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Syllabus

MTH 111.3 Engineering Mathematics I (3-2-0)

Evaluation:

	Theory	Practical	Total
Sessional	50	-	50
Final	50	-	50
Total	100	-	100

Course Objectives:

After the completion of this course students will be able to apply the concept of calculus (Differential and integral), analytica geometry and vector in their professional courses.

Chapter	Content	Hours
1	**Limit, Continuity and Derivative:** I. Limit, continuity and Derivative of a function with their properties II. Mean values Theorem with their application III. Higher order derivative IV. Indeterminate forms V. Asymptote VI. Curvature VII. Ideas of curve tracing	(15 Hrs)
2	**Integration with its Application:** I. Basic integration, standard integral, definite integral with their properties II. Fundamental theorem of integral calculus (without proof) III. Improper integral IV. Reduction formulae and use of beta Gamma functions V. Area bounded by curves VI. Approximate area by Simpsons and Trapezoidal rule, VII. Volume of solid revolution	(17Hrs)
3	**Two dimensional geometry:** I. Review (circle, Translation and rotation of axes) II. Conic section(parabola, ellipse, hyperbola), III. Central conics (Introduction only).	(7 Hrs)
4.	**Vector Algebra:** I. Review of vector and scalar quantity II. Space coordinates III. Product of two or more vectors IV. Reciprocal system of vectors and their properties V. Equations of lines and planes by vector methods	(6 Hrs)

Text Books

1. Calculus and analytical geometry: George B Thomas, Ross L. Finney
2. Engineering Mathematics I: Prof. D.D Sharma (Regmi), Toya Narayan Paudel, Hari Prasad Adhikari, Sukunda publication Bhotâhity , Kathmandu

Reference Books:

1. Calculus with analytical geometry: E.W. Swokóswski.
2. Coordinate Geometry: Lalji Prasad.
3. Vector Analysis: M.B. Singh
4. Integral Calculus: GD Panta.

Chapter 1

LIMIT, CONTINUITY AND DIFFERENTIABILITY

Theoretical Part

Variable

An element x is called variable if it is capable to take different value(s). Normally, the variable is noted by English small alphabet like x, y, z etc.

Function

A function f is a rule that assigns to each element x in a set D exactly one element in a set E. Then it is noted by f(x). Here D is domain set of f(x) and E is range set of f(x).

Domain and Range of f(x)

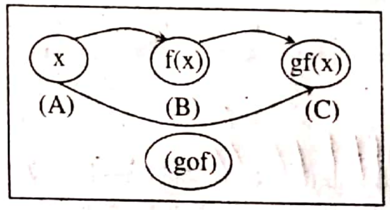
Let f(x) is a function. Then the set of elements D is called domain of f(x) if f(x) is defined for each x in D.

And the set of all possible values of f(x) for all x in D, is called range of f(x).

Composite of functions

Let f is a function from A to B and g is a function from B to C. Then the composite function of f and g is denoted by (gof) which is a function from A to C and is defined as,

$$(gof)(x) = g(f(x)) \quad \text{for } x \in A.$$



Meaning of $x \to a$

Set a = 1 and x is a variable then x moves to 1 from right sides and left side. Choose x, x = 2 and moves towards 1 then found x across 1.9, 1.8, ... 1.1, 1.01, 1.001, ... Such movement is known as x approaches to 1 from right side and noted as $x \to 1$ (x tends to 1 or x approaches to 1) from right side.

Likewise we observe for left approach of x to a.

Limit of a function:

A function f(x) is said to have a limit 'l' as $x \to a$ if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$.

In this case we write,

$$\lim_{x \to a} f(x) = l$$

Continuity of a function:

A function f(x) is said to be continuous at a point x = a if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).$$

In this case we write,

$$\lim_{x \to a} f(x) = f(a).$$

Differentiation of a function:

A function f(x) is said to be differentiable at x = a if for h > 0,

$$\lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right) = \lim_{h \to 0}\left(\frac{f(a-h)-f(a)}{-h}\right).$$

Note: Here, $\lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$ is known as right hand derivative (RHD) and $\lim_{h \to 0}\left(\frac{f(a-h)-f(a)}{-h}\right)$ is known as left hand derivative (LHD).

Theorem:

Differentiability of a function f(x) at x = a implies continuity but converse may not be always true.

OR If a function f(x) is differentiable at x = a show that it is continuous at x = a. By taking proper example show that the converse may not be true. [2005, Spring] [2008, Fall] [2002]

OR Prove that continuity is the necessary condition for a function to be differentiable but not sufficient. [2011 Spring]

OR Prove that the differentiability of a function at a point implies the continuity at that point. Give an example to show that the converse may not be true. [2018 Fall]

OR Prove that the continuity of a function at a point is necessary but not sufficient condition for the existence of the derivative of the function at that point. [2016 Fall]

Proof: Suppose that f(x) is differentiable at x = a. So, f '(a) exists and is finite.

That is, f '(a) = a finite value.

This means,

$$\lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right) = \lim_{h \to 0}\left(\frac{f(a-h)-f(a)}{-h}\right) = \text{finite quantity} \quad \ldots (1)$$

Then we need to show that f(x) is continuous at x = a.

Now,

$$f(a+h) - f(a) = \left(\frac{f(a+h)-f(a)}{h} \times h\right)$$

f(n),

Taking limit as $h \to 0$,

$$\lim_{h \to 0}\{f(a+h) - f(a)\} = \lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h} \times h\right)$$

$$= \text{finite quantity} \times 0 \qquad \text{[using (1)]}$$

$$= 0.$$

$$\Rightarrow \lim_{h \to 0} f(a+h) = f(a) \qquad \ldots\ldots (2)$$

Similarly, we get, $\lim_{h \to 0} f(a-h) = f(a)$ ...... (3)

From (2) and (3)

$$\lim_{h \to 0} f(a+h) = f(a) = \lim_{h \to 0} f(a-h).$$

let f(n) = |n|

Hence, f(x) is continuous at x = a, whenever f '(a) exists.

To show the converse may not be true, it is sufficient to show that a function f(x) is continuous at a point but not differentiable at that point. For this, consider a function f(x) = |x|.

Now, we wish to show f(x) is continuous at x = 0 but f(x) is not differentiable at x = 0.

Now, at x = 0,

let f(n) = |n|

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x| = |0^+| = 0.$$

$$\text{and LHS} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} |x| = |0^-| = 0.$$

also, $f(0) = |0| = 0.$

Thus, f(x) is continuous at x = 0.

For derivability at x = 0,

$$\text{RHD} = \lim_{h \to 0}\left(\frac{f(0+h)-f(0)}{h}\right) = \lim_{h \to 0}\frac{|h| - 0}{h} = \lim_{h \to 0}\frac{h}{h} = 1.$$

$$\text{and LHD} = \lim_{h \to 0}\left(\frac{f(0-h)-f(0)}{-h}\right) = \lim_{h \to 0}\frac{|-h| - 0}{-h} = \lim_{h \to 0}\frac{+h}{-h} = -1.$$

Thus, R.H.D. ≠ L.H.D. at x = 0. So, f(x) is not differentiable at x = 0, although it is continuous at x = 0.

Exercise 1.1

Find the domain and range of the functions:

1. $y = \frac{1}{x}$

Solution: Given that,

$$y = \frac{1}{x}.$$

Here, the given function y is defined for all the values of x except zero. So, the domain of the function is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$.

And, $\quad y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

Clearly, x is defined for all the values of y except zero. So, the range of the function is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$.

2. $y = \sqrt{x}$

Solution: Here, the given function y is defined only for $x \ge 0$.

Therefore domain of this function is $[0, \infty)$.

For the value of x, the function y takes any value except negative value. So the range of the given function is $[0, \infty)$.

3. $y = \sqrt{4 - x}$ **[2009, Spring][2008, Fall][2007, Spring]**

Solution: Here, the given function y is defined only when $(4 - x) \ge 0 \Rightarrow x \le 4$

Therefore, the domain of the given function is $(-\infty, 4]$.

For all values for x in the domain, y takes any value except negative value.

So, the range of that function is $[0, \infty)$.

4. $y = \sin x$

Solution : Here, the given function y is defined for any value of x so domain of this function is $(-\infty, \infty)$.

For the values of x in domain, y takes any value but the sine of function is bounded function and has oscillating value in between -1 to 1.

That means $|\sin x| \le 1$.

So, range of this function is $[-1, 1]$.

5. $y = \sqrt{x + 4}$ **[2011 Spring, Short]**

Solution: Here, the given function y is defined for all value of x for $(4 + x) \ge 0 \Rightarrow x \ge -4$. So, domain of the function is $[-4, \infty)$.

At $x = -4$, we get $y = 0$ and at $x = \infty$, we get $y = \infty$.

Thus, y takes all non-negative values.

Therefore, range of the given function y is $[0, \infty)$.

6. $y = \dfrac{1}{x - 2}$ **[2004, Spring]**

$y^2 = x+4$ $\quad x = y^2 - 4$ $\quad [0, \infty)$

Solution: Given that,

$$y = \frac{1}{x - 2}.$$

Here, the given function y is defined for each value of x except at $x = 2$.

Therefore, the domain of this function is $R - \{2\}$.

And,

$$y = \frac{1}{x - 2} \Rightarrow x = \frac{1}{y} + 2.$$

$u - x = y$ $\quad x = u - y$ $\quad u - y \ge 0$

This shows x is defined for all values of y except at $y = 0$.

So, range of the given function is $R - \{0\}$.

7. $y = x \sin x$

Solution: Here, the given function y is defined for all the values of x. So, the domain of this function is $(-\infty, \infty)$.

For the values of x, we have y takes any value. So, range of the given function is $(-\infty, \infty)$.

8. $y = (\sqrt{x})^2$

Solution: Here, the given function y is defined for $x \ge 0$. So, domain of the given function is $[0, \infty)$.

Since y has square form of x. So, it takes each non-negative values. So, range of the given function is $[0, \infty)$.

9. $y = 2\cos x$

Solution : Here, the given function y is defined for all values of x.

So, domain $= (-\infty, \infty)$.

The cosine function is bounded function having osculating value in $[-1, 1]$.

So, range $= 2[-1, 1] = [-2, 2]$.

$u - x \ge 0 \Rightarrow y \le x$

10. $y = -3\sin x$. **[2006, Fall]**

Solution : Here, the given function y is defined for all x.

So, domain $= (-\infty, \infty)$.

The sine function is bounded having osculating value in $[-1, 1]$.

So, range $= 3[-1, 1] = [-3, 3]$.

$u - x \ge 0$ $\quad u \ge x$ $\quad x \le y$ $\quad y \ge x$

11. $y = x^2 + 1$

Solution : Here, the given function y is defined for all x.

So, domain $= (-\infty, \infty)$.

Since, for $x \in (-\infty, \infty)^2$, x^2 has value in $[0, \infty)$. Thus, value of y is lie in $[0, \infty) + 1 = [1, \infty)$ for any value of x in $(-\infty, \infty)$.

So, range $= [1, \infty)$.

$x = \sqrt{y - 1}$ $\quad y - 1 \ge 0$ $\quad y \ge 1$ $\quad [1, \infty)$

12. $y = -x^2$

Solution: Here, the given function y is defined for all the values of x.

So, domain $= (-\infty, \infty)$.

Since, $(-\infty, \infty)^2$ has value in $[0, \infty)$. So, $y = -[0, \infty) = (-\infty, 0]$ for any value of x in $(-\infty, \infty)$.

So, range $= (-\infty, 0]$.

13. $y = \sqrt{x+1}$

Solution: Since, the function $\sqrt{(.)}$ defines only for non–negative value. So, is defined for all x with $x \geq -1$.

So, domain $= [-1, \infty)$.

And, $y = 0$ at $x = -1$. Also, $y \to \infty$ as $x \to \infty$.

So, range of $y = [0, \infty)$.

14. $y = 1 + \sqrt{x}$

Solution: Since, the function $\sqrt{(.)}$ defines only for non–negative value. So, the given function y is defined for all the values of x with $x \geq 0$. So, domain of the given function is $[0, \infty)$.

For all the values of $x \geq 0$, we get the value of $\sqrt{x}$ is in $[0, \infty)$. So,

$\therefore$ Range of $y = [0, \infty) + 1 = [1, \infty)$.

15. $y = (\sqrt{2x})^2$

Solution : Here, the given function y is defined for all the values of x only for non–negative values.

So, domain of $y = [0, \infty)$.

For the values of x, the function y takes all non–negative value.

So, range $= [0, \infty)$.

16. $y = -\frac{1}{x}$

Solution: Here,

$$y = -\frac{1}{x}$$

Clearly, y is defined for all the values of x except at $x = 0$.

So, domain $= R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$.

And,

$$y = -\frac{1}{x} \Rightarrow x = -\frac{1}{y}$$

Clearly, x is defined for all the values of y except at $y = 0$.

So, range $= R - \{0\}$.

17. $y = \sin 2x$

Solution : Since, the sine function is defined for all x with x is in $(-\infty, \infty)$.

So, $\sin 2x$ is defined for all x with x is in $(-\infty, \infty)$.

Therefore, domain $= (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$. So, sin 2x has osculating value in $[-1, 1]$.

So, range $= [-1, 1]$.

18. $y = \sin^2 x$

Solution: Since, the sine function is defined for all x with x is in $(-\infty, \infty)$.

So, $\sin^2 x$ is defined for all x with x is in $(-\infty, \infty)$.

Therefore, domain of $y = (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$. So, $\sin^2 x$ has osculating value in $[0, 1]$.

So, Range of $y = [0, 1]$.

19. $y = 1 + \sin x$

Solution : Since, the sine function is defined for all x with x is in $(-\infty, \infty)$.

So, $y = 1 + \sin x$ is defined for $1 + (-\infty, \infty)$ i.e. $(-\infty, \infty)$.

Therefore, domain of $y = (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$. So, $(1 + \sin x)$ has osculating value in $1 + [-1, 1]$ i.e. $[0, 2]$.

So, range of $y = [0, 2]$.

Exercise 1.2

1. Let $f(x) = x^4 - 5$ and $g(x) = \cos x$. Find (fog) (x) and (gof) (x).

Solution: We have, $f(x) = x^4 - 5$, $g(x) = \cos x$.

Then, $(fog)(x) = f[g(x)] = f(\cos x) = \cos^4 x - 5$

$(gof)(x) = g[f(x)] = g(x^4 - 5) = \cos(x^4 - 5)$.

2. Let $f(x) = x + 1$ and $g(x) = x^2$. Find (fog) (x) and (gof) (x).

Solution: We have, $f(x) = x + 1$ and $g(x) = x^2$

Then, $(fog)(x) = f[g(x)] = f(x^2) = x^2 + 1$

$(gof)(x) = g[f(x)] = g(x + 1) = (x + 1)^2$.

3. Let $f(x) = \tan x$, $g(x) = x^7$. Find (fog) (x) and (gof) (x).

Solution: We have, $f(x) = \tan x$ and $g(x) = x^7$

Then, $(fog)(x) = f[g(x)] = f(x^7) = \tan(x^7)$

$(gof)(x) = g[f(x)] = g(\tan x) = (\tan x)^7$.

Q. Let two functions $f: R \to R$ and $g: R \to R$ defined as $f(x) = x + 2$ and $g(x) = 3x^2$ for $x \in R$. Find fog(x) and gof(x). [2015 Spring short]

Exercise 1.3

Find the limit of the following function at specified points (if exists):

1. $f(x) = \begin{cases} 3x^2 - 1 & \text{when } x \le 2 \\ 4x + 3 & \text{when } x > 2 \end{cases}$ at $x = 2$.

Solution: At $x = 2$,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} (4x + 3) = 11$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (3x^2 - 1) = 11$$

Thus, L.H.L. = R.H.L. = 11. So the limit of f(x) exits at x = 2 and its limiting value is 11.

2. $f(x) = \begin{cases} 2x + 1 & \text{when } x \ge 1 \\ 4x^2 - 1 & \text{when } x < 1 \end{cases}$ at $x = 1$.

Solution: At $x = 1$,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2x + 1) = 3$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (4x^2 - 1) = 3$$

Thus, L.H.L. = R.H.L. = 3. So the limit of f(x) exits at x = 1 and its limiting value is 3.

3. $f(x) = \begin{cases} 3x + 2 & \text{when } x \ge 2 \\ 2x^2 + 1 & \text{when } x < 2 \end{cases}$ at $x = 2$.

Solution: At $x = 2$,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} (3x + 2) = 8$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (2x^2 + 1) = 9$$

Thus, L.H.L. ≠ R.H.L. So the limit of f(x) does exits at x = 2.

4. A function f(x) defined by f(x) $\begin{cases} x^2 & \text{when } x < 1 \\ 2.5 & \text{when } x = 1 \\ x^2 + 2 & \text{when } x > 1 \end{cases}$

Does $\lim_{x\to1} f(x)$ exist?

Solution: At $x = 1$,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2 + 2) = 3$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2 = 1.$$

Thus, L.H.L. ≠ R.H., so the limit of f(x) does exits at x = 1.

5. A function f(x) is defined as follows: $f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$. Find the value of $\lim_{x\to0} f(x)$.

Solution: At $x = 0$, $\text{R.H.L.} = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0.$

$$\text{L.H.L.} = \lim_{x\to0^-} f(x) = -x = -0 = 0.$$

Thus, L.H.L. = R.H.L. = 0. So the limit of f(x) exits at x = 0 and its limiting value is 0. Therefore, $\lim_{x\to0} f(x) = 0.$

Exercise 1.4

1. **At what point is the function**

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 \le x \le 1 \\ 1, & \text{for } x > 1 \end{cases} \quad \text{continuous?}$$

Solution: Here, the function f(x) is defined on (0, 1] At x = 0,

$$\text{R.H.L.} = \lim_{x\to0^+} f(x) = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{x\to0^-} f(x) = 0.$$

Here, R.H.L. ≠ L.H.L. is not continuous at x = 0

At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{x\to1^-} f(x) = 1.$$

Also, f(1) = 1

Here, L.H.L. = R.H.L. = f(1), so the given function f(x) is continuous only at x = 1.

2. **Let** $f(x) = \begin{cases} \left(\dfrac{x^2 - 1}{x - 1}\right) & \text{for } x \ne 1 \\ 2 & \text{for } x = 1 \end{cases}$. **Is f(x) continuous or not continuous at x = 1? Explain.**

Solution: Given function is

$$f(x) = \begin{cases} \left(\dfrac{x^2 - 1}{x - 1}\right) & \text{for } x \ne 1 \\ 2 & \text{for } x = 1 \end{cases}$$

And, at x = 1,

$$f(1) = 2,$$

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to1} f(x) = \lim_{x\to1} (x + 1) = 2.$$

Thus, L.H.L. = R.H.L. = f(1) = 2. So, the given function f(x) is continuous at x = 1. This means f(x) is continuous everywhere on R.

3. Define f(3) so that $f(x) = \dfrac{x^2-9}{x-3}$ is continuous at x = 3.

Solution: At x = 3,

$$\text{R.H.L.} = \lim_{x\to3^+}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to3^+}\left(\frac{(x-3).(x+3)}{x-3}\right) = \lim_{x\to3^+}(x+3) = 6.$$

$$\text{L.H.L.} = \lim_{x\to3^-}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to3^-}\left(\frac{(x-3).(x+3)}{x-3}\right) = \lim_{x\to3^-}(x+3) = 6.$$

In order that f(x) is continuous at x = 3, we must have,

f(3) = L.H.L. = R.H.L. = 6.

4. Let $f(x) = \begin{Bmatrix} x & \text{for } 0 \le x \le 1 \\ 2-x & \text{for } 1 < x < 2 \end{Bmatrix}$. Is f(x) continuous at x = 1?

Solution: Given function is

$$f(x) = \begin{Bmatrix} x & \text{for } 0 \le x \le 1 \\ 2-x & \text{for } 1 < x < 2 \end{Bmatrix}.$$

And, at x = 1,

f(1) = 1.

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2-x) = 1,$$

and, $$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x) = 1.$$

Therefore, L.H.L. = R.H.L. = f(1).

Thus, L.H.L. = R.H.L. = f(1) = 1. So, the given function f(x) is continuous at x = 1. This means f(x) is continuous everywhere on [0, 2).

5. What value should be assigned to 'a' to make the function.

(i) $f(x) = \begin{cases} x^2-1 & \text{for } x < 3 \\ 2ax & \text{for } x \ge 3 \end{cases}$ **is continuous at x = 3.**

Solution: Given function is

$$f(x) = \begin{cases} x^2-1 & \text{for } x < 3 \\ 2ax & \text{for } x \ge 3 \end{cases}$$

And, at x = 3,

f(3) = 2a(3) = 6a,

$$\text{R.H.L.} = \lim_{x\to3^+} f(x) = \lim_{x\to3^+} 2ax = 6a$$

and, $$\text{L.H.L.} = \lim_{x\to3^-} f(x) = \lim_{x\to3^-}(x^2-1) = 8.$$

By hypothesis, the given function f(x) is continuous at x = 3.

So, L.H.L. = R.H.L. = f(3).

$$\Rightarrow 6a = 8 \qquad \Rightarrow a = \frac{8}{6} = \frac{4}{3}.$$

Thus, for $a = \frac{4}{3}$, the given function f(x) is continuous at x = 3.

(ii) $f(x) = \begin{cases} x^3 & \text{for } x < 1/2 \\ ax^2 & \text{for } x \ge 1/2 \end{cases}$ **is continuous at $x = \frac{1}{2}$.**

Solution: Given function is

$$f(x) = \begin{cases} x^3 & \text{for } x < 1/2 \\ ax^2 & \text{for } x \ge 1/2 \end{cases}$$

And, at x = (1/2),

$$f(1/2) = a\left(\frac{1}{2}\right)^2 = \frac{a}{4}$$

$$\text{R.H.L.} = \lim_{x\to(1/2)^+} f(x) = \lim_{x\to(1/2)^+}(ax^2) = \frac{a}{4}$$

$$\text{L.H.L.} = \lim_{x\to(1/2)^-} f(x) = \lim_{x\to(1/2)^-}(x^3) = \frac{1}{8}$$

By hypothesis, the given function f(x) is continuous at x = 3.

So, L.H.L. = R.H.L. = f(1/2).

$$\Rightarrow \frac{a}{4} = \frac{1}{8} \quad \Rightarrow a = \frac{1}{2}.$$

Thus, for $a = \frac{1}{2}$, the given function f(x) is continuous at x = 3.

6. A function f(x) is defined as follows:

$$f(x) = \begin{cases} (1/2) - x & \text{when } 0 < x < (1/2) \\ 1/2 & \text{when } x = (1/2) \\ (3/2) - 2 & \text{when } (1/2) < x < 1 \end{cases}$$

Show that f(x) is discontinuous at $x = \frac{1}{2}$.

Solution: And, at x = (1/2),

$$\text{R.H.L.} = \lim_{x\to(1/2)^+} f(x) = \lim_{x\to(1/2)^+}\left(\frac{3}{2} - x\right) = \left(\frac{3}{2} - \frac{1}{2}\right) = 1$$

$$\text{L.H.L.} = \lim_{x\to(1/2)^-} f(x) = \lim_{x\to(1/2)^-}\left(\frac{1}{2} - x\right) = 0$$

Here, R.H.L. ≠ L.H.L. So, the limit of f(x) at $x = \frac{1}{2}$, does not exist. So, the given function f(x) is not continuous at $x = \frac{1}{2}$.

7. **A function f(x) is defined in (0, 3) such that**

$$f(x) = \begin{cases} x^2 & \text{for } 0 < x < 1 \\ x & \text{for } 1 \le x < 2 \\ \dfrac{x^3}{4} & \text{for } 2 \le x < 3 \end{cases}$$

Show that f(x) is continuous at x = 1 and x = 2.

Solution: And, at x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} x = 1$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2 = 1$$

and, f(1) = 1

This shows L.H.L. = R.H.L. = f(1). This means f(x) is continuous at x = 1. Therefore, the given function f(x) is continuous at x = 1.

Also, at x = 2,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} \left(\frac{x^3}{4}\right) = \frac{8}{4} = 2$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} x = 2$$

and, f(2) = 2

This shows L.H.L. = R.H.L. = f(2). This means f(x) is continuous at x = 2. Therefore, the given function f(x) is continuous at x = 2.

Thus, f(x) is continuous on (0, 3).

8. **A function f(x) is defined by** $f(x) = \begin{cases} x^2 & \text{when } x \ne 1 \\ 2 & \text{when } x = 1 \end{cases}$

Is f(x) continuous at x = 1?

Solution: Here f(x) is in piecewise form. And, clearly f(x) is continuous elsewhere at x = 1.

And, at x = 1,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to1} f(x) = \lim_{x\to1} x^2 = 1$$

and, f(1) = 2.

Here, L.H.L. = R.H.L. ≠ f(1). So the given function is not continuous at x = 1.

9. **A function f(x) is defined as follows:** $f(x) = \begin{cases} x^2 & \text{when } x < 1 \\ 2.5 & \text{when } x = 1 \\ x^2 + 2 & \text{when } x > 1 \end{cases}$

Is f(x) continuous at x = 1?

Solution: At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2 + 2) = 3$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2) = 1$$

Here, L.H.L. ≠ R.H.L. So, the given function f(x) is not continuous at x = 1.

10. **A function f(x) is defined by** $f(x) = \begin{cases} -x & \text{when } x \le 0 \\ x & \text{when } 0 < x < 1 \\ 2 - x & \text{when } x \ge 1 \end{cases}$.

Show that it is continuous at x = 0 and x = 1.

Solution: At x = 0,

$$\text{R.H.L.} = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0$$

$$\text{L.H.L.} = \lim_{x\to0^-} f(x) = \lim_{x\to0^-} (-x) = 0$$

and, f(0) = 0

Here, L.H.L. = R.H.L. = f(0).

So, the given function is continuous at x = 0.

And, at x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2 - x) = 1$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x) = 1$$

and, f(1) = 1.

Here, L.H.L. = R.H.L. = f(1).

So, the given function f(x) is continuous at x = 1.

Thus, the given function f(x) is continuous at x = 0 and x = 1.

Exercise 1.5

1. **Examine the continuity of the following functions at the specified points.**

(i) $f(x) = \begin{cases} (1 + 3x)^{1/x} & \text{when for } x \ne 0 \\ e^3 & \text{when } x = 0 \end{cases}$, at x = 0

Solution: given that, $f(x) = (1 + 3x)^{1/3}$ for $x \ne 0$.

So, f(x) has same value for R.H.L. and L.H.L. at x = 0.

At x = 0,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to0} f(x) = \lim_{x\to0} \left[(1 + 3x)^{1/3x}\right]^3 = e^3$$

$f(0) = e^3$

Thus L.H.L. = R.H.L. = f(0).

Hence, the given function f(x) is continuous at x = 0.

$\left[(1+0)^{3/0}\right]^3 =$

(ii) $f(x) = \begin{cases} -x & \text{for } x \le 0 \\ x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } x \ge 1 \end{cases}$, at $x = 1$.

Solution: At $x = 1$,

$$\text{R.H.L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x) = 1$$

$$\text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$$

and, $f(1) = 2 - 1 = 1$

Here, L.H.L. = R.H.L. = f(1).

So, the given function f(x) is continuous at $x = 1$.

(iii) $f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 1 \end{cases}$, at $x = 0$.

Solution: At $x = 0$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{x \to 0^-} f(x) = -1.$$

Here, L.H.L. ≠ R.H.L. So, f(x) is not continuous at $x = 0$.

(iv) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$, at $x = 0$.

Solution : Since $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \ne 0$.

So, f(x) has same value for R.H.L. and L.H.L., if they have finite value.

At $x = 0$,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \left(\sin\frac{1}{x}\right)$$

= a finite quantity oscillates from −1 to 1.

That means RHL has not necessarily equals to LHL.

That means f(x) is not continuous at $x = 0$.

(v) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$, at $x = 0$. **[2007, fall]**

Solution: Given that

$$f(x) = x \sin\frac{1}{x} \quad \text{for } x \ne 0.$$

So, f(x) has same value for R.H.L. and L.H.L, if they have finite value.

At $x = 0$,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \left[x \sin\frac{1}{x}\right]$$

= 0 (a finite quantity oscillating between −1 to 1)

= 0.

And, f(0) = 0.

Here, R.H.L. = L.H.L. = f(0) = 0. So, the given function f(x) is continuous at $x = 0$.

(vi) $f(x) = \begin{cases} (1 + x)^{1/x} & \text{for } x \ne 0 \\ 1 & \text{for } x = 0 \end{cases}$, at $x = 0$.

$(1+x)^{1/x} = e$

Solution: Given that

$$f(x) = (1 + x)^{1/x} \quad \text{for } x \ne 0.$$

So, f(x) has same value for R.H.L. and L.H.L. if they have finite value.

At $x = 0$,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x \to 0} (1 + x)^{1/x} = e.$$

and, $f(0) = 1$.

Here, L.H.L. = R.H.L. ≠ f(0). So, the given function f(x) is not continuous at $x = 0$.

(vii) $f(x) = \begin{cases} \dfrac{x - 1}{1 + e^{1/(x-1)}} & \text{for } x \ne 1 \\ 0 & \text{for } x = 1 \end{cases}$, at $x = 1$.

Solution: Since $f(x) = \left(\dfrac{x - 1}{1 + e^{1/(x-1)}}\right)$ for $x \ne 1$.

So, f(x) has same value for R.H.L. and L.H.L., if they have finite value.

At $x = 1$,

$$\text{R.H.L.} = \lim_{x \to 1^+} \left(\frac{x - 1}{1 + e^{1/(x-1)}}\right) = \frac{0}{1 + e^{\infty}} = \frac{0}{1 + \infty} = 0.$$

$$\text{L.H.L.} = \lim_{x \to 1^-} \left(\frac{x - 1}{1 + e^{1/(x-1)}}\right) = \frac{0}{1 + e^{-\infty}} = \frac{0}{1 + 0} = 0.$$

and, $f(1) = 0$.

Here, L.H.L. = R.H.L. = f(1) = 0.

So, the given function f(x) is continuous at $x = 1$.

2. Examine the continuity and derivability of the function f(x) defined as follows: **[2011 Fall] [2017 Fall]**

$$f(x) = \begin{cases} 1 & \text{when } x \in (-\infty, 0) \\ 1 + \sin x & \text{when } x \in (0, \pi/2) \text{ and } x = 0 \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & \text{when } x \in (\pi/2, \infty) \text{ and } x = \pi/2 \end{cases}$$

Solution: Here, we have to examine continuity and differentiability at $x = 0$ and at $x = \frac{\pi}{2}$.

$0 < x < \pi/2$

$-\infty < x < 0$

For the continuity at x = 0,

$$\text{R.H.L.} = \lim_{x\to 0^+} (1 + \sin x) = 1 + \sin 0 = 1.$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = 1.$$

and, $f(0) = 1 + \sin 0 = 1 + 0 = 1$. $\quad 0, \pi/2$

Here, L.H.L. = R.H.L. = f(0).

So, the given function f(x) is continuous at x = 0.

For derivability at x = 0,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(a+h) - f(a)}{h}\right) = \lim_{h\to 0}\left(\frac{f(0+h) - 1}{h}\right) = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(0-h) - f(0)}{-h}\right) = \lim_{h\to 0}\left(\frac{1-1}{-h}\right) = 0.$$

Here LHD ≠ RHD. So, the given function is not differentiable at x = 0.

For the continuity at $x = \frac{\pi}{2}$.

$$\text{R.H.L.} = \lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2^+}\left(2 + \left(x - \frac{\pi}{2}\right)^2\right) = 2.$$

$$\text{L.H.L.} = \lim_{x\to \pi/2^-} f(x) = \lim_{x\to \pi/2^-} (1 + \sin x) = 2.$$

and, $f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2.$ $\quad x > 0$

Here, L.H.L. = R.H.L. = $f\left(\frac{\pi}{2}\right) = 2$.

So, the given function is continuous at $x = \frac{\pi}{2}$.

For differentiability at $x = \frac{\pi}{2}$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f\left(\frac{\pi}{2} + h\right) - 2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2 - 2}{h}\right) = \lim_{h\to 0}\frac{h^2}{h} = 0.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(a-h) - f(a)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 + \sin\left(\frac{\pi}{2} - h\right) - 2}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 + \sin\frac{\pi}{2}\cdot\cos h + \cos\frac{\pi}{2}\cdot\sin h - 2}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 + \cos h - 2}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-1 + \cos h}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 - \cos h}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\sin^2\left(\frac{h}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\sin(h/2)}{(h/2)}\right)^2 \times \frac{1}{2} = 1 \times 0 = 0.$$

Here, LHD = RHD at $x = \frac{\pi}{2}$.

Thus, the given function f(x) is continuous everywhere and derivable at $x = \frac{\pi}{2}$ but not derivable at x = 0.

3. Determine whether f(x) is continuous and has a derivative at the origin, where

$$f(x) = \begin{cases} 2 + x & \text{if } x \ge 0 \\ 2 - x & \text{if } x < 0 \end{cases}.$$

Solution: At x = 0,

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(2 + x) = 2$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2 - x) = 2$$

and $f(0) = 2 + 0 = 2$.

Thus, L.H.L. = R.H.L. = f(0). So, f(x) is continuous at x = 2.

And,

$$\text{R.H.D.} = \lim_{h\to0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to0}\frac{2+(h)-2}{h} = \lim_{h\to0}\frac{h}{h} = 1.$$

$$\text{L.H.D.} = \lim_{h\to0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to0}\frac{2-(-h)-2}{-h} = \lim_{h\to0}\frac{-h}{h} = -1$$

Thus, the function f(x) is continuous at x = 0 but not derivable at x = 0.

4. Show that the function f(x) defined as below is continuous at x = 1 and x = 2 and it is derivable at x = 2 but not at x = 1.

$$f(x) = \begin{cases} x & \text{for } x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ -2+3x-x^2 & \text{for } x \ge 2 \end{cases}$$

[2003, fall] [2006, spring] [2012, Fall]

Solution: For continuity at x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2-x) = 1.$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1.$$

and, f(1) = 2 − 1 = 1.

Here, L.H.L. = R.H.L. = f(1) = 1. So, the given function f(x) is continuous at x = 1.

For derivability at x = 1,

$$\text{R.H.D.} = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{2-(1+h)-1}{h} = -1$$

$$\text{L.H.D.} = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{1-h-1}{-h} = 1$$

Here, L.H.D. ≠ R.H.D. So, the function is not differentiable at x = 1.

For continuity at x = 2,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} (-2+3x-x^2) = -2+6-4 = 0.$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (2-x) = 0.$$

and, f(2) = −2 + 6 − 4 = 0.

Here, L.H.L. = R.H.L. = f(2) = 0.

So, the given function f(x) is continuous at x = 2.

For derivability at x = 2,

$$\text{R.H.D.} = \lim_{h\to0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{-2+3(2+h)-(2+h)^2-0}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{-2+6+3h-4-4h-h^2}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h(3-4-h)}{k}\right) = -1.$$

$$\text{L.H.D.} = \lim_{h\to0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{2-(2-h)-0}{-h}\right) = \lim_{h\to0}\left(\frac{2-2+h}{-h}\right) = -1.$$

Here, L.H.D. = R.H.D. So, f(x) is derivable at x = 2.

Thus, the given function f(x) is continuous at x = 1 and x = 2 and it is derivable at x = 2 but not at x = 1.

5. Find f '(0) of $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$

Solution: Here we have to find the derivative value of f(x) at x = 0. So,

LHL = RHL = f '(0).

At x = 0,

$$\text{R.H.D.} = \lim_{h\to0}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h^2 \cdot \sin\left(\frac{1}{h}\right)-0}{h}\right)$$

$$= \lim_{h\to0}\left(h \cdot \sin\left(\frac{1}{h}\right)\right)$$

= 0 (finite oscillatory quantity lies between −1 to 1).

= 0.

Thus, f '(0) = L.H.D. = R.H.D. = 0.

6. If $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$. **Find f '(0).**

Solution: Here we have to find the derivative value of f(x) at x = 0. So,

LHL = RHL = f '(0).

At x = 0,

$$\text{R.H.D.} = \lim_{h\to0}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h^2 \cdot \cos\left(\frac{1}{h}\right)-0}{h}\right)$$

$$= \lim_{h\to0}\left(h \cdot \cos\left(\frac{1}{h}\right)\right)$$

$= (0)$ (finite oscillatory quantity lies between -1 to 1).

$= 0.$

Thus, $f'(0) = \text{L.H.D.} = \text{R.H.D.} = 0.$

7. If $f(x) = \begin{cases} x & \text{for } 0 \le x < 1/2 \\ 1-x & \text{for } 1/2 < x < 1 \end{cases}$. **Does** $f'\left(\frac{1}{2}\right)$ **exists?**

Solution: Here we have to find the derivative value of f(x) at $x = \frac{1}{2}$. So,

$$\text{LHL} = \text{RHL} = f'\left(\frac{1}{2}\right).$$

At $x = \frac{1}{2}$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{1-\left(\frac{1}{2}+h\right)-\frac{1}{2}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1-\frac{1}{2}-h-\frac{1}{2}}{h}\right) = -1.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f\left(\frac{1}{2}-h\right) - f\left(\frac{1}{2}\right)}{-h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{2}-h-\frac{1}{2}}{-h}\right) = 1.$$

This shows that, L.H.D. ≠ R.H.D. So, $f'\left(\frac{1}{2}\right)$ does not exists.

8. If $f(x) = \begin{cases} 3+2x & \text{for } -3/2 < x \le 0 \\ 3-2x & \text{for } 0 < x < 3/2 \end{cases}$. [2013 Spring]

Show that f(x) is continuous at x = 0 but f '(0) does not exists.

Solution: For continuity at x = 0,

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(3-2x) = 3.$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(3+2x) = 3.$$

and, $f(0) = 3 + 2(0) = 3.$

Here, L.H.L. = R.H.L. = f(0) = 3. This means f(x) is continuous at x = 0.

For differentiability,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(0+h)-f(0)}{h}\right) = \lim_{h\to 0}\left(\frac{3-2(h)-3}{h}\right) = -2.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(0-h)-f(0)}{-h}\right) = \lim_{h\to 0}\left(\frac{3+2(-h)-3}{-h}\right) = 2.$$

Here, L.H.D. ≠ R.H.D. This means f(x) is not differentiable at x = 0.

Thus, f '(0) does not exist.

9. If $f(x) = \begin{cases} 5x-4 & \text{for } 0 < x \le 1 \\ 4x^2-3x & \text{for } 1 < x < 2 \\ 3x+4 & \text{for } x \ge 2 \end{cases}$

Discuss the continuity of f(x) at x = 1 and 2, and the existence of f '(x) for these values.

Solution: For continuity at x = 1,

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x^2-3x) = 1$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5x-4) = 1$$

and, $f(1) = 5(1) - 4 = 1$

Here, L.H.L. = R.H.L.= f(1). So, f(x) is continuous at x = 1.

For derivability at x = 1,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4(1+h)^2-3(1+h)-1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4(1+2h+h^2)-3-3h-1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4+8h+4h^2-3-3h-1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(5+4h)}{h}\right) = 5.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{5(1-h)-4-1}{-h}\right) = \lim_{h\to 0}\left(\frac{5-5h-5}{-h}\right) = 5.$$

Thus, L.H.D. = R.H.D. This means f '(x) exists at x = 1.

For continuity at x = 2,

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x+4) = 10$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4x^2-3x) = 16-6 = 10.$$

and, $f(2) = 3(2) + 4 = 10.$

Here, L.H.L. = R.H.L. = f(2). So, f(x) is continuous at x = 2.

For derivability at x = 2,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(2+h)+4-10}{h}\right) = \lim_{h\to 0}\left(\frac{6+3h+4-10}{h}\right) = 3.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4(2-h)^2-3(2-h)-10}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{16-16h+4h^2-6+3h-10}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(-16+4h+3)}{-h}\right) = 18.$$

Thus, L.H.D. $\neq$ R.H.D. Hence f '(x) does not exist at x = 2.

10. If $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \le x \le 2 \\ x - \frac{x^2}{2} & \text{for } x > 2 \end{cases}$

Is f(x) continuous at x = 1 and 2? Does f '(x) exist for these values?

Solution: For continuity at x = 1,

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2-x) = 1$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1,$$

and, $f(1) = 1$.

Here, R.H.L. = L.H.L. = f(1). So, f(x) is continuous at x = 1.

For continuity at x = 2,

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(x - \frac{x^2}{2}\right) = 0$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2-x) = 0$$

and, $f(2) = 2 - 2 = 0$.

Here, L.H.L. = R.H.L. = f(2). So, f(x) is continuous at x = 2.

For derivability at x = 1,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2-(1+h)-1}{h}\right) = \lim_{h\to 0}\left(\frac{-h}{h}\right) = -1.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{-h}\right) = \lim_{h\to 0}\left(\frac{(1-h)-1}{-h}\right) = 1.$$

Thus, L.H.D. $\neq$ R.H.D. This means f '(1) does not exist.

For derivability at x = 2,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(2+h)-\frac{(2+h)^2}{2}-0}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4+2h-(4+2h+h^2)-0}{2h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4+2h-4-4h-h^2}{2h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(2-4-h)}{2h}\right) = -\frac{2}{2} = -1.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2-(2-h)-0}{-h}\right) = \lim_{h\to 0}\left(\frac{h}{-h}\right) = -1.$$

This shows that, L.H.D. = R.H.D.

So, f(x) is differntiable at x = 2 and f '(2) = –1.

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

LONG QUESTIONS

a. Show that the function

$$f(x) = \begin{cases} x & \text{for } x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ (-2+3x-x^2) & \text{for } x \ge 2 \end{cases}$$

is continuous at x = 1 but not differentiable at x = 1. [2009 Spring]

Solution: See Ex 1.5 Q 4 only at x = 1.

b. Define continuity of a function at a point. Let [2009, Fall][1999]

$$f(x) = \begin{cases} 4-x^2 & \text{for } x < 2 \\ x-2 & \text{for } x \ge 2 \end{cases}$$

Show that f(x) is continuous at x = 2 but not differentiable at x = 2.

Solution: First Part: See definition of continuity.

Second Part: Given that,

$$f(x) = \begin{cases} 4-x^2 & \text{for } x < 2 \\ x-2 & \text{for } x \ge 2 \end{cases}$$

For continuity at x = 2,

$$\text{Right hand limit (RHL)} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x-2) = 2-2 = 0.$$

$$\text{Left hand limit (LHL)} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4-x^2) = 4-4 = 0$$

And, $f(2) = 2 - 2 = 0$.

Since, RHL = LHL = f(2). So, f(x) is continuous at x = 2.

For differentiability at $x = 2$,

$$\text{Right hand derivative (RHD)} = \lim_{h\to0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{2+h-2-0}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h}{h}\right) = 1.$$

$$\text{Left hand derivative (LHD)} = \lim_{h\to0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{4-(2-h)^2-0}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{4-4+2h-h^2}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{4-h}{-1}\right) = -4.$$

Here, RHD $\neq$ LHD at $x = 2$. So, $f(x)$ is not differentiable at $x = 2$.
Thus $f(x)$ is continuous at $x = 2$ but not differentiable at $x = 2$.

c. If a function f(x) is defined by **[2015 Spring]**

$$f(x) = \begin{cases} 4-x^2 & \text{for } x < 2 \\ x-2 & \text{for } x \geq 2 \end{cases}$$

Show that f(x) is continuous at x = 2 but not differentiable at x = 2.

Solution: See second part from Final Exam Question (b).

d. Define continuity and differentiability of a function at a given point. If a function f(x) is defined by $f(x) = \begin{cases} x-2 & \text{for } x \geq 2 \\ 4-x^2 & \text{for } x < 2 \end{cases}$ **[2001]**

Show that it is continuous at x = 2 but it is not differentiable at x = 2.

Solution: First Part: See definition of continuity and differentiation.

Second Part: See Q. (b).

e. Define a limit and continuity of a function at a point. If a function f(x) be defined by $f(x) = \begin{cases} x^2-2 & \text{for } x \leq 2 \\ x^2-4x+6 & \text{for } x > 2 \end{cases}$

Show that it is continuous at x = 2 but not differentiable at x = 2. [2000]

Solution: First Part: See the definition of limit, continuity.

Second Part: Given that

$$f(x) = \begin{cases} x^2-2 & \text{for } x \leq 2 \\ x^2-4x+6 & \text{for } x > 2 \end{cases}$$

For continuity at $x = 2$,

$$\text{LHL} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-}(x^2-2) = 4-2 = 2.$$

$$\text{RHL} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+}(x^2-4x+6) = 4-8+6 = 2.$$

and $f(2) = 4-2 = 2$

Here, LHL = RHL = f(2). So, $f(x)$ is continuous at $x = 2$.

Differentiability at $x = 2$,

$$\text{LHD} = \lim_{h\to0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{(2-h)^2-2-2}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{4-4h+h^2-4}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{-h(4-h)}{-h}\right) = \lim_{h\to0}(4-h) = 4-0 = 4.$$

And,

$$\text{RHD} = \lim_{h\to0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{(2+h)^2-4(2+h)-2}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{4+4h+h^2-8-4h-2}{h}\right) = \lim_{h\to0}\left(\frac{-6}{h}+h\right) = \infty$$

Here, LHD $\neq$ RHD. So, $f(x)$ is not differentiable at $x = 2$.
Thus, $f(x)$ is continuous at $x = 2$ but not differentiable at $x = 2$.

f. Show that the function $f(x) = \begin{cases} x^2+2 & \text{for } x \leq 1 \\ 3x & \text{for } x > 1 \end{cases}$ **[2002]**

is continuous at x = 1 but it is not differentiable at x = 1.

Solution: We have to show that it is continuous at $x = 1$ but not differentiable at $x = 1$.

For continuity:

$$\text{Right hand limit (RHL)} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(3x) = 3.$$

$$\text{Left hand limit (LHL)} = \lim_{x\to1^-} f(x) = \lim_{x\to1^+}(x^2+2) = 1+2 = 3.$$

Functional value $f(1) = 1^2 + 2 = 3$,

Here, RHL = f(1) = LHL. So $f(x)$ is continuous at $x = 1$.

For differentiability:

$$\text{RHD} = \lim_{h\to0}\left(\frac{f(1+h)-f(1)}{h}\right) = \lim_{h\to0}\left(\frac{3(1+h)-3}{h}\right) = \lim_{h\to0}\frac{3h}{h} = 3.$$

$$LHD = \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(1-h)^2+2-3}{-h}\right) = \lim_{h\to 0}(2-h) = 2.$$

Here, RHD ≠ LHD. So, f(x) is not differentiable at x = 1.

g. Prove that the continuity of a function at a point is the necessary but not sufficient for the existence of the derivative of the function at that point. Is following function differentiable at x = 1? $f(x) = \begin{cases} x^2+2 & x \le 1 \\ 3x & 1 < x \end{cases}$

[2006, Fall][2008, Spring]

Solution: First Part: See Theorem on theory part.

Second Part: See Q. (e), final exam.

h. Define continuity of a function at a point. If f(x) = |x| show that f(x) is continuous at x = 0 but not differentiable at that point. [2003, Spring]

Solution: See definition.

Second part See the second part of the theorem.

i. Define differentiability of a function f(x) at x = a. If f(x) is defined as,

$$f(x) = \begin{cases} x^2+1 & \text{when } 0 \le x < 1 \\ x+1 & \text{when } 1 \le x < 2 \end{cases}$$

Show that f(x) is continuous at x = 1 but is not differentible at x = 1.

[2004, Fall]

Solution: First Part: See definition on page – 1.

Second Part: For continuity at x = 1,

$$LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2+1) = 1+1 = 2.$$

$$RHL = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 1+1 = 2.$$

and $f(1) = 1 + 1 = 2.$

Here, LHL = RHL = f(1). So, f(x) is continuous at x = 1.

For differentiability at x = 1,

$$LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^2+1-1}{-h} = \lim_{h\to 0}\frac{1-2h}{-h} = \infty.$$

So, LHD ≠ RHD. So, f(x) is not differentiable at x = 1.

j. When a function f(x) is said to be differentiable at a point? A function f is defined as, $f(x) = \begin{cases} 2x+1 & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ x^2+2 & \text{when } x > 1 \end{cases}$ [2004, Spring]

Show f(x) is continuous at x = 1. Also find f'(1) if exists.

Solution: First Part: See definition on page – 1.

Second Part: For continuity at x = 1,

$$LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+1) = 2(1)+1 = 3.$$

$$RHL = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+2) = 1+2 = 3.$$

and $f(1) = 3.$

Here, LHL = RHL = f(1). So, f(x) is continuous at x = 1.

For differentiability at x = 1,

$$LHD = \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2(1-h)+1-3}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2-2h+1-3}{-h}\right) = \lim_{h\to 0}(2) = 2.$$

and

$$RHD = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(1+h)^2+2-3}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1+2h+h^2+2-3}{h}\right) = \lim_{h\to 0}(2+h) = 2.$$

Here, LHD = RHD. So, f(x) is differentiable at x = 1. This means f '(1) exists and

$$f'(1) = LHD = RHD = 2.$$

k. Show that the function f(x) defined by [2018 Spring]

$$f(x) = \begin{cases} -x & \text{when } x \le 0 \\ x & \text{when } 0 < x < 1 \\ 2-x & \text{when } x \ge 1 \end{cases}$$ **[2005, Fall][2013, Fall]**

is continuous at x = 0 and x = 1, but is not differentiable at x = 1.

Solution: For continuity at x = 0,

$$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0.$$

$$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 1^+}(x) = 0.$$

and $f(0) = 0.$

Here, LHL = RHL = f(0). So, f(x) is continuous at x = 0.

For continuity at x = 1.

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1.$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1.$$

and $f(1) = 2 - 1 = 1$

Here, LHL = RHL = f(1). So, f(x) is continuous at x = 1.

For differentiability at x = 1,

$$\text{LHD} = \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{-h}\right) = \lim_{h\to 0}\left(\frac{(1-h)-1}{-h}\right) = \lim_{h\to 0}\left(\frac{h}{h}\right) = 1$$

and

$$\text{RHD} = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2-(1+h)-1}{h}\right) = \lim_{h\to 0}\left(\frac{-h}{h}\right) = -1.$$

Here, LHD ≠ RHD. So, f(x) is not differentiable at x = 1.

l. **Define continuity and differentiability of a function. Show that differentiability of a function f(x) = a, implies continuity but converse may not be true.**

Solution: See definition and prove from theory part.

m. **Define continuity and differentiability of a function. Show that the function** $f(x) = \begin{cases} x^2+2 & \text{for } x \le 1 \\ 3x & \text{for } x > 1 \end{cases}$ **[2014 Spring] is continuous at x = 1 but it is not differentiable at x = 1.**

Solution: See definition and solution of (e) from final exam question.

m. Examine the continuity and differentiability at x = 2 of the function f(x) defined as follows:

$$f(x) = \begin{cases} 2-x & \text{for } 0 < x < 2 \\ -2+3x-x^2 & \text{for } 2 \le x \le 4 \end{cases}$$ **[2015 Fall][2017 Spring]**

Solution: For continuity at x = 2,

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-2+3x-x^2) = -2+6-4 = 0.$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 0.$$

$f(2) = -2 + 6 - 4 = 0.$

Therefore, L.H.L. = R.H.L. = f(2) = 0.

So, the given function f(x) is continuous at x = 2.

For derivability at x = 2,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2+3(2+h)-(2+h)^2-0}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2+6+3h-4-4h-h^2}{h}\right)$$

$$= \lim_{h\to 0}\frac{h(3-4-h)}{k}$$

$= -1.$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{2-(2-h)-0}{-h} = \lim_{h\to 0}\frac{2-2+h}{-h} = -1.$$

Here, L.H.D. = R.H.D. So, f(x) is differentiable at x = 2.

This means f(x) is continuous at x = 2 and is differentiable at x = 2.

n. Define continuity and differentiability of a function f(x) at x = a. Find f '(0) if it exists, where $f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$. **[2016 Spring]**

Solution: See definition of continuity and differentiability from theory part.

Second Part: See Exercise 1.5 Q 6.

SHORT QUESTIONS

1. **Find the domain and range for $y = \sqrt{4-x^2}$. [2005, Fall][2003, Spring]**

Solution: Given that $y = \sqrt{4-x^2}$.

Here y is defined only for $(4-x^2) \ge 0 \Rightarrow x \in [-2, 2]$.

So, domain of y = [−2, 2].

And, for $x \in [-2, 0]$, we have, $0 \le y \le 2$. Also, for $x \in [0, 2]$, we have, $0 \le y \le 2$.

So, range of y = [0, 2].

2. **Find the domain and range of the function $f(x) = \sqrt{x^2-1}$. [2009, Fall]**

Solution: Since the function f(x) is defined for $(x^2-1) \ge 0 \Rightarrow x^2 \ge 1$

$\Rightarrow$ x is defined on $\Re$ except in (−1, 1).

So, domain of $f(x) = \Re - (-1, 1) = (-\infty, -1] \cup [1, \infty)$.

And, for $x \in (-\infty, -1]$ we get f(x) has value on $[1, \infty)$. Also, for $x \in [1, \infty)$ we get f(x) has value on $[1, \infty)$.

So, range of $f(x) = [1, \infty) \cup [1, \infty) = [1, \infty)$.

3. Give the domain and range of the function $f(x) = \sqrt{1-x^2}$. [1999][2000][2001]

Solution: Given that $y = \sqrt{1-x^2}$.

Here y is defined only for $(1-x^2) \geq 0 \Rightarrow x \in [-1, 1]$.

So, domain of $y = [-1, 1]$.

And, for $x \in [-1, 0]$, we have, $0 \leq y \leq 1$. Also, for $x \in [0, 1]$, we have $0 \leq y \leq 1$.

So, range of $y = [0, 1]$.

4. Let two functions $f: \mathfrak{R} \to \mathfrak{R}$ and $g: \mathfrak{R} \to \mathfrak{R}$ defined as $f(x) = x + 2$, $g(x) = 3x^2$, $x \in \mathfrak{R}$. Find fog(x) and gof(x). [2006, Spring]

Solution: Here, $f: \mathfrak{R} \to \mathfrak{R}$ and $g: \mathfrak{R} \to \mathfrak{R}$. Given that the functions are defined as,

$$f(x) = x + 2, \quad g(x) = 3x^2.$$

Now, $\text{fog}(x) = f(g(x)) = f(3x^2) = 3x^2 + 2$.

And, $\text{gof}(x) = g(f(x)) = g(x+2) = 3(x+2)^2 = 3x^2 + 12x + 12$.

5. Find the domain and range of the function $y = 3 + \sin x$.

Solution: Since, the sine function is defined for all x with x is in $(-\infty, \infty)$.

So, $y = 3 + \sin x$ is defined for all x with x is in $(-\infty, \infty)$.

So, domain of $y = (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$.

So, $3 + \sin x$ has osculating value in $3 + [-1, 1] = [2, 4]$.

So, range of the given function y is $[2, 4]$.

●●●

Chapter–2

HIGHER ORDER DERIVATIVE

Leibnitz's Theorem

Statement: Let $y = uv$; where u and v are functions of x. Then,

$$y_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots + {}^nC_r u_{n-r}v_r + \ldots + {}^nC_n uv_n.$$

where the suffixes of u and v denote the order of differentiation of u and v with respect to x.

Proof: Let

$$y = uv$$

Differentiating w. r. t. x, then

$$y_1 = u_1 v + uv_1$$

This means theorem is true for n = 1.

$$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + uv_2$$
$$= u_2 v + 2u_1 v_1 + uv_2$$
$$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 uv_2$$

This means theorem is true for n = 2.

Suppose the statement is true for n = m. So,

$$y_m = u_m v + {}^mC_1 u_{m-1}v_1 + {}^mC_2 u_{m-2}v_2 + \ldots\ldots + {}^mC_m uv_m$$

Differentiating w.r.t. x then,

$$y_{m+1} = u_{m+1}v + u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1}v_2 + {}^mC_2 u_{m-1} v_2 + {}^mC_2 u_{m-2}v_3 + \ldots + {}^mC_m u_1 v_m + {}^mC_m uv_{m+1}$$
$$\Rightarrow u_{m+1}v + ({}^mC_1 + 1)\, u_m v_1 + ({}^mC_1 + {}^mC_2)\, u_{m-1} v_2 + \ldots\ldots + ({}^mC_m)\, uv_{m+1}$$
$$= u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+2}C_2 u_{m-1} v_2 + \ldots + {}^{m+1}C_{m+1} uv_{m+1}$$

This means theorem is true for n = m + 1. So, by induction the statement is true for any positive integer.

Exercise 2

1. Find y_n in the following:

(i) $y = (a - bx)^m$

Solution: Let, $y = (a - bx)^m$

Differentiating, we get

$$y_1 = m(a - bx)^{m-1}(-b)$$
$$y_2 = m(m-1)(a - bx)^{m-2}(-b)^2$$
$$y_3 = m(m-1)(m-2)(a - bx)^{m-3}(-b)^3$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$y_n = m(m-1)(m-2)\ldots(m-(n-1))(-b)^n(a - bx)^{m-n}$$
$$= (-1)^n m(m-1)(m-2)\ldots(m-n+1)\, b^n (a - bx)^{m-n}$$

(ii) $y = \frac{1}{a-x} = (a-x)^{-1}$

Solution: We have, $y = \frac{1}{a-x} = (a-x)^{-1}$

Differentiating w.r.t. x,

$$y_1 = (-1)(a-x)^{-2}(-1) = (-1)^2 (a-x)^{-2} = (a-x)^{-2}.$$

$$y_2 = (-2)(a-x)^{-3}(-1) = (-1)^2 2!\,(a-x)^{-3} = 2!\,(a-x)^{-3}$$

Continuing the process up to n steps then,

$$y_n = n!\,(a-x)^{-(n+1)} = \frac{n!}{(a-x)^{n+1}}$$

(iii) $y = x^{2n}$

Solution: Let, $y = x^{2n}$

Differentiating, we get

$$y_1 = 2n\,x^{2n-1}$$

$$y_2 = 2n(2n-1)\,x^{2n-2}$$

$$y_3 = 2n(2n-1)(2n-2)\,x^{2n-3}$$

… … …

$$y_n = 2n(2n-1)(2n-2) \ldots \ldots (2n-(n-1))\,x^{2n-n}$$

$$= 2n(2n-1)(2n-2) \ldots (n+1)\,x^n$$

Multiplying numerator and denominator by $n(n-1)(n-2) \ldots 2.1$ we get

$$y_n = 2n(2n-1)(2n-2) \ldots (n+1)\,\frac{n(n-1)(n.2) \ldots 2.1}{n(n-1)(n-2) \ldots 2.1}\,x^n$$

Separating even and odd brackets

$$= \frac{\{2n[(2n-2)(2n-4) \ldots 6.4.2][(2n-1)(2n-3)(2n-5) \ldots 3.1]\}}{n!}\,x^n$$

$$= \frac{2^n\{n(n-1)(n-2) \ldots 3.2.1\}\{(2n-1)(2n-3)(2n-5) \ldots 3.1\}}{n!}\,x^n$$

$$= 2^n . \frac{n!}{n!}(2n-1)(2n-3)(2n-5) \ldots 3.1\,x^n$$

Thus, $y_n = 2^n\{1.3.5. \ldots (2n-1)\}\,x^n$

(iv) $y = \sqrt{x}$

Solution: Let, $y = \sqrt{x} = x^{1/2}$

Differentiating w.r.t. x, we get

$$y_1 = \frac{1}{2}x^{-(1/2)}$$

$$y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-(1/2)-1} = \frac{1}{2^2}(-1)(1)\,x^{-(1/2)-1}$$

$$y_3 = \frac{1}{2^2}(-1)(1)\left(-\frac{1}{2}-1\right)x^{-(1/2)-2} = \frac{1}{2^3}(-1)^2(1.3)\,x^{-(1/2)-2}$$

$$y_4 = \frac{1}{2^3}(-1)^2(1.3)\left(-\frac{1}{2}-2\right)x^{-(1/2)-3} = \frac{1}{2^4}(-1)^3(1.3.5)\,x^{-(1/2)-3}$$

… …

$$y_n = \frac{1}{2^n}(-1)^{n-1}[1.3.5 \ldots (2n-3)]\,x^{-(1/2)-(n-1)}$$

$$= (-1)^{n-1}\left(\frac{1.3.5 \ldots (2n-3)}{2^n x^{n-1/2}}\right)$$

(v) $y = \frac{1}{\sqrt{x}} = (x)^{-1/2}$

Solution: Let, $y = \frac{1}{\sqrt{x}} = (x)^{-(1/2)}$

Differentiating w.r.t. x, we get

$$y_1 = \frac{-1}{2}x^{-(1/2)-1} = (-1)\frac{1}{2}x^{-(1/2)-1}.$$

$$y_2 = (-1)\frac{1}{2}\left(-\frac{1}{2}-1\right)x^{-(1/2)-2} = \frac{1}{2^2}(-1)^2(1.3)\,x^{-(1/2)-2}$$

$$y_3 = \frac{1}{2^2}(-1)^2(1.3)\left(-\frac{1}{2}-2\right)x^{-(1/2)-3} = \frac{1}{2^3}(-1)^3(1.3.5)\,x^{-(1/2)-3}$$

$$y_4 = \frac{1}{2^3}(-1)^3(1.3.5)\left(-\frac{1}{2}-3\right)x^{-(1/2)-4} = \frac{1}{2^4}(-1)^4(1.3.5.7)\,x^{-(1/2)-4}$$

… …

$$y_n = \frac{1}{2^n}(-1)^n[1.3.5 \ldots (2n-1)]\,x^{-(1/2)-n}$$

$$= (-1)^n\left(\frac{1.3.5 \ldots (2n-1)}{2^n x^{n+1/2}}\right)$$

(vi) $y = 10^{3-2x}$

Solution: Let,

$$y = 10^{3-2x} = (10^3)10^{-2x} = (10^3)\,e^{-2x \log 10} \qquad [\because a^x = e^{x \log a}]$$

$$\Rightarrow \quad y = 10^3 e^{kx} \quad \ldots\ldots (i) \qquad \text{where, } -2\log 10 = k$$

Differentiating both sides of (i) then

$$y_1 = (10^3)\,k\,e^{kx}$$

$$y_2 = (10^3)\,k^2 e^{kx}$$

… … …

$$y_n = (10^3)\,k^n e^{kx}$$

$$\Rightarrow \quad y_n = (10^3)(-2\log 10)^n e^{(-2\log 10)x}$$

$$= (10^3)(-2)^n(\log 10)^n\,10^{-2x} \qquad (\because 10^{-2x} = e^{-2x \log 10})$$

$$= 10^{3-2x}(-2)^n(\log 10)^n$$

(viii) $y = \frac{x^n}{x-1}$

Solution: Since,

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + 1)$$

Here, $y = \frac{x^n}{x-1} = \frac{(x^n-1)+1}{x-1} = \frac{(x-1)(x^{n-1}+x^{n-2}+\ldots+1)+1}{(x-1)}$

$$\Rightarrow y = (x^{n-1} + x^{n-2} + \ldots + 1) + \frac{1}{x-1} \quad \ldots (i)$$

Differentiating w. r. t. x upto n steps

$$y_n = 0 + D^n\left(\frac{1}{x-1}\right) \qquad [\because D^m(x^n) = 0 \text{ if } m > n]$$

$$= D^n\left(\frac{1}{x-1}\right) = \frac{(-1)^n n!}{(x-1)^{n+1}} \qquad \left[\because D_n\left(\frac{1}{x+1}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$

(ix) $y = e^x \sin x . \sin 2x$

Solution: We know,

$$y = \frac{1}{2}e^x[2\sin 2x . \sin x] = \frac{1}{2}e^x[2\cos(2x-x) - \cos(2x+x)]$$

So, $y = \frac{1}{2}e^x \cos x - \frac{1}{2}e^x \cos 3x \quad \ldots (i)$

Differentiating both sides w.r. t x, upto n steps

$$y_n = \frac{1}{2}D^n(e^x \cos x) - \frac{1}{2}D^n(e^x \cos 3x).$$

Applying $D^n(e^{ax}\cos bx) = (a^2+b^2)^{n/2} e^{ax} \cos(bx + n\tan^{-1}\frac{b}{a})$

Then

$$y_n = \frac{1}{2}(1^2+1^2)^{n/2} e^x \cos(x + n\tan^{-1} 1)$$

$$-\frac{1}{2}.(1^2+3^2)^{n/2} e^x \cos\left(3x + n\tan^{-1}\frac{3}{1}\right)$$

$$= \frac{e^x}{2}[2^{n/2}\cos(x + n\tan^{-1}1) - 10^{n/2}\cos(3x + n\tan^{-1}3)]$$

$$\Rightarrow y_n = \frac{e^x}{2}\left[2^{n/2}\cos\left(x + \frac{n\pi}{4}\right) - 10^{n/2}\cos(3x + n\tan^{-1}3)\right]$$

(x) $y = e^{3x}\sin 4x \quad \ldots (i)$

Solution: We have, $y = e^{3x}\sin 4x \quad \ldots (i)$

Applying $D^n(e^{ax}\sin bx) = (a^2+b^2)^{n/2} e^{ax}\sin\left(x + n\tan^{-1}\frac{b}{a}\right)$

Then, differentiating upto n steps in (i), we get

$$y_n = D^n(e^{3x}\sin 4x)$$

$$= (3^2+4^2)^{n/2} e^{3x}\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

$$= e^{3x}(25)^{n/2}\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

$$\therefore \quad y_n = e^{3x} 5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

2. Find the n[th] derivatives of the following functions:

(i) $y = \frac{1}{x^2+16}$

Solution: Let,

$$y = \frac{1}{x^2+16} = \frac{1}{x^2-(4i)^2} = \frac{1}{(x-4i)(x+4i)}$$

$$y = \frac{1}{8i}\left[\frac{1}{x-4i} - \frac{1}{x+4i}\right] \quad \ldots\ldots (i)$$

Differentiating both sides w.r. t x upto n steps

$$y_n = \frac{1}{8i}\left[D^n\left(\frac{1}{x-4i}\right) - D^n\left(\frac{1}{x+4i}\right)\right]$$

Applying $D^n\left(\frac{1}{x+a}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}}$ Then,

$$y_n = \frac{1}{8i}\left[\frac{(-1)^n n!}{(x-4i)^{n+1}} - \frac{(-1)^n n!}{(x+4i)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{8i}[(x-4i)^{-(n+1)} - (x+4i)^{-(n+1)}]$$

Putting $x = r\cos\theta$, $4 = r\sin\theta$. So that $r = \sqrt{x^2+4^2}$, $\theta = \tan^{-1}\left(\frac{4}{x}\right)$ then,

$$y_n = \frac{(-1)^n n!}{8i . r^{n+1}}[r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)} - r^{-(n+1)}(\cos\theta + i\sin\theta)^{-(n+1)}]$$

Applying De Moivre's Theorem, we get,

$$y_n = \frac{(-1)^n n!}{8i . r^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta]$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{(-1)^n n!}{8i \times r^{n+1}} . 2i\sin(n+1)\theta$$

$$\therefore \quad y_n = \frac{(-i)^n n!}{4.(4/\sin\theta)^{n+1}}\sin(n+1)\theta.$$

$$y_n = \frac{(-1)^n n! \sin^{n+1}\theta \sin(n+1)\theta}{4^{n+2}} \quad \text{where } \theta = \tan^{-1}\left(\frac{4}{x}\right)$$

(ii) $y = \frac{x}{x^2+a^2}$

Solution: Let,

$$y = \frac{x}{x^2+a^2} = \frac{x}{(x-ai)(x+ai)} = \frac{1}{2}\left[\frac{1}{x+ai}+\frac{1}{x-ai}\right]$$

Differentiating w.r.t. x up to n steps, we get

$$y_n = \frac{1}{2}\left[d^n\left(\frac{1}{x+ai}\right)+D^n\left(\frac{1}{x-ai}\right)\right]$$

$$=\frac{1}{2}\left[\frac{(-1)^n n!}{(x+ai)^{n+1}}+\frac{(-1)^n n!}{(x-ai)^{n+1}}\right]$$

$$=\frac{1}{2}(-1)^n n!\,[(x+a)^{-(n+1)} = (x-ai)^{-(n+1)}]$$

Putting $x = r\cos\theta$, $a = r\sin\theta$ then $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

So, $$y_n = \frac{(-1)^n n!}{2}[(r\cos\theta = ir\sin\theta)^{-(n+1)} + (r\cos\theta - ir\sin\theta)^{-(n+1)}]$$

$$=\frac{(-1)^n n!}{2}r^{-(n+1)}[(\cos\theta+i\sin\theta)^{-(n+1)}+(\cos\theta-i\sin\theta)^{-(n+1)}]$$

$$=\frac{(-1)^n n!}{2r^{n+1}}[\cos(n+1)\theta - i\sin(n+1)\theta + \cos(n+1)\theta + i\sin(n+1)\theta]$$

$$=\frac{(-1)^n n!}{2r^{n+1}}2\cos(n+1)\theta$$

Since, $r = \frac{a}{\sin\theta}$ therefore,

$$y_n = (-1)^n n! \times \frac{\sin^{n+1}\theta}{a^{n+1}}\cos(n+1)\theta \quad ,\text{where } \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\ y_n = \frac{(-1)^n n!}{a^{n+1}}\sin^{n+1}\theta\cos(n+1)\theta.$$

(iii) $y = \frac{1}{x^2+x+1}$

Solution: We know,

$$y = \frac{1}{x^2+x+1} = \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad \text{...... (i)}$$

Applying, $D^n\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

Differentiating w.r.t. x, upto n steps in (i) we get,

$$y_n = D^n\left(\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$=\frac{(-1)^n n!}{\left(\frac{\sqrt{3}}{2}\right)^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta, \text{ where } \theta = \tan^{-1}\left(\frac{(\sqrt{3}/2)}{(1/2)+x}\right)$$

$$\therefore\ y_n = \frac{(-1)^n 2^{n+2} n!}{(\sqrt{3}^{n+2})}\sin^{n+1}\theta\sin(n+1)\theta \text{ ,where } \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)$$

(iv) $y = \frac{1}{(x^2+a^2)(x^2+b^2)}$

Solution: We have,

$$y = \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right]$$

Differentiating upto n steps, we get

$$y_n = \frac{1}{a^2-b^2}\left[D^n\left(\frac{1}{x^2+b^2}\right) - D^n\left(\frac{1}{x^2+a^2}\right)\right]$$

Now applying $D^n\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$ then,

$$y_n = \frac{(-1)^n n!}{a^{n+2}}\left[\frac{(-1)^n n!}{b^{n+2}}\sin^{n+1}\theta_1\sin(n+1)\theta_1 - \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta_2\sin(n+1)\theta_2\right]$$

where, $\theta_1 = \tan^{-1}\left(\frac{b}{x}\right)$, $\theta_2 = \tan^{-1}\left(\frac{a}{x}\right)$

Hence, $$y_n = \left[\frac{\sin^{n+1}\theta_1\sin(n+1)\theta_1}{b^{n+2}} - \frac{\sin^{n+1}\theta_2\sin(n+1)\theta_2}{a^{n+2}}\right]$$

where $\theta_1 = \tan^{-1}\left(\frac{b}{x}\right)$, $\theta_2 = \tan^{-1}\left(\frac{a}{x}\right)$

(v) $y = \cot^{-1}\left(\frac{x}{a}\right)$

$y = \cot^{-1}(x/a)$

$y_1 = -1$

Solution: We have, $y = \cot^{-1}\left(\frac{x}{a}\right)$ ... (i)

Differentiating both sides w. r. t. x, we get

$$y_1 = -\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a}$$

$$= -\frac{a}{x^2+a^2} = -\frac{a}{(x-ia)(x+ia)} = -\frac{1}{2i}\left[\frac{1}{x+ai} - \frac{1}{x-ai}\right]$$

$$\Rightarrow y_1 = \frac{1}{2i}\left[\frac{1}{x+ai} - \frac{1}{x-ai}\right]$$

Differentiating both sides upto (n – 1) steps,

$$y_n = \frac{1}{2i}\left[D^{n-1}\left(\frac{1}{x+ai}\right) - D^{n-1}\left(\frac{1}{x-ai}\right)\right] \quad \text{....... (ii)}$$

Using, $D^n\left(\frac{1}{x+a}\right) = \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}$ in (i) we get,

$$\Rightarrow D^{n-1}\left(\frac{1}{x+a}\right) = \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n} \text{ etc.}$$

Hence from eq^n (i), we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n} - \frac{(-1)^n(n-1)!}{(x-ai)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}[(x+ai)^{-n} - (x-ia)^{-n}]$$

Put $x = r\cos\theta$, $a = r\sin\theta$ then $\tan\theta = \frac{a}{x}$.

So, $y_n = \frac{(-1)^{n-1}(n-1)!}{2i}[r^{-n}(\cos\theta + i\sin\theta)^{-n} - r^{-n}(\cos\theta - i\sin\theta)^{-n}]$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}[\cos n\theta - i\sin n\theta - \cos n\theta - i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!\, r^{-n}}{2i}(-2i\sin n\theta)$$

Since, $r = \frac{a}{\sin\theta}$ then, $r^{-n} = \left(\frac{a}{\sin\theta}\right)^{-n} = \frac{\sin^n\theta}{a^n}$

So, $y_n = (-1)^n(n-1)!\frac{\sin^n\theta}{a^n}.\sin n\theta$

$$= \frac{(-1)^n(n-1)!}{a^n}\sin^n\theta\sin n\theta \quad \text{,where } \theta = \tan^{-1}\left(\frac{a}{x}\right)$$

3. Find y_n if

(i) $y = x^n e^x$

Solution: Let, $y = x^n e^x$

Applying Leibnitz's rule for higher derivative,

$$y_n = e^x x^n + {}^nC_1 e^x nx^{n-1} + {}^nC_2 e^x n(n-1)x^{n-2} + \dots + {}^nC_n e^x n!$$

$$= e^x x^n + n e^x nx^{n-1} + \left(\frac{n(n-1)}{2!}\right)e^x n(n-1)x^{n-2} + \dots + e^x n!$$

$$= e^x\left[x^n + \frac{(n)^2}{1!}x^{n-1} + \left(\frac{n^2(n-1)^2}{2!}\right)x^{n-2} + \dots + n!\right]$$

Thus, $y_n = e^x\left[x^n + \frac{(n)^2}{1!}x^{n-1} + \left(\frac{n^2(n-1)^2}{2!}\right)x^{n-2} + \dots + n!\right]$.

(ii) $y = e^x \log x$

Solution: Let, $y = e^x\log(x)$

Applying Leibnitz's rule for higher derivative,

$$y_n = D^n(e^x\log x)$$

$$= D^n(e^x)\log x + {}^nC_1 D^{n-1}(e^x).D(\log x) + {}^nc_2 D^{n-2}(e^x)D^2(\log x) + {}^nC_3 D^{n-3}(e^x).D^3(\log x) + \dots + e^x.D^n(\log x)$$

$$= e^x\log x + {}^nC_1 e^x.\frac{1}{x} + {}^nC_2 e^x\left(\frac{-1}{x^2}\right) + {}^nC_3 e^x\left(\frac{2}{x^3}\right) + \dots + e^x.\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$\left(\text{Since, } D^n(\log x)] = \frac{(-1)^{n-1}(n-1)!}{x^n}\right)$$

$$= e^x\left[\log x + \frac{{}^nC_1}{x} - \frac{{}^nC_2}{x^2} + \frac{{}^nC_3}{x^3}2! + \dots + \dots\frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

(iii) $y = e^{ax}\cos bx$

Solution: Let, $y = e^{ax}\cos bx$

Applying Leibnitz's rule with taking $u = e^{ax}$, $v = \cos bx$, we get

$$y_n = D^n(e^{ax}\cos bx)$$

$$= D^n(e^{ax}).\cos bx + {}^nC_1 D^{n-1}(e^{ax}).D(\cos bx) + {}^nC_2 D^{n-2}(e^{ax}).D^2(\cos bx) + \dots + e^{ax}D^n(\cos bx)$$

$$= a^n e^{ax}\cos bx + {}^nC_1 a^{n-1}e^{ax}.b\cos\left(bx + \frac{n\pi}{2}\right) + {}^nC_2 a^{n-2}e^{ax}b^2\cos\left(bx + \frac{2\pi}{2}\right) + \dots + e^{ax}b^n\cos\left(bx + \frac{n\pi}{2}\right)$$

$$= e^{ax}\left[a^n\cos bx + {}^nC_1 a^{n-1}b\cos\left(bx + \frac{n\pi}{2}\right) + {}^nC_2 a^{n-2}b^2\cos\left(bx + \frac{2\pi}{2}\right) + \dots + b^n\cos\left(bx + \frac{n\pi}{2}\right)\right]$$

4. If $y = \sin mx + \cos mx$, show that $y_n = m^n\{1 + (-1)^n\sin 2mx\}^{1/2}$

Solution: Let, $y = \sin mx + \cos mx$

Differentiating upto n times we get,

$$y_n = D^n(\sin mx) + D^n[\cos mx]$$

$$\Rightarrow y_n = m^n\sin\left(mx + \frac{n\pi}{2}\right) + m^n\cos\left(mx + \frac{n\pi}{2}\right)$$

Let us put $\theta = mx + \frac{n\pi}{2}$ then,

$$y_n = m^n(\sin\theta + \cos\theta) = m^n[(\sin\theta + \cos\theta)^2]^{1/2}$$

$$= m^n(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta]^{1/2}$$

$$= m^n[1 + \sin 2\theta]^{1/2}$$

Since, $2\theta = 2mx + n\pi$ then,

$$y_n = m^n[1 + \sin(2mx + n\pi)]^{1/2}$$

We know that, $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ then,

$$y_n = m^n[1 + \sin 2mx.(-1)^n + \cos 2mx.0]^{1/2}$$

$$y_n = m^n[1 + (-1)^n\sin 2mx]^{1/2}.$$

5. If $y = x^{n-1}\log(x)$, show that $ny_n = (n-1)!$.

Solution: Let, $y = x^{n-1}\log(x)$ ..... (i)

Differentiating y w. r. t. x, we get

$$y_1 = (n-1)x^{n-2}\log(x) + x^{n-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow xy_1 = (n-1)x^{n-1}\log(x) + x^{n-1}$$

$$= (n-1)y + x^{n-1}$$

Applying Leibnitz's rule for higher differentiation then

$$xy_n + {}^{n-1}C_1(1)(y_{n-1}) = (n-1)(y_{n-1}) + (n-1)(n-2)\ldots(2)(1)$$

$$\Rightarrow xy_n + (n-1)(y_{n-1}) = (n-1)(y_{n-1}) + (n-1)!$$

$$\Rightarrow xy_n = (n-1)!$$

6. If $y = e^{x^2}$, show that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$. [2013 Fall]

Solution: Let, $y = e^{x^2}$ ….. (i)

Differentiating w.r.t x, we get

$$y_1 = e^{x^2}\,2x$$

$$\Rightarrow y_1 = y\,2x = 2xy \quad \ldots\text{(ii)}$$

Applying Leibnitz's rule for higher differentiation then

$$y_{n+1} = 2[(x)(y_n) + {}^nC_1(1)(y_{n-1})]$$

$$y_{n+1} - 2xy_n - 2ny_{n-1} = 0 \qquad [\because {}^nC_1 = n]$$

7. If $y = e^{ax}\sin bx$, show that (i) $y_2 - 2ay_1 + (a^2+b^2)y = 0$

(ii) $y_{n+1} = 2ay_n - (a^2+b^2)y_{n-1}$

Solution: Let, $y = e^{ax}\sin bx$ … (i)

Differentiating both sides w.r.t. x, we get

$$y_1 = a\,e^{ax}\sin bx + e^{ax}\,b\cos bx$$

$$\Rightarrow y_1 = ay + be^{ax}\cos bx \quad \ldots\text{(ii)}$$

Again, differentiating w.r.t. x then,

$$y_2 = ay_1 + b[a\,e^{ax}\cos bx - e^{ax}\,b\sin bx]$$

$$= ay_1 + a[be^{ax}\cos bx) - b^2(e^{ax}\sin bx)$$

Since from (ii), $be^{ax}\cos bx = y_1 - ay$ therefore,

$$y_2 = ay_1 + a(y_1 - ay) - b^2y$$

$$\Rightarrow y_2 = ay_1 - a^2y - b^2y$$

$$\Rightarrow y_2 - 2ay_1 + (a^2+b^2)y = 0 \quad \ldots\text{(iii)}$$

This proves first part.

For second part:

Applying Leibnitz's rule for higher differentiation to (iii) then

$$y_{n+1} - 2ay_n + (a^2+b^2)y_{n-1} = 0$$

$$\Rightarrow y_{n+1} = 2ay_n - (a^2+b^2)y_{n-1}.$$

8. If $y = \log\left(x + \sqrt{a^2+x^2}\right)$, show that

(i) $(a^2+x^2)y_2 + xy_1 = 0$

(ii) $(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$. [2011 Fall]

Solution: Let $y = \log\left(x + \sqrt{a^2+x^2}\right)$

Differentiating we get,

$$y_1 = \left(\frac{1}{x+\sqrt{a^2+x^2}}\right)\left(1 + \frac{1}{2\sqrt{a^2+x^2}}(2x)\right)$$

$$= \left(\frac{1}{x+\sqrt{a^2+x^2}}\right)\left(\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}\right)$$

$$= \frac{1}{\sqrt{a^2+x^2}}$$

$$\Rightarrow \left(\sqrt{a^2+x^2}\right)y_1 = 1$$

$\frac{1}{2\sqrt{x}}$

Again, differentiating we get,

$$\left(\sqrt{a^2+x^2}\right)y_2 + y_1\left(\frac{x}{\sqrt{a^2+x^2}}\right) = 0$$

$$\Rightarrow (a^2+x^2)y_2 + xy_1 = 0$$

This completes the proof of (i)

For second part:

Applying Leibnitz's rule for higher differentiation then

$$(a^2+x^2)y_{n+2} + {}^nC_1(2x)\,y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1y_n = 0$$

$$\Rightarrow (a^2+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)\,y_n + xy_{n+1} + ny_n = 0$$

$$\Rightarrow (a^2+x^2)y_{n+2} + (2n+1)\,xy_{n+1} + n^2y_n = 0$$

This completes the proof of (ii)

$c(n,0) = \frac{n!}{0!}$

9. Find $y_n(0)$, if $y = e^{a\sin^{-1}x}$

Solution: Let $y = e^{a\sin^{-1}x}$ then, $y(0) = e^0 = 1$

Differentiating w. r. t. x then,

$$y_1 = e^{a\sin^{-1}x}\left(\frac{a}{\sqrt{1-x^2}}\right) = \frac{ay}{\sqrt{1-x^2}} \quad \ldots\text{(i)}$$

$\frac{1}{2\sqrt{n}}$

$$\Rightarrow (1-x^2)\,y_1^2 - a^2y^2 = 0.$$

Again differentiating w. r. t. x then, we get

$$(1-x^2)\,2y_1y_2 + y_1^2(-2x) - 2a^2\,yy_1 = 0$$

$$\Rightarrow (1-x^2)\,y_2 - xy_1 - a^2y = 0 \quad \ldots\text{(ii)}$$

$\frac{1}{2}$

Putting x = 0 in (i) $\quad y_1(0) = ay(0) = a$

Putting x = 0 in (ii) $\quad y_2(0) - 0 - a^2y(0) = 0$

$$\Rightarrow y_2(0) - a^2 = 0$$

$$\Rightarrow y_2(0) = a^2$$

Applying Leibnitz's Rule for higher differentiation, we get

$$(1-x^2)\,y_{n+2} + {}^nc_1\,y_{n+1}(-2x) + {}^nc_2y_n(-2) - [xy_{n+1} + {}^nc_1\,y_{n-1}] - a^2y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - 2nxy_{n+1} - n(n-1)\,y_n - xy^{n+1} - ny_n - a^2y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)xy_{n+1} - [n(n-1) + n + a^2]\,y_n = 0 \quad \ldots\text{(iii)}$$

Putting x = 0 in (iii) we get,

$$y_{n+2}(0) - (2n+1)\,.\,0 - (n^2+a^2)\,y_n(0) = 0$$

$$\Rightarrow y_{n+2}(0) = (n^2+a^2)\,y_n(0) \quad \ldots\text{(iv)}$$

Also, we have $y_1(0) = a$, $y_2(0) = a^2$

Putting n = 2, 4, 6, … in (iv) we get,

$$y_4(0) = (2^2+a^2)\,.\,y_2(0) = (2^2+a^2)\,a^2$$

$$y_6(0) = (4^2 + a^2) \cdot y_4(0) = (4^2 + a^2)(2^2 + a^2)a^2$$

$$y_8(0) = (6^2 + a^2) \cdot y_6(0) = (6^2 + a^2)(4^2 + a^2)(2^2 + a^2)a^2$$

... ...

$$\Rightarrow y_n(0) = [(n-2)^2 + a^2][(n-4)^2 + a^2] \ldots (4^2 + a^2)(2^2 + a^2)a^2$$

if n is even .... (*)

Again putting $n = 1, 3, 5 \ldots$ in (iv) we get

$$y_3(0) = (1^2 + a^2)a$$

$$y_5(0) = (3^2 + a^2)y_3(0) = (3^2 + a^2)(1^2 + a^2)a$$

$$y_7(0) = (5^2 + a^2)y_5(0) = (5^2 + a^2)(3^2 + a^2)(1^2 + a^2) \cdot a$$

... ...

$$y_n(0) = [(n-2)^2 + a^2][(n-4)^2 + a^2] \ldots [3^2 + a^2][1^2 + a^2]a.$$

if n is odd .... (**)

From (*) and (**), we get

$$y_n(0) = \begin{cases} [(n-2)^2 + a^2][(n-4)^2 + a^2] \ldots (4^2 + a^2)(2^2 + a^2)a^2 & \text{if n is even} \\ [(n-2)^2 + a^2][(n-4)^2] + a^2] \ldots (3^2 + a^2)(1^2 + a^2)a & \text{if n is odd} \end{cases}$$

10. If $y = (x + \sqrt{(1+x)^2})^m$, show that

(i) $(1 + x^2)y_2 + xy_1 - m^2y = 0$

(ii) $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

Solution: Let $y = \left(x + \sqrt{(1+x)^2}\right)^m$

Differentiating w.r.t. x, gives

$$y_1 = m\left(x + \sqrt{(1+x)^2}\right)^{m-1}\left(1 + \left(\frac{1}{2}\right)\frac{2x}{\sqrt{1+x^2}}\right)$$

$$= m\left(x + \sqrt{(1+x)^2}\right)^{m-1}\left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow y_1 = \frac{m\left(x + \sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$

$$\Rightarrow (1 + x^2)y_1^2 - m^2y^2 = 0$$

Again differentiating w.r.t. x, we get

$$(1 + x^2)2y_1y_2 + y_1^2(2x) - m^2(2y) - y_1' = 0$$

$$\Rightarrow (1 + x^2)y_2 + xy_1 - m^2y = 0 \quad \ldots\ldots (i)$$

This proves the first part

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1 + x^2)y_{n+2} + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2) + xy_{n+1} + {}^nC_1 y_{n.1} - m^2y_n = 0$$

$$\Rightarrow (1 + x^2)y_{n+2} + 2nxy_{n+1} + \left(\frac{n(n-1)}{2}\right)2y_n + xy_{n+1} + ny_n - m^2y_n = 0$$

$$\Rightarrow (1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + [n(n-1) + n - m^2]y_n = 0$$

$$\Rightarrow (1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0.$$

This completes the solution.

11. If $y^{1/m} + y^{-1/m} = 2x$, show that [2003, Fall]

$$(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

Solution: Let, $y^{1/m} + y^{-1/m} = 2x$ ... (i)

$\left(\frac{1}{m}\right) y^{\frac{1}{m}-1} \cdot y$

Differentiating (i) w.r.t. x then

$$\left(\frac{1}{m}\right)y^{1/m-1}y_1 - \left(\frac{1}{m}\right)y^{-(1/m)-1}y_1 = 2$$

$$\Rightarrow \left(\frac{y_1}{my}\right)(y^{1/m} - y^{-1/m}) = 2$$

$$\Rightarrow y_1(y^{1/m} - y^{-1/m}) = 2my$$

$$\Rightarrow y_1^2(y^{1/m} - y^{-1/m})^2 = 4m^2y^2$$

$$\Rightarrow y_1^2[(y^{1/m} + y^{-1/m})^2 - 4y^{1/m}y^{-1/m}] = 4m^2y^2$$

$$[\because (a-b)^2 = (a+b)^2 - 4ab]$$

Since, by (i), $y^{1/m} + y^{-1/m} = 2x$. So,

$$y_1^2[(2x)^2 - 4] = 4m^2y^2$$

$1 + \frac{1}{\sqrt{1+x^2}}$

$$\Rightarrow (x^2 - 1)4y_1^2 - 4m^2y^2 = 0$$

Again diff. w.r.t. x, we get

$$(x^2 - 1)2y_1y_2 + 2x \cdot y_1^2 - m^2 \cdot 2y \cdot y_1 = 0 \quad [\because 4 \neq 0.]$$

$$\Rightarrow 2y_1[(x^2 - 1)y_2 + xy_1 - m^2y] = 0$$

$\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}$

Since $2y_1 \neq 0$. So,

$$(x^2 - 1)y_2 + xy_1 - m^2y = 0 \quad \ldots (ii)$$

Applying Leibnitz's Rule for higher differentiation, we get

$$(x^2 - 1)y_{n+2} + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2) + xy_{n+1} + {}^nC_1 y_n(1) - m^2y_n = 0$$

$$\Rightarrow (x^2 - 1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2y_n = 0$$

$$\Rightarrow (x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + [n(n-1) + n - m^2]y_n = 0$$

$$\Rightarrow (x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0.$$

This completes the solution.

12. If $y = (\sin^{-1}x)^2$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$ and hence show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$.

[2007, Spring][2005, Spring]

Solution: We have, $y = (\sin^{-1}x)^2$

Differentiating w. r. t. x then, $y_1 = 2\sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right)$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = 2\sin^{-1}x$$

$$\Rightarrow (1 - x^2)y_1^2 - 4(\sin^{-1}x)^2 = 0$$

$$\Rightarrow (1 - x^2)y_1^2 - 4y = 0$$

Again differentiating w.r.t. x, we get

$$(1 - x^2) \cdot 2y_1y_2 + y_1^2 \cdot (-2x) - 4y_1 = 0$$

$$\Rightarrow 2y_1[(1 - x^2)y_2 - xy_1 - 2] = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - 2 = 0.$$

This proves the first part.

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - [xy_{n+1} + {}^nC_1 y_n(1)] - 0 = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \left(\frac{n(n-1)}{2}\right)2y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - [n^2 - n + n)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$$

13. If $y = \sin(m\sin^{-1}x)$, show that

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Proof: Let, $y = \sin(m\sin^{-1}x)$

Differentiating w.r.t. x, we get

$$y_1 = \cos(m\sin^{-1}x)(m)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2[1-\sin^2(m\sin^{-1}x)] = m^2[1-y^2]$$

$$\Rightarrow y_1^2(1-x^2) - m^2 + m^2y^2 = 0.$$

Again differentiating w.r.t. x, we get

$$(1-x^2)2y_1y_2 + y_1^2(-2x) - 0 + m^2 . 2yy_1 = 0$$

$$\Rightarrow 2y_1[(1-x^2)y_2 - xy_1 + m^2y] = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0 \quad \text{.... (i)}$$

This proves the first part.

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1-x^2)y_{n+2} + {}^nC_1y_{n+1}(-2x) + {}^nC_2 y_n(-2) - [xy_{n+1} + {}^nC_1 y_n] + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \left(\frac{n(n-1)}{2}\right)2y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - [n(n-1) + n - m^2]y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

This completes the solution.

14. If $\log(y) = \tan^{-1}x$, show that **[2011 Spring]**

(i) $(1+x^2)y_2 + (2x-1)y_1 = 0$

(ii) $(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$

Solution: We have, $\log(y) = \tan^{-1}x$

$$\Rightarrow y = e^{\tan^{-1}x} \quad \text{... (i)}$$

Differentiating w. r. t. x, then,

$$y_1 = e^{\tan^{-1}x}\left(\frac{1}{1+x^2}\right) = \frac{y}{1+x^2} \quad \{\text{by (i)}\}$$

$$\Rightarrow y_1(1+x^2) - y = 0$$

Again differentiating w.r.t. x, gives

$$y_2(1+x^2) + y_1(2x) - y_1 = 0$$

$$\Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0 \quad \text{.... (ii)}$$

This proves the first part.

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1+x^2)y_{n+2} + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2) + (2x-1)y_{n+1} + {}^nC_1 y_n(2) = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + \left(\frac{n(n-1)}{2}\right)2y_n + (2x-1)y_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + [n(n-1)+2n]y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0.$$

NOTE: Here, log is noted for ln (i.e. natural log, log = ln = $\log_e$).

15. If $y = e^{a\tan^{-1}x}$ and show that

(i) $(1+x^2)y_2 + (2x-a)y_1 = 0.$

(ii) $(1+x^2)y_{n+2} + (2nx+2x-a)y_{n+1} + n(n+1)y_n = 0.$

Solution: Let, $y = e^{a\tan^{-1}x}$

Diff. w.r.t. x then,

$$y_1 = e^{a\tan^{-1}x}\left(\frac{a}{1+x^2}\right)$$

$$\Rightarrow (1+x^2)y_1 = a e^{a\tan^{-1}x} = ay$$

Again diff. w.r.t. x then,

$$(1+x^2)y_2 + 2xy_1 = ay_1$$

$$\Rightarrow (1+x^2)y_2 + (2x-a)y_1 = 0$$

Applying Leibnitz's Rule for higher differentiation, we get

$$(1+x^2)y_{n+2} + {}^nC_1 y_{n+1}(2x) + {}^nC_2y_n(2) + (2x-a)y_{n+1} + {}^nC_1y_n(2) = 0.$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nx\,y_{n+1} + \left(\frac{n(n-1)}{2}\right)(2)y_n + (2x-a)y_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2xn+2x-a)y_{n+1} + (n(n-1)+2n)y_n = 0.$$

$$\Rightarrow (1+x^2)y_{n+2} = (2n.x+2x-a)y_{n+1} + n(n+1)y_n = 0.$$

This completes the solution.

c(n+1)

16. If $y = \tan^{-1}x$, show that **[2016 Spring][2014 Fall] [2008 Spring]**

(i) $(1+x^2)y_1 = 1.$

(ii) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0.$

Solution: Let $y = \tan^{-1}x$

Differentiating w. r. t. x then,

$$y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2)\, y_1 = 1.$$

Applying Leibnitz's rule for differentiation then,

$$(1+x^2)\, y_{n+1} + {}^nC_1\, y_n\, (2x) + {}^nC_2\, y_{n-1}\, (2) = 0$$

$$\Rightarrow (1+x^2)\, y_{n+1} + (n)(2x)\, y_n + \left(\frac{n(n-1)}{2}\right)(2)\, y_{n-1} = 0$$

$$\Rightarrow (1+x^2)\, y_{n+1} + 2nx\, y_n + n(n-1)\, y_{n-1} = 0.$$

This completes the solution.

17. If $y = \dfrac{ax^2 + bx + c}{(1-x)}$, show that $(1-x)\, y_3 = 3y_2$.

Solution: Let, $y(1-x) = ax^2 + bx + c$ ... (i)

Differentiating w.r.t. x then,

$$(1-x)\, y_1 + y(-1) = 2ax + b$$

Again, differentiating w. r. t. x then,

$$(1-x)\, y_2 - y_1 - y_1 = 2a$$

$$\Rightarrow (1-x)\, y_2 - 2y_1 = 2a$$

Again, differentiating w. r. t. x then,

$$(1-x)\, y_3 - y_2 - 2y_2 = 0$$

$$\Rightarrow (1-x)\, y_3 = 3y_2.$$

This completes the solution.

18. If $y = e^{-x} \cos x$, show that $y_4 + 4y = 0$

Solution: Let $y = e^{-x} \cos x$

Diff. w. r. t. x then,

$$y_1 = -e^{-x} \sin x - e^{-x} \cos x$$

$$\Rightarrow y_1 = -e^{-x} \sin x - y.$$

Again differentiating w.r.t. x then,

$$\begin{aligned} y_2 &= -e^{-x} \cos x + e^{-x} \sin x - y_1 \\ &= -y + e^{-x} \sin x + e^{-x} \sin x + y \\ &= 2e^{-x} \sin x. \end{aligned}$$

Again differentiating,

$$\begin{aligned} y_3 &= -2e^{-x} \sin x + 2e^{-x} \cos x \\ &= -2e^{-x} \sin x + 2y. \end{aligned}$$

Again differentiating,

$$\begin{aligned} y_4 &= -2e^{-x} \cos x + 2e^{-x} \sin x + 2y_1 \\ &= -2y + 2e^{-x} \sin x - 2e^{-x} \sin x - 2y \\ &= -4y. \end{aligned}$$

$$\Rightarrow y_4 + 4y = 0.$$

19. Show that $\dfrac{d^n}{dx^n}\left[\dfrac{\log x}{x}\right] = (-1)^n \dfrac{n!}{x^{n+1}}\left(\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \dots - \dfrac{1}{n}\right)$

Solution: Let $y = \dfrac{\log x}{x}$.

