Since the discriminant is negative ($D < 0$), the quadratic equation has no real solutions. This contradicts our initial assumption that there exists some $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$. Therefore, we conclude that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$. ⊛

Chapter 3

Induction

3.1 Proofs by Induction

Proofs by induction are a powerful method of proving statements that are asserted to be true for all natural numbers (or some well-defined subset of them). We remind ourselves:

Definition 3.1.1 (Natural Numbers). The set of positive integers: $\{1, 2, 3, \ldots\}$.

Sometimes, the natural numbers are defined to include 0, but we will not consider this definition here. A proof by induction is defined as follows:

Definition 3.1.2 (Proof by Induction). A *proof by induction* is a method of proving a statement by showing that if it holds for a natural number n, then it also holds for $n + 1$. The proof consists of two main steps:

1. **Base Case:** Prove that the statement holds for the initial value (usually $n = 1$).
2. **Inductive Step:** Assume the statement holds for some arbitrary natural number $n = k$, and then prove it holds for $n = k + 1$.

Proofs by induction are far more intuitive than their definition suggests. Let us first consider the concept of an inductive step: if we can show that the statement holds for $n = k + 1$, assuming it holds for $n = k$, then we can conclude that the statement holds for all natural numbers greater than or equal to k. We could define another variable, m,

$$m = k + 1$$

We know that if the statement holds for $n = k$, then it must also hold for $n = m$. But then, we can apply the same logic to m: if the statement holds for $n = m$, then it must also hold for $n = m + 1$. Of course:

$$m + 1 = (k + 1) + 1 = k + 2$$

Therefore, if the statement holds for $n = k$, it must also hold for $n = k + 2$. We can repeat this process indefinitely, showing that if the statement holds for $n = k$, it must also hold for all natural numbers greater than k.

The keen observer will note that we made an assumption at the beginning of this process: we assumed that the statement holds for $n = k$. What ensures that this assumption is valid? This is where the base case comes in. By proving the base case (that the statement holds for $n = 1$), we establish a starting point for our inductive step. From the base case, we know that the statement holds for $n = 1$. Therefore, by the inductive step, it must also hold for $n = 1 + 1 = 2$. Then, by applying the inductive step again, it must hold for $n = 3$, and so on. Thus, by proving both the base case and the inductive step, we can conclude that the statement holds for all natural numbers. As always, an example is better than an explanation:

Exercise. Prove by mathematical induction that the sum of the first n natural numbers is given by the formula:

$$S(n) = \frac{n(n+1)}{2} \tag{3.1}$$

where $S(n)$ is the sum of the first n natural numbers.

Answer. A quick note: this is the same problem that we saw in the direct proofs section. Almost always, there is more than one way to prove any statement. Let's proceed with the proof by induction:

Base Case: For our base case, we must prove that the statement holds for the smallest value of n. In this case, the statement concerns the natural numbers, so the smallest value of n is 1. We check that equation 3.1 holds for $n = 1$:

$$S(1) = \frac{1(1+1)}{2} = 1$$

The sum of the first 1 natural number is indeed 1, so the base case holds.

Inductive Step: For the inductive step, we assume that the statement holds for some arbitrary natural number $n = k$. This is our inductive hypothesis. We must then prove that the statement holds for $n = k + 1$. According to our inductive hypothesis, we have:

$$S(k) = \frac{k(k+1)}{2} \tag{3.2}$$

We must show that:

$$S(k+1) = \frac{(k+1)(k+2)}{2}$$

The sum of the first $k + 1$ natural numbers is simply the sum of the first k natural numbers plus $k + 1$:

$$S(k+1) = S(k) + (k+1)$$

From our inductive hypothesis (equation 3.2), we can substitute for $S(k)$:

$$\begin{aligned} S(k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

We see that the last line of this equation can be rewritten to match equation 3.1:

$$S(k+1) = \frac{(k+1)\left([k+1]+1\right)}{2}$$

Thus, we have shown that if the statement holds for $n = k$, it also holds for $n = k + 1$.

By the principle of mathematical induction, since we have proven both the base case and the inductive step, we conclude that the formula for the sum of the first n natural numbers holds for all natural numbers n. Since it holds for $n = 1$, it holds for $n = 2$, and so on, for all natural numbers.

⊛

Exercise. Prove by mathematical induction that $11^n - 6$ is divisible by 5 for all natural numbers $n \geq 1$.

Answer. We will prove this statement using mathematical induction.

Base Case: For our base case, we need to verify that the statement holds for $n = 1$:

$$11^1 - 6 = 11 - 6 = 5$$

Since 5 is divisible by 5, the base case holds.

Inductive Step: For the inductive step, we assume that the statement holds for some arbitrary natural number $n = k$. This is our inductive hypothesis. We must then prove that the statement holds for $n = k + 1$. According to our inductive hypothesis, we have:

$$11^k - 6 \text{ is divisible by } 5$$

This means there exists an integer m such that:

$$11^k - 6 = 5m \tag{3.3}$$

We need to show that:

$$11^{k+1} - 6 \text{ is divisible by } 5$$

We can express 11^{k+1} as $11^k \cdot 11$:

$$\begin{aligned} 11^{k+1} - 6 &= 11^k \cdot 11 - 6 \\ &= 11(11^k) - 6 \end{aligned}$$

Now, we can rewrite this expression using our inductive hypothesis (equation 3.3):

$$\begin{aligned} 11^{k+1} - 6 &= 11(11^k) - 6 \\ &= 11(5m + 6) - 6 \\ &= 55m + 66 - 6 \\ &= 55m + 60 \\ &= 5(11m + 12) \end{aligned}$$

Since $11m + 12$ is an integer, we have shown that $11^{k+1} - 6$ is divisible by 5.

By the principle of mathematical induction, since we have proven both the base case and the inductive step, we conclude that $11^n - 6$ is divisible by 5 for all natural numbers $n \geq 1$. ⊛