Differentiating w. r. t. x then,

$$y_1 = \frac{x\,\frac{1}{x} - \log x}{x^2} = \frac{(-1)\,.\,(\log x - 1)}{x^2}$$

Again differentiating w. r. t. x then,

$$y_2 = (-1)\left(\frac{x^2 . \frac{1}{x} - 2x\,(\log x - 1)}{x^3}\right) = \frac{(-1)^2\, 2!}{x^3}\left(\log x - 1 - \frac{1}{2}\right)$$

Again differentiating w. r. t. x then,

$$y_3 = (-1)^2\, 2!\left[\frac{x^3 . \frac{1}{x} - 3x^2\left(\log x - 1 - \frac{1}{2}\right)}{x^6}\right]$$

$$= \frac{(-1)^3 . 3!}{x^4}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3}\right).$$

Continuing the process upto n^{th} term then,

$$y_n = \frac{(-1)^n\, n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right).$$

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

1. State Leibnitz theorem. If $y = \log\left(x + \sqrt{a^2 + x^2}\right)$, show that $(a^2 + x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + n^2 y_n = 0$. **[2006, Fall]**

Solution: See statement of the theorem and Q.8 from above.

2. State Leibnitz's theorem. If $y = \left(x + \sqrt{1 + x^2}\right)^m$, show that

(i) $(1 + x^2)^2\, y_2 + xy_1 - m^2 y = 0$

(ii) $(1 + x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$ **[2007, Fall]**

Solution: See statement of the theorem and Q. 10 from above.

3. State Leibnitz theorem for successive derivative of the product of two functions. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)\, y_2 - xy_1 - 2 = 0$ and hence show that $(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$. **[2008,Fall]**

Solution: See statement of the theorem and see Q. 12 from above.

4. If $y = \sin^{-1}x$, show that [2015 Fall] [2009, Fall] [2009 Spring] [2002]

(i) $(1 - x^2)\, y_2 - xy_1 = 0$

(ii) $(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$. **[2015 Spring]**

Solution: We have, $y = \sin^{-1}x$.

Differentiating w. r. t. x, we get

$$y_1 = \frac{1}{\sqrt{1 - x^2}} \Rightarrow y_1^2 = \frac{1}{1 - x^2}$$

$\Rightarrow (1-x^2) y_1^2 = 1$ ....(1)

Differentiate w.r.t. x, we get

$$(1-x^2)\, 2y_1y_2 + y_1^2 (-2x) = 0$$

$$\Rightarrow (1-x^2)\, y_2 - xy_1 = 0 \quad \text{.... (2)}$$

Applying Leibnitz's rule for differentiation then,

$$(1-x^2)\, y_{n+2} + {}^nC_1 (-2x) y_{n+1} + {}^nC_2(-2) y_n - xy_{n+1} - {}^nC_1 y_n = 0$$

$$\Rightarrow (1-x^2)\, y_{n+2} + n(-2x)\, y_{n+1} - n(n-1)\, y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0 \quad \text{.... (iii)}$$

This completes the solution.

5. State Leibnitz theorem for successive derivative of the product of two functions. If $y = \sin^{-1}x$, show that [2017 Fall]

(i) $(1-x^2)\, y_2 - xy_1 = 0$

(ii) $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0.$

Solution: First part: See statement of Leibnitz theorem.

Second Part: See Final Exam Q. 4.

6. If $y = (x^2-1)^n$, prove that [1999] [2001] [2004 Spring]

i. $(x^2-1)\, y_2 + 2(1-n)xy_1 - 2ny = 0$ [2013 Spring]

ii. $(x^2-1) y_{n+2} + 2nxy_{n+1} - n(n+1) y_n = 0$ [2014 Spring]

Solution: Let, $y = (x^2-1)^n$

Differentiating w.r.t. x, we get

$$y_1 = n(x^2-1)^{n-1}(2x)$$

$$\Rightarrow y_1 = 2nx\,(x^2-1)^{n-1}$$

$$\Rightarrow y_1 = \frac{2nx(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow (x^2-1) y_1 = 2nxy$$

Again, differentiating w.r.t. x, we get

$$(x^2-1) y_2 + 2xy_1 = 2n(xy_1 + y)$$

$$\Rightarrow (x^2-1) y_2 + 2(n-1)xy_1 - 2ny = 0$$

Applying Leibnitz's rule for differentiation then,

$$(x^2-1)\, y_{n+2} + {}^nC_1(2x) y_{n+1} + {}^nC_2(2) y_n + 2(n-1)[xy_{n+1} + {}^nC_1(1) y_n] - 2ny_n = 0$$

$$\Rightarrow (x^2-1) y_{n+2} + 2nxy_{n+1} + \left(\frac{n(n-1)}{2}\right) 2y_n - 2(n-1)[xy_{n+1} + ny_n] - 2ny_n = 0$$

$$\Rightarrow (x^2-1) y_{n+2} + (2nx - 2nx + 2x) y_{n+1} + (n^2 - n - 2n^2 + 2n - 2n) y_n = 0$$

$$\Rightarrow (x^2-1) y_{n+2} + 2xy_{n+1} + (-n^2 - n) y_n = 0$$

$$\Rightarrow (x^2-1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0.$$

This completes the solution.

7. State Leibnitz's Theorem for highest derivatives. If $y = \sin^{-1}x$ prove that,

i. $(1-x^2)\, y_2 - xy_1 - 0$

ii. $(1-x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} - n^2 y_n = 0.$ [2000]

Solution: First Part: See Statement of Leibnitz,s theorem.

Second Part: See Q. 4, Final Exam.

8. If $y = a\cos(\log x) + b\,\text{sing}(\log x)$ prove that [2002] [2003, Spring]

(i) $x^2y_2 + xy_1 + y = 0$

(ii) $x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$ [2018 Fall][2006, Spring]

Solution: Let,

$$y = a\cos(\log x) + b\sin(\log x)$$

Differentiating w.r.t. x, we get

$$y_1 = -a\sin(\log x)\left(\frac{1}{x}\right) - b\cos(\log x)\left(\frac{1}{x}\right)$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating w.r.t. x,

$$y_1 + xy_2 = -a\cos(\log x)\left(\frac{1}{x}\right) - b\sin(\log x)\left(\frac{1}{x}\right)$$

$$\Rightarrow xy_1 + x^2y_2 = -\{a\cos(\log x) + b\sin(\log x)\}$$

$$\Rightarrow xy_1 + x^2y_2 = -y \quad \text{...(1)}$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Applying Leibnitz's rule for differentiation then,

$$x^2y_{n+2} + {}^nC_1(2x) y_{n+1} + {}^nC_2(2) y_n + xy_{n+1} + {}^nC_1(1) y_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + n(2x) y_{n+1} + \left(\frac{n(n-1)}{2}\right) 2y_n + xy_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - n + n + 1)\, y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$$

This completes the solution.

9. State Leibnitz's theorem for successive derivative of product of two functions $y = u.v.$ If $y = a\cos(\log x) + b\sin(\log x)$ show that

(i) $x^2y_2 + xy_1 + y = 0$

(ii) $x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$ [2017 Spring][2004, Fall]

Solution: First Part: See Statement of Leibnitz's theorem.

Second Part: See Q. 8, Final Exam.

10. State Leibnitz's theorem for successive derivative of the product of two functions and use it to get the n^{th} derivative of the differential equation: $(1+x^2)\, y_2 + (2x-1)\, y_1 = 0.$ [2005, Fall]

Solution: First Part: See Statement of Leibnitz's theorem.

Second Part: Given equation is,

$$(1+x^2) y_2 + (2x-1) y_1 = 0$$

Applying Leibnitz's rule for differentiation then,

$$(1+x^2) y_{n+2} + {}^nC_1(2x) y_{n+1} + {}^nC_2(2) y_n + (2x-1)\, y_{n+1} + {}^nC_1(2) y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + 2nx\, y_{n+1} + n(n-1) y_n + (2x-1)\, y_{n+1} + 2n\, y_n = 0$$

$$\Rightarrow (1+x^2)\, y_{n+2} + [2(n+1)x - 1]\, y_{n+1} + n(n+1) y_n = 0.$$

This completes the solution.

11. State Leibnitz's theorem for successive derivative of the product of two functions y = uv. If $y = \sin(m \sin^{-1} x)$, show that [2012 Fa...]

(i) $(1 - x^2) y_2 - xy_1 + m^2y = 0$

(ii) $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0$

Solution: First Part: See Statement of Leibnitz's theorem.

Second Part: See Q. No. 13 in above exercise.

12. State Leibnitz's theorem. If $y = e^{x^2}$, show that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$ [2016 Fa...]

Solution: First Part: See Statement of Leibnitz's theorem.

Second Part: See Q. No. 6 in above exercise.

SHORT QUESTION

a. Use Leibnitz's theorem to find y_n if $y = x^2e^x$. [2002] [2005, Fa...]

Solution: Let, $y = x^2e^x$

Applying Leibnitz's rule for differentiation with taking $u = e^x$, $v = x^2$ we have

$$y_n = x^2 e^x + {}^nC_1(2x)e^x + {}^nC_2 (2)e^x.$$

$$= e^xx^2 + n(2x)e^x + \left(\frac{n(n-1)}{2}\right)(2)e^x.$$

$$= e^xx^2 + 2nxe^x + (n^2 - n)e^x.$$

$$= e^x [x^2 + 2nx + n^2 - n]$$

b. If $y = \frac{1}{(x+1)}$ find y_n. [2005, Spring]

Solution: Let, $y = \frac{1}{(x+1)}$

$\Rightarrow (x + 1)y = 1$

Applying Leibnitz's rule for differentiation then

$$(x + 1)y_n + {}^nC_1 (1)y_{n-1} = 0$$

$$\Rightarrow (x + 1)y_n + ny_{n-1} = 0.$$

$$\Rightarrow y_n = \left(\frac{n}{x+1}\right) y_{n-1}$$

$$\Rightarrow y_n = ny(y_{n-1}).$$

●●●

Chapter 3

MEAN VALUE THEOREM

Continuous:

A function f(x) is continuous at a point x = a if it satisfies the condition

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

i.e. LHL = f.v. (functional value) = RHL

One–sided continuous:

A function f(x) is one sided continuous at x = a if it satisfies

(i) $\lim_{x\to a^-} f(x) = f(a)$ or, (ii) $\lim_{x\to a^+} f(x)$

Note that if (i) is satisfied then we call f(x) is continuous from left side and if (ii) is satisfied then we call f(x) is continuous from right side.

Note: A function f(x) is continuous on [a, b] means f(x) is continuous from right side at x = a and continuous from left side at x = b and f(x) is continuous on (a, b).

But there is no such functional value occurs in the case of differentiation. So, there is impossible to observe the differentiation at end point of an internal [a, b].

Extreme Value Theorem: If f(x) is continuous on [a, b] then f(x) attains its maxima and minima on [a, b].

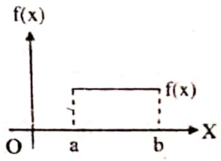
Rolle's Theorem

Statement: If $f(x)$ is continuous on a closed interval $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exists at least a point c in (a, b) such that $f'(c) = 0$.

Proof: Since $f(x)$ is continuous in closed interval $[a, b]$, then by extreme value theorem it should attain its maxima and minima in $[a, b]$. Let, M be the maximum value of f(x) in [a, b] and m be the minimum value of f(x) in [a, b].

Case I:

If M = m, then f(x) is constant in [a, b] i.e. f(x) = k for all x in [a, b] where k is some constant value. Therefore,



$f'(x) = 0$ for all x in (a, b).

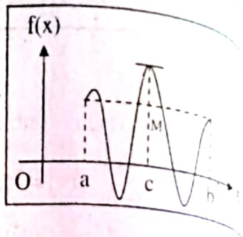
That is the theorem is true in this case.

So, at x = c,

$f'(c) = 0$ for c in (a, b).

Case II:

If $M \neq m$, then at least one of them is different from f(a) or f(b). Let $M \neq f(a)$. Then we wish to show that $f'(c) = 0$ for some c in (a, b).



Suppose that $f(c) = M$. Then

$$f(c-h) \leq f(c) \text{ and } f(c+h) \leq f(c).$$

So,

$$[f(c-h) - f(c)] \leq 0 \quad \text{and} \quad [f(c+h) - f(c)] \leq 0$$

Now, at $x = c$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(c+h) - f(c)}{h}\right), \text{ for } h \geq 0.$$

$$= \lim_{h\to 0}\left(\frac{f(c+h) - M}{h}\right) = \frac{-ve}{+ve} \leq 0 \quad \ldots \text{(i)}$$

$$\text{And, LHD} = \lim_{h\to 0}\left(\frac{f(c-h) - M}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(c-h) - M}{-h}\right) = \frac{-ve}{-ve} \geq 0 \quad \ldots \text{(ii)}$$

Since $f'(x)$ exists in (a, b), so we should have (i) and (ii) are equivalent which will be possible only when both (i) and (ii) are equal to zero.

Hence, $f'(c) = 0$ for some c in (a, b).

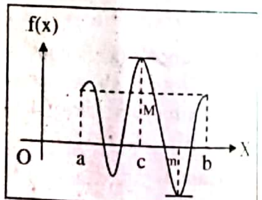
Case III

If m is different from f(a) and f(b) with $M \neq m$ and $f(c) = m$. Then as case–II, we obtain $f'(c) = 0$ for some c in (a, b).

Thus, in either case, we have $f'(c) = 0$ for some c in (a, b).

Geometrical interpretation of Rolle's theorem

If f(x) is continuous on [a, b] and differentiable on (a, b) and satisfied the condition f(a) = f(b) then it should attain some points in (a, b) at where the tangent line to the curve is parallel to x-axis.



The Rolle's theorem is useful to prove the following Mean Value Theorems for differentiation.

Lagrange's Mean Value Theorem

Statement: If a function f(x) is

(i) continuous on [a, b]

(ii) differentiable on (a, b),

then there exists at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{for } a \neq b.$$

Proof: Let us define a function

$$F(x) = f(x) + Kx \quad \ldots (1)$$

such that $F(a) = F(b)$. $\quad \ldots (2)$

Given that f(x) is continuous in [a, b] and differentiable in (a, b).

Since the linearity of two continuous (or differentiable) functions on a domain is again continuous (or differentiable) on that domain.

Since x is an algebraic linear form which is continuous and differentiable on the real line R i.e. on $(-\infty, \infty)$.

Therefore F(x) is continuous on [a, b] and is differentiable on (a, b).

Thus, F(x) satisfies all three conditions of the Rolle's Theorem then Rolle's theorem there exists at least one point $c \in (a, b)$ such that

$$F'(c) = 0 \quad \ldots (3)$$

Since, by (2) we have $F(a) = F(b)$.

$$\Rightarrow f(a) + Ka = f(b) + Kb.$$

$$\Rightarrow K = \frac{f(b) - f(a)}{a - b}, \text{for } a \neq b.$$

By (3) we have $F'(c) = 0$.

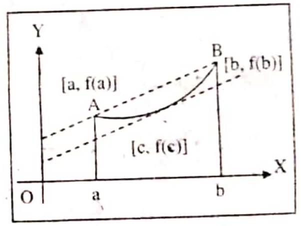
$$\Rightarrow f'(c) + K = 0.$$

$$\Rightarrow f'(c) = -K \Rightarrow f'(c) = -\frac{f(b) - f(a)}{a - b}, \text{for } a \neq b.$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}, \text{for } a \neq b.$$

Geometrical interpretation:

If f(x) is continuous on [a, b] and differentiable on (a, b) then there exists at least one point c in between (a, b) such that the tangent to the curve at (c, f(c)) is parallel to the chord joining the points (a, f(a)) and (b, f(b)).



Corollaries of Lagrange's Mean Value Theorem

Cor. (I) If $f'(x) > 0$ in [a, b] then f(x) is increasing on [a, b].

Proof:

Take x_1, x_2 in [a, b] with $x_1 < x_2$. Let the function f(x) is continuous in [a, b] and is differentiable in (a, b).

Thus f(x) satisfies both conditions of the Lagrange's Mean Value Theorem. Then by theorem we get, for $c \in (x_1, x_2)$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 \text{ [using given condition]}$$

$$\Rightarrow f(x_2) - f(x_1) > 0. \Rightarrow f(x_2) > f(x_1).$$

That means f(x) is increasing in [a, b].

Cor. (II) If f'(x) < 0 in [a, b] then f(x) is decreasing on [a, b].

Proof:

Take x_1, x_2 in [a, b] with $x_1 < x_2$. Let the function f(x) is continuous in [a, and is differentiable in (a, b).

Thus f(x) satisfies both conditions of the Lagrange's Mean Value Theore Then by theorem we get, for $c \in (x_1, x_2)$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0 \text{ [using given condition]}$$

$$\Rightarrow f(x_2) - f(x_1) < 0. \Rightarrow f(x_2) < f(x_1).$$

That means f(x) is decreasing in [a, b].

Cor. (III) If f(a) = f(b) then the Lagrange's Mean Value Theorem gives t **result of Rolle's Theorem.**

Proof:

Let, $f(a) = f(b)$

By given hypothesis, the function f(x) satisfies both conditions of t Lagrange's Mean Value Theorem. Then by theorem we get, for $c \in (a, b)$.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{f(b) - f(b)}{b - a} = f'(c) \qquad [\because \text{Given that } f(a) = f(b)]$$

$$\Rightarrow \frac{0}{b - a} = f'(c)$$

$$\Rightarrow f'(c) = 0.$$

This completes the proof.

Cauchy's Mean Value Theorem

Statement:

If f(x) and g(x) are two functions, which are

(i) continuous on [a, b]

(ii) differentiable on (a, b).

Then there exists at least one value $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

provided that $g'(x) \neq 0$ in (a, b) or $g(a) \neq g(b)$.

Proof:

Let us define a function

$$F(x) = f(x) - Kg(x) \qquad \dots(1)$$

such that $F(a) = F(b)$ $\qquad \dots(2)$

where K is some constant value.

Given that f(x) and g(x) are continuous in [a, b] and differentiable in (a, b).

Since the linearity of two continuous (or differentiable) functions on a domain is again continuous (or differentiable) on that domain.

Therefore F(x) is continuous on [a, b] and is differentiable on (a, b).

Thus, F(x) satisfies all three conditions of the Rolle's Theorem then Rolle's theorem there exists at least one point $c \in (a, b)$ such that

$$F'(c) = 0$$

$$\Rightarrow f'(c) - Kg'(c) = 0$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = -K \qquad \text{for } g'(c) \neq 0 \qquad \dots(3)$$

Since, by (2) we have F(a) = F(b).

$$\Rightarrow f(a) + Kg(a) = f(b) + Kg(b).$$

$$\Rightarrow K = \frac{f(b) - f(a)}{g(a) - g(b)} \Rightarrow K = -\frac{f(b) - f(a)}{g(b) - g(a)}, \text{for } g(a) \neq g(b).$$

Thus (3) becomes,

$$\frac{f'(c)}{g'(c)} = -\frac{f(b) - f(a)}{g(a) - g(b)}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \text{for } g'(c) \neq 0.$$

Note (1) *If we suppose* $g(x) = x$ *for CMVT. Then the CMVT gives the result,*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This means LMVT is only a particular case of CMVT.

Note (2): *Remember that the CMVT cannot be proved by applying the LMVT separately to the numerator and denominator to CMVT because the fixed but arbitrary point C given by LMVT may not same for* f(x) *and* g(x). *If we apply the LMVT to the numerator and denominator separately then we obtain,*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)} \text{ for } a < c_1, c_2 < b, c_1 \neq c_2$$

provided that $g'(c_2) \neq 0$.

Let f(x) is defined on a single form (i.e. non-piecewise form).

Exponential function: ($f(x) = e^{P(x)}/a^{f(x)}$)

If P(x) is defined function for $x \in D$ and a is constant value then f(x) is continuous on D.

Trigonometric function:

(i) $\sin(p(x))$ or $\cos(p(x))$: Let $p(x)$ is defined for all x in D. Th... $\sin(p(x))$ or $\cos(p(x))$ is continuous on D.

(ii) $\tan(P(x))$/cot/sec/cosec: Let $p(x)$ is defined for all x in D except at the point where the function takes ∞ value. Then the functions are continuous on D except the point $x = a$ such that the function has infinity (∞) value.

Polynomial function: ($f(x) = a_0 + a_1x + \ldots + a_nx^n$ where $a_0, a_1, \ldots, a_n$ are constant value).

Clearly $f(x)$ is defined/continuous on R.

Logarithm function: $f(x) = \log_e(P(x))$

Since the natural log (i.e. $\log_e$) is defined only for $P(x) > 0$ for $x \in D$ (not for $P(x) = 0$ and $P(x) < 0$), so $f(x)$ is defined/continuous on D

Rational function: $f(x) = \dfrac{P(x)}{Q(x)}$

The function $f(x)$ is defined/continuous only for $Q(x) \neq 0$ for $x \in D$

Root function: $f(x) = \sqrt[n]{P(x)}$

Since the root function is defined for non-negative value(s). So f(x) is defined/continuous on D if $P(x) \geq 0$ for $x \in D$.

Exercise 3.1

1. Verify Rolle's Theorem for,

(i) $f(x) = x^2 - 5x + 10$ for $x \in [2, 3]$

Solution: Given function is,

$$f(x) = x^2 - 5x + 10 \text{ for } x \in [2, 3]$$

Clearly $f(x)$ is a polynomial function, so it is continuous on [2, 3].

And, $f'(x) = 2x - 5$

which is again a polynomial. So, $f'(x)$ is continuous on (2, 3).

Therefore, $f(x)$ is differentiable on (2, 3).

Also,

$$f(2) = 2^2 - 5(2) + 10 = 4 - 10 + 10 = 4$$
$$f(3) = 3^2 - 5(3) + 10 = 9 - 15 + 10 = 4$$

Here, $f(2) = f(3)$.

Thus, $f(x)$ satisfies all three conditions of Rolle's theorem, So, by this theorem there is at least one point c in (2, 3) such that,

$$f'(c) = 0 \Rightarrow 2c - 5 = 0 \Rightarrow c = \frac{5}{2} \in (2, 3).$$

Therefore c prescribed by the Rolle's theorem is $c = \frac{5}{2} \in (2, 3)$.

Thus, $f(x)$ satisfies all conditions of Rolle's theorem. So, $f(x)$ verifies the Rolle's theorem.

(ii) $f(x) = \dfrac{\sin x}{e^x}$ for $x \in [0, \pi]$ **[2003, Fall, Short]**

Solution: Given function is,

$$f(x) = \frac{\sin x}{e^x} \text{ for } x \in [0, \pi].$$

Clearly, $\sin x$ and e^x both are continuous and differential on $\Re$. We know the quotient form of continuous functions is continuous provided the denominator is non-zero. Here

$$e^x \neq 0 \text{ for any x in } [0, \pi].$$

So, $f(x)$ is continuous on $[0, \pi]$.

And, $f'(x) = \dfrac{e^x[\cos x - \sin x]}{e^{2x}}$.

Since $e^{2x} \neq 0$ for any x in $(0, \pi)$.

So, $f'(x)$ is continuous on $(0, \pi)$. This means $f(x)$ is differentiable on $(0, \pi)$.

Also, $f(0) = \dfrac{\sin 0}{e^0} = \dfrac{0}{1} = 0$ and $f(\pi) = \dfrac{\sin \pi}{e^\pi} = \dfrac{0}{e^\pi} = 0$.

Here, $f(0) = f(\pi)$.

Thus, $f(x)$ satisfies all three conditions of Rolle's theorem, So, by this theorem there is at least one point c in $(0, \pi)$ such that,

$$f'(c) = 0 \Rightarrow \frac{e^c \cos c - e^c \sin c}{e^{2c}} = 0$$
$$\Rightarrow \cos c = \sin c$$
$$\Rightarrow \tan c = 1 = \tan\left(\frac{\pi}{4}\right) \Rightarrow c = \frac{\pi}{4} \in (0, \pi).$$

Therefore c prescribed by the Rolle's theorem is $c = \frac{\pi}{4} \in (0, \pi)$.

Thus, $f(x)$ satisfies all conditions of Rolle's theorem. So, $f(x)$ verifies the Rolle's theorem.

(iii) $f(x) = x(x + 3)\,e^{-x/2}$ for $x \in [-3, 0]$

Solution: Given function is,

$$f(x) = x(x + 3)\,e^{-x/2} \text{ for } x \in [-3, 0]$$

Clearly the linear functions x and (x + 3) are continuous and differentiable on $\Re$ and also the exponential function $e^{-x/2}$ is also continuous and differentiable on $\Re$. And, we know the product of continuous functions is again a continuous function and the product of differentiable functions is again a differentiable function. Since $f(x)$ is the product of polynomial and exponential function. So, $f(x)$ is continuous and differentiable on $[-3, 0]$.

Also, $f(-3) = -3(-3 + 3)\,e^{-3/2} = 0$ and $f(0) = 0\,(0 + 3)\,e^0 = 0$.

(i) $\sin(p(x))$ or $\cos(p(x))$: Let $p(x)$ is defined for all x in D. Then $\sin(p(x))$ or $\cos(p(x))$ is continuous on D.

(ii) $\tan(P(x))$/cot/sec/cosec: Let $p(x)$ is defined for all x in D except at the point where the function takes ∞ value. Then the functions are continuous on D except the point x = a such that the function has infinity (∞) value.

Polynomial function: $(f(x) = a_0 + a_1x + \dots + a_nx^n$ where $a_0, a_1 \dots, a_n$ one constant value).

Clearly f(x) is defined/continuous on R.

Logarithm function: $f(x) = \log_e(P(x))$

Since the natural log (i.e. $\log_e$) is defined only for $P(x) > 0$ for $x \in D$ (not for $P(x) = 0$ and $P(x) < 0$), so f(x) is defined/continuous on D.

Rational function: $f(x) = \dfrac{P(x)}{Q(x)}$

The function f(x) is defined/continuous only for $Q(x) \neq 0$ for $x \in D$.

Root function: $f(x) = \sqrt[n]{P(x)}$

Since the root function is defined for non-negative value(s). So, f(x) is defined/continuous on D if $P(x) \geq 0$ for $x \in D$.

Exercise 3.1

1. **Verify Rolle's Theorem for,**

(i) $f(x) = x^2 - 5x + 10$ **for** $x \in [2, 3]$

Solution: Given function is,

$$f(x) = x^2 - 5x + 10 \text{ for } x \in [2, 3]$$

Clearly f(x) is a polynomial function, so it is continuous on [2, 3].

And, $f'(x) = 2x - 5$

which is again a polynomial. So, f '(x) is continuous on (2, 3).

Therefore, f(x) is differentiable on (2, 3).

Also,

$$f(2) = 2^2 - 5(2) + 10 = 4 - 10 + 10 = 4$$
$$f(3) = 3^2 - 5(3) + 10 = 9 - 15 + 10 = 4$$

Here, $f(2) = f(3)$.

Thus, f(x) satisfies all three conditions of Rolle's theorem, So, by this theorem there is at least one point c in (2, 3) such that,

$$f'(c) = 0 \Rightarrow 2c - 5 = 0 \Rightarrow c = \frac{5}{2} \in (2, 3).$$

Therefore c prescribed by the Rolle's theorem is $c = \frac{5}{2} \in (2, 3)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(ii) $f(x) = \dfrac{\sin x}{e^x}$ **for** $x \in [0, \pi]$ **[2003, Fall, Short]**

Solution: Given function is,

$$f(x) = \frac{\sin x}{e^x} \text{ for } x \in [0, \pi].$$

Clearly, sinx and e^x both are continuous and differential on $\Re$. We know the quotient form of continuous functions is continuous provided the denominator is non-zero. Here

$$e^x \neq 0 \text{ for any x in } [0, \pi].$$

So, f(x) is continuous on $[0, \pi]$.

And, $f'(x) = \dfrac{e^x [\cos x - \sin x]}{e^{2x}}$.

Since $e^{2x} \neq 0$ for any x in $(0, \pi)$.

So, f '(x) is continuous on $(0, \pi)$. This means f(x) is differentiable on $(0, \pi)$.

Also, $f(0) = \dfrac{\sin 0}{e^0} = \dfrac{0}{1} = 0$ and $f(\pi) = \dfrac{\sin \pi}{e^\pi} = \dfrac{0}{e^\pi} = 0.$

Here, $f(0) = f(\pi)$.

Thus, f(x) satisfies all three conditions of Rolle's theorem, So, by this theorem there is at least one point c in $(0, \pi)$ such that,

$$f'(c) = 0 \Rightarrow \frac{e^c \cos c - e^c \sin c}{e^{2c}} = 0$$
$$\Rightarrow \cos c = \sin c$$
$$\Rightarrow \tan c = 1 = \tan\left(\frac{\pi}{4}\right) \Rightarrow c = \frac{\pi}{4} \in (0, \pi).$$

Therefore c prescribed by the Rolle's theorem is $c = \frac{\pi}{4} \in (0, \pi)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(iii) $f(x) = x(x + 3)\,e^{-x/2}$ **for** $x \in [-3, 0]$

Solution: Given function is,

$$f(x) = x(x + 3)\,e^{-x/2} \quad \text{for } x \in [-3, 0]$$

Clearly the linear functions x and (x + 3) are continuous and differentiable on $\Re$ and also the exponential function $e^{-x/2}$ is also continuous and differentiable on $\Re$. And, we know the product of continuous functions is again a continuous function and the product of differentiable functions is again a differentiable function. Since f(x) is the product of polynomial and exponential function. So, f(x) is continuous and differentiable on [-3, 0].

Also, $f(-3) = -3(-3 + 3)\,e^{-3/2} = 0$ and $f(0) = 0\,(0 + 3)\,e^0 = 0.$

Here, $f(-3) = f(0)$.

Thus, f(x) satisfies all the conditions of Rolle's theorem. Then by theorem there is at least one point $c \in (-3, 0)$ such that

$$f'(c) = 0 \Rightarrow (2c+3)e^{-c/2} + (c^2+3c)e^{-c/2}\cdot\left(-\frac{1}{2}\right) = 0$$

$$\Rightarrow 4c + 6 - c^2 - 3c = 0 \qquad \left[\because \frac{e^{-c/2}}{2} \neq 0\right]$$

$$\Rightarrow c^2 - c - 6 = 0$$

$$\Rightarrow (c-3)(c+2) = 0$$

$$\Rightarrow c = -2 \in (-3, 0) \text{ but } c = 3 \notin (-3, 0)$$

Therefore c prescribed by the Rolle's theorem is $c = -2 \in (-3, 0)$

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies Rolle's theorem.

$a > 0$,

(iv) $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right)$ **for** $x \in [a, b]$ **for** $a > 0$.

Solution: Given function is,

$$f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right) \text{ for } x \in [a, b] \text{ with } a > 0.$$

Clearly, $\left(\frac{x^2+ab}{(a+b)x}\right) > 0$ for $a > 0$.

Here, $a > 0$ and logarithm function is defined only for positive value. So f(x) is continuous on [a, b].

And,
$$f'(x) = \left(\frac{(a+b)x}{x^2+ab}\right)\cdot\frac{(ax+bx)2x - (x^2+ab)(a+b)}{(a+b)^2x^2}$$

$$= \frac{x\cdot(2x^2 - x^2 - ab)}{x^2(x^2+ab)}$$

$$= \frac{x^2 - ab}{x(x^2+ab)}$$

which is defined for $a > 0$ because $x(x^2+ab) > 0$ for any x in (a, b).

This means $f'(x)$ is continuous on (a, b) for $a > 0$.

That is f(x) is differentiable on (a, b).

Also,

$$f(a) = \log\left(\frac{x^2+ab}{(a+b)x}\right) = \log(1) = 0$$

and
$$f(b) = \log\left(\frac{b^2+ab}{(a+b)b}\right) = \log(1) = 0$$

Here, $f(a) = f(b)$.

Thus f(x) satisfies all three conditions of Rolle's theorem. Then by the theorem, there is at least one point $c \in (a, b)$ such that

$c^2 = ab$

$c = \sqrt{ab} \in (a, b)$

$$f'(c) = 0 \Rightarrow \frac{c^2 - ab}{c(c^2+ab)} = 0$$

$$\Rightarrow c^2 - ab = 0 \Rightarrow c = \pm\sqrt{ab} \in (a, b).$$

Therefore c prescribed by the Rolle's theorem is $c = \sqrt{ab} \in (a, b)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

Note: For this problem we should have $a > 0$. Otherwise, the problem may not verify the theorem. For, if $a = -1, b = 1$ then f(x) is undefined.

(v) $f(x) = (x-a)^m (x-b)^n$ **where m and n being positive integers in [a, b].**

Solution: Given function is,

$$f(x) = (x-a)^m (x-b)^n \qquad \text{for } x \in [a, b] \text{ and } m, n > 0.$$

Clearly $(x-a)^m (x-b)^n$ is a polynomial function. So, it is continuous on [a, b].

And,
$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$$

which is a polynomial function.

So, $f'(x)$ is continuous on (a, b). That means f(x) is differentiable on (a, b).

Also,
$$f(a) = (a-a)^m(a-b)^n = 0 \quad \text{and} \quad f(b) = (b-a)^m(b-b)^n = 0.$$

Here, $f(a) = f(b) = 0$.

Thus, f(x) satisfies all three conditions of Rolle's theorem. Then by the theorem, there is at least one point $c \in (a, b)$ such that

$f'(c) = 0$.

$$\Rightarrow f'(c) = m(c-a)^{m-1}(c-b)^n + n(c-a)^m(c-b)^{n-1} = 0$$

$$\Rightarrow m(c-b) + n(c-a) = 0 \qquad [\because (c-a)^{m-1}(c-b)^{n-1} \neq 0]$$

$$\Rightarrow c(m+n) - (mb+na) = 0 \Rightarrow c = \frac{mb+na}{m+n}.$$

This means c is the internal division point of the line joining a and b in the ratio m:n. So, $c \in (a, b)$.

Therefore c prescribed by the Rolle's theorem is $c = \frac{mb+na}{m+n} \in (a, b)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(vi) $f(x) = e^x(\sin x - \cos x)$ **in** $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Solution: Given function is,

$$f(x) = e^x(\sin x - \cos x) \qquad \text{for } x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Clearly sine, cosine and e^x are continuous on $\Re$ and are differentiable on ... Since the product of continuous functions and differentiable functions ... again continuous and differentiable. So, f(x) is also continuous on $\left[\frac{\pi}{4}, ...\right]$ and is differentiable on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

Also,

$$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right] = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0.$$

and,

$$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0.$$

Here, $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0.$

Thus, f(x) satisfies all three conditions of Rolle's Theorem. Then by ... theorem, there is at least one point $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that

$$f'(c) = 0.$$

Since, $f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x),$

So,

$$f'(c) = e^c[\sin c - \cos c + \cos c + \sin c] = 0.$$

$$\Rightarrow 2\sin c = 0 \qquad [\because e^c \neq 0]$$

$$\Rightarrow \sin c = 0 = \sin\pi$$

$$\Rightarrow c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right).$$

Therefore c prescribed by the Rolle's theorem is $c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, thus, f ... satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's Theorem.

2. Verify Lagrange's Mean Value Theorem for,

(i) $f(x) = x^2$ in $x \in [1, 2]$

f(x) = x² for (1,2)

Solution: Given function is,

$$f(x) = x^2 \quad \text{for } x \in [1, 2]$$

Clearly f(x) is a polynomial function which is continuous on [1, 2].

And,

$$f'(x) = 2x, \text{ which is polynomial function.}$$

So, f '(x) is continuous on (1, 2). Therefore, f(x) is differentiable on (1, 2).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem. Then by this theorem there is at least one point $c \in (1, 2)$ such that,

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} \Rightarrow 2c = \frac{4 - 1}{2 - 1} \Rightarrow 2c = 3 \Rightarrow c = \frac{3}{2} \in (1, 2).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = \frac{3}{2} \in (1, 2)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(ii) $f(x) = x(x - 1)(x - 2)$ in $x \in [0, ½]$

Solution: Given function is,

$$f(x) = x(x - 1)(x - 2) \quad \text{for } x \in [0, ½]$$

$$\Rightarrow f(x) = x^3 - 3x^2 + 2x \quad \text{for } x \in [0, ½]$$

Clearly, $f(x) = x^3 - 3x^2 + 2x$ is a polynomial function, is continuous on [0, ½].

And, $f'(x) = 3x^2 - 6x + 2$, which is polynomial. So, f '(x) is continuous.

So, f(x) is differentiable on (0, ½).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem. Then by the theorem there is at least one point $c \in (0, ½)$ such that

$$f'(c) = \frac{f(1/2) - f(0)}{(1/2) - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\frac{1}{2}} = \frac{3}{4} \qquad [\text{since, } f(0) = 0.]$$

$$\Rightarrow 12c^2 - 24c + 8 = 3$$

$$\Rightarrow 12c^2 - 24c + 5 = 0.$$

$$\Rightarrow c = \frac{+24 \pm \sqrt{24^2 - 240}}{24} = 1 \pm \frac{\sqrt{336}}{24} = 1 \pm 0.76$$

$$\Rightarrow c = 0.24 \in (0, ½).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = 0.24 \in (0, ½)$. Thus, f(x) satisfies all the conditions of Lagrange's Mean Value Theorem, so f(x) verifies the theorem.

(iii) $f(x) = Ax^2 + Bx + C$ for $x \in [a, b]$

Solution: Given function is,

$$f(x) = Ax^2 + Bx + C \quad \text{for } x \in [a, b].$$

Clearly $f(x) = Ax^2 + Bx + C$ is a polynomial, is continuous on [a, b].

And, $f'(x) = 2Ax + B$, which is linear polynomial function. This means f(x) is continuous. Therefore, f(x) is differentiable on (a, b).

Thus f(x) satisfies both conditions of Lagrange's Mean Value Theorem then by the theorem there is at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 2Ac + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b-a}$$

$$= \frac{A(b^2 - a^2) + B(b-a)}{b-a} = A(b+a) + B$$

$$\Rightarrow 2Ac = A(b+a)$$

$$\Rightarrow c = \frac{b+a}{2} \in (a, b).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = \frac{b+a}{2} \in (a, b)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(iv) $f(x) = e^x$ for $x \in [0, 1]$

Solution: Given function is,

$$f(x) = e^x \quad \text{for } x \in [0, 1]$$

Clearly f(x) is an exponential function, is continuous on [0, 1].

And, $f'(x) = e^x$ which is continuous. So, f(x) is differentiable on (0, 1). Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem then by the theorem there is at least one point $c \in (0, 1)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow e^c = \frac{e^b - e^a}{1-0} = e^1 - 1 = 2.72 - 1 = 1.72.$$

$$\Rightarrow c = \log(1.72) = 0.54 \in (0, 1).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = 0.54 \in (0, 1)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(v) $f(x) = \sqrt{x-1}$ for [1, 3].

Solution: Given function is,

$$f(x) = \sqrt{x-1} \text{ for } [1, 3].$$

Since, the root function is defined only for non-negative value. So, f(x) is defined for $x \geq 1$. Thus, f(x) is defined and is continuous on [1, 3].

And, $$f'(x) = \frac{1}{2\sqrt{x-1}}.$$

Clearly, $2\sqrt{x-1} > 0$ for $x > 1$. So, f '(x) is continuous on (1, 3). Therefore, f(x) is differentiable on (1, 3).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem then there is at least one point $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$\text{i.e.} \quad \frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}-0}{3-1} = \frac{\sqrt{2}}{2}.$$

$$\Rightarrow \sqrt{c-1} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow c - 1 = \frac{1}{2} = 0.5.$$

$$\Rightarrow c = 1.5 \in (1, 3).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = 1.5 \in (1, 3)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(vi) $f(x) = x + \frac{1}{x}$ for $x \in \left[\frac{-1}{2}, 2\right]$.

Solution: Given function is,

$$f(x) = x + \frac{1}{x} \text{ for } x \in \left[\frac{-1}{2}, 2\right].$$

Clearly $0 \in \left(-\frac{1}{2}, 2\right)$ and $f(0) = 0 + \frac{1}{0}$ which is undefined.

So, f(x) is not continuous at x = 0. Therefore, f(x) does not verify the Lagrange's Mean Value Theorem.

> **Note: The function $f(x) = x + \frac{1}{x}$ does not satisfy the condition(s) of Rolle's theorem and Lagrange's Mean Value Theorem in any interval that includes 0.**
>
> ***Reason: At $x = 0$, we get $f(0) = 0 + \frac{1}{0}$ which is undefined. So, $f(x)$ is not continuous at $x = 0$.***

3. Show that $|\sin b - \sin a| \leq |b - a|$, by using Lagrange's Mean Value Theorem.

Solution: Here we have to show $|\sin b - \sin a| \leq |b - a|$.

Clearly $\sin b$ and $\sin a$ is a sine function having angle b and a, respectively. So, assume that,

$$f(b) = \sin b \quad \text{and} \quad f(a) = \sin a$$

Then,

$$f(x) = \sin x.$$

So,

$$f'(x) = \cos x.$$

Clearly $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) for $a, b \in \Re$. Thus, $f(x)$ satisfies both conditions of Lagrange's Mean Value Theorem. Then by the theorem there is at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \cos c = \frac{\sin b - \sin a}{b-a}.$$

Since sine function has oscillatory value in between –1 to 1.

So, $-1 \le \cos c \le 1$, for any c in R.

$\Rightarrow |\cos c| \le 1$, for any c in (a, b).

So,

$$\left|\frac{\sin b - \sin a}{b-a}\right| = |\cos c| \le 1.$$

$$\Rightarrow \frac{|\sin b - \sin a|}{|b-a|} \le 1$$

$$\Rightarrow |\sin b - \sin a| \le |b-a|.$$

4. If $f(x) = \frac{1}{x}$, show that mean value theorem does not exist in [0, 1].

Solution: Let $f(x) = \frac{1}{x}$. At $x = 0$, we see $f(0) = \frac{1}{0}$ which is undefined. So, f(x) is not continuous on [0, 1]. Therefore, f(x) does not satisfy the condition of the mean value theorem on [0, 1].

Note: The function f(x) also does not satisfy the condition the Rolle's Theorem.

5. Does the function $f(x) = x$ in [0, 1] satisfy the hypothesis of Lagrange's Mean Value Theorem? If not, why not? If so, what value or values could c have?

Solution: Let $f(x) = x$ for $x \in [0, 1]$

Clearly f(x) is a polynomial function. So, f(x) is continuous on [0, 1].

And, $f'(x) = 1$ which is a constant function, is continuous on (0, 1).

So, f(x) is differentiable on (0, 1).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem then by the theorem there is at least one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e. $1 = \frac{1-0}{1-0} \Rightarrow 1 = 1$ which is always true.

That means for any value of c in (0, 1), f(x) verifies the Lagrange's mean value theorem.

6. If $f(x) = \tan x$ then $f(0) = 0$ and $f(\pi) = 0$. Is Rolle's theorem applicable to $f(x)$ in $[0, \pi]$? **[2008, Spring] [2009 Spring]**

Solution: Let. $f(x) = \tan x$ for $x \in [0, \pi]$.

Since $\frac{\pi}{2} \in [0, \pi]$, $f\left(\frac{\pi}{2}\right) = \tan\left(\frac{\pi}{2}\right) = \infty$.

So, f(x) is not continuous at $\frac{\pi}{2}$. Therefore, f(x) does not verify the Rolle's theorem.

Note: The function f(x) also does not verify the LMVT.

7. Show that $f(x) = x^2$ and $g(x) = 3x - 2$ in $x \in [1, 2]$ verify Cauchy's mean value theorem.

Solution: Given functions are,

$$f(x) = x^2 \quad \text{and} \quad g(x) = 3x - 2 \quad \text{for } x \in [1, 2].$$

Clearly f(x) and g(x) are polynomial functions. Since every polynomial function is continuous and differentiable, so f(x) and g(x) are continuous on [1, 2] and differentiable on (1, 2).

Also, $f'(x) = 2x$ and $g'(x) = 3 \neq 0$ for any x in (1, 2).

Thus, they satisfies all conditions of Cauchy's Mean Value Theorem then by the theorem there is at least one point c in (1, 2) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)} \quad \dots \text{(i)}$$

$$\Rightarrow \frac{2c}{3} = \frac{4-1}{4-1} = \frac{3}{3} = 1.$$

$$\Rightarrow c = \frac{3}{2} = 1.5 \in (1, 2).$$

This means the functions f(x) and g(x) verify the Cauchy's mean value theorem.

8. Show that $f(x) = x^2 - 4x$ is increasing in $(2, \infty)$ and decreasing in $(-\infty, 2)$.

Solution: Clearly $f(x) = x^2 - 4x$ is a polynomial function which is continuous on R and is differentiable on R.

By Lagrange's Mean Value Theorem we know that if f(x) is continuous on [a, b] and is differentiable on an interval (a, b) then there is at least one point c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

If a function is increasing then we should have $f(b) > f(a)$ for $b > a$. Therefore,

$$f'(c) = \frac{f(b) - f(a)}{b - a} > 0.$$

And similarly if f(x) is decreasing then we have

$$f'(c) < 0.$$

Here given function is,

$$f(x) = x^2 - 4x$$

Then, $f'(x) = 2x - 4$.

So, for any $x \in (2, \infty)$, we observe $f'(x) > 0$. Therefore, f(x) is increasing on $(2, \infty)$.

Also, for any $x \in (-\infty, 2)$, we observe $f'(x) < 0$. Therefore f(x) is decreasing on $(-\infty, 2)$.

9. Show that $f(x) = x$ and $g(x) = x^2 - 2x$ in $[0, 2]$ does not verify Cauchy's mean value theorem.

Solution: Here, $f(x) = x$ and $g(x) = x^2 - 2x$ for $x \in [0, 2]$.

And, $f'(x) = 1$ and $g'(x) = 2x - 2$.

Since, $1 \in (0, 2)$ and $g'(1) = 2 - 2 = 0$ that contradicts to the condition $g'(c) \neq 0$ for some $c \in (0, 2)$. Hence, the functions do not verify Cauchy's mean value theorem.

10. The function $f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 0 & \text{for } x = 1 \end{cases}$, is zero at $x = 0$ and $x = 1$, differentiable on $(0, 1)$ is never zero. How can this be? Does not Rolle's theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

Solution: Given that, $f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 0 & \text{for } x = 1 \end{cases}$

Clearly $f(x) = x$ and $f(x) = 0$ both are polynomial functions, are continuous on their respective domain. Here $f(x)$ has piecewise form, so we need to check the continuity of $f(x)$ at $x = 1$.

Here,

$$\text{LHL} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1.$$

and $f(1) = 0$.

This shows $f(x)$ is not continuous at $x = 1$. So, the given function $f(x)$ is not continuous on $[0, 1]$. However, $f(x) = x$ is differentiable on $(0, 1)$ being polynomial function and $f'(x) = 1$ for $x \in (0, 1)$.

That means $f'(x) \neq 0$ for all $x \in (0, 1)$.

That is, $f(x)$ is not continuous on $[0, 1]$ however it is differentiable on $(0, 1)$ and $f(0) = f(1)$. Being the lack of continuity of the function on closed interval, the Rolle's theorem does not work here.

11. For what value of a, m and b does the function,

$$f(x) = \begin{cases} 3 & \text{for } x = 0 \\ -x^2 + 3x + a & \text{for } 0 < x < 1 \\ mx + b & \text{for } 1 \le x \le 2 \end{cases}$$

Satisfy the hypothesis of the mean value theorem on the interval $[0, 2]$?

Solution: Given that,

$$f(x) = \begin{cases} 3 & \text{for } x = 0 \\ -x^2 + 3x + a & \text{for } 0 < x < 1 \\ mx + b & \text{for } 1 \le x \le 2 \end{cases}$$

satisfies the hypothesis (condition) of mean value theorem.

Therefore, $f(x)$ is continuous on $[0, 2]$ and is differentiable on $(0, 2)$.

Since, $f(x)$ is continuous at $x = 0$. So,

$$\text{RHL} = f(0) \quad \Rightarrow \lim_{x\to0^+} f(x) = f(0)$$

$f(0) = 3$

$$\Rightarrow \lim_{x\to0^+} (-x^2 + 3x + a) = 3$$

$$\Rightarrow a = 3$$

And, $f(x)$ is continuous at $x = 1$. So,

$$\text{LHL} = f(1) \quad \Rightarrow \lim_{x\to1^-} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to1^-} (-x^2 + 3x + a) = m + b$$

$$\Rightarrow -1 + 3 + 3 = m + b \qquad [\because a = 3]$$

$$\Rightarrow m + b = 5 \qquad \ldots \text{(i)}$$

$f(x) = f(1)$

Next, $f(x)$ is differential on $(0, 2)$. So, $f(x)$ is differential at $x = 1$.

That means

LHD = RHD at $x = 1$.

Here, $Lf'(1) = -2(1) + 3 = 1$

and, $Rf'(1) = m$

Since, $Lf'(1) = Rf'(1) \Rightarrow 1 = m$.

Then from (i), $b = 4$.

Thus, $a = 3$, $b = 4$ and $m = 1$.

12. Explain why Rolle's theorem is not applicable to the function $f(x) = 1 - (x - 1)^{2/3}$ in $0 \le x \le 2$.

Solution: Here, $f(x) = 1 - (x - 1)^{2/3} = 1 - \sqrt[3]{(x-1)^2}$ for $x \in [0, 2]$.

Clearly $(x - 1)^2$ is always positive. So, the cube root has a real positive value on $[0, 2]$. So, $f(x)$ is continuous in $[0, 2]$, being a linear form of a constant and defined root functions.

Also,

$$f'(x) = \frac{-2}{3(x-1)^{1/3}}$$

Here,

$$f'(1) = \frac{1}{0} \text{ which is undefined.}$$

Since $1 \in (0, 2)$. Therefore $f(x)$ is not differentiable at $x = 1$ that lies on $(0, 2)$.

Therefore, $f(x)$ does not applicable to verify the Rolle's Theorem.

Note: The function $f(x)$ also does not applicable to verify the LMVT.

13. Show the $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$ (for $a > 0$) by using Lagrange's Mean Value Theorem.

Solution: Here,

$$\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a} \qquad \text{for } a > 0$$

$$\Rightarrow \frac{1}{b} < \frac{\log(b) - \log(a)}{b-a} < \frac{1}{a} \qquad \ldots\ldots (i)$$

Comparing the term $\frac{\log(b) - \log(a)}{b-a}$ with $\frac{f(b) - f(a)}{b-a}$ then we get,

$$f(b) = \log(b), \quad f(a) = \log(a)$$

So, $f(x) = \log(x)$ for $x \in [a, b]$.

Clearly f(x) is continuous on [a, b] with a > 0.

Also, $f'(x) = \frac{1}{x}$ that is defined for non–zero value of x.

So, f(x) is differentiable on (a, b) with a > 0.

Thus, f(x) satisfies both condition of LMVT. So, by this theorem there ... least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{1}{c} = \frac{\log(b) - \log(a)}{b-a} \qquad \ldots\ldots (ii)$$

Since $c \in (a, b)$. So, a < c and c < b. Then,

$$\frac{1}{a} > \frac{1}{c} \text{ and } \frac{1}{c} > \frac{1}{b}$$

Therefore, (ii) becomes,

$$\frac{1}{b} < \frac{\log(b) - \log(a)}{b-a} < \frac{1}{a}$$

This is same to (i). So, the condition fulfills.

OTHER IMPORTANT QUESTION FROM FINAL EXAM

a. State and prove Lagrange's Mean Value Theorem. Let $f(x) = (x-2)^{2/3}$ with x in [1, 4] can Lagrange's Mean Value Theorem be applied in this function? If not why? **[2002] [2009 Spring]**

Solution: First Part: See Lagrange's Mean Value Theorem.

Second Part: Since, $f(x) = (x-2)^{2/3} = \sqrt[3]{(x-2)^2}$ for $x \in [1, 4]$.

Clearly $(x-2)^2$ is always positive. So, the cube root has a real positive value on R. So, f(x) is continuous in [1, 4], being defined root functions.

And $f'(x) = \frac{2}{3(x-2)^{1/3}}$.

But f '(2) is undefined and $x = 2 \in (1, 4)$.

Therefore f(x) is not differentiable on (1, 4).

Hence, f(x) does not verify the Lagrange's Mean Value Theorem.

State and prove Cauchy Mean value theorem. Also verify Cauchy Mean value theorem for $f(x) = x$, $g(x) = x^2$, $x \in [-2, 0]$. **[2009, Fall]**

Solution: First part: See Cauchy mean value theorem.

Second part: See Q. No. 7 in exercise 3.1.

c. State and prove Lagrange's Mean Value Theorem. Interpret it geometrically. **[2015 Spring][2014 Spring][2005, Fall] [2001] [1999]**

OR

State and prove Lagrange's Mean Value Theorem. What is its geometrical meaning. **[2005, Fall] [2001] [1999]**

Solution: See Lagrange's Mean Value Theorem and its geometrical meaning.

d. State and prove Rolle's theorem. If f(x) be defined by $f(x) = (x-2)^2$ for $x \in [1, 3]$, find $c \in (1, 3)$ such that $f'(c) = 0$. **[2000]**

Solution: First part : See Rolle's theorem.

Second Part: Given function is

$$f(x) = (x-2)^2 \qquad \text{for } x \in [1, 3]$$

Since f(x) is a polynomial function which is continuous on [1, 3].

Also, $f'(x) = 2(x-2)$

which is polynomial function. So, f '(x) is continuous on (1, 3).

So, f(x) is differentiable on (1, 3).

Also, $f(1) = (1-2)^2 = (-1)^2 = 1$ and $f(3) = (3-2)^2 = (1)^2 = 1$.

Thus, $f(1) = f(3)$.

Thus, f(x) satisfies all three conditions of Rolle's theorem then by this theorem there is at least one point $c \in (1, 3)$ such that

$$f'(c) = 0.$$

$$\Rightarrow 2(c-2) = 0 \quad \Rightarrow c = 2 \in (1, 3).$$

Thus f(x) satisfies all the conditions of the Rolle's Theorem. This proves that f(x) verifies the Rolle's theorem.

e. State and prove Cauchy's mean value theorem. **[2016 Spring][2016 Fall][2006, Fall] [2002]**

Solution: See Cauchy's mean value theorem.

f. State the prove Cauchy mean value theorem. Verify it for the function f(x) = x, g(x) = x on [–2, 0]. **[2003, Fall]**

Solution: First Part: See Cauchy's mean value theorem.

Second Part: Clearly the given functions f(x) = x and g(x) = x are polynomial functions, so they are continuous on [–2, 0] and are differentiable on (–2, 0).

Moreover, $g'(x) = 1 \neq 0$ for any x in (–2, 0).

Thus, the functions satisfy both conditions of the Cauchy's mean value theorem. Then by the theorem there is c in (–2, 0) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{1}{1} = \frac{0+2}{0+2} \Rightarrow \frac{1}{1} = \frac{1}{1} \Rightarrow 1 = 1$$ which is always true.

That means the Cauchy's mean value theorem is verified for each value on (–2, 0).

g. **State and prove Cauchy mean value theorem. Verify Rolle's theorem for $f(x) = x^2, x \in [-1, 1]$.** [2003, Spring]

Solution: First part: See Cauchy's mean value theorem.

Second Part:

Clearly, f(x) is polynomial function which is continuous on [–1, 1].

Also, $f'(x) = 2x$

which is a polynomial function. So, f '(x) is continuous on (–1, 1).

This means, f(x) is differentiable on (–1, 1).

Also, $f(-1) = 1 = f(1)$.

Thus, f(x) satisfies all three conditions of Rolle's theorem then by theorem there is $c \in (1, 3)$ such that

$f'(c) = 0.$

$\Rightarrow 2c = 0 \Rightarrow c = 0 \in (-1, 1).$

This proves that f(x) verifies the Rolle's theorem.

h. **State and prove Lagrange's Mean Value Theorem and use it to prove that $f(x) = x^2 + 1$ defined on [1, 4] is an increasing function.** [2004, Fall]

Solution: First part: See Lagrange's Mean Value Theorem.

Second Part:

Clearly, f(x) is a polynomial function which is continuous on [1, 4].

And, $f'(x) = 2x$

which is a polynomial function. So, f '(x) is continuous on (1, 4). Therefore f(x) is differentiable on (1, 4).

Thus, f(x) satisfies both conditions of the Lagrange's Mean Value Theorem. Now,

$f'(x) = 2x > 0$, for $x \in (1, 4)$.

That means f(x) is increasing function on (1, 4).

i. **State and prove Lagrange's Mean Value Theorem, and verify it for $f(x) = e^x$ on [0, 1].** [2004, Spring]

Solution: First part: See Lagrange's Mean Value Theorem.

Second Part: See Exercise 3.1, Q. 2(iv).

j. **State and prove Rolle's theorem. For $f(x) = (x)^{2/3}$ defined on [–8, 8], explain whether Rolle's theorem can be applied?** [2005, Spring]

Solution: First part: See Rolle's theorem.

Second Part: Here, $f(x) = (x)^{2/3} > 0$ on [–8, 8]. So, f(x) is continuous in [–8, 8].

And $f'(x) = \frac{2}{3x^{1/3}}$.

But $f'(0) = \frac{2}{0}$

Since $0 \in (-8, 8)$ and f '(0) is undefined, this means f(x) is not differentiable on (–8, 8).

Thus f(x) does not satisfy all the conditions of the Rolle's Theorem. Therefore, f(x) does not verify the Rolle's theorem.

Note: The function f(x) also does not verify the LMVT.

k. **State and prove Lagrange's Mean Value Theorem. Interpret it geometrically. Verify it for $f(x) = e^x$ on [0, 1].** [2006, Spring]

Solution: First and Second Part: See Lagrange's Mean Value Theorem and its geometrical meaning.

Third Part: See Exercise 3.1, Q. 2(iv).

l. **State and prove Lagrange's mean value theorem.** [2007, Fall]

Solution: See Lagrange's Mean Value Theorem.

m. **State and prove Lagrange's Mean Value Theorem. Use it to show that if f '(x) > 0 on [a, b] function f(x) is increasing on [a, b].** [2013 Fall] [2007, Spring]

Solution: First Part: See Lagrange's Mean Value Theorem.

Second Part: See corollary - I of LMVT.

n. **State and prove Cauchy Mean Value Theorem. Show that $f(x) = x + 2$ and $g(x) = x^2 + 4$ on [0, 2] verify Cauchy Mean Value Theorem.** [2008, Spring]

Solution: First Part: See Cauchy's mean value theorem.

Second Part: Clearly the given functions $f(x) = x + 2$ and $g(x) = x^2 + 4$ are polynomial functions which are continuous on [0, 2].

And, $f'(x) = 1$ and $g'(x) = 2x$.

Clearly both f '(x) and g'(x) are polynomial functions, are continuous on (0, 2). This means both functions are differentiable in (0, 2).

Also, $g'(x) = 2x \neq 0$ for $x \in (0, 2)$.

Thus, the functions satisfy both conditions of the Cauchy's mean value theorem. Then by this theorem there is at least one point c in (0, 2) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(0)}{g(2) - g(0)}$$

$$\Rightarrow \frac{1}{2c} = \frac{2-0}{4-0} \Rightarrow c = 1 \in (0, 2).$$

This shows that f(x) satisfies all the conditions of the Cauchy's Mean Value Theorem. Thus, f(x) verifies the Cauchy's Mean Value Theorem.

o. **State the prove Cauchy mean value theorem. Verify it for the function $f(x) = x$, $g(x) = x$ in [1, 3].** [2011 Spring]

Solution: First Part: See Cauchy's mean value theorem.

Second Part: Let $f(x) = x$ and $g(x) = x$. Clearly, both functions are polynomial functions. So, f(x) and g(x) are continuous on [1, 3] and differentiable on (1, 3).

Moreover, $g'(x) = 1 \neq 0$ for x in (1, 3).

Thus, the functions satisfy both conditions of the Cauchy's mean value theorem. Then by this theorem there is at least one point c in (1, 3) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(3) - f(1)}{g(3) - g(1)}$$

$$\Rightarrow \frac{1}{1} = \frac{3-1}{3-1} \Rightarrow \frac{1}{1} = \frac{2}{2} \Rightarrow 1 = 1 \text{ which is always true.}$$

This shows that f(x) satisfies all the conditions of the Cauchy's Mean Value Theorem. Thus, f(x) verifies the Cauchy's Mean Value Theorem.

p. State and prove Cauchy Mean value theorem. Does the theorem applicable to the functions $f(x) = x$ and, $g(x) = x^2 - 2x$ in the interval [0, 2]? Why? **[2018 Spring] [2009, Fall]**

Solution: First part: See Cauchy mean value theorem.

Second part: See Q. No. 9, exercise 3.1.

q. State and prove Rolle's theorem. **[2013 Spring]**

Solution: See theoretical part.

r. State and prove Rolle's theorem. Is Rolle's theorem applicable to the function $f(x) = \tan x$ in the interval? **[2014 Fall]**

Solution: First part: See Rolle's theorem.

Second part: The interval is not given, so the question should be restate.

s. State Lagrange's Mean Value Theorem. Is Lagrange's Mean Value Theorem applicable to the function $f(x) = |x|$ in the interval [–1, 1]? Give reasons. **[2015 Fall]**

Solution: First part: See Lagrange's Mean Value Theorem for statement.

Second Part:

Given function is $f(x) = |x|$ for [–1, 1].

Since, $Rf'(0) = 1$ and $Lf'(0) = -1$. This means f(x) is not differentiable at $x = 0$ and $0 \in (-1, 1)$.

This means f(x) is not applicable to Lagrange's Mean Value Theorem on [–1, 1] being not satisfying the essential condition of the theorem.

t. State and prove Lagrange's Mean Value Theorem. Show the $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$ (for a > 0) by using Lagrange's Mean Value Theorem. **[2018 Fall]**

Solution: First part: See Lagrange's Mean Value Theorem.

Second Part: See Exercise 3.1 Q.13.

u. State and prove Rolle's theorem. Verify the theorem for the function $f(x) = \log\left(\frac{x^2 + ab}{(a+b)x}\right)$ for $x \in [a, b]$ for a > 0. **[2017 Fall]**

Solution: First part: See Rolle's theorem.

Second part: See Exercise 3.1 Q.1(iv).

SHORT QUESTIONS

a. If $f(x) = x + \frac{1}{x}$ is a function defined on an interval $\left[\frac{1}{2}, 2\right]$ show that Rolle's theorem can be applied and find $c \in \left(\frac{1}{2}, 2\right)$ such that $f'(c) = 0$. **[1999] [2001]**

Solution: Here, $f(x) = x + \frac{1}{x}$ for $x \in \left[\frac{1}{2}, 2\right]$.

Clearly f(x) is continuous and derivable in $\Re$ except at $x = 0$. So, f(x) is continuous in $\left[\frac{1}{2}, 2\right]$.

And, $f'(x) = 1 - \frac{1}{x^2}$. This exists on R except at x = 0.

That means f(x) is differentiable in $\left(\frac{1}{2}, 2\right)$.

Also, $f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = 2 + \frac{1}{2} = f(2)$.

Thus f(x) satisfies all three conditions of Rolle's theorem then by this theorem there is at least one point $c \in \left(\frac{1}{2}, 2\right)$ such that

$$f'(c) = 0$$

$$\Rightarrow 1 - \frac{1}{c^2} = 0 \Rightarrow c^2 = 1 \Rightarrow c = 1 \in \left(\frac{1}{2}, 2\right).$$

This shows that f(x) is applicable to verify the Rolle's theorem.

b. Find $c \in (-1, 3)$ such that $f'(c) = 0$ due to Rolle's theorem for the function $f(x) = 2x - x^2$ defined on the interval [–1, 3]. **[2000]**

Solution: Clearly the given function $f(x) = 2x - x^2$ is a polynomial function. So, f(x) is continuous on [–1, 3] and is differentiable on (–1, 3).

Also, $f(-1) = -2 - 1 = -3$ and $f(3) = 6 - 9 = -3$.

Thus, $f(-1) = f(3)$.

Thus, f(x) satisfies all the conditions of Rolle's theorem then by this theorem there is at least one point $c \in (-1, 3)$ such that,

$$f'(c) = 0 \Rightarrow 2 - 2c = 0 \Rightarrow c = 1 \in (-1, 3).$$

This shows that f(x) satisfies all the conditions of the Rolle's Theorem. Thus, f(x) verifies the Rolle's Theorem.

c. If $f(x) = \sin x$ be defined in $[0, \pi]$, find 'c' prescribed by Rolle's theorem. [2002]

Solution: Let $f(x) = \sin x$ for $x \in [0, \pi]$.

Since the sine function is continuous and differentiable on the real line. So $f(x)$ is continuous on $[0, \pi]$ and is differentiable on $(0, \pi)$.

And, $f(0) = \sin 0 = 0 = \sin\pi = f(\pi)$.

Thus, $f(x)$ satisfies all three conditions of Rolle's theorem then by the theorem there is at least one point $c \in (0, \pi)$ such that $f'(c) = 0$.

$$\Rightarrow \cos c = 0 = \cos\frac{\pi}{2} \quad \Rightarrow c = \frac{\pi}{2} \in (0, \pi).$$

Thus, $c = \frac{\pi}{2}$ that is prescribed by the Rolle's theorem with $f(x) = \sin x$ in $[0, \pi]$.

d. If $f(x) = x^2 + 2x$, is a function defined on $[1, 3]$ find 'c' prescribed by Lagrange's Mean Value Theorem. [2004, Fall]

Solution: Clearly $f(x) = x^2 + 2x$ is a polynomial function. So, it is continuous on $[1, 3]$ and is differentiable on $(1, 3)$.

Thus, $f(x)$ satisfies both conditions of the Lagrange's Mean Value Theorem. Then by this theorem there is c in $(1, 3)$,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \Rightarrow 2c + 2 = \frac{9 - 1}{2} \Rightarrow 2c = 4 - 2 \Rightarrow c = 1 \in (0, 3).$$

Thus, $c = 1$ that prescribed by the Lagrange's Mean Value Theorem.

e. If $f'(x) > 0$ in $[a, b]$, use Lagrange's Mean Value Theorem to prove $f(x)$ is an increasing function. [2005, Fall]

Solution: See final exam question solution Q. m.

of ti n.m ...

g x_1, x_2 for

for $x_1 < x_2$

Chapter 4

INDETERMINATE FORMS

Definition of Indeterminate Form

In general, if we have a limit of the form $\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)$ where both $f(x)\to 0$, $g(x)\to 0$ as $x\to 0$ then the limit may or may not exist which we called an indeterminate form of $\frac{0}{0}$.

Likely, we observe other forms of indeterminate forms are $\frac{\infty}{\infty}$, $0\times\infty$, $\infty\times 0$, $\infty-\infty$, 0^0, ∞^0.

L'Hospital's Rule

Statement:

Let $f(x)$ and $g(x)$ are two functions such that $f(a) = g(a) = 0$ and also their derivatives $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $g'(a) \neq 0$. Then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) = \frac{f'(a)}{g'(a)}.$$

Proof: Let $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$. Then,

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) = \frac{\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}{\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)}$$

$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right)$$

$$= \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) \quad [\because f(a) = g(a) = 0]$$

Thus, $$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) \text{ for } g'(a) \neq 0.$$

Remember that the L'Hospital rule can be used only when the ratio is of indeterminate form i.e. $\frac{0}{0}$ form in limiting case.

Process to solve an indeterminate form

Form I: If we have the form is of type $\frac{0}{0}$ (in limiting) case) then we apply the L'Hospital rule.

Form II: Since, if $f(a) = \pm\infty$ and $g(a) = \pm\infty$ then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to\infty}\left(\frac{1/g(x)}{1/f(x)}\right)$$

c. **If f(x) = sin x be defined in [0, π], find 'c' prescribed by Rolle's theorem.** [2...]

Solution: Let $f(x) = \sin x$ for $x \in [0, \pi]$.

Since the sine function is continuous and differentiable on the real line, $f(x)$ is continuous on $[0, \pi]$ and is differentiable on $(0, \pi)$.

And, $f(0) = \sin 0 = 0 = \sin\pi = f(\pi)$.

Thus, f(x) satisfies all three conditions of Rolle's theorem then by the theorem there is at least one point $c \in (0, \pi)$ such that $f'(c) = 0$.

$$\Rightarrow \cos c = 0 = \cos\frac{\pi}{2} \Rightarrow c = \frac{\pi}{2} \in (0, \pi).$$

Thus, $c = \frac{\pi}{2}$ that is prescribed by the Rolle's theorem with $f(x) = \sin x$ in $[0, \pi]$.

d. **If $f(x) = x^2 + 2x$, is a function defined on [1, 3] find 'c' prescribed by Lagrange's Mean Value Theorem.** [2004, Fall]

Solution: Clearly $f(x) = x^2 + 2x$ is a polynomial function. So, it is continuous on [1, 3] and is differentiable on (1, 3).

Thus, f(x) satisfies both conditions of the Lagrange's Mean Value Theorem. Then by this theorem there is c in (1, 3),

$$f'(c) = \frac{f(3) - f(1)}{3-1} \Rightarrow 2c + 2 = \frac{9-1}{2} \Rightarrow 2c = 4 - 2 \Rightarrow c = 1 \in (0, 3).$$

Thus, c = 1 that prescribed by the Lagrange's Mean Value Theorem.

e. **If $f'(x) > 0$ in [a, b], use Lagrange's Mean Value Theorem to prove f(x) is an increasing function.** [2005, Fall]

Solution: See final exam question solution Q. m.

Chapter 4
INDETERMINATE FORMS

Definition of Indeterminate Form

In general, if we have a limit of the form $\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)$ where both $f(x)\to 0$, $g(x)\to 0$ as $x\to 0$ then the limit may or may not exist which we called an indeterminate form of $\frac{0}{0}$.

Likely, we observe other forms of indeterminate forms are $\frac{\infty}{\infty}$, $0\times\infty$, $\infty\times 0$, $\infty-\infty$, 0^0, ∞^0.

L'Hospital's Rule

Statement:

Let f(x) and g(x) are two functions such that f(a) = g(a) = 0 and also their derivatives $f'(x)$ and $g'(x)$ are continuous at x = a and $g'(a) \neq 0$. Then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) = \frac{f'(a)}{g'(a)}.$$

Proof: Let $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$. Then,

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) = \frac{\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}{\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)}$$

$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right)$$

$$= \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) \quad [\because f(a) = g(a) = 0]$$

Thus, $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$ for $g'(a) \neq 0$.

Remember that the L'Hospital rule can be used only when the ratio is of indeterminate form i.e. $\frac{0}{0}$ form in limiting case.

Process to solve an indeterminate form

Form I: If we have the form is of type $\frac{0}{0}$ (in limiting) case) then we apply the L'Hospital rule.

Form II: Since, if $f(a) = \pm\infty$ and $g(a) = \pm\infty$ then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to\infty}\left(\frac{1/g(x)}{1/f(x)}\right)$$

This shows that $\frac{\infty}{\infty}$ (in limiting case) is equivalent to $\frac{0}{0}$ (in limiting case).

So, if we have the form is of type $\frac{\infty}{\infty}$ (in limiting case) then we apply L'Hospital rule.

Form III: If the form is of type $(\infty, -\infty)$ (in limiting case) then process the problem to solve as,

$$\lim_{x\to a}[f(x)-g(x)] \quad [\text{This form is in } \infty-\infty \text{ type as } x\to a]$$

$$= \lim_{x\to a}\left[\frac{1}{1/f(x)}-\frac{1}{1/g(x)}\right] \quad \left[\text{This form is in } \frac{1}{0}-\frac{1}{0} \text{ type as } x\to a\right]$$

$$= \lim_{x\to a}\left[\frac{(1/g(x))-(1/f(x))}{1/(f(x)\,g(x))}\right] \quad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to a\right]$$

And, then apply L'Hospital rule.

Form IV: If the form is of type $0\times\infty$ or $\infty\times 0$ (in limiting case) then convert one numerator value to denominator. So that the problem becomes as in either $\frac{0}{0}$ (in limiting case) or $\frac{\infty}{\infty}$ (in limiting case). Then we apply the L'hospital rule.

Note: Remember that choose the function first and second as in the order ILATE (Inverse circular, Logarithm, Algebraic, Trigometric, Exponential) and then convert the second function to denominator.

Form V: If the form is in power type (i.e. in 0^0, ∞^0 or 1^∞) (in limiting case) then process the problem to solve as,

$$\lim_{x\to a}(f(x))^{g(x)} \quad [\text{This form is in either } 0^0 \text{ or } \infty^0 \text{ or } 1^\infty \text{ type as } x\to a]$$

Put,

$$y=(f(x))^{g(x)}$$

Then,

$$\log(y) = g(x)\log(f(x))$$

So,

$$\lim_{x\to a}\log(y) = \lim_{x\to a} g(x)\log(f(x))$$

$$\Rightarrow \log\left(\lim_{x\to a}(y)\right) = \lim_{x\to a} g(x)\log(f(x)) \qquad \text{.......(i)}$$

The right part of (i) will be in either $0\times\infty$ or $\infty\times 0$ (in limiting case) type. Therefore, the process the problem as in form (iv), this will give a solution, say u(a). That is,

$$\log\left(\lim_{x\to a}(y)\right) = u(a)$$

$$\Rightarrow \lim_{x\to a}(y) = e^{u(a)}$$

$$\Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{u(a)}$$

***Note:** Remember that 0^∞, $\infty.\infty$, $\infty+\infty$, ∞^∞, $\infty^{-\infty}$ (in limiting case) are not indeterminate form. So, we can not apply the L'Hopital rule for these forms. Note that $0^\infty = 0$, $\infty.\infty=\infty$, $\infty+\infty=\infty$, $\infty^\infty=\infty$ and $\infty^{-\infty}=0$.*

Exercise 4.1

1. Show that the following limits:

(i) $\lim_{x\to 2}\left(\frac{x^3-2x^2+2x-4}{x^2-5x+6}\right) = -6$

Solution: Here,

$$\lim_{x\to 2}\left(\frac{x^3-2x^2+2x-4}{x^2-5x+6}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 2}\left(\frac{3x^2-4x+2}{2x-5}\right) = \frac{3.4-4.2+2}{2\times 2-5} = -6.$$

Thus, $\lim_{x\to 2}\left(\frac{x^3-2x^2+2x-4}{x^2-5x+6}\right) = -6.$

(ii) $\lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right) = 2$ **[Short, 2004, Spring]**

Solution: Here,

$$\lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sec^2 x - 1}{1-\cos x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{2\sec^2 x\tan x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sec^2 x\sin x}{\cos x\sin x}\right)$$

$$= \lim_{x\to 0}(2\sec^3 x)$$

$$= 2 \qquad [\because \sec 0 = 1]$$

Thus, $\lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right) = 2.$

(iii) $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}$ **for n is positive integer.**

Solution: Here,

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to a}\left(\frac{nx^{n-1}-0}{1}\right) = na^{n-1}.$$

Thus, $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}$.

(iv) $\lim_{x\to 0}\left(\frac{x-\sin^{-1}x}{\sin^3 x}\right) = -\frac{1}{6}$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x-\sin^{-1}x}{\sin^3 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x-\sin^{-1}x}{x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\frac{1}{\sqrt{1-x^2}}}{3x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\frac{1}{2}(1-x^2)^{-3/2}-2x}{6x}\right) = \lim_{x\to 0}\left(\frac{-1}{6}(1-x^2)^{-3/2}\right) = -\frac{1}{6}$$

Thus, $\lim_{x\to 0}\left(\frac{x-\sin^{-1}x}{\sin^{3x}}\right) = -\frac{1}{6}$.

(v) $\lim_{x\to 0}\left(\frac{2\sin x-\sin 2x}{\tan^3 x}\right) = 1.$

Solution : Here,

$$\lim_{x\to 0}\left(\frac{2\sin x-\sin 2x}{\tan^3 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin x-\sin 2x}{x^3}\right) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin x-2\sin x\cos x}{x^3}\right)$$

$$= \lim_{x\to 0}\left[2\left(\frac{1-\cos x}{x^2}\right)\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin x}{2x}\right) = \lim_{x\to 0}(1) = 1. \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

Thus, $\lim_{x\to 0}\left(\frac{2\sin x-\sin 2x}{\tan^3 x}\right) = 1.$

(vi) $\lim_{x\to 1}\left(\frac{1+\log x-x}{1-2x+x^2}\right) = \frac{-1}{2}.$

Solution: Here,

$$\lim_{x\to 1}\left(\frac{1+\log x-x}{1-2x+x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{(1/x)-1}{-2+2x}\right)$$

$$= \lim_{x\to 1}\left(\frac{1-x}{-2x+2x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{-1}{-2+4x}\right) = \frac{-1}{2}.$$

Thus, $\lim_{x\to 1}\left(\frac{1+\log x-x}{1-2x+x^2}\right) = \frac{-1}{2}.$

(vii) $\lim_{x\to 0}\left(\frac{x\,e^x-\log(1+x)}{x^2}\right) = \frac{3}{2}.$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x\,e^x-\log(1+x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x+x\,e^x-\frac{1}{1+x}}{2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{(e^x+x\,e^x)(1+x)-1}{2x(1+x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x+2x\,e^x+x^2e^x-1}{2x+2x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x+2e^x+2xe^x+2xe^x+x^2e^x}{2+4x}\right) = \frac{3}{2}.$$

Thus, $\lim_{x\to 0}\left(\frac{x\,e^x-\log(1+x)}{x^2}\right) = \frac{3}{2}.$

(vii) $\lim_{x\to 0}\left(\frac{\cosh x-\cos x}{x\sin x}\right) = 1.$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{\cosh x-\cos x}{x\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cosh x-\cos x}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{\sinh x+\sin x}{2x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{\cosh x+\cos x}{2}\right) = \frac{\cosh 0+\cos 0}{2} = \frac{2}{2} = 1.$$

Thus, $\lim_{x\to 0}\left(\frac{\cosh x-\cos x}{x\sin x}\right) = 1.$

(ix) $\lim\limits_{t\to0}\left(\dfrac{\sin t^2}{t}\right) = 0.$

Solution: Here,

$$\lim_{t\to0}\left(\frac{\sin t^2}{t}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{t\to0}\left(\frac{\cos t^2 . 2t}{1}\right) = 0.$$

Thus, $\lim\limits_{t\to0}\left(\dfrac{\sin t^2}{t}\right) = 0.$

(x) $\lim\limits_{x\to0}\left(\dfrac{\sin 5x}{x}\right) = 5.$

Solution: Here,

$$\lim_{x\to0}\left(\frac{\sin 5x}{x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\left(5\,\frac{\sin 5x}{5x}\right) = \lim_{x\to0} 5 \quad \left(\lim_{x\to0}\frac{\sin x}{x} = 1\right)$$

$$= 5.$$

Thus, $\lim\limits_{x\to0}\left(\dfrac{\sin 5x}{x}\right) = 5.$

(xi) $\lim\limits_{\theta\to\pi}\left(\dfrac{\sin\theta}{\pi-\theta}\right) = 1.$

Solution : Here,

$$\lim_{\theta\to\pi}\left(\frac{\sin\theta}{\pi-\theta}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta\to\pi}\left(\frac{\cos\theta}{-1}\right) = \frac{\cos\pi}{-1} = \frac{-1}{-1} = 1.$$

Thus, $\lim\limits_{\theta\to\pi}\left(\dfrac{\sin\theta}{\pi-\theta}\right) = 1.$

(xii) $\lim\limits_{x\to\frac{\pi}{4}}\left(\dfrac{\sin x - \cos x}{x - \frac{\pi}{4}}\right) = \sqrt{2}.$ **[2007, Fall (Short)]**

Solution : Here,

$$\lim_{x\to\frac{\pi}{4}}\left(\frac{\sin x - \cos x}{x-\frac{\pi}{4}}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\frac{\pi}{4}}\left(\frac{\cos x + \sin x}{1}\right) = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{1}{\sqrt2} + \frac{1}{\sqrt2} = \frac{2}{\sqrt2} = \sqrt2.$$

Thus, $\lim\limits_{x\to\frac{\pi}{4}}\left(\dfrac{\sin x - \cos x}{x-\frac{\pi}{4}}\right) = \sqrt2.$

2. Prove the following:

(i) $\lim\limits_{x\to0^+}\left(\dfrac{\log\ (\sin x)}{\cot x}\right) = 0.$

Solution : Here,

$$\lim_{x\to0^+}\left(\frac{\log\ (\sin x)}{\cot x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to0^+}\left(\frac{\frac{\cos x}{\sin x}}{(-\operatorname{cosec}^2 x)}\right) = \lim_{x\to0^+}(-\cos x \sin x) = 0.$$

Thus, $\lim\limits_{x\to0^+}\left(\dfrac{\log\ (\sin x)}{\cot x}\right) = 0.$

(ii) $\lim\limits_{x\to0^+}[x \log (x)] = 0$

Solution: Here,

$$\lim_{x\to0^+}[x\log(x)] \qquad [\text{in } 0\times\infty \text{ form}]$$

$$= \lim_{x\to0^+}\left(\frac{\log x}{1/x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to0^+}\left(\frac{1/x}{-1/x^2}\right) = \lim_{x\to0^+}(-x) = 0.$$

Thus, $\lim\limits_{x\to0^+}[x\log(x)] = 0.$

(iii) $\lim\limits_{x\to0^+}[\log_{\tan x}\tan 2x] = 1.$

Solution: Here,

$$\lim_{x\to0^+}[\log_{\tan x}\tan 2x]$$

$$= \lim_{x\to0^+}\left(\frac{\log\tan 2x}{\log\tan x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to0^+}\left(\frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}}\right)$$

> Let $\log_x(y) = u$.
> $\Rightarrow\ y = x^u$.
> So, $\log(y) = u\log(x)$.
> $\Rightarrow u = \dfrac{\log(y)}{\log(x)}$.
> Thus,
> $\log_x(y) = \dfrac{\log(y)}{\log(x)}$.

$$= \lim_{x\to 0^+}\left(\frac{\frac{2}{\cos^2 2x}\times\frac{\cos 2x}{\sin 2x}}{\frac{1}{\cos^2 x}\times\frac{\cos x}{\sin x}}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{2\cos x \sin x}{\cos 2x \sin 2x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{1}{\cos 2x}\right) = \frac{1}{\cos 0} = 1.$$

Thus, $\lim_{x\to 0^+}[\log_{\tan x}\tan 2x] = 1.$

(iv) $\lim_{x\to a}\left[(a-x)\,.\,\tan\left(\frac{\pi x}{2a}\right)\right] = \frac{2a}{\pi}.$

Solution: Here,

$$\lim_{x\to a}\left[(a-x)\,.\,\tan\left(\frac{\pi x}{2a}\right)\right] \qquad \text{(in } 0\times\infty \text{ form)}$$

$$= \lim_{x\to a}\left(\frac{(a-x)}{\cot\left(\frac{\pi x}{2a}\right)}\right) \qquad \left(\text{in } \frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to a}\left(\frac{-1}{\left(\frac{-\pi}{2a}\right)\cdot\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}\right) = (-1)\cdot\left(\frac{-2a}{\pi}\right) = \frac{2a}{\pi}.$$

Thus, $\lim_{x\to a}\left[(a-x)\,.\,\tan\left(\frac{\pi x}{2a}\right)\right] = \frac{2a}{\pi}.$

(v) $\lim_{x\to\infty}\left(\frac{x^n}{e^x}\right) = 0.$

Solution: Here,

$$\lim_{x\to\infty}\left(\frac{x^n}{e^x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{n.x^{n-1}}{e^x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{n(n-1)\,x^{n-2}}{e^x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

Continuing the process up to n^{th} times then,

$$= \lim_{x\to\infty}\left(\frac{n(n-1)\ldots.2.1\,x^0}{e^x}\right) = \frac{n!}{e^\infty} = 0. \quad \left[\text{because } \infty \approx \frac{1}{0}\right]$$

Thus, $\lim_{x\to\infty}\left(\frac{x^n}{e^x}\right) = 0.$

(vi) $\lim_{x\to 0^+}\left(\frac{\log(x^2)}{\log(\cot^2 x)}\right) = -1.$

Solution: Here,

$$\lim_{x\to 0^+}\left(\frac{\log(x^2)}{\log(\cot^2 x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2x}{x^2}}{\frac{1}{\cot^2 x}\,.\,2\cot x\,(-\operatorname{cosec}^2 x)}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cot x}{-x\operatorname{cosec}^2 x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cos x\sin^2 x}{-x\sin x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-\sin 2x}{2x}\right) = -1 \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

Thus, $\lim_{x\to 0^+}\left(\frac{\log(x^2)}{\log(\cot^2 x)}\right) = -1.$

(vii) $\lim_{x\to 0^+}[x\log(\sin^2 x)] = 0.$

Solution: Here,

$$\lim_{x\to 0^+}[x\log(\sin^2 x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to 0^+}\left(\frac{\log(\sin^2 x)}{(1/x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2\sin x.\cos x}{\sin^2 x}}{-1/x^2}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-2\cos x.\,x^2}{\sin x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-2x^2}{\tan x}\right) = (-2)\lim_{x\to 0^+}(x) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= (-2)\times 0 = 0.$$

Thus, $\lim_{x\to 0^+}[x\log(\sin^2 x)] = 0.$

(viii) $\lim_{x\to\frac{\pi}{2}}\left[\sec x\,.\left(x\sin x - \frac{\pi}{2}\right)\right] = -1.$

Solution: Here,

$$\lim_{x\to\frac{\pi}{2}} \left[\sec x \cdot \left(x\sin x - \frac{\pi}{2}\right)\right] \qquad [\text{in } \infty \times 0 \text{ form}]$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{x\sin x - \frac{\pi}{2}}{\cos x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{\sin x + x\cos x}{-\sin x}\right) = \frac{\sin\frac{\pi}{2} + 0\,.\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1.$$

Thus, $\lim_{x\to\frac{\pi}{2}} \left[\sec x \cdot \left(x\sin x - \frac{\pi}{2}\right)\right] = -1.$

(iii) $\lim_{x\to 0^+} [x^m (\log x)^n] = 0,$ **m and n being position integer.**

Solution: Here,

$\lim_{x\to 0^+} [x^m (\log x)^n]$ for m and n being position integer

[in $0\times\infty$ form]

$$= \lim_{x\to 0^+} \frac{(\log x)^n}{x^{-m}} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \left(\frac{n(\log x)^{n-1}\cdot\frac{1}{x}}{-m\,x^{-m-1}}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{n(\log x)^{n-1}}{-m\,x^{-m}}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \left(\frac{n(n-1)(\log x)^{n-2}\cdot\frac{1}{x}}{(-m)(-m)\,x^{-m-1}}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{n(n-1)(\log x)^{n-2}}{(-m)^2\,x^{-m}}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

Continuing the process up to n^{th} times then,

$$= \lim_{x\to 0^+} \left(\frac{n(n-1)\ldots.2.1\,(\log x)^0}{(-m)^n\,x^{-m}}\right)$$

$$= \frac{n!}{(-m)^n} \lim_{x\to 0^+} \left(\frac{1}{x^{-m}}\right) = \frac{n!}{(-m)^n} \lim_{x\to 0^+} (x^m) = \frac{n!}{(-m)^n}\,.\,0 = 0.$$

Then, $\lim_{x\to 0^+} [x^m (\log x)^n] = 0.$

3. Prove the following limits:

(i) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = -\frac{1}{3}.$

Solution: Here,

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to 0} \left(\frac{\sin^2 x - x^2}{x^4\left(\frac{\sin^2 x}{x^2}\right)}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin^2 x - x^2}{x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \quad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0} \left(\frac{2\sin x\cos x - 2x}{4x^3}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin 2x - 2x}{4x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{\cancel{2\cos 2x - 2}}{12x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{-4\sin 2x}{24x}\right)$$

$$= \lim_{x\to 0} \left(\frac{-4}{12}\right) \qquad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \frac{-1}{3}.$$

Thus, $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \frac{-1}{3}.$

(ii) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}.$

Solution: Here,

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to 0} \left(\frac{e^x - 1 - x}{xe^x - x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - 1}{e^x + xe^x - 1} \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x}{e^x + e^x + xe^x} = \frac{e^0}{e^0 + e^0 + 0\,.\,e^0} = \frac{1}{1+1} = \frac{1}{2}.$$

Thus, $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}.$

(iii) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\frac{2}{3}$.

Solution: Here,

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left(\frac{\tan^2 x - x^2}{x^2\tan^2 x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\tan^2 x - x^2}{x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \quad \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x\,.\,\sec^2 x - 2x}{4x^3}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x\,(1+\tan^2 x)-2x}{4x^3}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x + 2\tan^3 x - 2x}{4x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\left(\frac{2\sec^2 x + 6\tan^2 x.\ \sec^2 x - 2}{12x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan^2 x + 6\tan^2 x.\ \sec^2 x}{12x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{\tan^2 x}{x^2}\right)\left(\frac{2+6\sec^2 x}{12}\right)=\left(\frac{2+6\sec^2 0}{12}\right) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$=\frac{8}{12}=\frac{2}{3}.$$

$\frac{\tan^2 x}{x^2}$ 2

Thus, $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\frac{2}{3}$.

(iv) $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]=\frac{1}{2}$.

Solution: Here,

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right] \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\left(\frac{1-\frac{1}{1+x}}{2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1+x-1}{2x(1+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{x}{2x(1+x)}\right)=\lim_{x\to 0}\left(\frac{1}{2(1+x)}\right)=\frac{1}{2}.$$

Thus, $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]=\frac{1}{2}$.

(v) $\lim\limits_{x\to 0^+}(x^x)=1$.

Solution: Let $y = x^x$

Taking log on both sides, we get

$$\log y = x\log x$$

2sec² - 2
2(sec² - 1)
2 tan²

Taking limit $\lim\limits_{x\to 0^+}$ on both sides, we get

$$\lim_{x\to 0^+}\log y=\lim_{x\to 0^+}[x\log x] \qquad (\text{in } 0\times\infty \text{ form}).$$

$$=\lim_{x\to 0^+}\left(\frac{\log x}{\left(\frac{1}{x}\right)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$=\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0.$$

Hence, $\lim\limits_{x\to 0^+}\log y=0 \Rightarrow \lim\limits_{x\to 0^+}y=e^0=1$.

Thus, $\lim\limits_{x\to 0^+}(x^x)=1$.

(vi) $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=1$.

Solution: Let $y=(\sin x)^{\tan x}$

Taking log on both sides, we get

$$\log y=\log(\sin x)^{\tan x}$$

$$\log y=\tan x\,.\,\log(\sin x)$$

Taking $\lim\limits_{x\to\frac{\pi}{2}}$ on both sides then we get,

$$\lim_{x\to\frac{\pi}{2}}\log y=\lim_{x\to\frac{\pi}{2}}[\tan x\log(\sin x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{\log(\sin x)}{\cot x}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}\right]=\lim_{x\to\frac{\pi}{2}}(-\cos x\sin x)=0.$$

$$\text{Now, } \lim_{x\to\frac{\pi}{2}} \log y = 0 \Rightarrow \lim_{x\to\frac{\pi}{2}} y = e^0 = 1.$$

$$\text{Therefore, } \lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = 1.$$

(vii) $\lim\limits_{x\to 0} (\cot x)^{\sin 2x} = 1.$

Solution: Let $y = (\cot x)^{\sin 2x}$

Taking log on both sides, we get

$$\log y = \sin 2x \log (\cot x)$$

Taking $\lim\limits_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} [\sin 2x \log (\cot x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to 0}\left[\frac{\log (\cot x)}{\operatorname{cosec} 2x}\right] \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{-2\operatorname{cosec} 2x \cdot \cot 2x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\tan x\left(\frac{1}{\sin^2 x}\right)}{2\left(\frac{1}{\sin 2x}\right)\times\left(\frac{1}{\tan 2x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{(x)\times\left(\frac{1}{x^2}\right)}{2\left(\frac{1}{2x}\right)\times\left(\frac{1}{2x}\right)}\right]$$

$$\left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{2x}{1}\right] = \frac{0}{1} = 0.$$

$$\text{Now, } \lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$$

$$\text{Thus, } \lim_{x\to 0} (\cot x)^{\sin 2x} = 1.$$

(viii) $\lim\limits_{x\to\pi^-} (\sin x)^{\tan x} = 1.$

Solution: Let $y = (\sin x)^{\tan x}$

Taking log on both sides, we get

$$\log y = \tan x . \log (\sin x)$$

Taking $\lim\limits_{x\to\pi^-}$ on both sides,

$$\lim_{x\to\pi^-} \log y = \lim_{x\to\pi^-} [\tan x . \log (\sin x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to\pi^-}\left(\frac{\log (\sin x)}{\cot x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\pi^-}\left(\frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}\right)$$

$$= \lim_{x\to\pi^-} (-\cos x \sin x) = \lim_{x\to\pi^-}\left(-\frac{1}{2}.\sin 2x\right)$$

$$= \frac{-1}{2}.\sin 2\pi = \frac{-1}{2}\times 0 = 0.$$

Now,

$$\lim_{x\to\pi^-} \log y = 0 \Rightarrow \lim_{x\to\pi^-} y = e^0 = 1.$$

$$\text{Thus, } \lim_{x\to\pi^-} (\sin x)^{\tan x} = 1.$$

(ix) $\lim\limits_{x\to 0} (\cot^2 x)^{\sin x} = 1.$

Solution: Let $y = (\cot^2 x)^{\sin x}$

Taking log on both sides, we get

$$\log y = \sin x \log (\cot^2 x) = \sin x \log (\tan x)^{-2} = -2 \sin x \log(\tan x)$$

$$\text{Taking } \lim_{x\to 0} \log y = (-2)\lim_{x\to 0} [\sin x \log (\tan x)]$$

$$= (-2)\lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)(x)\log (\tan x)\right]$$

$$= (-2)\lim_{x\to 0} [(x)\log (\tan x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= (-2)\lim_{x\to 0}\left[\frac{\log (\tan x)}{1/x}\right] \qquad (\text{in } 0\times\infty \text{ form})$$

$$\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= (-2)\lim_{x\to 0}\left[\frac{\left(\frac{1}{\tan x}\right)\sec^2 x}{-(1/x^2)}\right] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= (-2)\lim_{x\to 0}\left[\frac{\sec^2 x}{-(1/x)}\right] \qquad \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$= (2)\lim_{x\to 0}(x\sec^2 x)$$

$$= 0$$

Now,

$$\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$$

Therefore, $\lim_{x\to 0}(\cot^2 x)^{\sin x} = 1.$

(x) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x} = 1.$ **[2018 Spring][2001][199**

Solution: Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$

Taking log on both sides, we get

$$\log y = \tan x \log\left(\frac{1}{x^2}\right) = \tan x \log(x^{-2}) = (-2)\tan x \log(x)$$

Taking limit $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0}\log y = \lim_{x\to 0}[(-2)\tan x \log(x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to 0}[(-2)\, x \log(x)] \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)(\text{in } 0\times\infty \text{ form})$$

$$= (-2)\lim_{x\to 0}\left[\frac{\log(x)}{(1/x)}\right] \qquad \left[\text{in } \frac{\infty}{\infty}\text{ form}\right]$$

$$= (-2)\lim_{x\to 0}\left[\frac{1/x}{-(1/x^2)}\right]$$

$$= (2)\lim_{x\to 0}(x) = (2)(0) = 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x} = 1.$

(xi) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x} = 1.$ **[2007, Spring]**

Solution: Let $y = \left(\frac{\tan x}{x}\right)^{1/x}$

Taking log on both sides

$$\log y = \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$$

Taking $\lim_{x\to 0}$ both sides, we get

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\left[\frac{1}{x}\log\left(\frac{\tan x}{x}\right)\right] \qquad [\text{in } \infty\times 0 \text{ form}]$$

$$= \lim_{x\to 0}\left[\frac{\log\left(\frac{\tan x}{x}\right)}{x}\right] \qquad \left[\text{in } \frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\left[\left(\frac{x}{\tan x}\right)\frac{(x\sec^2 x - \tan x)}{x^2}\right]$$

$$= \lim_{x\to 0}\left(\frac{x\sec^2 x - \tan x}{x^2}\right) \left[\text{in } \frac{0}{0}\text{ form}\right]\left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$= \lim_{x\to 0}\left(\frac{\sec^2 x + 2x\sec^2 x \cdot \tan x - \sec^2 x}{2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2x\sec^2 x \cdot \tan x}{2x}\right)$$

$$= \lim_{x\to 0}[\sec^2 x \cdot \tan x] = 1 \cdot 0 = 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x} = 1.$

(xii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$ **[2009, Fall] [2008, Spring] [2006, Spring] [2013 Fall][2012 Fall][2005, Spring] [2004, Fall]**

Solution: Let $y = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Taking log on both sides

$$\log y = \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\log y = \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides,

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\left[\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)\right] \qquad [\text{in } \infty\times 0 \text{ form}]$$

$$= \lim_{x\to 0}\left[\frac{\log\left(\frac{\sin x}{x}\right)}{x^2}\right] \qquad \left[\text{in } \frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{x}{\sin x}\left(\frac{x\cos x-\sin x}{x^2}\right)}{2x}\right]$$

$$= \lim_{x\to 0}\left(\frac{x\cos x-\sin x}{2x^3}\right)\left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$= \lim_{x\to 0}\left(\frac{\cos x - x\sin x-\cos x}{6x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-x\sin x}{6x^2}\right) = \frac{-1}{6} \qquad \left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

Now, $\lim_{x\to 0}\log y = \frac{-1}{6} \Rightarrow \lim_{x\to 0} y = e^{-1/6}$

Thus, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

(xii) $\lim_{x\to 0^+}(\cot x)^{1/\log x} = e^{-1} = \frac{1}{e}$.

Solution: Let $y = (\cot x)^{1/\log x}$

Taking log on both sides,

$$\log y = \frac{1}{\log x}\log(\cot x)$$

Taking $\lim_{x\to 0^+}$ on both sides, we get

$$\lim_{x\to 0^+}\log y = \lim_{x\to 0^+}\left(\frac{\log(\cot x)}{\log x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{1/x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-x}{\cos x.\sin x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-1}{\cos x}\right) \qquad \left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$= -\frac{1}{\cos 0°} = -1.$$

Now, $\lim_{x\to 0^+} y = e^{-1} \Rightarrow \lim_{x\to 0^+}(\cot x)^{1/\log x} = e^{-1}$.

(xiv) $\lim_{x\to\pi/2}(\sec x - \tan x) = 0.$

Solution: Here,

$$\lim_{x\to\pi/2}(\sec x - \tan x) \qquad (\text{in } \infty-\infty \text{ form})$$

$$= \lim_{x\to\pi/2}\left(\frac{1-\sin x}{\cos x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\pi/2}\left(\frac{-\cos x}{-\sin x}\right) = \lim_{x\to\pi/2}(\cot x) = 0.$$

Thus, $\lim_{x\to\pi/2}(\sec x - \tan x) = 0.$

$\frac{1}{\cos x} - \frac{\sin x}{\cos x}$

$\frac{\cos x - \sin x}{\cos x}$

(xv) $\lim_{x\to 0}(x^{-1} - \cot x) = 0.$

Solution: Here,

$$\lim_{x\to 0}(x^{-1}-\cot x) \qquad (\text{in } \infty-\infty \text{ form})$$

$$= \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\tan x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan x - x}{x\tan x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan x - x}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$= \lim_{x\to 0}\left(\frac{\sec^2 x - 1}{2x}\right) = \lim_{x\to 0}\left(\frac{\tan^2 x}{2x^2}.x\right) = \lim_{x\to 0}\left(\frac{x}{2}\right) = 0.$$

Thus, $\lim_{x\to 0}(x^{-1}-\cot x) = 0.$

(xvi) $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right] = \frac{1}{2}.$

Solution: Here,

$$\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right] \qquad (\text{in } \infty-\infty \text{ form})$$

$$= \lim_{x\to 1}\left[\frac{x\log x - x + 1}{(x-1)\log x}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1}\left[\frac{\log x + 1 - 1}{\log x + (x-1)\times\frac{1}{x}}\right]$$

$$= \lim_{x\to 1}\left(\frac{x\log x}{x\log x + x - 1}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{\log x + 1}{\log x + 1 + 1}\right) = \frac{1}{2}.$$

Thus, $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right] = \frac{1}{2}.$

Sec x - tan x

(xvii) $\lim_{x\to 2}\left[\frac{4}{x^2-4}-\frac{1}{x-2}\right] = -\frac{1}{4}.$

$\lim_{x\to\pi/2}\left(\frac{1-\sin x}{\cos x}\right)$

Solution: Here,

$$\lim_{x\to 2}\left[\frac{4}{x^2-4}-\frac{1}{x-2}\right]$$

$$= \lim_{x\to 2}\left[\frac{4}{(x-2)(x+2)}-\frac{1}{x-2}\right]$$

$$= \lim_{x\to 2}\left[\frac{4-(x+2)}{(x-2)(x+2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{-(x-2)}{(x-2)(x+2)}\right] = \lim_{x\to 2}\left[\frac{-1}{(x+2)}\right] = \frac{-1}{4}.$$

Thus, $\lim_{x\to 2}\left[\frac{4}{x^2-4} - \frac{1}{x-2}\right] = \frac{-1}{4}.$

(xviii) $\lim_{x\to\infty}(x-\sqrt{x^2-9}) = 0.$

Solution: Put $x = \frac{1}{y}$ for $y \neq 0$, then $y \to 0$ as $x \to \infty$.

Here,

$$\lim_{x\to\infty}(x-\sqrt{x^2-9}) = \lim_{y\to 0}\left(\frac{1}{y} - \sqrt{\frac{1}{y^2}-9}\right)$$

$$= \lim_{y\to 0}\left(\frac{1-\sqrt{1-9y^2}}{y}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to 0}\left(\frac{-\left(\frac{1}{2}\right)\frac{1}{\sqrt{1-9y^2}}(-18y)}{1}\right)$$

$$= \lim_{y\to 0}\left(\frac{9y}{\sqrt{1-9y^2}}\right) = \frac{0}{1} = 0$$

Thus, $\lim_{x\to\infty}(x-\sqrt{x^2-9}) = 0.$

(xix) $\lim_{x\to\infty}[\sqrt{x^2+2x} - x] = 1.$

Solution: Put $x = \frac{1}{y}$ for $y \neq 0$, then $y \to 0$ as $x \to \infty$.

Here,

$$\lim_{x\to\infty}[\sqrt{x^2+2x} - x]$$

$$= \lim_{y\to 0}\left[\sqrt{\frac{1}{y^2}+2\left(\frac{1}{y}\right)} - \frac{1}{y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\sqrt{1+2y}-1}{y}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to 0}\left[\frac{\frac{1}{2}\frac{1}{\sqrt{1+2y}}(2)}{1}\right] = \lim_{y\to 0}\left[\frac{1}{\sqrt{1+2y}}\right] = 1.$$

Thus, $\lim_{x\to\infty}[\sqrt{x^2+2x} - x] = 1.$

(xx) $\lim_{x\to\infty}\frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \sqrt{\frac{3}{2}}.$

Solution: Here,

$$\lim_{x\to\infty}\frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \lim_{x\to\infty}\left[\frac{\sqrt{x}}{\sqrt{x}}\left(\frac{\sqrt{3+4/x}}{\sqrt{2+3/x}}\right)\right] = \frac{\sqrt{3+0}}{\sqrt{2+0}} = \sqrt{\frac{3}{2}}.$$

Thus, $\lim_{x\to\infty}\frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \sqrt{\frac{3}{2}}.$

lim (x - √(x²-9))
x→∞
x = 1/y
y - √(1/y² - 9)

(xxi) $\lim_{x\to 0} x^{2x} = 1.$

Solution: Let $y = x^{2x}$

Taking log on both side, we get

$$\log(y) = 2x \log x$$

Taking $\lim_{x\to 0}$ on both sides,

$$\lim_{x\to 0}\log y = \lim_{x\to 0}[2x\log x] \qquad [\text{ in } 0\times\infty \text{ form}]$$

$$= 2\lim_{x\to 0}\left(\frac{\log x}{(1/x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= 2\lim_{x\to 0}\left[\frac{1/x}{-1/x^2}\right] = (-2)\lim_{x\to 0}\left(\frac{1}{x}\right) = (-2)\times 0 = 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Therefore, $\lim_{x\to 0} x^{2x} = 1.$

x = 1/y

(xxii) $\lim_{x\to 0} x^{2\sin x} = 1$

Solution: Let $y = x^{2\sin x}$

Taking log on both sides, we get

$$\log y = 2\sin x \log x$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0}\log y = \lim_{x\to 0}[2\sin x\log x] \qquad [\text{in } 0\times\infty \text{ form}]$$

$$= (2)\lim_{x\to 0}\left(\frac{x\sin x}{x}.\log x\right)$$

$$= (2)\lim_{x\to 0}\left(\frac{\log x}{(1/x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= (2)\lim_{x\to 0}\left(\frac{1/x}{-1/x^2}\right) = (2)\lim_{x\to 0}(-x) = 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1$

Thus, $\lim_{x\to 0} x^{2\sin x} = 1.$

(xxiii) $\lim_{x\to 0}(\sin x)^{2\tan x} = 1$

Solution: Let $y = (\sin x)^{2\tan x}$

Taking log on both sides, we get

$$\log y = 2\tan x \log(\sin x)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} [2\tan x \log(\sin x)] \quad [\text{in } 0\times\infty \text{ form}]$$

$$= (2)\lim_{x\to 0}\left(\frac{x\tan x}{x}\cdot\log(\sin x)\right)$$

$$= (2)\lim_{x\to 0}\left(\frac{\log(\sin x)}{(1/x)}\right)\quad\left[\text{in }\frac{\infty}{\infty}\text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x}\right.$$

$$= (2)\lim_{x\to 0}\left(\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}\right)$$

$$= (2)\lim_{x\to 0}(-x\cos x)\left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$= (-2)\times 0$$

$$= 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0}(\sin x)^{2\tan x} = 1.$

(xxiv) $\lim_{x\to 1} x^{[1/(1-x)]} = \frac{1}{e}$

Solution: Let $y = x^{[1/(1-x)]}$

Taking log on both sides, we get

$$\log y = \frac{1}{1-x}\log x$$

Taking $\lim_{x\to 1}$ on both sides, we get

$$\lim_{x\to 1}\log y = \lim_{x\to 1}\left[\left(\frac{1}{1-x}\right)\log x\right]\quad[\text{in } 0\times\infty \text{ form}]$$

$$= \lim_{x\to 1}\left(\frac{\log(x)}{1-x}\right)\quad\left[\text{in }\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{1/x}{-1}\right) = -1.$$

Now, $\lim_{x\to 0}\log y = -1 \Rightarrow \lim_{x\to 0} y = e^{-1} = \frac{1}{e}.$

Thus, $\lim_{x\to 0} x^{[1/(1-x)]} = \frac{1}{e}.$

(xxv) $\lim_{x\to\infty}(\log x)^{1/x} = 1.$

Solution: Let $y = (\log x)^{1/x}$

Taking log on both sides, we get

$$\log y = \frac{\log x}{x}$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty}\log y = \lim_{x\to\infty}\left(\frac{\log x}{x}\right)\quad\left[\text{in }\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0.$$

Now, $\lim_{x\to\infty}\log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$

Thus, $\lim_{x\to\infty}(\log x)^{1/x} = 1.$

$\frac{1}{(1-x)}$ $\log$

$y = x$

$\log y = \frac{1}{(1-x)}$

(xxvi) $\lim_{x\to 0}(e^x + x)^{1/x} = e^2.$ **[2009 Spring]**

Solution: Put $y = (e^x + x)^{1/x}$

Taking log on both sides then,

$$\log y = \frac{1}{x}\log(e^x + x) = \frac{\log(e^x+x)}{x}$$

Taking $\lim_{x\to 0}$ on both sides then,

$$\log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}(\log y) = \lim_{x\to 0}\left(\frac{\log(e^x+x)}{x}\right)\quad\left(\text{in }\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{e^x+x}\cdot(e^x+1)\right]$$

$$= \frac{1+1}{1+0} = \frac{2}{1} = 2.$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}(\log y) = 2.$$

$$\Rightarrow \left(\lim_{x\to 0} y\right) = e^2.$$

$$\Rightarrow \lim_{x\to 0}(e^x+x)^{1/x} = e^2.$$

(xvii) $\lim_{x\to\infty}\left[1+\frac{1}{x^2}\right]^x = 1.$ **[2013 Spring]**

Solution:

Let $y = \left[1+\frac{1}{x^2}\right]^x$

Taking log both sides, we get

$$\log y = x\log\left(1+\frac{1}{x^2}\right)$$

Taking $\lim_{x\to\infty}$ on both sides

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty}\left[x \log\left(1+\frac{1}{x^2}\right)\right] \quad [\text{in } \infty \times 0 \text{ form}]$$

$$= \lim_{x\to\infty}\left[\frac{\log\left(1+\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2}{x\left(1+\frac{1}{x^2}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2x}{x^2+1}\right] \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2}{2x}\right] = \lim_{x\to\infty}\left[\frac{1}{x}\right] = 0.$$

Now, $\lim_{x\to\infty} \log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$

Thus, $\lim_{x\to\infty}\left[1+\frac{1}{x^2}\right]^x = 1.$

(xxviii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1.$ **[2016 Fall][2003, Spring**

Solution:

Let, $y = \left(\frac{\sin x}{x}\right)^{1/x}$

Taking log on both sides, we get

$$\therefore \log(y) = \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left[\frac{\log\left(\frac{\sin x}{x}\right)}{x}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{x\cos x - \sin x}{x^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{\cos x - x\sin x - \cos x}{2x}\right] = \lim_{x\to 0}\left[\frac{-\sin x}{2}\right] = 0$$

Now, $\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1.$

(xxix) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$ **[2015 Fall][2002]**

Solution:

Let $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$

Taking log both sides, we get

$$\log y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left[\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{x}{\tan x}\cdot\left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x\sec^2 x - \tan x}{2x^3}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{x}{\tan x} = 1\right).$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x + 2\sec^2 x \cdot \tan x \cdot x - \sec^2 x}{6x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x\sec^2 x \cdot \tan x}{6x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x}{3}\cdot\frac{\tan x}{x}\right] \quad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x}{3}\right].$$

$$= \frac{1}{3}.$$

Now,

$$\lim_{x\to 0} \log y = \frac{1}{3} \Rightarrow \lim_{x\to 0} y = e^{1/3}$$

Thus, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}.$

4. Show that

(i) $\lim_{x\to\infty}(1+2x)^{1/2\log(x)} = \infty.$

Solution: Let $y = (1+2x)^{1/2\log(x)}$

Taking log both sides, we get

$$\log y = \frac{1+2x}{2\log(x)}$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty}\left(\frac{1+2x}{2\log(x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{2}{2/x}\right) = = \lim_{x\to\infty}(x) = \infty.$$

Now, $\lim_{x\to\infty} \log y = \infty \Rightarrow \lim_{x\to\infty} y = e^{\infty} = \infty.$

Thus, $\lim_{x\to\infty} (1+2x)^{1/2\log(x)} = \infty.$

Answer to be corrected in the book.

(ii) $\lim_{x\to\infty} x^{1/x} = 1.$

Solution: Let $y = x^{1/x}$

Taking log on both sides, we get

$$\log y = \frac{1}{x}\log x$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty}\left(\frac{\log(x)}{x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0.$$

Now, $\lim_{x\to\infty} \log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$

Thus, $\lim_{x\to\infty} x^{1/x} = 1.$

(iii) $\lim_{x\to 0}\left(\frac{xe^x-(1+x)\log(1+x)}{x^2}\right) = \frac{1}{2}$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{xe^x-(1+x)\log(1+x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + xe^x - \log(1+x) - 1}{2x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x+e^x+xe^x-\frac{1}{1+x}}{2}\right)$$

$$= \frac{2e^0 + 0 \cdot e^0 - \frac{1}{1+0}}{2} = \frac{1}{2}.$$

Thus, $\lim_{x\to 0}\left(\frac{xe^x-(1+x)\log(1+x)}{x^2}\right) = \frac{1}{2}.$

(iv) $\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$

Solution: Put, $y = \lim_{x\to 0} (\cos x)^{1/x^2}$

Taking log on both sides, we get

$$\log y = \frac{1}{x^2}\log(\cos x)$$

Taking $\lim_{x\to 0}$ on both sides,

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left(\frac{\log(\cos x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\frac{\sin x}{\cos x}}{2x}\right)$$

$$= \lim_{x\to 0}\left(-\frac{\tan x}{2x}\right) = \frac{-1}{2} \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right).$$

Now, $\lim_{x\to 0} \log y = -\frac{1}{2} \Rightarrow \lim_{x\to 0} y = e^{-1/2}.$

Thus, $\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}.$

(v) $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan(\pi x/2a)} = e^{2/\pi}$

Solution: Let $y = \left(2-\frac{x}{a}\right)^{\tan(\pi x/2a)}$

Taking log on both sides, we get

$$\log y = \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

Taking $\lim_{x\to a}$ on both sides

$$\lim_{x\to a} \log y = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right) \qquad [\text{in } \infty\times 0 \text{ form}]$$

$$= \lim_{x\to a}\left(\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to a}\frac{\left(\frac{1}{2-(x/a)}\right)\cdot\left(-\frac{1}{a}\right)}{\left(-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\right)\cdot\frac{\pi}{2a}}$$

$$= \lim_{x\to a}\left(\frac{2a\sin^2\left(\frac{\pi x}{2a}\right)}{\pi\,(2a-x)}\right) = \frac{2a}{\pi a} = \frac{2}{\pi}.$$

Now, $\lim_{x\to a}\log y = \frac{2}{\pi} \Rightarrow \lim_{x\to a} y = e^{2/\pi}$

Thus, $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan(\pi x/2a)} = e^{2/\pi}$.

(vi) $\lim_{x\to 0}\left(\frac{x\cos x - \log(1+x)}{x^2}\right) = \frac{1}{2}$.

Solution : Here,

$$\lim_{x\to 0}\left(\frac{x\cos x - \log(1+x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\sin x - x\cos x - \sin x + \frac{1}{(1+x)^2}}{2}\right)$$

$$= \frac{0-0-0+1}{2} = \frac{1}{2}.$$

Thus, $\lim_{x\to 0}\left(\frac{x\cos x - \log(1+x)}{x^2}\right) = \frac{1}{2}$.

(vii) $\lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}\right) = \frac{1}{30}$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} + 2\cos x - 4}{5x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2\sin x}{20x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} - 2\cos x}{60x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x}{120x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} + 2\cos x}{120}\right) = \frac{e^0 + e^{-0} + 2\cos 0}{120} = \frac{4}{120} = \frac{1}{30}$$

Thus, $\lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}\right) = \frac{1}{30}$.

e Method

$$\lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{2\sinh x + 2\sin x - 4x}{x^5}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{2\cosh x + 2\cos x - 4}{5x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{2\sinh x - 2\sin x}{20x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{2\cosh x - 2\cos x}{60x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sinh x + 2\sin x}{120x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\cosh x + 2\cos x}{120}\right) = \frac{2+2}{120} = \frac{4}{120} = \frac{1}{30}.$$

Thus, $\lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}\right) = \frac{1}{30}$.

(viii) $\lim_{x\to 0}\left(\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}\right) = 0.$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\sin x - \frac{1}{1+x} + \cos x}{e^x - 1}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\cos x + \frac{1}{(1+x)^2} - \sin x}{e^x}\right) = \frac{-1 + \frac{1}{(1+0)^2} - 0}{1} = \frac{-1+1}{1} = 0.$$

Thus, $\lim_{x\to 0}\left(\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}\right) = 0.$

(ix) $\lim_{x\to 2}\left(\frac{x^2-4}{\sqrt{x+2} - \sqrt{3x-2}}\right) = -8$

Solution: Here,

$$\lim_{x\to 2}\left(\frac{x^2-4}{\sqrt{x+2} - \sqrt{3x-2}}\right)$$

$$= \lim_{x\to 2}\left(\frac{x^2-4}{(\sqrt{x+2} - \sqrt{3x+2})} \times \frac{(\sqrt{x+2} + \sqrt{3x-2})}{(\sqrt{x+2} + \sqrt{3x-2})}\right)$$

$$= \lim_{x\to 2}\left(\frac{(x^2-4)(\sqrt{x+2} + \sqrt{3x-2})}{x+2-3x+2}\right)$$

$$= \lim_{x\to 2}\left(\frac{(x-2)(x+2)(\sqrt{x+2} + \sqrt{3x-2})}{-2\,(x-2)}\right)$$

$$= \lim_{x\to2}\left[\left(-\frac{1}{2}\right)(x+2)\left(\sqrt{x+2}+\sqrt{3x-2}\right)\right]$$

$$= \left(-\frac{1}{2}\right)\times4\times(2+2) = -8.$$

Thus, $\lim_{x\to2}\left(\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right) = -8.$

(x) $\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right] = \frac{1}{2}$ **[2017 Fall]**

Solution: Here,

$$\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{y\to0}\left[\frac{1}{y}-\frac{1}{y^2}\log(1+y)\right] \quad \left[\text{put } y=\frac{1}{x} \text{ then as } x\to\infty \Rightarrow y\to0\right]$$

$$= \lim_{y\to0}\left[\frac{y-\log(1+y)}{y^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to0}\left[\frac{1-1/(1+y)}{2y}\right]$$

$$= \lim_{y\to0}\left[\frac{1+y-1}{(1+y)\,2y}\right] = \lim_{y\to0}\left[\frac{1}{(1+y)\,2}\right] = \frac{1}{2}.$$

Thus, $\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right] = \frac{1}{2}.$

5. If $\lim_{x\to0}\left(\frac{a\sin x - \sin 2x}{\tan^3 x}\right)$ **is finite show that a = 2 and limit is 1.**

[2011 Spring][2006, Fal

Solution: Here,

$$\lim_{x\to0}\left(\frac{a\sin x-\sin 2x}{\tan^3 x}\right)$$

$$= \lim_{x\to0}\left(\frac{a\sin x-\sin 2x}{x^3\cdot\left(\frac{\tan x}{x}\right)^3}\right)$$

$$= \lim_{x\to0}\left(\frac{a\sin x-\sin 2x}{x^3}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to0}\frac{\tan x}{x}=1\right)$$

$$= \lim_{x\to0}\left(\frac{a\cos x-2\cos x}{3x^2}\right) \quad \left[\frac{a-2}{0}\right]$$

[Givan that the limit is finite so we must have $a-2=0 \Rightarrow a=2$]. Then,

$$= \lim_{x\to0}\left(\frac{2\cos x-2\cos 2x}{3x^2}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\left(\frac{-2\sin x+4\sin 2x}{6x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\left(\frac{-2\cos x+8\cos 2x}{6}\right) = \frac{-2+8}{6} = 1.$$

Thus, the value of a is 2 and limit of the form is 1.

6. If $\lim_{x\to0}\left(\frac{x(1+a\cos x)-b\sin x}{x^3}\right) = 1$**. Show that** $a=-\frac{5}{2}$**, and** $b=-\frac{3}{2}$**.**

Solution: Here,

$$\lim_{x\to0}\left(\frac{x(1+a\cos x)-b\sin x}{x^3}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\left(\frac{1+a\cos x-ax\sin x-b\cos x}{3x^2}\right)$$

1 + a − b = 0

$$= \lim_{x\to0}\left(\frac{1+(a-b)\cos x-ax\sin x}{3x^2}\right)\left(\text{has } \frac{1+a-b}{0} = \frac{\text{finite}}{0} \text{ form}\right)$$

As the limit exist we must have $a-b+1=c$

$\Rightarrow a-b=-1$ ...... (i)

So, suppose that the above form has $\frac{0}{0}$ form. Then,

$$= \lim_{x\to0}\left(\frac{-(a-b)\sin x-a\sin x-ax\cos x}{6x}\right)$$

$$= \lim_{x\to0}\left(\frac{-(2a-b)\sin x-ax\cos x}{6x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\left(\frac{-(2a-b)\cos x-a\cos x+ax\sin x}{6}\right)$$

$$= \lim_{x\to0}\left(\frac{-(3a-b)\cos x+ax\sin x}{6}\right) = \left(\frac{-(3a-b)}{6}\right) = \frac{-3a+b}{6}$$

We have,

$$\lim_{x\to0}\left(\frac{x(1+a\cos x)-b\sin x}{x^3}\right)=1 \Rightarrow \frac{-3a+b}{6}=1$$

From (i), $a=b-1$

So, $-3(b-1)+b=6. \Rightarrow -3b+3+b=6$

$\Rightarrow -2b=3 \Rightarrow b=-\frac{3}{2}.$

Then, $a=-\frac{3}{2}-1=-\frac{5}{2} \Rightarrow a=-\frac{5}{2}.$

Thus, $a=-\frac{5}{2}$ and $b=-\frac{3}{2}$.

7. Show that a = 1, b = 2, c = 1, when $\lim_{x\to0}\left[\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}\right]=2$**.**

Solution: Here,

$$\lim_{x\to0}\left[\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}\right] \qquad \left(\frac{a-b+c=0}{0}\right)$$

$$= \lim_{x\to0}\left(\frac{ae^{2x}-be^x\cos x+ce^{-x}}{x^2}\right)\left[\text{in } \frac{a-b+c}{0} \text{ form}\right]\left[\lim_{x\to0}\frac{\sin x}{x}=1\right]$$

As the limit exists, we must have

$a-b+c=0$ ...... (i)

then the above form becomes as $\frac{0}{0}$ form. So,

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \qquad \left[\text{in } \frac{a-c}{0} \text{ form}\right]$$

As the limit exist, we must have get

$$a - c = 0, \quad a = c \qquad \text{........ (ii)}$$

then the above form becomes as $\frac{0}{0}$ form. So,

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{2}$$

$a - c \neq 0$

$$= \frac{a+b+c}{2} \qquad \text{......... (iii)}$$

Given that,

$$\lim_{x\to 0}\left[\frac{ae^x - b\cos x + ce^{-x}}{x\sin x}\right] = 2.$$

$$\Rightarrow \frac{a+b+c}{2} = 2 \Rightarrow a+b+c = 4 \qquad \text{......(iv)} \quad \text{[by (iii)]}$$

From (i) and (ii), we get

$$2a - b = 0 \Rightarrow b = 2a \qquad \text{... (v)}$$

From equations (ii), (iv) and (v) we get,

$$a + 2a + a = 4 \Rightarrow a = 1,$$

Therefore, $b = 2a = 2$ and $c = a = 1$.

Thus, $a = 1$, $b = 2$ and $c = 1$.

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

Long Questions

a. State L' Hospital's rule and evaluate: $\lim_{x\to\frac{\pi}{2}} \left(\frac{\tan x}{x}\right)^{1/x}$. **[2000]**

Solution: See statement of L' Hospital's rule and question is mistake. We should replace $\frac{\pi}{2}$ by 0 then see Q. 3(xi).

b. State the L' Hospital Rule and evaluate the limit: $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$. **[2014 Spring][2007, Fall] [2003, Fall]**

Solution: See statement of L' Hospital's rule and see Q. 3(i).

c. Evaluate: $\lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$. **[2005, Fall]**

Solution: Put $y = \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$

So,

$$\log(y) = \frac{\log\left(\frac{\pi}{2} - \tan^{-1} x\right)}{x}$$

So, taking $\lim_{x\to\infty}$ on both sides then,

$$\lim_{x\to\infty} \log(y) = \lim_{x\to\infty} \frac{\log\left(\frac{\pi}{2} - \tan^{-1} x\right)}{x} \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty} \left(\frac{1}{\frac{\pi}{2} - \tan^{-1} x}\right) \cdot \left(\frac{-1}{x^2+1}\right) \cdot 1$$

$$= \lim_{x\to\infty} \frac{-1}{\left(\frac{\pi}{2} - \tan^{-1} x\right)(x^2+1)} = \frac{-1}{\infty} = 0.$$

$-\frac{1}{\infty} \neq 0$

Thus, $\lim_{x\to\infty} \log(y) = \log\left(\lim_{x\to\infty} y\right) = 0$

$$\Rightarrow \lim_{x\to\infty}(y) = \lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x} = e^0 = 1.$$

d. Evaluate $\lim_{x\to 0} (\cos x)^{\cot^2 x}$ **[2018 Fall][2008, Fall]**

2017 Spring: State L.Hospital rule and evaluate the limit: $\lim_{x\to 0} (\cos x)^{\cot^2 x}$.

Solution: Let $y = (\cos x)^{\cot^2 x}$

Taking log on both sides we get,

$$\log y = \log(\cos x)^{\cot^2 x} = \cot^2 x \log \cos x$$

i.e. $\log y = \frac{\log(\cos x)}{\tan^2 x}$

Taking $\lim_{x\to 0}$ on both sides, we get

$$\lim_{x\to 0}(\log y) = \lim_{x\to 0}\left(\frac{\log(\cos x)}{\tan^2 x}\right) \qquad \left(\text{in } \frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0}\left(\frac{\log(\cos x)}{x^2}\right) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{-\sin x}{\cos x}}{2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\tan x}{2x}\right) = \frac{-1}{2}. \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

Hence,

$$\lim_{x\to 0}(\log y) = \frac{-1}{2} \Rightarrow \log\left(\lim_{x\to 0} y\right) = \frac{-1}{2}$$

$$\Rightarrow \lim_{x\to 0} y = e^{-1/2}$$

Thus, $\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$

e. **State L' Hospital's Theorem. Evaluate:** $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$. [2011 Fa

Solution: See statement of L' Hospital's rule and see Q. 3(xii).

f. **Define indeterminate forms. State L' Hospital's rule and show th** $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1$. [2014 Fa

Solution: See definition of indeterminate forms and statement of L' Hospi rule

Problem Part: See Q. 3(xxviii).

g. **Evaluate:** $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$ [2015 Spring

Solution:

Let $y = \left(\frac{\tan x}{x}\right)^{1/x}$

Taking log both sides, we get

$$\log y = \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\left[\frac{\log\left(\frac{\tan x}{x}\right)}{x}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{x}{\tan x}\cdot\left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{1}\right]$$

$$= \lim_{x\to 0}\left[\frac{x\sec^2 x - \tan x}{x^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{x}{\tan x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x + 2\sec^2 x \cdot \tan x \cdot x - \sec^2 x}{2x}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x\sec^2 x \cdot \tan x}{2x}\right]$$

$$= \lim_{x\to 0}[\sec^2 x \tan x],$$

$$= 0$$

Now,

$$\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1$$

Thus, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x} = 1$.

h. **State L' Hospital's rule for indeterminate forms. Evaluate:** $\left(\frac{\tan x}{x}\right)^{1/x^2}$. [2016 Spring]

...ee statement of L' Hospital's rule and see Q. 3(xxix).

Short Questions

a. **Evaluate:** $\lim_{x\to 0}\frac{x}{|x|}$. [2000]

Solution: Since we have |x| in denominator and we know the function |x| is not differentiable at x = 0. Therefore, the L'Hospital rule does not work here.

b. **Evaluate** $\lim_{x\to 0}\frac{x - \sin x}{x^3}$. [2002]

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x - \sin x}{x^3}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{1 - \cos x}{3x^2}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{6x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{1}{6}\right) \quad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \frac{1}{6}.$$

Thus, $\lim_{x\to 0}\left(\frac{x - \sin x}{x^3}\right) = \frac{1}{6}$.

c. **Evaluate:** $\lim_{x\to 0}\frac{\tan x - x}{x^3}$. [2005, Spring] [2004, Fall] [2002]

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x - \tan x}{x^3}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{1 - \sec^2 x}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\tan^2 x}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-1}{3}\right) \quad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= -\frac{1}{3}.$$

Thus, $\lim_{x\to 0}\left(\frac{x - \tan x}{x^3}\right) = -\frac{1}{3}$.

•••

Some Important Formulae

1. The sphere has volume $V = \frac{4}{3}\pi r^3$ and surface area $A = 4\pi r^2$ if r is the ra[dius] of sphere.
2. The volume of the circular cylinder is $V = \pi r^2 h$ where r = radius of the [base] and h = height of the cylinder.

   And base area of the circular cylinder $A = \pi r^2$

   Total surface area $S = 2\pi rh + \pi r^2 + \pi r^2 = 2\pi r\,[r + h]$
3. Important formulae for Cone
   - The volume of the circular cone is $V = \frac{1}{3}\pi r^2 h$

     where r is the radius of base and h the height of the cone.
   - Semi–vertical angle $\alpha = \tan^{-1}\left(\frac{r}{h}\right)$
   - Slant height $= \sqrt{r^2 + h^2}$
   - Curved surface $(S) = \pi r \times$ slant height

     $= \pi r\sqrt{r^2 + h^2}$

Extreme of f(x)

Let f(x) is defined on D. Then the value f(c) is maxima of f(x) on D if

$f(x) \le f(c)$ for all x in D.

And, f(d) is minima of f(x) on D if

$f(x) \ge f(d)$ for all x in D.

The maxima or minima of f(x) is known as extrema of f(x).

Extreme point:

Let f(x) is a function on D. Then the point α in D is called extreme point or critical point of f(x) on D if $f'(\alpha) = 0$ or $f'(\alpha)$ does not exist.

Criteria for Extreme Values

* The necessary condition for f(c) to be the extreme value of f(x) is $f'(c) = 0$, if it exists.
* If $f''(c) \ne 0$. Then f(c) is maximum if $f''(c) < 0$ and f(c) is minimum if $f''(c) > 0$.

Process to Find Extreme Value

Step I : Modeling the given problem.

Step II : With the help of given condition (constraint), develop the function [a]s a single variable.

Step III: Find f ' and f ".

Step IV : For critical point, set f ' = 0 and by solving it, find the critical point.

Step V : At the critical point, observe the value of f ".

Case A : If f " > 0 then f attains its minima at the critical point.

Case B : If f " < 0 then f attains its maxima at the critical point.

Step VI : Find the maximum or minimum value of f at the critical point.

Exercise 5.1

1. **Find the maximum and minimum values of the following functions:**

(i) $f(x) = x + \frac{1}{x}$

Solution: Given, $f(x) = x + \frac{1}{x}$

So, $f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$.

For extreme point, set

$$f'(x) = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1.$$

At x = 1,

$$f''(1) = \frac{2}{(1)^3} = 2 > 0.$$

So f(x) has minimum value at x = 1. And minimum value is f(1) = 2.

At x = –1

$$f''(-1) = \frac{2}{(-1)^3} = -2 < 0.$$

So f(x) has maximum value at x = –1. And maximum value is

$$f(-1) = -1 - 1 = -2.$$

Thus, f(x) has maximum value 2 at x = 1 and minimum value at x = –1.

(ii) $f(x) = x^3 - 3x^2 + 6x + 3$ **(Question wrong)**

Solution: Let $f(x) = x^3 - 3x^2 + 6x + 3$

So, $f'(x) = 3x^2 - 6x + 6$ and $f''(x) = 6x - 6$.

For extreme point, set

$$f'(x) = 0$$
$$\Rightarrow 3x^2 - 6x + 6 = 0$$
$$\Rightarrow x^2 - 2x + 2 = 0$$
$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

Thus, x has imaginary value. So, the solution is impossible.

(iii) $f(x) = \cos x$

Solution: Let, $f(x) = \cos x$

So, $f'(x) = -\sin x$ and $f''(x) = -\cos x$.

For extreme point, set

$$f'(x) = 0 \Rightarrow -\sin x = 0 = \sin n\pi \Rightarrow x = n\pi \text{ for } n \text{ is integer.}$$

At $x = 0$,

$$f''(x) = -1 < 0, \text{ so } f(x) \text{ has maximum value at } x = 0.$$

If n is even integer i.e. at $x = 2n\pi$,

$$f''(2n\pi) = -\cos(2n\pi) = -1 < 0.$$

So, f(x) has maxima at $x = 2n\pi$ and maximum value is

$$f(2n\pi) = \cos 2n\pi = 1.$$

If n is odd integer i.e. at $x = (2n + 1)\pi$,

$$\begin{aligned} f''((2n+1)\pi) &= -\cos((2n+1)\pi) \\ &= -\cos 2n\pi \cos\pi + \sin 2n\pi \sin\pi \\ &= -(1)(-1) + 0 \\ &= 1 > 0. \end{aligned}$$

So, f(x) has minima at $x = (2n + 1)\pi$ and maximum value is

$$\begin{aligned} f((2n+1)\pi) &= \cos(2n+1)\pi \\ &= \cos 2n\pi \cos\pi - \sin 2n\pi \sin\pi \\ &= (1)(-1) + 0 \\ &= -1 \end{aligned}$$

(iv) $f(x) = (1 + \cos x) \sin x$ at $x = \frac{\pi}{3}$

Solution: Given, $f(x) = (1 + \cos x) \sin x = \sin x + \frac{\sin 2x}{2}$

So, $f'(x) = \cos x + \cos 2x$ and $f''(x) = -\sin x - 2\sin 2x$.

At $x = \frac{\pi}{3}$,

$$f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2} - \frac{1}{2} = 0.$$

and

$$f''\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right) < 0.$$

f(x) has maximum value at $x = \frac{\pi}{3}$. And maximum value is,

$$f\left(\frac{\pi}{3}\right) = \left(1 + \cos\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \left(1 + \frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}.$$

2. **Find the maximum value of xy and minimum value of $x^2 + y^2$ such that $\frac{x}{2} + \frac{y}{3} = 1$.**

Solution: Here, $\frac{x}{2} + \frac{y}{3} = 1 \Rightarrow 3x + 2y = 6$

$$\Rightarrow y = \frac{6 - 3x}{2} \qquad \ldots \text{(i)}$$

Part I: We have,

$$f = xy = x\left(\frac{6-3x}{2}\right) = \frac{6x - 3x^2}{2} \qquad \ldots \text{(ii)}$$

So, $f'(x) = 3 - 3x$ and $f''(x) = -3 < 0$.

For extreme point, set

$$f'(x) = 0 \Rightarrow 3 - 3x = 0 \Rightarrow x = 1.$$

Also, $f''(x) = -6 < 0$.

So f has maxima at x = 1 and maximum value is

$$f = \frac{6-3}{2} = \frac{3}{2}. \qquad [\because \text{By (ii)}]$$

Part II: We have,

$$f(x) = x^2 + y^2 = x^2 + \left(\frac{6-3x}{2}\right)^2 \qquad \text{being } y = \frac{6-3x}{2}.$$

$$= x^2 + \frac{36 - 36x + 9x^2}{4}$$

$$= x^2 + 9 - 9x + \frac{9}{4}x^2$$

$$= \frac{13}{4}x^2 - 9x + 9$$

So,

$$f'(x) = \frac{13}{2}x - 9 \quad \text{and} \quad f''(x) = \frac{13}{2}$$

For extreme point,

$$f'(x) = 0 \Rightarrow \frac{13x}{2} - 9 = 0 \Rightarrow x = \frac{18}{13}.$$

Also, $f''(x) = \frac{13}{2} > 0$.

So f(x) has minima at $x = \frac{18}{13}$ and minimum value is

$$f = \frac{13}{4}\left(\frac{18}{13}\right)^2 - 9\left(\frac{18}{13}\right) + 9 = \left(\frac{81}{13}\right) - \frac{162}{13} + 9 = \frac{36}{13}.$$

Solution: Let $y=\left(\frac{1}{x}\right)^x$

Taking log on both sides, we get

$$\log y = x\log\left(\frac{1}{x}\right) = x\log(x^{-1}) = -x\log(x).$$

Now, differentiating w. r. t. x then we get

$$\frac{1}{y}y' = -x\left(\frac{1}{x}\right) - \log(x) = -\log(x) - 1$$

$$y' = -y\log(x) - y \quad \ldots (i)$$

Again differentiating w. r. t. x, then

$$y'' = -y'\log(x) - \frac{y}{x} - y' \quad \ldots (ii)$$

For extreme point, set

$$y' = 0 \Rightarrow -y\log(x) - y = 0$$
$$\Rightarrow \log(x) = 1$$
$$\Rightarrow x = e.$$

So, $y=\left(\frac{1}{e}\right)^e = (e^{-1})^e$

Put x = e in equation (ii) then,

$$y'' = -0 - \frac{(e^{-1})^e}{e} - 0 \quad [\because y' = 0]$$

$$\Rightarrow y'' = -\frac{(e^{-1})^e}{e} < 0 \quad [\because e > 0]$$

So $f(x) = y$ has maxima at $x = e$ and maximum value is,

$$f = \left(\frac{1}{e}\right)^e.$$

4. **Show that of all rectangles of given area the square has the smalle[st] perimeter.**

Solution: Let x is the length and y is the breadth of rectangle. Let A be the ar[ea] and P be the perimeter which is to be minimized.

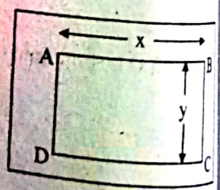
Since the area of the rectangle is given. So, the area is fixed. Therefore,

$$A = xy \Rightarrow y = \frac{A}{x}$$

and $P = 2(x+y) = 2\left(x+\frac{A}{x}\right) \quad \left[\because y = \frac{A}{x}\right]$

Then,

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} \quad \text{and} \quad \frac{d^2P}{dx^2} = \frac{4A}{x^3}$$



For extreme point, set

$$\frac{dP}{dx} = 0 \Rightarrow 2 - \frac{2A}{x^2} = 0 \Rightarrow x = \sqrt{A} \quad \text{[Being x is positive]}$$

and $y = \frac{A}{\sqrt{A}} = \sqrt{A}$.

At $x = \sqrt{A}$ $\quad \frac{d^2P}{dx^2} = \frac{4A}{(\sqrt{A})^3} > 0.$

So, P is minimum when $x = \sqrt{A} = y$. Thus, the rectangle is square when P is minimum.

Hence, the square has the smallest perimeter of all rectangles of given area.

5. **Show that largest rectangle with a given perimeter is square.**

Solution: Let P be the given perimeter. Also, let x be the length and y be the breadth of the rectangle.

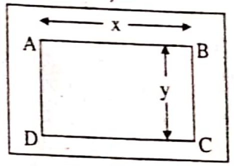
And, let A be the area which is to be maximized.

Here, the perimeter is given. So, the perimeter is fixed. Since,

$$P = 2(x+y) \Rightarrow y = \frac{P-2x}{2} \quad \ldots (i)$$

and $A = xy \Rightarrow A = \frac{x(P-2x)}{2}$ [from (i)]

$$\Rightarrow A = \frac{Px}{2} - x^2$$



Differentiating w. r. t. x then,

$$\frac{dA}{dx} = \frac{P}{2} - 2x \quad \text{and} \quad \frac{d^2A}{dx^2} = -2 < 0.$$

For extreme point, set

$$\frac{dA}{dx} = 0 \Rightarrow \frac{P}{2} - 2x = 0 \Rightarrow x = \frac{P}{4}.$$

Then $\quad y = \frac{P - 2\left(\frac{P}{4}\right)}{2} = \frac{P}{4}.$

Since, $\frac{d^2A}{dx^2} = -2 < 0$. So A is maximum when $x = \frac{P}{4}$ and $y = \frac{P}{4}$.

Here $x = \frac{P}{4} = y$, so the rectangle is a square.

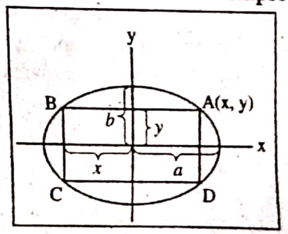
Thus the largest rectangle is square with a given perimeter.

6. **Prove that the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with area 2ab.**

Solution: Given equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = \frac{a^2b^2 - b^2x^2}{a^2} \quad \ldots (i)$$



So, the centre of ellipse is (0, 0).

Let A, B, C and D be the vertices of rectangle when co-ordinate of A (x, y) which lies on the ellipse i.e. sides of rectangle are $2x$ and $2y$. Then

Area of the rectangle $(A) = 4xy$

$$\Rightarrow A^2 = 16x^2y^2 = 16x^2\left(\frac{a^2b^2 - b^2x^2}{a^2}\right) \qquad \text{[Using (i)]}$$

$$= 16b^2x^2 - 16\left(\frac{b^2}{a^2}\right)x^4 \qquad \ldots(ii)$$

Here we have to show the rectangle is greatest. This means we have to observe that A^2 is maximum.

So, differentiating A^2 w. r. t. x then

$$\frac{dA^2}{dx} = 32\,b^2x - 64\left(\frac{b^2}{a^2}\right)x^3 \quad \text{and} \quad \frac{d^2A^2}{dx^2} = 32b^2 - 192\left(\frac{b^2}{a^2}\right)x^2$$

For extreme point of A^2 (i.e. A), set

$$\frac{dA^2}{dx} = 0 \Rightarrow 32b^2x - 64\left(\frac{b^2}{a^2}\right)x^3 = 0$$

$$\Rightarrow 1 - \frac{2x^2}{a^2} = 0 \Rightarrow x^2 = \frac{a^2}{2}.$$

And at $x^2 = \frac{a^2}{2}$,

$$\frac{d^2A^2}{dx^2} = 32b^2 - 192\left(\frac{b^2}{a^2}\right)\cdot\frac{a^2}{2} = -64\,b^2 < 0.$$

This means the area is maximum.

And the maximum area is,

$$A^2 = 16b^2x^2 - 16\left(\frac{b^2}{a^2}\right)x^4 = 16b^2\left(\frac{a^2}{2}\right)\left[1 - \left(\frac{1}{a^2}\right)\frac{a^2}{2}\right]$$

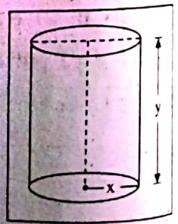
$$= 8a^2b^2\left(\frac{1}{2}\right) = 4\,a^2b^2.$$

$$\Rightarrow A = 2ab$$

Thus the area of greatest rectangle is 2ab.

7. **A cylindrical tin can closed at both ends of given capacity has to be constructed. Show that the amount of tin required will be minimum when the height is equal to the diameter.**

[2016 Spring][2015 Fall][2006, Fall] [2008, Fall]



Solution: Let, x = radius of the cylinder.

y = height of the cylinder.

Given that the capacity of the cylinder is given.

So, the volume (i.e. capacity) is constant.

Since,

$$V = \pi x^2 y \qquad \text{(which is constant)}$$

$$\Rightarrow y = \frac{V}{\pi x^2} \qquad \ldots(i)$$

Let S be the surface area (tin required) of the cylinder which is to be minimized.

We know that,

$$S = 2\pi xy + 2\pi x^2 \qquad [\because \text{cylindrical tin is closed at both ends}]$$

$$= 2\pi x\left(\frac{V}{\pi x^2}\right) + 2\pi x^2$$

$$\Rightarrow S = \frac{2V}{x} + 2\pi x^2$$

Differentiating w. r. t. x then

$$\frac{d(S)}{dx} = -\frac{2V}{x^2} + 4\pi x \quad \text{and} \quad \frac{d^2(S)}{dx^2} = \frac{4V}{x^3} + 4\pi$$

For extreme value of S, set

$$\frac{d(S)}{dx} = 0 \Rightarrow \frac{-2V}{x^2} + 4\pi x = 0 \Rightarrow \frac{2V}{x^2} = 4\pi x \Rightarrow x^3 = \frac{V}{2\pi}.$$

And, at $x^3 = \frac{V}{2\pi}$,

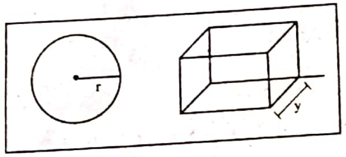
$$\frac{d^2(S)}{dx^2} = \frac{4V}{V/2\pi} + 4\pi = 8\pi + 4\pi = 12\pi > 0.$$

So, the surface area will be minimum (tin required) when $x^3 = \frac{V}{2\pi}$.

Here, $x^3 = \frac{V}{2\pi} \Rightarrow x^3 = \frac{\pi x^2 y}{2\pi} \Rightarrow 2x = y$

This shows that the diameter of the cylinder is equal to its height.

8. **The sum of the surfaces of a cube and a sphere is given, when the sum of their volume is least, show that the diameter of the sphere is equal to the edge of the cube.**



Solution: Let x = radius of the sphere

and y = edge of the cube.

Given,

S = sum of the surface area

which is given. So, S is fixed.

Let V be the sum of the volume which is to be minimized.

We know that,

$$S = 4\pi x^2 + 6y^2 \qquad \ldots\ldots (i)$$

So, $\frac{dS}{dx} = 8\pi x + 12y\frac{dy}{dx}$

Since S is fixed, so $\frac{dS}{dx} = 0$. Therefore,

$$0 = 8\pi x + 12y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{8\pi x}{12y} = -\frac{2\pi}{3}\left(\frac{x}{y}\right) \quad \ldots (ii)$$

Let V is sum of volume of the sphere and cube. That is,

$$V = \frac{4\pi x^3}{3} + y^3$$

So, $$\frac{dV}{dx} = 4\pi x^2 + 3y^2\frac{dy}{dx} = 4\pi x^2 - 2\pi xy \quad [\because \text{using (ii)}]$$

and, $$\frac{d^2V}{dx^2} = 8\pi x - 2\pi y - 2\pi x\frac{dy}{dx}$$

$$= 8\pi x - 2\pi y + \frac{4\pi^2x^2}{3y} \quad [\because \text{using (ii)}]$$

For critical point set,

$$\frac{dV}{dx} = 0 \Rightarrow 4\pi x^2 - 2\pi xy = 0$$

$$\Rightarrow y = 2x \quad [\because 2\pi x \neq 0]$$

At $y = 2x$,

$$\frac{d^2V}{dx^2} = 8\pi x - 4\pi x + \frac{4\pi^2x^2}{6x}$$

$$= 4\pi x + \frac{2\pi^2 x}{3} > 0.$$

This means the sum of volume is minimum when $2x = y$ i.e. the sum of volume of a cube and a sphere is least when edge of cube is equal to diameter (twice of radius) of sphere.

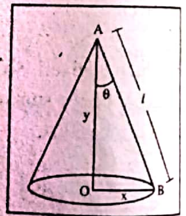
9. Show that the semi–vertical angle of the cone of maximum volume and given slant height, is $\tan^{-1}(\sqrt{2})$. **[2005, Spring]**

Solution: Let ABC be a cone.

Let l = slant height, x = radius of base

y = height of cone

θ = semi–vertical angle of the cone.



Now, from ΔAOB

$y = l\cos\theta$ and $x = l\sin\theta$

Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi l^2 \sin^2\theta \, l\cos\theta$$

$$\Rightarrow V = \frac{\pi}{3} l^3 \sin^2\theta\cos\theta = \left(\frac{\pi}{6}\right) l^3 \sin 2\theta \sin\theta$$

Differentiating w.r.t. θ then

$$\frac{dV}{d\theta} = \left(\frac{\pi}{6}\right) l^3 [2\cos 2\theta \sin\theta + \sin 2\theta\cos\theta]$$

and, $$\frac{d^2V}{d\theta^2} = \left(\frac{\pi}{6}\right) l^3 [-4\sin 2\theta\sin\theta + 2\cos 2\theta\cos\theta + 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta]$$

$$= \left(\frac{\pi}{6}\right) l^3 [4\cos 2\theta\cos\theta - 5\sin 2\theta\sin\theta]$$

For extreme point, set

$$\frac{dV}{d\theta} = 0 \Rightarrow 2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta = 0.$$

$$\Rightarrow 2\sin\theta(\cos^2\theta - \sin^2\theta + \cos^2\theta) = 0.$$

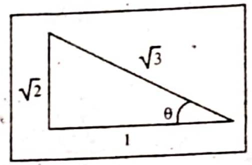
$$\Rightarrow \sin\theta(\cos^2\theta - \sin^2\theta + \cos^2\theta) = 0.$$

So, either $\sin\theta = 0 \Rightarrow \theta = 0$, that is not possible.

or, $2\cos^2\theta - \sin^2\theta = 0.$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}.$



Then, $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$.

At $\tan\theta = \sqrt{2}$,

$$\frac{d^2V}{d\theta^2} = \left(\frac{\pi}{6}\right) l^3 [4(\cos^3\theta - \sin^2\theta\cos\theta) - 10\sin^2\theta\cos\theta]$$

$$= \left(\frac{\pi}{6}\right) l^3 \left[4\left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) - 10\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \left(\frac{\pi}{6}\right) l^3 \left[\left(\frac{4}{3\sqrt{3}}\right) - \left(\frac{2}{3\sqrt{3}}\right) - \left(\frac{20}{3\sqrt{3}}\right)\right]$$

$$= \left(\frac{\pi}{6}\right) l^3 \left(\frac{-18}{3\sqrt{3}}\right) < 0$$

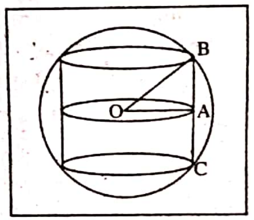
So, the volume is maximum when $\tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}(\sqrt{2})$.

10. Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r.

[2013 Spring][2013 Fall][1999] [2001]

Solution: Let O be the centre and r be the radius of the sphere. Let h be the height and R be the radius of the base of the cylinder.

Then let $OB = r$, $OA = R$ and $BC = h$. Let $\angle BOA = \theta$.



Then in ΔOAB, $\cos\theta = \frac{OA}{OB} \Rightarrow R = r\cos\theta$.

Also, $\sin\theta = \frac{AB}{OB} \Rightarrow h = 2r\sin\theta$.

Let S be the surface area of the cylinder.

Then $S = 2\pi Rh + 2\pi R^2$

$= 2\pi r\cos\theta \, 2r\sin\theta + 2\pi r^2\cos^2\theta$

$= 2\pi r^2(\sin 2\theta + \cos^2\theta) \quad \ldots (i)$

Differentiating (i) w. r. t. θ,

$$\frac{dS}{d\theta} = 2\pi(2\cos 2\theta - 2\sin\theta\cos\theta) = 2\pi(2\cos 2\theta - \sin 2\theta)$$

And, $\frac{d^2S}{d\theta^2} = 2\pi(-4\sin 2\theta - 2\cos 2\theta)$

For extreme value, set

$$\frac{dS}{d\theta} = 0$$

$\Rightarrow 2\pi(2\cos 2\theta - \sin 2\theta) = 0$

$\Rightarrow 2\cos 2\theta - \sin 2\theta = 0$

$\Rightarrow \tan 2\theta = 2.$

By triangular law,

$\tan 2\theta = 2 \Rightarrow \frac{p}{b} = \frac{2}{1}$

$\Rightarrow p = 2, b = 1.$

Then, $h = \sqrt{5}$.

So that $\sin 2\theta = \frac{p}{h} = \frac{2}{\sqrt{5}}$

and $\cos 2\theta = \frac{b}{h} = \frac{1}{\sqrt{5}}$.

Then

$$\sin 2\theta = \frac{2}{\sqrt{5}}, \quad \cos 2\theta = \frac{1}{\sqrt{5}}.$$

Then, at $\tan 2\theta = 2$,

$$\frac{d^2S}{d\theta^2} = 2\pi\left(-\frac{8}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right) = \frac{-20\pi}{\sqrt{5}} < 0.$$

So, the function has maximum value at $\tan 2\theta = 2$. And at that point,

$$S = 2\pi r^2(\sin 2\theta + \cos^2\theta)$$

$$= 2\pi r^2\left(\sin 2\theta + \frac{1+\cos 2\theta}{2}\right)$$

$$= 2\pi r^2\left(\frac{2}{\sqrt{5}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\right) = \pi r^2\left(\frac{5+\sqrt{5}}{\sqrt{5}}\right).$$

This is the required surface.

11. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a core is half that of the cone. **[2007, Fall]**

Solution: Given that a right circular cylinder is inscribed in a cone.

Let s = curved surface area of cylinder

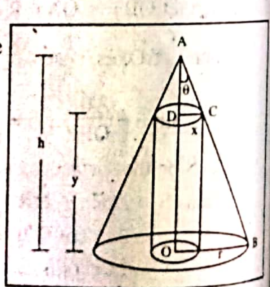
x = radius of the cylinder

y = height of the cylinder

h = height of the core

r = radius of the cone.

and, θ = semi–vertical angle of the cone



From the corresponding figure in ΔADC,

$x = AD\tan\theta$

$= (h - y)\tan\theta$

and,

$s = 2\pi y(h - y)\tan\theta$

$= 2\pi(hy - y^2)\tan\theta$

So, $\frac{ds}{dy} = 2\pi(h - 2y)\tan\theta$

and $\frac{d^2s}{dy^2} = -4\pi\tan\theta < 0.$

This means the curved surface area will be minimum at the critical point.

For the critical point, set

$\frac{ds}{dy} = 0 \Rightarrow 2\pi(h - 2y)\tan\theta = 0$

$\Rightarrow h - 2y = 0$ [Being $\tan\theta = 0 \Rightarrow \theta = 0$ which is impossible.]

$\Rightarrow h = 2y.$

This shows that the curved surface area will be minimum when height of the cylinder is half of height of the cone.

12. Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h. **[2014 Fall][2002][2007, Spring] [2000]**

OR A cylinder is inscribed in a given cone of height h. Find the height of the cylinder for which the volume is maximum. **[2006, Spring]**

Solution:

Let x = radius of the cylinder

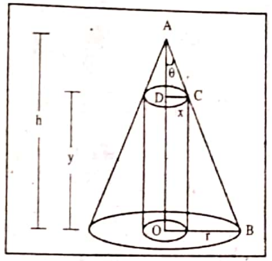
y = height of the cylinder

r = radius of the cone

V = volume of the cylinder

h = height of the cone.

and, θ = semi–vertical angle of the cone.



Form the corresponding figure in ΔADC,

$x = (h - y)\tan\theta$

Let v be the volume of the cylinder. So,

$V = \pi x^2 y = \pi(h - y)^2\tan^2\theta\, y$

$\Rightarrow V = \pi\tan^2\theta(h^2y - 2hy^2 + y^3)$

Then,

$\frac{dV}{dy} = \pi\tan^2\theta(h^2 - 4hy + 3y^2)$

and $\frac{d^2V}{dy^2} = \pi\tan^2\theta(-4h + 6y)$

For critical point, set,

$\frac{dV}{dy} = 0 \Rightarrow \pi\tan^2\theta(h^2 - 4hy + 3y^2) = 0$

$\Rightarrow 3y^2 - 4hy + h^2 = 0$

$\Rightarrow 3y^2 - 3hy - hy + h^2 = 0$

$\Rightarrow (3y - h)(y - h) = 0$

$$\Rightarrow y = h, \frac{h}{3}$$

When $y = h$, the cylinder cannot be inscribed in a given cone of height h

When $y = \frac{h}{3}$,

$$\frac{d^2V}{dy^2} = \pi \tan^2\theta \left(-4h + 6\left(\frac{h}{3}\right)\right) = -2h\pi \tan^2\theta < 0$$

Thus when $y = \frac{h}{3}$, the volume of the cylinder is maximum.

13. If 40 sq. feet of sheet metal are to be used in the construction of an open tank with a square base. Find the dimension in order that its capacity is greatest. [2008, Spring]

Solution: Let the given open tank has square base.

Let x = length = breadth of base

y = height of the tank

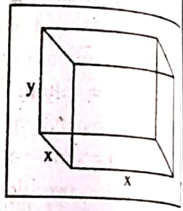
Given that 40 sq ft of steel metal are to be used in the construction of a open tank. So,

$$A = 40 \text{ sq. feet}$$

So, $A = x^2 + 4xy$ [being given tank is open tank]

$$\Rightarrow x^2 + 4xy = 40$$

$$\Rightarrow y = \frac{40 - x^2}{4x} \quad \ldots\ldots (i)$$



We know that the volume of the tank is,

$$V = x^2y = x^2\left(\frac{40 - x^2}{4x}\right)$$

$$\Rightarrow V = \frac{40x - x^3}{4} \quad \ldots (ii)$$

Here we have to find the dimension of the tank when the capacity is greatest i.e. maximum.

Differentiating (ii) w.r.t. x,

$$\frac{dV}{dx} = \frac{40 - 3x^2}{4} \quad \text{and} \quad \frac{d^2V}{dx^2} = \frac{-6x}{4} = \frac{-3x}{2}$$

For the extreme point, set

$$\frac{dV}{dx} = 0 \Rightarrow 40 - 3x^2 = 0 \Rightarrow x = \sqrt{\frac{40}{3}}.$$

At $x = \sqrt{\frac{40}{3}}$,

$$\frac{d^2V}{dx^2} = \left(\frac{-3}{2}\right)\left(\sqrt{\frac{40}{3}}\right) < 0.$$

s means the volume of the tank is maximum.

o,

$$y = \frac{40 - x^2}{4x} = \left(\sqrt{\frac{3}{40}}\right)\left(\frac{40 - \frac{40}{3}}{4}\right) = \left(\sqrt{\frac{3}{40}}\right)\left(\frac{20}{3}\right) = \sqrt{\frac{10}{3}}$$

Thus, capacity of the tank is greatest when its dimensions are

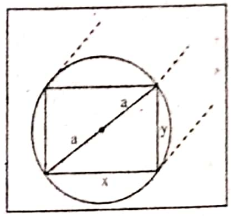
length = breadth = $\sqrt{\frac{40}{3}}$ feet, and height = $\sqrt{\frac{10}{3}}$ feet.

14. The strength of a beam varies jointly as its breadth and the square of the depth. Find the dimension of the strongest beam that can be cut from a circular wooden log of raidus a. [2012 Fall][2011 Fall][2003, Spring]

Solution: Let x = breadth of the beam

y = depth of the beam

So, given that the radius of circular wood a.



Then,

$$x^2 + y^2 = (2a)^2$$

$$\Rightarrow y^2 = 4a^2 - x^2 \quad \ldots\ldots (i)$$

Let,

S = strength at the beam

Given that the strength of the beam varies jointly as its breadth and the square of the depth. So,

$$S = \lambda xy^2 \quad \text{where } \lambda \text{ be a constant value.}$$

$$= \lambda x(4a^2 - x^2) = 4\lambda a^2x^2 - \lambda x^3 \quad \text{[Using (i)]}$$

Differentiating w.r.t. x then,

$$\frac{dS}{dx} = 4\lambda a^2 - 3\lambda x^2 \quad \text{and} \quad \frac{d^2S}{dx} = -6\lambda x.$$

For extreme point, set

$$\frac{dS}{dx} = 0 \Rightarrow 4\lambda a^2 = 3\lambda x^2 \Rightarrow x = \frac{2a}{\sqrt{3}}.$$

Now, from (i),

$$y^2 = 4a^2 - \frac{4a^2}{3} \Rightarrow y^2 = \frac{8a^2}{3} \Rightarrow y = \frac{2a\sqrt{2}}{\sqrt{3}}.$$

At $x = \frac{2a}{\sqrt{3}}$,

$$\frac{d^2S}{dx^2} = -6\lambda \frac{2a}{\sqrt{3}} < 0.$$

So, S will be maximum when $x = \frac{2a}{\sqrt{3}}$.

Thus, the dimension of the beam is,

breadth = $\frac{2a}{\sqrt{3}}$ and depth = $\frac{2a\sqrt{2}}{\sqrt{3}}$.

15. A cone is circumscribed to a sphere of radius r, show that when the volume of the cone is least its altitude is 4r and its semi–vertical angle is $\sin^{-1}\left(\frac{1}{3}\right)$.

Solution: Let the cone is circumscribed to a sphere of radius r.

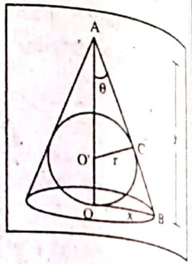
Let x = radius of base of the cone

y = height of the cone

which is circumscribed to a sphere of radius r.

Let θ = semi–vertical angle of the cone.

We know that volume of cone $(V) = \frac{1}{3}\pi x^2 y$



From figure, $\Delta ACO'$ and ΔAOB are similar. So,

$$\frac{HC}{OB} = \frac{AC}{AB} \Rightarrow \frac{r}{x} = \frac{y-r}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{r^2(x^2+y^2)}{x^2} = (y-r)^2$$

$$\Rightarrow r^2 + \frac{(yr)^2}{x^2} = y^2 - 2ry + r^2$$

$$\Rightarrow x^2 = \frac{y^2r^2}{(y^2-2ry)} = \frac{yr^2}{y-2r}$$

Since the volume of the cone is,

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\left(\frac{yr^2}{y-2r}\right)y = \frac{\pi}{3}\left(\frac{y^2r^2}{y-2r}\right).$$

Differentiating w.r.t. y then,

$$\frac{dV}{dy} = \frac{\pi}{3}\left[\frac{2yr^2(y-2r) - y^2r^2}{(y-2r)^2}\right]$$

$$= \frac{\pi}{3}\left[\frac{2y^2r^2 - 4yr^3 - y^2r^2}{(y-2r)^2}\right] = \frac{\pi}{3}\left[\frac{y^2r^2 - 4yr^3}{(y-2r)^2}\right].$$

and,

$$\frac{d^2V}{dy^2} = \frac{\pi}{3}\left(\frac{(2yr^2 - 4r^3)(y-2r)^2 - (y^2r^2 - 4yr^3).2(y-2r)}{(y-2r)^4}\right).$$

For extreme point,

$$\frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{3}\left[\frac{y^2r^2 - 4yr^3}{(y-2r)^2}\right] = 0$$

$$\Rightarrow y^2r^2 - 4yr^3 = 0$$

$$\Rightarrow y = 4r. \quad \text{[being y = 0 is impossible]}$$

Therefore at $y = 4r$,

$$\frac{d^2V}{dy^2} = \frac{\pi}{3}\left[\frac{(8r^3 - 4r^3)(4r-2r)^2 - (16r^4 - 16r^4).2(4r-2r)}{(4r-2r)^4}\right]$$

$$= \frac{\pi}{3}\left(\frac{16r^5}{(2r)^4}\right) > 0.$$

So volume (V) is least.

Thus, the cone has attitude 4r when its volume is least.

Also, from $\Delta O'AC$,

$$\sin\theta = \frac{O'C}{OA} \Rightarrow \sin\theta = \frac{r}{y-r} = \frac{r}{3r} = \frac{1}{3}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{3}\right).$$

16. For a given curved surface of right circular cone when the volume is maximum, prove that the semi–vertical angle is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Solution: Let θ be the semi–vertical angle and l be the slant height of the cone ABC with vertex at A.

Then radius of the cone = r = $l \sin\theta$.

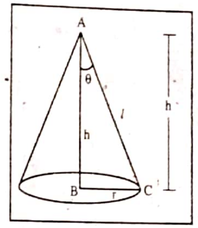
And height of the cone = h = $l \cos\theta$.

Let S be the surface area of the cone which is given. So, S is constant.

Since,

$$S = \pi l.r = \pi l^2 \sin\theta = \text{constant (given)}.$$

$$\Rightarrow l^2 = \frac{S}{\pi\sin\theta}.$$



And volume of the cone is,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta.$$

$$\Rightarrow V^2 = \frac{1}{9}\pi^2 l^6 \sin^4\theta\cos^2\theta$$

$$= \frac{1}{9}\pi^2\left(\frac{S^3}{\pi^3\sin^3\theta}\right)\sin^4\theta\cos^2\theta.$$

$$= \frac{1}{9\pi}S^3\sin\theta\cos^2\theta$$

$$= \frac{1}{9\pi}S^3[\sin\theta - \sin^3\theta].$$

So,

$$\frac{d(V^2)}{d\theta} = \frac{1}{9\pi}S^3[\cos\theta - 3\sin^2\theta\cos\theta]$$

$$= \frac{1}{9\pi}S^3[-2\cos\theta + 3\cos^3\theta].$$

And

$$\frac{d^2(V^2)}{d\theta^2} = \left(\frac{-1}{9\pi}\right)S^3[2\sin\theta - 9\cos^2\theta\sin\theta]$$

$$= \left(\frac{-1}{9\pi}\right)S^3[2\sin\theta - 9(1-\sin^2\theta)\sin\theta]$$

$$= \left(\frac{-1}{9\pi}\right)S^3[9\sin^3\theta - 7\sin\theta]$$

For extreme point, set

$$\frac{d(V^2)}{d\theta}=0$$

$$\Rightarrow \cos\theta\,(-2+3\cos^2\theta)=0.$$

$$\Rightarrow \cos\theta = 0 = \cos\frac{\pi}{2} \;\Rightarrow\; \theta=\frac{\pi}{2}$$

$$\text{or } -2+3\cos^2\theta=0 \;\Rightarrow\; -2+3-3\sin^2\theta=0.$$

$$\Rightarrow \sin^2\theta=\frac{1}{3}$$

$$\Rightarrow \sin\theta=\pm\frac{1}{\sqrt3}$$

At $\theta=\sin^{-1}\left(\frac{-1}{\sqrt3}\right)$,

$$\frac{d^2(V^2)}{d\theta^2}=\left(\frac{-1}{9\pi}\right)S^3\left[9\left(\frac{-1}{\sqrt3}\right)^3-7\left(\frac{-1}{\sqrt3}\right)\right]=\left(\frac{-1}{9\pi}\right)S^3\left(\frac{-2}{3\sqrt3}\right)>0.$$

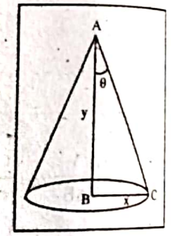
This means V is minimum, which is not of our interest.

At $\theta=\sin^{-1}\left(\frac{1}{\sqrt3}\right)$,

$$\frac{d^2(V^2)}{d\theta^2}=\left(\frac{-1}{9\pi}\right)S^3\left[9\left(\frac{1}{\sqrt3}\right)^3-7\left(\frac{1}{\sqrt3}\right)\right]=\left(\frac{-1}{9\pi}\right)S^3\left(\frac{2}{3\sqrt3}\right)<0.$$

So, the volume is maximum when $\theta=\sin^{-1}\left(\frac{1}{\sqrt3}\right)$.

17. For a given volume of a right cone, show that when the curve surface is minimum, semi–vertical angle is $\sin^{-1}\left(\frac{1}{\sqrt3}\right)$.



Solution: Let θ = semi–vertical angle
x = radius of the cone
y = height of the cone
V = volume of the cone
S = curved surface area of the cone.

Since the volume of the right circular cone is

$$V=\frac{\pi x^2y}{3}$$ which is given, so is constant.

$$\Rightarrow y=\frac{3V}{\pi x^2}.$$

And the curved surface area of the cone is

$$S=\pi x\sqrt{x^2+y^2}$$

$$\Rightarrow S^2=\pi^2x^2(x^2+y^2)$$

$$=\pi^2\left(x^4+x^2.\frac{9V^2}{\pi^2x^4}\right)$$

$$\Rightarrow S^2=\pi^2\left(x^4+\frac{9V^2}{\pi x^2}\right)$$

Then

$$\frac{d(S^2)}{dx}=\pi^2\left(4x^3-\frac{18V^2}{\pi^2x^3}\right) \text{ and } \frac{d^2(S^2)}{dx^2}=\pi^2\left(12x^2+\frac{54V^2}{\pi^2x^4}\right)$$

For extreme point, set

$$\frac{d(S^2)}{dx}=0 \;\Rightarrow\; \pi^2\left(4x^3-\frac{18V^2}{\pi^2x^3}\right)=0$$

$$\Rightarrow 4x^3=\frac{18V^2}{\pi^2x^3}$$

$$\Rightarrow x^6=\frac{9V^2}{2\pi^2} \;\Rightarrow\; x=\left(\frac{3V}{\pi\sqrt2}\right)^{1/3}$$

At $x=\left(\frac{3V}{\pi\sqrt2}\right)^{1/3}$

$$\frac{d^2(S^2)}{dx^2}=\pi^2\left(12x^2+\frac{54V^2}{\pi^2x^4}\right)>0,$$ being x is positive.

So S^2 (i.e. (S) is minimum.

Also,

$$y=\frac{3V}{\pi x^2}=\frac{3V}{\pi\left(\frac{3V}{\pi\sqrt2}\right)^{2/3}}=\left(\frac{6V}{\pi}\right)^{1/3}.$$

Here,

$$x^2+y^2=\left(\frac{3V}{\pi\sqrt2}\right)^{2/3}+\left(\frac{6V}{\pi}\right)^{2/3}$$

$$=\left(\frac{3V}{\pi}\right)^{2/3}\left(\frac{1}{2^{1/3}}+2^{2/3}\right)=\left(\frac{3V}{\pi}\right)^{2/3}\left(\frac{1+2}{2^{1/3}}\right)=\left(\frac{3V}{\pi}\right)^{2/3}\left(\frac{3}{2^{1/3}}\right)$$

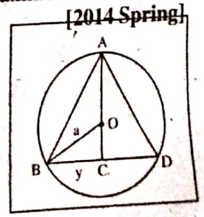
Now, from figure,

$$\sin\theta=\frac{x}{\sqrt{x^2+y^2}}=\left(\frac{3V}{\pi\sqrt2}\right)^{1/3}\left(\frac{\pi}{3V}\right)^{1/3}\left(\frac{2^{1/3}}{3}\right)^{1/2}=\frac{1}{\sqrt3}$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt3}$$

$$\Rightarrow \theta=\sin^{-1}\left(\frac{1}{\sqrt3}\right).$$

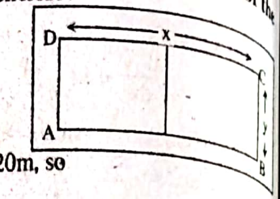
18. Find the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius a. [2014 Spring]



Solution: Let ABC is a cone with vertex at A that is inscribed in a sphere of radius a. Let,
y = radius of cone
x = height of the cone
V = volume of the cone

This shows the height of the tank is half the width.

21. The gardener having 120m of fancing wishes to endssed a rectangular plot of land and also ertest a fence across the land parallel to two of the sides what is the maximum area be can enclose?



Solution: Let ABCD is a garden that has

x = length of the garden

y = breadth of the garden

A = area of the garden

Given that the perimeter of the garden is 120m, so

$$2x + 2y + y = 120$$

$$\Rightarrow 2x + 3y = 120$$

$$\Rightarrow y = \frac{120 - 2x}{3}.$$

And the area of the garden is

$$A = xy = x\left(\frac{120 - 2x}{3}\right) = \frac{120x - 2x^2}{3}$$

Then,

$$\frac{dA}{dx} = \frac{120 - 4x}{3} \quad \text{and} \quad \frac{d^2A}{dx^2} = \frac{-4}{3} < 0.$$

So A is maximum at the critical point.

For extreme point, set

$$\frac{dA}{dx} = 0 \Rightarrow \frac{120 - 4x}{3} = 0 \Rightarrow x = 30.$$

When $x = 30$m we get $y = 20$m.

Hence the maximum value of A is,

$$A = 30 \times 20 = 600 \text{ m}^2.$$

22. The electric current i in a circuits varies according to the equation $i = 2t + \frac{200}{t}$, where i is in amperes and $t > 0$ is in seconds. Determine the minimum current.

Solution: Given that the current equation is,

$$i = 2t + \frac{200}{t}.$$

Then,

$$\frac{di}{dt} = 2 - \frac{200}{t^2}, \quad \text{and} \quad \frac{d^2i}{dt^2} = \frac{400}{t^3}.$$

For extreme point, set

$$\frac{di}{dt} = 0 \Rightarrow 2 - \frac{200}{t^2} = 0 \Rightarrow t^2 = 100 \Rightarrow t = 10.$$

Then at $t = 10$,

$$\frac{d^2i}{dt^2} = \frac{400}{1000} > 0.$$

So, the current is minimum when $t = 10$ and minimum current is,

$$i = 20 + 20 = 40.$$

23. In driving a motor boat, the petrol burnt per hour varies directly as the cube of its velocity. Find the most economical trip of the boat when going against a current of 2 km/hr.

Solution: Let, velocity of the boat relative to the water = v.

Current of water = 2 km/hr (given).

Distance to be traveled = d

Then, time required while going against the current $= \frac{d}{v - 2}$

Also, given that the petrol burnt per hour varies directly as cube of its velocity. i.e. $P \propto v^3$

Then, $P = kv^3$ (let) for some constant k.

Then total amount of total petrol burnt is,

$$P_T = \frac{d}{(v - 2)} \times kv^3$$

Now, we have to find the most economical trip of the boat. We know the trip will be most economical when the total amount of total petrol burnt will be least. That is, we should find P_T is minimum.

Here,

$$\frac{d(P_T)}{dv} = kd\left[\frac{(v-2)\,3v^2 - v^3}{(v-2)^2}\right] = kd\left(\frac{2v^3 - 6v^2}{(v-2)^2}\right) = 2kd\left(\frac{v^3 - 3v^2}{(v-2)^2}\right)$$

And,

$$\frac{d^2(P_T)}{dv^2} = 2kd\left[\frac{(v-2)^2(3v^2 - 6v) - (v^3 - 3v^2)\,2(v-2)}{(v-2)^4}\right]$$

For extreme point, set

$$\frac{d(P_T)}{dv} = 0 \Rightarrow \frac{2v^3 - 6v^2}{(v-2)^2} = 0$$

$$\Rightarrow v - 3 = 0 \qquad [\because 2v^2 \neq 0]$$

$$\Rightarrow v = 3.$$

At $v = 3$,

$$\frac{d^2(P_T)}{dv^2} = 2kd\left[\frac{(1)(9) - (0)\,2(1)}{(1)}\right] = 18kd > 0.$$

This means the least amount of petrol will burnt if the boat has velocity v = 3km/hr, which is the most economical trip of the boat.

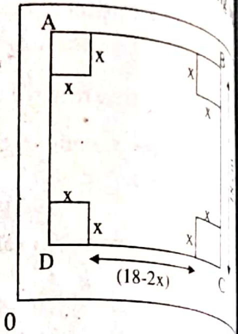
OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

1. A square piece of tin of side 18 cm is to be made into a box without lid, cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the value of box is maximum possible? **[2018 Fall] [2009, Fall]**

OR

A square piece of tin of side 18 cm is to be made into a box without cutting a square from each corner and folding up the flaps to form a box. In order to make the volume of the box maximum, what should the sides of the square to be cut off? [2005, Fall]

Solution: Let ABCD be square piece of tin of side 18 cm. Let x be the side of the square cut from each corner. By cutting x for each corner and folding the flaps to be form of the height x and side 18 – 2x.



If V be the volume of the box, then

V = base area × height

$$V = (18-2x)^2 (x) \quad .....(i)$$

Differentiating (i) w.r.t. to x, we get

$$\frac{dV}{dx} = (18-2x)^2 + 2(18-2x)(x)(-2) \quad ...(ii)$$

For the extreme point set

$$\frac{dV}{dx} = 0 \Rightarrow (18-2x)^2 - (4x)(18-2x) = 0$$

$$\Rightarrow (18-2x)(18-2x-4x) = 0$$

$$\Rightarrow (18-2x)(18-6x) = 0$$

$$\Rightarrow x = 9, 3.$$

If x = 9 then AB = 18 which is not possible because 2x = 18 is cut off piece from the side of the tin. So we should have x = 3.

Differentiating equation (ii) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = -4(18-2x) - 72 + 16x$$

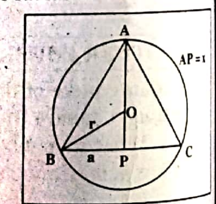
At x = 3, $\frac{d^2V}{dx^2} = -72 < 0.$

Therefore, V is maximum when x = 3.

Hence for the maximum value of the box, the side of the square to be cut off should be 3 cm.

2. **A cone is inscribed in a sphere of radius 'r'. Show that the volume of the cone will be maximum if the height of the cone is $\frac{4r}{3}$.** [2002][2004, Fall] [2018 Spring][2017 Fall]

Solution: Let O be the centre of sphere of radius r. Let ABC be a cone which inscribed in the sphere in such a way that A be the vertex of cone. Let AP = x be the height of the cone and BP = a, be the radius of the cone.



$$OP = AP - OA = x - r$$

Here,

$$(BP)^2 = (OB)^2 - (OP)^2$$
$$= r^2 - (x-r)^2$$
$$a^2 = 2rx - x^2$$

Let V be the volume of the cone. Then

$$V = \frac{1}{3}\pi (BP)^2 AP = \frac{1}{3}\pi a^2 x = \frac{1}{3}\pi(2rx - x^2)x$$

$$\Rightarrow V = \frac{\pi}{3}(2rx^2 - x^3) \quad .....(i)$$

Then

$$\frac{dV}{dx} = \frac{\pi}{3}(4rx - 3x^2) \quad \text{and} \quad \frac{d^2V}{dx^2} = \frac{\pi}{3}(4r - 6x)$$

For extreme point, set

$$\frac{dV}{dx} = 0 \Rightarrow \frac{\pi}{3}(4rx - 3x^2) = 0 \Rightarrow x = \frac{4r}{3}$$

being x = 0 is not possible.

At, $x = \frac{4r}{3}$,

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}\left(4r - 6\left(\frac{4r}{3}\right)\right) = \frac{\pi}{3}(-4r) = \frac{-4\pi r}{3} < 0$$

This means the volume of the cone is maximum.

Thus, the height of the cone i.e. x is $\frac{4r}{3}$ when volume of cone is greatest.

3. **Write down the criteria for extreme values of a function. Show that the maximum rectangle that can be inscribed in a circle is square.** [2003, Fall]

Solution: First Part: See criteria.

Second Part: See Q. 19.

4. **What is necessary condition for a function f(x) to be maximum or minimum at x = a? Does $f(x) = \frac{1}{x}$ reach to extrema at any point? If not why? Show that the largest rectangle with a given perimeter is a square.** [2004, Spring]

Solution: First Part: See criteria.

Second Part: See Q. 5.

5. **An oil can is to be made in the form of a right circular cylinder to contain one quart of oil. What dimension of the can will require the least amount of material?** [2015 Spring][2011, Spring]

Solution: Let r is radius and h is height of a right circular cylinder. Let V be volume of a right circular cylinder and A be total surface area of the cylinder.

Then,

$$V = \pi r^2 h = a^3 \text{ (let)}$$

$$\Rightarrow h = \frac{a^3}{\pi r^2} \quad ...(i)$$

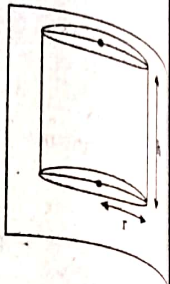
And,

$$A = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow A = 2\pi r^2 + \frac{2a^3}{r} \quad ...(ii) \text{ [using(i)]}$$

Then,
$$\frac{dA}{dr} = 4\pi r - \frac{2a^3}{r^2} \quad \text{and} \quad \frac{d^2A}{dr^2} = 4\pi + \frac{4a^3}{r^3}$$

For extreme point, set
$$\frac{dA}{dr} = 0 \Rightarrow 4\pi r - \frac{2a^3}{r^2} = 0$$
$$\Rightarrow r^3 = \frac{a^3}{2\pi}$$
$$\Rightarrow \frac{a^3}{r^3} = 2\pi$$



At $r^3 = \frac{a^3}{2\pi} \Rightarrow \frac{a^3}{r^3} = 2\pi$,
$$\frac{d^2A}{dr^2} = 4\pi + \frac{4a^3}{r^3} = 4\pi + 4(2\pi) = 4\pi + 8\pi = 12\pi > 0.$$

This means A is minimum i.e. least.

And at $r^3 = \frac{a^3}{2\pi} \Rightarrow \frac{a^3}{r^3} = 2\pi$,
$$h = \frac{a^3}{\pi r^2} = \frac{r}{\pi}\left(\frac{a^3}{r^3}\right) = \frac{r}{\pi}(2\pi) = 2r.$$

Thus, the dimensions of the can are
$$r^3 = \frac{a^3}{2\pi} = \frac{V}{2\pi} \quad \Rightarrow \quad r = \left(\frac{V}{2\pi}\right)^{1/3}$$
and $$h = 2r = 2\left(\frac{V}{2\pi}\right)^{1/3}.$$

Thus, the dimensions of the oil can that require the least amount of material to be made in the form of a right circular cylinder to contain one quart of oil are
$$r = \left(\frac{V}{2\pi}\right)^{1/3} \text{ and } h = 2\left(\frac{V}{2\pi}\right)^{1/3}.$$

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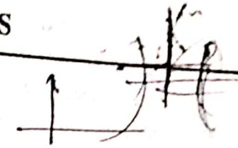
Chapter 6 ASYMPTOTES

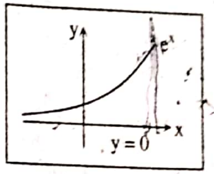


Theoretical Part

Definition of Asymptotes

A straight line is said to be an asymptotes to a curve if the perpendicular distance from a point P(x, y) on the curve to the straight line tends to zero as x or y or both approaches to infinity.

In figure, the perpendicular distance from e^x to the line y = 0 is decreasing when the curve moves to left and we can observe that the curve meats the line as $x \to -\infty$. So, y = 0 is an asymptotes of e^x.



Types of asymptotes

There are three types of asymptotes:

(i) **Horizontal Asymptotes (i.e. asymptotes parallel to x-axis):**

If
$$\lim_{x \to \infty} (y) = b \quad \text{or} \quad \lim_{x \to -\infty} (y) = b$$
then y = b is the horizontal asymptotes to the curve y = f(x).

(ii) **Vertical Asymptotes (i.e. asymptotes parallel to y-axis):**

If
$$\lim_{x \to a} (y) = (\pm\infty)$$
then x = a is the vertical asymptotes.

(iii) **Oblique Asymptotes**

An asymptotes of type y = mx + c with m ≠ 0, is known as oblique asymptotes. Such asymptote is neither parallel to x-axis nor to y-axis.

Process to find an asymptote to a curve is given in algebraic form.

Any curve may have horizontal, vertical and oblique asymptotes.

(i) **If the degree of equation is equal to degree of x that involves in the equation, then there is no asymptotes parallel to x-axis (i.e. horizontal asymptotes).**

If the degree of equation is not equal to degree of x then there may exist **horizontal asymptotes.** For such asymptotes, we process as:

Coefficient (s) of highest power of x involves in the equation = 0.

(ii) **If the degree of equation is equal to degree of y that involves in the equation, then there is no asymptotes parallel to y-axis (i.e. vertical asymptotes).**

If the degree of equation is not equal to degree of y then there may exist **vertical asymptotes.** For such asymptotes, we process as:

Coefficient (s) of highest power of y involves in the equation = 0

(iii) Let the given equation is of degree n. Let,

$$y = mx + c$$

be oblique asymptotes to the curve with value of m and c.

We define three functions with the help of given equation (taking x = 1, y = m).

$\phi_n(m)$ = terms which is of degree n.

$\phi_{n-1}(m)$ = terms which is of degree (n – 1).

$\phi_{n-2}(m)$ = terms which is of degree (n – 2).

To find the value of m, we set,

$$\phi_n(m) = 0$$

Solving this equation we obtain the real value of m.

The value of m may have repeated value and may have non-repeated value.

Case I: If m has repeated value for twice, then the value of c is obtained by solving the equation,

$$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

Case II: If m has non–repeated value then the value of c is obtained as

$$c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)}$$

where the prime (') denotes the differentiation.

Exercise 6.1

1. Find the asymptote of following curves:

(i) $y = \frac{3}{x-7}$

Solution: Here,

$$y = \frac{3}{x-7} \Rightarrow xy - 7y - 3 = 0.$$

Clearly the given equation is of degree 2. So, it may have at most 2 asymptotes.

Here, both x^2 and y^2 are absent. So, there may exist asymptotes parallel to x–axis and y–axis.

For horizontal asymptotes, equate the coefficient of highest power of x to zero.

i.e. coeff. of x = 0

or, y = 0

And for vertical asymptotes, equate the coefficient of highest power of y to zero.

i.e. coeff. of y = 0

or, x – 7 = 0.

Since the given curve cannot have more than two asymptotes, so the given curve has no oblique asymptotes.

Therefore, y = 0 and x = 7 are the required asymptotes of the given curve.

Alternative Method:

Here,

$$y = \frac{3}{x-7}$$

Here,

$$\lim_{x\to 7}(y) = \lim_{x\to 7}\left(\frac{3}{x-7}\right) = \infty.$$

This means x = 7 is vertical asymptotes to the given curve.

And,

$$\lim_{x\to\infty}(y) = \lim_{x\to\infty}\left(\frac{3}{x-7}\right) = 0$$

also,

$$\lim_{x\to-\infty}(y) = \lim_{x\to-\infty}\left(\frac{3}{x-7}\right) = 0$$

This means y = 0 be horizontal asymptotes to the given curve.

(ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution: Here,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 - a^2y^2 - a^2b^2 = 0 \quad \text{... (i)}$$

Clearly the given is of degree 2. So, it may have at most 2–asymptotes.

Here both x^2 and y^2 are present. So, there is no asymptote parallel to x–axis and y–axis.

Let, y = mx + c be an oblique asymptotes with value of m and c. For this with the help of given equation, set y = m and x = 1 then,

$$\phi_2(m) = b^2 - a^2m^2, \qquad \phi_1(m) = 0, \qquad \phi_0(m) = -a^2b^2.$$

For value m, set $\phi_2(m) = 0$

$$\Rightarrow b^2 - a^2m^2 = 0 \quad \Rightarrow m = \pm\frac{b}{a}.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = \frac{0}{-2a^2} = 0$$

Thus, $y = \pm\frac{b}{a}x \Rightarrow ay = \pm bx$ are the oblique asymptotes to the given curve.

Therefore, $ay = \pm bx$ are the asymptotes to the given curve.

(iii) $x^2y = 1$ **(for $x \neq 0$)**

Solution: Given curve is $x^2y - 1 = 0$ with $x \neq 0$.

Clearly the given curve is of third degree. So, it may have at most 3-asymptotes.

Also, x^3 and y^3 are absent. So, the vertical and horizontal asymptotes may exist.

For horizontal asymptotes, equate the coefficient of highest power of y with zero.

i.e. coeff. of $x^2 = 0$

or, $y = 0$

and for vertical asymptotes, equate the coefficient of highest power of y with zero.

i.e. coeff. of $y = 0$

or, $x^2 = 0 \Rightarrow x = 0$

which is not possible because $x \neq 0$.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m, \quad \phi_2(m) = 0, \quad \phi_1(m) = -1.$$

For value of m, equate $\phi_3(m)$ to zero.

i.e. $\phi_3(m) = 0 \Rightarrow m = 0.$

Here, m has non-repeated value. So, for the value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{0}{1} = 0$$

Thus $y = mx + c \Rightarrow y = 0$ be an oblique asymptotes.

Therefore, $y = 0$ is the asymptotes to the given curve.

(iv) $y = \dfrac{1}{(x+2)^2}$

Solution: Here,

$$y = \frac{1}{(x+2)^2} \Rightarrow x^2y + 4xy + 4y - 1 = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3-asymptotes.

Here x^3 and y^3 are absent, so there may exist vertical and horizontal asymptotes.

For vertical asymptotes, equate the coefficient of highest power of y, to zero. That is, coeff. of $y = 0$.

or, $x^2 + 4x + 4 = 0$

$\Rightarrow (x+2)^2 = 0.$

$\Rightarrow x = -2.$

...ertical asymptote is $x + 2 = 0$.

...orizontal asymptotes, equate the coefficient of highest power of x ...at is, coeff. of $x^2 = 0$

$y = 0$

So, the horizontal asymptotes to the given curve is $y = 0$.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m, \quad \phi_2(m) = 2m, \quad \phi_1(m) = 4.$$

For value of m, set $\phi_3(m) = 0$

$\Rightarrow m = 0$

Thus, m has non-repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m}{1} = 0 \quad [\text{since } m = 0]$$

So, $y = mx + c \Rightarrow y = 0$ is an oblique asymptote.

Therefore, $x + 2 = 0$, $y = 0$ are the asymptotes to the given curve.

(v) $x^2 - y^2 = 1.$

Solution: Given curve is,

$$x^2 - y^2 - 1 = 0.$$

Clearly this curve is of second degree. So, it may have at most 2-asymptotes.

Here x^2, y^2 are present so, there is no asymptotes parallel to x-axis and y-axis.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = 1 - m^2, \quad \phi_1(m) = 0, \quad \phi_0(m) = -1.$$

For value of m, set $\phi_2(m) = 0$.

$\Rightarrow 1 - m^2 = 0 \Rightarrow m = \pm 1.$

Here, m has non-repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = \frac{0}{-2m} = 0 \quad \text{for } m = \pm 1.$$

Thus, $y = mx + c \Rightarrow y = \pm x$ are oblique asymptotes.

Therefore, $y = \pm x$ are the asymptotes to the given curve.

(vi) $y = e^{1/x}$

Solved in the section, asymptotes of non-algebra functions.

(vii) $x^3 + y^3 = 3xy.$

Solution: Given curve is

$$x^3 + y^3 - 3xy = 0.$$

Clearly, given curve is of third degree. So, it may have at most three asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + m^3, \quad \phi_2(m) = -3m, \quad \phi_1(m) = 0.$$

For value of m, set $\phi_3(m) = 0$

$$\Rightarrow 1 + m^3 = 0 \Rightarrow m = -1$$

Thus, m has non–repeated value. So, for value of corresponding c is found by,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(-3m)}{3m^2} = -1$$

Thus, $y = -x - 1$ is an oblique asymptotes.

Therefore, $y = -x - 1$ is the asymptotes to the given curve.

(viii) $x^3 - y^3 = 6x^2$.

Solution: Given curve is

$$x^3 - y^3 - 6x^2 = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 - m^3, \quad \phi_2(m) = -6, \quad \phi_1(m) = 0.$$

For value of m, let $\phi_3(m) = 0$

$$\Rightarrow 1 - m^3 = 0 \Rightarrow m = 1.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{6}{-3m^2} = -2. \quad \text{[Being } m = 1\text{]}$$

Thus, $y = x - 2$ be an oblique asymptote.

Therefore, $x - y - 2 = 0$ be the asymptotes to the given curve.

(ix) $x^3 + y^3 = a^2x$

Solution: Given curve is

$$x^3 + y^3 - a^2x = 0.$$

Clearly, the given curve is of degree 3. So, it may have at most 3–asymptotes.

Here, both x^3 and y^3 are present. So, there is no vertical and horizontal asymptotes exist.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + m^3, \quad \phi_2(m) = 0, \quad \phi_1(m) = -a^2.$$

For value of m, set $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0$.

$$\Rightarrow m = 1$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(0)}{(3m^2)} = \frac{-0}{3} = 0.$$

Thus, $y = mx + c \Rightarrow y = x$ is an oblique asymptotes.

Therefore, $x = y$ is the asymptotes to the given curve.

(x) $x^2 - y^2 - 6x + 4y + 1 = 0$.

Solution: Given curve is

$$x^2 - y^2 - 6x + 4y + 1 = 0.$$

Clearly, the given curve is of degree 2. So, it may have at most 2–asymptotes.

Here, both x^2 and y^2 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = 1 - m^2, \quad \phi_1(m) = -6 + 4m, \quad \phi_0(m) = 0.$$

For value of m, set $\phi_2(m) = 0 \Rightarrow 1 - m^2 = 0$

$$\Rightarrow m = \pm 1.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = -\frac{(-6 + 4m)}{(-2m)} = \frac{-3 + 2m}{m}$$

For $m = 1$, the corresponding value of c is,

$$c = \frac{-3 + 2}{1} = -1,$$

and for $m = -1$, the corresponding value of c is,

$$c = \frac{-3 - 2}{-1} = 5.$$

Thus, $y = mx + c \Rightarrow y = x - 1, y = -x + 5$ are oblique asymptotes.

Therefore, $x - y + 1 = 0, x + y = 5$ are the asymptotes to the given curve.

(xi) $x^3 - 4y^3 + 3x^2 + y - x + 3 = 0$.

Solution: Given curve be $x^3 - 4y^3 + 3x^2 + y - x + 3 = 0$.

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 - 4m^3, \quad \phi_2(m) = 3, \quad \phi_1(m) = m - 1.$$

For value of m, let $\phi_3(m) = 0 \Rightarrow 1 - 4m^3 = 0$

$$\Rightarrow m^3 = \frac{1}{4} \Rightarrow m = \left(\frac{1}{4}\right)^{1/3} = (2)^{-2/3}$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3}{-12m^2} = \frac{1}{4\,(2)^{-4/3}} = \frac{2^{4/3}}{4} = \frac{2\,(2)^{1/3}}{4} = \frac{1}{2^{2/3}} = (2)^{-2/3}$$

Thus, $y = mx + c \Rightarrow y = (2)^{-2/3}x + (2)^{-2/3}$ be an oblique asymptotes.

Therefore, $y = (2)^{-2/3}x + (2)^{-2/3}$ be the asymptotes to the given curve.

(xii) $y = x^2 - 1$.

Solution: Given curve is

$$y - x^2 + 1 = 0.$$

Clearly, the given curve is of second degree. So, it may have at mo[st] 2-asymptotes.

Here, x^2 is present but y^2 is absent. So, there is no horizontal asympt[ote] exist.

And for vertical asymptotes, equate the coefficient of highest power of y, t[o] zero. That is, coeff. of $y = 0$

$$\Rightarrow \quad 1 = 0 \qquad \text{which is impossible.}$$

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For thi[s] with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = -1.$$

And for value of m, set

$$\phi_2(m) = 0 \Rightarrow -1 = 0, \qquad \text{which is impossible.}$$

Therefore, the given curve has no asymptotes.

2. Find the asymptotes of the following curves

(i) $y^2 - x^2 - 2x - 2y - 3 = 0$.

Solution: Given curve is

$$y^2 - x^2 - 2x - 2y - 3 = 0.$$

Clearly, the given curve is of second degree. So, it may have at most 2–asymptotes.

And, the equation includes both x^2 and y^2, so there is no asymptotes exist parallel to x–axis and y–axis.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = m^2 - 1, \quad \phi_1(m) = -2 - 2m, \quad \phi_0(m) = -3.$$

For value of m, set

$$\phi_2(m) = 0 \quad \Rightarrow m^2 - 1 = 0$$
$$\Rightarrow m = \pm 1.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = -\frac{-2-2m}{2m} = \frac{1+m}{m}$$

For, $m = 1$, the corresponding value of c is,

$$c = 2.$$

And, for, $m = -1$, the corresponding value of c is,

$$c = 0.$$

Thus, $y = mx + c \Rightarrow y = x + 2$, $y = -x$ be the oblique asymptotes.

Therefore, $x - y + 2 = 0$, $x + y = 0$ are the asymptotes to the given curve.

(ii) $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$.

Solution: Given curve is

$$x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3-asymptotes.

And the curve includes both x^3 and y^3. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + 3m - m^2 - 3m^3, \quad \phi_2(m) = 1 - 2m + 2m^2, \quad \phi_1(m) = 4.$$

For value of m, set

$$\phi_3(m) = 0 \quad \Rightarrow 1 + 3m - m^2 - 3m^3 = 0$$
$$\Rightarrow (1 + 3m)(1 - m^2) = 0$$
$$\Rightarrow m = \pm 1, -\frac{1}{3}$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(1 - 2m + 2m^2)}{-3 - 2m - 9m^2} = \frac{1 - 2m + 2m^2}{3 + 2m + 9m^2}$$

For $m = 1$, the corresponding value of c is,

$$c = \frac{1-2+3}{3+2+9} = \frac{4}{14} = \frac{2}{7}$$

$1 - m^2$

And, for $m = -1$, the corresponding value of c is,

$$c = \frac{1+2+3}{3+2+9} = \frac{6}{14} = \frac{3}{7}$$

Also, for $m = \frac{1}{3}$, the corresponding value of c is,

$$c = \frac{1 - \frac{2}{3} + \frac{2}{9}}{3 + \frac{2}{3} + \frac{9}{9}} = \frac{9-6+2}{27+6+9} = \frac{6}{42} = \frac{1}{7}.$$

Thus, the oblique asymptotes are

$$y = x + \frac{1}{14}, \; y = -x + \frac{6}{14} \text{ and } \quad y = \frac{x}{3} + \frac{6}{42}.$$

Therefore, the asymptotes to the given curve are

$$7y = 7x + 1, \; 7y + 7x = 3, \; 21y = 7x + 3.$$

(iii) $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$.

Solution: Given curve is

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3-asymptotes. Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3,$$

$$\phi_2(m) = -14m + 7m^2$$
$$\phi_1(m) = 4 + 5m.$$

For the value of m, set

$$\phi_3(m) = 0 \quad \Rightarrow \quad 3 + 2m - 7m^2 + 2m^3 = 0$$
$$\Rightarrow \quad 2m^3 - 2m^2 - 5m^2 + 5m - 3m + 3 = 0$$
$$\Rightarrow \quad (m-1)(2m^2 - 5m - 3) = 0$$
$$\Rightarrow \quad (m-1)(2m+1)(m-3) = 0$$
$$\Rightarrow \quad m = 1, 3, -\frac{1}{2}$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{7m(2-m)}{2 - 14m + 6m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{7(2-1)}{2 - 14 + 6} = \frac{7}{-6}$$

And, for m = 3, the corresponding value of c is,

$$c = \frac{21(2-3)}{2 - 42 + 54} = \frac{-21}{14} = -\frac{3}{2}$$

Also, for $m = \frac{-1}{2}$, the corresponding value of c is,

$$c = \frac{(-7/2)(2 + 1/2)}{2 + 7 + 3/2} = -\frac{35}{42} = -\frac{5}{6}$$

Thus, $y = mx + c \Rightarrow 6y = 6x - 7,\ 2y = 6x - 3,\ 6y = -3x - 5$ are oblique asymptotes to the given curve.

Therefore, $6y - 6x + 7 = 0,\ 2y - 6x + 3 = 0,\ 3x + 6y + 5 = 0$ are the asymptotes to the given curve.

(iv) $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0.$

Solution: Given curve is

$$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0.$$

Clearly given curve is of degree 3. So, it may have at most three asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptotes exist.

For oblique asymptotes, let y = mx + c be required asymptotes.

For this, set y = m and x = 1. Then with the help of given curve set,

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3, \quad \phi_2(m) = -4m^2 + 2m, \quad \phi_1(m) = -5m.$$

For the value of m, set

$$\phi_3(m) = 0 \quad \Rightarrow \quad 1 + 2m - m^2 - 2m^3 = 0$$
$$\Rightarrow \quad 1 - m + 3m - 3m^2 + 2m^2 - 2m^3 = 0$$
$$\Rightarrow \quad (1-m)(1 + 3m + 2m^2) = 0$$
$$\Rightarrow \quad (1-m)(1+m)(1+2m) = 0$$

$$\Rightarrow \quad m = 1, -1, -\frac{1}{2}.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{4m^2 - 2m}{2 - 2m - 6m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{4-2}{2-2-6} = \frac{2}{-6} = \frac{1}{-3}.$$

And, for m = –1, the corresponding value of c is,

$$c = \frac{4+2}{2+2-6} = \frac{6}{-2} = -3.$$

Also, for $m = \frac{-1}{2}$, the corresponding value of c is,

$$c = \frac{1+1}{2 + 1 - 3/2} = \frac{4}{-3}$$

Thus, $y = mx + c \Rightarrow 3y = 3x - 1,\ y = -x - 3,\ 3x + 6y + 4 = 0$ are oblique asymptotes to the given curve.

Therefore, $3y = 3x - 1,\ y = -x - 3,\ 3x + 6y + 4 = 0$ are the asymptotes to the given curve.

(v) $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$

Solution: Given curve is

$$x^4 - y^4 - 3x^2y + 3xy^2 + xy = 0.$$

Clearly, the given curve is of fourth degree. So, it may have at most 4-asymptotes.

Here, x^4 and y^4 are present. So, there is no vertical and horizontal asymptote.

Let, y = mx + c be an oblique asymptotes with value of m and c. For this with the help of given equation, set y = m and x = 1 then,

$$\phi_4(m) = 1 - m^4, \qquad \phi_3(m) = 3m + 3m^2, \qquad \phi_2(m) = m.$$

For value of m, let $\phi_3(m) = 0$

$$\Rightarrow \quad 1 - m^4 = 0$$
$$\Rightarrow \quad 1 - m^2 = 0 \quad \text{[since } m^2 = -1 \text{ does not give any real value of m]}$$
$$\Rightarrow \quad m = \pm 1.$$

Thus, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{-3m(1+m)}{-4m^3} = \frac{3(1+m)}{4m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{3(1+1)}{4(1)} = \frac{6}{4} = \frac{3}{2}$$

And, for m = –1, the corresponding value of c is,

$$c = \frac{3(1-1)}{4(1)} = 0$$

Thus, $y = mx + c \Rightarrow 2y = 2x + 3, y = -x$ are oblique asymptotes.

Therefore, $2x - 2y + 3 = 0, x + y = 0$ are the asymptotes to given curve.

(vi) $4x^4 - 5x^2y^2 + y^3 - 3x^2y + 5x - 8 + y^4 = 0$

Solution: Given curve be $4x^4 - 5x^2y^2 + y^3 - 3x^2y + 5x - 8 + y^4 = 0$.

Clearly, the given curve is of fourth degree. So, it may have at most 4-asymptotes.

Here, x^4 and y^4 are present. So, there does not exist any vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For the with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_4(m) = 4 - 5m^2 + m^4, \quad \phi_3(m) = m^3 - 3m, \quad \phi_2(m) = 0.$$

For value of m, set

$$\phi_4(m) = 0 \Rightarrow 4 - 5m^2 + m^4 = 0$$
$$\Rightarrow (4 - m^2)(1 - m^2) = 0$$
$$\Rightarrow m = \pm 1, \pm 2$$

Here, m has non-repeated value. So, for the value of corresponding c,

$$c = -\frac{\phi_3(m)}{\phi_4'(m)} = -\frac{(m^3 - 3m)}{-10m + 4m^3} = \frac{-m^2 + 3}{-10 + 4m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{3 - 1}{-10 + 4} = \frac{2}{-6} = -\frac{1}{3}.$$

And, for m = –1, the corresponding value of c is,

$$c = \frac{3 - 1}{-10 + 4} = -\frac{1}{3}.$$

Also, for m = 2, the corresponding value of c is,

$$c = \frac{3 - 4}{-10 + 16} = -\frac{1}{6}$$

Also, for m = –2, the corresponding value of c is,

$$c = \frac{3 - 4}{-10 + 16} = -\frac{1}{6}$$

Therefore, $y = mx + c \Rightarrow 3y = \pm 3x - 1, 6y = \pm 12x - 1$ are oblique asymptotes.

Hence, $3y \pm 3x + 1 = 0, 6y \pm 12x + 1 = 0$ are the asymptotes to the given curve.

(vii) $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ **[2011 Fall] [2003, Fall][2004, Spring] [2016 Fall][2018 Fall][2015, Spring]**

Solution: Given curve is

$$x^2(x - y)^2 - a^2(x^2 + y^2) = 0.$$
$$\Rightarrow x^4 - 2x^3y + x^2y^2 - a^2x^2 - a^2y^2 = 0.$$

Clearly, the given curve is of fourth degree. So, it may have at most 4 asymptotes.

Here x^4 is present. So, there does not exist any horizontal asymptotes. But y^4 is absent. So, there may exist vertical asymptotes. Therefore, for vertical asymptotes, equate the coefficient of highest power of y, to zero. i.e. coeff. of $y^2 = 0$

$$\Rightarrow x^2 - a^2 = 0 \quad \Rightarrow x = \pm a$$

These are required vertical asymptotes.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_4(m) = 1 - 2m + m^2, \quad \phi_3(m) = 0, \quad \phi_2(m) = -a^2(1 + m^2)$$

For value of m, set $\phi_4(m) = 0 \Rightarrow 1 - 2m + m^2 = 0$

$$\Rightarrow (m - 1)^2 = 0$$
$$\Rightarrow m = 1, 1$$

Here, m has repeated value. So, for value of the corresponding c,

$$\frac{c^2}{2!}\phi_4''(m) + \frac{c}{1!}\phi_3'(m) + \phi_2(m) = 0$$
$$\Rightarrow \frac{c^2}{2}(2) + c(0) - a^2(1 + m^2) = 0$$

For m = 1, the corresponding value of c is,

$$c^2 - 2a^2 = 0 \Rightarrow c = \pm a\sqrt{2}$$

So, $y = mx + c \Rightarrow y = x \pm a\sqrt{2}$ are oblique asymptotes.

Hence, $x = \pm a$, $y = x \pm a\sqrt{2}$ are the asymptotes to the given curve.

(viii) $x^2y^2 - 4(x - y)^2 + 2y - 3 = 0$

Solution: Given curve is

$$x^2y^2 - 4(x - y)^2 + 2y - 3 = 0.$$
$$\Rightarrow x^2y^2 - 4x^2 - 4y^2 + 8xy + 2y - 3 = 0$$

Clearly, the given curve is of fourth degree. So, it may have at most 4-asymptotes.

Here both x^4 and y^4 are absent. So, there may exist horizontal and vertical asymptotes.

For horizontal asymptotes, equate the coefficient of highest power of x, to zero. That is, coeff. of $x^2 = 0$.

$$\Rightarrow y^2 - 4 = 0 \quad \Rightarrow y = \pm 2.$$

And for vertical asymptotes, equate the coefficient of highest power of y to zero. That is, coeff. of $y^2 = 0$.

$$\Rightarrow x^2 - 4 = 0 \quad \Rightarrow x = \pm 2.$$

Since the curve does not have more than 4–asymptotes and we have already found $x = \pm 2$, $y = \pm 2$ as the asymptotes to the given curve, so the oblique asymptote does not exist.

Hence $x = \pm 2$, $y = \pm 2$ are the asymptotes to the given curve.

(ix) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$ **[2017 Spring][2018 Spring]**

Solution: Given curve is

$$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$$

Clearly given curve is of degree 3. So, it may have at most 3–asymptotes. Here both x^3 and y^3 are present. So, there does not exist any horizontal and vertical asymptotes.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + 3m - 4m^3, \quad \phi_2(m) = 0, \quad \phi_1(m) = -1 + m$$

For value of m, set

$$\phi_3(m) = 0 \Rightarrow 1 + 3m - 4m^3 = 0$$
$$\Rightarrow 1 - m + 4m - 4m^2 + 4m^2 - 4m^3 = 0$$
$$\Rightarrow (1 - m)(1 + 4m + 4m^2) = 0$$
$$\Rightarrow (1 - m)(1 + 2m)^2 = 0$$
$$\Rightarrow m = 1, -\frac{1}{2}, -\frac{1}{2}$$

For $m = 1$ (non–repeated value of m), the corresponding value of c is,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2} = -\frac{0}{-9} = 0.$$

And, for $m = \frac{-1}{2}$ (repeated value of m), to find the corresponding value of c,

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$$
$$\Rightarrow \frac{c^2}{2}(-24m) + c(0) + (-1 + m) = 0$$
$$\Rightarrow 6c^2 - \frac{3}{2} = 0.$$
$$\Rightarrow c^2 = \frac{3}{12} = \frac{1}{4} \Rightarrow c = \pm\frac{1}{2}$$

Thus, $y = mx + c \Rightarrow y = x, y = -\frac{1}{2}x \pm \frac{1}{2}$ are oblique asymptotes.

Hence, $x = y, x + 2y \pm 1 = 0$ are asymptotes to the given curve.

(x) $y^3 + x^2y + 2xy^2 - y + 1 = 0$ **[2013 Fall][2012 Fall]**

Solution: Given curve is

$$y^3 + x^2y + 2xy^2 - y + 1 = 0.$$

Clearly given curve is of degree 3. So, it may have at most 3–asymptotes. Here y^3 is present. So, the vertical horizontal do not exist. And, x^3 is absent. So, there may exist horizontal asymptotes. For this, we equate the coefficient of highest power of x, to zero. That is,

$$\text{coeff. of } x^2 = 0$$
$$\Rightarrow y = 0.$$

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m^3 + m + 2m^2, \quad \phi_2(m) = 0, \quad \phi_1(m) = -m$$

For value of m, set

$$\phi_3(m) = 0 \Rightarrow m^3 + m + 2m^2 = 0$$
$$\Rightarrow m(1 + 2m + m^2) = 0$$
$$\Rightarrow m(m + 1)^2 = 0$$
$$\Rightarrow m = 0, -1, -1$$

Thus, m has non–repeated value 0 and repeated value –1.

For $m = 0$ (non–repeated value of m), the corresponding value of c is found by,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3m^2 + 1 + 4m} = -\frac{0}{1} = 0$$

For $m = -1$ (repeated value of m), to find the corresponding value of c,

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$$
$$\Rightarrow \frac{c^2}{2}(6m + 4) + 0 + (-m) = 0 \quad [\text{since } \phi_2(m) = 0]$$
$$\Rightarrow -c^2 + 1 = 0$$
$$\Rightarrow c = \pm 1$$

Thus, $y = mx + c \Rightarrow y = 0, y = -x \pm 1$ are oblique asymptotes.

Hence, $y = 0, x + y \pm 1 = 0$ are asymptotes to the given curve.

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

LONG QUESTIONS

1. **Define asymptotes of the curves with different types. Find asymptotes of $x^3 - y^3 = 3y(x + y)$.** **[2009 Spring]**

Solution: First Part: See definition of asymptotes.

Second Part: Given curve be

$$x^3 - y^3 = 3y(x + y)$$

Clearly, the curve is of degree 3. So, it may have at most 3–asymptotes. Also, both x^3 and y^3 are present. So, there is no asymptotes parallel to x–axis and y–axis.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 - m^3, \quad \phi_2(m) = -3m(1 + m), \quad \phi_1(m) = 0$$

For value of m, set

$$\phi_3(m) = 0 \Rightarrow 1 - m^3 = 0 \Rightarrow m = 1.$$

For $m = 1$ (non–repeated value of m), the corresponding value of c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3m(1 + m)}{-3m^2} = -\frac{3(1 + 1)}{-3} = 2.$$

Thus, $y = mx + c \Rightarrow y = x + 2$ be oblique asymptotes to the given curve.
Hence, $y = x + 2$ is the asymptotes to the given curve.

2. **Define asymptotes of a curve. Find the asymptotes of the curve $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$** [2000]

Solution: Given curve be

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$$

Clearly, the curve is of degree 4. So, it may have at most 4–asymptotes. Also, both x^4 and y^4 are absent. So, there may exist both vertical and horizontal asymptotes.

For vertical asymptotes, equate the coefficient of highest power of y in the given equation to zero.

That is, coeff. of $y^2 = 0 \Rightarrow x^2 - x = 0$

$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0, x = 1$$

And for horizontal asymptotes, equate the coefficient of height power of x in the given equation, to zero.

That is, coeff. of $x^2 = 0 \Rightarrow y^2 - y = 0$

$$\Rightarrow y = 0, y = 1$$

Since the given curve does not have more than 4 asymptotes and the curve has asymptotes $x = 0$, $x = 1$, $y = 0$, $y = 1$. So, the curve does not have any oblique asymptotes.

Hence, $x = 0, x = 1, y = 0, y = 1$ are the asymptotes to the given curve.

3. **Find all the asymptote of the curve $x^3 + y^3 = 3axy$.** [2013 Spring][2002]

Solution: See Exercise 6.1 Q 1(vii).

4. **Define asymptote of a curve. Find the asymptotes of $y^3 - x^2y + 2y^2 + 4y + x = 0$.** [2004, Fall]

Solution: First Part: See definition of asymptotes.

Second Part: The given equation of curve is

$$y^3 - x^2y + 2y^2 + 4y + x = 0$$

This is a curve of degree 3. It has at most three asymptotes. Here x^3 is absent but y^3 present.

To find horizontal asymptotes we solve,

$$\text{coefficient of } x^2 = 0$$
$$\Rightarrow -y = 0 \quad \Rightarrow y = 0 \quad .....(i)$$

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m^3 - m, \quad \phi_2(m) = 2m^2, \quad \phi_1(m) = 4m + 1.$$

Then for the value of m, we solve

$$\phi_3(m) = 0 \quad \text{i.e. } m^3 - m = 0$$
$$\Rightarrow m(m^2 - 1) = 0 \Rightarrow m = 0, \pm 1.$$

Thus, values of m are 0, 1, –1.

Since the value of m is non–repeated. So, for value of corresponding c, we use

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-2m^2}{3m^2 - 1}$$

For $m = 0$ (non–repeated value of m), the corresponding value of c is,

$$c = \frac{0}{-1} = 0.$$

For $m = 1$ (non–repeated value of m), the corresponding value of c is found by,

$$c = \frac{-2}{3-1} = -1$$

For $m = -1$ (non–repeated value of m), the corresponding value of c is found by,

$$c = \frac{-2}{3-1} = -1$$

Thus, we get, $y = 0$, $y = x - 1$ and $y = -x - 1$ are the oblique asymptotes.
Hence, the lines $y = 0$, $y = x - 1$ and $y = -x - 1$ are the required asymptotes of the given curve.

5. **Define asymptotes of the curves with different types. Find asymptotes of $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0$.** [2006, Fall]

Solution: First Part: See definition of asymptotes.

Second Part: See Q. 2(iv) from above exercise.

6. **Define asymptotes of a curve. Find the asymptotes of $y^3 + x^2y + 2xy^2 - y + 1 = 0$.** [2008, Fall]

Solution: First Part: See definition of asymptotes.

Second Part: See Q. 2(x) from above exercise.

7. **Find the asymptotes of the curve $y^3 - x^2y + 2y^2 + 4y + x = 0$.** [2008, Spring]

Solution: See second part of Q. 5 from final exam question.

8. **Define asymptotes of a curve. Find asymptotes of $x^3 + y^3 = 3axy$.** [2011 Spring]

Solution; First Part: See definition of asymptotes.

Second Part: See Q. No. 3 of final exam questions.

9. **Define asymptotes of a curve and classify them. Find asymptotes of the curve: $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$.** [2014 Fall]

Solution: First Part: See definition of asymptotes and its types.

Second Part: See Q. No. 2(v).

10. **Define asymptotes of a curve and classify them. Find asymptotes of the curve: $x^2(x-y)^2 - a^2(x^2 + y^2) = 0$.** [2014 Spring]

Solution: First Part: See definition of asymptotes and its types.

Second Part: See Q. No. 2(vii).

also, $\lim_{x\to a}(y) = \lim_{x\to a}(\log(x)) = \log(a) \neq \infty$.

This means $y = \log(x)$ has vertical asymptotes $y = 0$ but horizontal asymptotes.

(iv) $y = \sin x$

Here

$$\lim_{x\to\infty}(\sin x) = \text{has oscillator value in } [-1, 1]$$

and $$\lim_{x\to a}(\sin x) = \sin(a) \neq \pm\infty$$

This means $y = \sin x$ has no horizontal and vertical asymptotes.

Note: Same as cosx.

(v) $y = \tan x$

Here,

$$\lim_{x\to\infty}(y) = \lim_{x\to\infty}\tan x = \text{gives no information}$$

and $$\lim_{x\to a}(y) = \lim_{x\to\infty}\tan(x) = \tan(a).$$

Since $\tan(a)\to\infty$ as $a\to\frac{\pi}{2}$ and $\tan(a)\to(-\infty)$ as $a\to\frac{-\pi}{2}$.

In general $\tan(a)\to\infty$ as $a\to\frac{(2n+1)\pi}{2}$

Therefore, tanx has vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{-\pi}{2}$ and in general

$$x = \frac{(2n+1)\pi}{2}, x = \frac{-(2n+1)\pi}{2}$$

and has no horizontal asymptotes.

Note: Similar for cotx, secx, cosecx.

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Chapter 7

CURVATURE

Theoretical Part

Definition of Curvature

The rate of change of direction of the curve with respect to arc, is called the curvature. It is denoted by κ.

Definition of Radius of Curvature

The reciprocal of curvature is called the radius of curvature. It is denoted by ρ.

List of formulae for radius of curvature

1. If the curve is given in (s, ψ) then

$$\rho = \frac{ds}{d\psi}$$

2. If the curve is given in (x, y) then

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2} \neq 0$$

or, $$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy}, x_2 = \frac{d^2x}{dy^2} \neq 0$$

3. If the curve is given in polar form i.e. in (r, θ) then

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \quad \text{where } r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}$$

4. If the curve is in pedal equation $p = f(r)$ then

$$\rho = r\frac{dr}{dp}$$

5. If the curve is in parametric form i.e. $x = \phi(t)$, $y = \psi(t)$ then

$$\rho = \frac{((x')^2+(y')^2)^{3/2}}{(x')(y'')-(x'')(y')}$$

where, $x' = \frac{dx}{dt}$, $x'' = \frac{d^2x}{dt^2}$, $y' = \frac{dy}{dt}$, $y'' = \frac{d^2y}{dt^2}$

Newton's Method:

(i) If the curve is in (x, y) and the tangent to the curve at $(0, 0)$ be x–axis then

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^2}{2y}\right)$$

(ii) If the curve is in (x, y) and the tangent to the curve at $(0, 0)$ be y–axis then

$$\rho = \lim_{y\to 0} \left(\frac{x^2}{2y}\right)$$

Series method/Method of expansion

If the curve is in (x, y) and the tangent to the curve at (0, 0) and we have $f'(0) = p$, $f''(0) = q$, then the radius of curvature of the curve at (0, 0) is

$$\rho = \frac{(1+p^2)^{3/2}}{q}$$

Process to finding p, q

Substitute $y = xp + \left(\frac{x^2}{2!}\right)q + \ldots$ in the given equation, so that the equation becomes in x. Then we equate the coefficient of like terms (at least two coefficients should be observed), we get two equations p, q solving then equation we get the value of q and q.

Exercise 7.1

1. Find the radius of curvature at (s, ψ) on the following curve

(i) $s = c\tan\psi$

Solution: Here, the given curve is

$$s = c\tan\psi$$

So, $\frac{ds}{d\psi} = c\sec^2\psi$

Thus, radius of curvature of the given curve is,

$$\rho = c\sec^2\psi.$$

(ii) $s = 8a\sin^2\frac{\psi}{6}$ **[2018 Spring (Short)] [2014 Spring (Short)]**

Solution: Here, the given curve is

$$s = 8a\sin^2\frac{\psi}{6}$$

So, $\frac{ds}{d\psi} = 8a\,2\sin\frac{\psi}{6}\cdot\cos\frac{\psi}{6}\cdot\frac{1}{6} \Rightarrow \frac{ds}{d\psi} = \frac{8a}{6}\sin\frac{2\psi}{6} = \frac{4a}{3}\sin\frac{\psi}{3}.$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{4a}{3}\sin\frac{\psi}{3}.$$

(iii) $s = c\log(\sec\psi)$

Solution: Here, the given curve is

$$s = c\log(\sec\psi)$$

$\frac{ds}{d\psi} = \frac{d}{d\psi}(c\log(\sec\psi)) \Rightarrow \frac{ds}{d\psi} = c\frac{1}{\sec\psi}\times\sec\psi\,.\tan\psi$

Thus, radius of curvature of the given curve is,

$$\rho = c\tan\psi.$$

(iv) $s = a\log\left(\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)$

Solution: Here,

$$s = a\log\left(\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)$$

So, $\rho = \frac{ds}{d\psi} = \frac{a\,.\sec^2\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}\frac{1}{2}$

$$= \frac{a}{2\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}$$

$$= \frac{a}{\sin\left(2\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)} = \frac{a}{\sin\left(\frac{\pi}{2}+\psi\right)} = \frac{a}{\cos\psi} = a\sec\psi.$$

Thus, radius of curvature of the given curve is,

$$\rho = a\sec\psi.$$

2. Find the radius of curvature at (x, y) for the curves.

(i) $y^2 = 4ax$ **[2014 Fall (short)]**

Solution: Here, $4ax = y^2$ (given).

$\therefore 4a\frac{dx}{dy} = 2y \quad \Rightarrow x_1 = \frac{dx}{dy} = \frac{y}{2a}$

$(1+x_1^2)^{3/2}$

x_2

Again, Differentiating w. r. t. x, we get

$$x_2 = \frac{d^2y}{dx^2} = \frac{1}{2a}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{\left(1+\frac{y^2}{4a^2}\right)^{3/2}}{1/2a}$$

$$= \frac{2a\,(y^2+4a^2)^{3/2}}{8a^3}$$

$$= \frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$= \frac{(4ax+4a^2)^{3/2}}{4a^2} = \frac{8a^{3/2}}{4a^2}(a+x)^{3/2} = \frac{2}{a^{1/2}}(a+x)^{3/2}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{2}{a^{1/2}}(a+x)^{3/2}.$$

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{y^2}{2y}\right).$$

Series method/Method of expansion

If the curve is in (x, y) and the tangent to the curve at (0, 0) and we set $f'(0) = p$, $f''(0) = q$, then the radius of curvature of the curve at (0, 0) is

$$\rho = \frac{(1+p^2)^{3/2}}{q}$$

Process to finding p, q

- Substitute $y = xp + \left(\frac{x^2}{2!}\right)q + \ldots$ in the given equation, so that the equation becomes in x. Then we equate the coefficient of like terms (at least two coefficients should be observed), we get two equations p, q solving then equation we get the value of q and q.

Exercise 7.1

1. Find the radius of curvature at (s, ψ) on the following curve

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$$s = c\tan\psi$$

So, $\dfrac{ds}{d\psi} = c\sec^2\psi$

Thus, radius of curvature of the given curve is,

$$\rho = c\sec^2\psi.$$

(ii) $s = 8a\sin^2\frac{\psi}{6}$ **[2018 Spring (Short)] [2014 Spring (Short)]**

Solution: Here, the given curve is

$$s = 8a\sin^2\frac{\psi}{6}$$

So, $\dfrac{ds}{d\psi} = 8a\,2\sin\frac{\psi}{6}\cdot\cos\frac{\psi}{6}\cdot\frac{1}{6} \Rightarrow \frac{ds}{d\psi} = \frac{8a}{6}\sin\frac{2\psi}{6} = \frac{4a}{3}\sin\frac{\psi}{3}.$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{4a}{3}\sin\frac{\psi}{3}.$$

(iii) $s = c\log(\sec\psi)$

Solution: Here, the given curve is

$$s = c\log(\sec\psi)$$

So, $\dfrac{ds}{d\psi} = \dfrac{d}{d\psi}(c\log(\sec\psi)) \Rightarrow \dfrac{ds}{d\psi} = c\dfrac{1}{\sec\psi}\times\sec\psi\,.\tan\psi$

Thus, radius of curvature of the given curve is,

$$\rho = c\tan\psi.$$

(iv) $s = a\log\left(\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)$

Solution: Here,

$$s = a\log\left(\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)$$

So, $\rho = \dfrac{ds}{d\psi} = \dfrac{a\,.\sec^2\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}\dfrac{1}{2}$

$$= \frac{a}{2\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}$$

$$= \frac{a}{\sin\left(2\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right)} = \frac{a}{\sin\left(\frac{\pi}{2}+\psi\right)} = \frac{a}{\cos\psi} = a\sec\psi.$$

Thus, radius of curvature of the given curve is,

$$\rho = a\sec\psi.$$

2. Find the radius of curvature at (x, y) for the curves.

(i) $y^2 = 4ax$ **[2014 Fall (short)]**

Solution: Here, $4ax = y^2$ (given).

$\therefore 4a\dfrac{dx}{dy} = 2y \quad \Rightarrow x_1 = \dfrac{dx}{dy} = \dfrac{y}{2a}$

$(1+x_1^2)^{3/2}$ / x_2

Again, Differentiating w. r. t. x, we get

$$x_2 = \frac{d^2y}{dx^2} = \frac{1}{2a}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{\left(1+\frac{y^2}{4a^2}\right)^{3/2}}{1/2a}$$

$$= \frac{2a(y^2+4a^2)^{3/2}}{8a^3}$$

$$= \frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$= \frac{(4ax+4a^2)^{3/2}}{4a^2} = \frac{8a^{3/2}}{4a^2}(a+x)^{3/2} = \frac{2}{a^{1/2}}(a+x)^{3/2}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{2}{a^{1/2}}(a+x)^{3/2}.$$

Q. Find the radius of curvature of curve $y^2 = 4ax$ at (0, 0).
[2016 Spring (short)] [2017 Spring (short)]

(ii) $y = \log(\sin x)$.

Solution: Here, $y = \log(\sin x)$

Differentiating w. r. t. x, then

$$y_1 = \frac{1}{\sin x}\cos x = \cot x \quad \text{and} \quad y_2 = -\operatorname{cosec}^2 x$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2 x)^{3/2}}{-\operatorname{cosec}^2 x} = \frac{\operatorname{cosec}^3 x}{-\operatorname{cosec}^2 x} = -\operatorname{cosec} x$$

Since ρ is non–negative.

Thus, radius of curvature of the given curve is,

$$\rho = \operatorname{cosec} x.$$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Here,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

Differentiating w. r. t. x, then

$$2b^2x + 2a^2y.y_1 = 0$$

$$\Rightarrow y_1 = -\frac{b^2x}{a^2y}$$

Again differentiating w. r. t. x, then

$$2b^2 + 2a^2y_1^2 + 2a^2y \,.\, y_2 = 0$$

$$\Rightarrow y_2 = \frac{-b^2 - a^2y_1^2}{a^2y} = -\frac{1}{a^2y}\left[b^2 + a^2\frac{b^4x^2}{a^4y^2}\right]$$

$$= -\frac{1 \,.\, b^2}{a^2y \,.\, a^4y^2}[a^4y^2 + a^2b^2x^2]$$

$$= -\frac{a^2b^2}{a^6y^3}(a^2y^2 + b^2x^2)$$

$$= -\frac{a^2b^2}{a^6y^3}\,.\,a^2b^2 \quad = -\frac{b^4}{a^2y^3}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = -\frac{a^2y^3}{b^4}\left[1 + \frac{b^4x^2}{a^4y^2}\right]^{3/2} = -\frac{a^2y^3}{b^4a^3y^3}[a^4y^2 + b^4x^2]^{3/2}$$

$$\Rightarrow \rho = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4} \quad [\because \rho \text{ is not negative}]$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4}$$

(iv) $x^{2/3} + y^{2/3} = a^{2/3}$

Solution: Here, $x^{2/3} + y^{2/3} = a^{2/3}$.

Differentiating w. r. t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

Now, $1 + y_1^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{1}{x} \Rightarrow 1 + y_1^2 = a^{2/3}x^{-2/3}$ … (A)

Differentiating (A) w. r. t. x, we get

$$0 + 2y_1y_2 = a^{2/3}\left(\frac{-2}{3}\right)x^{-5/3}$$

$$\therefore y_2 = -a^{2/3}\cdot\frac{x^{-5/3}}{3y_1}$$

$$= -a^{2/3}\frac{x^{-5/3}\,.\,x^{1/3}}{y^{1/3}} \quad \left[\because y_1 = \frac{-y^{1/3}}{x^{1/3}}\right]$$

$$= \frac{a^{2/3}}{3}x^{-4/3}y^{-1/3}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(a^{2/3}x^{-2/3})^{3/2}}{\frac{a^{2/3}}{3}x^{-4/3}y^{-1/3}} \quad \text{[Using (A)]}$$

$$= \frac{3ax^{-1}}{a^{2/3}x^{-4/3}y^{-1/3}} = 3a^{1/3}x^{1/3}y^{1/3} = 3(axy)^{1/3}.$$

Thus, radius of curvature of the given curve is,

$$\rho = 3(axy)^{1/3}.$$

(v) $y = 4\sin x - \sin 2x$ at $x = \pi/2$ [2008, Fall(short)]

Solution: Here, $y = 4\sin x - \sin 2x$

So, $y_1 = \frac{dy}{dx} = 4\cos x - 2\cos 2x$

And, $y_2 = \frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$

Then at $x = \frac{\pi}{2}$, $y_1 = 0 - 2(-1) = 2$ and, $y_2 = 4\sin\frac{\pi}{2} = -4$.

Now the radius of the curvature of the given curve at $x = \pi/2$ is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(5)^{3/2}}{4}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{(5)^{3/2}}{4}$$

$-\operatorname{cosec} x$

(vi) $9x^2 + 4y^2 = 36x$ at (2, 3) [2006, Fall(short)]

Solution: Here, $9x^2 + 4y^2 = 36x$

Now, differentiating w. r. t. x, then

$$18x + 8y \cdot y_1 = 36 \quad \Rightarrow y_1 = \frac{36 - 18x}{8y}$$

and $$18 + 8y_1 \cdot y_1 + 8y \cdot y_2 = 0 \quad \Rightarrow y_2 = -\frac{18 + 8y_1^2}{8y}$$

At (2, 3)

$$y_1 = \frac{36 - 36}{24} = 0 \quad \text{and} \quad y_2 = -\frac{18 + 8 \cdot 0}{24} = -\frac{3}{4}$$

Now the radius of the curvature of the given curve at (2, 3) is

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{1}{-3/4} = -\frac{4}{3}$$

Since ρ is always non-negative. So, $\rho = \frac{4}{3}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{4}{3}.$$

(vii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where y = x cuts it.

[2003, Spring] [2007, Spring]

Solution: Here, $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

i.e. $y = a + x - 2\sqrt{ax}$. $\sqrt{a}-\sqrt{y}$

Differentiating w. r. t. x, then

$\sqrt{x}+\sqrt{x}=\sqrt{a}$

$$y_1 = 1 - 2\sqrt{a} \cdot \frac{1}{2\sqrt{x}} = 1 - \frac{\sqrt{a}}{\sqrt{x}}$$

$\sqrt{a}=\sqrt{2x}$ and,

$$y_2 = \frac{\sqrt{a}}{2\,(x)^{3/2}}$$

$a=2x$

Solving the given line y = x and the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then we get x = a/4 and y = a/4.

$x=a/2$

Therefore, at (a/4, a/4),

$$y_1 = 1 - \frac{\sqrt{a}}{\sqrt{a/4}} = 1 - 2 = -1 \quad \text{and} \quad y_2 = \frac{\sqrt{a}}{2\,(a/4)^{3/2}} = \frac{4}{a}.$$

Now, the radius of curvature of the given curve at (a/4, a/4) is,

$$\rho = \frac{\{1 + (y_1)^2\}^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4/a} = \frac{2a\sqrt{2}}{4} = \frac{a}{\sqrt{2}}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{a}{\sqrt{2}}.$$

3. Find the radius of curvature at any point to the following curves:

(i) x = a cosθ, y = a sinθ

Solution: Here, x = a cosθ, y = a sinθ.

So, x' = −a sinθ, y' = a cosθ

And, x" = −a cosθ, y" = −a sinθ

Now, the radius of curvature of the given curve is,

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(a^2\sin^2\theta + a^2\cos^2\theta)^{3/2}}{-a^2\cos^2\theta - a^2\sin^2\theta} = \frac{(a^2)^{3/2}}{-a^2} = \frac{a^3}{-a^2} = a.$$

Thus, radius of curvature of the given curve is,

$$\rho = a.$$

Q. Find the radius of curvature of the curves: x = r cosθ, y = r sinθ.

[2017 Spring Short]

Solution: See above solution with replacing a by r.

(ii) $x = at^2$, y = 2at

Solution: Here, $x = at^2$, y = 2at.

So, y' = 2a, x' = 2at

And, y" = 0, x" = 2a.

Now, the radius of curvature of the given curve is,

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(4a^2 + 4a^2t^2)^{3/2}}{2a \cdot 2a} = \frac{8a^3}{4a^2}(1+t^2)^{3/2} = 2a(1+t^2)^{3/2}.$$

Thus, radius of curvature of the given curve is,

$$\rho = 2a(1 + t^2)^{3/2}.$$

(iii) x = a(cost + tsint), y = a(sint − tcost)

Solution: Here,

x = a(cost + tsint), y = a(sint − tcost)

Differentiating we get,

$\dot{x} = a(-\sin t + \sin t + t\cos t)$, and $\dot{y} = a(\cos t - \cos t + t\sin t)$

$\Rightarrow \dot{x} = at\cos t$ $\quad \Rightarrow \dot{y} = at\sin t$

and $\ddot{x} = a\cos t - at\sin t$ $\quad \Rightarrow \ddot{y} = a\sin t + at\cos t$

Now, radius of curvature of given curve is,

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

$$= \frac{(a^2t^2\cos^2 t + a^2t^2\sin^2 t)^{3/2}}{at\cos t\,(a\sin t + at\cos t) - (a\cos t - at\sin t)\,at\sin t}$$

$$= \frac{(a^2t^2)^{3/2}}{a^2t[\cos t\sin t + t\cos^2 t - \cos t\sin t + t\sin^2 t]}$$

$$= \frac{a^3t^3}{a^2t^2(\cos^2 t + \sin^2 t)}$$

$$= at$$

Thus, radius of curvature of the given curve is,

$$\rho = at.$$

(iv) $x = a\cos^3\theta,\ y = a\sin^3\theta$, at $\theta = \pi/4$

Solution: Here, $x = a\cos^3\theta,\ y = a\sin^3\theta$

Differentiating w. r. t. θ then,

$$\dot{x} = -3a\cos^2\theta\sin\theta \qquad \dot{y} = 3a\sin^2\theta\cos\theta$$

and
$$\ddot{x} = +6a\cos\theta\sin^2\theta - 3a\cos^3\theta$$
$$\ddot{y} = 6a\sin\theta\cos^2\theta - 3a\sin^3\theta$$

at $\theta = \frac{\pi}{4}$,

$$\dot{x} = -3a\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}} = -\frac{3a}{2\sqrt{2}} \qquad \dot{y} = \frac{3a}{2\sqrt{2}}$$

$$\ddot{x} = \frac{6a}{2\sqrt{2}} - \frac{3a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}} \qquad \ddot{y} = \frac{6a}{2\sqrt{2}} - \frac{3a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}}$$

Now, radius of curvature of given curve at $\theta = \frac{\pi}{4}$ is,

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \frac{\left[\left(-\frac{3a}{2\sqrt{2}}\right)^2+\left(\frac{3a}{2\sqrt{2}}\right)^2\right]^{3/2}}{-\frac{9a^2}{8}-\frac{9a^2}{8}}$$

$$= -\frac{8}{18a^2}\left[2\left(\frac{3a}{2\sqrt{2}}\right)^2\right]^{3/2}$$

$$= -\frac{8}{18a^2}\cdot\frac{27a^3}{16\sqrt{2}}\cdot 2^{3/2}$$

$$= -\frac{3a}{2}$$

Since ρ is non–negative. So, $\rho = \frac{3a}{2}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{3a}{2}.$$

4. Find the radius of curvature at any point (r, θ) for the following curves.

(i) $r = a(1-\cos\theta)$

Solution: Here, $r = a(1-\cos\theta)$

Differentiating w. r. t. θ, then,

$$r_1 = a\sin\theta \quad \text{and,} \quad r_2 = a\cos\theta$$

Now, radius of curvature of given curve is,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(a^2-2a^2\cos\theta+a^2\cos^2\theta+a^2\sin^2\theta)^{3/2}}{a^2-2a^2\cos\theta+a^2\cos^2\theta+2a^2\sin^2\theta-a^2\cos\theta(1-\cos\theta)}$$

$$= \frac{(a^2-2a^2\cos\theta+a^2)^{3/2}}{a^2+a^2\cos^2\theta+2a^2\sin^2\theta-3a^2\cos\theta+a^2\cos^2\theta}$$

$$= \frac{(2a^2)^{1/2}(1-\cos\theta)^{1/2}}{a^2+2a^2-3a^2\cos\theta}$$

$$= \frac{a\sqrt{2}\,.\,\sqrt{r/a}}{3a^2(1-\cos\theta)}$$

$$= \frac{\sqrt{2r}}{3r\sqrt{a}} = \frac{1}{3}\sqrt{\frac{2}{ar}}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{3}\sqrt{\frac{2}{ar}}.$$

$$\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

(ii) $r^2 = a^2\cos 2\theta$

Solution: Here, $r^2 = a^2\cos 2\theta$

Differentiating w.r.t. θ then,

$$2r.\,r_1 = -2a^2\sin 2\theta$$

$$\Rightarrow r_1 = -\frac{a^2\sin 2\theta}{r} = -\frac{ra^2\sin 2\theta}{r^2} = \frac{r\,a^2\sin 2\theta}{a^2\cos 2\theta} = -r\tan 2\theta$$

and,

$$r_2 = -r_1\tan 2\theta - 2r\sec^2 2\theta$$
$$= r\tan^2 2\theta - 2r\sec^2 2\theta$$
$$= r\tan^2 2\theta - 2r - 2r\tan^2 2\theta$$
$$= -r\tan^2 2\theta - 2r$$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\tan^2 2\theta)^{3/2}}{r^2+2r^2\tan^2 2\theta+r^2(2+\tan^2 2\theta)}$$

$$= \frac{r(1+\tan^2 2\theta)^{3/2}}{1+2\tan^2 2\theta+2+2\tan^2 2\theta}$$

$$= \frac{r\sec^3 2\theta}{3+3\tan^2 2\theta}$$

$$= \frac{r\sec^3 2\theta}{3\sec^2 2\theta}$$

$$= \frac{r\sec 2\theta}{3}$$

$$= \frac{a}{3}\sec 2\theta\sqrt{\cos 2\theta}$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{a}{3}\sec 2\theta\sqrt{\cos 2\theta}.$$

(iii) $r = ae^{\theta\cot\alpha}$ [2018 Fall (Short)] [2011 Fall (Short)] [2011 Spring

Solution: Here, $r = ae^{\theta\cot\alpha}$

So, $r_1 = a\cot\alpha\, e^{\theta\cot\alpha} = r\cot\alpha$

And, $r_2 = a\cot^2\alpha\, e^{\theta\cot\alpha} = r\cot^2\alpha$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha} = \frac{r^3(1+\cot^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)} = r\,\mathrm{cosec}\alpha.$$

Thus, radius of curvature of the given curve is,

$$\rho = r\,\mathrm{cosec}\alpha.$$

(iv) $r^2\cos 2\theta = a^2$

Solution: Here, $r^2\cos 2\theta = a^2$

So, $-2r^2\sin 2\theta + 2\cos 2\theta\, r\,.\, r_1 = 0$

$\Rightarrow r_1 = r\tan 2\theta.$

And, $r_2 = 2r\sec^2 2\theta + \tan 2\theta\,.\, r\tan 2\theta$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{(r^2 + r^2\tan^2 2\theta)^{3/2}}{r^2 + 2r^2\tan^2 2\theta - 2r^2\sec^2 2\theta - r^2\tan^2 2\theta}$$

$$= \frac{r^3(\sec^2 2\theta)^{3/2}}{r^2(1 + 2\tan^2 2\theta - 2\sec^2 2\theta)}$$

$$= r\frac{\sec^2 2\theta}{-\sec^2 2\theta}$$

$$= -r\sec 2\theta$$

$$= -r/\cos 2\theta \quad = -r/a^2/r^2 \quad = -r^3/a^2$$

Since $\rho \neq -$ ve. So, $\rho = r^3/a^3$.

Thus, radius of curvature of the given curve is,

$$\rho = r^3/a^3.$$

(v) $r = a\sin n\theta$, at origin

Solution: Given that, $r = a\sin n\theta$

So, $r_1 = an\cos n\theta$ and $r_2 = -an^2\sin n\theta$

At origin,

$$r = 0,\ r_1 = an,\ r_2 = 0$$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{(a^2n^2)^{3/2}}{2a^2n^2} = \frac{a^3n^3}{2a^2n^2} = \frac{an}{2}$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{an}{2}.$$

5. Find the radius of curvature at origin of the following curves as:

(i) $y = x^4 - 4x^3 - 18x^2$ [2013 Spring (Short)]

Solution: Here, equating the lowest order term in the equation, to zero. That is, $y = 0$ i.e. the x–axis is tangent at the origin. So the radius of curvature of the given curve is,

$$\rho = \lim_{y\to 0}\left\{\frac{x^2}{2y}\right\}$$

Now, dividing both sides of given equation $y = x^4 - 4x^3 - 18x^2$ by $2y$ and then taking limit as $x\to 0$, $y\to 0$ both sides, we get,

$$\lim_{\substack{x\to 0\\ y\to 0}}(2) = \lim_{\substack{x\to 0\\ y\to 0}}\left(x^2\left\{\frac{x^2}{2y}\right\}\right) - \lim_{\substack{x\to 0\\ y\to 0}}\left(4x\left\{\frac{x^2}{2y}\right\}\right) - 18\lim_{\substack{x\to 0\\ y\to 0}}\left\{\frac{x^2}{2y}\right\}.$$

$\Rightarrow 2 = 0(\rho) - 0(\rho) - 18\rho.$

$\Rightarrow \rho = \frac{1}{-36}.$

Since $\rho \neq -$ve. So, $\rho = \frac{1}{36}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{36}.$$

Alternative Method:

Here, $y = x^4 - 4x^3 - 18x^2$

So, $\frac{dy}{dx} = 4x^3 - 12x^2 - 36x$.

And, $\frac{d^2y}{dx^2} = 12x^2 - 24x - 36.$

At origin, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = -36$

Now, the radius of the given curve is,

$$\rho = \left\{\frac{1 + \left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^{2/3}}\right\}^{3/2} = \left\{\frac{1 - 0^2}{(-36)^{2/3}}\right\}^{3/2} = \frac{1}{-36}.$$

Since $\rho \neq -$ve. So, $\rho = \frac{1}{36}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{36}.$$

(ii) $3x^2 + 4y^2 = 2x$

Solution: Here, equating the lowest order term in the equation, to zero. That is, $x = 0$ i.e. the y–axis is tangent at the origin. So the radius of curvature of the given curve is,

$$\rho = \lim_{y\to 0}\left\{\frac{y^2}{2x}\right\}$$

Now, dividing both sides of given equation $3x^2 + 4y^2 = 2x$ by $2x$ and taking limit as $x\to 0$, $y \to 0$ both sides, we get,

$$\frac{3}{2}\lim_{\substack{x\to 0\\ y\to 0}}(x) + 4\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{2x}\right) - 1 = 0.$$

$$\Rightarrow 0 + 4\rho - 1 = 0.$$

$$\Rightarrow 4\rho = 1$$

$$\Rightarrow \rho = \frac{1}{4}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{4}.$$

(iv) $x^3 + y^3 = 3axy$ [2017 Fall]

Solution: Here, equating the lowest order term in the equation, to zero. That is, $xy = 0$. That is the x-axis and y-axis are tangent at the origin. Thus,

$$\lim_{x\to 0}\left\{\frac{x^2}{2y}\right\} = \rho \quad \text{and} \quad \lim_{y\to 0}\left\{\frac{y^2}{2y}\right\} = \rho$$

Given equation is,

$$x^3 + y^3 = 3axy$$

Dividing both sides of this equation by $2xy$ we get

$$\left\{\frac{x^2}{2y}\right\} + \left\{\frac{y^2}{2x}\right\} = \frac{3a}{2}$$

Taking limit at $x\to 0$, $y\to 0$ both side, we get

$$\lim_{\substack{x\to 0\\ y\to 0}}\left\{\frac{x^2}{2y}\right\} + \lim_{\substack{x\to 0\\ y\to 0}}\left\{\frac{y^2}{2x}\right\} = \frac{3a}{2}.$$

$$\Rightarrow \rho + \rho = \frac{3a}{2}$$

$$\Rightarrow 2\rho = \frac{3a}{2} \quad \Rightarrow \rho = \frac{3a}{4}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{3a}{4}.$$

(v) $x^2 + y^2 + 6x + 8y = 0$

Solution: Here, $x^2 + y^2 + 6x + 8y = 0$ ... (i)

Putting $y = xp + \frac{x^2}{2}q + \ldots$ in equation (i), we get

$$\text{or} \quad x^2 + \left(xp + \frac{x^2}{2}q + \ldots\right)^2 + 6x + 8\left(xp + \frac{x^2}{2}q + \ldots\right) = 0.$$

$$\text{or} \quad x(6 + 8p) + x^2(1 + p^2 + 4q) + \ldots = 0.$$

Equating the coefficient of x and x^2, we get

$$6 + 8p = 0 \quad \Rightarrow p = \frac{-3}{4}$$

and

$$1 + p^2 + 4q = 0 \Rightarrow 4q = (-1 - p^2) = \left(-1 - \frac{9}{16}\right) = \frac{-25}{16}$$

$$\Rightarrow q = \frac{-25}{64}$$

Now, the radius of curvature of the given curve is,

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \left|\frac{(1 + 9/16)^{3/2}}{-25/64}\right| = \left|\frac{125\times 64}{-25\times 64}\right| = 5.$$

(vi) $y^2 - 3xy - 4x^2 + 5x^3 + x^4y - y^5 = 0$

Solution: Here,

$$y^2 - 3xy - 4x^2 + 5x^3 + x^4y - y^5 = 0 \quad \ldots (i)$$

Putting $y = xp + \frac{x^2}{2}q + \ldots$ in equation (i), we get

$$\left(xp + \frac{x^2}{2}q + \ldots\right)^2 - 3x\left(xp + \frac{x^2}{2}q + \ldots\right) - 4x^2 + 5x^3 + x^4\left(xp + \frac{x^2}{2}q + \ldots\right) - \left(xp + \frac{x^2}{2}q + \ldots\right)^5 = 0.$$

$$\Rightarrow \quad x^2(p^2 - 3p - 4) + x^3\left(pq - \frac{3q}{2} + 5\right) + \ldots = 0$$

Equating the coefficient of x^2 and x^3, we get

$$p^2 - 3p - 4 = 0$$

$$\Rightarrow (p-4)(p+1) = 0$$

$$\Rightarrow p = 4, -1.$$

and,

$$pq - \frac{3q}{2} + 5 = 0$$

At $p = 4$, $\quad 4q - \frac{3q}{2} + 5 = 0 \Rightarrow 5q = -10 \Rightarrow q = -2.$

At $p = -1$, $\quad -q - \frac{3q}{2} + 5 = 0 \Rightarrow 5q = 10 \Rightarrow q = 2$

When $(p, q) = (4, -2)$ then

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+15)^{3/2}}{-2} = -\frac{(17)^{3/2}}{2} = \frac{17\sqrt{17}}{2}.$$

When $(p, q) = (-1, 2)$ then

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

Since, $\rho \neq$ –ve. Then, $\rho = \sqrt{17}$ and $\sqrt{2}$

(vii) $3x^2 + xy + y^2 - 4x = 0$

Solution: Here, $3x^2 + xy + y^2 - 4x = 0$ ... (i)

Here, equating the lowest order term in the equation, to zero. That means, $x = 0$ i.e., the x-axis is tangent at the origin.

Therefore,

$$\rho = \lim_{y\to 0}\left\{\frac{y^2}{2x}\right\}$$

Dividing both sides of (i) by 2x we get,

$$\frac{3}{2}x + \frac{y}{2} + \left\{\frac{y^2}{2x}\right\} - 2 = 0$$

Taking limit at $x \to 0$ and $y \to 0$ both side we get

$$\frac{3}{2}\lim_{\substack{x\to 0\\ y\to 0}}(x) + \frac{1}{2}\lim_{\substack{x\to 0\\ y\to 0}}(y) + \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{2x}\right) - 2$$

Or, $0 + 0 + \rho - 2 = 0$

$\Rightarrow \rho = 2.$

Thus, radius of curvature of the given curve is,

$\rho = 2.$

(viii) $3x^3 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$

Solution: Here, $3x^3 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$ ... (i)

Here, equating the lowest order term in the equation, to zero. That means, $x = 0$ i.e., the y-axis is tangent at the origin.

Thus,

$$\rho = \lim_{y\to 0}\left\{\frac{y^2}{2x}\right\}$$

Dividing both side of (i) by 2x we get

Taking limit at $x \to 0$, $y \to 0$ both side we get,

$$\frac{3}{2}\lim_{\substack{x\to 0\\ y\to 0}}(x^3) - \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^4}{x}\right) + \frac{5}{2}\lim_{\substack{x\to 0\\ y\to 0}}(xy) + \lim_{\substack{x\to 0\\ y\to 0}}(y) - 2\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{2x}\right) + 2 = 0$$

$\Rightarrow 0 - 0 + 0 + 0 - 2\rho + 2 = 0$

$\Rightarrow -2\rho + 2 = 0$

$\Rightarrow \rho = 1.$

Thus, radius of curvature of the given curve is,

$\rho = 1.$

(ix) $x^2 + 6y^2 + 2x - y = 0$

Solution: Here, $x^2 + 6y^2 + 2x - y = 0$ ... (i)

Putting $y = xp + \frac{x^2}{2}q + ...$ in equation (i), we get

$$\text{or}\quad x^2 + 6\left(xp + \frac{x^2}{2}q + ...\right)^2 + 2x - \left(xp + \frac{x^2}{2}q + ...\right) = 0.$$

$$\text{or}\quad x(2-p) + x^2\left(1 + 6p^2 - \frac{q}{2}\right) + ... = 0.$$

Equating the coefficient of x and x^2, we get

$2 - p = 0 \Rightarrow \quad p = 2.$

And, $1 + 6p^2 - \frac{q}{2} = 0 \Rightarrow \quad q = 2(1 + 6p^2) = 2(1 + 24) = 50.$

Now the radius of curvature of the given curve is,

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{(1 + 4)^{3/2}}{50} = \frac{5\sqrt{5}}{50} = \frac{1}{2\sqrt{5}}.$$

(x) $x^4 + y^2 = 6a(x + y)$

Solution: Here, $x^4 + y^2 = 6a(x + y)$

$x^4 + y^2 - 6a(x + y) = 0$ ... (i)

Putting $y = xp + \frac{x^2}{2}q + ...$ in equation (i), we get

$$x^4 + \left(xp + \frac{x^2}{2}q + ...\right)^2 - 6a\left(x + xp + \frac{x^2}{2}q + ...\right) = 0.$$

$$\Rightarrow \quad x(-6a - 6ap) + x^2\left(p^2 - 6a\frac{q}{2}\right) + ... = 0.$$

Equating the coefficient of x and x^2, we get

$-6a - 6ap = 0 \quad \Rightarrow \quad p = -1.$

And, $p^2 - 6a\frac{q}{2} = 0 \Rightarrow q = \frac{2p^2}{6a} = \frac{1}{3a}.$

Now the radius of curvature of the given curve is,

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{(1 + 1)^{3/2}}{1/3a} = 6\sqrt{2}\,a.$$

6. **Show that the radius of curvature of the curve $y = x^2(x - 3)$ at the points, where the tangent is parallel to x–axis are $\pm\frac{1}{6}$.**

Solution: Given curve be

$y = x^2(x - 3)$

$\Rightarrow y = x^3 - 3x^2$ ... (i)

Diff. w.r.t. x then,

$y_1 = 3x^2 - 6x$ and $y_2 = 6x - 6.$

Since by hypothesis, the tangent is parallel to x–axis.

So, $y_1 = 0 \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x - 2) = 0$

$\Rightarrow x = 0, 2.$

Then by (i), $y = 0, -4.$

Thus, the tangents are at (0, 0) and (2, –4).

At (0, 0), $y_1 = 0$ and $y_2 = -6.$

At (2, –4), $y_1 = 0$ and $y_2 = 6.$

Now, the radius f curvature of (i) is,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Therefore, at (0, 0), $\rho = \frac{(1+0^2)^{3/2}}{-6} = \frac{-1}{6}$

And at (2, –4), $\rho = \frac{(1+0^2)^{3/2}}{6} = \frac{1}{6}$

Thus (i) has radius of curvature $\frac{-1}{6}$ at (0, 0) and $\frac{1}{6}$ at (2, –4).

7. If ρ_1 and ρ_2 be the radius of curvature at the ends of focal chord of the parabola $y^2 = 4ax$, prove that $\rho_1^{(-2/3)} + \rho_2^{(-2/3)} = (2a)^{-2/3}$.

[2005, Spring]

Solution: Given curve is, $y^2 = 4ax$ ... (i)

Since the general point of the parabola (i) be $x = at^2$

$y = 2at$

So, $\dot{x} = 2at$, $\dot{y} = 2a$

and $\ddot{x} = 2a$, $\ddot{y} = 0$

Now, radius of curvature of (i) is

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \frac{(4a^2t^2+4a^2)^{3/2}}{0-4a^2} = \frac{8a^3(t^2+1)^{3/2}}{-4a^2}$$

$\Rightarrow \rho = 2a(1+t^2)^{3/2}$ ..... (ii)

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are ends of the chord of the parabola (i), then, $t_1.t_2 = -1$.

Therefore, (ii) become

at P, $\rho_1 = 2a(1+t_1^2)^{3/2}$ and at Q, $\rho_2 = 2a(1+t_2^2)^{3/2}$

Now,

$$\frac{1}{\rho_1^{2/3}} + \frac{1}{\rho_2^{2/3}} = (2a)^{-2/3}[(1+t_1^2)^{-1} + (1+t_2^2)^{-1}]$$

$$= (2a)^{-2/3}\left(\frac{1}{1+t_1^2} + \frac{1}{1+t_2^2}\right)$$

$$= (2a)^{-2/3}\left(\frac{1+t_1^2+1+t_1^2}{1+t_1^2+t_2^2+t_1^2t_2^2}\right)$$

$$= (2a)^{-2/3}\left(\frac{2+t_1^2+t_2^2}{1+t_1^2+t_2^2+1}\right) \quad \text{[since } t_1.t_2 = -1]$$

$$= (2a)^{-2/3}\left(\frac{2+t_1^2+t_2^2}{2+t_1^2+t_2^2}\right)$$

$$= (2a)^{-2/3}$$

8. Show that of the curvature at any point of a circle is constant.

Solution: Let the equation of circle be

$x^2 + y^2 = a^2$ ... (i)

$2x + 2y \cdot y_1 = 0 \Rightarrow y_1 = -\frac{x}{y}$

and $2 + 2y_1.y_1 + 2y.y_2 = 0$

$\Rightarrow y_2 = \frac{-1-y_1^2}{y_1} = \frac{-y^2-x^2}{y^3} = -\frac{a^2}{y^3}$.

Now,

$$\rho = \left|\frac{(1+y_1^2)^{3/2}}{y_2}\right| = \left|\frac{(1+x^2/y^2)^{3/2}}{-a^2/y^3}\right| = \left|\frac{(x^2+y^2)^{3/2}}{-a^2}\right| = \left|\frac{a^3}{-a^2}\right| = a.$$

$\Rightarrow \rho = a$

Hence, the curvature be $\frac{1}{\rho} = \frac{1}{a}$ which is a constant.

9. Find ρ at (r, θ) on $r = a(1-\cos\theta)$ and show that it varies as $\sqrt{r}$.

Solution: Given that $r = a(1-\cos\theta)$

So, $r_1 = a\sin\theta$, $r_2 = a\cos\theta$

Now,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{a^3(1-2\cos\theta+\cos^2\theta+\sin^2\theta)^{3/2}}{a^2[1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a\,2\sqrt{2}(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$$

$$= \frac{\sqrt{a}\,.\,2\sqrt{2}\,r^{3/2}}{3r} = \left(\frac{2\sqrt{2a}}{3}\right)\sqrt{r}$$

This shows that ρ varies as $\sqrt{r}$.

10. Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y–axis.

Solution: Given that, $y = e^x$

So, $y_1 = e^x = y_2$. Thus, $y = y_1 = y_2 = e^x$

At y–axis, the x–ordinate is 0. And, $y = e^0 = 1$.

At $x = 0$, the curve $y = e^x$ crosses the y–axis at (0, 1).

At (0, 1),

$$y = y_1 = y_2 = e^0 = 1.$$

Now, the radius of curvature of $y = e^x$ at the point where y cross the y-axis i.e. at (0, 1) is,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}.$$

$y = e^0$

$y = 1$

$(0, 1)$

$y = 1$

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

LONG QUESTIONS

1. Find the radius of curvature at origin of the following curves, $y^2 - 2xy - 3x^2 - 4x^3 - x^2y^2 = 0$. [2009, Fall]

Solution: Given curve is

$$y^2 - 2xy - 3x^2 - 4x^3 - x^2y^2 = 0 \quad .....(1)$$

Now, equate the coefficient of lowest power of x and y to zero then we get y = 0 and x = 0.

Therefore, the tangent at origin to (1), are both x–axis and y–axis. So,

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}, \qquad \rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2y}$$

$\frac{y^2}{2n}$ x^2

Here dividing (i) by 2xy then

$$\frac{y}{2x} - 1 - \frac{3x}{2y} - \frac{4x^2}{2y} - \frac{xy}{2} = 0 \quad ......(2)$$

As $x\to 0$, $y \to 0$, the first and third term have $\frac{0}{0}$ form. So, applying L.Hospital rule on these term then, gives either zero or unvalid result. So omitting the value of these terms with limiting case (2) becomes,

$$-1 - 4\rho = 0$$

$$\Rightarrow \rho = -\frac{1}{4}$$

Hence, the radius of curvature of (1) be $\rho = -\frac{1}{4}$.

2. Find the radius of curvature at the right extremity of major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [1999] [2001]

Solution: First Part: See definition of radius of curvature.

Second Part: See from exercise Q. 2(iii)

3. Define a curvature of a curve at a point. Find the curvature and radius of curvature of the curve, $x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$ at the point x = a. [2000]

Solution: First Part: See definition of radius of curvature.

Second Part: Given curve is-

$$x^2(y-a) - \frac{a^3}{\sqrt{5}} = 0 \quad ...(i)$$

So, $$x^2 + 2x.\, x_1(y-a) = 0$$

$$\Rightarrow x_1 = -\frac{x}{2(y-a)} \quad ......(ii)$$

and,

$$x_2 = \frac{2(y-a)(-x_1) - (-x)2}{4(y-a)^2} \quad .....(iii)$$

At x = a, (i) gives,

$$y = a\left(1 + \frac{1}{\sqrt{5}}\right)$$

(ii) gives,

$$x_1 = -\frac{\sqrt{5}}{2}$$

and (iii) gives,

$$x_2 = \frac{5(a - \sqrt{5})}{4a^2}$$

Now, radius of curvature of (i) be

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} = \frac{(1+5/4)^{3/2}}{5(a-\sqrt{5})} = \frac{a^2(9)^{3/2}}{10(a-\sqrt{5})} = \frac{27a^2}{10(a-\sqrt{5})}$$

Hence, the radius of curvature of (i) be

$$\rho = \frac{27a^2}{10(a-\sqrt{5})}.$$

5. Find the radius of curvature of the ellipse $9x^2 + 4y^2 = 36$ at (2, 0). [2002]

The Newton's method does not work here being to find the curvature at not origin.

Solution: Given curve is

$$9x^2 + 4y^2 = 36 \quad ...(i)$$

So, $$18x.x_1 + 8y = 0$$

$$\Rightarrow x_1 = -\frac{4y}{9x}$$

and, $$18x_1^2 + 18x.x_2 + 8 = 0$$

$$\Rightarrow x_2 = \frac{-8 - 18x_1^2}{18x}$$

At (2, 0)

$$x_1 = 0 \text{ and } x_2 = -\frac{8}{36} = -\frac{2}{9}$$

Now, radius of curvature of (i) at (2, 0) be

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} = \frac{1}{-\frac{2}{9}} = -\frac{9}{2}$$

6. Define curvature of a given curve and find radius of curvature of the curve, $4x^2 - 3xy + y^2 - 3y = 0$ at (0, 0). [2002]

Solution: First part: See definition.

Second part: Given curve is

$$4x^2 - 3xy + y^2 - 3y = 0 \quad ......(i)$$

Equate the lowest order term in the equation, to zero. So, $-3y = 0 \Rightarrow y = 0$.

This shows that the tangent at origin is x–axis.

So, $$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

Now, dividing (i) by 2y and taking $x \to 0$, $y \to 0$ then

$$\lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{4x^2}{2y} - \frac{3x}{2} + \frac{y}{2} - \frac{3}{2}\right) = 0$$

$$\Rightarrow 4\rho - 0 + 0 - \frac{3}{2} = 0 \Rightarrow \rho = \frac{3}{8}.$$

Hence, radius of curvature to (1) be $\rho = \frac{3}{8}$.

7. Define radius of curvature. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where the line $y = x$ cuts it. [2005, Fall]

Solution: See definition and Q. 2(vii) from exercise.

8. What do you mean by curvature and radius of curvature of a given curve $y = f(x)$. Find the radius of curvature of $y^2 = 4x$ at (0, 0). [2006, Fall]

Solution: First Part: See definition of radius of curvature.

Second Part: Here, the equation of parabola is

$$y^2 = 4x$$

Differentiating w.r.t. x, we get $\frac{dy}{dx} = \frac{2}{y}$ at origin which is infinite at (0, 0).

Thus, $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$ is not applicable.

So, we use the formula

$$\rho = \frac{(1 + x_1^2)^{3/2}}{x_2} \quad \text{......(i)}$$

where,

$$x_1 = \frac{dx}{dy} = \frac{2y}{4} = \frac{y}{2} \quad \text{and} \quad x_2 = \frac{d^2x}{dy^2} = \frac{1}{2}.$$

Therefore, radius of curvature

$$\rho = \frac{\left[1 + \left(\frac{y}{2}\right)^2\right]^{3/2}}{\frac{1}{2}} \quad \text{at } (0, 0)$$

$$= \frac{1}{1/2} = 2$$

9. Define curvature and radius of curvature of a function at a given point. Find the radius of curvature at (0, 0) of $4x^2 - 3xy + y^2 - 3y = 0$. [2006, Spring]

Solution: First Part: See definition of radius of curvature.

Second Part: See final exam question Q6.

10. Show that the radius of curvature at the extremity of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the half of the latus rectum. [2007, Fall]

Solution: We have, given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.......(i)}$$

Here, right extremity of major axis is (a, 0).

We know, radius of curvature

$$\rho = \frac{(1 + x_1^2)^{3/2}}{x_2} \quad \text{.....(ii)}$$

Differentiating equation (i) w.r.t. to y, we get

$$\frac{2x}{a^2}\frac{dx}{dy} + \frac{2y}{b^2} = 0 \Rightarrow \frac{dx}{dy} = -\frac{ya^2}{xb^2}$$

$$\Rightarrow x_1 = 0 \quad \text{at } (a, 0)$$

Also, we get

$$\frac{d^2x}{dy^2} = -\frac{a^2}{b^2}\left[\frac{x - y\frac{dx}{dy}}{x^2}\right]$$

$$\Rightarrow x_2 = -\frac{a^2}{b^2}\left(\frac{a - 0}{a^2}\right) = -\frac{a}{b^2}$$

From (ii), we get

$$\rho = \frac{1}{\left(-\frac{a}{b^2}\right)} = \frac{-b^2}{a}.$$

Therefore, required radius of curvature, $\rho = \frac{b^2}{a} = \frac{1}{2}$ of latus rectum.

SHORT QUESTIONS

1. Find radius of curvature of the curve $x = a\cos\theta$, $y = a\sin\theta$. [2008, Spring] [2009 Spring]

Solution: Given that, $x = 0\cos\theta$, $y = a\sin\theta$

Then,

$$\dot{x} = -a\sin\theta, \quad \dot{y} = a\cos\theta$$

And $\quad \ddot{x} = -a\cos\theta, \quad \ddot{y} = -a\sin\theta$

y² = 4x

Now,

y² = 4x

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} = \frac{(a^2\sin^2\theta - a^2\cos^2\theta)^{3/2}}{a^2\sin^2\theta + a^2\cos^2\theta} = \frac{a^3}{a^2} = a.$$

2. Find radius of curvature of $y^2 = 4x$ at (4, 4). [2004, Fall]

Solution: Given curve is

2y =

$$y^2 = 4x$$

y² = 2y

So, $4x_1 = 2y$ and $4x_2 = 2$.

Now, radius of curvature of given curve is

$$\rho = \frac{(1 + x_1^2)^{3/2}}{x_2} = \frac{[1 + (2y/4)^2]^{3/2}}{2/4} = \frac{2}{8}(4 + y^2)^{3/2}$$

At (4, 4)

$$\rho = \frac{2}{8}(4 + 16)^{3/2} = 10\sqrt{5}.$$

e. Find radius of curvature of $y^2 = 4x$ at (0, 0). [2012 Fall]

Solution: See problem part of Q. No. 8, Exam Question Solution – 2006 Fall.

f. Find radius of curvature of $y = x^2 + 4$ at (0, 4). [2015 Fall (Short)][2013 Fall (Short)]

Solution: Given curve is

$$y = x^2 + 4.$$

So, $y_1 = 2x$ and $y_2 = 2$.

At (0, 4), $y_1 = 0$ and $y_2 = 2$.

Now, radius of curvature of given curve at (0, 4) is

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{[1 + 0]^{3/2}}{2} = \frac{1}{2}$$

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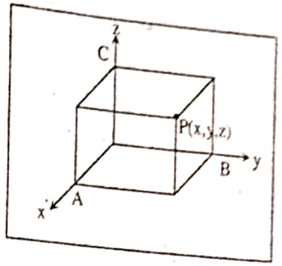
SPACE IN COORDINATE

Cartesian coordinate

Here, OX, OY, OZ are mutually perpendicular coordinate axes. Here a point P(x, y, z) is Cartesian coordinate whose origin is (0, 0, 0).



Cylindrical Coordinate

A point in space which has circular base, is cylindrical coordinate. A point P(r, θ, z) is cylindrical coordinate.

Relation between Cartesian and Cylindrical Coordinate

Let P(x, y, z) be Cartesian coordinate and P(r, θ, z) be cylindrical coordinate, that are related as

$$x = r\cos\theta,\ y = r\sin\theta,\quad z = z.$$

This implies,

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\ z = z.$$

Spherical coordinate

A point in space which has symmetrical form with origin, is spherical coordinate is denoted as (ρ, ϕ, θ).

Spherical Coordinate

Let P(x, y, z) be Cartesian coordinate and P(ρ, ϕ, θ) be spherical coordinate that are relted as

$$x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi.$$

This implies

$$\rho = \sqrt{x^2 + y^2 + z^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\quad \phi = \cos^{-1}\left(\frac{z}{\rho}\right).$$

Exercise 8.1

1. Find cylindrical and spherical coordinates system of a point whose cartesian coordinate is (1, 0, 0).

Solution: Given Cartesian coordinate is

$$(x, y, z) = (1, 0, 0).$$

So, $x = 1, y = 0, z = 0.$

Then,

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 0} = 1$$

At (4, 4)

$$\rho = \frac{2}{8}(4+16)^{3/2} = 10\sqrt{5}.$$

e. **Find radius of curvature of $y^2 = 4x$ at (0, 0).** [2012 Fall]

Solution: See problem part of Q. No. 8, Exam Question Solution – 2006 Fall

f. **Find radius of curvature of $y = x^2 + 4$ at (0, 4).** [2015 Fall (Short)][2013 Fall (Short)]

Solution: Given curve is

$$y = x^2 + 4.$$

So, $y_1 = 2x$ and $y_2 = 2.$

At (0, 4), $y_1 = 0$ and $y_2 = 2.$

Now, radius of curvature of given curve at (0, 4) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+0]^{3/2}}{2} = \frac{1}{2}$$

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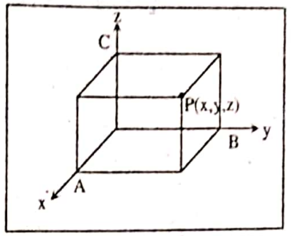
Chapter 8 SPACE IN COORDINATE

Cartesian coordinate

Here, OX, OY, OZ are mutually perpendicular coordinate axes. Here a point P(x, y, z) is Cartesian coordinate whose origin is (0, 0, 0).



Cylindrical Coordinate

A point in space which has circular base, is cylindrical coordinate. A point P(r, θ, z) is cylindrical coordinate.

Relation between Cartesian and Cylindrical Coordinate

Let P(x, y, z) be Cartesian coordinate and P(r, θ, z) be cylindrical coordinate, that are related as

$$x = r\cos\theta,\ y = r\sin\theta,\quad z = z.$$

This implies,

$$r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\ z = z.$$

Spherical coordinate

A point in space which has symmetrical form with origin, is spherical coordinate is denoted as (ρ, ϕ, θ).

Spherical Coordinate

Let P(x, y, z) be Cartesian coordinate and P(ρ, ϕ, θ) be spherical coordinate that are relted as

$$x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi.$$

This implies

$$\rho = \sqrt{x^2+y^2+z^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\quad \phi = \cos^{-1}\left(\frac{z}{\rho}\right).$$

Exercise 8.1

1. **Find cylindrical and spherical coordinates system of a point whose cartesian coordinate is (1, 0, 0).**

Solution: Given Cartesian coordinate is

$$(x, y, z) = (1, 0, 0).$$

So, $x = 1, y = 0, z = 0.$

Then,

$$r = \sqrt{x^2+y^2} = \sqrt{1+0} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$z = 0$$

So, the cylindrical coordinate is

$$(r, \theta, z) = (1, 0, 0).$$

and

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 0 + 0} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = 0$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{0}{1}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

So, the spherical coordinate is

$$(\rho, \phi, \theta) = \left(1, \frac{\pi}{2}, 0\right).$$

2. Find cylindrical and spherical coordinates system of a point whose cartesian coordinate is (0, 0, 1).

Solution: Similar to Q.1.

3. Find cartesian and spherical coordinates system of a point whose cylindrical coordinate is ($\sqrt{2}$, 0, 1).

Solution: Given cylindrical coordinate is

$$(r, \theta, z) = (\sqrt{2}, 0, 1).$$

Then

$$r = \sqrt{2}, \theta = 0, x = 1.$$

Here

$$x = r\cos\theta = \sqrt{2}\cos 0 = \sqrt{2}$$

$$y = r\sin\theta = \sqrt{2}\sin 0 = 0$$

$$z = 1$$

So, Cartesian coordinate is

$$(x, y, z) = (\sqrt{2}, 0, 1)$$

And

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2 + 0 + 1} = \sqrt{3}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{\sqrt{2}}\right) = \tan^{-1}(0) = 0.$$

So spherical coordinate is

$$(\rho, \phi, \theta) = \left(\sqrt{3}, \cos^{-1}\left(\frac{1}{\sqrt{3}}\right), 0\right)$$

4. Find cartesian and spherical coordinates system of a point whose cylindrical coordinate is (1, 0, 0).

Solution: Similar to Q 3.

5. Find cartesian and cylindrical coordinates system of a point whose spherical coordinate is $\left(\sqrt{3}, \frac{\pi}{3}, \frac{-\pi}{2}\right)$.

Solution: Given spherical coordinate is

$$(\rho, \phi, \theta) = \left(\sqrt{3}, \frac{\pi}{3}, \frac{-\pi}{2}\right)$$

So

$$\rho = \sqrt{3}, \phi = \frac{\pi}{3}, \theta = \frac{-\pi}{2}$$

Here,

$$x = \rho\sin\phi\cos\theta = (\sqrt{3})\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{-\pi}{2}\right) = (\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)(0) = 0$$

$$y = \rho\sin\phi\sin\theta = (\sqrt{3})\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{-\pi}{2}\right) = (\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)(-1) = -\frac{3}{2}$$

$$z = \rho\cos\phi = \sqrt{3}\cos\left(\frac{\pi}{3}\right) = (\sqrt{3})\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

So, the cartesian coordinate is

$$(x, y, z) = \left(0, \frac{-3}{2}, \frac{\sqrt{3}}{2}\right)$$

and

$$r = \sqrt{x^2 + y^2} = \sqrt{0 + \frac{9}{4}} = \frac{3}{2}$$

So, the cylindrical coordinate is

$$(r, \theta, z) = \left(\frac{3}{2}, \frac{-\pi}{2}, \frac{\sqrt{3}}{2}\right)$$

6. Find cartesian and cylindrical coordinates system of a point whose spherical coordinate is $\left(\sqrt{2}, \pi, \frac{3\pi}{2}\right)$.

Solution: Similar to 5.

7. Translate the equations from the given coordinate system (cartesian, cylindrical or spherical) into equations in the other two system.

(i) $x^2 + y^2 + z^2 = 4$

Solution: Given Cartesian equation is

$$x^2 + y^2 + z^2 = 4 \quad .....(i)$$

$$\Rightarrow \rho^2 = 4.$$

$$\Rightarrow \rho = 2.$$

Also, we have

$$r^2 = x^2 + y^2 \quad .....(ii)$$

Then (i) gives,

$$r^2 + z^2 = 4 \quad .....(iii)$$

Clearly (ii) in spherical equation of (i) and (iii) is cylindrical equation of (i)

(ii) $z^2 = r^2$

Solution: Given cylindrical equation is,

$$z^2 = r^2 \quad .....(i)$$

$$\Rightarrow z^2 = x^2 + y^2$$

$$\Rightarrow 2z^2 = x^2 + y^2 + z^2 = \rho^2$$

$$\Rightarrow \sqrt{2}\,z = \rho$$

$$\Rightarrow \sqrt{z}\ \rho\cos\phi = \rho$$

$$\Rightarrow \cos\phi = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \phi = \frac{\pi}{4} \quad ......(ii)$$

Also (i) implies

$$z^2 = r^2 = x^2 + y^2$$

$$\Rightarrow x^2 + y^2 - z^2 = 0 \quad .....(iii)$$

Here (ii) is spherical equation of (i) and (iii) is Cartesian equation of (i).

(iii) $x = y$

Solution: Given Cartesian equation is

$$x = y \quad ......(i)$$

$$\Rightarrow \rho\sin\phi\cos\theta = \rho\sin\phi\sin\theta$$

$$\Rightarrow \cos\theta = \sin\theta$$

$$\Rightarrow \tan\theta = 1 = \tan(\pi/4)$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad ......(ii)$$

And $\quad x = y$

$$\Rightarrow r\cos\theta = r\sin\theta$$

$$\Rightarrow \tan\theta = 1 = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad .....(ii)$$

Here (ii) is spherical equation of (i) and (iii) is cylindrical equation of (i).

(iv) $\theta = 0$

Solution: Given equation is

$$\theta = 0 \quad .....(i)$$

Since, θ is coordinate element of cylindrical and spherical coordinate. So, (i) is cylindrical as well as spherical equation.

And,

$$\theta = 0^{\circ} \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{y}{x} = \tan 0 = 0 \Rightarrow y = 0.$$

This is cartesian equation of (i).

(v) $\rho = 1$

Solution: Given spherical equation is

$$\rho = 1 \quad .....(i)$$

$$\Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$$

$$\Rightarrow x^2 + y^2 + z^2 = 1 \quad .....(ii)$$

This is cartesian equation of (i).

Since we know

$$r^2 = x^2 + y^2$$

So, (ii) gives

$$r^2 + z^2 = 1 \quad .....(iii)$$

This is in cylindrical coordinate system.

(vi) $\rho\sin\phi\cos\theta = 2$

Solution: Given spherical equation is

$$\rho\sin\phi\cos\theta = 2 \quad .....(i)$$

$$\Rightarrow y = 2 \quad ....(ii)$$

This is in Cartesian coordinate system

and form (ii)

$$r\cos\theta = 2 \quad ....(iii)$$

This is in cylindrical coordinate system.

(vii) $\rho = 2\cos\phi$

Solution: Given equation is spherical coordinate system is

$$\rho = 2\cos\phi \quad ......(i)$$

$$\Rightarrow \rho = 2\left(\frac{z}{\rho}\right)$$

$$\Rightarrow \rho^2 = 2z$$

$$\therefore x^2 + y^2 + z^2 = 2z \quad ....(ii)$$

This is in Cartesian coordinate system.

(viii) $x^2 + 6xy + y^2 + z^2 = 1$

Solution: Given equation in cartesian coordinate system is

$$x^2 + 6xy + y^2 + z^2 = 1 \quad .....(i)$$

$$\Rightarrow (x^2 + y^2 + z^2) + 6xy = 1$$

$\Rightarrow \rho^2 + 6\rho \sin\phi \cos\theta \, \rho \sin\phi \sin\theta = 1$

$\Rightarrow \rho^2(1 + 6\sin^2\phi \cos\theta \sin\theta) = 1$

$\Rightarrow \delta^2(1 + 3\sin^2\phi \sin 2\theta) = 1$ ......(ii)

This is in spherical coordinate system. And

$x^2 + 6xy + y^2 + z^2 = 1$

$\Rightarrow (x^2 + y^2) + 6xy + z^2 = 1$

$\Rightarrow r^2 + 6r^2 \cos\theta \sin\theta + z^2 = 1$

$\Rightarrow r^2(1 + 3\sin 2\theta) + z^2 = 1$

This is in cylindrical coordinate system.

8. Transform the equations (by using spherical polar coordinates)

(i) $x^2 + y^2 + z^2 = 2z$

Solution: Given equation is

$x^2 + y^2 + z^2 = 2z$

$\Rightarrow \rho^2 + 2\rho \cos\phi$

$\Rightarrow \rho = 2\cos\phi$

This is spherical polar coordinate system.

(ii) $x^2 + y^2 - 3z^2 = 0$

Solution: Given equation is

$x^2 + y^2 - 3z^2 = 0$

$\Rightarrow x^2 + y^2 + z^2 - 4z^2 = 0$

$\Rightarrow \rho^2 - 4\rho^2\cos^2\phi = 0$

$\Rightarrow \rho^2(1 - 4\cos^2\phi) = 0$

This is in spherical polar coordinate system.

9. Change to spherical polar and cylindrical polar coordinates:

(i) $x^2 + y^2 = 5$ (ii) $x^2 - z^2 = 4$.

Solution: Similar to Q.7 (i)

10. Transform equation $x^2 + y^2 = x$ to cylindrical coordinates.

Solution: Given equation is

$x^2 + y^2 = x \Rightarrow r^2 = r\cos\theta$

$\Rightarrow r = \cos\theta$

This is in cylindrical coordinate system.

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Chapter 9 TWO DIMENSIONAL GEOMETRY

List of formulae for transform

1. If the axes are shifted parallel to the axes then
$x = x_1 + h, \quad y = y_1 + k$
where (x, y) be coordinate of given axes, (x_1, y_1) be coordinate of new axes and (h, k) be the origin of new (shifted) axes.

2. If the axes are translated with an angle θ remaining same origin (θ, 0) then
$x = x_1\cos\theta - y_1\sin\theta, \quad y = x_1\sin\theta + y_1\cos\theta$
where (x, y) be coordinate of given axes, (x_1, y_1) be coordinate of new axes and (h, k) be the origin of new (shifted) axes.

Exercise 9.1

1. Transform the equation $x^2 - y^2 + 2x + 4y = 0$ by transferring the origin to (–1, 2) the coordinate axes remaining parallel.

Solution: Given equation is

$x^2 - y^2 + 2x + 4y = 0$ ... (i)

Here, origin transformed to point (–1, 2). Then, x is replaced by x – 1 and y is replaced by y + 2 in equation (i) then,

$(x - 1)^2 - (y + 2)^2 + 2(x - 1) + 4(y + 2) = 0$

$\Rightarrow x^2 - 2x + 1 - y^2 - 4y - 4 + 2x - 2 + 4y + 8 = 0$

$\Rightarrow x^2 - y^2 + 3 = 0$

This is the required equation.

2. Transform the equation $x^2 - 3y^2 + 4x + 6y = 0$ by transferring the origin to the point (–2, 1) the co–ordinate axes remaining parallel.

Solution: Given equation is

$x^2 - 3y^2 + 4x + 6y = 0$ ... (i)

Here, origin transformed to point (–2, 1). Then, x is replaced by x – 2 and y is replaced by y + 1 in equation (i) then,

$(x - 2)^2 - 3(y + 1)^2 + 4(x - 2) + 6(y + 1) = 0$

$\Rightarrow x^2 - 4x + 4 - 3y^2 - 6y - 3 + 4x - 8 + 6y + 6 = 0$

$\Rightarrow x^2 - 3y^2 - 1 = 0$

This is the required equation.

3. Translate the axes so as to change the equation $3x^2 - 2xy + 4y^2 + 8x - 10y + 8 = 0$ into an equation with linear terms missing.

Solution: Given equation is

$3x^2 - 2xy + 4y^2 + 8x - 10y + 8 = 0$ ... (i)

Transform origin to (h, k) when axes remaining parallel, then x becomes (x + h) and y becomes (y + k) in equation (i) then,

$3(x+h)^2 - 2(x+h)(y+k) + 4(y+k)^2 + 8(x+h) - 10(y+k) + 8 = 0$

$\Rightarrow 3(x^2 + 2xh + h^2) - 2xy - 2hy - 2kx - 2hk + 4y^2 + 8yk + 4k^2 + 8x + 8h$
$- 10y - 10k + 8 = 0$

$\Rightarrow 3x^2 - 2xy + 4y^2 + x(6h - 2k + 8) + y(-2h + 8k - 10) + 3h^2 - 2hk + 4k^2$
$+ 8h - 10k + 8 = 0$ ... (ii)

Since, in this equation linear term is missing. So, coefficient of x = 0 and coefficient of y is 0.

i.e. $6h - 2k + 8 = 0$ and $-2h + 8k - 10 = 0$

Solving we get, $k = 1$ and $h = -1$.

Substituting these values in equation (ii) then,

$$3x^2 - 2xy + 4y^2 + 3 + 2 + 4 - 8 - 10 + 8 = 0$$

$$\Rightarrow 3x^2 - 2xy + 4y^2 - 1 = 0.$$

This is the required equation.

4. Transform the equation $x^2 + 2cxy + y^2 = a^2$, by turning the rectangular axes through the angle $\frac{\pi}{4}$.

Solution: Given, $\theta = \frac{\pi}{4}$ and given equation is

$$x^2 + 2cxy + y^2 = a^2 \quad \text{... (i)}$$

By hypothesis, the axes are turned with an angle $\frac{\pi}{4}$. So, replace, x by $x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} = \frac{x-y}{\sqrt{2}}$ and y by $x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} = \frac{x+y}{\sqrt{2}}$ in equation (i) then,

$$\left(\frac{x-y}{\sqrt{2}}\right)^2 + 2c\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + \left(\frac{x+y}{\sqrt{2}}\right)^2 = a^2$$

$$\Rightarrow \frac{(x-y)^2}{2} + c(x^2 - y^2) + \frac{(x+y)^2}{2} = a^2$$

$$\Rightarrow (x^2 - 2xy + y^2) + 2cx^2 - 2cy^2 + (x^2 + 2xy + y^2) = 2a^2$$

$$\Rightarrow x^2(1+c) + y^2(1-c) = a^2$$

This is the required equation.

5. What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point (–2, –3), the new axes being inclined at an angle $\frac{\pi}{4}$, with the old?

Solution: When the axes turn by an angle θ without change in origin, then we replace,

$$x = x\cos\theta - y\sin\theta \quad \text{and} \quad y = x\sin\theta + y\cos\theta.$$

Again when axes are transferred from origin to point (h, k) we replace,

$$x = x + h \quad \text{and} \quad y = y + k.$$

Here, after changing origin and turning the axes,

$$x = (x+h)\cos\theta - (y+k)\sin\theta$$
$$y = (x+h)\sin\theta + (y+k)\cos\theta$$

Given equation is

$$2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0 \quad \text{... (i)}$$

This is transforms the axes parallel with (h, k) = (–2, –3) and turned with an angle $\theta = \frac{\pi}{4}$. Therefore the equation (i) transforms as,

$$x = (x-2)\cos\frac{\pi}{4} - (y-3)\sin\frac{\pi}{4} = \frac{x-2-y+3}{\sqrt{2}} = \frac{x-y+1}{\sqrt{2}}$$

$$y = (x-2)\sin\frac{\pi}{4} + (y-3)\cos\frac{\pi}{4} = \frac{x-2+y-3}{\sqrt{2}} = \frac{x+y-1}{\sqrt{2}}$$

Then equation (ii) becomes,

$$2\left(\frac{x-y+1}{\sqrt{2}}\right)^2 + 4\left(\frac{x-y+1}{\sqrt{2}}\right)\left(\frac{x+y-5}{\sqrt{2}}\right) - 5\left(\frac{x+y-5}{\sqrt{2}}\right)^2 +$$
$$20\left(\frac{x-y+1}{\sqrt{2}}\right) - 22\left(\frac{x+y-5}{\sqrt{2}}\right) - 14 = 0.$$

$\Rightarrow (x-y+1)^2 + 2(x-y+1)(x+y-5) - \frac{5}{2}(x+y-5)^2 + 10\sqrt{2}$
$(x-y+1) - 11\sqrt{2}(x+y-5) - 14 = 0.$

$\Rightarrow 2(x-y+1)^2 + 4(x-y+1)(x+y-5) - 5(x+y-5)^2 +$
$20\sqrt{2}(x-y+1) - 22\sqrt{2}(x+y-5) - 28 = 0.$

$\Rightarrow 2(x^2 + y^2 + 1 - 2xy + 2x - 2y) + 4(x^2 + xy - 5x - xy - y^2 + 5y + x +$
$y - 5) - 5(x^2 + y^2 + 25 + xy - 5y - 5x) + 20\sqrt{2}x - 20\sqrt{2}y + 20\sqrt{2} -$
$22\sqrt{2}x - 22\sqrt{2}y + 110\sqrt{2} - 28 = 0.$

$\Rightarrow x^2 - 7y^2 - 9xy + (3 - 2\sqrt{2})x + (45 - 42\sqrt{2})y - 171 + 130\sqrt{2} = 0.$

This is required equation.

6. By transforming to parallel axes through a properly chosen point (h, k) prove that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ can be reduced to one containing only terms of second degree.

Solution: Given equation is

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad \text{... (i)}$$

Let $x = x + h$ and $y = y + k$.

Then equation (i) then,

$$12(x+h)^2 - 10(x+h)(y+k) + 2(y+k)^2 + 11(x+h) - 5(y+k) + 2 = 0.$$

$\Rightarrow 12[x^2 + 2xh + h^2] - 10(xy + xk + hy + hk) + 2(y^2 + 2yk + k^2) +$
$(11x + 11h) - (5y + 5k) + 2 = 0.$

$\Rightarrow 12x^2 + 24xh + 12h^2 - 10xy - 10xk - 10hy - 10hk + 2y^2 + 4yk + 2k^2$
$+ 11x + 11h - 5y - 5k + 2 = 0.$

$\Rightarrow 12x^2 + 2y^2 - 10xy + x(24h - 10k + 11) - y(10h + 4k - 5) +$
$(12h^2 - 10hk + 2k^2 + 11h - 5k + 2) = 0.$

We have, in this equation linear term and constant term is absent. If ... only the term of second degree then,

Coefficient of $x = 0$, constant term $= 0$

$$\Rightarrow 24h - 10k + 11 = 0$$

Coefficient of $y = 0$

$$\Rightarrow 10h + 4k - 5 = 0$$

Solving these two equations we get

$$h = -\frac{3}{2}, \text{ and } \qquad k = -\frac{5}{2}.$$

Hence by transforming the axes to origin (h, k) i.e. $\left(-\frac{3}{2}, -\frac{5}{2}\right)$, the equation becomes,

$$12\left(x - \frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)\left(y - \frac{5}{2}\right) + 2\left(y - \frac{5}{2}\right) + 11\left(x - \frac{3}{2}\right) - 5\left(y - \frac{5}{2}\right) + 2 = \ldots$$

$$\Rightarrow 12\left(x^2 - 3x + \frac{9}{4}\right) - 10\left(xy - \frac{15x}{2} - \frac{3y}{2} + \frac{15}{4}\right) + 2\left(y^2 - 5y + \frac{25}{4}\right) + \ldots$$

$$-\frac{33}{2} - 5y + \frac{25}{2} + 2 = 0$$

$$\Rightarrow 12x^2 - 10xy + 2y^2 - 36x + 25x + 11x + 15y - 10y - 5y + \frac{54}{2} \ldots$$

$$+ \frac{50}{4} - \frac{33}{2} + \frac{25}{2} + 2 = 0$$

$$\Rightarrow 12x^2 - 10xy + 2y^2 = 0.$$

This is the required equation.

7. What does the equation of the straight lines $7x^2 + 4xy + 4y^2 = 0$ become when the axes are turned through 45°, the origin remaining fixed?

Solution: Here, given equation is

$$7x^2 + 4xy + 4y^2 = 0 \qquad \ldots (i)$$

When the axes are turned through $\theta = 45°$ then, x and y becomes as,

$$x = x\cos\theta - y\sin\theta = \frac{x - y}{\sqrt{2}} \quad \text{and} \quad y = x\sin\theta + y\cos\theta = \frac{x + y}{\sqrt{2}}$$

Putting values of x and y in equation (i) then

$$7\left(\frac{x - y}{\sqrt{2}}\right)^2 + 4\left(\frac{x - y}{\sqrt{2}}\right)\left(\frac{x + y}{\sqrt{2}}\right) + 4\left(\frac{x + y}{\sqrt{2}}\right)^2 = 0$$

$$\Rightarrow \frac{7}{2}(x - y)^2 + 2(x^2 - y^2) + 2(x + y)^2 = 0$$

$$\Rightarrow 7x^2 - 14xy + 7y^2 + 4x^2 - 4y^2 + 4x^2 + 8xy + 4y^2 = 0$$

$$\Rightarrow 15x^2 - 6xy + 7y^2 = 0.$$

This is the required equation.

Q. Transform to parallel axes through the point (3, –4) the equation $x^2 - y^2 + 2x - 3y = 0$.

Solution: Similar to Q. 1.

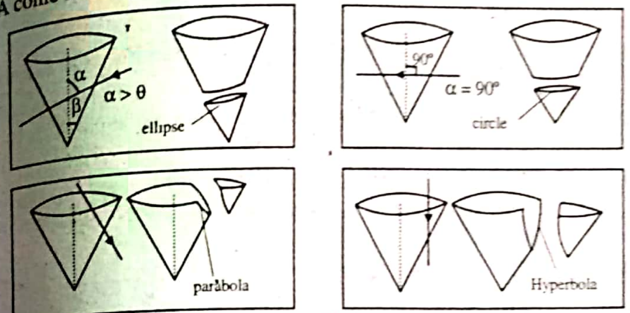
PARABOLA

Definition of conic section

A curve that obtained by the intersection of a cone and a plane, is known as the conic section.

Classification of conic section

A conic section can be classified as:



(i) If the plane cut a cone perpendicularly to the axis of the cone then it gives a circle.

(ii) If the plane cut a cone at an angle greater than the semi–vertical angle then it gives an ellipse.

(iii) If the plane cut a cone with the plane is parallel to the generator and if it does not pass through the vertex then it gives a parabola.

(iv) If the plane cut a cone at an angle less than the semi–vertical angle then it gives an hyperbola.

Eccentricity of Conic Section

The constant ratio between the distance from a fixed point and distance from a fixed straight line of a conic section is called eccentricity of the conic section. It is denoted by e.

Classification of Conic Section as their Eccentricity

As, the value of eccentricity, the conic section is classified as,

(i) If $e = 1$, then the conic section is parabola.

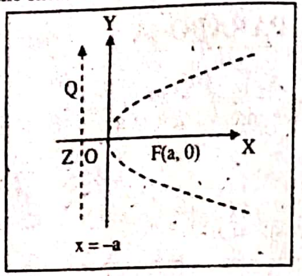
(ii) If $e < 1$, then the conic section is called ellipse.

(iii) If $e > 1$ then the conic section is called hyperbola.

(iv) If $e = 0$ then the conic is called circle.

Definition of parabola

A parabola is a locus of points in a plane which is equi–distance from a fixed point and fixed line. The fixed point is called focus point of the parabola and the fixed line is to line of directrix.

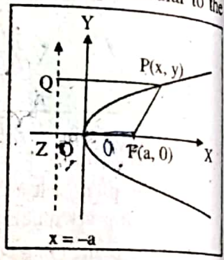


Standard Equation of Parabola with vertex at (0, 0) and Focus at F(a, 0).

Let the parabola has vertex at V(0, 0) and focus at F(a, 0). Here y–value in vertex and focus is fixed, so the equation of directrix of the parabola is

$$x = -a.$$

Let P(x, y) be any point on the parabola and let PQ is perpendicular to the line of directrix. Then by definition of parabola,



$$PF = PQ$$

$$\Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = \left|\pm\frac{x+a}{\sqrt{(1)^2}}\right|$$

$$\Rightarrow (x-a)^2 + y^2 = (x+a)^2$$

$$\Rightarrow y^2 = (x+a)^2 - (x-a)^2$$

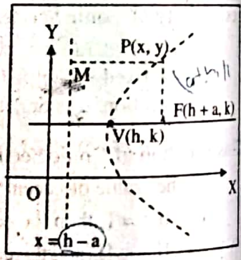
$$\Rightarrow y^2 = 4ax$$

Thus, $y^2 = 4ax$ is the equation of the parabola having focus (a, 0) and directrix $x = -a$.

Equation of the parabola having vertex (h, k) and focus (h + a, k)

(i.e. axis being parallel to x–axis)

Let the parabola has vertex at V(h, k) and focus at F(h+a, k). Here y–value in vertex and focus is fixed, so the equation of directrix of the parabola is



$$x = h - a.$$

Let P(x, y) be any point on the parabola and let PQ is perpendicular to the line of directrix. Then by definition of parabola,

$$PF = PQ$$

i.e $\sqrt{(x-(h+a))^2 + (y-k)^2} = \left|\pm\frac{x-(h-a)}{\sqrt{(1)^2}}\right|$

$$(x-(h+a))^2 + (y-k)^2 = (x-(h-a))^2$$

$$\Rightarrow (y-k)^2 = (x-(h-a))^2 - (x-(h+a))^2$$

$$\Rightarrow (y-k)^2 = (x-h+a)^2 - (x-h-a)^2$$

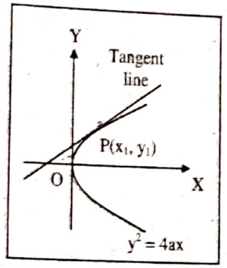
$$\Rightarrow (y-k)^2 = 4a(x-h)$$

This is the equation of parabola having vertex (h, k) and focus (h + a, k) and directrix $x = h - a$.

Equation of Tangent at (x_1, y_1) to the Parabola $y^2 = 4ax$.

The equation of parabola in standard form is

$$y^2 = 4ax \quad \ldots (i)$$



Differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2a}{y} \quad \ldots (ii)$$

At (x_1, y_1),

$$y_1^2 = 4ax_1 \quad \ldots (iii)$$

$$\frac{dy_1}{dx_1} = \frac{2a}{y_1}$$

Since the equation of a tangent line to (i), is passing through the point (x_1, y_1) is,

$$y - y_1 = m(x - x_1) \quad \ldots (iv)$$

where m be the slope of the line.

Since, the point (x_1, y_1) be the common point to the line (iv) and the given parabola (i). So,

$$m = \frac{dy_1}{dx_1} = \frac{2a}{y_1}.$$

Then (iv) becomes,

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow yy_1 = 2ax + y_1^2 - 2ax_1$$

$$\Rightarrow yy_1 = 2ax + 2ax_1 \quad \text{[Using (iii)]}$$

$$\Rightarrow yy_1 = 2a(x + x_1)$$

Thus the equation of tangent at (x_1, y_1) on the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

Note: Equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1).$$

Condition for Tangency that a line y = mx + c touches the given parabola $y^2 = 4ax$.

The given equation of line and parabola are

$$y = mx + c \quad \ldots (i)$$

$$y^2 = 4ax \quad \ldots (ii)$$

$$(mx + c)^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx + c^2 = 4ax$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0 \quad \dots \text{(iii)}$$

which is quadratic in x.

Since the line (i) is tangent to the curve (ii) then the discriminate value of (iii) is zero. Then,

$$(2mc - 4a)^2 - 4.m^2c^2 = 0 \qquad [B^2 - 4AC = 0]$$

$$\Rightarrow 4(mc - 2a)^2 . 4m^2c^2 = 0$$

$$\Rightarrow (mc - 2a)^2\, m^2c^2 = 0$$

$$\Rightarrow m^2c^2 - 4mca + 4a^2 - m^2c^2 = 0$$

$$\Rightarrow -4mca + 4a^2 = 0$$

$$\Rightarrow 4a^2 = 4mca$$

$$\Rightarrow c = \frac{a}{m}$$

This is the condition that the line (i) is tangent to the curve (ii).

For point of contact,

Since line (i) is tangent to the parabola (ii), so the discriminate value of (iii) is zero i.e. $B^2 - 4AC = 0$

Therefore, (iii) gives

$$\boxed{x = \frac{-B}{2A}}$$

$$\text{i.e. } x = \frac{-(2mc - 4a)}{2m^2} = \frac{2a - mc}{m^2} = \frac{2a - a}{m^2} = \frac{a}{m^2} \quad \left(\because c = \frac{a}{m} \Rightarrow mc = a\right)$$

Then (i) gives

$$y = mx + c = m\left(\frac{a}{m^2}\right) + \frac{a}{m} = \frac{2a}{m}$$

So, the print of contact of (i) and (ii) is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Condition for Tangency that a line $lx + my + n = 0$ touches the given parabola $y^2 = 4ax$.

Q. Prove that the line $lx + my + n = 0$ touches $y^2 = 4ax$ if $ln = am^2$.

We have, given equation of line and parabola is

$$lx + my + n = 0 \quad \dots \text{(i)}$$

$$y^2 = 4ax \quad \dots \text{(ii)}$$

From (i) and (ii), we get

$$\left(-\frac{n + lx}{m}\right)^2 = 4ax$$

$$\Rightarrow n^2 + 2nlx + l^2x^2 = 4am^2x$$

$$\Rightarrow l^2x^2 + (2nl - 4am^2)x + n^2 = 0 \quad \dots \text{(iii)}$$

which is quadratic in x.

Since the line (i) is tangent to (ii) then the discriminate of (iii) is zero. That is,

$$(2nl - 4am^2)^2 - 4l^2n^2 = 0$$

$$\Rightarrow 4l^2n^2 - 16nlam^2 + 16a^2m^4 - 4l^2n^2 = 0$$

$$\Rightarrow 16a^2m^4 = 16nlam^2$$

$$\Rightarrow am^2 = ln$$

This shows that the line $lx + my + n = 0$ touches $y^2 = 4ax$ when $ln = am^2$.

For point of contact,

Since line (i) is tangent to the parabola (ii), so the discriminate value of (iii) is zero i.e. $B^2 - 4AC = 0$

Therefore, (iii) gives

$$\boxed{x = \frac{-B}{2A}}$$

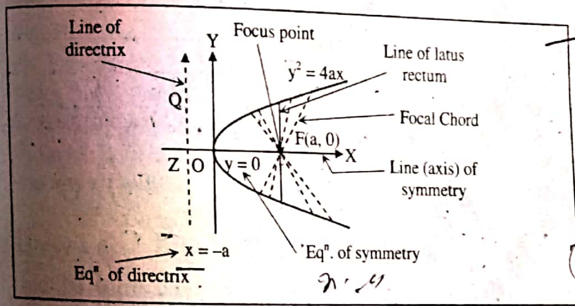
$$\text{i.e. } x = \frac{-(2nl - 4am^2)}{2l^2} = \frac{-(2nl - 4ln)}{2l^2} = \frac{n}{l}.$$

Then (i) gives

$$l\left(\frac{n}{l}\right) + my + n = 0 \Rightarrow y = \frac{-2n}{m}$$

Therefore the point of contact of (i) and (ii) is $\left(\frac{n}{l}, \frac{-2n}{m}\right)$.

Parabola: $y^2 = 4ax$.

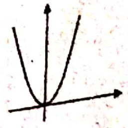


Types of Parabola

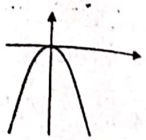
A. $y^2 = 4ax$ (a > 0)

B. $y^2 = 4ax$ (a < 0)

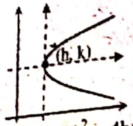
C. $x^2 = 4by$ $(b > 0)$



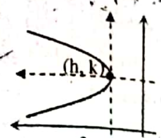
D. $x^2 = 4by$ $(b < 0)$



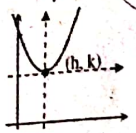
E. $(y-k)^2 = 4a(x-h)$ $(a > 0)$



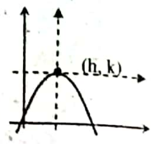
F. $(y-k)^2 = 4a(x-h)$ $(a < 0)$



G. $(x-h)^2 = 4b(y-k)$ $(b > 0)$



H. $(x-h)^2 = 4b(y-k)$ $(b < 0)$



(b) **When the axis is parallel to x–axis then latus rectum is 4a and other results are**

Eq^n. of parabola	$y^2 = 4ax$	$x^2 = 4by$	$(y-k)^2 = 4a(x-h)$	$(x-h)^2 = 4b(y-k)$
Vertex	(0, 0)	(0, 0)	(h, k)	(h, k)
Focus	(a, 0)	(0, b)	(h+a, k)	(h, k+b)
Length of latus rectum	$\lvert 4a\rvert$	$\lvert 4b\rvert$	$\lvert 4a\rvert$	$\lvert 4b\rvert$
Eq^n. of directrix	$x = -a$	$y = -b$	$x = h - a$	$y = k - b$
line of symmetry	$y = 0$	$x = 0$	$y = k$	$x = h$

Note: The point $(at^2, 2at)$ clearly satisfies that equation of the parabola $y^2 = 4ax$. This point is called the point 't'. So the points t_1 and t_2 on the parabola mean the point $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. Here t is called parameter.

Exercise 9.2

1. **Sketch the parabola, showing the focus, vertex and directrix:**

(i) **V(0, 0), F(0, 2)**

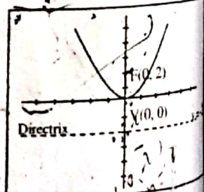
Solution: Here, the x ordinate has same value and y varies. So, the focus lies on the line parallel to y-axis.

Here, V(h, k) = V(0, 0).

And F(h, k + a) = F(0, 2).

Then, a = 2.

Since, y ordinate varies and a has positive value, so the parabola has up openward.



Also, equation of directrix be, $y = -a \Rightarrow y = -2$.

With the help of these information, the sketch of the parabola is as:

(ii) **V(0, 0), F(–2, 0)**

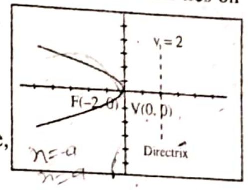
Solution: Here, the y ordinate has same value and x varies. So, the focus lies on the line parallel to x–axis.

Here, V(h, k) = V(0, 0).

And F(h + a, k) = F(– 2, 0).

Then, a = – 2.

Since, x ordinate varies and a has negative value, so the parabola has left openward.



Also, equation of directrix be, $x = -a \Rightarrow x = -(-2) \Rightarrow x = 2$.

With the help of these information, the sketch of the parabola is as:

(iii) **V(0, 0), F(0, –4)** (iv) **V(–2, 3), F(–2, 4)** (vii) **V(1, –3), F(1, 0)**

Process as (i)

(v) **V(0, 3), F(–1, 3)** (vi) **V(–3, 2), F(0, 1)**

Process as (ii)

2. **Sketch the following parabola with showing the focus, vertex and directrix where V is vertex and L is directrix. Also, find the equation of the parabola.**

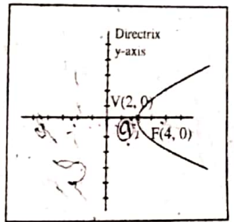
(i) **V(2, 0), L is the y–axis.**

Solution: Here, vertex V(h, k) = V(2, 0).

and directrix be y–axis (i.e. x = 0).

Since we know 'a' is the distance between V and directrix.

So, a = 2.



Also, the directrix be y–axis, so the vertex and focus lie on the line that is parallel to x–axis.

Therefore, F(h + a, k) = F(2 + 2, 0) = F(4, 0).

Now, the equation of parabola is

$(y-k)^2 = 4a(x-h) \Rightarrow y^2 = 8(x-2)$.

With the help of these information, the sketch of the parabola is as:

(ii) **V(1, –2), L is the x–axis**

(iii) **V(–3, 1), L is the line x = 1.**

(iv) **V(–2, –2), L is the line y = – 3,**

(v) **V(0, 1), L is the line x = –1.**

(vi) **V(0, 1), L is the line y = 2.**

Process to solve as (i).

3. **Find the locus, vertex and directrix of the parabola and sketch.**

(i) $y^2 = 8x$

Solution: We have, $y^2 = 8x$.

Comparing this equation with $(y-k)^2 = 4a(x-h)$ we get

$h = 0, k = 0$ and $a = 2$.

Now, vertex be, $V(h, k) = V(0, 0)$.

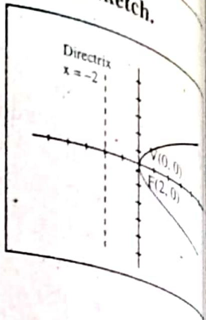
Focus be, $F(h+a, k) = F(2, 0)$.

And the directrix is,

$x - h = -a$

$\Rightarrow x = -2 \Rightarrow x + 2 = 0$.

With the help of these information, the sketch of the parabola is as:



(ii) $x^2 = 100y$ (iii) $y^2 + 36x = 0$

Process as in (i).

4. **Find the vertex, axis of symmetry, focus and directrix of the given parabola and sketch.**

(i) $x^2 + 8y - 2x = 7$

Solution: Given equation is

$x^2 + 8y - 2x = 7$

$\Rightarrow x^2 - 2.1.x + (1)^2 = -8y + 7 + 1$

$\Rightarrow (x-1)^2 = -8(y-1)$.

Comparing the equation with $(x-h)^2 = 4a(y-k)$ we get,

$h = 1, k = 1$ and $a = -2$.

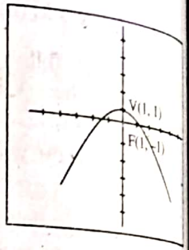
Now, vertex $V(h, k) = V(1, 1)$

axis of symmetry is, $(x-1)^2 = 0 \Rightarrow x = 1$.

Equation of directrix is, $y - 1 = -a \Rightarrow y - 1 = 2 \Rightarrow y = 3$.

and focus $= F(h, k+a) = F(1, -1)$.

With the help of these information, the sketch of the parabola is as:



(ii) – (x) Process as in (i).

5. **Find the length of latus rectum of the curve $y^2 = 4px$.**

Solution: Here, $y^2 = 4px$

Comparing the equation with $y^2 = 4ax$ then we get,

$4a = 4p \Rightarrow a = p$.

We know, the length of latus rectum $= 4a = 4p$.

6. **A double ordinate of the parabola $y^2 = 2ax$ is of length $4a$, prove that the lines joining the vertex to its ends are at right angles.** [2008, Spring]

Solution: Given parabola is $y^2 = 2ax$.

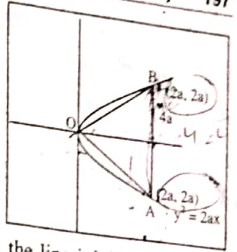
And length of double ordinate is $4a$.

So, $2y = 4a \Rightarrow y = 2a$.

For ends of double ordinate, put $y = 2a$ in $y^2 = 2ax \Rightarrow (2a)^2 = 2ax$.

$\Rightarrow x = 2a$.

Thus, the co-ordinate of end points of double ordinate are $A(2a, -2a)$ and $B(2a, 2a)$.



Since the vertex of given parabola is (0, 0). So, the line joining vertex and ends of ordinate are OA and OB. Here,

$$\text{Slope of OA } (m_1 \text{ say}) = \frac{-2a-0}{2a-0} = -1$$

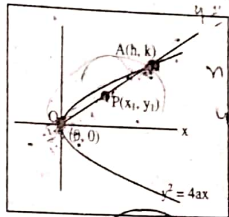
$$\text{Slope of OB } (m_2 \text{ say}) = \frac{2a-0}{2a-0} = 1$$

Now, $m_1.m_2 = -1$. This proves that OA and OB are perpendicular to each other.

7. **Find the locus of mid-points of the chord on $y^2 = 4ax$ through vertex. Prove that it is parabola. Find its latus rectum.**

Solution: Let $P(x_1, y_1)$ be the mid-point of the chord on $y^2 = 4ax$ passing through vertex (0, 0).

Let, A is the point where the line OP cuts the parabola, say the coordinate of A is (h, k). Then the slope of OA is,



$$\text{Slope of OA} = \frac{k-0}{h-0} = \frac{k}{h}$$

Here slope of the chord joining (0, 0) and $P(x_1, y_1)$ is,

$$\text{Slope of OP} = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1} \quad \ldots (i)$$

Since the point A lies on the parabola $y^2 = 4ax$. Therefore, it satisfies the equation of parabola $y^2 = 4ax$ then

$k^2 = 4ah$.

Thus,

$$\text{Slope of OA} = \frac{k}{h} = \frac{k}{k^2/4a} = \frac{4a}{k} = \frac{4a}{2y_1} \quad \text{[Being P be mid-point of OA]}$$

Since the line OP and OA are segments of a line. So, their slope should be equal. That is,

Slope of OP = Slope of OA

$$\Rightarrow \frac{y_1}{x_1} = \frac{4a}{2y_1}$$

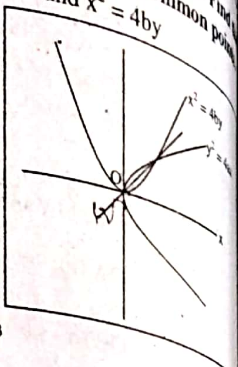
$$\Rightarrow y_1^2 = 2ax_1.$$

Hence, the locus of x_1 and y_1 is $y^2 = 2ax$.

And, the length of latus rectum of $y^2 = 2ax$ is $2a$.

8. Find the points common to the parabolas $y^2 = 4ax$ and $x^2 = 4by$. Find the equation of the common chord that passes through the common points.

Solution: For point of intersection between $y^2 = 4ax$ and $x^2 = 4by$ eliminating y we get,



$$\left[\frac{x^2}{4b}\right]^2 = 4ax$$

$$\Rightarrow x^4 = 64ab^2x$$

$$\Rightarrow x^4 - 64ab^2x = 0$$

$$\Rightarrow x(x^3 - 64ab^2) = 0$$

Either, $x = 0$ or $x^3 - 64ab^2 = 0$

$$\Rightarrow x = 4a^{1/3}b^{2/3}$$

Then, $y = \frac{x^2}{4b} = \frac{(4a^{1/3}b^{2/3})^2}{4b} = 4b^{1/3}a^{2/3}$

Hence the common point to the both curves are (0, 0) $(4a^{1/3}b^{2/3}, 4b^{1/3}a^{2/3})$.

Now, equation of chord is,

$$y - 0 = \frac{4b^{1/3}a^{2/3} - 0}{4a^{1/3}b^{2/3} - 0}(x - 0)$$

$$\Rightarrow y = a^{1/3}b^{-1/3}x$$

$$\Rightarrow b^{1/3}y = a^{1/3}x$$

This is the required equation of chord.

Thus, the equation of the common chord that passes through the common points is $b^{1/3}y = a^{1/3}x$.

9. Obtain the co-ordinate of the focus and the equation of the directrix of the parabola $x^2 - 8x + 2y - 10 = 0$.

Solution: Given parabola is,

$$x^2 - 8x + 2y - 10 = 0$$

$$\Rightarrow x^2 - 8x + 16 = -2y + 10 + 16$$

$$\Rightarrow (x - 4)^2 = -2(y - 13)$$

Comparing the equation with $(x - h)^2 = -4a(y - k)$ we get

$h = 4$, $k = 13$ and $a = -½$

Now, vertex of the parabola is $V(h, k) = V(4, 13)$

focus is $F(h, k + a) = F(4, 13 - ½) = F(4, 25/2)$.

and equation of directrix is,

$$y - k = -a$$

$$\Rightarrow y - 13 = ½ \Rightarrow 2y - 27 = 0.$$

Thus, the co-ordinate of the focus is $F(4, 25/2)$ and the equation of directrix of the parabola $x^2 - 8x + 2y - 10 = 0$ is $2y - 27 = 0$.

10. Find the vertex, focus, latus rectum, axis and directrix of the parabola $x^2 - y - 2x = 0$.

Solution: Given equation is,

$$x^2 - y - 2x = 0 \quad \Rightarrow x^2 - 2x + 1 = y + 1$$

$$\Rightarrow (x - 1)^2 = (y + 1)$$

Comparing the equation with $(x - h)^2 = 4a(y - k)$, we get

$h = 1, k = -1$ and $a = ¼$

Now, vertex is $V(h, k) = V(1, -1)$

focus, $F(h, k + a) = F(1, -1 + ¼) = F(1, -3/4)$

length of latus rectum $= 4a = 1$

axis of symmetry be, $x = 1$

And equation of directrix is,

$$y - k = -a$$

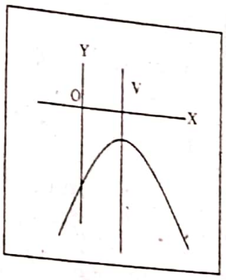
$$\Rightarrow y = -1 - ¼$$

$$\Rightarrow y = -5/4$$

$$\Rightarrow 4y + 5 = 0.$$

11. Discuss the equation $2x^2 + 5y - 3x + 4 = 0$ and sketch the curve.

Solution: Given equation is,



$$2x^2 + 5y - 3x + 4 = 0$$

$$\Rightarrow x^2 + \frac{5y}{2} - \frac{3}{2}x + 2 = 0$$

$$\Rightarrow x^2 - \frac{3}{2}x + \frac{9}{16} = -5/2\,y - 2 + \frac{9}{16}$$

$$\Rightarrow \left(x - \frac{3}{4}\right)^2 = -\frac{5}{2}y - \frac{23}{16}$$

$$\Rightarrow \left(x - \frac{3}{4}\right)^2 = -\frac{5}{2}\left[y + \frac{23}{40}\right] \quad \ldots (i)$$

Comparing equation (i) with $(x - h)^2 = -4a(y - k)$, we get

$h = ¾$, $k = -23/40$ and $a = -5/8$

Now, vertex $V(h, k) = V(3/4, -23/40)$.

axis of symmetry be, $(x - ¾)^2 = 0 \Rightarrow x = ¾$

equation of directrix is, $y = k - a$

$$\Rightarrow y = -\frac{23}{40} - \frac{5}{8} = \frac{-23 - 25}{40} = \frac{-6}{5}$$

Focus is $(h, k - a) = [3/4, -6/5]$

Then, latus rectum $= 4a = 4 \times \frac{5}{8} = \frac{5}{2}$.

With these information, the sketch of the parabola is as:

12. Find the vertex, axis, focus and latus rectum of the parabola $4y^2 + 12x - 20y + 6y = 0$.

Solution: Given equation is,

$$4y^2 + 12x - 20y + 67 = 0$$

$$\Rightarrow y^2 - 5y = -3x - \frac{67}{4} \quad \Rightarrow y^2 - 2\times\frac{5}{2}y + \frac{25}{4}$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2 = -3x - \frac{21}{2}$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2 = -3\left(x+\frac{7}{2}\right) \quad \ldots (i)$$

Comparing equation (i) with $(y-k)^2 = -4a(x-h)$, we get

$$h = \frac{-7}{2}, k = \frac{5}{2} \text{ and } a = \frac{-3}{4}.$$

Now, vertex $V(h, k) = \left(-\frac{7}{2}, \frac{5}{2}\right)$

axis of symmetry is, $\left(y-\frac{5}{2}\right) = 0 \Rightarrow y = \frac{5}{2}$

focus $= F(h-a, k) = \left(-\frac{7}{2} - \frac{3}{4}, \frac{5}{2}\right) = \left(-\frac{17}{4}, \frac{5}{2}\right)$

length of latus rectum $= 4a = 4\times\frac{3}{4} = 3.$

13. Find the equation of the parabola having focus (–3, 0), directrix $x + 5 = 0$. [2011 Spring, Sh...]

Solution: Given that the focus of the parabola is,

$F(-3, 0)$

and the equation of directrix of the parabola is,

$x + 5 = 0$

$\Rightarrow x = -5.$

This tells us the symmetry line of the parabola is parallel to x-axis. Therefore, the focus of this parabola is

$F(h+a, k) = F(-3, 0)$

This implies $h + a = -3, k = 0.$

We know the distance between the focus and directrix is,

$2a = |(-3) - (-5)| = 2$

$\Rightarrow a = 1.$

Therefore, $h = -3 - a = -3 - 1 = -4, k = 0.$

Then, the vertex of the parabola is

$V(h, k) = V(-4, 0).$

Since the line of symmetry line is parallel to x-axis and $a > 0$. So, the equation of parabola is

$(y-k)^2 = 4a(x-h)$

$\Rightarrow (y-0)^2 = 4\times 1\,(x+4)$

$\Rightarrow y^2 = 4(x+4).$

This is the equation of the parabola.

14. Find equation of parabola with ends of latus rectum (–1, 5), (–1, –11) and vertex at (–5, –3).

Solution: Given that the vertex is,

$V(h, k) = (-5, -3)$

Therefore, $h = -5$ and $k = -3.$ ... (i)

Also given that the ends of latus rectum are (–1, 5), (–1, –11). We know the focus is the mid-point of the end points of the latus rectum. So,

Focus = Mid-point of ends of latus rectum

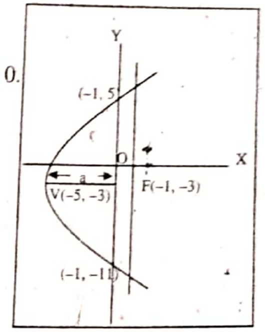
$$= \left(\frac{-1-1}{2}, \frac{5-11}{2}\right) = F(-1, -3)$$

Here, both vertex and focus have same y value, so

$F(h+a, k) = F(-1, -3)$ ... (ii)

Therefore,

$h + a = -1 \Rightarrow -5 + a = -1 \Rightarrow a = 4 > 0.$



Since the y value of the vertex and focus is same. So the symmetry line of the parabola is parallel to x axis. Therefore the equation of the parabola is,

$(y-k)^2 = 4a(x-h)$

$\Rightarrow (y+3)^2 = 4\times 4(x+5)$

$\Rightarrow y^2 + 6y + 9 = 16x + 80$

$\Rightarrow y^2 - 16x + 6y - 71 = 0$

Therefore, $y^2 + 6y - 16x - 71 = 0$ is the equation of the parabola.

15. Find equation of parabola passing through (3, 3), (6, 5) and (6, –3), and its axis being parallel to the x-axis.

Solution: Given that the parabola is passing through the points (3, 3), (6, 5) and (6, –3). So, while plotting them the parabola was found to be concave right.

Now, the equation of parabola with vertex (h, k) and concave right is

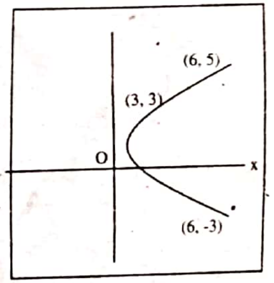
$(y-k)^2 = 4a(x-h)$ ... (1)

that passes through (3, 3), (6, 5) and (6, –3), then

$(3-k)^2 = 4a(3-h)$ ... (2)

$(5-k)^2 = 4a(6-h)$ .... (3)

$(-3-k)^2 = 4a(6-h)$ ... (4)



Dividing equation (3) by equation (4), we get

$(5-k)^2 = (3+k)^2$

$\Rightarrow 25 - 10k + k^2 = 9 + 6k + k^2$

$\Rightarrow 25 - 9 = 6k + 10k \quad \Rightarrow 16 = 16k$

$\Rightarrow k = 1.$

Putting the value of k in equation (2) and (3)

$4 = 4a(3-h)$ ... (5)

$16 = 4a(6-h)$ ... (6)

Solving (5) and (6), then

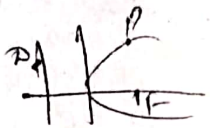
$h = 2$ and $a = 1$.

Hence, the equation (i) becomes

$$(y-1)^2 = 4 \times 1(x-2)$$

$$\Rightarrow y^2 - 2y - 4x + 9 = 0$$

This is the equation of the parabola.



16. Find the equation of parabola whose focus (a, b) and directrix $\frac{x}{a} + \frac{y}{b} = 1$.

Solution: Given that the focus is F(a, b) and directrix is $bx + ay = ab$.

Let P(x, y) be any point on the parabola. Then by definition,

length of PF = perpendicular distance from P to directrix.

$$\Rightarrow \sqrt{(x-a)^2 + (y-b)^2} = \pm \frac{bx + ay - ab}{\sqrt{b^2 + a^2}}$$

$$\Rightarrow (b^2 + a^2)(x^2 + y^2 - 2ax - 2by + a^2 + b^2) = b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2ab^2x - 2a^2by$$

$$\Rightarrow b^2x^2 + a^2x^2 + b^2y^2 + a^2y^2 - 2ab^2x - 2a^3x - 2b^3y - 2a^2by + a^2b^2 + a^4 + b^4 + a^2b^2 = b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2ab^2x - 2a^2by$$

$$\Rightarrow a^2x^2 - 2abxy + b^2y^2 - 2a^3x - 2b^3y + a^4 + b^4 + a^2b^2 = 0$$

$$\Rightarrow (ax - by)^2 - 2(a^3x + b^3y) + a^4 + b^3 + a^2b = 0.$$

This is the equation of required parabola.

Exercise 9.3

1. Find the equation of tangent and normal at the extremities of the latus rectum of the parabola $y^2 = 12x$. [2004, Spring]

Solution: Since the coordinate of end points of latus rectum of the parabola $y^2 = 4ax$ be $(a, \pm 2a)$. Therefore, the end points of latus rectum of the parabola $y^2 = 12x$ are (3, 6) and (3, –6).

Now, equation of tangent at extremity (3, 6) is,

$$\boxed{yy_1 = 2a(x + x_1)}$$

i.e. $y \cdot 6 = 2 \cdot 3(x+3) \Rightarrow y = x + 3$.

And, the equation of normal at extremity (3, 6) is,

$$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

i.e. $y - 6 = -\frac{6}{2\times 3}(x-3) \Rightarrow y - 6 = -x + 3 \Rightarrow x + y = 9$.

Also, the equation of tangent at extremity (3, –6) is,

$$\boxed{yy_1 = 2a(x + x_1)}$$

i.e. $y(-6) = 2 \cdot 3(x+3) \Rightarrow -y = x + 3 \Rightarrow x + y + 3 = 0$.

And, the equation of normal at extremity (3, –6) is,

$$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

i.e. $y + 6 = -\frac{(-6)}{2\times 3}(x-3) \Rightarrow y + 6 = x - 3 \Rightarrow x - y = 9$.

Thus, the equation of tangent to the parabola $y^2 = 12x$ at the extremities of the latus rectum (3, 6) is $y = x + 3$ and at (3, –6) is $x + y + 3 = 0$ and the equation of normal is at (3, 6) is $x + y = 9$ and at (3, –6) is $x - y = 9$.

2. A tangent to $y^2 = 4x$ makes an angle of 45° with $2x + y = 0$. Find its equation and the point of contact.

Solution: Since, the equation of tangent to $y^2 = 4x$ is

$$y = mx + \frac{1}{m} \quad \dots (i) \qquad [\text{Since } a = 1]$$

which makes an angle 45° with $y = -2x$ ... (ii)

Since, by definition $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$ where m_1 and m_2 are slope of two intersecting lines at an angle θ. Therefore

$$\tan 45° = \frac{m - (-2)}{1 + m(-2)} \Rightarrow 1 = \frac{m+2}{1-2m} \Rightarrow 1 - 2m = m + 2 \Rightarrow m = \frac{-1}{3}.$$

Then (i) becomes,

$$y = \left(-\frac{1}{3}\right)x + \frac{1}{-1/3} \Rightarrow y = -\frac{1}{3}x - 3 \quad \dots (iii)$$

For point of contact between equation (iii) and $y^2 = 4x$, eliminating y from these equation then,

$$\left(-\frac{x}{3} - 3\right)^2 = 4x$$

$$\Rightarrow (x+9)^2 = 36x$$

$$\Rightarrow x^2 + 18x + 81 = 36x$$

$$\Rightarrow x^2 - 18x + 81 = 0 \Rightarrow (x-9)^2 = 0 \Rightarrow x = 9.$$

Putting the value of x in (iii),

$$y = -\frac{1}{3}(9) - 3 \text{ then } y = -6.$$

Hence, the point of contact is (9, – 6).

3. Find the condition that the line $y = mx + c$ may touch the parabola $y^2 = 4a(x + a)$.

Solution: We have, given equation of line and parabola is

$$y = mx + c \quad \dots (i) \quad \text{and} \quad y^2 = 4a(x + a) \quad \dots (ii)$$

Since (i) touches (ii) so

$$(mx + c)^2 = 4a(x + a)$$

$$\Rightarrow m^2x^2 + 2mxc + c^2 = 4ax + 4a^2$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + (c^2 - 4a^2) = 0 \quad \dots (iii)$$

which is quadratic in x. Since (i) touches (ii) so the discriminant term (iii) is equal to zero.

$$\boxed{b^2 - 4ac = 0}$$

i.e. $(2mc - 4a)^2 - 4m^2(c^2 - 4a^2) = 0$

$\Rightarrow 4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 + 16m^2a^2 = 0$

$\Rightarrow 16mca = 16a^2 + 16m^2a^2.$

$\Rightarrow c = \frac{-16a^2(1+m^2)}{16ma} \quad \Rightarrow c = \frac{a(1+m^2)}{m} \quad \Rightarrow c = am + \frac{a}{m}$

Thus the line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = am + \frac{a}{m}$.

4. Find the equation of the common tangent of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution: We have, equation of tangent on $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \qquad \dots (i)$$

Since (i) is tangent to the parabola $y^2 = 4ax$ and also the tangent is common to the parabola $x^2 = 4by$. So, the tangent (i) must satisfy the parabola

$$x^2 = 4by. \qquad \dots (ii)$$

Therefore, eliminating y from (1) and (2), we get,

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow x^2 - 4bmx - \frac{4ab}{m} = 0 \qquad \dots (iii)$$

which is quadratic in x,

Since the line (i) is tangent on (ii), so the discriminate value of (iii) is zero.

i.e. $(-4bm)^2 - 4 \cdot 1\left(-\frac{4ab}{m}\right) = 0$

$\Rightarrow 16b^2m^2 = -\frac{16ab}{m}$

$\Rightarrow m^3 = -\frac{a}{b} \quad \Rightarrow m = -\left(\frac{a}{b}\right)^{1/3}$

Substituting the value of m in equation (i), then

$$y = -\left(\frac{a}{b}\right)^{1/3} x + \frac{a}{(-a/b)^{1/3}}$$

$$y = -\frac{a^{1/3}}{b^{1/3}}x - b^{1/3}a^{2/3}$$

$\Rightarrow y\,b^{1/3} = -a^{1/3}x - b^{2/3} \cdot a^{2/3}$

$\Rightarrow a^{1/3}x + b^{1/3} \cdot y + a^{2/3}b^{2/3} = 0.$

Thus, the equation of the common tangent of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0.$

Prove that the line $lx + my + n = 0$ touches $y^2 = 4ax$ if $ln = am^2$.

Solution: See the theory part before Exercise 9.2.

Find the equation of tangent on $y^2 = 25x$ through (4, 10).

Solution: Given equation of parabola is,

$$y^2 = 25x \qquad \dots (i)$$

which has a tangent at (4, 10).

Comparing equation (i) with $y^2 = 4ax$ we get $a = \frac{25}{4}$.

Now the equation of tangent on $y^2 = 25x$ at (4, 10) is

$$yy_1 = 2a(x + x_1)$$

$\Rightarrow y \cdot 10 = 2 \times \frac{25}{4}(x + 4)$

$\Rightarrow 40y = 50(x + 4) \Rightarrow 4y = 5x + 20.$

Thus, the equation of tangent on $y^2 = 25x$ that passes through the point (4, 10) is $5x - 4y + 20 = 0$.

Find the equations of the tangents to the parabola $y^2 = 9x$ which passes through (4, 10).

Solution: Given parabola is

$$y^2 = 9x \qquad \dots (i)$$

Comparing equation (i) with $y^2 = 4ax$ then we get, $a = \frac{9}{4}$.

Now, the equation of tangent to $y^2 = 9x$ is

$$y = mx + \frac{9/4}{m} \qquad \dots (ii)$$

Since the tangent line (ii) passes through (4, 10), then

$$10 = 4m + \frac{9}{4m}$$

$\Rightarrow 40m = 16m^2 + 9$

$\Rightarrow 16m^2 - 40m + 9 = 0 \Rightarrow (4m - 9)(4m - 1) = 0.$

Either, $4m - 9 = 0 \Rightarrow m = \frac{9}{4}$

or $4m - 1 = 0 \Rightarrow m = \frac{1}{4}$.

Thus, the equation of tangents is,

$y = \frac{9}{4}x + \frac{9/4}{9/4} \Rightarrow 4y = 9x + 4.$

and $4y = x + 36.$

Thus, the equations of the tangents to the parabola $y^2 = 9x$ which passes through (4, 10) are $4y = 9x + 4$ and $4y = x + 36$.

Find its point of contact.

Solution: Given equation of parabola is,

$$y^2 = 16x \quad \dots (i)$$

Comparing equation (i) with $y^2 = 4ax$ then we get, $a = 4$.

Now, the equation of tangent to $y^2 = 16x$ is

$$y = mx + \frac{4}{m} \quad \dots (ii)$$

Given that the parabola makes an angle 60° with x-axis i.e. y = 0,

$m = \tan 60° = \sqrt{3}$.

Then, the equation (ii) becomes,

$$y = \sqrt{3}\,x + \frac{4}{\sqrt{3}} \Rightarrow \sqrt{3}y = \sqrt{3}x + 4 \quad \dots (iii)$$

This is the equation of required tangent to the given parabola satisfying given conditions.

Now, from equation (i) and (ii)

y=mx+a/m

$$\left(\frac{3x+4}{\sqrt{3}}\right)^2 = 16x$$

$$\Rightarrow 9x^2 + 24x + 16 = 48x$$

$$\Rightarrow 9x^2 - 24x + 16 = 0 \quad \Rightarrow (3x-4)^2 = 0$$

$$\Rightarrow 3x - 4 = 0 \Rightarrow x = \frac{4}{3}.$$

And, $y^2 = 16\left(\frac{4}{3}\right) \Rightarrow y = \pm\frac{8}{\sqrt{3}}$

Thus, the tangent to the parabola $y^2 = 16x$ makes an angle 60° with x-axis has its point of contact is $\left(\frac{4}{3}, \pm\frac{8}{\sqrt{3}}\right)$.

9. Find the equation of the tangents and normal at the ends of the latus rectum of $y^2 = 4ax$.

Solution: Given equation of parabola is,

$$y^2 = 4ax \quad \dots (i)$$

whose length of latus rectum = 4a

and the coordinates of its ends of latus rectum are (a, 2a) and (a, –2a).

Now the equation of tangent at (a, 2a) is,

$$y \cdot 2a = 2a(x + a) \Rightarrow x - y + a = 0.$$

and equation of tangent at (a, –2a) is,

$$y(-2a) = 2a(x + a)$$

$$\Rightarrow -2ay = 2a(x + a) \Rightarrow -y = x + a \Rightarrow x + y + a = 0.$$

equation of normal at (a, 2a) is

$$(y - 2a) = \frac{-2a}{2a}(x - a)$$

$$\Rightarrow y - 2a = -(x - a) \Rightarrow x + y - 3a = 0.$$

and equation of normal at (a, –2a) is,

$$(y + 2a) = -\frac{-2a}{2a}(x - a)$$

$$\Rightarrow y + 2a = (x - a) \Rightarrow x - y - 3a = 0.$$

Thus, the equation of the tangents at the ends of the latus rectum of $y^2 = 4ax$ at (a, 2a) is $x - y + a = 0$ and at (a, –2a) is $x + y + a = 0$ and the equation of the normal at (a, 2a) is $x + y - 3a = 0$ and at (a, –2a) is $x - y - 3a = 0$.

10. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution: Given equation of parabola is

$$y^2 = 8x \quad \dots (i)$$

Comparing it with $y^2 = 4ax$ we get $a = 2$.

Now the equation of tangent to (i) is,

$$y = m_1x + a/m_1$$

$$\Rightarrow y = m_1x + 2/m_1 \quad \dots (ii)$$

Given that the tangent (ii) makes an angle of 45° with the straight line,

$$y = 3x + 5 \quad \dots (iii)$$

Here, using the formula of angle between two lines,

$$\tan\theta = \frac{m - m_1}{1 + (m)(m_1)}$$

where m_1 be the slope of required line.

$$\Rightarrow \tan 45° = \frac{3 - m_1}{1 + (3)(m_1)}$$

$$\Rightarrow 1 + 3m_1 = 3 - m_1 \Rightarrow 4m_1 = 2 \Rightarrow m_1 = \frac{1}{2}.$$

y=mx+

Thus, the equation of tangent is,

$$y = \frac{1}{2}x + \frac{2}{1/2}$$

$$y = \frac{x-8}{2}$$

$$\Rightarrow 2y - x = 8 \Rightarrow x = 2y - 8 \quad \dots (iv)$$

And, for the point of contact, solving (i) and (iv),

$$y^2 = 4(2y - 8)$$

$$\Rightarrow y^2 - 16y + 64 = 0$$

$$\Rightarrow (y - 8)^2 = 0 \Rightarrow y = 8.$$

Then, $x = 2 \times 8 - 8 = 8$

Hence, the point of contact is (8, 8).

11. Show that the straight line $7x + 6y - 13 = 0$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$. Find the point of contact.

Solution: Given equation of tangent is

$$7x + 6y - 13 = 0 \quad \dots (i)$$

and given equation of parabola is

$$y^2 - 7x - 8y + 14 = 0$$
$$\Rightarrow y^2 - (-6y + 13) - 8y + 14 = 0$$
$$\Rightarrow y^2 - 2y + 1 = 0$$
$$\Rightarrow (y-1)^2 = 0$$
$$\Rightarrow y = 1$$

Then (i) gives

$$7x + 6 - 13 = 0$$
$$\Rightarrow 7x = 7$$
$$\Rightarrow x = 1$$

This shows the given line (i) and curve (ii) meet at a single point (1, This means line (i) is tangent to curve (ii) and their point of contac (1, 1).

12. Find the value of λ, when the line $x - y + 1 = 0$ is a tangent to parabola $y^2 = \lambda x$.

Solution: Here, given equation of tangent is,

$$x - y + 1 = 0 \Rightarrow y = x + 1 \quad \ldots (i)$$

and equation of parabola is, $y^2 = \lambda x \quad \ldots (ii)$

Comparing the equation (i) with $y = mx + c$, we get

$$m = 1 \text{ and } c = 1.$$

and comparing the equation (ii) with $y^2 = 4ax$, then we get

$$a = \frac{\lambda}{4}.$$

Since the line (i) is tangent to the parabola (ii). So, we must have,

$$c = \frac{a}{m} \Rightarrow 1 = \frac{\lambda}{4} \Rightarrow \lambda = 4.$$

Thus, for $\lambda = 4$, the line $x - y + 1 = 0$ is a tangent to the parabola $y^2 = \lambda x$

13. Find the equation of the tangents to the parabola $y^2 = 7x$, which perpendicular to the line $4x + y = 0$. Also, find the point of contact.

[2014 Spring]

Solution: Here, given equation of parabola is

$$y^2 = 7x \quad \ldots (i)$$

Comparing (i) with $y^2 = 4ax$ we get

$$a = \frac{7}{4}$$

and given that the equation of straight line is

$$4x + y = 0 \Rightarrow y = -4x \quad \ldots (ii)$$

Comparing (ii) with $y = m_1x + c$ we get

$$m_1 = -4, c = 0.$$

Since the equation of tangent to (i) is,

$$y = mx + \frac{a}{m}$$
$$\Rightarrow y = mx + \frac{7}{4m} \quad \ldots (iii)$$

By hypothesis the line (ii) is perpendicular to the tangent (iii). So, the product of their slopes should equal to –1.

i.e. $m(-4) = -1 \Rightarrow m = 1/4$.

Substituting this value in (iii) then it becomes,

$$\Rightarrow y = \frac{1}{4}x + 7.$$
$$\Rightarrow 4y = x + 28$$
$$\Rightarrow x - 4y + 28 = 0 \quad \ldots (iv)$$

This is the required equation of tangent to (i).

For the point of contact, solving equation (i) and (iii) we get,

$$y^2 = 7(4y - 28)$$
$$\Rightarrow y^2 - 28y + 196 = 0$$
$$\Rightarrow (y - 14)^2 = 0$$
$$\Rightarrow y = 14.$$

And $x = 4y - 28 = 4 \times 14 - 28 = 28.$

Hence, the point of contact is (28, 14).

14. Find the equation of tangents to the parabola $y^2 = 5x$ passing through the point (5, 13). Also, find the point of contact of the tangents.

Solution: Given parabola is

$$y^2 = 5x \quad \ldots (i)$$

Comparing the equation (i) with $y^2 = 4ax$, we get

$$a = 5/4.$$

And, the equation of tangent to (i) is,

$$y = mx + \frac{a}{m}$$
$$\Rightarrow y = mx + \frac{5}{4m} \quad \ldots (ii)$$

Now the equation of a line that passes through the point (5, 13) be,

$$y - 13 = m(x - 5)$$
$$\Rightarrow y = mx + 13 - 5m \quad \ldots (iii)$$

Since line (ii) and (iii) both are tangent line to (i) so they must be identical. That is,

$$13 - 5m = \frac{5}{4m}$$
$$\Rightarrow 20m^2 - 52m + 5 = 0$$
$$\Rightarrow 10m(2m - 5) - 1(2m - 5) = 0$$
$$\Rightarrow (2m - 5)(10m - 1) = 0$$

$\Rightarrow m = \frac{5}{2}, \frac{1}{10}.$

Substituting value in equation (ii) then

$y - 13 = \frac{5}{2}(x - 5)$ and $y - 13 = \frac{1}{10}(x - 5)$

$\Rightarrow 5x - 2y + 1 = 0$ $\Rightarrow x - 10y + 125 = 0$

$\Rightarrow x = \frac{2y - 1}{5}$ ... (iv) $\Rightarrow x = 10y - 125$ ... (v)

For the point of contact, solving equation (i), (iv) and (i), (v) then,

$y^2 = 5\left(\frac{2y-1}{5}\right)$ and $y^2 = 5(10y - 125)$

$\Rightarrow y^2 = 2y - 1$ $\Rightarrow y^2 - 50y + 625 = 0$

$\Rightarrow y^2 - 2y + 1 = 0$ $\Rightarrow (y - 25)^2 = 0$

$\Rightarrow (y - 1)^2 = 0$ $\Rightarrow y = 25$

$\Rightarrow y = 1.$

Putting value y = 1 in equation (iv) and y = 25 in (v) then,

$x = \frac{2y-1}{5} = \frac{2-1}{5} = \frac{1}{5}$ and $x = 10 \times 25 - 125 = 125.$

Thus the point of contact are (1/5, 1) and (125, 25) for tangent to $5x - 2y + 1 = 0$ and $x - 10y + 125 = 0$, respectively.

15. Show that the normal to the parabola $y^2 = 8x$ at (2, 4) meets the parabola again in (18, –12).

Solution: Given parabola is

$y^2 = 8x$ ... (i)

Comparing equation (i) with $y^2 = 4ax$, we get

$a = 2.$

Now, equation of normal to parabola at (2, 4) is

$(y - y_1) = -\frac{y_1}{2a}(x - x_1)$

$\Rightarrow (y - 4) = \frac{-4}{2 \times 2}(x - 2)$

$\Rightarrow y - 4 = -x + 2$ $\Rightarrow x + y = 6$ ... (ii)

If the normal again meet the parabola then the point (18, –12) should satisfy the equation (i) and (ii), we get

$y^2 = 8x$ and $x + y = 6$

$\Rightarrow (12)^2 = 8 \times 18$ $\Rightarrow 18 - 12 = 6$

$\Rightarrow 144 = 144$ (true). $\Rightarrow 6 = 6$ (true).

Hence, the normal again meets the parabola at (18, –12).

16. Show that the line $lx + my + n = 0$ touches the parabola $y^2 = 4a(x - b)$ if $am^2 = bl^2 + nl$.

Solution: Given parabola is,

$y^2 = 4a(x - 6)$ ... (i)

and given line is,

$lx + my + n = 0$

$\Rightarrow y = -\frac{(lx + n)}{m}$ ... (ii)

Solving equation (i) and (ii) by eliminating 'y',

$\left[-\frac{(lx + n)}{m}\right]^2 = 4a(x - 6)$

$\Rightarrow l^2x^2 + 2nlx + n^2 = 4am^2x - 4bm^2a$

$\Rightarrow l^2x^2 + (2nl - 4am^2)x + (4abm^2 + n^2) = 0$ ... (iii)

This is a quadratic in x.

Since equation (ii) is tangent to (i), so discriminate of (3) must be equal to zero. Therefore,

$(2nl - 4am^2)^2 - 4l^2(4abm^2 + n^2) = 0$

$\Rightarrow 4n^2l^2 - 16alm^2n + 16a^2m^4 - 16abl^2m^2 - 4l^2n^2 = 0$

$\Rightarrow a^2m^4 = abl^2m^2 + alm^2n$ [Since $16 \neq 0$]

$\Rightarrow a^2m^4 = am^2(bl^2 + ln)$

$\Rightarrow am^2 = bl^2 + ln$

Hence, the line (ii) touches the parabola (i) only if $am^2 = bl^2 + nl$.

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM PARABOLA

1. Find vertices and foci, line of symmetry and directrics of the parabolas. (i) $y_1^2 - 4y - 4x = 0$ (ii) $x^2 - 2x - 8y - 15 = 0$. [2002]

Solution: (i) We have equation of parabola is

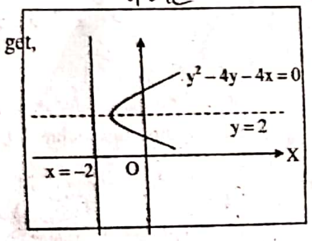
$y^2 - 4y - 4x = 0$

$\Rightarrow y^2 - 4y = 4x$

$\Rightarrow (y - 2)^2 = 4x + 4$

$\Rightarrow (y - 2)^2 = 4(x + 1)$

(h, k) ~ f (h+a, k)
x = h−a
y = k



Comparing (i) with $(y - k)^2 = 4a(x - h)$ we get,

$h = -1, k = 2$ and $a = 1.$

So, vertex (h, k) = (–1, 2)

Focus (h + a, k) = (–1 + 1, 2) = (0, 2)

Axis of symmetry is $y - k = 0$ $\Rightarrow y = 2$

Equation of directrix is

$x - h = -a$

$\Rightarrow x = -1 - 1$

$\Rightarrow x = -2.$

h−a

(ii) We have,

$x^2 - 2x - 8y - 15 = 0$

$\Rightarrow (x-1)^2 = 8y + 16$

$\Rightarrow (x-1)^2 = 8(y+2)$

This is the equation of parabola. Comparing it with $(x-h)^2 = 4a(y-k)$ ...(i) then we get

$h = 1, k = -2, a = 2$

Then its vertex is $(h, k) = (1, -2)$.

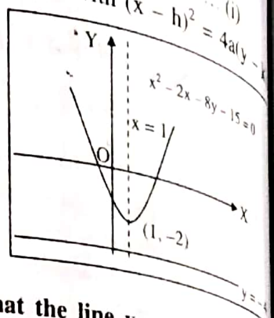
Focus is $(h, k + a) = (1, -2 + 2) = (1, 0)$

Axis of symmetry is, $x - 1 = 0$

Equation of directrix is

$y + 2 = -2$

$\Rightarrow y = -4.$



2. Define parabola. Fine the condition that the line $y = mx + c$ may be tangent to the parabola $y^2 = 4ax$. [2003, Fall]

Solution: First Part: See definition of parabola.

Second Part: See the condition of tangency (Theory Part).

3. **Define parabola. Find the equation of parabola whose vertex is at (h, k) and whose axis is parallel to x–axis.** [2003, Spring]

Solution: See definition of parabola.

See standard equation of parabola with vertex (h, k).

5. **Derive standard equation of parabola $y^2 = 4ax$. Find vertex, focus, axis of symmetry, directrix of the curve: $x^2 - 2y + 8x + 10 = 0$.** [2006, Fall]

Solution: First Part: See standard equation of parabola.

Second Part: Given curve is,

$x^2 - 2y + 8x + 10 = 0$

$\Rightarrow (x^2 + 8x + 16) = 2y - 10 + 16$

$\Rightarrow (x + 4)^2 = 2(y + 3)$

Comparing it with $(x - h)^2 = 4a(y - k)$ then we get,

$h = -4, k = -3, a = \frac{1}{2}$

Now, vertex, $V(h, k) = V(-4, -3)$

and the line of symmetry is, $x + 4 = 0 \Rightarrow x = -4.$

So, the focus is associated with y. Therefore, focus is,

$F(h, k + a) = F(-4, -3 + \frac{1}{2}) = F(-4, -5/2)$

And, equation of directrix is,

$\boxed{y - k = -a}$

$y + 3 = -\frac{1}{2}$

$\Rightarrow 2y + 7 = 0.$

y = k − a

6. **Find the condition that the line $lx + my = p$ may be a tangent to the parabola $y^2 = 4ax$.** [2007, Spring]

Solution: Given line is

$lx + my = p$ .....(1)

and the curve

$y^2 = 4ax$ .....(2)

Since (1) is tangent to (2) then

$y + 2 = 4a\left(\frac{p - my}{l}\right)$

$\Rightarrow ly^2 = 4ap - 4amy$

$\Rightarrow ly^2 + 4amy - 4ap = 0$ ....(3)

Since the line (1) is tangent to (2). So, the discriminate value of (3) is zero.

That is, $\boxed{b^2 - 4ac = 0}$

$\Rightarrow (4am)^2 - 4l(-4ap) = 0$

$\Rightarrow 16a\,[am^2 + lp] = 0$

$\Rightarrow am^2 + lp = 0 \quad [\because 16a \neq 0]$

This is the required condition.

Find the condition that the line $lx + my + n = 0$ may be a tangent to the parabola $y^2 = 4ax$. Also find the point of contact. [2014 Fall][2013 Spring]

Solution: For first part, see Q. 6 with replacing p by –n.

And for point of contact, we have

$x = \frac{-B}{2A} = \frac{-4am}{2l} = \frac{-2am}{l}$

Then, $y = \frac{-n}{m} - \frac{-2am}{m} = \frac{2a + n}{m}$.

Thus the point of contact are (x, y) given in above.

7. **Define conic section by their eccentricity and classify them. Derive standard equation of parabola $y^2 = 4ax$.** [2017 Fall][2008, Fall]

Solution: See the definition of conic section and its classification.

See standard equation of parabola.

9. **Define a conic section and its eccentricity and classify them. Derive the equation of a parabola whose focus is (a, 0) and directrix is $x = -a$.** [2001] [1999]

Solution: See the definition of conic section and its classification.

Second Part: Given that focus of a parabola is (a, 0) and directrix is $x = -a$.

Since the directrix is $x = -a$ that means the line of symmetry is parallel to x–axis.

So, $h = \frac{a + (-a)}{2} = \frac{0}{2} = 0, k = 0$

And length from vertex to focus $= a - 0 = a$.

$\Rightarrow \ x^2 - 2x = 8y + 15$

$\Rightarrow \ (x-1)^2 = 8y + 16$

$\Rightarrow \ (x-1)^2 = 8(y+2)$ ...(i)

This is the equation of parabola. Comparing it with $(x-h)^2 = 4a(y-k)$ then we get

$h = 1, k = -2, a = 2$

Then its vertex is $(h, k) = (1, -2)$.

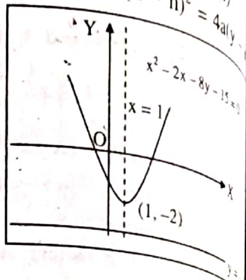
Focus is $(h, k+a) = (1, -2+2) = (1, 0)$

Axis of symmetry is, $x - 1 = 0$

Equation of directrix is

$y + 2 = -2$

$\Rightarrow \ y = -4.$



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Solution: First Part: See standard equation of parabola.

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So, the focus is associated with y. Therefore, focus is,

$F(h, k+a) = F(-4, -3 + \frac{1}{2}) = F(-4, -5/2)$

And, equation of directrix is,

$\boxed{y - k = -a}$

$y + 3 = -\frac{1}{2}$

$\Rightarrow \ 2y + 7 = 0.$

$y = k - a$

6. Find the condition that the line $lx + my = p$ may be a tangent to the parabola $y^2 = 4ax$. [2007, Spring]

Solution: Given line is

$lx + my = p$ .....(1)

and the curve

$y^2 = 4ax$ .....(2)

Since (1) is tangent to (2) then

$$y^2 = 4a\left(\frac{p - my}{l}\right)$$

$\Rightarrow \ ly^2 = 4ap - 4amy$

$\Rightarrow \ ly^2 + 4amy - 4ap = 0$ ....(3)

Since the line (1) is tangent to (2). So, the discriminate value of (3) is zero.

That is, $\boxed{b^2 - 4ac = 0}$

$\Rightarrow \ (4am)^2 - 4l(-4ap) = 0$

$\Rightarrow \ 16a\,[am^2 + lp] = 0$

$\Rightarrow \ am^2 + lp = 0$ $\quad [\because 16a \neq 0]$

This is the required condition.

Find the condition that the line $lx + my + n = 0$ may be a tangent to the parabola $y^2 = 4ax$. Also find the point of contact. [2014 Fall][2013 Spring]

Solution: For first part, see Q. 6 with replacing p by −n.

And for point of contact, we have

$$x = \frac{-B}{2A} = \frac{-4am}{2l} = \frac{-2am}{l}$$

Then, $$y = \frac{-n}{m} - \frac{-2am}{m} = \frac{2a + n}{m}$$

Thus the point of contact are (x, y) given in above.

7. Define conic section by their eccentricity and classify them. Derive standard equation of parabola $y^2 = 4ax$. [2017 Fall][2008, Fall]

Solution: See the definition of conic section and its classification.

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Solution: See the definition of conic section and its classification.

Second Part: Given that focus of a parabola is (a, 0) and directrix is $x = -a$.

Since the directrix is $x = -a$ that means the line of symmetry is parallel to x–axis.

So, $\quad h = \frac{a + (-a)}{2} = \frac{0}{2} = 0, k = 0$

And length from vertex to focus $= a - 0 = a$.

Thus, equation of parabola be

$$(y-k)^2 = 4a(x-h)$$

$$\Rightarrow (y-0)^2 = 4a(x-0)$$

$$\Rightarrow y^2 = 4ax$$

This is the required equation.

11. What do you mean by eccentricity of a conic section? Classify the... Find the vertex, axis, focus and directrix of the parabo... $y^2 + 6y + 2x + 5 = 0$. [200...]

Solution: First Part: See definition of eccentricity and its classification.

Second Part: Given parabola is

$$y^2 + 6y + 2x + 5 = 0$$

$$\Rightarrow y^2 + 6y + 9 = -2x - 5 + 9$$

$$\Rightarrow (y+3)^2 = -2(x-2) \qquad \dots (1)$$

Comparing (1) with $(y-k)^2 = 4a(x-h)$ then we get,

$$h = 2,\ k = -3,\ a = -\frac{1}{2}$$

Now, vertex, $V(h, k) = V(2, -3)$

axis of symmetry is,

$$\boxed{y - k = 0}$$

$$y + 3 = 0.$$

focus $F(h + a, k) = F(2 - \frac{1}{2}, -3) = F(3/2, -3)$

equation of the directrix is, $\boxed{x - h = -a}$

$$x - 2 = \frac{1}{2}$$

$$\Rightarrow 2x - 5 = 0.$$

12. Find the equation of tangent at (2, ¼) on the parabola $y^2 = 16x$.

Solution: Given equation of parabola is,

$$y^2 = 16x \qquad \dots (i)$$

which has a tangent at (2, ¼).

Comparing equation (i) with $y^2 = 4ax$ then $a = 4$.

Now the equation of tangent on $y^2 = 16x$ at (1, 1/4) is,

$$yy_1 = 2a(x + x_1)$$

$$\Rightarrow \frac{y}{4} = 2 \times 4\,(x + 2)$$

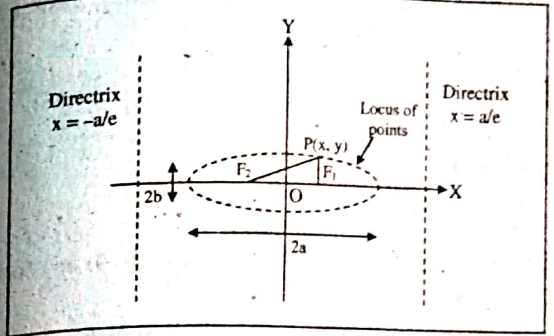
$$\Rightarrow y = 32(x+2) \Rightarrow y = 32x + 64.$$

Thus, the equation of tangent on $y^2 = 16x$ at the point (2, ¼).

Ellipse

Definition:

An ellipse is the locus of points in a plane whose distance from two fixed points in the plane has a contact sum. The fixed points are called foci of the ellipse.



Standard Equation of Ellipse Having Centre at (0, 0)

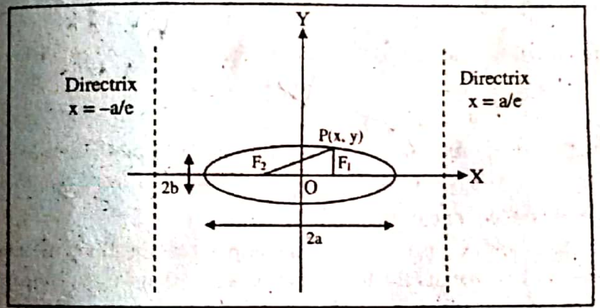
Derive the equation of an ellipse in its standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[2004, Fall] [2002]**

Let O(0, 0) be the centre of the ellipse. Let $F_1(c, 0)$ and $F_2(-c, 0)$ be foci of the ellipse and let P(x, y) be any point of the ellipse. Then by definition of the ellipse, the sum of distance of P from F_1 and F_2, is constant. That is,

$$PF_1 + PF_2 = 2a$$

i.e. $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$



Squaring both side we get,

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$\Rightarrow x^2 - 2xc + c^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2$$

$$\Rightarrow 4a\sqrt{(x+c)^2 + y^2} = 4(a^2 + xc)$$

Again, squaring both sides, we get

$$a^2[(x+c)^2 + y^2] = a^4 + 2a^2xc + x^2c^2$$

$\Rightarrow a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2$

$\Rightarrow (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \ldots (i)$

In any triangle, we have the sum of two sides is always greater than the third side. So, in figure in ΔPF_1F_2,

$(PF_1 + PF_2) > F_1F_2$

$\Rightarrow 2a > 2c \Rightarrow a > c \Rightarrow a^2 > c^2 \Rightarrow (a^2 - c^2) > 0.$

So, let, $b^2 = a^2 - c^2$

Thus equation (i) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of ellipse having center at (0, 0) and foci at $(\pm c, 0)$ where $b^2 = a^2 - c^2$.

Q. Derive the equation of an ellipse with centre at (0, 0). [2016 Spring]

Note: Equation of ellipse with centre at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Equation of tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The equation of ellipse in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots (i)$$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

At (x_1, y_1), the ellipse is,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \ldots\ldots (ii)$$

$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$

Differentiating equation (i) w. r. t. x, we get

$$\frac{dy_1}{dx_1} = \frac{-b^2x_1}{a^2y_1}$$

Since the equation of line that passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \quad \ldots (iii)$$

where m be the slope of the line.

Since the point (x_1, y_1) be the common point of the line (iii) and the given ellipse (i). This means the line is a tangent to (i) at (x_1, y_1) and so, the slope m of (iii) is same as the slope $\frac{dy_1}{dx_1}$ of (i).

That is,

$$m = \frac{dy_1}{dx_1} = \frac{-b^2x_1}{a^2y_1}$$

Therefore, the line (iii) becomes

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1. \quad \text{[Using (ii)]}$$

Thus, the equation of tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Note: Equation of normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

Condition for tangency that a line y = mx + c touches the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [2015 Spring][2017 Fall]

Let the equation of the given line is

$$y = mx + c \quad \ldots (i)$$

and the equation of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \quad \ldots (ii)$$

Eliminating y from equation (i) and (ii), we get

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

$$b^2x^2 + a^2(m^2x^2 + 2mxc + c^2) = a^2b^2$$

$$\Rightarrow b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow (b^2 + a^2m^2)x^2 + 2a^2mcx + (a^2c^2 - a^2b^2) = 0 \quad \ldots (iii)$$

This is quadratic in x.

Since, (iii) be the common value of (i) and (ii). And, (i) is tangent on (ii). So, its discriminate term of (iii) should be equal to zero. So,

$$(2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$$

$$\Rightarrow a^4m^2b^2 + a^2b^4 = a^2b^2c^2$$

$$\Rightarrow a^2m^2 + b^2 = c^2.$$

This is the required condition for tangency.

For point of contact,

$$x = -\frac{B}{2A} \quad [\because \text{Being the discriminate term is zero i.e. } B^2 - 4AC = 0]$$

$$= -\frac{2a^2mc}{2(b^2 + a^2m^2)} = -\frac{2a^2mc}{c^2} = -\frac{a^2m}{c}$$

Then (i) gives,

$$y = mx + c = m\left(-\frac{ma^2}{c}\right) + c = -\frac{m^2a^2 - c^2}{c} = \frac{b^2}{c}$$ [Using (...)]

Thus, the point of contact of (i) and (ii) is $\left(-\frac{ma^2}{c}, \frac{b^2}{c}\right)$.

Q. Find the condition that the line y = mx + c may touch the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find the point of contact. [2018 Spring]

Condition of tangency of the line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find the point of contact. [2006, Fall] [2008, Fall]

Given line is

$$lx + my + n = 0$$

$$\Rightarrow y = -\left(\frac{l}{m}\right)x - \frac{n}{m} \quad \ldots (i)$$

and the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \quad \ldots (ii)$$

Eliminating y from given equations (i) and (ii)

$$b^2x^2 + a^2\left(-\frac{lx+n}{m}\right)^2 = a^2b^2$$

$$\Rightarrow m^2b^2x^2 + a^2(l^2x^2 + 2lxn + n^2) = a^2b^2m^2$$

$$\Rightarrow (m^2b^2 + a^2l^2)x^2 + 2a^2lnx + (a^2n^2 - a^2b^2m^2) = 0 \quad \ldots (iii)$$

Since given line is tangent on the ellipse, so the discriminate term of (iii) is zero.

i.e. $(2a^2ln)^2 - 4(m^2b^2 + a^2l^2)(a^2n^2 - a^2b^2m^2) = 0$

$\Rightarrow 4a^4l^2n^2 - 4\{m^2b^2(a^2n^2 - a^2b^2m^2) + a^2l^2(a^2n^2 - a^2b^2m^2\} = 0$

$\Rightarrow m^4b^4 + a^2b^2l^2m^2 = m^2n^2b^2$

$\Rightarrow a^2l^2 + m^2b^2 = n^2.$

This is the required condition.

And, for point of contact,

$$x = -\frac{B}{2A} \quad [\because \text{Being the discriminate term is zero i.e. } B^2 - 4AC = 0]$$

$$= -\frac{2a^2ln}{2(m^2b^2 + a^2l^2)}$$

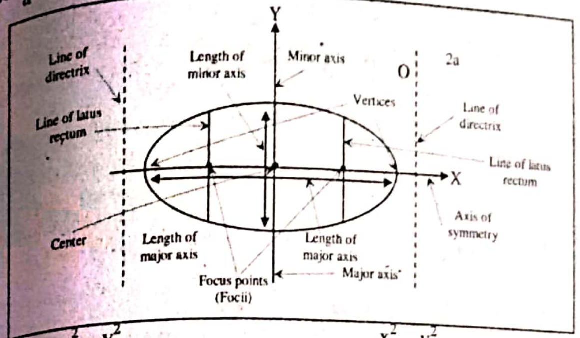
$$\Rightarrow x = -\frac{2a^2ln}{2n^2} = -\frac{a^2l}{n}$$

Then (i) gives,

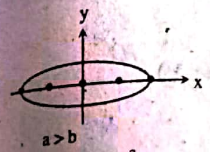
$$y = -\frac{lx - n}{m} = -\frac{l\left(-\frac{a^2l}{n}\right) - n}{m} \Rightarrow y = \frac{a^2l^2 - n^2}{mn} = \frac{-b^2m^2}{mn} = \frac{-b^2m}{n}$$

Thus the point of contact is $\left(-\frac{a^2l}{n}, -\frac{b^2m}{n}\right)$.

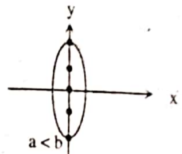
Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$



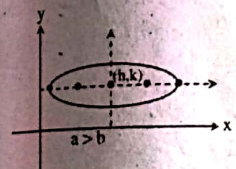
A. Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$



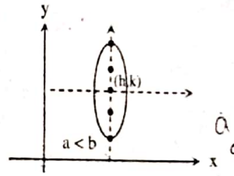
B. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$



C. $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$



D. $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$



Eq^n. of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$		$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$	
	$a > b > 0$	$0 < a < b$	$a > b > 0$	$0 < a < b$
Center	(0, 0)	(0, 0)	(h, k)	(h, k)
Vertex	$(\pm a, 0)$	$(0, \pm b)$	$(h \pm a, k)$	$(h, k \pm b)$
Major axis	2a	2b	2a	2b
Minor axis	2b	2a	2b	2a
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Eccentricity (e)	$\sqrt{1-\frac{b^2}{a^2}}$	$\sqrt{1-\frac{a^2}{b^2}}$	$\sqrt{1-\frac{b^2}{a^2}}$	$\sqrt{1-\frac{a^2}{b^2}}$

Focus	$(\pm ae, 0)$	$(0, \pm be)$	$(h \pm ae, k)$	$(h, k ...$
Eq^n of directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$	$x = h \pm \frac{a}{e}$	$y = k ...$
Line of symmetry	$x = 0, y = 0$	$x = 0, y = 0$	$x = h, y = k$	$x = h, y ...$

Exercise 9.4

1. Find the equation of ellipse which has centre C, focus F and major axis a or b and calculate eccentricity.

(i) C(0, 0), F(0, 2), b = 4

Solution: Given that, C(0, 0) and F(0, 2). Here the y value in centre and focus vary. So, C(h, k) = C(0, 0) and F(h, k + c) = F(0, 2) then,

$$h = 0, k = 0 \text{ and } c = 2 \text{ along y-axis.}$$

Therefore, the major axis is parallel to y–axis. So, $b > a$

Since, $c^2 = b^2 - a^2$ and given that $b = 4$.

Therefore, $a^2 = b^2 - c^2 = 16 - 4 = 12 \Rightarrow a = \sqrt{12}$

Now the equation of ellipse with centre (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{12} + \frac{y^2}{16} = 1.$$

$c^2 = b^2 - a^2$

and its eccentricity is, $e = \frac{c}{b} = \frac{2}{4} = \frac{1}{2}$.

(ii) C(0, 2), b = 3, F(0, 0)

Solution: Given that, C(0, 2) and F(0, 0). Here the y value in centre and focus vary. So, C(h, k) = C(0, 2) and F(h, k + c) = F(0, 0) then,

$$h = 0, k = 2 \text{ and } c = -2 \quad \text{along y-axis.}$$

Therefore, the major axis is parallel to y–axis. So, $b > a$.

Since, $c^2 = b^2 - a^2$ and given that $b = 4$.

Therefore, $a^2 = b^2 - c^2 = 9 - 4 = 5 \Rightarrow a = \sqrt{5}$.

Now the equation of ellipse with centre (0, 0) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

$$\Rightarrow \frac{(x-0)^2}{5} + \frac{(y-2)^2}{1} = 1$$

$$\Rightarrow \frac{x^2}{5} + \frac{(y-2)^2}{9} = 1.$$

$e = \frac{c}{b}$

and its eccentricity is, $e = \frac{c}{b} = \frac{2}{3}$.

$a > b$

(iii) C(2, 2), F(–1, 2), a = $\sqrt{10}$.

Solution: Given that, C(2, 2) and F(–1, 2). Here the x value in centre and focus is vary. So, C(h, k) = C(0, 2) and F(h, k + c) = F(0, 0) then,

$h + c, k$

Solution: Here, C(h, k) = C(2, 2) and F(h + c, k) = F(–1, 2). Then, from centre and focus, we have

$$h = 2, k = 2 \text{ and } c = 3 \text{ along x–axis.}$$

Therefore, the major axis is parallel to x–axis. So, $a > b$.

Since, $c^2 = a^2 - b^2$ and given that $a = \sqrt{10}$.

Therefore, $b^2 = a^2 - c^2 = 10 - 9 = 1 \Rightarrow b = 1$.

Now the equation of ellipse with centre (2, 2) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

$$\Rightarrow \frac{(x-2)^2}{10} + \frac{(y-2)^2}{1} = 1$$

and its eccentricity is, $e = \frac{c}{a} = \frac{3}{\sqrt{10}}$.

$e = \frac{c}{a}$

2. The end points of the major and minor axes of ellipse are (1, 1), (3, 4), (1, 7) and (–1, 4). Find equation of ellipse and find its focus.

Solution: Given that the end points of major and minor axes of ellipse are (1, 1), (3, 4), (1, 7) and (–1, 4). Therefore,

$$2b = |7 - 1| = 6 \Rightarrow b = 3 \quad \text{and} \quad 2a = |-1 - 3| = 4 \Rightarrow a = 2.$$

Here, $b > a$. Then, $c^2 = \sqrt{b^2 - a^2} = \sqrt{9 - 4} = \sqrt{5}$.

Since the center of the ellipse is the mid-point of the axis. So,

$$h = \frac{1+1}{2} = 1, \quad k = \frac{1+7}{2} = 4.$$

That is $\quad C(h, k) = C(1, 4)$.

Therefore, equation of ellipse is,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \Rightarrow \frac{(x-1)^2}{4} + \frac{(y-4)^2}{9} = 1.$$

Here, $b = 3 > 2 = a$. So, the foci lie on the line parallel to y–axis. Therefore,

$$F(h, k \pm c) = F(1, 4 \pm \sqrt{5}).$$

3. Find center, vertices and foci of the ellipse

(i) $x^2 + 5y^2 + 4x = 1$

Solution: Given equation is,

$$x^2 + 5y^2 + 4x = 1.$$

$$\Rightarrow (x+2)^2 + 5y^2 = 5.$$

$$\Rightarrow \frac{(x+2)^2}{5} + \frac{y^2}{1} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$$h = -2, \quad k = 0, \quad a^2 = 5 \text{ and } b^2 = 1.$$

Here, $a > b$. So, $c^2 = a^2 - b^2 = 5 - 1 \Rightarrow c^2 = 4$.

Therefore, centre (h, k) = C(–2, 0).

$c^2 = a^2 - b^2$

Since a > b. And, the major axis is parallel to x–axis. Therefore, the fo[ci lie] on the line that is parallel to x–axis. So,

Foci are at $F(h \pm c, k) = F(-2 \pm 2, 0)$.

Vertices are at $V(h \pm a, k) = V(-2 \pm \sqrt{5}, 0)$.

(ii) $x^2 + 2y^2 - x - 4y + 1 = 0$

Solution: Given equation is,

$$x^2 + 2y^2 - x - 4y + 1 = 0.$$

$$\Rightarrow \frac{(x - 1/2)^2}{5/4} + \frac{(y-1)^2}{5/2} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$$h = 1/2, k = 1, \quad a^2 = 5/4 \text{ and } b^2 = 5/2.$$

Here, b > a. So, $c = \sqrt{b^2 - a^2} = \sqrt{5/4}$.

Therefore, centre $(h, k) = C(-1, -4)$.

Since b > a. And, the major axis is parallel to y–axis. Therefore, the foci [lie] on the line that is parallel to y–axis. So,

Foci are at $F(h, k \pm c) = F(1/2, 1 \pm \sqrt{5/4})$.

Vertices are at $V(h, k \pm b) = V(1/2, 1 \pm \sqrt{5/2})$.

(iii) $25(x-3)^2 + 4(y-1)^2 = 100$

Solution: The given equation becomes as,

$$25(x-3)^2 + 4(y-1)^2 = 100$$

$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{25} = 1$$

Comparing it with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$$h = 3, \ k = 1, \ a^2 = 4 \Rightarrow a = 2 \text{ and } b^2 = 25 \Rightarrow b = 5.$$

Here, b > a, so $c = \sqrt{b^2 - a^2} = \sqrt{5^2 - 2^2} = \sqrt{21}$

Therefore, centre of the ellipse is $C(h, k) \Rightarrow C(3, 1)$

Since b > a. And, the major axis is parallel to y–axis. Therefore, the foci li[e] on the line that is parallel to y–axis. So,

foci of the ellipse are at $F(h, k \pm c) = F(3, 1 \pm \sqrt{21})$.

Vertices are $V(h, k \pm b) = V(3, 1 \pm 5)$.

(iv) $x^2 + 10x + 25y^2 = 0$ **[2018 Spring Short] [2018 Fall Short][2012 Fal[l]**

Solution: The given equation is,

$$x^2 + 10x + 25y^2 = 0.$$

$$\Rightarrow x^2 + 2(5)x + (5)^2 + (5y)^2 - 25 = 0$$

$$\Rightarrow (x+5)^2 + (5y)^2 = 25$$

$$\Rightarrow \frac{(x+5)^2}{25} + \frac{y^2}{1} = 1$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$$h = -5, \ k = 0, \ a^2 = 25 \Rightarrow a = 5 \text{ and } b^2 = 1 \Rightarrow b = 1.$$

Here, a > b. So, $c^2 = a^2 - b^2 = 25 - 1 = 24. \Rightarrow c = 2\sqrt{6}$

Therefore, Center of the ellipse is $C(h, k) \Rightarrow C(-5, 0)$

Since a > b. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to x–axis. So,

foci of the ellipse are at $F(h \pm c, k) = (-5 \pm 2\sqrt{6}, 0)$.

Vertices of the ellipse is at $V(h \pm a, k) = F(-5 \pm 5, 0)$.

$c = \sqrt{b^2 - a^2}$

ae

(v) $x^2 + 9y^2 - 4x + 18y + 4 = 0$

Solution: The given equation is,

$$x^2 + 9y^2 - 4x + 18y + 4 = 0$$

$$\Rightarrow (x)^2 - 2.2\,x + (2)^2 + 9\{y^2 + 2.1y + (1)^2\} + 4 - 4 - 1 = 0$$

$$\Rightarrow (x-2)^2 + 9(y+1)^2 = 1$$

$$\Rightarrow \frac{(x-2)^2}{1} + \frac{(y+1)^2}{1/9} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$$h = 2, \ k = -1, \ a^2 = 1 \Rightarrow a = 1 \text{ and } b^2 = \frac{1}{9} \Rightarrow b = \frac{1}{3}.$$

So, a > b, then $c = \sqrt{a^2 - b^2} = \sqrt{\frac{8}{9}} = 2\frac{\sqrt{2}}{3}$.

Therefore, Center of the ellipse is $C(h, k) = C(2, -1)$

Since a > b. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to x–axis. So,

foci of the ellipse are $F(h, k \pm c) = F(2, \pm 2\frac{\sqrt{2}}{3}, -1)$

Vertices is at $V(h, k \pm b) = V(2, \pm\frac{1}{3}, -1)$.

(vi) $4x^2 + y^2 - 16x + 4y + 16 = 0$

Solution: The given equation is,

$$4x^2 + y^2 - 16x + 4y + 16 = 0$$

$$\Rightarrow 4(x^2 - 4x + 4) + (y^2 + 4y + 4) - y = 0$$

$$\Rightarrow 4(x-1)^2 + (y+2)^2 = 4$$

$$\Rightarrow \frac{(x-2)^2}{1} + \frac{(y+2)^2}{4} = 1$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then,

$$h = 2, \ k = -2, \ a^2 = 1 \Rightarrow a = 1 \text{ and } b^2 = 4 \Rightarrow b = 2.$$

Here, b > a, so $c^2 = b^2 - a^2 = 4 - 1 \Rightarrow c = \sqrt{3}$.

Therefore, Center of the ellipse, $C(h, k) = C(2, -2)$

Since $b > a$. And, the major axis is parallel to y–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are $F(h, k \pm c) = F(2, -2 \pm \sqrt{5})$

Vertices of the ellipse $V(h, k \pm b) = V(2, -2 \pm 2)$.

(vii) $9x^2 + 16y^2 + 18x - 96y + 4 = 0$ [2017 Spring]

Solution: The given equation is,

$$9x^2 + 16y^2 + 18x - 96y + 4 = 0$$

$$\Rightarrow 9(x^2 + 2x + 1) + 16(y^2 - 2.y.3 + 3^2) = 144$$

$$\Rightarrow 9(x + 1)^2 + 16(y - 3)^2 = 144$$

$$\Rightarrow \frac{(x+1)^2}{16} + \frac{(y-3)^2}{9} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then,

$h = -1, k = 3, a^2 = 16$ and $b^2 = 9$.

Here, $a > b$. So, $c^2 = a^2 - b^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$.

Hence, centre, $C(h, k) = C(-1, 3)$

Since $a > b$. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are at $F(h \pm c, k) = F(-1 \pm \sqrt{7}, 3)$

Vertices of the ellipse are at $V(h \pm a, k) = V(-1 \pm 4, 3)$.

(viii) $16(x - 2)^2 + 9(y + 3)^2 = 144$ [2016 Fall]

Solution: The given equation is,

$$16(x - 2)^2 + 9(y + 3)^2 = 144$$

$$\Rightarrow \frac{(x-2)^2}{9} + \frac{(y+3)^2}{16} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then, we get

$h = 2, k = -3, a^2 = 9 \Rightarrow a = 3$ and $b^2 = 16 \Rightarrow b = 4$.

Here, $b > a$. So $c = \sqrt{b^2 - a^2} = \sqrt{4^2 - 3^2} = \sqrt{7}$.

Therefore, center of the ellipse, $C(h, h) = C(2, -3)$.

Since $b > a$. And, the major axis is parallel to y–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are at $F(h, k \pm c) = F(2, -3 \pm \sqrt{7})$.

Vertices of the ellipse are at $V(h, k \pm b) = V(2, -3 \pm 4)$.

4. Find the equation of an ellipse whose axes lies along the co-ordinate axes and which posses through (4, 3) and (–1, 4).

Solution: Given that the ellipse has axes along the coordinate axes. So, the centre of ellipse should be at origin.

Therefore, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots (1)$$

Since the given ellipse (1) passes through the point (4, 3). So,

$$\frac{16}{a^2} + \frac{9}{b^2} = 1$$

$$\Rightarrow \frac{1}{a^2} = \frac{1}{16}\left(1 - \frac{9}{b^2}\right) \quad \ldots (2)$$

Again the ellipse (1) passes through the point (–1, 4). So,

$$\frac{1}{a^2} + \frac{16}{b^2} = 1$$

$$\Rightarrow \frac{1}{16} - \frac{9}{16b^2} + \frac{16}{b^2} = 1 \quad \text{[using (2)]}$$

$$\Rightarrow \frac{1}{b^2}\left(\frac{-9}{16} + 16\right) = 1 - \frac{1}{16}$$

$$\Rightarrow b^2 = \frac{247}{15}$$

Therefore, (2) gives,

$$\frac{1}{a^2} = \frac{1}{16}\left(1 - \frac{9 \times 15}{247}\right) = \frac{112}{16 \times 247} = \frac{7}{247}$$

$$\Rightarrow a^2 = \frac{247}{7}.$$

Hence (1) becomes,

$$\frac{7x^2}{247} + \frac{15y^2}{247} = 1.$$

$$\Rightarrow 7x^2 + 15y^2 = 247.$$

5. Find the eccentricity and the co-ordinate of the foci of the ellipse $2x^2 + 3y^2 - 1 = 0$.

Solution: Given ellipse is

$$2x^2 + 3y^2 = 1$$

$$\Rightarrow \frac{x^2}{1/2} + \frac{y^2}{1/3} = 1 \quad \ldots (i)$$

a>b
c=√(a²-b²)
e=c/a
c=a·e
e=c/a

Comparing the equation (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, we get,

$h = 0, k = 0, a^2 = \frac{1}{2}$ and $b^2 = \frac{1}{3}$.

Here, $a > b$. So, $c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{2} - \frac{1}{3}} = \frac{1}{\sqrt{6}}$.

Therefore, eccentricity (e) $= \frac{c}{a} = \frac{\frac{1}{\sqrt{6}}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$.

Since, a > b, so the foci lie on the line parallel to x–axis. Therefore, foci of the ellipse is $F(h \pm c, k) = F\left(0 \pm \frac{1}{\sqrt{6}}, 0\right) = F\left(\pm\frac{1}{\sqrt{6}}, 0\right)$

6. **Show that $2x^2 + y^2 = 3x$ represents an ellipse; find its eccentricity and co–ordinates of foci.**

Solution: Given equation is,

$$2x^2 + y^2 = 3x$$

$$\Rightarrow 2\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) - \frac{9}{8} + y^2 = 0$$

$$\Rightarrow \frac{(x-3/4)^2}{1/2} + y^2 = \frac{9}{8}$$

$$\Rightarrow \frac{(x-3/4)^2}{9/16} + \frac{y^2}{9/8} = 1$$

$$h = \frac{3}{4}, k = 0, a^2 = \frac{9}{16}, b^2 = \frac{9}{8}$$

Here,

$$b > a. \text{ So, } c^2 = b^2 - a^2 = \frac{9}{8} - \frac{9}{16} = \frac{9}{16}$$

$$f(h, k \pm c) = \left(\frac{3}{4}, \frac{3}{4}\right)$$

$$e = \frac{c}{b} = \frac{9/16}{9/8} = \frac{1}{2}$$

F(0,3) x+7=0

7. **Find the equation of the ellipse whose focus, directrix and eccentricity are given as:**

a. **F(0, 3), x + 7 = 0 and $e = \frac{1}{3}$**

b. **F(–1, 1), x – y + 3 = 0 and $e = \frac{1}{2}$**

c. **F(2, 5), x + y = 1 and $e = \frac{2}{3}$**

Solution:

a. Given that the ellipse has focus F(0, 3), equation of directrix is x + 7 = 0 and eccentricity is $e = \frac{1}{3}$. Let P(x, y) be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

$$\Rightarrow \frac{1}{3} = \frac{\sqrt{(x-0)^2 + (y-3)^2}}{\pm\frac{x+7}{1}}$$

Squaring on both sides we get,

$$\Rightarrow \frac{1}{9}(x+7)^2 = x^2 + (y-3)^2$$

$$\Rightarrow x^2 + 14x + 49 = 9x^2 + 9(y^2 - 6y + 9)$$

$$\Rightarrow 8x^2 + 9y^2 - 14x - 54y + 32 = 0.$$

This is the equation of the required ellipse.

b. Given that the ellipse has focus F(–1, 1), equation of directrix is x – y + 3 = 0 and eccentricity is $e = \frac{1}{2}$. Let P(x, y) be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

$$\Rightarrow \frac{1}{2} = \frac{\sqrt{(x+1)^2 + (y-1)^2}}{\pm\frac{x-y+3}{\sqrt{2}}}$$

x+7=0

Squaring both sides, we get

$$\frac{1}{4} \times \frac{1}{2}(x-y+3)^2 = (x+1)^2 + (y-1)^2$$

$$\Rightarrow \frac{1}{8}(x^2 + y^2 + 6x - 6y - 2xy + 9) = (x^2 + 2x + 1) + (y^2 - 2y + 1)$$

$$\Rightarrow x^2 + y^2 + 6x - 6y - 2xy + 9 = 8x^2 + 16x + 8y^2 - 16y + 16.$$

$$\Rightarrow 7x^2 + 7y^2 + 10x - 10y + 2xy + 7 = 0.$$

This is the equation of the required ellipse.

c. Given that the ellipse has focus F(2, 5), equation of directrix is x + y – 1 = 0 and eccentricity is $e = \frac{2}{3}$. Let P(x, y) be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

$$\Rightarrow \frac{2}{3} = \frac{\sqrt{(x-2)^2 + (y-5)^2}}{\pm\frac{x+y-1}{\sqrt{2}}}$$

Squaring both sides, we get

$$\frac{4}{9} \times \frac{1}{2}(x+y-1)^2 = (x-2)^2 + (y-5)^2$$

$$\Rightarrow 2(x^2 + y^2 + 1 + 2xy - 2x - 2y) = 9[(x^2 - 4x + 4) + (y^2 - 10y + 25)]$$

$$\Rightarrow 2x^2 + 2y^2 + 2 + 4xy - 4x - 4y = 9x^2 - 36x + 225 + 36 + 9y^2 - 90y.$$

$$\Rightarrow 7x^2 + 7y^2 - 4xy - 32x - 86y + 259 = 0$$

This is the equation of required ellipse.

Find the equation of ellipse whose foci are at (–2, 4) and (4, 4); length of major axis is 10. Also, find the eccentricity.

Solution: Here, given two foci of an ellipse are (–2, 4) and (4, 4).

Since, the centre is the mid–point of foci, so

$$\text{Centre } (h, k) = \left(\frac{-2+4}{2}, \frac{4+4}{2}\right) = (1, 4).$$

(1, 4)

This implies, $h = 1$ and $k = 4$.

Since the foci have same y value therefore, the foci lie on the line parallel to x–axis. That is the major axis of the ellipse is the axis parallel to x–axis.

Therefore,

$$F(h + c, k) = F(4, 4).$$

That means, $h + c = 4 \Rightarrow c = 3$ $[\because h = 1.]$

Clearly the major axis of the ellipse is the axis. So,

major axis = 2a.

Given that length of major axis = 10.

Therefore,

$$2a = 10 \Rightarrow a = 5.$$

Then, $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$

Now, the equation of ellipse is,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \frac{(x-1)^2}{25} + \frac{(y-4)^2}{16} = 1$$

$$\Rightarrow 16x^2 - 32x + 16 + 25y^2 - 200y + 400 = 400$$

$$\Rightarrow 16x^2 + 25y^2 - 32x - 200y + 16 = 0$$

This is the equation of the required ellipse.

And, eccentricity $(e) = \frac{c}{a} = \frac{3}{5}$.

9. **Find the equation of ellipse referred to its axes as the axes of co-ordinates and foci along x–axis with latus rectum of length 4 and distance between foci is $4\sqrt{2}$.**

Solution: Since the ellipse whose axes are co–ordinates axis has centre at origin. So, the equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots \text{(i)}$$

$2c = 4\sqrt{2}$

Also, given that the distance between foci is $4\sqrt{2}$. Therefore,

$$2c = 4\sqrt{2} \Rightarrow c = 2\sqrt{2}.$$

Given that the foci lie along x–axis, so major axis is x–axis. Therefore, a > b.

Again given that the length of latus rectum is 4.

i.e. $\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a.$

Since a > b. So, we have,

$$c^2 = a^2 - b^2$$

$$\Rightarrow (2\sqrt{2})^2 = a^2 - (2a)$$

$$\Rightarrow a^2 - 2a - 8 = 0$$

$$\Rightarrow (a + 2)(a - 4) = 0$$

(h±c, k)

Since a can not measure in negative. So, $a - 4 = 0 \Rightarrow a = 4$.

Then, $b^2 = 2a = 8$.

Hence (i) becomes,

$$\frac{x^2}{16} + \frac{y^2}{8} = 1.$$

This is the equation of required ellipse.

10. **Find the equation of the ellipse having origin at centre, major axis as x–axis, latus rectum is 3 and eccentricity is $\frac{1}{\sqrt{2}}$.**

Solution: The equation of ellipse having origin at centre is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots \text{(i)}$$

a>b

Given that the major axis as x–axis, therefore, a > b.

And, length of latus rectum is 3. So, $\frac{2b^2}{a} = 3. \Rightarrow b^2 = \frac{3a}{2}$.

Also, being a > b, the eccentricity is,

$$e = \frac{c}{a} = \frac{1}{\sqrt{2}}. \Rightarrow c = \frac{a}{\sqrt{2}}.$$

Here, the major axis as x–axis, so we have $c^2 = a^2 - b^2$.

$$b^2 = a^2 - c^2$$

$$\Rightarrow \frac{3a}{2} = a^2 - \frac{a^2}{2} \Rightarrow \frac{3a}{2} = \frac{a^2}{2} \Rightarrow a = 3.$$

$c^2 = a^2 - b^2$

Therefore, $b^2 = \frac{3a}{2} = \frac{3 \times 3}{2} = \frac{9}{2}$.

Hence (i) becomes,

$$\frac{x^2}{9} + \frac{2y^2}{9} = 1.$$

This is the equation of the required ellipse.

Exercise 9.5

1. **Find the equation of tangent and normal at the point (4, 3) on the ellipse $3x^2 + 4y^2 = 84$.**

Solution: Given, ellipse is

$$3x^2 + 4y^2 = 84 \quad \dots \text{(i)}$$

Differentiating (i) w. r. t. x, we get

$$6x + 8y.\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-6x}{8y} = -\frac{3x}{4y}.$$

$\frac{dy}{dx} = m = -\frac{3 \times 4}{4 \times 3} = -1.$

Now, equation of tangent to (i) that passes through point (4, 3) is

$$y - 3 = -1(x - 4)$$

$\left[\because \text{eq}^n \text{ of normal at } (x_1, y_1) \text{ is } y - y_1 = \frac{-1}{m}(x - x_1)\right]$

$\Rightarrow x + y = 7.$

And equation of normal to (i) at (4, 3) is,

$y - 3 = -(-1)(x - 4)$. [$\because$ eq^n of tangent at (x_1, y_1) is $y - y_1 = m(x - x_1)$]

$\Rightarrow y - 3 = x - 4$

$\Rightarrow x - y = 1.$

2. Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the point of contact. [2006, Fall] [2008, Fall]

Solution: See theory part of ellipse before Ex. 9.4

3. Show that the line $y = x \pm \frac{\sqrt{5}}{\sqrt{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$. Find the point of contact.

Solution: Given line is, $y = x \pm \frac{\sqrt{5}}{\sqrt{6}}$ ... (i)

Comparing the equation (i) with $lx + my + n = 0$ we get,

$$l = 1, m = -1, n = \pm\frac{\sqrt{5}}{\sqrt{6}}$$

And, the equation of ellipse is, $2x^2 + 3y^2 = 1$

$$\Rightarrow \frac{x^2}{1/2} + \frac{y^2}{1/3} = 1 \quad \text{... (ii)}$$

Comparing eq^n. (ii) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$a^2 = \frac{1}{2} \text{ and } b^2 = \frac{1}{3}$$

For line (i) to be tangent on (ii) is

$$a^2l^2 + b^2m^2 = n^2$$

$$\Rightarrow \frac{1}{2} \times (1)^2 + \frac{1}{3} \times (-1)^2 = \frac{5}{6}$$

$$\Rightarrow \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \quad \Rightarrow \frac{5}{6} = \frac{5}{6} \quad \text{This is true.}$$

So, (i) is tangent to (ii).

Also, for point of contact,

$$x = -\frac{a^2 l}{n} = \frac{-\frac{1}{2} \cdot 1}{\pm\frac{\sqrt{5}}{\sqrt{6}}} = \pm\frac{\sqrt{6}}{2\sqrt{5}} \pm \sqrt{\frac{6}{20}} = \pm\sqrt{\frac{3}{10}}$$

$$y = -\frac{b^2 m}{n} = \frac{-\frac{1}{3} \cdot (-1)}{\pm\frac{\sqrt{5}}{\sqrt{6}}} = \pm\sqrt{\frac{6}{45}} = \pm\sqrt{\frac{2}{15}}$$

Hence, point of contact is $\left(\pm\sqrt{\frac{3}{10}}, \pm\sqrt{\frac{2}{15}}\right)$.

4. Find the equation of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$.

Solution: Given ellipse is

$$4x^2 + 3y^2 = 5 \quad \text{... (i)}$$

Since the tangent line to (i) is parallel to $y = 3x + 7$. So, the equation of tangent to (i) is,

$$3x - y + k = 0 \quad \text{... (ii)}$$

where k is a scalar.

Since (ii) is tangent on (i). So,

$$4x^2 + 3(3x + k)^2 = 5$$

$$\Rightarrow 4x^2 + 27x^2 + 18xk + 3k^2 = 5$$

$$\Rightarrow 31x^2 + 18k + 3k^2 - 5 = 0$$

which is quadratic in x and its discriminate term is zero, being (ii) is tangent to (i). That is,

$$(18k)^2 - 4(31)(3k^2 - 5) = 0$$

$$\Rightarrow 324k^2 - 372k^2 + 620 = 0$$

$$\Rightarrow 48k^2 = 620$$

$$\Rightarrow k^2 = \frac{620}{48} \Rightarrow k = \pm\sqrt{\frac{155}{12}}$$

Therefore, the equation of tangents are

$$3x - y \pm \sqrt{\frac{155}{12}} = 0.$$

5. Find the condition for the line $y = mx + c$ is tangent on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [2012 Fall][2011 Spring][2009, Fall][2002] [2005, Fall]

Solution: See the theory part, before 9.4.

6. Show that $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$ is normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Solution: Given ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow b^2x^2+a^2y^2=a^2b^2 \quad \ldots (i)$$

Differentiating (i) w.r.t. x, we get

$$\Rightarrow 2b^2x+2a^2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

So, slope of tangent on (i) at (x_1, y_1) is,

$$m=\frac{dy_1}{dx_1}=-\frac{b^2x_1}{a^2y_1}$$

Then, the slope of normal on (i) at (x_1, y_1) is $m=\frac{a^2y_1}{b^2x_1}$

Hence, the equation of normal to (i) at (x_1, y_1) is,

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$\Rightarrow \left(\frac{b^2}{y_1}\right)y-b^2=\left(\frac{a^2}{x_1}\right)x-a^2$$

$$\Rightarrow \frac{xa^2}{x_1}-\frac{yb^2}{y_1}=a^2-b^2.$$

This completes the requirement.

7. **Show that the line $3x+4y+\sqrt{7}=0$ touches the ellipse $3x^2+4y^2=1$. Also find the point of contact.**

Solution: Given line is

$$3x+4y+\sqrt{7}=0$$

$$\Rightarrow 4y=-3x-\sqrt{7} \Rightarrow y=-\frac{3x}{4}-\frac{\sqrt{7}}{4} \quad \ldots (i)$$

Comparing this equation with $y = mx + c$, we get

$$m=-\frac{3}{4} \quad \text{and} \quad c=-\frac{\sqrt{7}}{4}$$

Again, given equation of ellipse is

$$3x^2+4y^2=1$$

$$\Rightarrow \frac{x^2}{1/3}+\frac{y^2}{1/4}=1 \quad \ldots (ii)$$

Comparing the equation (ii) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a^2=\frac{1}{3}, \quad \text{and } b^2=\frac{1}{4}$$

We have the condition to touch the ellipse by the line is,

$$a^2m^2+b^2=c^2$$

$$\Rightarrow \frac{1}{3}\left(-\frac{3}{4}\right)^2+\frac{1}{4}=\left(-\frac{\sqrt{7}}{4}\right)^2$$

$$\Rightarrow \frac{1}{3}\times\frac{9}{16}+\frac{1}{4}=\frac{7}{16}$$

$$\Rightarrow \frac{3+4}{16}=\frac{7}{16} \Rightarrow \frac{7}{16}=\frac{7}{16}.$$ This is true.

This shows that (i) is tangent to (ii).

And, the point of contact is,

$$(x_1, y_1)=\left(-\frac{ma^2}{c}, \frac{b^2}{c}\right)=\left(\frac{-\frac{1}{3}\frac{3}{4}}{-\frac{\sqrt{7}}{4}}, \frac{\frac{1}{4}}{-\frac{\sqrt{7}}{4}}\right)=\left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$$

8. **Find the value of λ, when the straight line $y = x + \lambda$ touches the ellipse $2x^2+3y^2=6$.**

Solution: Given equation of ellipse is

$$2x^2+3y^2=6$$

$$\Rightarrow \frac{x^2}{3}+\frac{y^2}{2}=1 \quad \ldots (i)$$

Comparing eq^n. (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a^2=3 \quad \text{and} \quad b^2=2$$

Given straight line is $y = x + \lambda$ ... (ii)

Comparing with $y = mx + c$, we get

$$m = 1, \quad c = \lambda$$

For the line to touch the ellipse is,

$$a^2m^2+b^2=c^2$$

$$\Rightarrow 3.1+2=\lambda^2 \Rightarrow 5=\lambda^2 \Rightarrow \lambda=\pm\sqrt{5}.$$

9. **Find the equation of tangents to the ellipse $x^2 + 3y^2 = 3$, which are parallel to the line $4x - y + 8 = 0$. Also, find the point of contact.**

Solution: Given ellipse is,

$$x^2+3y^2=3 \quad \ldots (i)$$

And, the equation of line parallel to $4x - y + 8 = 0$ is

$$4x-y+k=0 \quad \ldots (ii)$$

where k is a constant.

Since (ii) is tangent on (i). So,

$$x^2+3(4x+k)^2=3$$

$$\Rightarrow x^2+48x^2+24xk+3k^2=3$$

$$\Rightarrow 49x^2+24k.x+(3k^2-3)=0 \quad \ldots (iii)$$

which is quadratic in x. Since (i) touches the line (ii). So, the discriminate term of (iii) should be equal to zero.

i.e. $(24k)^2=4.49(3k^2-3)$

$$\Rightarrow 576k^2-588k^2+588=0$$

$\Rightarrow 32k^2 = 588 \Rightarrow k^2 = 49 \Rightarrow k = \pm 7$

Therefore (ii) becomes

$4x - y \pm 7 = 0.$

For points of contact

a. When the line $4x - y + 7 = 0$ touches the ellipse $x^2 + 3y^2 = 3$ ... point of contact is,

$$x = -\frac{B}{2A} = \frac{-24k}{2\times 49} = -\frac{12}{7}$$

and, $$y = -4x + 7 = -\frac{48}{7} + 7 = \frac{1}{7}$$

Thus the point of contact be $\left(-\frac{12}{7}, \frac{1}{7}\right)$.

b. When the line $4x - y + 7 = 0$ touches the ellipse $x^2 + 3y^2 = 3$ ... point of contact is,

$$x = -\frac{B}{2A} = -\frac{24k}{2\times 49} = -\frac{24\times(-7)}{2\times 49} = \frac{12}{7}$$

and, $$y = 4x - 7 = \frac{48}{7} - 7 = -\frac{1}{7}$$

Thus the point of contact be $\left(\frac{12}{7}, -\frac{1}{7}\right)$.

OTHER IMPORTANT QUESTIONS FROM FINAL EXA[M] ELLIPSE

1. **Find the equation of an ellipse having a focus at (1, 0) and directri[x] the side is $x + y + 3 = 0$ and eccentricity is $\frac{3}{4}$.** [1999] [20..]

Solution: Let P(x, y) be a point on the ellipse which has a focus at F(1, 0), e= ...

and the equation of directrix is, $x + y + 3 = 0$.

By definition of eccentricity,

$$e = \frac{\text{length of PF}}{\text{perpendicular distance from P to the directrix}}$$

$$\Rightarrow \frac{3}{4} = \frac{\sqrt{(x-1)^2 + y^2}}{\pm\frac{x+y+3}{\sqrt{1+1}}} = \pm\frac{\sqrt{2}\sqrt{(x-1)^2+y^2}}{(x+y+3)}$$

$$\Rightarrow \frac{9}{16} = \frac{2[(x-1)^2 + y^2]}{(x+y+3)^2}$$

$$\Rightarrow 9(x^2 + y^2 + 9 + 2xy + 6x + 6y) = 32(x^2 + 1 - 2x + y^2)$$

$$\Rightarrow 23x^2 + 23y^2 - 18xy - 118x - 54y - 49 = 0$$

[T]his is equation of required ellipse.

Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [2008, Spring]

Solution: See the first part of Q.2, Exercise 9.5.

Define eccentricity of the conic section. Derive the standard equation of ellipse in its standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [2014 Spring][2003, Fall]

Solution: See definition of eccentricity.

See standard equation of ellipse.

Find the centre, vertices, foci, eccentricity and length of latus rectum of the ellipse $9x^2 + 16y^2 + 18x - 96y + 9 = 0$. [2003, Spring]

Solution: Given curve is

$$9x^2 + 16y^2 + 18x - 96y + 9 = 0$$

$$\Rightarrow 9(x^2 + 2x + 1) + 16(y^2 - 6y + 9) - 144 = 0$$

$$\Rightarrow \frac{(x+1)^2}{16} + \frac{(y-3)^2}{9} = 1$$

Comparing it with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ then

$a = 4, b = 3, h = -1, k = 3$

Here $a = 4 > 3 = b$. So, the foci lie on the line that is parallel to x–axis. So,

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

Now,

centre of ellipse, $C(h, k) = C(-1, 3)$

vertices of ellipse, $V(h \pm a, k) = V(-1 \pm 4, 3)$

foci of ellipse, $F(h \pm c, k) = F(-1 \pm \sqrt{7}, 3)$

eccentricity (e) $= \frac{c}{a} = \frac{\sqrt{7}}{4}$

and length of latus rectum $= \frac{2b^2}{a} = \frac{18}{4} = \frac{9}{2}$.

fo (h, k±c) =

5. **Show that $25x^2 + 9y^2 - 100x + 54y - 44 = 0$ represents an ellipse. Then, find its centre, and foci.** [2004, Spring]

Solution: Given curve is

$$25x^2 + 9y^2 - 100x + 54y - 44 = 0$$

$$\Rightarrow 25(x^2 - 4x + 4) + 9(y^2 + 6y + 9) - 44 - 100 - 81 = 0$$

$$\Rightarrow 25(x-1)^2 + 9(y+3)^2 - 225 = 0$$

$$\Rightarrow \frac{(x-2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Comparing it with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ then

$a = 3, b = 5, h = 2, k = -3$

This shows that the curve represents an ellipse.

c = b > a
b > c
c = b² − a²
centre (h, k) =

Here, $a = 3 < 5 = b$. So, the foci lie on the line that is parallel to y-axis. So,

$$c = \sqrt{b^2 - a^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

Now,

centre of ellipse $C(h, k) = C(2, -3)$

foci of ellipse $F(h, k \pm c) = F(2, -3 \pm 4)$.

6. Define conic section and classify them with respect to eccentricity. Obtain the equation of the tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [2005, Spring]

Solution: See definition of conic section.

See equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

7. Define eccentricity of a conic section and classify conic sections. Find the condition that the line $lx + my + n = 0$ may be a tangent to the ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [2006, Spring] [2009 Spring]

Solution: See definition part.

See the first part of Q2, Ex.9.5.

8. Find the center, vertices, eccentricity and foci of the ellipse, $9x^2 + 16y^2 + 18x - 96y + 9 = 0$. [2007, Spring]

Solution: See Q. 4, 2003 Spring.

9. Define conic section and derive the standard equation of ellipse. [2018 Fall][2015 Fall]

Solution: See definition of conic section.

See standard equation of ellipse.

10. Find center, foci, vertices of the conic section: $4x^2 + y^2 - 16x + 4y + 16 = 0$. Also, Sketch the conic section. [2013 Spring]

Solution: For problem part, see Exercise 9.4 Q. 3(vi)

With this information the sketch of the conic section is aside figure.

11. Find the equation of the tangents to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which is parallel to the line $x = y + 4$.

Solution: Given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow 9x^2 + 4x^2 = 36 \quad \ldots (i)$$

Since the tangent line to (i) is parallel to $x = y + 4$. So, the equation of tangent to (i) is,

$$x - y + k = 0 \quad \ldots (ii)$$

where k is a scalar.

Since (ii) is tangent on (i). So,

$$9x^2 + 4(x + k)^2 = 36$$

$$\Rightarrow 9x^2 + 4x^2 + 8xk + 4k^2 = 36$$

$$\Rightarrow 13x^2 + 8xk + 4k^2 - 36 = 0$$

which is quadratic in x and its discriminate term is zero, being (ii) is tangent to (i). That is,

$$(8k)^2 - 4(13)(4k^2 - 36) = 0$$

$$\Rightarrow 3k^2 - (13)(k^2 - 9) = 0 \quad \text{[Being } 16 \neq 0\text{]}$$

$$\Rightarrow 3k^2 - 13k^2 + 117 = 0$$

$$\Rightarrow 10k^2 = 117$$

$$\Rightarrow k = \pm\sqrt{\frac{117}{10}}$$

Therefore, the equation of tangents are $x - y \pm \sqrt{\frac{117}{10}} = 0$.

12. Obtain the vertices, center, coordinates of foci, eccentricity of the following ellipse $9x^2 + 4y^2 + 36x - 8y + 4 = 0$. 2011 Fall]

Solution: The given equation is,

$$9x^2 + 4y^2 + 36x - 8y + 4 = 0$$

$$\Rightarrow 9(x + 2)^2 + 4(y - 1)^2 - 36 - 4 + 4 = 0$$

$$\Rightarrow \frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1.$$

$a^2l^2 + b^2m^2 = n^2$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then,

$h = -2$, $k = 1$, $a^2 = 4$ and $b^2 = 9$.

Here, $a < b$. So, $c^2 = b^2 - a^2 = 9 - 4 = 5$.

Hence, centre, $C(h, k) = C(-2, 1)$

Since $a < b$. And, the major axis is parallel to y-axis. Therefore, the foci lie on the line that is parallel to y-axis. So,

Vertices of the ellipse are at $V(h, k \pm b) = V(-2, 1 \pm 3)$.

coordinate of centre of the ellipse are at $F(h, k) = F(-2, 1)$

coordinates of foci of the ellipse are at $F(h, k \pm c) = F(-2, 1 \pm \sqrt{5})$

eccentricity of the ellipse is, $e = \frac{c}{b} = \frac{\sqrt{5}}{3}$.

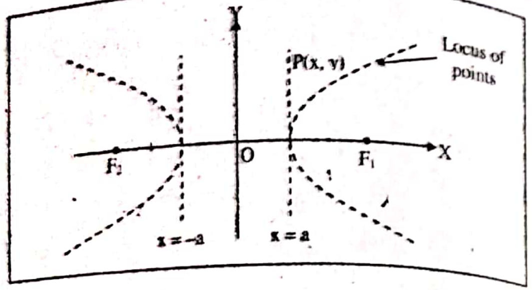
$x - y + y$

$x - y + k = 0$

Hyperbola

Definition:

A hyperbola is the locus of points in a plane whose distance from two fixed point in the plane have a constant difference. The fixed points are called foci of the hyperbola.



Standard Equation of Hyperbola having centre at (0, 0)

Derive the equation of the hyperbola in its standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

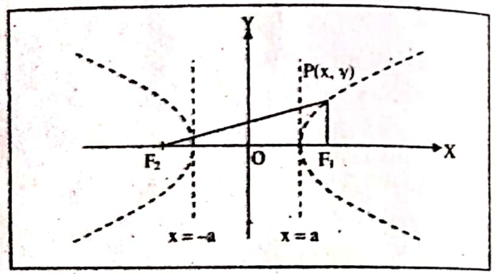
[2011 Spring][2009 Spring] [2009, Fall] [2002] [2006, Spring] [2004, Spring]

Let O(0, 0) be the centre of the hyperbola. Let $F_1(c, 0)$ and $F_2(-c, 0)$ be foci of the hyperbola and let P(x, y) be any point of the hyperbola. The difference of distance of foci form P is constant. That is,

$$PF_1 - PF_2 = 2a$$

i.e. $\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a$

$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$



Squaring both side we get,

$$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$\Rightarrow x^2 - 2xc + c^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2$

$\Rightarrow 4a\sqrt{(x+c)^2 + y^2} = -4(a^2 + xc)$

Again, squaring both sides, we get

$$a^2[(x+c)^2 + y^2] = a^4 + 2a^2xc + x^2c^2$$

$\Rightarrow a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2$

$\Rightarrow (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$ ... (i)

In any triangle, we have the sum of two sides is always greater than the third side. So, in figure in ΔPF_1F_2,

$$(PF_1 - PF_2) < F_1F_2$$

$\Rightarrow 2a < 2c \Rightarrow a < c \Rightarrow a^2 < c^2 \Rightarrow (c^2 - a^2) > 0.$

So, let, $b^2 = c^2 - a^2$

Thus equation (i) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the equation of hyperbola having center (0, 0) and foci at $(\pm c, 0)$, where $b^2 = c^2 - a^2$.

Note: Equation of hyperbola with centre at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Equation of tangent at (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

[2000]

The equation of hyperbola in standard form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{..... (i)}$$

At (x_1, y_1), the hyperbola is,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \text{..... (ii)}$$

Differentiating equation (i) w. r. t. x, we get

$$\frac{dy_1}{dx_1} = \frac{b^2x_1}{a^2y_1}$$

Since the equation of line that passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \quad \text{... (iii)}$$

where m be the slope of the line.

Since the point (x_1, y_1) be the common point of the line (iii) and the given ellipse (i). This means the line is a tangent to (i) at (x_1, y_1) and so, the slope m of (iii) is same as the slope $\frac{dy_1}{dx_1}$ of (i).

That is,

$$m = \frac{dy_1}{dx_1} = \frac{b^2x_1}{a^2y_1}$$

Therefore, the line (iii) becomes

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$

$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$ [Using (ii)]

Thus, the equation of tangent at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Note: Equation of normal at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1).$$

Condition for Tangency that a line $y = mx + c$ touches the given hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let the equation of the given line is

$$y = mx + c \quad \dots (i)$$

and the equation of the given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 - a^2y^2 = a^2b^2 \quad \dots (ii)$$

Eliminating y from equation (i) and (ii), we get

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$b^2x^2 - a^2(m^2x^2 + 2mxc + c^2) = a^2b^2$$

$$\Rightarrow b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow (b^2 - a^2m^2)x^2 - 2a^2mcx - (a^2c^2 + a^2b^2) = 0 \quad \dots (iii)$$

This is quadratic in x.

Since, (iii) be the common value of (i) and (ii). And, (i) is tangent on (ii). So, its discriminate term of (iii) should be equal to zero. So,

$$(-2a^2mc)^2 - 4(b^2 - a^2m^2)(-a^2c^2 - a^2b^2) = 0$$

$$\Rightarrow a^4m^2b^2 - a^2b^4 = a^2b^2c^2$$

$$\Rightarrow a^2m^2 - b^2 = c^2.$$

This is the required condition for tangency.

For point of contact,

Since the tangent to a curve touches the curve at a single point. So, for the point of contact of (i) and (ii), we observe the discriminate value of (iii) is zero. Therefore, from (iii)

$$x = -\frac{B}{2A} = -\frac{2a^2mc}{2(b^2 + a^2m^2)} = -\frac{2a^2mc}{c^2} = -\frac{a^2m}{c}$$

Then (i) gives,

$$y = mx + c = m\left(-\frac{ma^2}{c}\right) + c = -\frac{m^2a^2 - c^2}{c} = \frac{-b^2}{c} \quad \text{[Using (iii)]}$$

Thus, the point of contact of (i) and (ii) is $\left(-\frac{ma^2}{c}, \frac{-b^2}{c}\right)$.

Show that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2l^2 - b^2m^2 = n^2$. [2011 Fall]

OR

Find the condition the line $lx + my + n = 0$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [2018 Fall][2015 Fall]

Solution: Given equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (i)$$

Given line is

$$lx + my + n = 0$$

$$\Rightarrow y = -\frac{(lx + n)}{m} \quad \dots (ii)$$

Eliminating y from equation (i) and (ii), then

$$\frac{x^2}{a^2} - \frac{\left(\frac{-lx + n}{m}\right)^2}{b^2} = 1$$

$$\Rightarrow b^2m^2x^2 - a^2l^2x^2 - 2a^2lxn - a^2n^2 = a^2m^2b^2$$

$$\Rightarrow (b^2m^2 - a^2l^2)x^2 - 2a^2lnx - a^2n^2 - a^2m^2b^2 = 0 \quad \dots (iii)$$

If equation (ii) is tangent on (i), then the discriminate term of (iii) should be zero.

$$\text{i.e. } (-2a^2ln)^2 + 4(b^2m^2 - a^2l^2)(a^2n^2 + a^2m^2b^2) = 0$$

$$\Rightarrow a^4l^2n^2 + a^2b^2m^2n^2 + a^2b^4m^4 - a^4n^2l^2 - a^4m^2l^2b^2 = 0$$

$$\Rightarrow n^2 + b^2m^2 - a^2l^2 = 0$$

$$\Rightarrow -b^2m^2 + a^2l^2 = n^2 \quad [\because a^2b^2m^2 \neq 0]$$

This shows that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when $a^2l^2 - b^2m^2 = n^2$.

And, for point of contact,

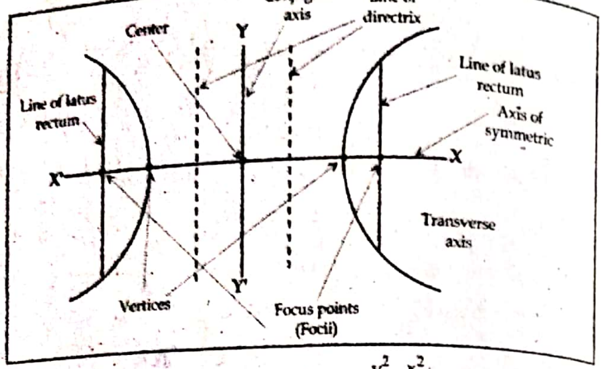
$$x = -\frac{B}{2A} = -\frac{2a^2ln}{2(m^2b^2 - a^2l^2)} = -\frac{2a^2ln}{-2n^2} = \frac{a^2l}{n}.$$

Then (i) gives,

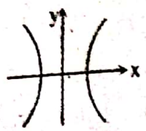
$$y = -\frac{lx - n}{m} = -\frac{l\left(\frac{a^2l}{n}\right) - n}{m} \Rightarrow y = -\frac{a^2l^2 - n^2}{mn} = \frac{b^2m^2}{mn} = \frac{b^2m}{n}.$$

Thus the point of contact is $\left(\frac{a^2l}{n}, \frac{b^2m}{n}\right)$.

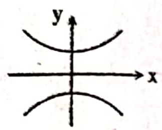
Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$



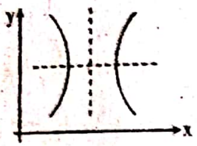
A. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



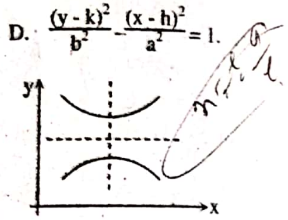
B. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



C. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$



D. $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$



Eq^n. of hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Center	(0, 0)	(0, 0)	(h, k)	(h, k)
Vertex	$(\pm a, 0)$	$(0, \pm b)$	$(h \pm a, k)$	$(h, k \pm b)$
Transverse axis	2a	2b	2a	2b
Conjugate axis	2b	2a	2b	2a
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Eccentricity (e)	$\sqrt{1+\frac{b^2}{a^2}}$	$\sqrt{1+\frac{a^2}{b^2}}$	$\sqrt{1+\frac{b^2}{a^2}}$	$\sqrt{1+\frac{a^2}{b^2}}$
Focus	$(\pm ae, 0)$	$(0, \pm be)$	$(h \pm ae; k)$	$(h, k \pm be)$
Eq^n. of directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
Line of transverse	y = 0	x = 0	y = k	x = h
Line of symmetric	x = 0, y = 0	x = 0, y = 0	x = h, y = k	x = h, y = k

If the centre of hyperbola is at (h, k) and transverse axis is parallel to x-axis then equation of hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$e = \frac{c}{a}$

In this case,

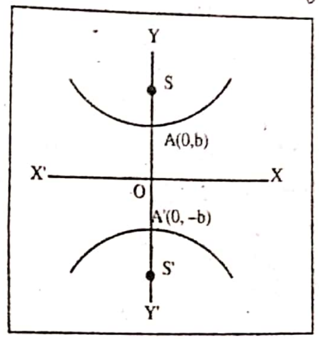
(a) vertex $(h \pm a, k)$
(b) foci $(h + ae, k)$
(c) length of transverse axis = 2a
(d) length of conjugate axis = 2b
(e) eccentricity $e = \sqrt{\frac{a^2+b^2}{a^2}}$
(f) equation of directrix $x = h \pm \frac{a}{e}$
(g) length of latus rectum = $\frac{2b^2}{a}$

If the transverse axis is parallel to y-axis and centre is at origin then;

2b

2a

(a) equation of hyperbola: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
(b) vertex $(0, \pm b)$ y
(c) transverse axis = 2b
(d) conjugate axis = 2a
(e) equation of directrix: $y = \pm\frac{b}{e}$
(f) foci $(0, \pm be)$
(g) eccentricity $e = \sqrt{\frac{b^2+a^2}{b^2}}$
(h) latus rectum: $\frac{2a^2}{b}$



Exercise 9.6

1. **Sketch each of the following hyperbolas:**

(i) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (ii) $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (iii) $\frac{x^2}{9} - \frac{y^2}{16} = -1$

Solution:

(i) Given hyperbola is,

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \ldots(i)$$

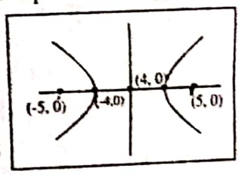
$(0, \pm be)$

Comparing the equation (i) with the standard equation of hyperbola, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ then we get,

$$a = 4, b = 3.$$

Thus, a > b. Then

$$c = \sqrt{a^2 + b^2} = \sqrt{25} = 5$$



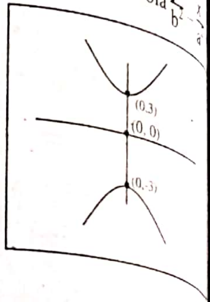
Here, centre (h, k) = (0, 0)
foci are $(\pm c, 0) = (\pm 5, 0)$
vertices are $(\pm a, 0) = (\pm 4, 0)$.

(ii) Given hyperbola is

$$\frac{y^2}{9}-\frac{x^2}{16}=1 \quad \dots (i)$$

Comparing the equation (i) with standard equation of hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}$ then we get,

$$a = 3, b = 4.$$



Thus, a < b. Then

$$c=\sqrt{a^2+b^2}=\sqrt{25}=5$$

Now, centre (h, k) = (0, 0)

foci are $(0, \pm c) = (0, \pm 5)$

vertices are $(0, \pm b) = (0, \pm 3)$.

(iii) Given hyperbola is

$$\frac{x^2}{9}-\frac{y^2}{16}=-1 \Rightarrow \frac{y^2}{9}-\frac{x^2}{16}=1$$

The question is reduced to (ii).

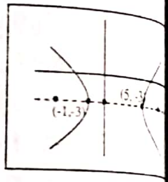
2. Find the center, vertices, foci, eccentricity of the following hyper... and sketch curve

(i) $4(x-2)^2-9(y+3)^2=36$. **[2017 Spring Short][2016 Fall Sh...**

Solution: Here, the given equation of hyperbola is

$$4(x-2)^2-9(y+5)^2=36$$

$$\Rightarrow \frac{(x-2)^2}{9}-\frac{(y+3)^2}{4}=1 \quad \dots (i)$$



Comparing the equation (i) with the equation of hyperbola

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,

$$h = 2, k = 3, a = 3 \text{ and } b = 2.$$

Here, a > b. Then,

$$c=\sqrt{a^2+b^2}=\sqrt{13}.$$

Now, centre (h, k) = (2, –3); foci $(h \pm c, k) = (2 \pm \sqrt{13}, -3)$;

vertices $(h \pm a, k) = (2 \pm 3, -3)$; eccentricity (e) $=\frac{c}{a}=\frac{\sqrt{13}}{3}$.

(ii) $5x^2-4y^2+20x+8y=4$

Solution: Given equation of hyperbola is

$$5x^2-4y^2+20x+8y=4$$

$$\Rightarrow 5(x^2+4x+4)-4(y^2-2y+1)=20+4-4$$

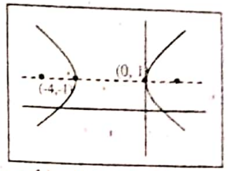
$$\Rightarrow 5(x+2)^2-4(y-1)^2=20$$

$$\Rightarrow \frac{(x+2)^2}{4}-\frac{(y-1)^2}{5}=1 \quad \dots (i)$$

Comparing the equation (i) with the equation of hyperbola

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,

$$h=-2, k=1, a=\sqrt{4}=2, b=\sqrt{5}.$$



Here, a > b. Then,

$$c=\sqrt{a^2+b^2}=\sqrt{5+3}=3.$$

Then, centre (h, k) = (–2, 1); foci $(h \pm c, k) = (-2 \pm 3, 1)$;

vertices $(h \pm a, k) = (-2 \pm 2, 1)$; eccentricity (e) $=\frac{c}{a}=\frac{3}{2}$.

Q. $4y^2 = x^2 - 4x$ -2±1,1) 0,1) 4,1) **[2013 Fall]**

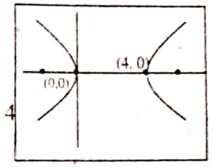
Solution: Given equation of hyperbola is

$$4y^2 = x^2 - 4x$$

$$\Rightarrow x^2-4x-4y^2=0$$

$$\Rightarrow x^2-4x+4-4y^2=4 \quad \Rightarrow (x-2)^2-4(y^2)=4$$

$$\Rightarrow \frac{(x-2)^2}{4}-\frac{y^2}{1}=1.$$



Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,

$$h=2, k=0, a=2, b=1.$$

Here, a > b. Then,

$$c=\sqrt{a^2+b^2}=\sqrt{4+1}=\sqrt{5}.$$

Now, centre (h, k) = (2, 0); foci $(h \pm c, k) = (2 \pm \sqrt{5}, 0)$;

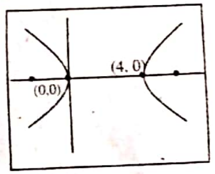
vertices $(h \pm a, k) = (2 \pm \sqrt{5}, 0)$; eccentricity, $e=\frac{c}{a}=\frac{\sqrt{5}}{2}$.

(iii) $x^2-y^2-2x+4y=4$ **[2015 Spring Short]**

Solution: Given equation of conic is

$$x^2-y^2-2x+4y=4$$

$$\Rightarrow (x-1)^2-(y-2)^2=1$$



Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get.

$$h=1, k=2, a=1, b=1.$$

Here, a = b. Then,

$$c=\sqrt{a^2+b^2}=\sqrt{1+1}=\sqrt{2}.$$

Now, centre (h, k) = (1, 2); foci $(h \pm c, k) = (1 \pm \sqrt{2}, 2)$;

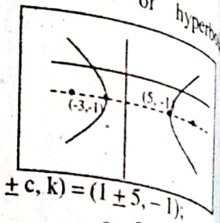
vertices $(h \pm a, k) = (1 \pm 1, 2)$; eccentricity, $e=\frac{c}{a}=\frac{\sqrt{2}}{1}=\sqrt{2}$.

(iv) $9x^2-16y^2-18x-32y-151=0$

Solution: Given equation of hyperbola is

$$9x^2-16y^2-18x-32y-151=0$$
$$\Rightarrow 9(x^2-2x+1)-16(y^2+2y+1)=151-16+9$$
$$\Rightarrow 9(x-1)^2-16(y+1)^2=144$$
$$\Rightarrow \frac{(x-1)^2}{4^2}-\frac{(y+1)^2}{3^2}=1$$

Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,



$$h=1,\ k=-1,\ a=4,\ b=3.$$

Here, a > b. Then,

$$c=\sqrt{a^2+b^2}=\sqrt{16+9}=5.$$

Now, centre $(h, k)=(1,-1)$; foci $(h\pm c, k)=(1\pm 5,-1)$;

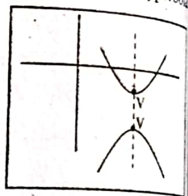
vertices $(h\pm a, k)=(1\pm 4,-1)$; eccentricity $(e)=\frac{c}{a}=\frac{5}{4}$.

(v) $4(y+3)^2-9(x-2)^2=1$

Solution: Given equation of hyperbola is

$$4(y+3)^2-9(x-2)^2=1$$
$$\Rightarrow \frac{(y+3)^2}{1/4}-\frac{(x-2)^2}{1/9}=1. \quad \ldots (i)$$

Comparing the equation (i) with the equation of hyperbola $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$, we get



$$h=-3,\ k=2,\ b^2=\frac{1}{4}\ \text{and}\ a^2=\frac{1}{9}.$$

Then, $c=\sqrt{a^2+b^2}=\sqrt{\frac{1}{4}+\frac{1}{9}}=\frac{\sqrt{13}}{6}$

Hence, Centre $(h, k)=(2,-3)$ Vertices $(h, k\pm b)=(2,-3\pm\frac{1}{2})$

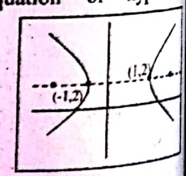
Foci $(h, k\pm c)=(2,-3\pm\frac{\sqrt{13}}{6})$ Eccentricity $(e)=\frac{c}{b}=\frac{\sqrt{13/6}}{1/2}=\frac{\sqrt{13}}{3}$

(vi) $4x^2=y^2-4y+8$

Solution: Given equation of hyperbola is

$$4x^2=y^2-4y+8 \Rightarrow 4x^2-(y-2)^2=4$$
$$\Rightarrow \frac{x^2}{1^2}-\frac{(y-2)^2}{2^2}=1 \quad \ldots (i)$$

Comparing the equation (i) with the equation of hyperbola $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$, we get



$$h=0,\ k=2,\ a=1,\ b=2.$$
$$c=\sqrt{a^2+b^2}=\sqrt{5}.$$

Now, centre $(h, k)=(0, 1)$ vertices $(h\pm a, k)=(\pm 1, 2)$

foci $(h\pm a, k)=(\pm\sqrt{5}, 2)$ eccentricity $(e)=\frac{c}{a}=\frac{\sqrt{5}}{1}=\sqrt{5}$

3. **Find the equation of the straight lines which are tangents both to the parabola $y^2=8x$ and the hyperbola $3x^2-y^2=3$.**

Solution: Here, equation of parabola is

$$y^2=8x \quad \ldots (i)$$

Comparing (i) with $y^2=4ax$ then we get

$$a=2$$

Therefore, the equation of tangent to the parabola (i) is

$$y=mx+\frac{2}{m} \quad \ldots (ii)$$

And, given that the equation of hyperbola is

$$3x^2-y^2=3 \Rightarrow \frac{x^2}{1}-\frac{y^2}{3}=1 \quad \ldots (iii)$$

Comparing equation (iii) with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $a^2=1$, $b^2=3$.

Now, equation of tangent to hyperbola (iii) is

$$y=mx\pm\sqrt{m^2-3} \quad \ldots (iv) \quad [\because c=\sqrt{a^2m^2-b^2}]$$

According to question, the tangent to (i) is also a tangent is (iii). This means the line (ii) and (iv) are identical, so

$$mx+\frac{2}{m}=mx\pm\sqrt{m^2-3}$$
$$\Rightarrow \frac{2}{m}=\pm\sqrt{m^2-3}$$
$$\Rightarrow \frac{4}{m^2}=m^2-3 \quad \text{[Squaring both sides.]}$$
$$\Rightarrow 4=m^4-3m^2$$
$$\Rightarrow m^4-3m^2-4=0$$
$$\Rightarrow m^4-4m^2+m^2-4=0$$
$$\Rightarrow m^2(m^2-4)+1(m^2-4)=0$$
$$\Rightarrow (m^2+1)(m^2-4)=0.$$

i.e. either $m^2-4=0$ or $m^2+1=0$

$\Rightarrow m=\pm 2$, otherwise m gives imaginary values.

So, putting the value of m in equation (ii) then we get,

$y=2x+1$

and $y=-2x-1 \Rightarrow y+2x+1=0$

Thus, $y=2x+2$ and $y+2x+1=0$ are equation of tangent lines.

4. **Find the equation of hyperbola whose focus, directrix and eccentricity respectively are**

(a) **(2, 1), $x + 2y = 1$, $e = \sqrt{2}$**

Solution: Given that he equation of directrix of the hyperbola is

$$x + 2y = 1 \qquad \dots (i)$$

Also given that the focus of the hyperbola is F(2, 1) and eccentricity is $e = \sqrt{2}$. Let P(x, y) be any point on the locus of the hyperbola. Then we know the eccentricity is the ratio of the distance between hyperbola and focus point and perpendicular distance from hyperbola to its directrix. That is,

$$e = \left|\frac{\text{distance form P to F}}{\text{Perpendicular distance from P to } x + 2y = 1}\right|$$

i.e. (e) (Perpendicular distance from P to x +2y = 1) = distance form P to F.

$$\Rightarrow \sqrt{2}\left|\frac{x + 2y - 1}{\sqrt{1^2 + 2^2}}\right| = \sqrt{(x-2)^2 + (y-1)^2}$$

$$\Rightarrow 2(x + 2y - 1)^2 = 5[(x-2)^2 + (y-1)^2]$$

$$\Rightarrow 2x^2 + 8y^2 + 2 + 8xy - 4x - 8y = 5x^2 - 20x + 20 + 5y^2 - 10y + 5$$

$$\Rightarrow 3x^2 - 3y^2 - 8xy - 16x - 2y + 3 = 0.$$

(b) **(6, 0), $4x - 3y = 6$, $e = \frac{5}{4}$**

(c) **(0, 4), $y + 3 = 0$, $e = \frac{4}{3}$**

Solution: Similar to (a)

5. **Find equation of hyperbola with center origin, conjugate axis is 3 and distance between two foci is 5.**

Solution: Let the transverse axis (i.e. conjugate axis) be y–axis.

Hence, equation of hyperbola with centre at origin is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \dots (i)$$

Given that the length of conjugate axis is 3. That is,

$$2b = 3 \Rightarrow b = \frac{3}{2}.$$

2b=3

And, distance between two foci is 5. That is,

$$2c = 5 \quad \Rightarrow \quad c = \frac{5}{2}$$

2c=5

Then, $\quad a^2 = c^2 - b^2 = \frac{25}{4} - \frac{9}{4} = 4$

So, (i) becomes

$$\frac{x^2}{4} - \frac{y^2}{9/4} = 1 \quad \Rightarrow \quad 9x^2 - 16y^2 = 36.$$

Therefore, $9x^2 - 16y^2 = 36$ is the equation of hyperbola.

6. **Find the equation of hyperbola whose foci (4, 2), (8, 2) and eccentricity 2.**

Solution: Given foci of the hyperbola are (4, 2), (8, 2).

Since, centre is the mid–point of foci, so

$$C(h, k) = \left(\frac{4+8}{2}, \frac{2+2}{2}\right) = (6, 2).$$

This gives, $h = 6, k = 2.$

And, the distance between two foci is 4. That is,

$$2c = |4 - 8| = 4 \quad \Rightarrow c = 2.$$

And, given that the eccentricity is,

$$e = 2.$$

The hyperbola has foci with fixed y value. So, the transverse axis is parallel to x–axis. So,

$$e = \frac{c}{a} \Rightarrow 2 = \frac{2}{a} \quad \Rightarrow a = 1.$$

Therefore, $\quad b = \sqrt{c^2 - a^2} = \sqrt{3}$

Hence, the equation of the hyperbola is,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

i.e. $\quad \frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1.$

7. **Find the equation of the hyperbola with vertices are at $(0, \pm 6)$ and eccentricity is $e = \frac{5}{3}$. Also, find its foci.**

Solution: Given, vertices of the hyperbola are $V(0, \pm 6)$.

Since the centre is the mid–point of the vertices. Therefore, the centre of the hyperbola is,

$$C(h, k) = C\left(\frac{0+0}{2}, \frac{6-6}{2}\right) = C(0, 0).$$

That is, $\quad h = 0, k = 0.$

Here x–coordinate of the vertices are fixed. So, the transverse axis is parallel to y–axis.

Therefore,

$$V(h, k \pm b) = V(0, 0 \pm 6)$$

This implies $\pm b = \pm 6 \Rightarrow b = 6.$ $\qquad [\because h = k = 0]$

Also, given that eccentricity is,

$$e = \frac{5}{3} \Rightarrow \frac{c}{b} = \frac{5}{3} \Rightarrow \frac{5}{3} = \frac{c}{6} \Rightarrow c = 10.$$

e = 5/3

Then, $\quad a = \sqrt{c^2 - b^2} = \sqrt{64} = 8.$

Now, the equation of hyperbola is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

i.e. $\frac{(y-0)^2}{36} - \frac{(x-0)^2}{64} = 1 \Rightarrow \frac{x^2}{64} - \frac{y^2}{36} + 1 = 0.$

And, the foci of the hyperbola are $F(h, k \pm c) = F(0, \pm 10)$.

8. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = ...$ Find the equation of hyperbola having eccentricity 2.

Solution: Given ellipse is

$$\frac{x^2}{26} + \frac{y^2}{9} = 1 \quad \dots (i)$$

Comparing equation (i), with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ then we get

$$a = 5, \quad b = 3, \quad h = 0 \text{ and } k = 0.$$

Then, centre $C(h, k) = C(0, 0)$.

In the given ellipse (i) we get $a > b$. So, the foci lie on the line parallel to x-axis. So, the foci of the ellipse is $F(h \pm c, k) = F(\pm 5, 0)$.

Then, $c = 5$.

Given that the foci of required hyperbola coincide with the foci of (i). So the foci of the hyperbola are

$$F(\pm 5, 0) = F(h \pm c, k).$$

This implies $h = 0, k = 0$ and $c = 5$.

Clearly these foci have fixed y value. So, the transverse axis of the hyperbola is the axis parallel to x-axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \dots (ii)$$

and its eccentricity is

$$e = \frac{c}{a} \quad \dots (iii)$$

Given that the eccentricity of the hyperbola is 2. That is,

$$e = \frac{c}{a}. \Rightarrow 2 = \frac{5}{a} \Rightarrow a = \frac{5}{2} \Rightarrow a^2 = \frac{25}{4}.$$

Then, $b^2 = c^2 - a^2 = 25 - \frac{25}{4} = \frac{75}{4}$.

Now the equation (ii) becomes

$$\frac{4x^2}{25} - \frac{4y^2}{75} = 1$$

$$\Rightarrow 12x^2 - 4y^2 = 75$$

This is equation of the hyperbola.

9. Show that the line $y = x + 2$ touches to the hyperbola $5x^2 - 9y^2 = 45$. Find the point of contact.

Solution: Given that the equation of hyperbola is,

$$5x^2 - 9y^2 = 45 \quad \dots (i)$$

And the line is, $y = x + 2 \quad \dots (ii)$

Eliminating y from equation (i) and (ii) then

$$5x^2 - 9(x+2)^2 = 45$$

$$\Rightarrow 5x^2 - 9x^2 - 36x - 36 = 45$$

$$\Rightarrow 4x^2 + 36x + 81 = 0$$

$$\Rightarrow x = \frac{-36 \pm \sqrt{(36)^2 - 4(4)(81)}}{8} = \frac{-36}{8} = \frac{-9}{2}.$$

Then (ii) gives

$$y = \frac{-9}{2} + 2 = \frac{-5}{2}.$$

This shows (ii) touches (i) at a single point $\left(-\frac{9}{2}, -\frac{5}{2}\right)$. This means (ii) is tangent to (i) and the point of contact between them is $\left(-\frac{9}{2}, -\frac{5}{2}\right)$.

10. Find the equation of hyperbola with center origin and passing through (2, 1) and (4, 3).

Solution: Given that the center of hyperbola is (0, 0).

Let the equation of hyperbola having center at origin is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (i)$$

Given that the hyperbola (ii) is passing through points (2, 1) and (4, 3). Therefore,

$$\frac{4}{a^2} - \frac{1}{b^2} = 1 \quad \dots (ii) \quad \text{and} \quad \frac{16}{a^2} - \frac{9}{b^2} = 1 \quad \dots (iii)$$

Solving equation (ii) and (iii) then we get,

$$a^2 = \frac{5}{2} \quad \text{and} \quad b^2 = \frac{5}{8}.$$

Therefore (i) becomes,

$$\frac{2x^2}{5} - \frac{8y^2}{5} = 1 \Rightarrow 2x^2 - 8y^2 = 5.$$

11. Find the value of λ, when the line $y = 2x + \lambda$ is tangent to the hyperbola $3x^2 - y^2 = 3$.

Solution: Given hyperbola is,

$$3x^2 - y^2 = 3 \quad \dots (i)$$

and given line is

$$y = 2x + \lambda \quad \dots (ii)$$

Eliminating y from equation (i) and (ii), we get

$$3x^2 - (2x + \lambda)^2 = 3$$

$$\Rightarrow 3x^2 - 4x^2 - 4xs\lambda - \lambda^2 = 0$$

$$\Rightarrow -x^2 - 4\lambda x - \lambda^2 = 3$$

$$\Rightarrow x^2 + 4\lambda x + \lambda^2 + 3 = 0 \quad \dots (iii)$$

which is quadratic in x. Since, the equation (ii) is tangent to (i), so discriminate value of (iii), is zero. That is,

$$(4\lambda)^2 - 4(1)(\lambda^2 + 3) = 0$$

$$\Rightarrow 16\lambda^2 - 4\lambda^2 - 12 = 0 \Rightarrow 12\lambda^2 = 12 \Rightarrow \lambda = \pm 1.$$

Thus for $\lambda = \pm 1$, the line (ii) is tangent to the hyperbola (i).

12. Find the equation of the tangents to the hyperbola $3x^2 - 4y^2 = 12$, which are perpendicular to the line $y = x + 2$. Also, find the point of contact. [2017 Spring]

Solution: Given that the equation of line is,

$$y = x + 2 \quad \dots (i)$$

Comparing it with the line $y = mx + c$ then we get

$$m = 1, c = 2.$$

Since we have the equation to tangent on hyperbola is,

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad \dots (ii)$$

By given condition the line (i) is perpendicular to (ii). So, by condition of normality we get,

$$m_1 . m_2 = -1 \Rightarrow m_1 = -\frac{1}{1} = -1.$$

Also, given hyperbola is,

$$3x^2 - 4y^2 = 12$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{3} = 1 \quad \dots (iii)$$

Comparing it with the standard equation of hyperbola then we get

$$a^2 = 4, \ b^2 = 3, h = 0, k = 0.$$

Then, the equation (ii) becomes,

$$y = -1(x) \pm \sqrt{4 . 1 - 3}$$

$$\Rightarrow y = -x \pm 1. \quad \dots (iv)$$

Therefore, $x + y = \pm 1$ are the equation of required tangents to (iii).

For point of contact of $x + y \pm 1 = 0$ and curve $3x^2 - 4y^2 = 12$ is,

$$x = -\frac{B}{2A} = 4 \qquad x = -\frac{B}{2A} = -4$$

So, $y = -3$ $\qquad y = 3$

This, the point of contact (4, –3) and (–4, 3).

13. Show that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2l^2 - b^2m^2 = n^2$. [2016 Spring][2011 Fall]

OR

Find the condition the line $lx + my + n = 0$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [2015 Fall]

: See theory part of hyperbola before Ex. 9.6.

14. Let e and e_1 are the eccentricities of a hyperbola and its conjugate, show that $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.

Solution: Let e and e_1 are the eccentricities of a hyperbola and its conjugate, so

$$e = \frac{c}{a} \text{ and } e_1 = \frac{c}{b}.$$

Now,

$$\frac{1}{e^2} + \frac{1}{e_1^2} = \frac{1}{(c/a)^2} + \frac{1}{(c/b)^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1.$$

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

1. Find centre, vertices, foci and eccentricity of the hyperbola $9(x - 2)^2 - 4(y + 3)^2 = 36$. [1999, 2001]

Solution: Given equation is

$$9(x - 2)^2 - 4(y + 3)^2 = 36$$

$$\Rightarrow \frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{9} = 1$$

Comparing it with $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ then we get,

$$h = 2, k = -3, a = 2, b = 3$$

Then, $c^2 = a^2 + b^2 = 4 + 9 \Rightarrow c = \sqrt{13}$.

Now the center of hyperbola is C(h, k) = C(2, 3)

vertices, $V(h \pm a, k) = V(2 \pm 2, -3)$.

foci are $F(h \pm c, k) = F(2 \pm \sqrt{13}, -3)$

eccentricity is, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$.

2. Find the center, vertices, foci and eccentricity of the hyperbola $4x^2 - 5y^2 - 16x + 10y + 31 = 0$. [2000]

Solution: Given that,

$$4x^2 - 5y^2 - 16x + 10y + 31 = 0$$

$$\Rightarrow 4(x^2 - 4x + 4) - 5(y^2 - 2y + 1) - 16 + 5 + 31 = 0$$

$$\Rightarrow 4(x - 2)^2 - 5(y - 1)^2 + 20 = 0$$

$$\Rightarrow \frac{(y - 1)^2}{4} - \frac{(x - 2)^2}{5} = 1$$

Comparing it with $\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1$ then we get

$$h = 2, k = 1, b = 2, a = \sqrt{5}$$

Here, y has positive value. So, the foci lie on the line parallel to y-axis.

And $c = \sqrt{a^2 + b^2} = \sqrt{5 + 16} = \sqrt{21}$

Now,

Centre of hyperbola, C(h, k) = C(2, 1)

Vertices of hyperbola, $V(h, k \pm b) = V(2, 1 \pm 2)$

foci of hyperbola, $F(h, k \pm c) = F(2, 1 \pm \sqrt{21})$

and eccentricity$(e) = \frac{2a^2}{b} = \frac{10}{2} = 5.$

3. Find the centre, vertices, foci and eccentricity of the hyperb[ola] $4y^2 = x^2 - 4x$. [2003, Fa[ll]]

Solution: Given curve is

$$4y^2 = x^2 - 4$$
$$\Rightarrow x^2 - 4x + 4 - 4y^2 = 4$$
$$\Rightarrow \frac{(x-2)^2}{4} - \frac{y^2}{1} = 1$$

Comparing it with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ then we get

$a = 2, b = 1, h = 2, k = 0$

Here, x has positive value. So, the foci lie on the line parallel to x-axis.

And, $c = \sqrt{a^2 + b^2} = \sqrt{4+1} = \sqrt{5}$.

Now,

Centre of hyperbola, $C(h, k) = C(2, 0)$

Vertices of hyperbola, $V(h \pm a, k) = V(2 \pm 2, 0)$

Foci of hyperbola, $F(h \pm c, k) = F(2 \pm \sqrt{5}, 0)$ $\quad e = \frac{c}{a} = \frac{\sqrt{5}}{2}$

Eccentricity of hyperbola, $e = \frac{2b^2}{a} = \frac{2}{2} = 1.$

4. Define conic section and classify them. Find the condition that the lin[e] $y = mx + c$ may be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [2004, Fall]

Solution: First Part: See the definition of conic section.

Second Part: See theory part of hyperbola before Ex. 9.6

5. Define eccentricity of a conic section, and derive the equation of [a] hyperbola in its standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [2016 Fall][2015 Spring]

[2014 Fall][2012 Fall][2005 Fall]

Solution: See definition of eccentricity.

See standard equation of hyperbola.

6. Find the condition that the line $y = mx + c$ may be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [2005, Spring]

Solution: See theory part of hyperbola before Ex. 9.6.

7. Define hyperbola and derive its standard equation in the form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [2018 Spring][2007, Spring]

[So]lution: See definition of hyperbola.

See standard equation of hyperbola.

Define conic section with different eccentricities. Derive the standard equation of hyperbola. [2008, Spring]

[So]lution: See definition of conic section as their eccentricity.

See standard equation of hyperbola.

Find the equation of a conic section with focus at (2, 0), directrix $x = 4$ and [ec]centricity $e = 1$.

2013 Spring

[So]lution

[G]iven that $e = 1$. So, the conic section is a parabola. Also, given that focus is at [(2,] 0) and equation of directrix of the parabola is $x = 4$. Then, the vertex of the [pa]rabola is

$$v(h, k) = V\left(\frac{2+a}{2}, 0\right) = V(3, 0)$$

[An]d $a = \frac{2-4}{2} = -1$

[No]w, the equation of parabola is,

$$(y-k)^2 = 4a(x-h)$$
$$(y-0)^2 = 4(-1)(x-3)$$
$$y^2 = -4x + 12$$
$$4x + y^2 - 12 = 0$$

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check!!

Chapter 10

CURVE SKETCHING

Guidelines for Sketching a Curve

To sketch the curve of given function y = f(x) by hand, following informatio about curve must know:

A. **Domain: Find the domain of the given function**, it means value of x so th f(x) is exist.

B. **Intercepts:** It means where curve meet x–axis and y–axis. For x–interce (i.e. it meet x–axis) put y = 0 and find value of x and for y intercept (i.e. it meet y–axis) put x = 0 and find value of y.

C. **Symmetry:** For curve symmetric about y–axis, f(x) = f(–x). For cur symmetric about origin, f(–x) = – f(x).

D. **Asymptote:** [In general rational function and logarithmic and exponenti functions has asymptote].

Horizontal asymptotes: If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$ the y = L is horizontal asymptote of curve y = f(x).

Vertical asymptotes: A line x = a is vertical asymptote of y = f(x) if eithe

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

Note that graph of asymptotes must be dashed lines.

E. **Interval of increasing and decreasing:** Find the critical points by usin f'(x) = 0 and f'(x) = ∞ and find the interval of increasing an decreasing.

F. **Local maxima and minima points.** Find the points where local maximu and minimum value of f(x) occurs. This is obtained from tabl formed from E. Note that local maximum point is occure at th point where sign is changed from + to – ve and local minimum obtained when sign is changed from –ve to +ve.

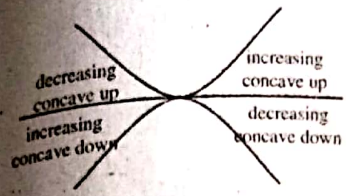
G. **Concavity and point of inflection:** By computing f"(x) = 0 and f"(x) = find the interval of concave up and concave down and find th point of inflection. Note that point where concavity change (changes from +ve to –ve or –ve to +ve), is point of inflection.

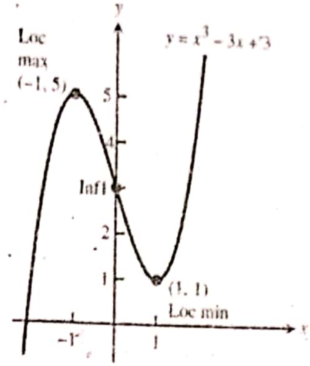
H. **Using the above information A–G.** Sketch the graph by free hand.

Example 1: Sketch the graph of $f(x) = x^3 - 3x + 3$.

Solution: Given curve is,

$$f(x) = x^3 - 3x + 3$$
$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$





For critical points;

$$f'(x) = 0.$$
$$\Rightarrow \quad 3(x-1)(x+1) = 0.$$
$$\Rightarrow \quad x = 1, -1.$$

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of f'	+ve	–ve	+ve
Nature of f	Increasing	decreasing	Increasing

Thus, the maxima occur at $x = -1$, and $(-1, f(-1)) = (-1, 5)$ is maximum point. And minima occur at $x = 1$ i.e. $(1, f(1)) = (1, 1)$ is minimum point.

Again, $f''(x) = 6x$.

For point of inflection,

$$f''(x) = 0 \quad \Rightarrow \quad 6x = 0$$
$$\Rightarrow \quad x = 0.$$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of f"	–ve	+ve
Nature of f	Concave down	Concave up

Thus, point of inflection occur at $x = 0$. So $(0, f(0)) = (0, 3)$ is the point of inflection.

Summarizing above two tables,

$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Increasing	Decreasing	Decreasing	Increasing
Concave down	Concave down	Concave up	Concave up

Example 2: Sketch the graph of $f(x) = x^{4/3} - 4x^{1/3}$

Solution: Given curve is,

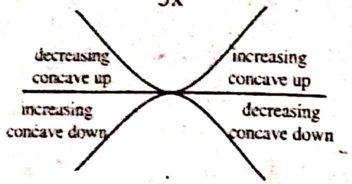
$$f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$$

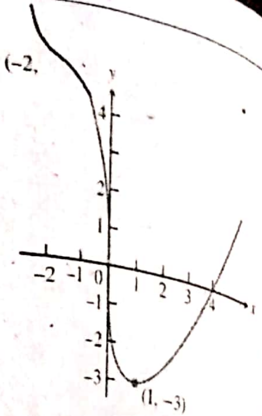
So, $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$

$$= \frac{4}{3}x^{1/3} \cdot x^{-2/3} \cdot x^{2/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}[x-1]$$

$$= \frac{4(x-1)}{3x^{2/3}}$$





For critical point,

$$f'(x) = 0 \text{ and } f'(x) = \infty.$$

Therefore, the critical points are $x = 0, 1$.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'	–ve	–ve	+ve
Nature of f	Decreasing	Decreasing	Increasing

This shows the minima occur at $x = 1$. So, the minima point is $(1, f(1)) = (1, -3)$.

Again,

$$f''(x) = \frac{4}{9} \times x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-2/3}x^{-5/3} \cdot x^{5/3} + \frac{8}{9}x^{-5/3}$$

$$= \frac{4}{9}x^{-5/3}(x+2)$$

$$= \frac{4(x+2)}{9x^{5/3}}$$

For point of inflection,

$$f''(x) = 0 \text{ and } f''(x) = \infty$$

which gives $x = -2$ and $x = 0$.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
Sign of f''	+ve	–ve	+ve
Nature of f	Concave up	Concave down	Concave up

Here, point of inflections are at $x = -2$ and at $x = 0$

$\therefore$ Point of inflections are (0, 0) and (–2, 7.56)

Summarizing above tables

$(-\infty, -2)$	$(-2, 0)$	$(0, 1)$	$(1, \infty)$
Decreasing	Decreasing	Decreasing	Increasing
Concave up	Concave down	Concave up	Concave up

Example 3: Use the guidelines to sketch the curve $y = \frac{2x^2}{x^2 - 1}$.

Solution:

A. For domain, set of all real number except 1 and –1.
i.e. domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

B. For the x–intercept: Put y = 0 we get x = 0.
For the y–intercept: Put x = 0 we get y = 0.
Thus curve meet the x–axis and y–axis at origin.

C. Since $f(-x) = f(x)$, so it is symmetrical about x–axis.

D. To find asymptotes;

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2.$$

Thus y = 2 is vertical asymtote.

Here $\lim_{x \to 1} f(x) = \infty$ and $\lim_{x \to -1} f(x) = \infty$.

Thus x = +1 and x = –1 are. Horizontal asymptote.

E. For interval of increasing and decreasing,

$$f'(x) = \frac{(x^2 + 1) 4x - 2x^2 \times 2x}{(x^2 + 1)^2} = -\frac{4x}{(x^2 - 1)^2}$$

For critical points $f'(x) = 0$ and $f'(x) = \infty$

We get x = 0 and x = $\pm$ 1

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'(x)	+ve	+ve	–ve	–ve
Nature of f(x)	Increasing	Increasing	Decreasing	Decreasing

F. Sign is changing from +ve to –ve at point x = 0, so
Maxima occur at x = 0, so maximum point is (0, y(0)), i.e. (0, 0).

G. For concavity and point of inflection.

$$f''(x) = \frac{(x^2 - 1)^2 \times -4 + 4x\, 2(x^2 - 1) \times 2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

For point of inflection f''(x) = 0 and f''(x) = ∞, we get $x = \pm 1$.

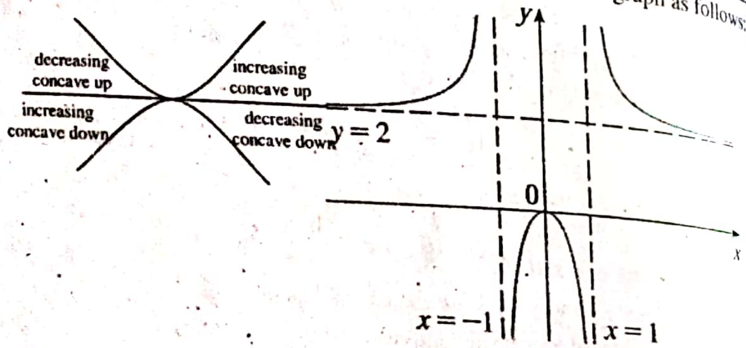
Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of f'(x)	+ve	–ve	+ve
Nature of f(x)	concave up	concave down	concave up

Even signs are changes at x = 1 and x = –1, but they are not point of inflection because they are not lies on domain.

H. Summarizing above two table in E and G.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Nature of f(x)	Increasing	Increasing	Decreasing	Decreasing
f(x)	Concave up	Concave down	Concave down	Concave up

Using this table with information we can sketch the graph as follows:



Example 4: Sketch the graph of $f(x) = \frac{x^2}{\sqrt{x+1}}$ by using the guidelines.

Solution:

A. Domain for the function is $x + 1 \geq 0$ i.e. $x \geq -1$

So domain is $[-1, \infty)$.

B. Intercepts: The x–intercept is x = 0 (Put y = 0 in equation $y = \frac{x^2}{\sqrt{x^2+1}}$)

y–intercept is y = 0 (Put x = 0 in equation $y = \frac{x^2}{\sqrt{x^2+1}}$)

∴ Curve meet x–axis and y–axis at only one point (0, 0).

C. Symmetry: It is not symmetrical about y–axis and origin.

(Here $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$).

D. Asymptote:

Since $\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^3}+\frac{1}{x^4}}} = \infty$ (not finite)

∴ There is no horizontal asymptote.

Since $\lim_{x\to 1^{+}} + \frac{x^2}{\sqrt{x+1}} = \infty$. Thus x = – 1 is vertical asymptote.

E. Interval of increasing and decreasing:

$$f'(x) = \frac{2x\sqrt{x+1} - x^2 \cdot \frac{1}{2\sqrt{x+1}}}{(x+1)} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

Let f '(x) = 0 then x = 0 and $x = -\frac{4}{3}$ (This lies outside of domain so no need to take)

f '(x) = ∞ then x = – 1.

Thus, x = 0, –1.

Interval	$(-1, 0)$	$(0, \infty)$
Sign of f'(x)	–ve	+ve
Nature of f(x)	Decreasing	Increasing

F. Minima occure at x = 0. So minimum point is (0, f(0)) i.e. (0, 0).

G. Concavity and point of inflection:

$$f''(x) = \frac{2(x+1)^{3/2} \cdot (6x+4) - x(3x+4) \cdot 3(x+1)^{1/2}}{4(x+1)^3}$$

$$= \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$$

Let f ''(x) = 0 then $3x^2 + 8x + 8 = 0$ gives not real number ($b^2 - 4ac = -32$)

f ''(x) = ∞ then x = – 1

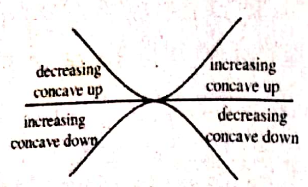
Thus for x = – 1

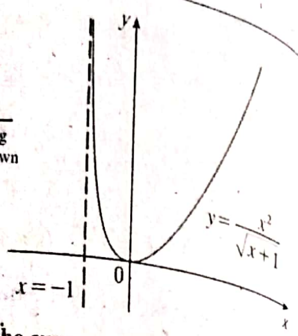
Interval	$(-1, \infty)$
Sign of f''(x)	+ve
Nature of f(x)	Concave upward

H. Summarizing tables in E and G

Interval	$(-1, 0)$	$(0, \infty)$
Nature of	Decreasing	Increasing
f(x)	Concave	Concave

Using this table with information can sketch the graph as follows:





Example 5: Use the guideline to sketch the curve $y = \ln(4 - x^2)$.

Solution:

A. Domain, $4 - x^2 > 0$ i.e. $x^2 - 4 < 0 \Rightarrow (x - 2)(x + 2) < 0$

$\therefore$ Domain is $(-2, 2)$

B. Intercepts: For x–intercept; Put $y = 0$ then

$$\ln(4 - x^2) = 0 \Rightarrow 4 - x^2 = 1 \Rightarrow x = \pm\sqrt{3}$$

So curve meet x–axis at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$.

For y–intercept, put $x = 0$, we get $y = \ln 4$.

So curve meet y–axis at $(0, \ln 4)$.

C. Symmetry: Here $f(-x) = f(x)$, so curve is symmetrical about y-axis.

D. For vertical asymptote;

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \text{ and } \lim_{x \to -2^+} \ln(4 - x^2) = -\infty.$$

Thus $x = -2$ and $x = 2$ are vertical asymptotes.

There is not horizontal asymptotes because $\lim_{x \to \pm\infty} \ln(4 - x^2) = \infty$.

E. For interval of increasing and decreasing

$$f'(x) = \frac{-2x}{4 - x^2}$$

For $f'(x) = 0$ then $x = 0$ and $f'(x) = \infty$ then $x = \pm 2$

So, $x = 0, 2$ and -2.

Interval	(–2, 0)	(0, 2)
Sign of f'(x)	+ve	–ve
Nature of f(x)	Increasing	Decreasing

F. Maximum point is at $x = 0$, so maximum point is $(0, f(0))$

So, $(0, \ln 4)$ is maximum point.

G. For interval of concavity and point of inflection

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

$f''(x) = 0$ then x not real.

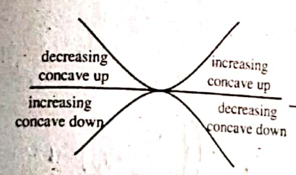
$f''(x) = \infty$ then $x = \pm 2$.

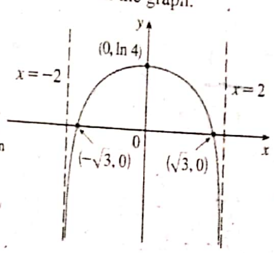
Interval	(–2, 2)
Sign of f''(x)	–ve
Nature of f(x)	Concave down

H: Summarizing the table on E and G

Interval	(–2, 0)	(0, 0)
Nature of	Increasing	Decreasing
f(x)	Concave down	Concave down

Using this table with information we can sketch the graph:





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Chapter 11

INTEGRAL CALCULUS

Table of Standard Results

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, for $(n \neq 1)$
2. $\int \frac{1}{x} dx = \log|x| + C$
3. $\int dx = x + C$
4. $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$
5. $\int a^x dx = \frac{a^x}{\log_e a} + C$
6. $\int \sin x\, dx = -\cos x + C$
7. $\int \cos x\, dx = \sin x + C$
8. $\int \sec^2 x\, dx = \tan x + C$
9. $\int \mathrm{cosec}^2 x\, dx = -\cot x + C$
10. $\int \sec x \tan x\, dx = \sec x + C$
11. $\int \mathrm{cosec}\, x \cot x\, dx = -\mathrm{cosec}\, x + C$
12. $\int \sinh x\, dx = \cosh x + C$
13. $\int \cosh x\, dx = \sinh x + C$
14. $\int \mathrm{sech}^2 x\, dx = \tanh x + C$
15. $\int \mathrm{cosech}^2 x\, dx = -\coth x + C$
16. $\int \mathrm{sech}\, x \tanh x\, dx = -\mathrm{sech}\, x + C$
17. $\int \mathrm{cosech}\, x \coth x\, dx = -\mathrm{cosech}\, x + C$
18. $\int \tan x\, dx = \log|\sec x| + C$
19. $\int \cot x\, dx = \log|\sin x| + C$
20. $\int \mathrm{cosec}\, dx = \log\left|\tan\frac{x}{2}\right| + C = \log|\mathrm{cosec}\, x - \cot x| + C$
21. $\int \sec x dx = \log|\sec x + \tan x| + C$
22. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C \quad (x > a)$
24. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C. \quad (x < a)$
25. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$
26. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C.$
27. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$
28. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
29. $\int \sqrt{x^2+a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + C$
$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + C$

30. $\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + C$
$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + C$
31. $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
32. $\int e^{ax}\sin bx\, dx = e^{ax}\left[\frac{a\sin bx - b\cos bx}{a^2+b^2}\right] + C$
33. $\int e^{ax}\cos bx\, dx = e^{ax}\left[\frac{a\cos bx + b\sin bx}{a^2+b^2}\right] + C$

TECHNIQUE OF SOME INTEGRALS

A. Integrals of type $\int \sin^m x . \cos^n x\, dx$ with m, n are positive integers and,

(i) if m is even and n is odd then put $\sin x = t$ (a new variable), so that the integrand reduces to in algebraic form.

(ii) if m is odd and n is even then put $\cos x = t$.

(iii) if m and n both are even then either use simple trigonometric formulae or reduction formula.

(iv) if m and n both are odd then put either $\sin x = t$ or $\cos x = t$ and then process for integration

B. If the exponential function 'e' has power different from x then put t = (power of e). And then process to integrate.

Example: If we have,

$$I = \int \left(x e^{x^2}\right) dx \quad \text{and} \quad I = \int \left(\frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\right) dx$$

then put $x^2 = t$ (in first integral) and $\sin^{-1} x = t$ (in second integral) and then process to integrate.

C. If the integrand is trigonometric function with its angle different from x then we put, t = (angle of the trigonometric function)

Example: If $I = \int e^x \sec^2(e^x)\, dx$ then put $t = e^x$

If $I = \int \left(\frac{\tan(\sqrt{x})}{\sqrt{x}}\right) dx$ then put $t = \sqrt{x}$.

D. If the integrand consists the terms as $a^2 - x^2$, $a^2 + x^2$ or $x^2 + a^2$, $x^2 - a^2$ then,

(i) If the integrand is of type $a^2 - x^2$ then put $x = a\sin\theta$

(ii) If the integrand is of type $a^2 + x^2$ or $x^2 + a^2$ then put $x = a\tan\theta$

(iii) If the integrand is of type $x^2 - a^2$ then put $x = a\sec\theta$.

E. If the integral of type $\int (f(x))^n f'(x)\, dx$ where f(x) is any function put,

f(x) = t (new variable).

$$I = \frac{1}{p}\int \frac{-\frac{1}{t^2}}{\frac{1}{t}\sqrt{\frac{a}{p^2}\left(\frac{1}{t}-q\right)^2+\frac{b}{p}\left(\frac{1}{t}-q\right)+c}}\,dt$$

$$= -\int \frac{dt}{t\sqrt{\frac{a}{t^2}(1-2t+qt^2)+\frac{bp}{t}(1-qt)+cp}}$$

$$= -\int \frac{dt}{\sqrt{t^2(aq-bpq+cp)+(-2a+bp)\,t+a}}$$

which is same form as in (G).

N. The integrals of type $I = \int \frac{dx}{a+b\cos x}$ or $I = \int \frac{dx}{a+b\sin x}$

Or $I = \int \frac{dx}{a+b\sin x+c\cos x}$

then put $\tan\left(\frac{x}{2}\right) = t$ so that,

$$dx = \frac{2dt}{1+y^2},\quad \sin x = \frac{2y}{1+y^2} \quad \text{and} \quad \cos x = \frac{1-y^2}{1+y^2}$$

and process to solve.

[*such method is developed by a German mathematician Karl Weierstrass*].

O. If the integrals is of type $I = \int \frac{dx}{a\sin x+b\cos x}$

then put $a = r\cos\theta$, $b = r\sin\theta$ then,

$I = \int \frac{dx}{r.\sin(x+\theta)}$ and integrate it.

> **Remember:** In such type choose the value of a and b so that the denominator should become as $r(\sin x\cos\theta+\cos x\sin\theta)$.
> **OR** The problem can be solved by using $a = r\sin\theta$, $b = r\cos\theta$.

P. If the integrals of type

$$I = \int\left(\frac{p\sin x+q}{a\sin x+b\cos x+c}\right)dx \quad \text{or } I = \int\left(\frac{p\cos x+q}{a\sin x+b\cos x+c}\right)dx$$

or $$I = \int\left(\frac{p\sin x+q\cos x+r}{a\sin x+b\cos x+c}\right)dx$$

then put,

$$\text{numerator} = A\frac{d}{dx}(\text{denominator}) + B\,.\,\text{denominator}$$

with A and B are constants.

Then the integral becomes,

$$I = A\cdot\int\frac{d(\text{denominator})}{\text{denominator}}\,dx + B\int dx$$

$= A\log(\text{denominator}) + B.x. + c$(integrating constant).

Example See Exercise 11.2, B(8).

Q. The integrals of type

$$I = \int\frac{dx}{a+b\cos^2 x} \quad \text{or} \quad I = \int\frac{dx}{a+b\sin^2 x}$$

or $$I = \int\frac{dx}{a\cos^2 x+b\sin^2 x} \quad \text{or} \quad I = \int\frac{dx}{(a\cos x+b\sin x)^2}$$

then divide the numerator and denominator by $\cos^2 x$ and then put $\tan x = t$. And process to solve.

R. The integral of type $I = \int\frac{dx}{\sin x\,(1+\cos x)}$

then multiply numerator and denominator by sinx and then put $\cos x = t$. So that the integrad reduces to the algebraic form which can be evaluated by resolving into partial fractions.

Example:

$$I = \int\frac{dx}{\sin x(1+\cos x)} = \int\frac{\sin x\,.\,dx}{\sin^2 x\,(1+\cos x)}$$

Put $\cos x = t$ then,

$$I = \int\frac{-dt}{(1-t^2)(1+t)}$$

which can be solved by using partial fractions.

Note: If $I = \int\frac{dx}{\cos x\,(1+\sin x)}$ then multiply the numerator and denominator by cosx and then put $\sin x = t$.

SOME FORMULAE

A. Integrating by parts

If the integrand consist two different types of functions then we use the formula.

If the integrand of the form $\int f_1(x)\,.\,f_2(x)$ then,

$$\int f_1(x)\,.\,f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int\left(\frac{d\,f_1(x)'}{dx}\int f_2(x)\,dx\right)dx$$

Among different functions, normally we use the rule ILATE to choose the first and second functions as its order where,

I = Inverse circular function — L = Logarithmic function

A = Algebraic function — T = Trigonometric function

E = Exponential function

Note: Some time the integrand consists only a single function as $\int \log(x)$, however it is not possible to integrate by simple process. In such case multiply the integrand by 1 so that the integrand becomes a multiple of functions given function and a constant function (i.e. 1). Then we apply integration by parts with choosing the constant as second function.

Thus, normally we use the rule ILATEC.

Generalization of the rule:

$$\int f.\,g\,dx = f\,g' - f_1 g'' + f_2 g''' - f_3 g^{iv} + f_4 g^{v} + \ldots$$

where f is the first function and g is the second function. Also, the suffix denotes for derivative and 'prime of g (i.e. power of g)' denotes number of times of integration of g.

B. $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$

C. Integration by Partial Function

Case of proper rational fractions

a. Linear and non–repeated denominator:

For each factor of the form (ax + b), there should be a single fraction of the form $\frac{A}{ax+b}$, where A is a constant.

Example: If $I = \int \frac{x\,dx}{(x+3)(x+1)}$

Let $\frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$ for A, B are constants.

$\Rightarrow \quad x = A(x+1) + B(x+3) = (A+B)x + (A+3B)$

equating the coefficient of x and the constant term then we get,

$A + B = 1,\ A + 3B = 0$

Solving we get,

$A = \frac{3}{4}, \quad B = \frac{1}{3}$

Then,

$$I = \frac{3}{4}\int \frac{dx}{x+3} + \frac{1}{3}\int \frac{dx}{x+1}$$

$$= \frac{3}{4}\log(x+3) + \frac{1}{3}\log(x+1) + C$$

b. Linear and repeated denominator:

For each factor of the form $(ax + b)^n$, there should be a single fraction as $\frac{A_n}{(ax+b)^n}$ for n = 1, 2, ...

i.e. $\frac{f(x)}{(ax+b)^n} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$

c. Quadratic and non–repeated denominator:

For each factor of the form $ax^2 + bx + c$ (with $a \neq 0, c \neq 0$), there should be a single fraction of the form,

$\frac{Ax+B}{ax^2+bx+c}$, with A and B are constants.

d. Quadratic and repeated denominator

For each factor of the form $(ax + bx + c)^n$ there should be a single fraction of the form

$$\frac{f(x)}{(ax^2+bx+c)^n} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

D. Case of Improper Rational Fraction

If the integrand functions has numerator and denominator with degree of num^r ≥ degree of deno^r then the function is called improper function.

If the given rational fraction is in improper form then process as

given fraction = Quotient + $\frac{\text{Remainder}}{\text{divisor}}$ (i.e. proper)

where the form $\frac{\text{Remainder}}{\text{divisor}}$ is a proper fraction.

So that, $\int$ given fraction = $\int$ quotient + $\int$ proper fraction.

Exercise 11.1

A. Evaluate the following integrals

1. $\int \left(\frac{x^3+5x^2-3}{x+2}\right) dx$

Solution: Here,

$$\int \left(\frac{x^3+5x^2-3}{x+2}\right) dx$$

$$= \int \left[x^2 + 3x - 6 + \frac{9}{x+2}\right] dx$$

$$= \int x^2\,dx + 3\int x\,dx - 6\int dx + 9\int \frac{dx}{x+2}$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} - 6x + 9\log|x+2| + C.$$

2. $\int \left(t + \frac{1}{t}\right)^2 dt$

Solution: Here,

$$\int \left(t+\frac{1}{t}\right)^2 dt = \int t^2\,dt + 2\int dt + \int t^{-2}\,dt$$

$$= \frac{t^3}{3} + 2t + \frac{t^{-1}}{-1} + c = \frac{t^3}{3} + 2t - \frac{1}{t} + C.$$

3. $\int \left(6\operatorname{cosec}^2 x - \frac{1}{x^2}\right) dx$

Solution: Here,

$$\int \left(6\operatorname{cosec}^2 x - \frac{1}{x^2}\right) dx$$

$$= 6\int \operatorname{cosec}^2 x\, dx - \int x^{-2}\, dx$$

$$= -6\cot x - \frac{x^{-1}}{-1} + C \quad = -6\cot x + \frac{1}{x} + C.$$

4. $\int \left(\frac{x^4 + x^2 + 1}{x+1}\right) dx$

Solution: Here,

$$\int \left(\frac{x^4 + x^2 + 1}{x+1}\right) dx$$

$$= \int \left(x^3 - x^2 + 2x - 2 + \frac{3}{x+1}\right) dx$$

$$= \int x^3\, dx - \int x^2\, dx + 2\int x\, dx - 2\int dx + 3\int \frac{dx}{x+1}$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{2x^2}{2} - 2x + 3\log|x+1| + C$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + x^2 - 2x + 3\log|x+1| + C$$

5. $\int (\tan^2 x - 3x^2)\, dx$

Solution: Here,

$$\int (\tan^2 x - 3x^2)\, dx$$

$$= \int (\sec^2 x - 1)\, dx - 3\int x^2\, dx$$

$$= \int \sec^2 x\, dx - \int dx - 3\int x^2\, dx$$

$$= \tan x - x - \frac{3x^3}{3} + C$$

$$= \tan x - x - x^3 + C.$$

6. $\int \frac{\cos 2x\, dx}{\cos^2 x \sin^2 x}$

Solution: Here,

$$\int \frac{\cos 2x\, dx}{\cos^2 x \sin^2 x} dx$$

$$= \int \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x \,.\, \sin^2 x}\right) dx$$

$$= \int \frac{\cos^2 x}{\cos^2 x \,.\, \sin^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x \,.\, \sin^2 x} dx$$

$$= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x}$$

$$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx$$

$$= -\cot x - \tan x + C.$$

7. $\int \sin^2 x \cos^2 x\, dx$

Solution: Here,

$$\int \sin^2 x \cos^2 x\, dx$$

$$= \frac{1}{4}\int (2\sin x \cos x)^2\, dx$$

$$= \frac{1}{4}\int \sin^2 2x\, dx$$

$$= \frac{1}{4}\int \left(\frac{1 - \cos 4x}{2}\right) dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x\, dx \quad = \frac{1}{8}.x - \frac{1}{8}.\frac{\sin 4x}{4} + C$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

$$= \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$$

B. **Evaluate the following integrals:**

1. $\int \left(\frac{16x}{8x^2 + 2}\right) dx$

Solution: Here,

$$I = \int \left(\frac{16x}{8x^2 + 2}\right) dx$$

Put $8x^2 + 2 = y$ then $16x.dx = dy$. Therefore,

$$I = \int \frac{dy}{y} = \log|y| + c \quad = \log|8x^2 + 2| + C.$$

2. $\int \frac{\cos x}{\sqrt{1 + \sin x}} dx$

Solution: Here,

$$I = \int \frac{\cos x}{\sqrt{1 + \sin x}} dx$$

Put, $1 + \sin x = y$ then $\cos x.\, dx = dy$

Therefore,

$$I = \int \frac{dy}{\sqrt{y}} = \int y^{-1/2}\,dy = \frac{y^{-1/2+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{y^{1/2}}{\frac{1}{2}} + C$$

$$= 2\sqrt{y} + C$$

$$= 2\sqrt{1+\sin x} + C.$$

3. $\int (x.e^{x^2})\,dx$

Solution: Here,

$$I = \int (x\,e^{x^2})\,dx$$

Put $x^2 = y$ then $2x.dx = dy \Rightarrow x\,dx = \frac{dy}{2}$. Therefore,

$$I = \frac{1}{2}\int e^y\,dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2} + C.$$

4. $\int \left(\frac{x}{\sqrt{8x^2+1}}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{8x^2+1}}\right) dx$$

Put, $8x^2 + 1 = y$ then $16x.dx = dy \Rightarrow x\,dx = \frac{dy}{16}$. Therefore,

$$I = \frac{1}{16}\int \frac{dy}{\sqrt{y}} = \frac{1}{16}\int y^{-1/2}\,dy$$

$$= \frac{y^{1/2}}{16.\frac{1}{2}} + C = \frac{\sqrt{y}}{8} + C = \frac{\sqrt{8x^2+1}}{8} + C.$$

5. $\int \frac{\sin x}{3+4\cos x}\,dx$

Solution: Here,

$$I = \int \frac{\sin x}{3+4\cos x}\,dx$$

Put $3 + 4\cos x = y$. Then $-4\sin x.dx = dy \Rightarrow \sin x.dx = -\frac{dy}{4}$

Therefore,

$$I = -\frac{1}{4}\int \frac{dy}{y} = -\frac{1}{4}\log|y| + C = -\frac{1}{4}\log|3+4\cos x| + C.$$

6. $\int e^x \sec^2(e^x)\,dx$

Solution: Here,

$$I = \int e^x \sec^2(e^x)\,dx$$

Put $e^x = y$ then $e^x.\,dx = dy$. Therefore,

$$I = \int \sec^2 y\,dy = \tan y + C = \tan(e^x) + C.$$

7. $\int \frac{dx}{\sqrt{x}(1+x)}$

Solution: Here,

$$I = \int \frac{dx}{\sqrt{x}\,(1+x)}$$

Put $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}\,dx = dy$. Therefore,

$$I = 2\int \frac{dy}{1+y^2} = 2\tan^{-1}(y) + C = 2\tan^{-1}(\sqrt{x}) + C$$

8. $\int e^x\sqrt{3+4e^x}\,dx$

Solution: Here,

$$I = \int e^x\sqrt{3+4e^x}\,dx$$

Put, $3 + 4e^x = y$. Then $4e^x.dx = dy$. Then,

$$I = \int \sqrt{y}\,.\,\frac{dy}{4} = \frac{1}{4}\int y^{1/2}\,dy = \frac{1}{4}.\frac{y^{3/2}}{\frac{3}{2}} + C.$$

$$\Rightarrow I = \frac{(3+4\,e^x)^{3/2}}{6} + C.$$

9. $\int \frac{dx}{x-\sqrt{x}}$

Solution: Here,

$$I = \int \frac{dx}{x-\sqrt{x}}$$

Put, $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}\,dx = dy \Rightarrow dx = 2y\,dy$. Then,

$$I = \int \frac{2y\,dy}{y^2-y} = 2\int \frac{dy}{y-1}$$

Again, put $y - 1 = z$ then $dy = dz$. Therefore,

$$I = 2\int \frac{dz}{z} = 2\log|z| + C$$

$$= 2\log|y-1| + C = 2\log\left|\sqrt{x}-1\right| + C.$$

10. $\int \frac{dx}{x\log x}$ **for x > 0.**

Solution: Here,

$$I = \int \frac{dx}{x.\log x} \qquad \text{for } x > 0$$

Put, $\log x = y$ then $\frac{1}{x}dx = dy$. Therefore,

$$I = \int \frac{dy}{y} = \log|y| + C = \log|\log x| + C.$$

Note: *If $x \le 0$ then the function is not integrable.*

11. $\int \frac{e^x}{1+e^x}dx$

Solution: Here,

$$I = \int \frac{e^x}{1+e^x}dx$$

Put, $1 + e^x = y$. Then $e^x.dx = dy$. Then,

$$I = \int \frac{dy}{y} = \log|y| + C$$

$$= \log|e^x + 1| + C = \log(e^x + 1) + C.$$

12. $\int e^{3x}\,dx$

Solution: Here,

$$I = \int e^{3x}\,dx$$

Put $3x = y$. Then $3.dx = dy \Rightarrow dx = \frac{dy}{3}$. Then,

$$I = \frac{1}{3}\int e^y\,dy = \frac{e^y}{3} + C = \frac{e^{3x}}{3} + C.$$

13. $\int \left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)dx$$

Put, $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}dx = dy \Rightarrow \frac{dx}{\sqrt{x}} = 2\,dy$. Then,

$$I = 2\int \tan y\,dy = 2\log|\sec y| + c = 2\log\left|\sec(\sqrt{x})\right| + c$$

$$\Rightarrow I = \log\left|\sec^2(\sqrt{x})\right| + c.$$

14. $\int \frac{x\,dx}{\sqrt{1-4x^2}}$

Solution: Here,

$$I = \int \frac{x\,dx}{\sqrt{1-4x^2}} \quad \text{for } |x| < ½.$$

Put $1 - 4x^2 = y$ then $-8x.dx = dy \Rightarrow x\,dx = -\frac{dy}{8}$. Then,

$$I = -\frac{1}{8}\int \frac{dy}{\sqrt{y}} = -\frac{1}{8}\int y^{-1/2}\,dy = -\frac{1}{8}\left(\frac{y^{1/2}}{1/2}\right) + C.$$

$$\Rightarrow I = -\frac{(1-4x)^{1/2}}{4} + C.$$

$\frac{\sin^2 2x}{1+\cos 2x}$

15. $\int \left(\frac{\sin^2 2x}{1+\cos 2x}\right)dx$

Solution: Here,

$$I = \int \left(\frac{\sin^2 2x}{1+\cos 2x}\right)dx = \int \left(\frac{4\sin^2 x\cos^2 x}{2\cos^2 x}\right)dx = 2\int \sin^2 x\,dx$$

$$\Rightarrow I = 2\int \left(\frac{1-\cos 2x}{2}\right)dx = \int dx - \int \cos 2x\,dx$$

$$= x - \frac{\sin 2x}{2} + C.$$

$(2\sin x.\cos x)^2$

$4\sin^2 x.\cos^2 x$

16. $\int \frac{dy}{y\sqrt{y^2-1}}$

$\frac{\sin^2 2x}{1+\cos 2x}$ $4x - x^2$

Solution: Here,

$$I = \int \frac{dy}{y\sqrt{y^2-1}}$$

Put $y = \sec\theta$, then $dy = \sec\theta.\tan\theta\,d\theta$. So,

$$I = \int \frac{\sec\theta.\tan\theta.d\theta}{\sec\theta.\tan\theta} = \int 1.d\theta = \theta + C = \sec^{-1} y + C.$$

17. $\int \frac{x\,dx}{(3x^2+4)^3}$.

Solution: Here,

$$I = \int \frac{x\,dx}{(3x^2+4)^3}$$

Put, $3x^2 + 4 = y$, then $6x\,dx = dy$. So,

$$I = \frac{1}{6}\int \frac{dy}{y^3} = \frac{1}{-12}\cdot\frac{1}{y^2} + C = \frac{-1}{12\,(3x^2+4)^2} + C.$$

18. $\int \frac{x^2\,dx}{\sqrt{x^3+5}}$

Solution: Here,

$$I = \int \frac{x^2\,dx}{\sqrt{x^3+5}}$$

Put, $x^3 + 5 = y$, then $3x^2.dx = dy$. So,

$$I = \frac{1}{3}\int \frac{dy}{\sqrt{y}} = \frac{2}{3}\sqrt{y} + c = \frac{2}{3}\sqrt{x^3+5} + C.$$

19. $\int 3^{4x}\,dx$

Solution: Here,

$$I = \int 3^{4x}\,dx$$

Put, $4x = y$, then $4\,dx = dy$. So,

$$I = \frac{1}{4}\int 3^y\,dy = \frac{1}{4}\left(\frac{3^y}{\log 3}\right) + C = \frac{1}{4}\left(\frac{3^{4x}}{\log 3}\right) + C.$$

20. $\int \cos^2 x \sin x\,dx$

Solution: Here,

$$I = \int \cos^2 x.\sin x\,dx$$

Put $\cos x = y$, then $-\sin x\,dx = dy$. So,

$$I = \int (y^2)(-dy) = -\frac{y^3}{3} + C = -\frac{\cos^3 x}{3} + C.$$

21. $\int \cot^3 x\,\mathrm{cosec}^2 x\,dx$

Solution: Here,

$$I = \int \cot^3 x.\mathrm{cosec}^2 x.\,dx$$

Put, $\cot x = y$, then $\mathrm{cosec}^2 x.\,dx = -dy$. So,

$$I = -\int y^3\,dy = -\frac{y^4}{4} + C = -\frac{\cot^4 x}{4} + C.$$

22. $\int \frac{dx}{\sqrt{e^{2x}-1}}$ **[2011 Fall]**

Solution: Here,

$$I = \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{(e^x)^2-1}}$$

Put, $e^x = \sec\theta$ then $e^x.\,dx = \sec\theta.\tan\theta\,d\theta \Rightarrow dx = \frac{\sec\theta.\tan\theta\,d\theta}{\sec\theta} = \tan\theta\,d\theta.$

Then,

$$I = \int \frac{\sec\theta.\tan\theta.\,d\theta}{\sqrt{\sec^2\theta-1}}$$

$$= \int \frac{\tan\theta.\,d\theta}{\tan\theta} = \int d\theta = \theta + C = \sec^{-1} y + C = \sec^{-1}(e^x) + C.$$

23. $\int \frac{x^8\,dx}{(1-x^3)^{1/3}}$

Solution: Here,

$$I = \int \frac{x^8\,dx}{(1-x^3)^{1/3}}$$

Put, $(1-x^3) = y^3$ then, $(-3x^2)\,dx = 3y^2dy \Rightarrow x^2dx = -y^2dy$. So,

$$I = \int \frac{(1-y^3)^2.(-y^2\,dy)}{y}$$

$$= -\int y\,(1-y^3)^2.\,dy$$

$$= -\int y\,dy + 2\int y^4dy - \int y^7\,dy$$

$$= -\frac{y^2}{2} + \frac{2}{5}y^5 - \frac{y^8}{8} + C.$$

$$= \frac{1}{2}(1-x^3)^{2/3} + \frac{2}{5}(1-x^3)^{5/3} - \frac{1}{8}(1-x^3)^{8/3} + C.$$

24. $\int e^{\tan 3x}\sec^2 3x\,dx$

Solution: Here,

$$I = \int e^{\tan 3x}.\sec^2 3x\,dx$$

Put, $\tan 3x = y$ then $3\sec^2 3x\,dx = dy$. Then,

$$I = \frac{1}{3}\int e^y\,dy = \frac{1}{3}e^y + C = \frac{1}{3}e^{\tan 3x} + C.$$

25. $\int x^{1/3}\sqrt{x^{4/3}-1}\,dx$

Solution: Here,

$$I = \int x^{1/3}\sqrt{x^{4/3}-1}\,dx$$

Put, $x^{4/3} - 1 = y$ then $\frac{4}{3}x^{1/3}.dx = dy \Rightarrow x^{1/3}.dx = \frac{3}{4}dy$. Then,

$$I = \frac{3}{4}\int \sqrt{y}\,dy = \frac{3}{4}\cdot\frac{2}{3}y^{3/2} + C = \frac{1}{2}(x^{4/3} - 1)^{3/2} + C.$$

26. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$

Solution: Here,

$$I = \int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$$

Let, $\sqrt{x+1} = y$. Then, $\frac{1}{2\sqrt{x+1}}dx = dy \Rightarrow \frac{1}{\sqrt{x+1}}dx = 2\,dy$.

Then,

$$I = 2\int e^y\,dy = 2e^y + C = 2e^{\sqrt{x+1}} + C.$$

27. $\int \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$

Solution: Here,

$$I = \int \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$$

Put, $1+\cos\theta = y$ then $-\sin\theta\,d\theta = dy$. Then,

$$I = -\int \frac{dy}{\sqrt{y}} = -2\sqrt{y} + C = -2\sqrt{1+\cos\theta} + C.$$

28. $\int \frac{\sqrt{\tan^{-1}x}}{2(1+x^2)}dx$

Solution: Here,

$$I = \int \frac{\sqrt{\tan^{-1}x}}{2(1+x^2)}dx$$

Put, $\tan^{-1}x = y$ then $\frac{1}{1+x^2}dx = dy$. Then,

$$I = \int \left(\frac{\sqrt{y}}{2}\right)dy = \frac{1}{3}(y)^{3/2} + C = \frac{1}{3}(\tan^{-1}x)^{3/2} + C.$$

29. $\int \left(\sin x\sqrt{1-\cos 2x}\right)dx$

Solution: Here,

$$I = \int \left(\sin x\sqrt{1-\cos 2x}\right)dx$$

$$= \int \left(\sin x\sqrt{2\sin^2 x}\right)dx$$

$$= \sqrt{2}\int \sin x \sin x\,dx$$

$$= \sqrt{2}\int \sin^2 x\,dx$$

$$= \sqrt{2}\int \left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{\sqrt{2}}\left(x - \frac{\sin 2x}{2}\right) + C.$$

30. $\int x\sin^3(x^2)\cos(x^2)\,dx$

Solution: Here,

$$I = \int x\sin^3(x^2)\cos(x^2)\,dx$$

Put $\sin(x^2) = y$ then $2x.\cos(x^2).dx = dy$. Then,

$$I = \int y^3\frac{dy}{2} = \frac{y^4}{8} + C = \frac{(\sin(x^2))^4}{8} + C.$$

31. $\int \sin^3 x\cos^4 x\,dx$

Solution: Here,

$$I = \int \sin^3 x\cos^4 x.\,dx = \int (1-\cos^2 x)\cos^4 x.\sin x\,dx$$

Let, $\cos x = y$ then $(-\sin x).dx = dy$. Then,

$$I = -\int (y^4 - y^6)\,dx = \frac{y^7}{7} - \frac{y^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.$$

32. $\int \frac{\cos(\log x)}{x}dx$

Solution: Here,

$$I = \int \frac{\cos(\log x)}{x}dx$$

Let, $\log x = y$ then, $\frac{1}{x}dx = dy$. Then,

$$I = \int \cos y.\,dy = \sin y + C = \sin(\log x) + C.$$

33. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx$

Solution: Here,

$$I = \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$$

$$= \int \left(\frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\right) dx$$

$$= \int \left(\frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\right) dx$$

$$= \int \left(\frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\right) dx$$

$$= \left(\frac{1}{a-b}\right)\left[\int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx\right]$$

$$= \left(\frac{1}{a-b}\right)\frac{2}{3}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C.$$

34. $\displaystyle\int \frac{3e^{2x}+3e^{4x}}{e^x+e^{-x}}\,dx$

Solution: Here,

$$I = \int \frac{3e^{2x}+3e^{4x}}{e^x+e^{-x}}\,dx = \int \frac{3e^{2x}(1+e^{2x})}{e^{-x}(e^{2x}+1)}\,dx$$

$$= 3\int e^{3x}\,dx$$

$$= 3\left(\frac{1}{3}\right)e^{3x} + C$$

$$= e^{3x} + C.$$

35. $\displaystyle\int \log(x^2+1)\,2x(1+x^2)^{-1}\,dx$

Solution: Here,

$$I = \int 2x \cdot \frac{\log(x^2+1)}{(x^2+1)}\,dx$$

Let, $\log(x^2+1) = y$ then, $\frac{1}{x^2+1}.2x.dx = dy$. Then,

$$I = \int y\,.\,dy = \frac{y^2}{2} + C = \frac{(\log(x^2+1))^2}{2} + C.$$

36. $\displaystyle\int \frac{\sin 2x}{(\sin^4 x + \cos^4 x)}\,dx$

Solution: Here,

$$I = \int \frac{\sin 2x}{(\sin^4 x+\cos^4 x)}\,dx = \int \frac{\tan x \sec^2 x}{(\tan^4 x + 1)}\,dy$$

Let, $\tan x = y$ then, $\sec^2 x\,dx = dy$. Then,

$$I = 2\int \frac{y\,dy}{y^4+1}$$

Again let, $y^2 = z$ then, $2y.dy = dz$. Then,

$$I = \int \frac{dz}{z^2+1}$$

$$= \tan^{-1}(z) + C = \tan^{-1}(y^2) + C = \tan^{-1}(\tan^2 x) + C.$$

37. $\displaystyle\int \frac{1}{(3\tan x + 1)\cos^2 x}\,dx$

Solution: Here,

$$I = \int \frac{\sec^2 x}{(3\tan x + 1)}\,dx$$

Let, $3\tan x + 1 = y$ then $3\sec^2 x\,dx = dy$. Then,

$$I = \frac{1}{3}\int \frac{dy}{y} = \frac{1}{3}\log|y| + C = \frac{1}{3}\log|3\tan x + 1| + C.$$

38. $\displaystyle\int \frac{1}{x\log x \log(\log x)}\,dx$ for $x > 0$.

Solution: Here,

$$I = \int \frac{1}{x\log x\log(\log x)}\,dx \quad \text{for } x > 0.$$

Put, $\log x = y$ then $\frac{1}{x}dx = dy$. Then,

$$I = \int \frac{1}{y\log(y)}\,dy$$

Again put, $\log(y) = z$, then $\frac{1}{y}dy = dz$. Then,

$$I = \int \frac{dz}{z} = \log(z) + C$$

$$= \log(\log y) + C = \log\{\log(\log x)\} + C.$$

39. $\displaystyle\int \sec^4 x\,dx$

Solution: Here,

$$I = \int \sec^4 x\,dx = \int \sec^2 x.\sec^2 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx.$$

$$\Rightarrow I = \int \sec^2 x\,dx + \int \tan^2 x\,.\sec^2 x\,dx$$

Put, $\tan x = y$ then $\sec^2 x\,dx = dy$. Then,

$$I = \int dy + \int y^2 dy = y + \frac{y^3}{3} + C \quad = \tan x + \frac{\tan^3 x}{3} + C.$$

40. $\int \tan^5 x\, dx$

Solution: Here,

$$I = \int \tan^5 x\, dx$$

$$= \int \tan^2 x \cdot \tan^3 x\, dx$$

$$= \int (\sec^2 x - 1) \tan^3 x\, dx$$

$$= \int \tan^3 x \cdot \sec^2 x\, dx - \int \tan^3 x\, dx$$

$$= \int \tan^3 x \cdot \sec^2 x\, dx - \int (\sec^2 x - 1) \tan x\, dx$$

$$= \int \tan^3 x \cdot \sec^2 x\, dx - \int \tan x \cdot \sec^2 x\, dx + \int \tan x\, dx$$

$$= \int \tan^3 x \cdot \sec^2 x\, dx - \int \tan x \cdot \sec^2 x\, dx + \log|\sec x| + C_1$$

Put, $\tan x = y$ then $\sec^2 x\, dx = dy$. Then,

$$I = \int y^3 dy - \int y\, dy + \log|\sec x| + C_1$$

$$= \frac{y^4}{4} - \frac{y^2}{2} + \log|\sec x| + C_1 + C_2$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + C_3 \quad \text{for } C_1 + C_2 = C_3$$

$$= \frac{(\sec^2 x - 1)^2}{4} - \left(\frac{\sec^2 x - 1}{2}\right) + \log|\sec x| + C_3$$

$$= \frac{\sec^4 x + 1 - 2\sec^2 x - 2\sec^2 x + 2}{4} + \log|\sec x| + C_3$$

$$= \frac{\sec^4 x}{4} - \sec^2 x + \log|\sec x| + C_3 + \frac{3}{4}$$

$$= \frac{\sec^4 x}{4} - \sec^2 x + \log|\sec x| + C \quad \text{for } C = C_3 + \frac{3}{4}$$

TECHNIQUE OF SOME INTEGRALS

❖ Integrals of type

$$I = \int \frac{px + q}{ax^2 + bx + c}\, dx, \quad \text{or} \quad I = \int \frac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}$$

or $I = \int (px + q)\sqrt{ax^2 + bx + c}\, dx$

then put $px + q = \frac{kd}{dx}(ax^2 + bx + c) + m$

$\Rightarrow \quad px + q = k(2ax + b) + m \quad \ldots (i)$

and then comparing the coefficient of x and constant term and then use it in (*i*). After then the integral changes to of type (G).

❖ Integrals of type $I = \int \frac{dx}{(ax + b)\sqrt{cx + d}}$ then

Put $cx + d = t^2$

then $$I = \int \frac{2t\, dt}{c\left[a\left(\frac{t^2 - d}{c}\right) + b\right]\sqrt{t^2}}$$

$$\Rightarrow \quad I = \frac{2}{a}\int \frac{dt}{t^2 + (bc - ad)}$$

which is similar form as (G).

❖ If the integrals of type $I = \int \frac{dx}{(px + q)\sqrt{ax^2 + bx + c}}$

then put $(px + q) = \frac{1}{t}$ then

$$I = \frac{1}{p}\int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{a}{p^2}\left(\frac{1}{t} - q\right)^2 + \frac{b}{p}\left(\frac{1}{t} - q\right) + c}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{a}{t^2}(1 - 2t + qt^2) + \frac{bp}{t}(1 - qt) + cp}}$$

$$= -\int \frac{dt}{\sqrt{t^2(aq - bpq + cp) + (-2a + bp)t + a}}$$

which is same form as in (G).

❖ The integrals of type $I = \int \frac{dx}{a + b\cos x}$ or $I = \int \frac{dx}{a + b\sin x}$

Or $I = \int \frac{dx}{a + b\sin x + c\cos x}$,

then put $\tan\left(\frac{x}{2}\right) = t$ so that,

$$dx = \frac{2dt}{1 + y^2}, \quad \sin x = \frac{2y}{1 + y^2} \quad \text{and} \quad \cos x = \frac{1 - y^2}{1 + y^2}$$

and process to solve.

[***such method is developed by a German mathematician Karl Weierstrass***].

- If the integrals is of type $I = \int \frac{dx}{a\sin x + b\cos x}$ then, then put $a = r\cos\theta$, $b = r\sin\theta$ then, $I = \int \frac{dx}{r.\sin(x+\theta)}$ and integrate it.

Remember: In such type choose the value of a and b so that the denominator should become as $r(\sin x\cos\theta + \cos x\sin\theta)$.
OR The problem can be solved by using $a = r\sin\theta$, $b = r\cos\theta$.

Some Formulae

Case of Improper Rational Fraction

If the given rational fraction is in improper form then process as

$$\text{given fraction} = \text{Quotient} + \frac{\text{Remainder}}{\text{divisor}} \text{ (i.e. proper)}$$

where the form $\frac{\text{Remainder}}{\text{divisor}}$ is a proper fraction.

So that, $\int \text{given fraction} = \int \text{quotient} + \int \text{proper fraction}$.

Exercise 11.2

A. Evaluate the integrals:

1. $\int \frac{dx}{\sqrt{x^2-2x+5}}$

Solution: Here,

$$I = \int \frac{dx}{\sqrt{x^2-2x+5}}$$
$$= \int \frac{dx}{\sqrt{x^2-2x+1+4}} = \int \frac{dx}{\sqrt{(x-1)^2+9^2}}$$

Put, $x - 1 = y$ then $dx = dy$. Then,

$$I = \int \frac{dy}{\sqrt{y^2+2^2}}$$
$$= \log\left|y + \sqrt{y^2+2^2}\right| + C = \log|x - 1 + \sqrt{x^2-2x+5}| + C.$$

2. $\int \left(\frac{x}{x^2-2x+5}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x}{x^2-2x+5}\right) dx = \int \left(\frac{\frac{1}{2}(2x-2+2)}{x^2-2x+5}\right) dx$$
$$= \frac{1}{2}\int \left(\frac{(2x-2)}{x^2-2x+5}\right) dx + \frac{2}{2}\int \frac{dx}{x^2-2x+1+4}$$

Put, $x^2 - 2x + 5 = y$ then $(2x - 1)\, dx = dy$. Then,

$$I = \frac{1}{2}\int \frac{dy}{y} + \int \frac{dx}{(x-1)^2+2^2}$$
$$= \frac{1}{2}\log|y| + \int \frac{dx}{\alpha^2+2^2} + C_1 \qquad [\text{setting } x - 1 = \alpha]$$
$$= \frac{1}{2}\log|x^2-2x+5| + \frac{1}{2}\tan^{-1}\left(\frac{\alpha}{2}\right) + C$$
$$= \log\left(\sqrt{x^2-2x+5}\right) + 2\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$
$$= \log\left(\sqrt{x^2-2x+5}\right) + 2\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

3. $\int \left(\frac{x}{\sqrt{x^2-2x+5}}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{x^2-2x+5}}\right) dx$$
$$= \frac{1}{2}\int \left(\frac{2x-2+2}{\sqrt{x^2-2x+5}}\right) dx$$
$$= \frac{1}{2}\int \left(\frac{2x-2}{\sqrt{x^2-2x+5}}\right) dx + \frac{2}{2}\int \frac{dx}{\sqrt{x^2-2x+1+4}}$$
$$= \frac{1}{2}\int \left(\frac{2x-2}{\sqrt{x^2-2x+5}}\right) dx + \int \frac{dx}{(x-1)^2+2^2}$$

For first integral, put $x^2 - 2x + 5 = y$ then $(2x - 2)\, dx = dy$.
And for second integral, put $x - 1 = \alpha$ then $dx = d\alpha$. Then,

$$I = \frac{1}{2}\int \frac{dy}{\sqrt{y}} + \int \frac{d\alpha}{\sqrt{\alpha^2+2^2}}$$
$$= \frac{1}{2}\cdot\frac{y^{1/2}}{1/2} + \log\left|\alpha + \sqrt{\alpha^2+2^2}\right| + C$$
$$= \sqrt{x^2-2x+5} + \log\left|x - 1 + \sqrt{x^2-2x+5}\right| + C.$$

4. $\int \left(\frac{x+1}{3+2x-x^2}\right) dx$

Solution: Here,

$$I = \int\left(\frac{x+1}{3+2x-x^2}\right)dx = \int\left(\frac{\frac{1}{2}(2x+2-2)+1}{3+2x-x^2}\right)dx$$

2-2x

$$= \left(\frac{-1}{2}\right)\int\left(\frac{2-2x}{3+2x-x^2}\right)dx + 2\int\frac{dx}{3+2x-x^2}$$

Put $3+2x-x^2 = y$ then $(2-2x)\,dx = dy$. Then,

$$I = \left(\frac{-1}{2}\right)\int\frac{dy}{y} + 2\int\frac{dx}{4-(x-1)^2}$$

Put $x - 1 = \alpha$ then $dx = d\alpha$. Then,

$$I = \left(\frac{-1}{2}\right)\int\frac{dy}{y} + 2\int\frac{dy}{2^2-\alpha^2}$$

$$= \left(\frac{-1}{2}\right)\log|y| + 2\left(\frac{1}{2(2)}\right)\log\left|\frac{2+\alpha}{2-\alpha}\right| + C$$

$$= \left(\frac{-1}{2}\right)\log|3+2x-x^2| + \frac{1}{2}\log\left|\frac{2+x-1}{2-x+1}\right| + C$$

$$= \left(\frac{-1}{2}\right)\log|3+2x-x^2| + \frac{1}{2}\log\left|\frac{x+1}{3-x}\right| + C.$$

5. $\displaystyle\int\left(\frac{3}{\sqrt{15-6x-x^2}}\right)dx$

Solution: Here,

$$I = \int\left(\frac{3}{\sqrt{15-6x-x^2}}\right)dx = (3)\int\frac{dx}{\sqrt{24-(x+3)^2}}$$

Put $x + 3 = y$ then $dx = dy$. Then,

$$I = (3)\int\frac{dy}{\sqrt{24-y^2}} = 3\sin^{-1}\left(\frac{y}{\sqrt{24}}\right) + C = 3\sin^{-1}\left(\frac{x+3}{\sqrt{24}}\right) + C.$$

6. $\displaystyle\int\left(\frac{a-x}{\sqrt{2ax-x^2}}\right)dx$

Solution: Here,

$$I = \int\left(\frac{a-x}{\sqrt{2ax-x^2}}\right)dx = \left(\frac{1}{2}\right)\int\left(\frac{2a-2x}{\sqrt{2ax-x^2}}\right)dx$$

Put, $2ax - x^2 = y$ then $(2a-2x)\,dx = dy$. Then,

$$I = \left(\frac{1}{2}\right)\int\frac{dy}{\sqrt{y}} = \left(\frac{1}{2}\right)\left(\frac{y^{1/2}}{1/2}\right) + C = \sqrt{2ax-x^2} + C.$$

7. $\displaystyle\int\frac{dx}{\sqrt{x^2+2x}}$

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...ion: Here,

$$I = \int\frac{dx}{\sqrt{x^2+2x}} = \int\frac{dx}{\sqrt{(x+1)^2-1^2}}$$

Put, $x + 1 = y$ then $dx = dy$. Then,

$$I = \int\frac{dy}{\sqrt{y^2-1^2}}$$

$$= \log\left|y + \sqrt{y^2-1}\right| + C$$

$$= \log\left|x + 1 + \sqrt{(x+1)^2-1}\right| + C$$

$$= \log\left|x + 1 + \sqrt{x^2+2x}\right| + C.$$

8. $\displaystyle\int\left(\frac{(1-x)}{\sqrt{8+2x-x^2}}\right)dx$

Solution: Here,

$$I = \int\left(\frac{(1-x)}{\sqrt{8+2x-x^2}}\right)dx = \frac{1}{2}\int\left(\frac{(2-2x)}{\sqrt{8+2x-x^2}}\right)dx$$

Put $8 + 2x - x^2 = y$ then $(2-2x)\,dx = dy$. Then,

$$I = \frac{1}{2}\int\frac{dy}{\sqrt{y}}$$

$$= \frac{1}{2}\int y^{-1/2}\,dy = \frac{1}{2}\left(\frac{y^{1/2}}{\frac{1}{2}}\right) + C = \sqrt{8+2x-x^2} + C.$$

9. $\displaystyle\int\left(\frac{x}{\sqrt{x^2+4x+5}}\right)dx$ [2002] [2009, Fall]

Solution: Here,

$$I = \int\left(\frac{x}{\sqrt{x^2+4x+5}}\right)dx = \frac{1}{2}\int\left(\frac{2x+4-4}{\sqrt{x^2+4x+5}}\right)dx$$

$$= \int\left(\frac{2x+4}{\sqrt{x^2+4x+5}}\right)dx - \frac{4}{2}\int\frac{dx}{\sqrt{(x+2)^2+1}}$$

For first integral, put $x^2 + 4x + 5 = y$ then $(2x+4)\,dx = dy$. Also, for second integral, put $x + 2 = \alpha$ then $dx = d\alpha$. Then,

$$I = \frac{1}{2}\int\frac{dy}{\sqrt{y}} - 2\int\frac{dy}{\sqrt{\alpha^2+1^2}}$$

$$= \frac{1}{2}\cdot\frac{y^{1/2}}{\frac{1}{2}} - 2\,.\,\log\left|\alpha + \sqrt{\alpha^2+1^2}\right| + C$$

$$= \sqrt{y} - 2\log\left|x + 2 + \sqrt{(x+2)^2+1}\right| + C$$

$$= \sqrt{x^2+4x+5} - 2\log\left|x + 2 + \sqrt{x^2+4x+5}\right| + C.$$

10. $\displaystyle\int\frac{dx}{(x+3)\sqrt{x^2+6x+10}}$

Solution: Here,

$$I = \int\frac{dx}{(x+3)\sqrt{x^2+6x+10}}$$

Put $x + 3 = \frac{1}{y}$ then $dx = -\frac{1}{y^2}dy$. Then,

$$I = \int \frac{(-1)/y^2}{\frac{1}{y}\sqrt{\left(\frac{1-3y}{y}\right)^2 + 6\left(\frac{1-3y}{y}\right) + 10}}\,dy$$

$$= \int \frac{dy}{y\left(\frac{1}{y}\right)\sqrt{(1-3y)^2 + 6y(1-3y) + 10y^2}}$$

$$= -\int \frac{dy}{\sqrt{1 - 6y + 9y^2 + 6y - 18y^2 + 10y^2}}$$

$$= -\int \frac{dy}{\sqrt{y^2+1}} = -\log\left|y + \sqrt{y^2+1}\right| + C$$

$$= -\log\left|\frac{1}{x+3} + \sqrt{\left(\frac{1}{x+3}\right)^2 + 1}\right| + C$$

$$= -\log\left|\frac{1}{x+3} + \frac{1}{x+3}\sqrt{1 + (x+3)^2}\right| + C$$

$$= -\log\left|\frac{1 + \sqrt{x^2 + 6x + 10}}{x+3}\right| + C.$$

11. $\int \left(\frac{(2x+3)}{4x^2+4x+5}\right)dx$

Solution: Here,

$$I = \int \left(\frac{(2x+3)}{4x^2+4x+5}\right)dx$$

$$= \frac{1}{4}\int \left(\frac{8x+4+8}{4x^2+4x+5}\right)dx$$

$$= \frac{1}{4}\int \left(\frac{8x+4}{4x^2+4x+5}\right)dx + \frac{8}{4}\int \frac{dx}{(2x)^2 + 2.2x.1 + 1 + 4}$$

$$= \frac{1}{4}\int \left(\frac{8x+4}{4x^2+4x+5}\right)dx + 2\int \frac{dx}{(2x+1)^2 + 2^2}$$

For first integral, put $4x^2 + 4x + 5 = y$ then $(8x + 4)\,dx = dy$. Also, for second integral, put $2x + 1 = \alpha$ then $2dx = d\alpha$. Then,

$$I = \frac{1}{4}\int \frac{dy}{y} + \int \frac{d\alpha}{\alpha^2 + 2^2}$$

$$= \frac{1}{4}\log|y| + \frac{1}{2}\tan^{-1}\left(\frac{\alpha}{2}\right) + C$$

$$= \frac{1}{4}\log|4x^2 + 4x + 5| + \frac{1}{2}\tan^{-1}\left(\frac{2x+1}{2}\right) + C.$$

12. $\int \left(\frac{x}{\sqrt{x^2+4x+13}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{x^2+4x+13}}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{2x+4-4}{\sqrt{x^2+4x+13}}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{2x+4}{\sqrt{x^2+4x+13}}\right)dx - \frac{4}{2}\int \frac{dx}{\sqrt{(x+2)^2+3^2}}$$

For first integral, put $x^2 + 4x + 13 = y$ then $(2x + 4)\,dx = dy$. Also, for second integral, put $x + 2 = \alpha$ then $dx = d\alpha$. Then,

$$I = \frac{1}{2}\int \frac{dy}{\sqrt{y}} - 2\int \frac{d\alpha}{\sqrt{\alpha^2+3^2}}$$

$$= \frac{1}{2}\cdot\frac{y^{1/2}}{1/2} - 2\log\left|\alpha + \sqrt{\alpha^2+3^2}\right| + C$$

$$= \sqrt{y} - 2\log\left|x + 2 + \sqrt{x^2+4x+13}\right| + C$$

$$= \sqrt{x^2+4x+13} - 2\log\left|x + 2 + \sqrt{x^2+4x+13}\right| + C.$$

13. $\int \left(\frac{2x^2+3x+4}{x^2+6x+10}\right)dx$

Solution: Here,

$$I = \int \left(\frac{2x^2+3x+4}{x^2+6x+10}\right)dx$$

$$= \int \left(\frac{2(x^2+6x+10) - 9x - 16}{x^2+6x+10}\right)dx$$

$$= 2\int dx - \int \left(\frac{9x+16}{x^2+6x+10}\right)dx$$

$$= 2\int dx - \int \left(\frac{\frac{9}{2}(2x+6) - 11}{x^2+6x+10}\right)dx$$

$$= 2\int dx - \frac{9}{2}\int \left(\frac{2x+6}{x^2+6x+10}\right)dx + 11\int \frac{dx}{(x+3)^2+1^2}$$

Put $x^2 + 6x + 10 = y$ then $(2x + 6)\,dx = dy$. Then,

$$\int \left(\frac{2x+6}{x^2+6x+10}\right)dx = \int \frac{dy}{y} = \log|y| + C_1$$

$$= \log|x^2 + 6x + 10| + C_1.$$

Therefore,

$$I = 2\int dx - \frac{9}{2}\log|x^2 + 6x + 10| + 11\int \frac{dx}{(x+3)^2+1^2} + C_1$$

$$= 2x - \frac{9}{2}\log|x^2 + 6x + 10| + 11\tan^{-1}(x+3) + C.$$

$$I = \int \frac{dx}{4 + 5\sin x}$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right) \cdot \frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1 + y^2}$.

Also, $\sin x = \frac{2y}{1 + y^2}$.

Then,

$$I = \int \frac{2\,dy/(1 + y^2)}{4 + 5 \cdot \frac{2y}{1 + y^2}} = 2\int \frac{dy}{4y^2 + 10y + 4} = \frac{1}{2}\int \frac{dy}{y^2 + \frac{5}{2}y + 1}$$

$$\Rightarrow I = \frac{1}{2}\int \frac{dy}{\left(y + \frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

Put, $y + \frac{5}{4} = t$ then $dy = dt$. Then,

$$I = \frac{1}{2}\int \frac{dt}{t^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2 \cdot \frac{3}{4}} \log\left|\frac{t - \frac{3}{4}}{t + \frac{3}{4}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{y + \frac{5}{4} - \frac{3}{4}}{y + \frac{5}{4} + \frac{3}{4}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{\tan(x/2) + 1}{2\,[\tan(x/2) + 2]}\right| + C.$$

4. $\int \frac{dx}{1 - \cos x + \sin x}$ **[2018 Spring][2013 Spring][2004, Spring]**

Solution: Here,

$$I = \int \frac{dx}{1 - \cos x + \sin x}$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right) \cdot \frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1 + y^2}$

Also,

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1 - y^2}{1 + y^2} \text{ and } \sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2y}{1 + y^2}$$

Now,

$$I = \int \frac{2dy/(1 + y^2)}{1 - \frac{1 - y^2}{1 + y^2} + \frac{2y}{1 + y^2}}$$

$$= 2\int \frac{dy}{1 + y^2 - 1 + y^2 + 2y}$$

$$= 2\int \frac{dy}{2y^2 + 2y} = \frac{2}{2}\int \frac{dy}{y^2 + y} = \int \frac{dy}{\left(y + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$\Rightarrow I = \frac{1}{2 \cdot \frac{1}{2}}\log\left|\frac{y + \frac{1}{2} - \frac{1}{2}}{y + \frac{1}{2} + \frac{1}{2}}\right| + C$$

$$= \log\left|\frac{y}{y + 1}\right| + C = \log\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2} + 1}\right| + C$$

$$= -\log\left|\frac{\tan\frac{x}{2} + 1}{\tan\frac{x}{2}}\right| + C = -\log\left|1 + \cot\frac{x}{2}\right| + C.$$

5. $\int \frac{dx}{4\sin x + 3\cos x + 13}$ **[2017 Spring]**

Solution: Here,

$$I = \int \frac{dx}{4\sin x + 3\cos x + 13}$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right) \cdot \frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1 + y^2}$

Also,

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1 - y^2}{1 + y^2} \text{ and } \sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2y}{1 + y^2}$$

Now,

$$I = \int \frac{2dy/(1 + y^2)}{4\frac{2y}{1 + y^2} + 3\frac{1 - y^2}{1 + y^2} + 13}$$

$$= 2\int \frac{dy}{8y + 3 - 3y^2 + 13 + 13y^2}$$

$$= 2\int \frac{dy}{12y^2 + 8y + 16} = \frac{1}{6}\int \frac{dy}{y^2 + \frac{2}{3}y + \frac{4}{3}}$$

$$= \int \frac{dy}{(y + 1/3)^2 + (\sqrt{11/3})^2}$$

$$\Rightarrow I = \frac{3}{\sqrt{11}} \tan^{-1}\left(\frac{3y+1}{\sqrt{11}}\right) + C$$

$$\Rightarrow I = \frac{3}{\sqrt{11}} \tan^{-1}\left(\frac{1}{\sqrt{11}}\left(3\tan\left(\frac{x}{2}\right)+1\right)\right) + C.$$

6. $\int \frac{dx}{5+4\cos x}$ **[2015 Fall][2008, Spring][2009 Spring]** [2003, Spring]

Solution: Here,

$$I = \int \left(\frac{1}{5+4\cos x}\right) dx$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy \Rightarrow dx = \frac{2dy}{1+y^2}$

Also,

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-y^2}{1+y^2}$$

Then,

$$I = \int \frac{2dy/(1+y^2)}{5+4.\left(\frac{1-y^2}{1+y^2}\right)} = 2\int \frac{dy}{y^2+9}$$

$$= 2.\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right) + C$$

$$= 2\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) + C.$$

7. $\int \frac{dx}{3\sin x + 4\cos x}$

Solution: Here,

$$I = \int \frac{dx}{3\sin x + 4\cos x}$$

Put, $3 = r\cos\theta$, $4 = r\sin\theta$. Then, $r^2 = 3^2 + 4^2 = 25$. And, $\tan\theta = \frac{4}{3}$.

Now,

$$I = \int \frac{dx}{r(\sin x\cos\theta + \cos x\sin\theta)}$$

$$= \frac{1}{5}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{5}\int \operatorname{cosec}(x+\theta)\,dx$$

Put, $x + \theta = y$ then $dx = dy$. So,

$$I = \frac{1}{5}\int \operatorname{cosec} y\,dy$$

$$= \frac{1}{5}\log\left|\tan\frac{y}{2}\right| + C = \frac{1}{5}\log\left|\tan\left(x+\frac{\theta}{2}\right)\right| + C$$

$$= \frac{1}{5}\log\left|\tan\left[\frac{x}{2}+\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right]\right| + C.$$

8. $\int \left(\frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\right) dx$ **[2017 Fall]**

Solution: Let

$$I = \int \left(\frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\right) dx \quad \dots (i)$$

Here,

$$2\sin x + 3\cos x = A.\,d(3\sin x + 4\cos x) + B(3\sin x + 4\cos x)$$
$$= A(3\cos x - 4\sin x) + B(3\sin x + 4\cos x)$$
$$= (-4A + 3B)\sin x + (3A + 4B)\cos x$$

Equating the coefficient of sinx and cosx then,

$-A + 3B = 2$ and $3A + 4B = 3$

Solving we get,

$A = \frac{1}{25}$ and $B = \frac{18}{25}$

Then (i) becomes,

$$I = \frac{1}{25}\int \frac{d(3\sin x + 4\cos x)}{3\sin x + 4\cos x}dx + \frac{18}{25}\int \frac{3\sin x + 4\cos x}{3\sin x + 4\cos x}dx$$

$$= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}\int dx + C_1$$

$$= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}x + C_1 + C_2.$$

$$= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}x + C.$$

TECHNIQUE OF SOME INTEGRALS

A. Integrating by parts

If the integrand consist two different types of functions then we use the formula.

If the integrand of the form $\int f_1(x) . f_2(x)$ then,

$$\int f_1(x) . f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int\left(\frac{d\,f_1(x)}{dx}\int f_2(x)\,dx\right)dx$$

Among different functions, normally we use the rule ILATE to choose the first and second functions as its order where,

I = Inverse circular function L = Logarithmic function
A = Algebraic function T = Trigonometric function
E = Exponential function

Note: Some time the integrand consist only a single function as $\int \log(x)\,dx$ however it is not possible to integrate by simple process. In such case we multiply the integrand by 1 so that the integrand becomes a multiple of two functions given function and a constant function (i.e. 1). Then we apply integration by parts with choosing the constant as second function.

Thus, normally we use the rule ILATEC.

Generalization of the rule:

$$\int f.\,g\,dx = f\,g' - f_1 g'' + f_2 g''' - f_3 g^{iv} + f_4 g^{v} + \ldots$$

where f is the first function and g is the second function. Also, the suffix denotes for derivative and 'prime of g (i.e. power of g)' denotes number of times of integration of g.

B. $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$

Exercise 11.3

Evaluate the integrals:

1. $\int x \sin x\,dx$

Solution: Here,

$I = \int x \sin x\,dx.$

Taking x as 1st function and sinx as 2nd function and we use the integration by parts then,

$$I = uv_1 - u'v_2 + u^2v_3 - \ldots + C$$
$$= -x\cos x + 1.\sin x + C$$
$$= -x\cos x + \sin x + C.$$

2. $\int x^2 \sin x\,dx$

Solution: Here,

$I = \int x^2 \sin x\,dx$

Taking x^2 as 1st function and sinx as 2nd function and we use the integration by parts then,

$$I = uv_1 - u^1v_2 + u^2v_3 - u^3v_4 + \ldots + C$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Q. Integrate: $\int x \sin^2 x\,dx$ [2016 Spring Short]

3. $\int x \log x\,dx$

Solution: Here,

$I = \int x \log x\,dx$

Taking log x as 1st function and x as 2nd function and we use the integration by parts then,

$$I = uv_1 - u'v_2 + u^2v_3 - \ldots + C$$
$$= \log x.\frac{x^2}{2} - \frac{1}{x}.\frac{x^3}{6} + C = \log x.\frac{x^2}{2} - \frac{x^2}{6} + C.$$

4. $\int x^n \log(ax)\,dx,\ n \neq -1$

Solution: Here,

$I = \int x^n \log(ax)\,dx$ for $n \neq -1$

Taking log (ax) as 1st function and x^n as 2nd function and we use the integration by parts then,

$$I = \log(ax)\int x^n dx - \int\left[\frac{d}{dx}(\log ax)\int x^n dx\right]dx$$

$$= \log(ax).\left(\frac{x^{n+1}}{n+1}\right) - \int \frac{a}{ax}.\left(\frac{x^{n+1}}{n+1}\right)dx + C_1$$
$$= \left(\frac{1}{n+1}\right)[x^{n+1}.\log(ax) - \int x^n dx] + C_1$$
$$= \left(\frac{1}{n+1}\right)\left[x^{n+1}.\log(ax) - \frac{x^{n+1}}{n+1}\right] + C$$
$$= \left(\frac{x^{n+1}}{n+1}\right)\left[\log(ax) - \frac{1}{n+1}\right] + C.$$

Q. **Integrate:** $\int x^3 \log(x)\,dx$, [2018 Spring short]

5. $\int \log(x)\,dx$

Solution: Here,

$I = \int \log x\,dx = \int 1.\log x\,dx$

Taking log(x) as 1st function and 1 as 2nd function and we use the integration by parts then,

$$I = \log(x)\int 1\,dx - \int\left[\frac{d}{dx}(\log x)\int 1\,dx\right]dx$$
$$= \log(x).x - \int \frac{1}{x}.x\,dx + C_1$$
$$= x\log(x) - x + C$$
$$= x[\log(x) - 1] + C.$$

6. $\int x^5 e^x\,dx$ [2017 Fall Short]

Solution: Here,

$I = \int x^5 e^x\,dx$

Taking x^5 as 1st function and e^x as 2nd function and then applying integration by parts then,

$$I = uv_1 - u'v_2 + u^2v_3 - u^3v_4 + u^4v_5 - u^5v_6 + \ldots + C$$
$$= x^5.e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= e^x[x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120] + C.$$

7. $\int x^5 \sin x\,dx$

Solution: Here,

$I = \int x^5 \sin x\,dx$

Taking x^5 as 1st function and sin x as 2nd function and then applying integration by parts then,

$$I = uv_1 - u^1v_2 + u^2v_3 - u^3v_4 + u^4v_5 - u^5v_6 + \ldots + C$$
$$= -x^5\cos x + 5x^4\sin x + 20x^3\cos x - 60x^2\sin x - 120x\cos x + 120\sin x + C$$
$$= \cos x(-x^5 + 20x^3 - 120x) + \sin x(5x^4 - 6x^2 + 120) + C.$$

8. $\int \frac{(1-x^2)^{3/2}}{x^6}dx$

Solution: Here,

$$I = \int \frac{(1-x^2)^{3/2}}{x^6}\, dx$$

Put $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. Then,

$$I = \int \frac{\cos^3\theta}{\sin^6\theta}\cos\theta\, d\theta = \int \cot^4\theta \csc^2\theta\, d\theta$$

Set $\cot\theta = y$ then $(-\csc\theta)\, d\theta = dy$. Then,

$$I = -\int y^4\, dy$$
$$= -\frac{y^5}{5} + C$$
$$= -\frac{1}{5}\cot^5\theta + C$$
$$= -\frac{(1-x^2)^{3/2}}{x^6} + C.$$

$\sin\theta = x \Rightarrow \frac{p}{h} = x$
$\Rightarrow p = x, h = 1.$
Then $b = \sqrt{h^2 - p^2} = \sqrt{1-x^2}$
So, $\cot\theta = \frac{b}{p} = \frac{\sqrt{1-x^2}}{x}$

9. $\int \left(\frac{\log(x)}{x^2}\right) dx$ for $x > 0$.

Solution: Here,

$$I = \int \left(\frac{\log(x)}{x^2}\right) dx \quad \text{for } x > 0.$$

Taking $\log(x)$ as 1^{st} function and x^{-2} as 2^{nd} function, and then appl[ying] integration by parts then,

$$I = \log(x)\int \frac{1}{x^2}dx - \int\left(\frac{d}{dx}\log x \int x^{-2}dx\right)dx$$
$$= -\log(x)\cdot\frac{1}{x} + \int \frac{1}{x}\cdot(x^{-1})\, dx + C_1$$
$$= -\frac{\log x}{x} - x^{-1} + C$$
$$= -\frac{1}{x}(\log x + 1) + C.$$

10. $\int \sin^{-1}x\, dx$

Solution: Here,

$$I = \int \sin^{-1}x\, dx = \int 1.\sin^{-1}x\, dx$$

Taking $\sin^{-1}x$ as 1^{st} function and 1 as 2^{nd} function and then appl[ying] integration by parts then,

$$I = \sin^{-1}x\int 1\, dx - \int\left(\frac{d}{dx}\sin^{-1}x\int dx\right)dx$$
$$= \sin^{-1}x.\, x - \int \frac{1}{\sqrt{1-x^2}}\cdot x\, dx + C_1$$

$$= x\sin^{-1}x + \frac{1}{2}\int\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx + C_1$$

Put, $1 - x^2 = y^2$ then $(-x)dx = ydy$. Then,

$$\int\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx = \int \frac{2y\, dy}{y}$$
$$= 2\int dy = 2y + C_2 = 2\sqrt{1-x^2} + C_2$$

Therefore,

$$I = x\sin^{-1}x + \frac{1}{2}\cdot 2\sqrt{1-x^2} + C_1 + C_2$$
$$= x.\sin^{-1}x + \sqrt{1-x^2} + C \quad ,\text{for } C_1 + C_2 = C.$$

Q. **Integrate:** $\int \tan^{-1}x\, dx$

11. $\int x \sin x \sin(2x)\sin(3x)\, dx$ [2016 Fall Short]

Solution: Here,

$$I = \int x\sin x \sin(2x)\sin(3x)\, dx$$

Since, $2\sin(2x)\sin(3x) = \cos(2x - 3x) - \cos(2x + 3x)$

$$= \cos x - \cos(5x) \quad [\because \cos(-\theta) = \cos\theta]$$

Then,

$$I = \frac{1}{2}\int x\sin x[\cos x - \cos(5x)]\, dx$$
$$= \frac{1}{4}\int x\sin(2x)\, dx - \frac{1}{2}\int x\sin x\cos(5x)\, dx$$

Also, $2\sin x\cos(5x) = \sin(x + 5x) + \sin(x - 5x)$
$= \sin(6x) - \sin(4x)$

Therefore,

$$I = \frac{1}{4}\int x\sin(2x)\, dx - \frac{1}{4}\int x\sin(6x)\, dx + \frac{1}{4}\int x\sin(4x)\, dx.$$

Taking x as 1^{st} function and sine as 2^{nd} function and then applying integration by parts then,

$$I = \frac{1}{4}\left[x\int \sin 2x\, dx - \int\left(\frac{d}{dx}x\int \sin 2x\, dx\right)dx\right] +$$
$$\frac{1}{4}\left[x\int \sin 6x\, dx - \int\left(\frac{d}{dx}x\int \sin 6x\, dx\right)dx\right] +$$
$$\frac{1}{4}\left[x\int \sin 4x\, dx - \int\left(\frac{d}{dx}x\int \sin 4x\, dx\right)dx\right]$$
$$= \frac{1}{4}\left[-x\frac{\cos 2x}{2} + \int\frac{\cos 2x}{2}dx\right] - \frac{1}{4}\left[-x\frac{\cos 6x}{6} + \int\frac{\cos 6x}{6}dx\right] + \frac{1}{4}\left[-x\frac{\cos 4x}{4} + \int\frac{\cos 4x}{4}dx\right]$$

$$= \frac{1}{4}\left[-\frac{x\cos(2x)}{2}+\frac{\sin(2x)}{4}+\frac{x\cos(6x)}{6}-\frac{\sin(6x)}{36}-\frac{x\cos(4x)}{4}+\frac{\sin(4x)}{16}\right]+C$$

$$= -\frac{x}{8}\left[\cos(2x)+\frac{\cos(4x)}{2}-\frac{\cos(6x)}{3}\right]+\frac{1}{16}\left[\sin(2x)+\frac{\sin(4x)}{4}-\frac{\sin(6x)}{9}\right]+C.$$

12. $\int\left(\frac{x+\sin x}{1+\cos x}\right)dx$ **[2003, Fall] [2002]**

Solution: Here,

$$I = \int\left(\frac{x+\sin x}{1+\cos x}\right)dx = \int\frac{x+2\sin(x/2).\cos(x/2)}{2\cos^2(x/2)}dx$$

$$= \frac{1}{2}\int x\sec^2\left(\frac{x}{2}\right)dx+\frac{2}{2}\int\frac{\sin(x/2).\cos(x/2)}{\cos^2(x/2)}dx$$

$$= \frac{1}{2}\int x\sec^2\left(\frac{x}{2}\right)dx+\int\tan\left(\frac{x}{2}\right)dx.$$

Taking x as 1[st] function and $\sec^2\frac{x}{2}$ as 2[nd] function and then applying integration by parts then,

$$I = \frac{1}{2}\left[x\int\sec^2\frac{x}{2}dx-\int\left(\frac{d}{dx}x\int\sec^2\frac{x}{2}dx\right)dx\right]^2+\log|\sec(x/2)|+C_1$$

$$= \frac{1}{2}\left[x.\frac{\tan(x/2)}{(1/2)}-2\int\tan(x/2).dx\right]^2+\log|\sec(x/2)|+C_1$$

$$= x\tan\left(\frac{x}{2}\right)-2\log|\sec(x/2)|+2\log|\sec(x/2)|+C$$

$$= x.\tan\left(\frac{x}{2}\right)+C.$$

13. $\int e^x(\tan x-\log\cos x)\,dx$

Solution: Here,

$$I = \int e^x(\tan x-\log\cos x)\,dx$$

Put $-\log(\cos x) = f(x)$ then $\frac{\sin x}{\cos x}dx = f'(x)\,dx \Rightarrow \tan x\,dx = f'(x)\,dx.$

Therefore,

$$I = \int e^x[f(x)+f'(x)]\,dx = e^x f(x)+C$$

$$= e^x(-\log\cos x)+C$$

$$= e^x.\log\sec x+C.$$

14. $\int\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$

Solution: Here,

$$I = \int\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$$

Put $\tan^{-1}x = y$ then $\frac{1}{1+x^2}dx = dy$. Then,

$$I = \int\frac{\tan y.y}{\sqrt{1+\tan^2 y}}dy$$

$$= \int\frac{\tan y.y}{\sec y}dy$$

$$= \int y\sin y\,dy$$

$$= -y\cos y+\sin y+C \quad [\because \text{using Q. 1}]$$

$$= -\frac{\tan^{-1}x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}+C = \frac{1}{\sqrt{1+x^2}}(x-\tan^{-1}x)+C.$$

15. $\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution: Here,

$$I = \int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Put, $x = \tan y$ then $dx = \sec^2 y\,dy$, then,

$$\frac{2x}{1+x^2} = \frac{2\tan y}{1+\tan^2 y} = \frac{2\tan y}{\sec^2 y} = 2\sin y\cos y = \sin 2y$$

Now,

$$I = \int\sin^{-1}(\sin 2y).\sec^2 y\,dy$$

$$= 2\int y.\sec^2 y\,dy$$

Taking y as 1[st] function and $\sec^2 y$ as 2[nd] function, and then applying integration by parts then,

$$I = 2[uv_1-u'v_2]+C$$

$$= 2[y\tan y-\log\sec y]+C = 2y\tan y+2\log\cos y+C$$

$$= 2x.\tan^{-1}x+2\log\left(\frac{1}{\sqrt{1+x^2}}\right)+C$$

$$= 2x\tan^{-1}x-\log(\sqrt{1+x^2})^2+C$$

$$= 2x\tan^{-1}x-\log(1+x^2)+C.$$

16. $\int x\cos^3 x\sin x\,dx$

Solution: Here,

$$I = \int x\cos^3 x\sin x\,dx$$

$$= \int x\left(\frac{\cos 3x+3\cos x}{4}\right)\sin x\,dx$$

$$= \frac{1}{4}\int x\cos 3x\sin x\,dx+\frac{3}{4}\int x\cos x\sin x\,dx$$

$$= \frac{1}{8}\int x\,[\sin(3x + x) - \sin(3x - x)]\,dx + \frac{3}{8}\int x \sin(2x)\,dx$$

$$= \frac{1}{8}\int x \sin(4x)\,dx - \frac{1}{8}\int x \sin(2x)\,dx + \frac{3}{8}\int x \sin(2x)\,dx$$

$$= \frac{1}{8}\int x \sin(4x) + \frac{2}{8}\int x \sin(2x)\,dx$$

Taking x as 1st function and sine function as 2nd function, and then applying integration by parts then,

$$= \frac{1}{8}\left[x\int \sin 4x\,dx - \int\left(\frac{d}{dx}x\int \sin 4x\,dx\right)dx\right] + \frac{2}{8}\left[x\int \sin 2x\,dx - \int\left(\frac{d}{dx}x\int \sin 2x\,dx\right)dx\right]$$

$$= \frac{1}{8}\left[\frac{-x\cos 4x}{4} + \frac{\sin 4x}{16} - x\cos 2x + \frac{\sin 2x}{2}\right] + C.$$

17. $\int \cos(\log x)\,dx$

Solution: Let, $I = \int \cos(\log x)\,dx$

Put $\log x = y$ then $\frac{1}{x}dx = dy \Rightarrow dx = e^y\,dy$. Then,

$$I = \int \cos y\,.\,e^y\,dy$$

Taking cos y as 1st function and e^y as 2nd function, and then applying integration by parts then,

$$I = \cos y\int e^y\,dy - \int\left(\frac{d}{dy}\cos y\int e^y\,dy\right)dy$$

$$= \cos y\,.\,e^y + \int \sin y\,e^y\,dy + C_1$$

$$= e^y\cos y + \sin y\int e^y\,dy - \int\left(\frac{d}{dy}\sin y\int e^y\,dy\right)dy + C_1$$

$$= e^y\cos y + \sin y\,e^y - \int \cos y\,.\,e^y\,dy + C_2$$

$$= e^y(\cos y + \sin y) + C_2 - I$$

$$\Rightarrow I = \frac{1}{2}e^y(\cos y + \sin y) + C$$

$$= \frac{1}{2}x\,[\cos(\log x) + \sin(\log x)] + C.$$

18. $\int e^x \sin x\,dx$

Solution: Let,

$$I = \int \sin x\,e^x\,dx$$

$$= e^x\sin x - \int\left(\frac{d}{dx}\sin x\int e^x\,dx\right)dx + C_1$$

$$= e^x\sin x - \int \cos x\,.\,e^x\,dx + C_1$$

$$= e^x\sin x - \cos x\,.\,e^x - \int \sin x\,.\,e^x\,dx + C_2$$

$$= e^x(\sin x - \cos x) + C_2 - I$$

$$\Rightarrow I = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

19. $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Solution: Here,

$$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

Put $x = \cos\theta$ then $dx = -\sin\theta\,d\theta$.

Since,

$$\frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}$$

So that,

$$I = \int \tan^{-1}\sqrt{\tan^2\frac{\theta}{2}}\,.\,(-\sin\theta)\,d\theta$$

$$= -\int \tan^{-1}\left(\tan\frac{\theta}{2}\right).\sin\theta\,d\theta = -\frac{1}{2}\int \theta\sin\theta\,d\theta.$$

And then process as Q. 1.

$\int \sin\sqrt{x}\,dx$

Solution: Here,

$$I = \int \sin\sqrt{x}\,dx$$

Put $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}dx = dy \Rightarrow dx = 2y\,dy$. Then,

$$I = 2\int y\sin y\,dy.$$

And then process a Q.1.

TECHNIQUE OF SOME INTEGRALS

Integration by Partial Function

Case of proper rational fractions

b. Linear and non–repeated denominator:

For each factor of the form $(ax + b)$, there should be a single frac[tion of] the form $\frac{A}{ax+b}$, where A is a constant.

Example: If $I = \int \frac{x\,dx}{(x+3)(x+1)}$

Let $\frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$,

for A, B are constants.

Then $x = A(x+1) + B(x+3)$ equating the coefficient of x and constant term then we get,

$A + B = 1,\ A + 3B = 0$

Solving we get,

$A = \frac{3}{4},\quad B = \frac{1}{3}$

Then,

$$I = \frac{3}{4}\int \frac{dx}{x+3} + \frac{1}{3}\int \frac{dx}{x+1}$$

$$= \frac{3}{4}\log(x+3) + \frac{1}{3}\log(x+1) + C$$

b. Linear and repeated denominator:

For each factor of the form $(ax + b)^n$, there should be a single fraction $\frac{A_n}{(ax+b)^n}$ for $n = 1, 2, \ldots$

i.e. $\frac{f(x)}{(ax+b)^n} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$

c. Quadratic and non–repeated denominator:

For each factor of the form $ax^2 + bx + c$ (with $a \neq 0$, $c \neq 0$), there should [be a] single fraction of the form,

$\frac{Ax+B}{ax^2+bx+c}$, with A and B are constants.

d. Quadratic and repeated denominator

For each factor of the form $(ax + bx + c)^n$ there should be a single fract[ion of] the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{ax^2+bx+c} + \ldots + \frac{A_nx+B_n}{ax^2+bx+c}$$

Case of Improper Rational Fraction

If the given rational fraction is in improper form then process as

given fraction = Quotient + $\frac{\text{Remainder}}{\text{divisor}}$ (i.e. proper)

where the form $\frac{\text{Remainder}}{\text{divisor}}$ is a proper fraction.

So that, $\int$ given fraction = $\int$ quotient + $\int$ proper fraction.

Exercise 11.4

Integrate the following functions w.r.t. x:

1. (i) $\frac{x}{(x-3)(x+1)}$ [2018 Fall (short)][2015 Fall (short)]

Solution: Here,

$$\frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$\Rightarrow x = A(x+1) + B(x-3) = (A+B)x + (A-3B)$

Equating the coefficient of x and the constant term then,

$A + B = 1,\ A - 3B = 0$

Solving we get, $A = \frac{3}{4}, B = \frac{1}{4}$

Now,

$$\int \frac{x}{(x-3)(x+1)}dx = \frac{3}{4}\int \frac{dx}{x-3} + \frac{1}{4}\int \frac{dx}{x+1}$$

$$= \frac{3}{4}\log|x-3| + \frac{1}{4}\log|x+1| + C$$

$$= \frac{1}{4}\log|(x-3)^3| + \frac{1}{4}\log|x+1| + C$$

$$= \frac{1}{4}\log|(x+1)(x-3)^3| + C.$$

(ii) $\frac{5x-3}{(x+1)(x-3)}$

Solution: Here,

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$\Rightarrow 5x - 3 = A(x+3) + B(x+1)$

$= (A+B)x + (B-3A)$

Equating the coefficient of x and the constant term then,

$A + B = 5,\ B - 3A = -3$

Solving we get, $A = 2$ and $B = 3$

Now,

$$\int \frac{5x-3}{(x+1)(x-3)}dx = 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3}$$

$$= 2\log|x+1| + 3\log|x-3| + C.$$

(iii) $\dfrac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}$

Solution: Here,

$$\frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)} = \frac{A}{x+3} + \frac{B}{x+4} + \frac{C}{x+5}$$

$\Rightarrow (x-1)(x-2) = A(x+4)(x+5) + B(x+3)(x+5) + C(x+3)(x+4)$

$\Rightarrow x^2 - 3x + 2 = (A+B+C)x^2 + (9a + 8B + 7C)x + (20A + 15BG - 12C)$

Equating the coefficient of x^2, x and the constant term then,

$A + B + C = 1,\ \ 9A + 8B + 7C = -3,\ \ 20A + 15B + 12C = 2$

Solving we get, A = 10, B = –30, C = 21.

Now,

$$\int \frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}dx = 10\int \frac{dx}{x+3} - 30\int \frac{dx}{x+4} + 21\int \frac{dx}{x+5}$$

$$= 10\log|x+3| - 30\log|x+4| + 21\log|x+5| + c$$

(iv) $\dfrac{x+1}{(x^2-13x+42)}$

Solution: Here,

$$\frac{x+1}{(x^2-13x+42)} = \frac{x+12}{x^2-7x-6x+42} = \frac{x+12}{(x-7)(x-6)}$$

$$\Rightarrow \frac{x+12}{(x-6)(x-7)} = \frac{A}{x-6} + \frac{B}{x-7}$$

$\Rightarrow\ x + 12 = A(x-7) + B(x-6) = (A+B)x + (-7A - 6B)$

Equating the coefficient of x and the constant term then,

$A + B = 1$ and $-7A - 6B = 12$

Solving we get, $A = -18$ and $B = +19$

Now,

$$\int \frac{x+1}{(x^2-13x+42)}dx = -18\int \frac{dx}{x-6} + 19\int \frac{dx}{x-7}$$

$$= -18\log|x-6| + 19\log|x-7| + C.$$

2. (i) $\dfrac{x+4}{(x+1)^2}$

Solution: Here,

$$\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$\Rightarrow x + 4 = A(x+1) + B = Ax + (A+B)$

Equating the coefficient of x and the constant term then,

$A = 1,\ A + B = 4$

Solving we get, $A = 1, B = 3$

Now,

$$\int \frac{x+4}{(x+1)^2}dx = \int \frac{dx}{x+1} + 3\int \frac{dx}{(x+1)^2}$$

$$= \int \frac{dx}{x+1} + 3\int (x+1)^{-2}dx$$

$$= \log|x+1| + 3\left(\frac{-1}{x+1}\right) + C = \log|x+1| - \frac{3}{x+1} + C$$

(ii) $\dfrac{x^2-4}{(x^2+1)(x^2+3)}$

Solution: Here,

$$\frac{x^2-4}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

$\Rightarrow x^2 - 4 = (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$

$= (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D).$

Equating the coefficient of x^3, x^2, x and the constant term then,

$A + C = 0,\ \ 3A + C = 0,\ \ B + D = 1,\ \ 3B + D = -4$

Solving we get,

$A = 0 = C,\ B = -\dfrac{5}{2},\ D = \dfrac{7}{2}$

Now,

$$\int \frac{x^2-4}{(x^2+1)(x^2+3)}dx = -\frac{5}{2}\int \frac{dx}{x^2+1} + \frac{7}{2}\int \frac{dx}{x^2+3}$$

$$= -\frac{5}{2}\tan^{-1}(x) + \frac{7}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$= -\frac{5}{2}\tan^{-1}(x) + \frac{7}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c.$$

x^3

(iii) $\dfrac{x^2}{(x-a)(x-b)}$

$x^2 - ax - bx + ab$

Solution: Here,

$\dfrac{x^2}{(x-a)(x-b)}$, which is improper quotient form. That is the variable has same power in numerator as in denominator. So,

$$\frac{x^2}{(x-a)(x-b)} = \frac{x^2 - x(a+b) + ab}{x^2 - x(a+b) + ab} + \frac{x(a+b) - ab}{(x-a)(x-b)}$$

$xa + xb - ab$

$$= 1 + \frac{x(a+b) - ab}{(x-a)(x-b)}$$

Here,

$$\frac{x(a+b) - ab}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$\Rightarrow\ x(a+b) - ab = A(x-b) + B(x-a)$

$= (A+B)x + (-Ab - aB)$

Equating the coefficient of x and the constant term then,

$A + B = a + b,\ \ Ab + Ba = ab$

Solving we get,

$$A = \frac{a^2}{a-b}, \quad B = -\frac{b^2}{a-b}$$

Now,

$$\int \frac{x^2}{(x-a)(x-b)}dx = \int dx + \frac{a^2}{a-b}\int \frac{dx}{x-a} - \frac{b^2}{a-b}\int \frac{dx}{x-b}$$

$$= x + \frac{a^2}{a-b}|x-a| - \frac{b^2}{a-b}\log|x-b| + C.$$

(iv) $\frac{2x}{(x^2+1).(x^2+3)}$

Solution: Here,

$$\frac{2x}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

$$\Rightarrow \quad 2x = (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D).$$

Equating the coefficient of x^3, x^2, x and the constant term then,

$A + C = 0, \quad 3A + C = 2, \quad B + D = 0$ and $3B + D = 0$

Solving we get, $B = 0 = D, \quad A = 1$ and $C = -1$.

Now,

$$\int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{x^2+3}$$

$$= \frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \frac{1}{2}\int \frac{2x\,dx}{x^2+3}$$

$$= \frac{1}{2}(\log|x^2+1| - \log|x^2+3|) + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C.$$

3. (i) $\frac{1}{1+3e^x+2e^{2x}}$

Solution: Here,

$$I = \int \frac{dx}{1+3e^x+2e^{2x}}$$

$(2y^2+3y+1)$

Put, $e^x = y$ then $e^x\,dx = dy \Rightarrow dx = \frac{dy}{y}$. Then,

$$I = \int \frac{dy}{y(1+3y+2y^2)} = \int \frac{dy}{y(y+1)(2y+1)} \quad \ldots (1)$$

Here,

$$\frac{1}{y(y+1)(2y+1)} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{2y+1}$$

$$\Rightarrow \quad 1 = A(y+1)(2y+1) + By(2y+1) + C(y+1)y$$

$$= (2A+2B+C)y^2 + (3A+B+C)y + A$$

Equating the coefficient of y^2, y and the constant term then,

$2A + 2B + C = 0, \quad 3A + B + C = 0$ and $A = 1$.

Solving we get, $A = 1, \; B = 1$ and $C = -4$.

Now, (i) becomes

$$I = \int \frac{dy}{y} + \int \frac{dy}{y+1} - 4\int \frac{dy}{2y+1}$$

$$= \int \frac{dy}{y} + \int \frac{dy}{y+1} - 2\int \frac{2dy}{2y+1}$$

$$= \log|y| + \log|y+1| - 2\log|2y+1| + C$$

$$= \log(e^x) + \log(e^x+1) - 2\log(2e^x+1) + C$$

$$= x + \log(e^x+1) - 2\log(1+2e^x) + C.$$

(ii) $\frac{x}{(x-1)^2(x+2)}$

Solution: Here,

$$I = \int \frac{x}{(x-1)^2(x+2)}dx$$

Set,

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\Rightarrow \quad x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$= (A+C)x^2 + (A+B-2C)x + (-2A+2B+C)$$

Equating the coefficient of x^2, x and the constant term then,

$A + C = 0, \quad A + B - 2C = 1, \quad -2A + 2B + C = 0.$

Solving we get,

$$A = \frac{2}{9}, \quad B = \frac{1}{3}, \quad C = -\frac{2}{9}$$

Now,

$$I = \frac{2}{9}\int \frac{dy}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$$

$$= \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int (x-1)^{-2}dx - \frac{2}{9}\int \frac{dx}{x+2}$$

$$= \frac{2}{9}\log(x-1) - \frac{1}{3(x-1)} - \frac{2}{9}\log(x+2) + c$$

$$\Rightarrow \quad I = \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c.$$

(iii) $\frac{x^2+8}{x^2-5x+6}$

Solution: Here,

$$\frac{x^2+8}{x^2-5x+6}$$

This is the improper fraction. So, divide the term at first. So that,

$$\frac{x^2+8}{x^2-5x+6} = \frac{x^2-5x+6}{x^2-5x+6} + \frac{5x+2}{x^2-5x+6}$$

$$= 1 + \frac{5x+2}{(x-2)(x-3)}$$

Here,

$$\frac{5x+2}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Rightarrow 5x+2 = A(x-3) + B(x-2) = (A+B)x - (3A+2B)$$

Equating the coefficient of x and the constant term then,

$$A + B = 5 \text{ and } -(3A+2B) = 2$$

Solving we get, $A = -12$ and $B = 17$

Now,

$$\int \frac{x^2+8}{x^2-5x+6}dx = \int dx - 12\int\frac{dx}{x-2} + 17\int\frac{dx}{x-3}$$

$$= x - 12\log|x-2| + 17\log|x-3| + c.$$

(iv) $\dfrac{1}{x(x^2+x+1)}$

Solution: Here,

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow 1 = A(x^2+x+1) + (Bx+C)x = (A+B)x^2 + (A+C)x + A$$

Equating the coefficient of x^2, x and the constant term then,

$$A = 1, \quad A + B = 0, \quad A + C = 0$$

Solving we get, $A = 1, B = C = -1$

Now,

$$\int\frac{dx}{x(x^2+x+1)} = \int\frac{dx}{x} + \int\frac{-x-1}{x^2+x+1}dx$$

$$= \int\frac{dx}{x} - \int\frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{x^2+x+1}dx$$

$$= \int\frac{dx}{x} - \frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$= \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \left(\frac{1}{2}\right)\log\left|\frac{x^2}{x^2+x+1}\right| - \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \left(\frac{1}{2}\right)\log\left|\frac{x^2}{x^2+x+1}\right| - \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

4. (i) $\dfrac{x^2}{(x^2+1)(x^2+4)}$

Solution: Here,

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$\Rightarrow x^2 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D).$$

Equating the coefficient of x^3, x^2, x and the constant term then,

$$A + C = 0, \quad 4A + C = 0, \quad B + D = 1 \quad \text{and} \quad 4B + D = 0$$

Solving we get, $A = 0 = B$, $B = \frac{-1}{3}$ and $D = \frac{4}{3}$.

Now,

$$\int\frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{-1}{3}\int\frac{dx}{x^2+1} + \frac{4}{3}\int\frac{dx}{x^2+4}$$

$$= \frac{-1}{3}\tan^{-1}(x) + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \frac{-1}{3}\tan^{-1}(x) + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + c.$$

$$= \frac{1}{3}\left(2\tan^{-1}\left(\frac{x}{2}\right) - \tan^{-1}(x)\right) + c.$$

(ii) $\dfrac{1}{x[6(\log x)^2 + 7\log(x) + 2]}$

Solution: Here,

$$I = \int\frac{1}{x[6(\log x)^2 + 7\log(x) + 2]}dx$$

Put $\log x = y$ then $\frac{1}{x}dx = dy$. Then,

$$I = \int\frac{dy}{6y^2+7y+2} = \int\frac{dy}{6y^2+3y+4y+2}$$

$$\Rightarrow I = \int\frac{dy}{(3y+2)(2y+1)}$$

Here,

$$\frac{1}{(3y+2)(2y+1)} = \frac{A}{3y+2} + \frac{B}{2y+1}$$

$$\Rightarrow 1 = A(2y+1) + B(3y+2) = (2A+3B)y + (A+2B)$$

Comparing the coefficient of y and the constant term then,

$$2A + 3B = 0, \quad A + 2B = 1$$

Solving we get, $A = -3$ and $B = +2$

Therefore,

$$I = -3\int\frac{dy}{3y+2} + 2\int\frac{dy}{2y+1}$$

$$= -\log|3y+2| + 2\log|2y+1| + c$$

$$= \log\left|\frac{2y+1}{3y+2}\right| + c$$

$$= \log\left|\frac{2\log x+1}{3\log x+2}\right| + c.$$

(iii) $\dfrac{\cos x}{(\sin x+2)(2\sin x+3)}$

Solution: Here,

$$I = \int \frac{\cos x}{(\sin x+2)(2\sin x+3)}\,dx$$

Put $\sin x = y$ then $\cos x\,dx = dy$. So that,

$$I = \int \frac{dy}{(y+2)(2y+3)}$$

Here,

$$\frac{1}{(y+2)(2y+3)} = \frac{A}{y+2} + \frac{B}{2y+3}$$

$$\Rightarrow\ 1 = A(2y+3) + B(y+2) = (2A+B)\,y + (3A+2B)$$

Equating the coefficient of y and the constant term then

$$2A + B = 0,\quad 3A + 2B = 1$$

Solving we get, $A = -1$ and $B = 2$

Then,

$$I = -\int\frac{dy}{y+2} + \int\frac{2\,dy}{2y+3}$$

$$= -\log|y+2| + \log|2y+3| + c$$

$$= \log\left|\frac{2y+3}{y+2}\right| + c$$

$$= \log\left|\frac{2\sin x+3}{\sin x+2}\right| + c.$$

(iv) $\dfrac{x^3-5x}{(x^2-9)(x^2+1)}$

Solution: Let,

$$I = \int\frac{x^3-5x}{(x^2-9)(x^2+1)}\,dx$$

x^2-3^2

$(x+3)(x-3)$

Here,

$$\frac{x^3-5x}{(x^2-9)(x^2+1)} = \frac{Ax+B}{x^2-9} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow\ x^3 - 5x = (Ax+B)(x^2+1) + (Cx+D)(x_2-9)$$

$$= (A+C)x^3 + (B-9D)x^2 + (A-9C)x + (B-9D).$$

Equating the coefficient of x^3, x^2, x and the constant term from sides then.

$$A + C = 1,\qquad B + D = 0$$

$$A - 9C = -5,\quad B - 9D = 0$$

Solving we get,

$$A = \frac{2}{5},\ C = \frac{3}{5},\ B = 0,\ D = 0$$

Thus,

$$I = \frac{2}{5}\int\frac{x}{x^2-9}\,dx\,dx + \frac{3}{5}\int\frac{x}{x^2+1}\,dx$$

$$= \frac{2}{5}\cdot\frac{1}{2}\log|x^2-9| + \frac{3}{5}\cdot\frac{1}{2}\log|x^2+1| + c$$

$$= \frac{1}{5}\log|x^2-9| + \frac{3}{10}\log|x^2+1| + c.$$

(v) $\dfrac{1}{(e^x-1)^2}$

Solution: Here, $I = \displaystyle\int\frac{dx}{(e^x-1)^2}$

Put $e^x = y$ then $e^x\,dx = dy \Rightarrow dx = \dfrac{dy}{y}$. Then,

$$I = \int\frac{dy}{y(y-1)^2}$$

Here,

$$\frac{1}{y(y-1)^2} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{(y-1)^2}$$

$\log a - \log b$

$\log(a/b)$

$$\Rightarrow\ 1 = A(y-1)^2 + By(y-1) + Cy.$$

$$= (A+B)\,y^2 + (-2A - B + C)\,y + A$$

Equating the coefficient of y^2, y and the constant term then.

$$A + B = 0,\quad -2A - B + C = 0,\quad A = 1.$$

Solving we get, $A = 1$, $B = -1$ and $C = 1$

Therefore,

$$I = \int\frac{dy}{y} - \int\frac{dy}{y-1} + \int\frac{dy}{(y-1)^2}$$

$$= \log|y| - \log|y-1| - (y-1)^{-1} + c$$

$$= \log\left|\frac{y}{y-1}\right| - \left|\frac{1}{y-1}\right| + c$$

$$= \log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + c.$$

(vi) $\dfrac{7x^2+3x+1}{x^2+x}$

Solution: Here,

$$I = \int\left(\frac{7x^2+3x+1}{x^2+x}\right)dx$$

$$= \int \left(7 + \frac{-4x+1}{x^2+x}\right) dx$$

$$= 7\int dx - \int \left(\frac{4x+2-2-1}{x^2+x}\right) dx$$

$$= 7\int dx - 2\int \left(\frac{2x+1}{x^2+x}\right) dx + 3\int \frac{dx}{x(x+1)}$$

Here,

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$\Rightarrow \quad 1 = A(x+1) + Bx = (A+B)x + A$$

Equating the coefficient of x and the constant term then,

$$A + B = 0, A = 1$$

So that, $A = 1, B = -1$.

Therefore,

$$I = 7\int dx - 2\int \frac{2x+1}{x^2+x} dx + 3\int \frac{dx}{x} - 3\int \frac{dx}{x+1}$$

$$= 7x - 2\log|x^2+x| + 3\log|x| - 3\log|x+1| + c.$$

Miscellaneous Exercise

1. Show that $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x} dx = \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}x + C$

Solution: Repeated question to exercise 11.2 Q. No. B(8).

2. Show that $\int \left(\frac{e^x(1+\sin x)}{1+\cos x}\right) dx = e^x \tan\left(\frac{x}{2}\right) + C$ [2016 Fa

Solution: Let,

$$I = \int \left(\frac{e^x(1+\sin x)}{1+\cos x}\right) dx$$

$$= \int e^x \left(\frac{1 + 2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right) dx$$

$$= \int e^x \left(\frac{1}{2}\sec^2(x/2) + \tan(x/2)\right) dx$$

Set $f(x) = \tan\left(\frac{x}{2}\right)$ then $f'(x) = \sec^2\left(\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{1}{2}\sec^2(x/2)$

So, $I = \int e^x [f'(x) + f(x)] dx$

$$= e^x f(x) + C$$

$$= e^x \tan\left(\frac{x}{2}\right) + C.$$

4. Show that $\int \frac{dx}{x\sqrt{x^4-1}} = \frac{1}{2}\sec^{-1}(x^2) + C$

Solution: Let,

$$I = \int \frac{dx}{x\sqrt{x^4-1}}$$

Put $x^2 = y$ then $2x\,dx = dy$. So, $x^2\frac{dx}{x} = \frac{dy}{2} \Rightarrow \frac{dx}{x} = \frac{dy}{2y}$

Then,

$$I = \int \frac{dy}{2y\sqrt{y^2-1}}$$

Set, $y = \sec\theta$ then $dy = \sec\theta\tan\theta\, d\theta$. So that,

$$I = \int \left(\frac{\sec\theta\tan\theta}{2\sec\theta\sqrt{\sec^2\theta - 1}}\right) d\theta$$

$$= \frac{1}{2}\int \frac{\tan\theta}{\tan\theta} d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\sec^{-1}(x^2) + C.$$

5. Show that $\int \frac{x^2+1}{x^4+1} dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + C$

Solution: Let,

$$I = \int \frac{x^2+1}{x^4+1} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx \quad \text{[dividing by } x^2\text{]}$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2})^2} dx$$

Put, $x - \frac{1}{x} = y$ then $\left(1 + \frac{1}{x^2}\right) dx = dy$. Then,

$$I = \int \frac{dy}{y^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x - (1/x)}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + C.$$

6. Show that $\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x} dx = \frac{1}{a-b}\log(a\sin^2 x + b\cos^2 x) + C$

Solution: Let,

$$I = \int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x}\,dx$$

$$= \int \frac{2\sin x\cos x}{a\sin^2 x + b\cos^2 x}\,dx$$

$$= \left(\frac{1}{a-b}\right)\int \frac{2a\sin x\cos x - 2b\cos x\sin x}{a\sin^2 x + b\cos^2 x}\,dx$$

$$= \left(\frac{1}{a-b}\right)\int \frac{d(a\sin^2 x + b\cos^2 x)}{a\sin^2 x + b\cos^2 x}$$

$$= \left(\frac{1}{a-b}\right)\log(a\sin^2 x + b\cos^2 x) + C.$$

7. Show that $\int \frac{dx}{\sqrt{1+\sin x}} = \sqrt{2}\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right) + C$

Solution: Let,

$$I = \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= \int \frac{dx}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$$= \int \frac{dx}{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}}$$

$$= \int \frac{dx}{\cos\frac{x}{2} + \sin\frac{x}{2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\frac{\pi}{4}\cos\frac{x}{2} + \cos\frac{\pi}{4}\sin\frac{x}{2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$$

Put $\frac{x}{2}+\frac{\pi}{4} = y$ then $\frac{dx}{2} + dy$. Then,

$$I = \frac{2}{\sqrt{2}}\int \operatorname{cosec} y\,dy$$

$$= \sqrt{2}\log\left(\tan\frac{y}{2}\right) + C$$

$$= \sqrt{2}\log\left(\tan\left(\frac{\frac{\pi}{4}+\frac{x}{2}}{2}\right)\right) + C = \sqrt{2}\log\left(\tan\left(\frac{\pi}{8}+\frac{x}{4}\right)\right) + C.$$

8. Show that $\int \frac{\sin x\,dx}{\sqrt{1+\sin x}} = 2\sqrt{1-\sin x} - \sqrt{2}\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right) + C$

Solution: Let,

$$I = \int \frac{\sin x}{\sqrt{1+\sin x}}\,dx$$

$$= \int \frac{1+\sin x - 1}{\sqrt{1+\sin x}}\,dx$$

$$= \int \sqrt{1+\sin x}\,dx - \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= I_1 - I_2 \qquad \text{.....(i)}$$

Here,

$$I_1 = \int \sqrt{1+\sin x}\,dx$$

$$= \int \sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\,dx$$

$$= \int \cos\frac{x}{2}.\,dx + \int \sin\frac{x}{2}\,dx$$

$$= \left(\frac{\sin(x/2)}{1/2}\right) + \left(\frac{-\cos(x/2)}{1/2}\right) + C_1$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C_1$$

$$= 2\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2} + C_1 = 2\sqrt{1-2\sin\frac{x}{2}\cos\frac{x}{2}} + C_1$$

$$= 2\sqrt{1-\sin x} + C_1$$

And, $I_2 = \int \frac{dx}{\sqrt{1+\sin x}}$

$$= \sqrt{2}\log\left(\tan\left(\frac{\pi}{8}+\frac{x}{4}\right)\right) + C_2 \quad \text{[by Q.7]}$$

Thus, (i) becomes,

$$I = 2\sqrt{1-\sin x} - \sqrt{2}\log\left(\tan\left(\frac{\pi}{8}+\frac{x}{4}\right)\right) + C \qquad \text{for } C = C_1 + C_2$$

9. Show that $\int \frac{(1+\cos x)}{\sin x.\cos x}\,dx = \log\left(\tan x.\tan\frac{x}{2}\right) + C$

Solution: Let,

$$I = \int \left(\frac{1+\cos x}{\sin x\cos x}\right)dx$$

$$= \int \left(\frac{1}{\sin x\cos x} + \frac{1}{\sin x}\right)dx$$

$$= 2\int \frac{1}{\sin 2x}\,dx + \int \operatorname{cosec} x\,dx$$

$$= 2\int \text{cosec}\, 2x\, dx + \int \text{cosec}x\, dx$$

$$= 2\log\left(\tan\frac{2x}{2}\right)\cdot\frac{1}{2} + \log\tan\left(\frac{x}{2}\right) + C$$

$$= \log(\tan x) + \log\left(\tan\frac{x}{2}\right) + C$$

$$= \log\left(\tan x.\tan\frac{x}{2}\right) + C.$$

10. Show that $\int \frac{dx}{a^2 - b^2\cos^2x} = \frac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a}{\sqrt{a^2-b^2}}\tan x\right)$ for a>b.

Solution: Let,

$$I = \int \frac{dx}{a^2 - b^2\cos^2x} \quad \text{for } a > b$$

$$= \int \frac{\sec^2x}{a^2\sec^2x - b^2}dx \quad [\therefore \text{ dividing by } \cos^2x]$$

$$= \int \frac{\sec^2x}{a^2 + a^2\tan^2x - b^2}dx.$$

$$= \int \frac{\sec^2x}{a^2\tan^2x - (b^2 - a^2)}dx$$

Put a tanx = y then $a\sec^2x\, dx = dy$. Then,

$$I = \frac{1}{a}\int \frac{dy}{y^2 + (a^2 - b^2)} = \frac{1}{a}\cdot\frac{1}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{y}{\sqrt{a^2-b^2}}\right) + C$$

$$= \frac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a\tan x}{\sqrt{a^2-b^2}}\right) + C.$$

11. Show that $\int \frac{dx}{a^2\cos^2x + b^2\sin^2x} = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + C$

Solution: Let,

$$I = \int \frac{dx}{a^2\cos^2x + b^2\sin^2x}$$

$$= \int \frac{\sec^2x\, dx}{a^2 + b^2\tan^2x} \quad [\text{dividing by } \cos^2x]$$

Put b tanx = y then $b\sec^2x\, dx = dy$. Then,

$$I = \frac{1}{b}\int \frac{dy}{y^2 + a^2} = \frac{1}{b}\cdot\frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right) + C$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right) + C.$$

12. Show that $\int \frac{\log x}{(1+\log x)^2}dx = \frac{x}{1+\log x} + C$ **for x > 0.**

Solution: Let,

$$I = \int \frac{\log x}{(1+\log x)^2}dx$$

$$= \int \frac{1 + \log x - 1}{(1+\log x)^2}dx$$

$$= \int \frac{1+\log x}{(1+\log x)^2}dx - \int \frac{dx}{(1+\log x)^2}$$

$$= \int \frac{dx}{1+\log x} - \int \frac{dx}{(1+\log x)^2}$$

$$= \int 1(1+\log x)^{-1}dx - \int (1+\log x)^{-2}dx$$

$$= (1+\log x)^{-1}.x + \int (1+\log x)^{-2}\frac{1}{x}.x\, dx - \int (1+\log x)^{-2}dx + C$$

[∵ applying by parts for first integral]

$$= \frac{x}{1+\log x} + \int \frac{dx}{(1+\log x)^2} - \int \frac{dx}{(1+\log x)^2} + C$$

$$= \frac{x}{1+\log x} + C.$$

13. Show that $\int \sqrt{\sec x - 1}\, dx = -2\log\left[\cos\frac{x}{2} + \sqrt{\frac{\cos x}{2}}\right] + C$

Solution: Let,

$$I = \int \sqrt{\sec x - 1}\, dx$$

$$= \int \sqrt{\frac{1-\cos x}{\cos x}}dx$$

$$= \int \sqrt{\frac{2\sin^2(x/2)}{2\cos^2(x/2) - 1}}dx$$

$$= \int \frac{\sqrt{2}.\sin(x/2)}{\sqrt{2\cos^2(x/2) - 1}}dx$$

Put $\sqrt{2}\cos\left(\frac{x}{2}\right) = y$ then $-\sqrt{2}.\sin\left(\frac{x}{2}\right).\frac{1}{2}dx = dy$. Then

$$I = \int \frac{2\, dy}{\sqrt{y^2 - 1}} = 2\log(y + \sqrt{y^2 - 1}) + C$$

$$= 2\log\left(\sqrt{2}\cos(x/2) + \sqrt{\cos x}\right) + C.$$

14. Show that $\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}dx = 2x + \log(2\cos x + \sin x + 3) + C.$

Solution: Let,

$$I = \int \frac{5\cos x + 6}{2\cos x + \sin x + 3}dx \quad \text{.... (i)}$$

Here,

$$(5\cos x + 6) = A\, d(2\cos x + \sin x + 3) + B(2\cos x + \sin x + 3)$$
$$= A(-2\sin x + \cos x) + B(2\cos x + \sin x + 3)$$

Equating the coefficient of sinx, cosx and the constant term from both sides then,

$-2B + B = 0, \quad A + 2B = 5, \quad 3B = 6$

Solving we get,

$A = 1, \quad B = 2.$

Therefore, (1) becomes,

$$I = \int \frac{d(2\cos x + \sin x + 3)}{2\cos x + \sin x + 3} + 2\int \frac{2\cos x + \sin x + 3}{2\cos x + \sin x + 3}\,dx$$

$$= \log(2\cos x + \sin x + 3) + 2\int dx + C.$$

$$= \log(2\cos x + \sin x + 3) + 2x + C.$$

Properties of Definite Integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad$ for $a < c < b$
4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
5. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if f is even} \\ 0 & \text{if f is odd} \end{cases}$
6. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

WI

Exercise 11.5

Show that:

1. $\int_0^{\pi/2} \frac{\sin\theta\,d\theta}{\sin\theta + \cos\theta} = \frac{\pi}{4}$ [2013 Fall Short]

Solution: Here,

$$I = \int_0^{\pi/2} \frac{\sin\theta\,d\theta}{\sin\theta + \cos\theta} \quad \dots \text{(i)}$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)}\,d\theta \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta}\,d\theta \quad \dots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\sin\theta\,d\theta}{\sin\theta + \cos\theta} + \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta}\,d\theta = \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2}.$$

Thus, $I = \frac{\pi}{4}$.

2. $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}} = \frac{\pi}{4}$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \dots \text{(i)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \dots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Thus, $I = \frac{\pi}{4}$.

3. $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$ **[2017 Spring][2009 Spring] [2003, Spring] [2007, Fall]**

Solution: Here,

$$I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

Put $x = a\sin\theta$ then $dx = a\cos\theta\, d\theta$.

When $x = 0 \Rightarrow \theta = 0$, and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{a\cos\theta}{a\sin\theta + a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta \quad \dots (i)$$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-\theta\right) d\theta}{\sin\left(\frac{\pi}{2}-\theta\right) + \cos\left(\frac{\pi}{2}-\theta\right)} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta \quad \dots (ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta} d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

Thus, $I = \frac{\pi}{4}$.

4. $\int_0^{\pi/2} \frac{\sqrt{\cot x}\,dx}{1+\sqrt{\cot x}} = \frac{\pi}{4}$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{\sqrt{\cot x}\,dx}{1+\sqrt{\cot x}} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

Same to Q. No. 2 equation (i).

Thus from the Q. No 2, we have

$I = \frac{\pi}{4}$.

5. $\int_0^{\pi} \frac{x\tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$

Solution: Here,

$$I = \int_0^{\pi} \frac{x\tan x}{\sec x + \cos x} dx$$

$$= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx \quad \dots (i)$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx \quad \dots (ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$$

Put, $\cos x = y$ then $-\sin x\, dx = dy$.

When $x = 0 \Rightarrow y = 1$ and $x = \pi \Rightarrow y = -1$. Then,

$$2I = \pi \int_1^{-1} \frac{-dy}{1+y^2}$$

$$= \pi \int_{-1}^{1} \frac{dy}{1+y^2} = \pi\left[\tan^{-1} y\right]_{-1}^{1}$$

$$= \pi\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi^2}{2}.$$

$$\Rightarrow I = \frac{\pi^2}{4}.$$

6. $\int_0^a \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{a-x}} = \frac{a}{2}$ **[2012 Fall]**

Solution: Here,

$$I = \int_0^a \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{a-x}} \quad \dots (i)$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \quad \dots (ii) \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding (i) and (ii) then,

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx = \int_0^a dx = [x]_0^a = a.$$

$$\Rightarrow I = \frac{a}{2}.$$

7. $\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log 2$ [2002] [2003, Spring] [2004, Fall] [2011 Spring][2011 Fall][2004, Spring][2006, Fall][2008, Spring] [2016 Spring][2016 Fall][2014 Fall][2013 Fall]

Solution: Here,

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/4} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan\theta}{\tan\frac{\pi}{4}\cdot\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4} \log(2)\,d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \log(2)\int_0^{\pi/4} d\theta - I.$$

$$\Rightarrow 2I = \log(2)\int_0^{\pi/4} d\theta = \log(2)\,[\theta]_0^{\pi/4} = \frac{\pi}{4}\cdot\log(2)$$

$$\Rightarrow I = \frac{\pi}{8}\log(2).$$

8. $\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} = \frac{\pi^2}{4}$ **(Repeated Question to Q. No. 5)**

[2015 Fall][2013 Spring][2011 Spring]

9. $\int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x} = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$ [2018 Fall]

Solution: Here,

$$I = \int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x} \quad \dots (i)$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)}{\cos x+\sin x}\,dx \quad \dots (ii)$$

Adding (i) and (ii) then,

$$2I = \frac{\pi}{2}\cdot\int_0^{\pi/2} \frac{dx}{\cos x+\sin x}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin\frac{\pi}{4}\cdot\cos x+\cos\frac{\pi}{4}\cdot\sin x}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left(\operatorname{cosec}\left(\frac{\pi}{4}+x\right)\right)-\cot\left(\frac{\pi}{4}+x\right)\right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left(\operatorname{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right)-\log\left(\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}}[\log(-\sqrt{2}-1) - \log(\sqrt{2}-1)]$$

$$= \frac{\pi}{2\sqrt{2}} \log\left(\frac{-\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \log\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \log\left\{\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right) \times \frac{(1+\sqrt{2})}{(1+\sqrt{2})}\right\}$$

$$= \frac{\pi}{2\sqrt{2}} \log \frac{(1+\sqrt{2})^2}{1}$$

$$= \frac{2\pi}{2\sqrt{2}} \log(1+\sqrt{2})$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{2}} \log(1+\sqrt{2}).$$

10. $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log(2).$ [2017 Fall]

Solution: Here,

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

Put $x = \tan\theta$ then $dx = \sec^2\theta\, d\theta$.

When $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{4}$. Then,

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

(Same Question to Q. No. 7)

11. Show that $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x.\cos x}\,dx = 0$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x.\cos x}\,dx \quad \ldots (i)$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right).\cos\left(\frac{\pi}{2}-x\right)}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x.\sin x}\,dx \quad \ldots (ii)$$

Now, adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x.\cos x}\,dx = 0.\int_0^{\pi/2} \frac{dx}{1+\sin x.\cos x} = 0$$

$$\Rightarrow I = 0.$$

12. $\displaystyle\int_0^{\pi/2} \log(\tan x)\,dx = 0$

Solution: Here,

$$I = \int_0^{\pi/2} \log(\tan x)\,dx \quad \ldots (i) \quad \text{for } \tan x > 0$$

$$= \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \log(\cot x)\,dx \quad \ldots (ii)$$

Now, adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\,dx$$

$$= \int_0^{\pi/2} \log(\tan x.\cot x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(1)\,dx = \log(1)\int_0^{\pi/2} dx = 0.\int_0^{\pi/2} dx = 0.$$

$$\Rightarrow I = 0.$$

13. $\int_0^{\pi/2} \sin 2x\,(\log \tan x)\,dx = 0.$

Solution: Here,

$$I = \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx \qquad \ldots (i)$$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \sin 2x \log(\cot x)\,dx \qquad \ldots (ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \sin 2x \log(\tan x.\cot x)\,dx = \int_0^{\pi/2} \sin 2x \log(1)\,dx.$$

$$\Rightarrow 2I = \log(1)\int_0^{\pi/2} \sin 2x\,dx = 0.\int_0^{\pi/2} \sin 2x\,dx \qquad [\because \log(1) = 0]$$

$$\Rightarrow I = 0.$$

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14. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$ [2006, Spring] [2007, Spring]

Solution: Here,

$$I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx$$

Put, $x = \sin\theta$ then $dx = \cos\theta\,d\theta$.

When $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{\log(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int_0^{\pi/2} \log(\sin\theta)\,d\theta \qquad \ldots (i)$$

$$= \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/2} \log(\cos\theta)\,d\theta \qquad \ldots (ii)$$

Now, adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \log(\sin\theta.\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2\theta}{2}\right) d\theta = \int_0^{\pi/2} \log(\sin 2\theta)\,d\theta - \log(2)\int_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \log(\sin 2\theta)\,d\theta + \frac{\pi}{2}.\log\left(\frac{1}{2}\right)$$

Put $2\theta = y$ then $2d\theta = dy$. When $\theta = 0 \Rightarrow y = 0$ and $\theta = \frac{\pi}{2} \Rightarrow y = \pi$.

Then,

$$2I = \frac{1}{2}\int_0^{\pi} \log(\sin y)\,dy + \frac{\pi}{2}.\log\left(\frac{1}{2}\right)$$

$$= \int_0^{\pi/2} \log(\sin y)\,dy + \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

$$= I + \frac{\pi}{2}\log\left(\frac{1}{2}\right) \qquad [\because \text{using (i)}]$$

$$\Rightarrow I = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

-1 log2

log2 -1

+log(1/2)

15. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}\,dx = \frac{\pi^2}{16}$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}\,dx \qquad \ldots (i)$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\cos^4\left(\frac{\pi}{2} - x\right) + \sin^4\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x \sin x}{\sin^4 x + \cos^4 x}\,dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$\Rightarrow\ 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x \sec^2 x}{\tan^4 x + 1}\,dx \qquad [\text{dividing by } \cos^4 x]$$

Put $\tan^2 x = y$ then $2\tan x.\ \sec^2 x\, dx = dy$.

When $x = 0 \Rightarrow y = 0$ and $x = \frac{\pi}{2} \Rightarrow y = \infty$. Then,

$$2I = \frac{\pi}{4}\int_0^{\infty} \frac{dy}{y^2+1}$$

$$= \frac{\pi}{4}[\tan^{-1} y]_0^{\infty} = \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0) = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{8}.$$

$$\Rightarrow I = \frac{\pi^2}{16}.$$

16. $\displaystyle\int_0^1 \cot^{-1}(1 - x + x^2)\,dx = \frac{\pi}{2} - \log(2).$ **[2018 Spring][2014 Spring]**

Solution: Here,

$$I = \int_0^1 \cot^{-1}(1 - x + x^2)\,dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right)dx \qquad \left[\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \int_0^1 \tan^{-1}\left(\frac{x + (1-x)}{1 - x(1-x)}\right)dx$$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\,dx \qquad \left[\because \tan^{-1}\left(\frac{A+B}{1-AB}\right) = \tan^{-1} A + \tan^{-1} B\right]$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= I_1 + I_2$$

Here,

$$I_2 = \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1}(1-(1-x))\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 \tan^{-1} x\,dx$$

Then,

$$I = I_1 + I_2 = \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx$$

$$= 2\int_0^1 1.\tan^{-1} x\,dx$$

$$= 2.\left\{[\tan^{-1} x.\,x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\right\} \qquad [\because \text{integrating by parts}]$$

$$= 2.\tan^{-1}(1) - \int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= 2.\frac{\pi}{4} - [\log(1+x^2)]_0^1 = \frac{\pi}{2} - \log(2) \qquad [\because \log(1) = 0]$$

$$\Rightarrow I = \int_0^1 \cot^{-1}(1 - x + x^2)\,dx = \frac{\pi}{2} - \log 2.$$

Miscellaneous Exercise

Evaluate the following integral:

1. $\displaystyle\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$

Solution: Let $I = \displaystyle\int \frac{\cos 2x - \cos 2x}{\cos x - \cos\alpha}\,dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\,dx$$

$$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}\,dx$$

$$= 2\int (\cos x + \cos\alpha)\,dx$$

$= 2\left[\int \cos x\, dx + \cos\alpha \int dx\right]$

$= 2(\sin x + x\cos\alpha) + C.$

2. $\displaystyle\int \frac{dx}{(e^x + e^{-x})^2}$

Solution: Let,

$$I = \int \frac{dx}{(e^x - e^{-x})^2}$$

$$= \int \frac{e^{2x}}{(e^{2x} - 1)^2}\, dx$$

Put $e^{2x} - 1 = y$ then $2e^{2x}\, dx = dy$. Then,

$$I = \frac{1}{2}\int \frac{dy}{y^2}$$

$$= \frac{1}{2}\left(\frac{y^{-1}}{-1}\right) + C$$

$$= -\frac{1}{2(e^{2x} - 1)} + C.$$

3. $\displaystyle\int \sqrt{\frac{a + x}{x}}\, dx$ **[2013 Fall][2011 Spring]**

Solution: Let,

$$I = \int \sqrt{\frac{a + x}{x}}\, dx$$

Put $x = a\tan^2\theta$ then $dx = 2a\tan\theta.\sec^2\theta\, d\theta$. Then,

$$I = \int \sqrt{\frac{a + a\tan^2\theta}{a\tan^2\theta}}\,.\,2a\tan\theta\,.\,\sec^2\theta\, d\theta$$

$$= \int \sqrt{\frac{\sec^2\theta}{\tan^2\theta}}\,.\,2a\sec^2\theta\tan\theta\, d\theta$$

$$= \int \frac{2a\sec^3\theta\tan\theta\, d\theta}{\tan\theta} = 2a\int \sec^3\theta\, d\theta$$

$$= 2a\int \sqrt{1 + \tan^2\theta}\,.\sec^2\theta\, d\theta$$

Set $\tan\theta = y$ then $\sec^2\theta\, d\theta = dy$. Then,

$$I = 2a\int \sqrt{1 + y^2}\, dy$$

$$= 2a\left[\frac{y\sqrt{1 + y^2}}{2} + \frac{1}{2}\log\left(y + \sqrt{1 + y^2}\right)\right] + C_1$$

$$= 2a\left[\frac{\tan\theta\,.\sec\theta}{2} + \frac{1}{2}\log(\tan\theta + \sec\theta)\right] + C_1$$

$$= a\left[\sqrt{\frac{x}{a}}\,.\sqrt{\frac{x + a}{a}} + \log\left(\sqrt{\frac{x}{a}} + \sqrt{\frac{x + a}{a}}\right)\right] + C_1$$

$$= \sqrt{x(x + a)} + a\log\left(\sqrt{x} + \sqrt{x + a}\right) + C$$

$$\left[\because C = a\log\left(\frac{1}{\sqrt{a}}\right) + C_1\right]$$

4. $\displaystyle\int \frac{x^2 - 1}{x^4 - x^2 + 1}\, dx$

Solution: Let,

$$I = \int \frac{x^2 - 1}{x^4 - x^2 + 1}\, dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}}\, dx \quad \text{[dividing by } x^2\text{]}$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 3}\, dx$$

Put $x + \frac{1}{x} = y$ then $\left(1 - \frac{1}{x^2}\right) dx = dy$. Then,

$$I = \int \frac{dy}{y^2 - 3} = \frac{1}{2\sqrt{3}}\log\left(\frac{y - \sqrt{3}}{y + \sqrt{3}}\right) + C$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{x + \frac{1}{x} - \sqrt{3}}{x + \frac{1}{x} + \sqrt{3}}\right) + C$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{x^2 - \sqrt{3}\,x + 1}{x^2 + x\sqrt{3} + 1}\right) + C.$$

5. $\displaystyle\int \frac{x^2}{x^4 + 1}\, dx$

Solution: Let,

$$I = \int \frac{x^2}{x^4 + 1}\, dx = \frac{1}{2}\int \frac{x^2 + 1 + x^2 - 1}{x^4 + 1}\, dx$$

$$= \frac{1}{2}\int \frac{x^2 + 1}{x^4 + 1}\, dx + \frac{1}{2}\int \frac{x^2 - 1}{x^4 + 1}\, dx$$

$$= \frac{1}{2}\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx + \frac{1}{2}\int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx$$

[dividing by x^2]

$$= \frac{1}{2}\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2}\, dx + \frac{1}{2}\int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2}\, dx$$

Put $x-\frac{1}{x}=u$ then $\left(1+\frac{1}{x^2}\right)dx = du$.

Also, put $x+\frac{1}{x}=v$ then $\left(1-\frac{1}{x^2}\right)dx = dv$. Then,

$$I = \frac{1}{2}\int\frac{du}{u^2+2}+\frac{1}{2}\int\frac{dv}{v^2-2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left(\frac{v-\sqrt{2}}{v+\sqrt{2}}\right)+C$$

$$= \frac{1}{2\sqrt{2}}\log\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\log\left(\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right)+C.$$

6. $\int\frac{\cos x}{\cos x+\sin x}dx$

Solution: Let,

$$I = \int\frac{\cos x}{\cos x+\sin x}dx \qquad .....(i)$$

Here,

$\cos x = A\, d(\cos x+\sin x) + B(\cos x + \sin x)$

$\quad = A(-\sin x+\cos x)+B(\cos x+\sin x)$

Equating the coefficient of sinx and cosx from both sides then,

$-A+B=0$ and $A+B=1$

Solving we get,

$A=\frac{1}{2}, B=\frac{1}{2}$

Therefore, (i) becomes,

$$I = \frac{1}{2}\int\frac{d(\cos x+\sin x)}{\cos x+\sin x}+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x+\sin x}dx$$

$$= \frac{1}{2}\int\frac{d(\cos x+\sin x)}{\cos x+\sin x}+\frac{1}{2}\int dx$$

$$= \frac{1}{2}\log(\cos x+\sin x)+\frac{1}{2}x+C.$$

7. $\int\log\left(\sqrt{x^2+1}\right)dx.$

Solution: Let,

$$I = \int\log\left(\sqrt{x^2+1}\right)dx$$

$$= \frac{1}{2}\int\log(x^2+1)\,dx$$

$$= \frac{1}{2}\int 1\log(x^2+1)\,dx$$

$$= \frac{1}{2}\left[x\log(x^2+1)-\int x.\frac{1}{x^2+1}2x\,dx\right]+C_1$$

[Applying by parts]

$$= \frac{1}{2}x\log(x^2+1)-\int\frac{x^2}{x^2+1}dx+C_1$$

$$= \frac{1}{2}x\log(x^2+1)-\int\frac{x^2+1}{x^2+1}dx+\int\frac{dx}{x^2+1}+C_1$$

$$= \frac{x}{2}\log(x^2+1)-x+\tan^{-1}(x)+C_1+C_2$$

$$= \frac{x}{2}\log(x^2+1)-x+\tan^{-1}x+C \qquad \text{for } C=C_1+C_2$$

$\int\tan^{-1}\left(\sqrt{x+1}\right)dx$

Solution: Let,

$$I = \int\tan^{-1}\left(\sqrt{x+1}\right)dx$$

Put $\sqrt{x+1}=y$ then $\frac{1}{2\sqrt{x+1}}dx = dy \Rightarrow dx = 2y\,dy$. Then,

$$I = \int 2y\tan^{-1}y\,dy$$

$$= 2\left[\tan^{-1}y\frac{y^2}{2}-\int\frac{1}{y^2+1}\cdot\frac{y^2}{2}dy\right]+C_1$$

$$= y^2\tan^{-1}y-\int\frac{y^2+1-1}{y^2+1}dy+C_1$$

$$= y^2\tan^{-1}y-\int dy+\int\frac{dy}{y^2+1}+C_1$$

$$= y^2\tan^{-1}y-y+\tan^{-1}y+C_1+C_2$$

$$= (x+1)\tan^{-1}\left(\sqrt{x+1}\right)-\sqrt{x+1}+\tan^{-1}\left(\sqrt{x+1}\right)+C$$

for $C=C_1+C_2$

$$= (x+2)\tan^{-1}\left(\sqrt{x+1}\right)-\left(\sqrt{x+1}\right)+C$$

9. $\int_0^{\pi/2}\frac{\cos x}{\sqrt{1+\cos x}}dx$

Solution: Let,

$$I = \int_0^{\pi/2}\frac{\cos x}{\sqrt{1+\cos x}}dx$$

$$= \int_0^{\pi/2}\frac{2\cos^2(x/2)-1}{\sqrt{2}.\cos(x/2)}dx$$

$$= \sqrt{2}\int_0^{\pi/2} \cos(x/2)\,dx - \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sec(x/2)\,dx$$

$$= 2\sqrt{2}\left[\sin\frac{x}{2}\right]_0^{\pi/2} - \frac{2}{\sqrt{2}}\left[\log\left(\sec\frac{x}{2} + \tan\frac{x}{2}\right)\right]_0^{\pi/2}$$

$$= 2\sqrt{2}.\sin\left(\frac{\pi}{4}\right) - \frac{2}{\sqrt{2}}\left[\log\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) - \log(\sec 0 + \tan 0)\right]$$

$$= 2\sqrt{2}.\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}\log(\sqrt{2}+1) \qquad [\text{since } \log(1) = 0]$$

$$= 2 - \sqrt{2}\log(\sqrt{2}+1).$$

10. $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}$.

Solution: Let,

$$I = \int_0^{\pi/2} \frac{dx}{2+\cos x}.$$

Set $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right).\frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1+y^2}$.

And, $\cos x = \frac{1-y^2}{1+y^2}$

Also, $x = 0 \Rightarrow y = 0$ and $x = \frac{\pi}{2} \Rightarrow y = 1$. Then,

$$I = \int_0^1 \frac{2\frac{dy}{1+y^2}}{2+(1-y^2)/(1+y^2)}$$

$$= \int_0^1 \frac{2dy}{2+2y^2+1-y^2}$$

$$= \int_0^1 \frac{2\,dy}{y^2+3}$$

$$= 2.\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{y}{\sqrt{3}}\right)\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

11. $\displaystyle\int_1^e \frac{dx}{x\sqrt{1-(\log x)^2}}$

Solution: Let,

$$I = \int_0^e \frac{dx}{x\sqrt{1-(\log x)^2}}$$

$\log(x) = y$ then $\left(\frac{1}{x}\right)dx = dy$. Also, $x = 1 \Rightarrow y = 0$, $x = e \Rightarrow y = 1$. Then,

$$I = \int_0^1 \frac{dy}{\sqrt{1-y^2}} = [\sin^{-1}(y)]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

12. $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

Solution: Let,

$$I = \int \frac{x+1}{x^2(x-1)}\,dx \qquad \text{... (i)}$$

Here,

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\Rightarrow x+1 = Ax(x-1) + B(x-1) + Cx^2$$

Set $x = 0$ then $B = -1$ and set $x = 1$ then $C = 2$.

Also, set $x = 2$ then $3 = 2A - 1 + 8 \Rightarrow A = -2$.

Therefore (i) becomes,

$$I = -2\int\frac{dx}{x} - \int\frac{dx}{x^2} + 2\int\frac{dx}{x-1}$$

$$= -2\log x - \left(\frac{x^{-1}}{-1}\right) + 2\log(x-1) + C$$

$$= 2\log\left(\frac{x-1}{x}\right) + \frac{1}{x} + C$$

13. $\displaystyle\int \frac{8}{x^4+2x^3}\,dx$

Solution: Let,

$$I = \int \frac{8}{x^4+2x^3}\,dx. \qquad \text{... (i)}$$

Here,

$$\frac{8}{x^4+2x^3} = \frac{8}{x^3.(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2}$$

$$\Rightarrow 8 = Ax^2(x+2) + Bx(x+2) + C(x+2) + Dx^3$$

$$\Rightarrow 8 = (A+D)x^3 + (2A+B)x^2 + (2B+C)x + 2C.$$

Equating the coefficient of like terms then we get,

$A + D = 0,\quad 2A + B = 0,\quad 2B + C = 0,\quad 2C = 8.$

Solving the equations we obtain

$A = 1,\quad B = -2,\quad C = 4,\quad D = -1.$

Then (i) reduces to

$$I = \int\frac{dx}{x} - 2\int\frac{dx}{x^2} + 4\int\frac{dx}{x^3} - \frac{dx}{x+2}$$

$$= \log(x) - 2\left(\frac{x^{-1}}{-1}\right) + 4\left(\frac{x^{-2}}{-2}\right) - \log(x+2) + C$$

$$= \log\left(\frac{x}{x+2}\right) + \frac{2}{x} - \frac{2}{x^2} + C$$

14. $\int x\tan^2 x\,dx$

Solution: Let,

$$I = \int x\tan^2 x\,dx$$

$$= \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$$

$$= x\tan x - \int \tan x\,dx - \frac{x^2}{2} + C_1$$

$$= x\tan x - \log(\sec x) - \frac{x^2}{2} + C$$

$$= x\tan x - \log(\cos x)^{-1} - \frac{x^2}{2} + C$$

$$= x\tan x + \log(\cos x) - \frac{x^2}{2} + C.$$

15. $\int (x+1)^2 e^x\,dx$

Solution: Let,

$$I = \int (x+1)^2 e^x\,dx$$

$$= (x+1)^2 e^x - 2(x+1)e^x + 2e^x + C$$

$$= (x+1)^2 e^x - 2x e^x + C.$$

16. $\int \cos\sqrt{x}\,dx$

2(n-0)

Solution: Let,

$$I = \int \cos\sqrt{x}\,dx.$$

Put $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}dx = dy \Rightarrow dx = 2y\,dy$. Then,

$$I = \int \cos y .\, 2y\,dy = 2[y\sin y - (1)(-\cos y)] + C$$

$$= 2[y\sin y + \cos y] + C$$

$$= 2\left[\sqrt{x}\sin(\sqrt{x}) + \cos(\sqrt{x})\right] + C.$$

Improper Integral

A definite integral in which the integral function does not exist at least a point in and on the given limits, is known as improper integral.

Classification of Improper Integrals

For convenience, let us classify the improper integrals into four types as:

(i) $\int_a^{\infty} f(x)\,dx,\quad a \le x < \infty$ (ii) $\int_{-\infty}^{b} f(x)\,dx,\quad -\infty < x \le b$

(iii) $\int_a^b f(x)\,dx$, $a < x \le b$ where f(x) does not exist at x = a

(iv) $\int_a^b f(x)\,dx$, $a \le x \le b$ where f(x) does not exist at x = b.

Process to solve

We process the above forms so the forms reduce to a definite integral and we integrate the integral. The above forms can be treated as:

(i) $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$ (ii) $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$

(iii) $\int_a^b f(x)\,dx$ (when f(x) does not exist at x = a) = $\lim_{c\to a^+}\int_c^b f(x)\,dx$

(iv) $\int_a^b f(x)\,dx$ (when f(x) ideas not exist at x = b) = $\lim_{c\to b^-}\int_a^c f(x)\,dx.$

Exercise 11.6

Evaluate the following improper integral if it exists.

1. $\int_0^{\infty} \frac{dx}{1+x^2}$, [2014 Spring (Short)]

Solution: Let,

$$I = \int_0^{\infty}\frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty}[\tan^{-1}(x)]_0^b = \lim_{b\to\infty}\tan^{-1}(b) = \frac{\pi}{2}.$$

2. $\int_0^{\infty}\frac{x\,dx}{x^2+4}$

Solution: Here,

$$I = \int_0^{\infty}\frac{x\,dx}{x^2+4} = \lim_{b\to\infty}\int_0^b \frac{x\,dx}{x^2+4}$$

$$= \lim_{b\to\infty}\frac{1}{2}\int_0^b \frac{d(x^2)}{x^2+4}$$

$$= \frac{1}{2}\lim_{b\to\infty}[\log(x^2+4)]_0^b$$

$$= \frac{1}{2} \lim_{b\to\infty} [\log (b^2+4) - \log (4)]$$

Since $\lim_{b\to\infty} \log (b^2+4)$ does not exist. So, $\int_0^{\infty} \frac{x\,dx}{x^2+4}$ does not exist.

3. $\int_2^{\infty} \frac{x\,dx}{x^2-1}$

Solution: Here,

$$I = \int_2^{\infty} \frac{x\,dx}{x^2-1} = \lim_{b\to\infty} \frac{1}{2} \int_2^{b} \frac{2x\,dx}{x^2-1}$$

$$= \frac{1}{2} \lim_{b\to\infty} [\log (x^2-1)]_2^b$$

$$= \frac{1}{2}\left[\lim_{b\to\infty} \log (b^2-1) - \log(3)\right]$$

Since $\lim_{b\to\infty} \log (b^2-1)$ does not exist. So, $\int_2^{\infty} \frac{x\,dx}{x^2-1}$ does not exist.

4. $\int_0^{\infty} xe^{-x^2}\,dx$

Solution: Here,

$$I = \int_0^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty} \int_0^{b} xe^{-x^2}\,dx$$

Put $x^2 = t$ then $2x\,dx = dt$. And $x = 0 \Rightarrow t = 0$, $x = b \Rightarrow t = b^2$. Then,

$$I = \lim_{b\to\infty} \frac{1}{2} \int_0^{b^2} e^{-t}\,dt = \frac{1}{2} \lim_{b\to\infty} \left[\frac{e^{-t}}{-1}\right]_0^{b^2} = \frac{1}{2}\left(\frac{e^{-\infty} - e^0}{-1}\right) = \frac{1}{2}.$$

[since $e^{-\infty} = 0$]

5. $\int_{-1}^{1} \frac{dx}{x^3}$ **[2013 Fall (Short)]**

Solution: Here,

$$I = \int_{-1}^{1} \frac{dx}{x^3} = \lim_{h\to 0} \int_{-1}^{-h} \frac{dx}{x^3} + \lim_{h\to 0} \int_{h}^{1} \frac{dx}{x^3}$$

$$= \lim_{h\to 0} \left[\frac{x^{-4}}{-4}\right]_{-1}^{-h} + \lim_{h\to 0} \left[\frac{x^{-4}}{-4}\right]_{h}^{1}$$

$$= \lim_{h\to 0} \left[\frac{(-h)^{-4} - (-1)^{-4}}{-4} + \frac{(1)^{-4} - (h)^{-4}}{-4}\right]$$

$$= \lim_{h\to 0} \left(\frac{1}{-4}\right)\left[\frac{1}{(-h)^4} - \frac{1}{(-1)^4} + \frac{1}{(1)^4} - \frac{1}{(h)^4}\right]$$

$$= -\frac{1}{4} \lim_{h\to 0} \left(\frac{1}{h^4} - 1 + 1 - \frac{1}{h^4}\right)$$

$$= -\frac{1}{4} \lim_{h\to 0} (0)$$

$$= 0.$$

Thus, $\int_{-1}^{1} \frac{dx}{x^3} = 0.$

6. $\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$

Solution: Let,

$$I = \int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$$

Since the integral does not exist only at $x = \infty$ and $x = -\infty$. Then,

$$I = \lim_{a\to\infty} \int_{-a}^{a} \frac{x\,dx}{x^4+1} = \lim_{a\to\infty} \int_{-a}^{a} \frac{x\,dx}{x^4+1}$$

Put $x^2 = y$ then $2x\,dx = dy$. Also, $x = 0 \Rightarrow y = 0$ and $x = a \Rightarrow y = a^2$. Then,

$$I = \lim_{a\to\infty} \int_{-a^2}^{a^2} \frac{dy}{y^2+1}$$

$$= \lim_{a\to\infty} [\tan^{-1} y]_{-a^2}^{a^2} = \lim_{a\to\infty} [\tan^{-1}(a^2) - \tan^{-1}(-a^2)]$$

$$= \lim_{a\to\infty} [2\tan^{-1}(a^2)] = 2\tan^{-1}(\infty) = \pi.$$

Thus, $\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1} = \pi.$

7. $\int_1^{\infty} \frac{\log x}{x^2}\,dx$

Solution: Let,

$$I = \int_1^{\infty} \frac{\log(x)}{x^2}\,dx$$

Put $\log(x) = y$ then $\frac{dx}{x} = dy$. Also, $x = 1 \Rightarrow y = 0$ and $x = \infty \Rightarrow y = \infty$. Then,

$$I = \int_0^{\infty} \frac{y}{e^y}\,dy = \lim_{b\to\infty} \int_0^{b} \frac{y}{e^y}\,dy$$

$$= \lim_{b\to\infty} \int_0^{b} ye^{-y}\,dy$$

$$= \lim_{b\to\infty} \left[y\left(\frac{e^{-y}}{-1}\right) - \frac{e^{-y}}{(-1)^2}\right]_0^b$$

$$= \lim_{b\to\infty} [-be^{-b} + 0 - e^{-b} + 1]$$

$$= 1.$$

[2012 Fall (Short)]

8. $\int_1^{\infty} \frac{x\,dx}{(1+x^2)^2}$

Solution: Let,

$$I = \int_1^{\infty} \frac{x\,dx}{(1+x^2)^2}$$

Put $x = \tan\theta$ then $dx = \sec^2\theta\, d\theta$.

Also, $x = 1 \Rightarrow \theta = \frac{\pi}{4}$ and $x = \infty \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_{\pi/4}^{\pi/2} \frac{\tan\theta}{(1+\tan^2\theta)^2}\sec^2\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\sin\theta\cos\theta)\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{\sin 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_{\pi/4}^{\pi/2} = \frac{-1}{4}[\cos(\pi) - \cos(\pi/2)] = \frac{-1}{4}[-1-0] = \frac{1}{4}$$

Thus, $\int_1^\infty \frac{x\, dx}{(1+x^2)^2} = \frac{1}{4}$.

9. $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution: Let,

$$I = \int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Put $x = \sin\theta$ then $dx = \cos\theta\, dy$.

Also, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{\theta}{\cos\theta}\cos\theta\, d\theta = \int_0^{\pi/2}\theta\, d\theta = \left[\frac{\theta^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}.$$

10. $\int_0^\pi \frac{\sin x}{\cos^2 x}dx$

Solution: Let,

$$I = \int_0^\pi \frac{\sin x}{\cos^2 x}dx$$

Since $\cos\frac{\pi}{2} = 0$. So, the integrand is undefined at $x = \frac{\pi}{2}$. Therefore,

$$I = \lim_{h\to 0}\left[\int_0^{\frac{\pi}{2}-h}\frac{\sin x}{\cos^2 x}dx + \int_{\frac{\pi}{2}+h}^{\pi}\frac{\sin x}{\cos^2 x}dx\right]$$

Put $\cos x = y$ then $(-\sin x)\, dx = dy$.

Also, $x = 0 \Rightarrow y = 1$, $x = \frac{\pi}{2} - h \Rightarrow y = \cos\left(\frac{\pi}{2}-h\right) = \sin h$.

And, $x = \frac{\pi}{2} + h \Rightarrow y = \cos\left(\frac{\pi}{2}+h\right) = -\sin h$, $x = \pi \Rightarrow y = 0$. Then,

$$I = \lim_{h\to 0}\left[\int_1^{\sin h}\left(\frac{-dy}{y^2}\right) + \int_{-\sin h}^{0}\left(\frac{-1}{y^2}dy\right)\right]$$

$$= \lim_{h\to 0}\left\{\left[-\frac{y^{-1}}{-1}\right]_1^{\sin h} + \left[\frac{-y^{-1}}{-1}\right]_{-\sin h}^{0}\right\}$$

$$= \lim_{h\to 0}\left[\left(\frac{1}{\sin h}-1\right) + \left(\frac{1}{0}+\frac{1}{\sin h}\right)\right]$$

Since $\frac{1}{0}$ is undefined. Therefore, I does not give fixed and acceptable value.

11. $\int_0^\pi \frac{dx}{1+\cos x}$

Solution: Let,

$$I = \int_0^\pi \frac{dx}{1+\cos x}$$

Clearly $\cos\pi = -1$. So, the integrand is undefined at $x = \pi$. Therefore,

$$I = \lim_{h\to 0}\int_0^{\pi-h}\frac{dx}{1+\cos x}$$

$$= \lim_{h\to 0}\int_0^{\pi-h}\frac{dx}{2\cos^2(x/2)} = \frac{1}{2}\lim_{h\to 0}\int_0^{\pi-h}\sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\lim_{h\to 0}\left[\frac{\tan x/2}{1/2}\right]_0^{\pi-h}$$

$$= \lim_{h\to 0}\tan\left(\frac{\pi}{2}-\frac{h}{2}\right)$$

$$= \lim_{h\to 0}\cot\left(\frac{h}{2}\right)$$

$$= \cot 0$$

$$= \text{doesnot exist.}$$

Thus, the value of I doesnot exist.

12. $\int_0^a \sqrt{\frac{a-x}{x}}dx$

Solution: Let,

$$I = \int_0^a \sqrt{\frac{a-x}{x}}dx$$

Clearly, the integrand is undefined at $x = 0$. Therefore,

$$I = \lim_{h\to 0}\int_h^a \sqrt{\frac{a-x}{x}}dx.$$

Put $x = t^2$ then $dx = 2t\, dt$. Also, $x = h \Rightarrow t = \sqrt{h}$, $x = a \Rightarrow t = \sqrt{a}$.

Then,

$$I = \lim_{h\to 0} \int_{\sqrt{h}}^{\sqrt{a}} \sqrt{\frac{a-t^2}{t^2}}\, 2t\, dt$$

$$= \lim_{h\to 0} 2 \int_{\sqrt{h}}^{\sqrt{a}} \sqrt{a-t^2}\, dt$$

$$= 2 \lim_{h\to 0} \left[\frac{t\sqrt{a-t^2}}{2} + \left(\frac{a}{2}\right)\sin^{-1}\left(\frac{t}{\sqrt{a}}\right)\right]_{\sqrt{h}}^{\sqrt{a}}$$

$$= 2 \lim_{h\to 0}\left[\left(0 + \left(\frac{a}{2}\right)\sin^{-1}(1)\right) - \left(\frac{\sqrt{h}\sqrt{a-h}}{2} + \left(\frac{a}{2}\right)\sin^{-1}\left(\sqrt{\frac{h}{a}}\right)\right)\right]$$

$$= 2\left[\left(0 + \left(\frac{a}{2}\right)\left(\frac{\pi}{2}\right)\right) - \left(\frac{0}{2} + 0\right)\right]$$

$$= 2\left[\frac{a\pi}{2}\right] = a\pi.$$

Thus, $\int_0^a \sqrt{\frac{a-x}{x}}\, dx = a\pi.$

13. $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$

Solution: Let,

$$I = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$$

Clearly the integrand is undefined at x = 1. Therefore,

$$I = \lim_{h\to 0} \int_{-1}^{1-h} \sqrt{\frac{1+x}{1-x}}\, dx.$$

Put $x = \cos\theta$ then $dx = -\sin\theta\, d\theta$.

Since,

$$\frac{1+x}{1-x} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)} = \cot^2(\theta/2).$$

So that,

$$I = \lim_{h\to 0} \int_{-1}^{1-h} \cot(\theta/2)\,(-\sin\theta)\, d\theta$$

$$= \lim_{h\to 0} \int_{-1}^{1-h} \cot(\theta/2)\,(-2)\sin(\theta/2)\cos(\theta/2)\, d\theta$$

$$= (-1) \lim_{h\to 0} \int_{-1}^{1-h} 2\cos^2(\theta/2)\, d\theta$$

$$= (-1) \lim_{h\to 0} \int_{-1}^{1-h} [1+\cos\theta]\, d\theta$$

$$= (-1) \lim_{h\to 0} [\theta + \sin\theta]_{-1}^{1-h}$$

$$= (-1) \lim_{h\to 0} [\{1-h+\sin(1-h)\} - \{-1+\sin(-1)\}]$$

$$= (-1)\, [\{1+\sin(1)\} - \{-1-\sin(1)\}]$$

$$= (-2)\, [1+\sin(1)].$$

Thus, $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = (-2)\, [1+\sin(1)].$

$\int_0^2 \frac{dx}{(1-x)^2}$

...tion: Let,

$$I = \int_0^2 \frac{dx}{(1-x)^2}$$

Clearly the integrand is undefined at x = 1. Therefore,

$$I = \lim_{h\to 0}\left[\int_0^{1-h} \frac{dx}{(1-x)^2} + \int_{1+h}^{2} \frac{dx}{(1-x)^2}\right]$$

$$= -\lim_{h\to 0}\left[\int_0^{1-h} (1-x)^{-2}(-dx) + \int_{1+h}^{2} (1-x)^{-2}(-dx)\right]$$

$$= -\lim_{h\to 0}\left\{\left[\frac{(1-x)^{-1}}{-1}\right]_0^{1-h} + \left[\frac{(1-x)^{-1}}{-1}\right]_{1+h}^{2}\right\}$$

$$= -\lim_{h\to 0}\left\{\left[\frac{1}{x-1}\right]_0^{1-h} + \left[\frac{1}{x-1}\right]_{1+h}^{2}\right\}$$

$$= -\lim_{h\to 0}\left\{\left(\frac{1}{-h}\right) - \left(\frac{1}{-1}\right) + \left(\frac{1}{1}\right) - \left(\frac{1}{h}\right)\right\}$$

$$= -\lim_{h\to 0}\left(2 - \frac{2}{h}\right)$$ which does not exist.

Therefore, the given integral I does not exist.

Define **Improper integral** and evaluate the improper integral $\int_0^2 \frac{dx}{(1-x)^2}$

...tion: See definition and Q. 14.

$\int_0^1 \log(x)\, dx$

...tion: Let,

$$I = \int_0^1 \log(x)\, dx$$

Since log(0) is undefined. Therefore, the integrand is undefined at x = 0. Therefore,

$$I = \lim_{h\to 0} \int_h^1 1.\log x\, dx$$

$$= \lim_{h\to 0}\left\{[\log x.\, dx]_h^1 - \int_h^1 \frac{1}{x}.\, x dx\right\}$$

$$= \lim_{h\to 0}\left[-h\log h - [x]_h^1\right] \qquad [\because \log(1) = 0]$$

$$= \lim_{h\to 0}[-h\log h - 1 + h] \quad = -0 - 1 + 0 \quad = -1.$$

16. $\int_0^\infty e^{-ax}\cos bx\,dx \quad$ **for a > 0**

Solution: Let,

$$I = \int_0^\infty e^{-ax}\cos bx\,dx \quad \text{for } a > 0$$

We know,

$$\int e^{ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx) + C$$

Clearly,

$$I = \lim_{m\to\infty}\int_0^m e^{-ax}\cos bx\,dx$$

$$= \lim_{m\to\infty}\left[\frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx)\right]_0^m$$

$$= \lim_{m\to\infty}\left[\frac{e^{-am}}{a^2+b^2}(-a\cos bm + b\sin bm) - \frac{1}{a^2+b^2}(-a)\right]$$

$$= \frac{a}{a^2+b^2} \qquad \text{as } e^{-am} \to 0,\ m\to\infty$$

Hence, $I = \dfrac{a}{a^2+b^2}$

Q. Find the formula for $\int \sec^n$ n dx and then evaluate $\int \sec^3 x\,dx$ [2013, Spring]

Solution: Let,

$$I_n = \int \sec^n x\,dx$$

$$= \int \sec^{n-2} x \sec^2 x\, d$$

$$= \sec^{n-2} x \tan - (n-2)\int (\sec^{n-3} x \sec x \tan x \times \tan x\,dx)$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\tan^2 x)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\left(\int \sec^n x\,dx - \int \sec^{n-2} x\,dx\right)$$

$$= \sec^{n-2} x \tan x - (n-2)\,I_n + (n-1)\,I_{n-2}$$

$$\Rightarrow (1 + n - 2)I_n = \sec^{n-2} x \tan x + (n-2)\,I_{n-2}$$

$$\therefore\ I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

This is the required reduction formula for $\int \sec^n x\,dx$

Again, $\int \sec^3 x\,dx = I_3 = \dfrac{\sec^3 x \tan x}{4} + \dfrac{3}{4} I_1$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\log(\sec x + \tan x) + C$$

Saborne lagey aaj ki yaad do me

Exercise 11.7

1. **Find the reduction formula for $\int \cos^n x\,dx$ and then evaluate $\int \cos^7 x\,dx$.** [2015 Spring][2012 Fall]

Solution: Let $I_n = \int \cos^n x\,dx$

$$= \int \cos^{n-1} x \cos x\,dx$$

Taking $\cos^{n-1}x$ as first function and cosx as second function and then applying by parts then,

$$= \cos^{n-1} x.\sin x - (n-1)\int \cos^{n-2} x\,(-\sin x)\sin x\,dx$$

$$= \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\,dx$$

$$= \sin x\cos^{n-1} x + (n-1)\left[\int \cos^{n-2} x\,dx - \int \cos^n x\,dx\right]$$

$$= \sin x\cos^{n-1} x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

$$\Rightarrow (1 + n - 1)\,I_n = \sin x\cos^{n-1} x + (n-1)\,I_{n-2}$$

$$\Rightarrow I_n = \frac{\sin x\cos^{n-1} x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}$$

For n = 7,

$$I_7 = \int \cos^7 x\,dx$$

$$= \frac{\sin x\cos^6 x}{7} + \frac{6}{7} I_5$$

$$= \frac{\sin x\cos^6 x}{7} + \frac{6}{7}\left[\frac{\sin x\cos^4 x}{5} + \left(\frac{4}{5}\right) I_3\right]$$

$$= \frac{\sin x\cos^6 x}{7} + \frac{6\sin x\cos^4 x}{35} + \frac{24}{35}\left[\frac{\sin x\cos x}{3} + \frac{2}{3} I_1\right]$$

$$= \frac{\sin x\cos^6 x}{7} + \frac{6\sin x\cos^4 x}{35} + \frac{24}{105}\sin x\cos x + \frac{48}{105}\sin x + C.$$

2. **Find the reduction formula for $\int_0^{\pi/2} \cos^n x\,dx$ and then evaluate $\int_0^{\pi/2} \cos^7 x\,dx$.**

Solution: Let $\quad J_n = \int_0^{\pi/2} \cos^n x\,dx$

By Q.1., $\quad J_n = \left[\dfrac{\sin x\cos^{n-1} x}{n}\right]_0^{\pi/2} + \left(\dfrac{n-1}{n}\right) J_{n-2}$

$$= \left(\frac{n-1}{n}\right) J_{n-2} \qquad [\because \cos(\pi/2) = 0 \text{ and } \sin 0 = 0]$$

For n = 7,

$$J_7 = \frac{6}{7} J_5 = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot J_1$$

$$= \frac{48}{105}\int_0^{\pi/2} \cos x\,dx$$

$$= \frac{48}{105}[\sin x]_0^{\pi/2} = \frac{48}{105} = \frac{16}{35}.$$

3. Find the reduction formula for $\int \cot^n x\, dx$ and then evaluate $\int \cot^7 x\, dx$.

Solution: Let $I_n = \int \cot^n x\, dx$

$$= \int \cot^{n-2} x \cot^2 x\, dx$$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$$

$$= \cot^{n-2} x(-\cot x) - (n-2)\int \cot^{n-3} x\,(-\operatorname{cosec}^2 x)(-\cot x)\, dx - I_{n-2}$$

$$= -\cot^{n-1} x - (n-2)\int \cot^{n-2} x(\cot^2 x + 1)\, dx - I_{n-2}$$

$$= -\cot^{n-1} x - (n-2)\,[I_n + I_{n-2}] - I_{n-2}$$

$$\Rightarrow (1 + n - 2)\, I_n = -\cot^{n-1} x - (n-2+1)\, I_{n-2}$$

$$\Rightarrow (n-1)\, I_n = -\cot^{n-1} x - (n-1)\, I_{n-2}$$

$$\Rightarrow I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

In particular, for $n = 7$,

$$I_7 = -\frac{\cot^6 x}{6} - I_5 = -\frac{\cot^6 x}{6} - \left[-\frac{\cot^4 x}{4} - \left[-\frac{\cot^2 x}{2} - I_1\right]\right]$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \int \cot x\, dx$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log(\sin x) + C.$$

4. Find the reduction formula for $\int \operatorname{cosec}^n x\, dx$ and then evaluate $\int \operatorname{cosec}^5 x\, dx$.

Solution: Let,

n-2

$$I_n = \int \operatorname{cosec}^n x\, dx$$

$$= \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\, dx$$

$$= \operatorname{cosec}^{n-2} x(-\cot x) - (n-2)\int \operatorname{cosec}^{n-3} x(-\operatorname{cosec} x \cot x)(-\cot x)\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x - 1)\, dx$$

In $$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\, I_n + (n-2)\, I_{n-2}$$

$$\Rightarrow I_n = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

In particular for $n = 5$,

$$I_5 = \int \operatorname{cosec}^5 x\, dx$$

$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4} I_3$$

$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4}\left[-\frac{\cot x \operatorname{cosec} x}{2} + \frac{1}{2} I_1\right]$$

$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\int \operatorname{cosec} x\, dx$$

$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log\left(\tan\frac{x}{2}\right) + C.$$

5. If $J_n = \int_0^{\pi/4} \tan^n x\, dx$ then show that $J_n = \frac{1}{n-1} - J_{n-2}$ and then find the value of J_5.

Solution: Let $I_n = \int \tan^n x\, dx$

$$= \int \tan^{n-2} x \tan^2 x\, dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

$$\Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

This is the reduction formula for $I_n = \int \tan^n x\, dx$

Again, let

$$J_n = \int_0^{\pi/4} \tan^n x\, dx$$

$$= \left(\frac{\tan^{n-1} x}{n-1}\right)_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\, dx$$

$$J_n = \frac{1}{n-1} - J_{n-2}$$

This is the required reduction formula for $J_n = \int_0^{\pi/4} \tan^n x\, dx$.

Also at $n = 5$,

$$I_5 = \int_0^{\pi/4} \tan^5 x\, dx$$

$$= \frac{1}{4} - I_3 = \frac{1}{4} - \frac{1}{2} + I_1 = \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \tan x\, dx$$

$$= \frac{1}{4} - \frac{1}{2} + [\log(\sec x)]_0^{\pi/4}$$

$$= -\frac{1}{4} + \left[\log\left(\sec\frac{\pi}{4}\right) - \log(\sec 0)\right]$$

$$= -\frac{1}{4} + [\log\sqrt{2} - 0]$$

$$= \frac{1}{2}\log(2) - \frac{1}{4}$$

6. Find the reduction formula for $\int \cos^m x \cos nx\, dx$ and show that $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$.

Solution: Let, $I_{m,n} = \int \cos^m x \cos nx \, dx$

$$= \cos^m x \frac{\sin nx}{n} - m\int \cos^{m-1} x \frac{\sin nx}{n}(-\sin x)\,dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin nx \sin x \, dx$$

Since, $2\sin nx \sin x = \cos(n-1)x - \cos(n+1)x$

$$= \cos(n-1)x - \cos nx \cos x + \sin nx \sin x$$

$\Rightarrow \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$

So,

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\,[\cos(n-1)x - \cos nx \cos x]\,dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}[I_{m-1,n-1} - I_{m,n}]$$

$$\Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

For m = n then,

$$\therefore \quad I_{n,n} = \frac{\cos^n x \sin nx}{2n} + \frac{n}{2n} I_{n-1,n-1}$$

$$\Rightarrow \quad I_{n,n} = \frac{\cos^n x \sin nx}{2n} + \frac{1}{2} I_{n-1,n-1}$$

And, let

$$J_{n,n} = \int_0^{\pi/2} \cos^n x \cos nx \, dx$$

$$= [I_{n,n}]_0^{\pi/2} = \frac{1}{2}[I_{n-1,n-1}]_0^{\pi/2} \qquad [\text{Being } \cos(\pi/2) = 0]$$

$$= \frac{1}{2^n}[I_{0,0}]_0^{\pi/2}$$

$$= \frac{1}{2^n}\int_0^{\pi/2} \cos^0 x \cos(0x)\,dx = \frac{1}{2^n}[x]_0^{\pi/2} = \frac{\pi}{2^{n+1}}$$

7. Find the reduction formula for $\int \cos^m x \sin nx \, dx$ and then evaluate $\int \cos^2 x \sin 3x \, dx$.

Solution: Let, $I_{m,n} = \int \cos^m x \sin nx \, dx$

$$= \cos^m x\left(\frac{\cos nx}{-n}\right) - \int m \cos^{m-1} x\,(-\sin x)\left(\frac{\cos nx}{-n}\right) dx$$

$$= -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n}\int \cos^{m-1} x\,(\sin x \cos nx)\,dx \qquad \ldots (i)$$

We have,

$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\Rightarrow \quad \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

From (i),

$$I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n}\int \cos^{m-1} x\,[\sin nx \cos x - \sin(n-1)x]\,dx$$

$$\Rightarrow I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n}\int \sin nx \cos^m x \, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\,dx$$

$$\Rightarrow I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

This is the required reduction formula for

$I_{m,n} = \int \cos^m x \sin nx \, dx.$

And, in particular, let m = 2, n = 3 then,

$$I_{2,3} = -\frac{\cos^2 x \cos 3x}{5} + \frac{2}{5} I_{1,2}$$

$$= -\frac{\cos^2 x \cos 3x}{5} + \frac{2}{5}\left[-\frac{\cos x \cos 2x}{3} + \frac{1}{3} I_{0,1}\right]$$

$$= -\frac{\cos^2 x \cos 3x}{5} - \frac{2\cos x \cos 2x}{15} + \frac{2}{15}\int \sin x \, dx$$

$$= -\frac{\cos^2 x \cos 3x}{5} - \frac{2\cos x \cos 2x}{15} + \frac{2}{15}\cos x + C.$$

Show that $\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = \frac{1}{3} + \frac{5\pi}{64}$.

Solution: By Q.No. 7, We have,

$$I_{m,n} = \int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

So, $\quad J_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx \, dx$

$$= -\left[\frac{\cos^m x \cos nx}{m+n}\right]_0^{\pi/2} + \frac{m}{m+n}[I_{m-1,n-1}]_0^{\pi/2}$$

$$= \frac{1}{m+n} + \frac{m}{m+n}[I_{m-1,n-1}]_0^{\pi/2}$$

In particular, let m = 5, n = 3 then,

$$J_{5,3} = \frac{1}{8} + \frac{5}{8}[I_{4,2}]_0^{\pi/2}$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{6} + \frac{4}{6}\left[\frac{1}{4} + \frac{3}{4}[I_{2,0}]_0^{\pi/2}\right]\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{48} + \frac{15}{48}\int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{16}{48} + \frac{15}{48}\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{3} + \frac{5}{16}\left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{3} + \frac{5}{32} \cdot \frac{\pi}{2} \qquad = \frac{1}{3} + \frac{5\pi}{64}.$$

10. Find the reduction formula for $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$.

Solution: Already contained in Q.No. 8.

11. If $I_n = \int \sinh^n x\, dx$ then show that $I_n = \frac{\sinh^{n-1}x \cosh x}{n} - \left(\frac{n-1}{n}\right) I_{n-2}$

Solution: Let, $I_n = \int \sinh^n x\, dx$

$$= \int \sinh^{n-1} x \sinh x\, dx$$
$$= \sinh^{n-1}x \cosh x - (n-1)\int \sinh^{n-2} x \cosh x \cosh x\, dx$$
$$= \sinh^{n-1}x \cosh x - (n-1)\int \sinh^{n-2} x \cosh^2 x\, dx$$
$$= \sinh^{n-1}x \cosh x - (n-1)\int \sinh^{n-2} x (1 + \sinh^2 x)\, dx$$
$$= \sinh^{n-1}x \cosh x - (n-1) I_{n-2} - (n-1) I_n$$
$$\Rightarrow I_n = \frac{\sinh^{n-1}x \cosh x}{n} - \left(\frac{n-1}{n}\right) I_{n-2}$$

Beta and Gamma Function

Definition: An integral of the form $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for $m > 0, n > 0$, known as function and it is denoted by $\beta(m, n)$.

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad \text{for } n > 0$$

Properties:

1. $\beta(m, n) = \beta(n, m)$

Proof: Here,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Put $1 - x = y$ then $-dx = dy$. So,

$$\beta(m, n) = \int_1^0 (1-y)^{m-1} y^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\, dy$$
$$= \beta(n, m)$$

2. $\Gamma(n+1) = n\Gamma(n)$

Proof: Here,

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{(n+1)-1}\, dx$$
$$= \int_0^\infty e^{-x} x^n\, dx$$
$$= \left[x^n\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty - \int_0^\infty nx^{n-1}\left(\frac{e^{-x}}{-1}\right) dx \quad [\because \text{Integrating by parts}]$$
$$= n\int_0^\infty x^{n-1} e^{-x}\, dx$$
$$= n\Gamma(n)$$

3. $\Gamma(1) = 0! = 1$

Proof: Here,

$$\Gamma(1) = \Gamma(0+1) = 0!$$

And,

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = \left(\frac{0-1}{-1}\right) = 1.$$

4. $\Gamma(m+1) = m!$

Proof: Here,

$$\Gamma(m+1) = (m)\Gamma(m)$$
$$= m\Gamma((m-1)+1)$$
$$= m(m-1)\Gamma(m-1)$$
$$= m(m-1)(m-2)\Gamma(m-3)$$
$$= m(m-1)(m-2)\ldots(m-(m-1))\Gamma(m-(m-1))$$
$$= m(m-1)(m-2)\ldots 1\Gamma(1)$$
$$= m(m-1)(m-2)\ldots 1 \quad [\because \Gamma(1) = 1]$$
$$= m!$$

5. $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof: Here,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \Big[(1-x)^{n-1}\frac{x^m}{m} - (n-1)(-1)(1-x)^{n-2}\frac{x^{m+1}}{m(m+1)} + (n-1)(-1)^2(n-2)(1-x)^{n-3}\frac{x^{m+2}}{m(m+1)(m+2)} - \ldots + (-1)^{n+1}(n-1)(n-2)\ldots(n-(n-1))(-1)^{n-1}(1-x)^{n-n}\frac{x^{m+(n-1)}}{m(m+1)(m+2)\ldots(m+(n-1))}\Big]_0^1$$

[∵ Applying successive integration]

$$= 0 - 0 + 0 - 0 + \ldots + 0 + \Big[(-1)^{n+1}(n-1)(n-2)\ldots(-1)(-1)^{n-1}(1-x)^0 \cdot \frac{x^{m+n-1}}{m(m+1)(m+2)\ldots(m+n-1)}\Big]_0^1$$

$$= \frac{(-1)^{2n}(n-1)(n-2)\ldots\ldots 2.1}{m(m+1)(m+2)\ldots\ldots(m+n-1)}$$

$$= \frac{(n-1)(n-2)\ldots\ldots 2.1\,(m-1)(m-2)\ldots.2.1}{1.2\ldots..(m-2)(m-1)m(m+1)(m=2)\ldots.(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)!}$$

$$= \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

[$\because \Gamma(m+1) = m\Gamma(m) = m(m-1)\Gamma(m-1) = \ldots m(m-1)\ldots 2.1 = m!$]

6. $\Gamma(m)\Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$ **for** $0 < m < 1$.

Proof: Here,

$$\Gamma(m)\Gamma(1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(m+1-m)}$$

$$= \beta(m, 1-m)$$

$$= \int_0^1 x^{m-1}(1-x)^{1-m-1}\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{-m}\,dx$$

Put $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$. So,

$$\Gamma(m)\Gamma(1-m) = \int_0^{\pi/2} \sin^{2m-2}\theta\cos^{-2m}\theta\, 2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{1-2m}\theta\, d\theta$$

$$= \Gamma\left(\frac{2m-1+1}{2}\right)\Gamma 1-2m$$

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Proof: Here,

$$I = \int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1/2}(1-\sin^2\theta)^{n-1/2}\sin\theta\cos\theta\, d\theta$$

Put $\sin^2\theta = x$ then $2\sin\theta\cos\theta\, d\theta = dx$. So,

$$I = \frac{1}{2}\int_0^1 x^{m-1/2}(1-x)^{n-1/2}\,dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Proof: Here,

$$I = \int_0^\infty e^{-x^2}\,dx$$

Put $x^2 = t$ then $2xdx = dt \Rightarrow dx = \dfrac{dt}{2\sqrt{t}}$. So,

$$I = \int_0^\infty e^{-t}\frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t}t^{-1/2}\,dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \quad [\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}]$$

$\int e^{-t}\frac{dt}{2\sqrt{t}}$

$\frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/2} dt = \frac{1}{2}\Gamma$

Exercise 11.8

1. Evaluate the followings:

(i) $\Gamma(9)$

Solution: Here,

$$\Gamma(9) = \Gamma(8-1) = 8\Gamma(8) = 8.7.6.5.4.3.2.1\,\Gamma(1)$$
$$= 40320.1 \qquad [\because \Gamma(1) = 1]$$
$$= 40320$$

(ii) $\Gamma\left(\frac{11}{2}\right)$

Solution: Here,

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{945}{32}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

(iii) $\dfrac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(7)}$

Solution: Here,

$$\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(7)} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{6.5.4.3.2.1} = \frac{7\pi}{1024}\cdot \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

2. Evaluate

(i) $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$

Solution: Here,

$$\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$$
$$= \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\Gamma\left(\frac{4+3+2}{2}\right)} \qquad \left[\because \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}\right]$$
$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{9}{2}\right)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).1!}{2.\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{2}{35}.$$

Thus, $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx = \frac{2}{35}$

(ii) $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx$

Solution: Here,

$$\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{2\Gamma\left(\frac{6+8+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{2\Gamma(8)}$$
$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2(7!)}$$
$$= \frac{5\pi}{4096}$$

Thus, $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx = \frac{5\pi}{4096}.$

Evaluate the following integrals using beta and gamma functions:

$\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$

Solution: Here,

$$I = \int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$$

Put $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$, $x = 1 \Rightarrow \theta = \frac{\pi}{2}$.

Then,

$$I = \int_0^{\pi/2} \frac{\sin^6\theta}{\cos\theta}\cos\theta\, d\theta$$
$$= \int_0^{\pi/2} \sin^6\theta\, d\theta$$
$$= \int_0^{\pi/2} \sin^6\theta \cos^0\theta\, d\theta$$
$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{7}{2}\right)\left(\frac{1}{2}\right)}{2\Gamma(4)} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2(3!)} = \frac{5\pi}{32}$$

$$\left[\because \Gamma(m-1) = (m)! \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Thus, $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx = \frac{5\pi}{32}$

(ii) $\int_0^{2a} x^5\sqrt{(2a-x)x}\,dx$

Solution: Let, $I = \int_0^{2a} x^5\sqrt{(2a-x)x}\,dx$

Put $x = 2a\sin^2\theta$ then $dx = 2a.\,2\sin\theta\cos\theta\,d\theta$.

Also, $x = 0 \Rightarrow \theta = 0$, $x = 2a \Rightarrow \theta = \frac{\pi}{2}$.

Therefore,

$$I = \int_0^{\pi/2} (2a)^5 \sin^{10}\theta\sqrt{2a(1-\sin^2\theta)\,2a\sin^2\theta}\;2a.\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2^8a^7\sin^{12}\theta\cos^2\theta\,d\theta$$

$$= 2^8a^7\frac{\Gamma\left(\frac{12+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{12+2+2}{2}\right)}$$

2^7

$$= 2^8a^7\frac{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\Gamma(8)}$$

$$= a^7\frac{11.9.7.5.3.\pi}{7!} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{33a^7\pi}{16}$$

Thus, $\int_0^{2a} x^5\sqrt{(2a-x)x}\,dx = \frac{33a^7\pi}{16}$.

(iii) — (viii) Process as in (ii) with substituting the value of x as recommende[d] in book.

4. Evaluate the following integrals by using beta and gamma functions:

(i) $\int_0^{\pi/8} \cos^3 4x\,dx$

Solution: Let, $I = \int_0^{\pi/8} \cos^3 4x\,dx$ 4x = y

Put $4x = y$ then $dx = \frac{dy}{4}$. Also, $x = 0 \Rightarrow y = 0$, $x = \frac{\pi}{8} \Rightarrow y = \frac{\pi}{2}$. So, that,

$$I = \int_0^{\pi/2} \cos^3 y\frac{dy}{4} = \frac{1}{4}\int_0^{\pi/2}\cos^3 y\sin^0 y\,dy$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{3+0+2}{2}\right)}$$

$$= \frac{1}{8}\cdot\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{1}{8}\cdot\frac{1.\Gamma\left(\frac{1}{2}\right)}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{1}{12}$$

Thus, $\int_0^{\pi/8} \cos^3 4x\,dx = \frac{1}{12}$

(ii) — (iv) Process as in (i)

5. Evaluate $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$.

Solution: Here,

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} \qquad \left[\because \Gamma m\Gamma(1-m) = \frac{\pi}{\sin m\pi} \text{ for } 0<m>1\right]$$

$$= \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}.$$

6. Show that B(n, m) B(m + n, l) = B(m, l) B(m + l, n).

Solution: Here,

B(n, m) B(m + n, l) = B(m, l) B(m + l, n)

$$\Rightarrow \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}\cdot\frac{\Gamma(m+n)\Gamma(l)}{\Gamma(m+n+l)} = \frac{\Gamma(m)\Gamma(l)}{\Gamma(m+l)}\cdot\frac{\Gamma(m+l)\Gamma(n)}{\Gamma(m+l+n)}$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} = \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)}$$

This is true.

So, B(n, m) B(m + n, l) = B(m, l) B(m + l, n).

> **Note:** Similarly,
> B(n, m) B(m + n, l) = B(l, n) B(l + n, m)
> Thus, the required form is proved.

7. Show that $\Gamma\left(\frac{2n+1}{2}\right) = \frac{1}{2^n}(2n-1)(2n-3)\ldots5.3.1\sqrt{\pi}$

Solution: Here,

$$\Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n-1}{2}\right)\Gamma\left(\frac{2n-1}{2}\right) \qquad [\because \Gamma(m+1) = m\Gamma(m)]$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\ldots\left(\frac{2(3)+1}{2}\right)\left(\frac{2(2)+1}{2}\right)\left(\frac{2(1)+1}{2}\right)\Gamma\left(\frac{2(1)+1}{2}\right)$$

$$= \frac{1}{2^{n-1}}(2n-1)(2n-3)\ldots7.5.3\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^{n-1}}(2n-1)(2n-3)\ldots 5.3.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2^{n}}(2n-1)(2n-3)\ldots 5.3.1\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

This completes the solution.

8. Show that $\int_{-1}^{1} (1+x)^p (1-x)^q\, dx = 2^{p+q+1}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$ **for p>–1, q>–1.**

Solution: Here, $I = \int_{-1}^{1} (1+x)^p (1-x)^q\, dx$

Set $1 + x = 2y$ then $dx = 2dy$. Also, $x = -1 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$.
Then,

$$I = \int_0^1 (2y)^p (1-(2y-1))^q (2dy)$$

$$= \int_0^1 (2y)^p (2-2y)^q\, 2dy$$

$$= 2^{p+q+1}\int_0^1 y^p (1-y)^q\, dy$$

$$= 2^{p+q+1}\beta(p+1, q+1) \qquad \text{for } (p+1) > 0, (q+1) > 0$$

$$= 2^{p+q+1}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} \qquad \text{for } p > -1, q > -1$$

This is required solution.

9. Show that $\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$ **for m > –1, n > –1.**

Solution: Here, $I = \int_a^b (x-a)^m (b-x)^n\, dx$

Set, $x - a = (b-a)y$. Then $x = a \Rightarrow y = 0$, $x = b \Rightarrow y = 1$.
And, $dx = (b-a)\, dy$. Then,

$$I = \int_0^1 (b-a)^m y^m (b-a-(b-a)y)^n (b-a)\, dy$$

$$= \int_0^1 (b-a)^m y^m (b-a)^n (1-y)^n (b-a)\, dy$$

$$= (b-a)^{m+n+1}\int_0^1 y^m (1-y)^n\, dy$$

$$= (b-a)^{m+n+1}\beta(m+1, n+1) \qquad \text{for } (m+1) > 0, (n+1) > 0$$

$$= (b-a)^{m+n+1}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \qquad \text{for } m > -1, n > -1.$$

Show that $\int_0^\infty e^{-x^2} x^\lambda\, dx = \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right)$ **for λ > –1.**

Solution: Here, $I = \int_0^\infty e^{-x^2} x^\lambda\, dx$

Set $x^2 = t$ then $2x\, dx = dt \Rightarrow dx = \frac{dt}{2\sqrt{t}}$

Also, $x = 0 \Rightarrow t = 0$, $x = \infty \Rightarrow t = \infty$. Then,

$$I = \int_0^\infty e^{-t} t^{\lambda/2}\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{((\lambda-1)/2)}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{((\lambda+1)/2+1)}\, dt = \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right) \quad \text{for } \frac{\lambda+1}{2} > 0$$

$$= \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right) \quad \text{for } \lambda > -1.$$

11. Show that $\int_0^\infty e^{-x^4} x^2\, dx \int_0^\infty e^{-x^4}\, dx = \frac{\pi}{8\sqrt{2}}$.

Solution: Here,

$$I = \int_0^\infty e^{-x^4} x^2\, dx \int_0^\infty e^{-x^4}\, dx$$

Set, $x^4 = t$ then $4x^3\, dx = dt \Rightarrow dx = \frac{dt}{4t^{3/4}}$.

Also, $x = 0 \Rightarrow t = 0$, $x \to \infty \Rightarrow t \to \infty$. Then,

$$I = \int_0^\infty e^{-t} t^{1/2}\frac{dt}{4t^{3/4}}\int_0^\infty e^{-t}\frac{dt}{4t^{3/4}}$$

$$= \frac{1}{16}\int_0^\infty e^{-t} t^{-1/4}\, dt \int_0^\infty e^{-t} t^{-3/4}\, dt$$

$$= \frac{1}{16}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \qquad [\because \text{using definition of gamma function}]$$

$$= \frac{1}{16}\pi\sqrt{2} \qquad [\because \text{By Q.5}]$$

$$= \frac{\pi}{8\sqrt{2}}.$$

12. Evaluate $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$.

Solution: Here,

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$$

$$= \left[\Gamma\left(\frac{8}{9}\right)\Gamma\left(\frac{1}{9}\right)\right]\left[\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{2}{9}\right)\right]\left[\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{3}{9}\right)\right]\left[\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{4}{9}\right)\right]$$

$$= \left[\Gamma\left(\frac{8}{9}\right)\Gamma\left(1-\frac{8}{9}\right)\right]\left[\Gamma\left(\frac{7}{9}\right)\Gamma\left(1-\frac{7}{9}\right)\right]\left[\Gamma\left(\frac{6}{9}\right)\Gamma\left(1-\frac{6}{9}\right)\right]\left[\Gamma\left(\frac{5}{9}\right)\Gamma\left(1-\frac{5}{9}\right)\right]$$

$$= \frac{\pi}{\sin\left(\frac{8\pi}{9}\right)} \cdot \frac{\pi}{\sin\left(\frac{7\pi}{9}\right)} \cdot \frac{\pi}{\sin\left(\frac{6\pi}{9}\right)} \cdot \frac{\pi}{\sin\left(\frac{5\pi}{9}\right)}$$

$\left[\because \text{applying } \Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi} \text{ for } 0 < m > 1\right]$

$$= \frac{\pi}{\sin 160^\circ} \cdot \frac{\pi}{\sin 140^\circ} \cdot \frac{\pi}{\sin 120^\circ} \cdot \frac{\pi}{\sin 100^\circ}$$ n(n+1)/2

$\sum^{n}$ r
r=1

Some formulae for Ex. 11.9

1. $\sum_{r=1}^{n} 1 = n$

2. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

3. $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

5. $\sum_{p=1}^{n} r^p = \frac{a(r^n-1)}{r-1}$ where a be the first term of the series. $r^p = \frac{a(r^n-1)}{r-a}$

6. $\sum_{r=1}^{n} \cos(a+rh) = \cos\left(a + \frac{nh}{2}\right) \frac{\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$

where a be the lower limit for integration

7. $\sum_{r=1}^{n} \sin(a+rh) = \sin\left(a + \frac{nh}{2}\right) \frac{\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$

$$\boxed{\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh) \quad ,\text{where } nh = b-a.}$$

Exercise 11.9

Evaluate the following integrals by the method of summation (i.e. by **definition)**

1. $\int_1^2 x^2\,dx$ **[2017 Spring]**

Solution:

Comparing the given integral $\int_1^2 x^2\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = x^2$, $a = 1$, $b = 2$. So, $nh = b - a = 1$.

Then, $f(a + rh) = (a + rh)^2$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$\int_1^2 x^2\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (1+rh)^2$$

$$= \lim_{h\to 0} h\left[\sum_{r=1}^{n} 1 + 2h\sum_{r=1}^{n} r + h^2 \sum_{r=1}^{n} r^2\right]$$

$$= \lim_{h\to 0} h\left[n + 2h \cdot \frac{n(n+1)}{2} + h^2 \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0}\left[nh + nh(nh+h) + \frac{nh(nh+h)(2nh+h)}{6}\right]$$

$$= \lim_{h\to 0}\left[1 + 1(1+h) + \frac{1(1+h)(2.1+h)}{6}\right]$$

$$= 1 + 1(1+0) + \frac{1(1+0)(2+0)}{6}$$

$$= \frac{3+3+1}{3} = \frac{7}{3}.$$

Thus, $\int_1^2 x^2\,dx = \frac{7}{3}$.

2. $\int_a^b e^{-x}\,dx$ **[2018 Fall][2017 Fall]**

Solution:

Comparing the given integral $\int_a^b e^{-x}\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = e^{-x}$, $a = a$, $b = b$. So, $nh = b - a$

Also, $f(a + rh) = e^{-(a+rh)} = e^{-a} \cdot e^{-rh}$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$\int_a^b e^{x}\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (e^{-a} \cdot e^{-rh})$$

$$= e^{-a} \lim_{h\to 0} h \sum_{r=1}^{n} (e^{-h})^r$$

$$= e^{-a} \lim_{h\to 0} h \frac{e^{-h}[(e^{-h})^n - 1]}{e^{-h} - 1}$$

$$= e^{-a} \lim_{h\to 0} h \frac{e^{-h}(e^{-nh} - 1)}{e^{-h} - 1}$$

$$= e^{-a} \lim_{h\to 0} \frac{e^{-h}(e^{-(b-a)} - 1)}{e^{-h} - 1} . h$$

$$= e^{-a}(e^{a-b} - 1) \lim_{h\to 0} \frac{e^{-h}}{e^{-h} - 1} . h$$

$$= (e^{-b} - e^{-a}) . \lim_{h\to 0} \frac{-h}{e^{-h} - 1} (-1) e^{-h}$$

$$= (e^{-a} - e^{-b}) . 1 \qquad \left[\text{Being } \frac{e^x - 1}{x} = 1\right]$$

$$= e^{-a} - e^{-b}.$$

Thus, $\int_a^b e^{-x} dx = e^{-a} - e^{-b}$.

3. $\int_1^4 (x^2 - x)\, dx$

Solution:

Comparing the given integral $\int_1^4 (x^2 - x)\, dx$ with the integral $\int_a^b f(x)\, dx$ then we get,

$f(x) = (x^2 - x)$, $a = 1$, $b = 4$. So, $nh = b - a = 3$

Also, $f(a + rh) = f(1 + rh)$

$= (1 + rh)^2 - (1 + rh) = r^2h^2 + rh.$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

Therefore,

$$\int_1^4 (x^2 - x)\, dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(1 + rh)$$

$$= \lim_{h\to 0} h \left[\sum_{r=1}^{n} rh + \sum_{r=1}^{n} r^2h^2\right]$$

$$= \lim_{h\to 0} h \left[h \sum_{r=1}^{n} r + h^2 \sum_{r=1}^{n} r^2\right]$$

$$= \lim_{h\to 0} \left[h \frac{n(n+1)}{2} + h^2 \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0} \left[\frac{2h(nh+h)}{2} + \frac{nh(nh+h)(2nh+h)}{6}\right]$$

$$= \lim_{h\to 0} \left[\frac{3(3+h)}{2} + \frac{3(3+h)(6+h)}{6}\right]$$

$$= \frac{3(3+0)}{2} + \frac{(3+0)(6+0)}{2} = \frac{9+18}{2} = \frac{27}{2}$$

Thus, $\int_1^4 (x^2 - x)\, dx = \frac{27}{2}$.

$\int_0^1 x^{1/2} dx$ [2018 Spring][2016 Spring]

Solution: Given integral is,

$$\int_0^1 x^{1/2} dx \qquad \dots (i)$$

Put, $x = y^2$ then $dx = 2y\, dy$. Also, $x = 0 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$.

Then,

$$\int_0^1 \sqrt{x}\, dx = 2\int_0^1 y^2\, dy \qquad \dots (ii)$$

Comparing the given integral $\int_0^1 y^2\, dy$ with the integral $\int_a^b f(y)\, dy$ then we get,

$f(y) = y^2$, $a = 0$, $b = 1$. So, $nh = b - a = 1$

Also, $f(a + rh) = f(rh) = (rh)^2$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

Therefore,

$$2\int_0^1 y^2\, dy = 2 \lim_{h\to 0} h \sum_{r=1}^{n} r^2h^2$$

$$= 2 \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= 2 \lim_{h\to 0} \frac{nh(nh+h)(2nh+h)}{6}$$

$$= 2 \lim_{h\to 0} \frac{1(1+h)(2+h)}{6} = 2\frac{(1+0)(2+0)}{6} = \frac{2}{3}.$$

Thus, $\int_0^1 x^{1/2} dx = \frac{2}{3}$.

Note: The ab-initio method does not work here being $a = 0$.

5. $\int_0^1 (a_1x + b_1)\, dx$

Solution:

Comparing the given integral $\int_0^1 (a_1x+b_1)dx$ with the integral $\int_a^b f(x)dx$ then we get,

$f(x) = a_1x + b_1$, $a = 0$, $b = 1$. So, $nh = b - a = 1$.

Also, $f(a + rh) = a_1(a+rh) + b_1 = a_1rh + b_1$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

$a_1rh + b$

Therefore,

$$\int_0^1 (a_1x + b_1)\, dx = \lim_{h\to 0} h \sum_{r=1}^{n} (a_1rh + b_1)$$

$$= \lim_{h\to 0} h\left(a_1h \sum_{r=1}^{n} r + b_1 \sum_{r=1}^{n} 1\right)$$

$$= \lim_{h\to 0}\left[a_1h^2 . \frac{n(n+1)}{2} + b_1h \,.\, n\right]$$

$$= \lim_{h\to 0}\left(\frac{a_1}{2} . nh\,(nh + h) + b_1nh\right)$$

$$= \lim_{h\to 0}\left(\frac{a_1}{2} .1\,(1 + h) + b_1 \,.\, 1\right)$$

$$= \frac{a_1}{2}(1 + 0) + b_1 \;= \frac{a_1}{2} + b_1.$$

Thus, $\int_0^1 (a_1x + b_1)\, dx = \frac{a_1}{2} + b_1.$

6. $\int_a^b \cos x\, dx$ **for** $a > 0$.

Solution:

Comparing the given integral $\int_a^b \cos x\, dx$ with the integral $\int_a^b f(x)\, dx$ then we get,

$f(x) = \cos x$, $a = a$, $b = b$. So, $nh = b - a$.

Also, $f(a + rh) = \cos(a + rh)$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

Therefore,

$$\int_a^b \cos x\, dx = \lim_{h\to 0} n \sum_{r=1}^{n} \cos(a + rh)$$

$$= \lim_{h\to 0} h \cos\left(a + \frac{nh}{2}\right) . \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} h \cos\left(a + \frac{b-a}{2}\right) . \frac{\sin\frac{b-a}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} \cos\left(\frac{a+b}{2}\right) . \sin\left(\frac{b-a}{2}\right) . \frac{h}{\sin\frac{h}{2}}$$

$$= \frac{1}{2} . 2\cos\left(\frac{a+b}{2}\right) . \sin\left(\frac{b-a}{2}\right) \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} . 2$$

$$= \left[\sin\left(\frac{a+b}{2} + \frac{b-a}{2}\right) + \sin\left(\frac{a+b}{2} - \frac{b-a}{2}\right)\right] . 1$$

$$= \sin b + \sin a.$$

Thus, $\int_a^b \cos x\, dx = \sin b + \sin a.$

7. $\int_0^{\pi/2} \cos x\, dx$

Solution:

Comparing the given integral $\int_0^{\pi/2} \cos x\, dx$ with the integral $\int_a^b f(x)\, dx$ then we get,

$f(x) = \cos x$, $a = 0$, $b = \frac{\pi}{2}$. So, $nh = b - a = \frac{\pi}{2}$

Also, $f(a + rh) = \cos(a + rh)$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

Therefore,

$$\int_0^{\pi/2} \cos x\, dx = \lim_{h\to 0} h \sum_{r=1}^{n} \cos(a + rh)$$

$$= \lim_{h\to 0} h \cdot \cos\left(\frac{a+nh}{2}\right) \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} \cos\frac{\pi}{4} \cdot \sin\frac{\pi}{4} \cdot \frac{h}{\sin\frac{h}{2}} \quad [\because a = 0]$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \cdot 2 = \frac{1}{2} \cdot 2 \cdot 1 = 1.$$

Thus, $\int_0^{\pi/2} \cos x \, dx = 1.$

8. $\int_0^{\pi/2}$ **sinx dx**

Solution:

Comparing the given integral $\int_0^{\pi/2} \sin x \, dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$$f(x) = \sin x,\ a = 0,\ b = \frac{\pi}{2}.\ \text{So, } nh = b - a = \frac{\pi}{2}.$$

Also, $f(a + rh) = \sin(a + rh).$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_0^{\pi/2} \sin x \, dx = \lim_{h\to 0} h \sum_{r=1}^{n} \sin(a + rh)$$

$$= \lim_{h\to 0} h \cdot \sin\left(a + \frac{nh}{2}\right) \frac{\sin\frac{nh}{2}}{\sin(h/2)}$$

$$= \lim_{h\to 0} h \sin\frac{\pi}{4} \cdot \sin\frac{\pi}{4} \cdot \frac{1}{\sin(h/2)} \quad [\because a = 0,\ nh = \tfrac{\pi}{2}]$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \lim_{h\to 0} \frac{(h/2)}{\sin(h/2)} \cdot 2$$

$$= \frac{1}{2} \cdot 2 \cdot 1 \quad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$= 1.$$

Thus, $\int_0^{\pi/2} \sin x \, dx = 1.$

9. $\int_a^b \frac{dx}{x^2}$ **(a > 0)**

Solution: Put $x^{-1} = y$ then $-\frac{dx}{x^2} = dy.$

Also, $x = a \Rightarrow y = \frac{1}{a}$ and $x = b \Rightarrow y = \frac{1}{b}$. Then,

$$I = \int_a^b \frac{dx}{x^2} = -\int_{1/a}^{1/b} dy$$

Comparing the given integral $\int_{1/a}^{1/b} dy$ with the integral $\int_{a_1}^{b_1} f(y)\,dy$ then we get,

$$f(y) = 1,\ a_1 = \frac{1}{a} \text{ and } b_1 = \frac{1}{b}.\ \text{So, } nh = b_1 - a_1 = \frac{1}{b} - \frac{1}{a}$$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$I = -\int_{1/a}^{1/b} dy = -\lim_{h\to 0} h \cdot \sum_{r=1}^{n} 1$$

$$= -\lim_{h\to 0} hn = -\lim_{h\to 0}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}.$$

Thus, $\int_a^b \frac{dx}{x^2} = \frac{b-a}{ab}.$

10. $\int_a^b \frac{dx}{\sqrt{x}}$ **(a > 0).**

Solution: Put, $\sqrt{x} = y$ then $\frac{1}{2} x^{-1/2} dx = dy.$

When $x = a \Rightarrow y = \sqrt{a}$ and $x = b \Rightarrow y = \sqrt{b}$. Then,

$$I = \int_a^b \frac{dx}{\sqrt{x}} = \int_{\sqrt{a}}^{\sqrt{b}} 2\,dy = 2\int_{\sqrt{a}}^{\sqrt{b}} dy$$

Comparing the given integral $\int_{\sqrt{a}}^{\sqrt{b}} dy$ with the integral $\int_a^b f(y)\,dy$ then we get,

$$f(y) = 1,\ a_1 = \sqrt{a},\ b_1 = \sqrt{b}.\ \text{So, } nh = b_1 - a_1 = \sqrt{b} - \sqrt{a}.$$

Also, $f(a_1 + rh) = 1.$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$I = 2\int_{\sqrt{a}}^{\sqrt{b}} dy = 2\lim_{h\to 0} h.\sum_{r=1}^{n} f(a_1+rh)$$

$$= 2\lim_{h\to 0} . h. \sum_{r=1}^{n} . 1$$

$$= 2\lim_{h\to 0} . h . n$$

$$= 2\lim_{h\to 0} (\sqrt{b}-\sqrt{a}) = 2(\sqrt{b}-\sqrt{a}).$$

Thus, $\int_a^b \frac{dx}{\sqrt{x}} = 2(\sqrt{b}-\sqrt{a}).$

11. $\int_a^b \sqrt{x}\,dx \ (a>0).$

Solution: Put $x^{3/2} = y$. Then $\frac{3}{2}x^{1/2}\,dx = dy \Rightarrow \sqrt{x}\,dx = \frac{2dy}{3}$.

And, $x = a \Rightarrow y = a^{3/2}$, $x = b \Rightarrow y = b^{3/2}$. Then,

$$I = \int_a^b \sqrt{x}\,dx = \frac{2}{3}\int_{a^{3/2}}^{b^{3/2}} dy$$

Comparing the given integral $\int_{a^{3/2}}^{b^{3/2}} dy$ with the integral $\int_a^b f(y)\,dy$ then we get,

$f(x) = 1$, $a_1 = a^{3/2}$, $b_1 = b^{3/2}$. So, $nh = b_1 - a_1 = b^{3/2} - a^{3/2}$

Also, $f(a_1 + rh) = 1$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$I = \frac{2}{3}\int_{a^{3/2}}^{b^{3/2}} dy = \frac{2}{3}\lim_{h\to 0} h\sum_{r=1}^{n} f(a_1+rh)$$

$$= \frac{2}{3}\lim_{h\to 0} h.\sum_{r=1}^{n} .1$$

$$= \frac{2}{3}\lim_{h\to 0} h.n$$

$$= \frac{2}{3}\lim_{h\to 0} (b^{3/2}-a^{3/2}) = \frac{2}{3}(b^{3/2}-a^{3/2}).$$

Thus, $\int_a^b \sqrt{x}\,dx = \frac{2}{3}(b^{3/2}-a^{3/2}).$

12. $\int_0^{\pi/4} \sin^2 x\,dx$

Solution: Here,

$$I = \int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\int_0^{\pi/4} dx - \frac{1}{2}\int_0^{\pi/4}\cos 2x\,dx$$

$$= \frac{1}{2}(I_1 - I_2)$$

Comparing the given integral $\int_0^{\pi/4} dx$ and $\int_0^{\pi/4}\cos 2x\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

for I_1, $f(x) = 1$, $a = 0$, $b = \frac{\pi}{4}$. So, $nh = b - a = \frac{\pi}{4}$.

Also, $f(a+rh) = 1$.

And for I_2, $f(x) = \cos 2x$, $a = 0$, $b = \frac{\pi}{4}$. So, $nh = b - a = \frac{\pi}{4}$.

Also, $f(a+rh) = \cos 2(a+rh)$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$I = \frac{1}{2}(I_1 - I_2) = \frac{1}{2}\int_0^{\pi/4} dx - \frac{1}{2}\int_0^{\pi/4}\cos 2x\,dx$$

$$= \frac{1}{2}\lim_{h\to 0} h\sum_{r=1}^{n} 1 - \frac{1}{2}\lim_{h\to 0} h\sum_{r=1}^{n}\cos 2(a+rh)$$

$$= \frac{1}{2}\lim_{h\to 0} h.n - \frac{1}{2}\lim_{h\to 0} h\cos 2\left(a+\frac{nh}{2}\right)\frac{\sin 2\left(\frac{nh}{2}\right)}{\sin 2\left(\frac{h}{2}\right)}$$

$$= \frac{1}{2}\lim_{h\to 0} \cdot \frac{\pi}{4} - \frac{1}{2}\lim_{h\to 0} h.\cos\frac{\pi}{4}.\sin\frac{\pi}{4}.\frac{1}{\sinh}$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\lim_{h\to 0}\frac{h}{\sinh}$$

$$= \frac{\pi}{8} - \frac{1}{4} \cdot 1 = \frac{\pi}{8} - \frac{1}{4}.$$

Thus, $\int_0^{\pi/4} \sin^2 x\, dx = \frac{\pi}{8} - \frac{1}{4}.$

13. $\int_0^{\pi/4} \cos^2 x\, dx$

Solution: Here,

$$I = \int_0^{\pi/4} \cos^2 x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} dx + \frac{1}{2}\int_0^{\pi/4} \cos 2x\, dx$$

Process as Q. 12 then will obtain $I = \frac{\pi}{8} + \frac{1}{4}.$

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

LONG QUESTIONS

1. $\int \frac{x+2}{x^2 - 13x + 42}\, dx.$ [2004, Fall]

Solution: Given that,

$$I = \int \frac{(x+2)}{x^2 - 13x + 42}\, dx = \int \frac{(x+2)}{(x-7)(x-6)}\, dx$$

Here,

$$\frac{x+2}{(x-7)(x-6)} = \frac{A}{x-7} + \frac{B}{x-6}$$

$$\Rightarrow x + 2 = A(x-6) + B(x-7)$$

$$= (A+B)x + (-6A - 7B).$$

Comparing the coefficient of x and the constant term from both sides then

$A + B = 1, -6A = 7B = 2$

Solving, $A = 9, B = -8$

Then,

$$I = 9\int \frac{dx}{x-7} - 8\int \frac{dx}{x-6}$$

$$= 9\log|x-7| - 8\log|x-6| + C.$$

2. $\int \frac{dx}{x + \sqrt{x}}$ [2015 Spring short][2004, Spring]

Solution: Given that,

$$I = \int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

Put $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}\, dx = dy$. Then,

$$I = \int \frac{2y\, dy}{y(y+1)} = 2\int \frac{dy}{y+1} = 2\log(y+1) + C$$

$$\Rightarrow I = 2\log(\sqrt{x} + 1) + C.$$

3. $\int \frac{dx}{\sin x + \cos x}$ [2005, Fall]

Solution: Let,

$$I = \int \frac{dx}{\sin x + \cos x}$$

$$= \int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)}$$

Put $x + \frac{\pi}{4} = y$ then $dx = dy$. Then,

$$I = \frac{1}{\sqrt{2}}\int \frac{dy}{\sin y}$$

$$= \frac{1}{\sqrt{2}}\int \operatorname{cosec} y\, dy = \frac{1}{\sqrt{2}}\log\left(\tan\left(\frac{y}{2}\right)\right) + C$$

$$= \frac{1}{\sqrt{2}}\log\tan\left(\frac{x + \frac{\pi}{4}}{2}\right) + C$$

$$= \frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + C.$$

4. $\int \frac{(x+2)}{\sqrt{4x - x^2}}\, dx.$ [2018 Spring][2005, Fall]

Solution: Let,

$$I = \int \frac{(x+2)}{\sqrt{4x - x^2}}\, dx$$

$$= -\frac{1}{2}\int \frac{(4-2x)-4-4}{\sqrt{4x-x^2}}\,dx$$

Put $4x - x^2 = y^2$ then $(4-2x)\,dx = 2y\,dy$. Then,

$$I = -\frac{1}{2}\int \frac{2y\,dy}{y} + \int \frac{8dy}{\sqrt{4x-x^2}}$$

$$= -\int dy + 8\int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= -y + 8\sin^{-1}\left(\frac{x-2}{2}\right) + C = -\sqrt{4x-x^2} + 8\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

5a. $\int \frac{dx}{4-5\sin x}dx.$ [2000]

Solution: Let

$$I = \int \frac{dx}{4-5\sin x}$$

Set $\tan\frac{x}{2} = y$ then $\sec^2\frac{x}{2}\cdot\frac{dx}{2} = dy \Rightarrow dx = \frac{2dy}{1+y^2}$.

Also,

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2y}{1+y^2}$$

Now,

$$I = \int \frac{2\frac{dy}{1+y^2}}{4-5\left(\frac{2y}{1+y^2}\right)}$$

$$= \int \frac{2\,dy}{4+4y^2-10y} = \frac{2}{4}\int \frac{dy}{y^2 - 2\frac{5}{4}y + \frac{25}{16} - \frac{25}{16} + 1}$$

$$= \frac{1}{2}\int \frac{dy}{\left(y-\frac{5}{4}\right)^2 - \frac{9}{16}}$$

Set $y - \frac{5}{4} = t$ then $dy = dt$. Then,

$$I = \frac{1}{2}\frac{dt}{t^2 - \frac{9}{16}} = \frac{1}{2}\cdot\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{t-\frac{3}{4}}{t+\frac{3}{4}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t-3}{4t+3}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4\left(y-\frac{5}{4}\right)-3}{4\left(y-\frac{5}{4}\right)+3}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4y-5-12}{4y-5+12}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4y-17}{4y+7}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4\tan\frac{x}{2}-17}{4\tan\frac{x}{2}+7}\right| + C$$

$\int \frac{dx}{5-13\sin x}dx.$ [2016 Spring][2016 Fall][2011 Fall]

Solution: Let

$$I = \int \frac{dx}{5-13\sin x}$$

Set $\tan\frac{x}{2} = y$ then $\sec^2\frac{x}{2}\cdot\frac{dx}{2} = dy \Rightarrow dx = \frac{2dy}{1+y^2}$.

Also,

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2y}{1+y^2}$$

Now,

$$I = \int \frac{2\frac{dy}{1+y^2}}{5-13\left(\frac{2y}{1+y^2}\right)}$$

$$= \int \frac{2\,dy}{5+5y^2-26y} = \frac{2}{5}\int \frac{dy}{y^2 - 2\frac{13}{5}y + \frac{169}{25} - \frac{169}{25} + 1}$$

$$= \frac{2}{5}\int \frac{dy}{\left(y-\frac{13}{5}\right)^2 - \frac{144}{25}}$$

Set $y - \frac{13}{5} = t$ then $dy = dt$. Then,

$$I = \frac{2}{5}\int \frac{dt}{t^2-(12/5)^2} = \frac{2}{5}\cdot\frac{5}{24}\log\left|\frac{t-(12/5)}{t+(12/5)}\right| + C$$

$$= \frac{1}{12}\log\left|\frac{5t-12}{5t+12}\right| + C$$

$$= \frac{1}{12}\log\left|\frac{5(y-(13/5))-12}{5(y-(13/5))+12}\right| + C$$

$$= \frac{1}{12}\log\left|\frac{5y-25}{5y-1}\right| + C$$

$$= \frac{1}{12}\log\left|\frac{5\tan(x/2)-25}{5\tan(x/2)+1}\right| + C$$

6. $\int \frac{dx}{4+5\cos x}$ [2004, Fall]

Solution: Let $I = \int \frac{dx}{4+5\cos x}$

Set $\tan\frac{x}{2} = y$ then $dx = 2\frac{dy}{1+y^2}$. Then,

$$I = \int \frac{2dy/(1+y^2)}{4+5(1-y^2)/(1+y^2)}$$

$$= \int \frac{2dy}{4+4y^2+5-5y^2}$$

$$= 2\int \frac{dy}{9-y^2} = 2\cdot\frac{1}{6}\log\left|\frac{3+y}{3-y}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right| + C.$$

7a. $\int \frac{dx}{2+3\cos x}$ [2007, Fall]

Solution: Let $I = \int \frac{dx}{2+3\cos x}$

Set $\tan\left(\frac{x}{2}\right) = y$ then $dx = 2\frac{dy}{1+y^2}$ and, $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-y^2}{1+y^2}$

Now,

$$I = \int \frac{2\,dy/(1+y^2)}{2+3(1-y^2)/(1+y^2)}$$

$$= \int \frac{2dy}{2+2y^2+3-3y^2}$$

$$= 2\int \frac{dy}{5-y^2} = 2\cdot\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}+y}{\sqrt{5}-y}\right| + C$$

$$= \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}+\tan\frac{x}{2}}{\sqrt{5}-\tan\frac{x}{2}}\right| + C.$$

$\int \frac{dx}{5+\cos x}$ [2011 Spring]

Solution: Let $I = \int \frac{dx}{5+\cos x}$

Set $\tan\frac{x}{2} = y$ then $dx = 2\frac{dy}{1+y^2}$. Then,

$$I = \int \frac{2dy/(1+y^2)}{5+(1-y^2)/(1+y^2)}$$

$$= \int \frac{2dy}{6+4y^2}$$

$$= \frac{1}{2}\int \frac{dy}{y^2+3/2} = \frac{\sqrt{2}}{2\sqrt{3}}\tan^{-1}\left(\frac{y\sqrt{2}}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\tan\left(\frac{x}{2}\right)\right) + C.$$

$\int \frac{1}{2+\cos x+\sin x}dx.$ [2002]

Solution: Let $I = \int \frac{dx}{2+\cos x+\sin x}$

Set $\tan\left(\frac{x}{2}\right) = y$ then $dx = 2\frac{dy}{1+y^2}$.

Also, $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-y^2}{1+y^2}$ and $\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2y}{1+y^2}$

Now,

$$I = \int \frac{2\,dy/(1+y^2)}{2+\frac{1-y^2}{1+y^2}+\frac{2y}{1+y^2}}$$

$$= 2\int \frac{dy}{2+2y^2+1-y^2+2y}$$

$$= 2\int \frac{dy}{3+2y+y^2} = 2\int \frac{dy}{(y+1)^2+(\sqrt{2})^2}$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y+1}{\sqrt{2}}\right) + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{\sqrt{2}}\right) + C$$

9. $\int \frac{x^4+4x^3+2x+1}{x^2+3x+2}dx.$ [2000]

Solution: Let $I = \int \frac{x^4 + 4x^3 + 2x + 1}{x^2 + 3x + 2} dx$.

$$= \int \left[x^2 + x - 5 + \frac{15x - 9}{(x + 2)(x + 1)}\right] dx$$

$$= \int x^2 dx + \int x dx - 5 \int dx + \int \frac{15x - 9}{(x + 2)(x + 1)} dx$$

Here,

$$\frac{15x - 9}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}$$

$$\Rightarrow 15x - 9 = A(x + 2) + B(x + 1)$$

$$= (A + B)x + (2A + B).$$

Comparing the coefficient of x and the constant term we get,

$$A + B = 15 \quad \text{and} \quad 2A + B = -9.$$

Solving we get, $A = -24$, $B = 39$.

Therefore,

$$I = \int x^2 dx + \int x dx - 5 \int dx - 24 \int \frac{dx}{x + 1} + 39 \int \frac{dx}{x + 2}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - 5x - 24 \log(x + 1) + 39 \log(x + 2) + C.$$

10. $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}}$ **[2005, Spring] [2006, Fall]**

Solution: Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$.

Set $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. Then,

$$I = \int \frac{\sin\theta \cdot \theta}{\cos\theta} \cdot \cos\theta\, d\theta$$

$$= \int \theta \cdot \sin\theta\, d\theta.$$

Then see Q.No. 1, Exercise 11.4 with replacing θ by x.

11. $\int \frac{\cos x\, dx}{\sqrt{2 \sin^2 x + 3 \sin x + 4}}$ **[2017 Fall][2006, Fall][2008, Fal**

Solution: Let $I = \int \frac{\cos x\, dx}{\sqrt{2\sin^2 x + 3\sin x + 4}}$

Set $\sin x = y$ then $\cos x\, dx = dy$. Then,

$$I = \int \frac{dy}{\sqrt{2y^2 + 3y + 4}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{y^2 + \frac{3}{2}y + 2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{y^2 + 2 \cdot \frac{3}{4} y + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 + 2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{\left(y + \frac{3}{4}\right)^2 + \left(\sqrt{\frac{23}{16}}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \log \left| y + \frac{3}{4} + \sqrt{y^2 + \frac{3}{2}y + 2} \right| + C_1.$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{4y + 3 + 2\sqrt{2}\sqrt{2y^2 + 3y + 4}}{4} \right| + C_1.$$

$$= \frac{1}{\sqrt{2}} \log \left| 4\sin x + 3 + 2\sqrt{2}\sqrt{2\sin^2 x + 3\sin x + 4} \right| + C.$$

$\frac{\sqrt{y^2+3y+2}}{\sqrt{2}}$

12. $\int \frac{dx}{1 + 3 \sin^2 x}$ [2006, Fall] [2002]

Solution: Let $I = \int \frac{dx}{1 + 3\sin^2 x}$

$$= \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x} dx \quad [\because \text{dividing num}^r \text{ and den}^r \text{ by } \cos^2 x]$$

$$= \int \frac{\sec^2 x}{1 + 4\tan^2 x} dx.$$

Put, $\tan x = y$ then $\sec^2 x\, dx = dy$. Then,

$$I = \int \frac{dy}{1 + 4y^2} = \frac{1}{4} \int \frac{dy}{y^2 + 1/4} = \frac{1}{4} \cdot \frac{1}{1/2} \tan^{-1}\left(\frac{y}{1/2}\right) + C$$

$$\Rightarrow I = \frac{1}{2} \tan^{-1}(2y) + C$$

$$= \frac{1}{2} \tan^{-1}(2\tan x) + C.$$

13. $\int \frac{x^2 + x + 1}{\sqrt{1 - x^2}} dx.$ [2005, Spring]

Solution: Let, $I = \int \frac{x^2 + x + 1}{\sqrt{1 - x^2}} dx$

Put $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. Then,

$$I = \int \frac{\sin^2\theta + \sin\theta + 1}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \int (\sin^2\theta + \sin\theta + 1) \, d\theta$$

$$= \int \left(\frac{1 - \cos 2\theta}{2} + \sin\theta + 1\right) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) - \cos\theta + \theta + C$$

$$= \frac{1}{2}(\theta - \sin\theta \cos\theta) - \cos\theta + \theta + C$$

$$= \frac{1}{2}\theta - \left(\frac{\sin\theta}{2} + 1\right)\cos\theta + C$$

$$= \frac{3}{2}\sin^{-1}x - \left(\frac{x}{2} + 1\right)\sqrt{1 - x^2} + C$$

14. $\int \frac{xe^x}{(x+1)^2} dx$ **[2008, Spring]**

Solution: Let, $I = \int \frac{xe^x}{(x+1)^2} dx = \int \frac{(x + 1 - 1)\, e^x}{(x+1)^2} dx$

$$= \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x \, dx$$

Since we have $\int [f(x) + f'(x)]\, e^x \, dx = f(x)\, e^x + C.$

So, $I = \frac{1}{x+1} e^x + C$

15. $\int \frac{e^{2x} + 1}{e^{2x} - 1} dx$ **[2008, Spring]**

Solution: Let, $I = \int \frac{e^{2x} + 1}{e^{2x} - 1} dx = \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

Put $e^x - e^{-x} = t$ then $(e^x + e^{-x})\, dx = dt$. So,

$$I = \int \frac{dt}{t} = \log(t) + C = \log(e^x - e^{-x}) + C.$$

16. $\int \frac{dx}{4 - 5\sin^2 x}$ **[2014, Fall]**

Solution: Let $I = \int \frac{dx}{4 - 5\sin^2 x}$

$$= \int \frac{\sec^2 x}{4\sec^2 x - 5\tan^2 x} dx \quad [\because \text{dividing num}^r \text{ and den}^r \text{ by } \cos^2 x]$$

$$= \int \frac{\sec^2 x}{1 - \tan^2 x} dx$$

Put, $\tan x = y$ then $\sec^2 x \, dx = dy$. Then,

$$I = \int \frac{dy}{1 - y^2} = \frac{1}{2}\log\left|\frac{1+y}{1-y}\right| + C = \frac{1}{2}\log\left|\frac{1 + \tan x}{1 - \tan x}\right| + C$$

DEFINITE INTEGRALS – LONG QUESTIONS

(a) $\int_0^4 \frac{x + 2}{\sqrt{4x - x^2}} dx.$ **[1999] [2001] [2003, Spring]**

Solution: Let

$$I = \int_0^4 \frac{x+2}{\sqrt{4x - x^2}} dx$$

$$= -\frac{1}{2}\int_0^4 \frac{4 - 2x - 8}{\sqrt{4x - x^2}} dx$$

$$= -\frac{1}{2}\int_0^4 \frac{4 - 2x}{\sqrt{4x - x^2}} dx + 4\int_0^4 \frac{dx}{\sqrt{4x - x^2}}$$

$$= -\frac{1}{2}\int_0^4 \frac{4 - 2x}{\sqrt{4x - x^2}} dx + 4\int_0^4 \frac{dx}{\sqrt{4 - (x-2)^2}}$$

Set, $\sqrt{4x - x^2} = y$ for first integral. So, $\frac{4 - 2x}{\sqrt{4x - x^2}} dx = dy.$

Also, $x = 0 \Rightarrow y = 0$, $x = 4 \Rightarrow y = 0$.

Therefore, the first integral has zero value being both upper and lower limit is 0. Thus,

$$I = 4\int_0^4 \frac{dx}{\sqrt{4 - (x-2)^2}} = 4\left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^4$$

$$= 4[\sin^{-1}(1) - \sin^{-1}(-1)]$$

$$= 4\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)$$

$$= 4\left(\frac{2\pi - 3\pi}{4}\right)$$

$$= \pi.$$

(b) $\int_0^{\pi/2} \cos^4 x \sin^3 x \, dx.$ **[2016 Fall Short][1999] [2001]**

Solution: Let,

$$I = \int_0^{\pi/2} \cos^4 x \sin^3 x \, dx$$

Set, $\cos x = y$ then $- \sin x \, dx = dy$.

Also, $x = 0 \Rightarrow y = 1, x = \frac{\pi}{2} \Rightarrow y = 0$. Then,

$$I = \int_1^0 y^4 (1 - y^2)(-dy)$$

$$= \int_0^1 y^4 dy - \int_0^1 y^6 dy = \left[\frac{y^5}{5} - \frac{y^7}{7}\right]_0^1 = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}.$$

(c) $\int_1^2 \sqrt{2x - x^2}\, dx$. [1999] [2001]

Solution: Let,

$$I = \int_1^2 \sqrt{2x - x^2}\, dx = \int_1^2 \sqrt{1 - (x-1)^2}\, dx$$

$$= \left[\frac{(x-1)\sqrt{2x - x^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_1^2$$

$$= \frac{1}{2}[\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}.$$

(d) $\int_0^{\pi/2} \log \sin x\, dx$. [2002]

Solution: Let,

$$I = \int_0^{\pi/2} \log(\sin x)\, dx \quad \ldots\ldots(i)$$

$$= \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(x-a)\, dx\right]$$

$$= \int_0^{\pi/2} \log \cos x\, dx \quad \ldots\ldots(ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)]\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} \log(2)\, dx$$

Set, $2x = \theta$ then $2dx = d\theta$. Also, $x = 0 \Rightarrow \theta = 0$; $x = \frac{\pi}{2} \Rightarrow \theta = \pi$. Then,

$$2I = \frac{1}{2}\int_0^{\pi} \log(\sin\theta)\, d\theta - \log(2)\int_0^{\pi/2} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin\theta)\, d\theta - (\log 2)\frac{\pi}{2}$$

$$= I - \log(2)\frac{\pi}{2} \quad \text{[using (i)]}$$

$$\Rightarrow 2I - I = \log(2)^{-1}\frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{2}\log\left(\frac{1}{2}\right).$$

(e) $\int_0^a \frac{x^4\, dx}{\sqrt{a^2 - x^2}}$ [2003, Fall]

Solution: Let,

$$I = \int_0^a \frac{x^4\, dx}{\sqrt{a^2 - x^2}}$$

Set, $x = a\sin\theta$ then $dx = -a\cos\theta\, d\theta$.

Also, $x = 0 \Rightarrow \theta = 0$, $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{a^4 \sin^4\theta\,(-a\cos\theta)}{a\cos\theta}\, d\theta$$

$$= -a^4 \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= -a^4 \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2}\right)^2 d\theta$$

$$= -\frac{a^4}{4}\int_0^{\pi/2} (1 - 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= -\frac{a^4}{4}\int_0^{\pi/2} \left(1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) d\theta$$

$$= -\frac{a^4}{8}\int_0^{\pi/2} (3 - 4\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= -\frac{a^4}{8}\left[3\theta - \frac{4\sin 2\theta}{2} + \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= -\frac{a^4}{8}\left[\frac{3\pi}{2} - \frac{4}{2}\sin\pi + \frac{\sin 2\pi}{4}\right] \quad [\because \sin 0 = 0]$$

$$= -\frac{a^4}{8}\left(\frac{3\pi}{2}\right) \quad [\because \sin n\pi = 0]$$

$$= -\frac{3\pi a^4}{16}.$$

(f) $\int_0^2 x\sqrt{x^4 + 2x^2 + 1}\, dx.$ **[2000]**

Solution: Let,

$$I = \int_0^2 x\sqrt{x^4 + 2x^2 + 1}\, dx$$

$$= \int_0^2 x\sqrt{(x^2+1)^2}\, dx$$

$$= \int_0^2 x(x^2+1)\, dx$$

$$= \int_0^2 (x^3 + x)\, dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = \frac{16}{4} + \frac{4}{2} = 4 + 2 = 6.$$

(g) $\int_1^2 (x-3)\sqrt{9x^2 + 12x + 13}\, dx.$ **[2000]**

Solution: Let,

$$I = \int_1^2 (x-3)\sqrt{9x^2 + 12x + 13}\, dx$$

Put, $x - 3 = k\, d(9x^2 + 12x + 13) + m$

$\Rightarrow x - 3 = k(18x + 12) + m$

Equating we get, $1 = 18k$ and $-3 = 12k + m.$

Solving we get,

$$k = \frac{1}{18}, \quad \text{and} \quad m = -\frac{11}{3}$$

Then,

$$I = \int_1^2 \left[\frac{1}{18} d(9x^2 + 12x + 13) - \frac{11}{3}\right]\sqrt{9x^2 + 12x + 13}\, dx$$

$$= \frac{1}{18}\int_1^2 (18x + 1)\sqrt{9x^2 + 12x + 13}\, dx - \frac{11}{3}\int_1^2 \sqrt{9x^2 + 12x + 13}\, dx$$

Put $9x^2 + 12x + 13 = y$ **for first integral.**

Then, $18x + 12 = dy$. Also, $x = 1 \Rightarrow y = 34$, $x = 2 \Rightarrow y = 73.$

So,

$$I = \frac{1}{18}\int_{34}^{73} \sqrt{y}\, dy - \frac{11}{3}\int_1^2 \sqrt{9x^2 + 12x + 13}\, dx$$

$$= \frac{1}{18}\int_{34}^{73} \sqrt{y}\, dy - \frac{11}{3}\int_1^2 \sqrt{(3x+2^2) + 9}\, dx$$

Set, $3x + 2 = t$ for second integral. So, $3dx = dt$.

Also, $x = 1 \Rightarrow t = 5$, $x = 2 \Rightarrow t = 8$.

Now,

$$I = \frac{1}{18}\int_{34}^{73} \sqrt{y}\, dy - \frac{11}{3}\int_5^8 \sqrt{t^2 + 9}\, \frac{dt}{3}$$

$$= \frac{1}{18}\left[\frac{y^{3/2}}{3/2}\right]_{34}^{73} - \frac{11}{3}\cdot\frac{3}{2}\left[\sqrt{t^2+9}\right]_5^8$$

$$= \frac{1}{27}[73^{3/2} - 34^{3/2}] - \frac{11}{2}[\sqrt{73} - \sqrt{34}]$$

$$= \frac{1}{27}(73\sqrt{73} - 34\sqrt{34}) - \frac{11}{2}(\sqrt{73} - \sqrt{34})$$

$$= \sqrt{73}\left(\frac{73}{27} - \frac{11}{2}\right) - \sqrt{34}\left(-\frac{34}{27} + \frac{11}{2}\right)$$

$$= \sqrt{73}\left(-\frac{151}{54}\right) - \sqrt{34}\left(\frac{229}{54}\right)$$

$$= -\frac{(151\sqrt{73} + 229\sqrt{34})}{54}.$$

(h) $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$ **[2005, Fall]**

Solution: See Q.2, Exercise 11.5.

(i) $\int_0^{\pi/2} \frac{dx}{2 + \cos x}$ **[2007, Spring]**

Solution: Let,

$$I = \int_0^{\pi/2} \frac{dx}{1 + 2\cos x}$$

Set, $\tan\left(\frac{x}{2}\right) = y$ then $dx = \frac{2\, dy}{1 + y^2}$. Also, $x = 0 \Rightarrow y = 0$, $x = \frac{\pi}{2} \Rightarrow y = 1.$

And, $\cos x = \frac{1 - y^2}{1 + y^2}$.

Now,

$$I = \int_0^1 \frac{2\, dy}{1 + y^2 + 2(1 - y^2)}$$

$$= 2\int_0^1 \frac{dy}{3 - y^2} \quad = 2\cdot\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3} + y}{\sqrt{3} - y}\right| + C$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{3} + \tan\frac{x}{2}}{\sqrt{3} - \tan\frac{x}{2}} \right| + C.$$

(j) $\int_0^4 x \sin^2 x \, dx$ [2007, Fall]

Solution: Let,

$$I = \int_0^4 x \sin^2 x \, dx$$

$$= \int_0^4 x \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^4 x\, dx - \frac{1}{2}\int_0^4 x \cos 2x \, dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^4 - \frac{1}{2}\left[x\frac{\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^4$$

$$= \frac{16}{4} - \frac{1}{2}\left[\frac{4\sin 8}{2} + \frac{\cos 8}{4} - \frac{1}{4}\right]$$

$$= \frac{16}{4} + \frac{1}{8} - \frac{1}{8}(8\sin 8 + \cos 8)$$

$$= \frac{1}{8}(33 - 8\sin 8 - \cos 8).$$

(k) $\int_0^{\pi/2} \frac{dx}{1 + 2\cos x}$ [2006, Spring]

Solution: Let, $I = \int_0^{\pi/2} \frac{dx}{1 + 2\cos x}$

Process as in (i).

(l) Show that $\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log \cos\theta \, d\theta = \frac{\pi}{2} \log \frac{1}{2}$ [2005, Spring][2017 Spring]

Solution: Let,

$$I = \int_0^{\pi/2} \log \sin\theta \, d\theta \quad \text{.......(i)}$$

$$\Rightarrow I = \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - \theta\right)\right] d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \log \cos\theta \, d\theta \quad \text{....(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log \sin\theta \, d\theta + \int_0^{\pi/2} \log \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (\log \sin\theta + \log \cos\theta) \, d\theta$$

$$= \int_0^{\pi/2} \log (\sin\theta \cos\theta) \, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin\theta\cos\theta}{2}\right) d\theta$$

$$= \int_0^{\pi/2} [\log(\sin 2\theta) - \log 2] \, d\theta$$

$$= \int_0^{\pi/2} \log(\sin 2\theta) \, d\theta - \log 2 \int_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \log(\sin 2\theta) \, d\theta - \log 2 [\theta]_0^{\pi/2}$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin 2\theta) \, d\theta - \frac{\pi}{2}\log 2 \quad \text{....(iii)}$$

Putting $2\theta = t$, we get $d\theta = \frac{dt}{2}$.

And $\theta = 0 \Rightarrow t = 0$ and $\theta = \frac{\pi}{2} \Rightarrow t = \pi$.

$$\Rightarrow \int_0^{\pi/2} \log \sin 2\theta \, d\theta = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt$$

$$= \int_0^{\pi/2} \log \sin t \, . \, dt$$

$$= I \quad \text{[By (i)]}$$

Now (iii) becomes,

$$2I = I - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\frac{\pi}{2} . \log 2 = \frac{\pi}{2}\log\frac{1}{2}$$

Thus we get,

$$\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log \cos\theta \, d\theta = \frac{\pi}{2}\log\frac{1}{2}$$

(m) $\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx.$ [2015 Spring][2012 Fall]

Solution: Let,

$$I = \int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$$

$$= \int_0^{\pi/2} \sin^2 x \sin^2 x \cos^2 x\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 2x}{2} \sin^2 2x\, dx$$

$$= \frac{1}{16}\int_0^{\pi/2} (1-\cos 2x)(1-\cos 4x)\, dx$$

$$= \frac{1}{16}\int_0^{\pi/2} (1-\cos 2x - \cos 4x + \cos 2x \cos 4x)\, dx$$

$$= \frac{1}{16}\int_0^{\pi/2} \left(1-\cos 2x - \cos 4x + \frac{1}{2}(\cos 6x + \cos 2x)\right) dx$$

$$= \frac{1}{16}\left[x - \frac{\sin 2x}{2} - \frac{\sin 4x}{4} + \frac{\sin 6x}{12} + \frac{\sin 2x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{16}\left[\frac{\pi}{2}\right] \qquad [\because \sin n\pi = 0 \text{ for n is integer}]$$

$$= \frac{\pi}{32}$$

DEFINITE INTEGRAL – LIMIT AS A SUMMATION

(a) $\int_0^2 x^2\, dx$ [2002] [2008, Spring]

Solution:

Comparing the given integral $\int_0^2 x^2\, dx$ with the integral $\int_a^b f(x)\, dx$ then we get,

$f(x) = x^2$, $a = 0$, $b = 2$. So, $nh = b - a = 2 - 0 = 2$.

Also, $f(a + rh) = (a + rh)^2 = r^2h^2$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

Therefore,

$$\int_0^2 x^2 dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a + rh)$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} (rh)^2$$

$$= \lim_{h\to 0} h^3 \sum_{r=1}^{n} r^2 = \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h\to 0} \frac{nh\,(nh+h)(2nh+h)}{6}$$

$$= \lim_{h\to 0} \frac{2(2+h)(4+h)}{6}$$

$$= \frac{2(2+0)(4+0)}{6} = \frac{8}{3}.$$

Thus, $\int_0^2 x^2 dx = \frac{8}{3}$.

(b) $\int_0^1 x^3\, dx$ [2015 Spring][2009, Fall] [2006, Spring]

Solution:

Comparing the given integral $\int_0^1 x^3\, dx$ with the integral $\int_a^b f(x)\, dx$ then we get,

$f(x) = x^3$, $a = 0$, $b = 1$. So, $nh = b - a = 1 - 0 = 1$.

Also, $f(a + rh) = (a + rh)^3 = r^3h^3$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\, dx = f(a + rh).$$

Therefore,

$$\int_0^1 x^3 dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a + rh)$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} r^3h^3$$

$$= \lim_{h\to 0} h^4 \sum_{r=1}^{n} r^3 = \lim_{h\to 0} h^4 \left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{h\to 0} \left(\frac{nh\,(nh+h)}{2}\right)^2$$

$$= \lim_{h\to 0} \left(\frac{1(1+h)}{2}\right)^2$$

$$= \left(\frac{1+0}{2}\right)^2 = \frac{1}{4}.$$

Thus, $\int_0^1 x^3 dx = \frac{1}{4}$.

(c) $\int_a^b e^{mx}\,dx$ **[2014 Spring][2013 Spring][2013 Fall][2003, Fall]**

Solution:

Comparing the given integral $\int_a^b e^{mx}\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = e^{mx}$, a = a, b = b. So, nh = b – a

Also, $f(a + rh) = e^{m(a + rh)} = e^{am} \cdot e^{mrh}$

$f(a+r) = e^{m(a+rh)}$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

$\int_a^b e^{mn}$

Therefore,

$$\int_a^b e^{mx}\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (e^{am} \cdot e^{mrh})$$

$a =$

$$= e^{am} \lim_{h\to 0} h \sum_{r=1}^{n} (e^{mh})^r$$

$$= e^{am} \lim_{h\to 0} h \frac{e^{mh}[(e^{mh})^n - 1]}{e^{mh} - 1}$$

$$= e^{am} \lim_{h\to 0} h \frac{e^{mh}(e^{mnh} - 1)}{e^{mh} - 1}$$

$$= e^{am} \lim_{h\to 0} \frac{e^{mh}(e^{m(b-a)} - 1)}{e^{mh} - 1} \cdot h$$

e^{mh}

$e^{m}\left(\frac{e^{amh} - 1}{e^{m} - 1}\right)$

$$= e^{am}(e^{mb - ma} - 1) \lim_{h\to 0} \frac{e^{mh}}{e^{mh} - 1} \cdot h$$

$$= (e^{mb} - e^{am}) \cdot \lim_{h\to 0} \frac{mh}{e^{mh} - 1} \left(\frac{e^{mh}}{m}\right)$$

$$= (e^{mb} - e^{am})\left(\frac{1}{m}\right) \qquad \left[\text{Being } \frac{e^x - 1}{x} = 1\right]$$

$$= \frac{e^{mb} - e^{am}}{m}.$$

$\left(\frac{1}{m}\right)$

Thus, $\int_a^b e^{mx}\,dx = \frac{e^{mb} - e^{am}}{m}$.

(d) $\int_a^b e^x\,dx$ **[2014 Fall][2007, Spring] [2005, Spring]**

Solution:

Comparing the given integral $\int_a^b e^x\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = e^x$, a = a, b = b. So, nh = b – a

Also, $f(a + rh) = e^{(a + rh)} = e^a \cdot e^{rh}$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_a^b e^x\,dx = e^a \lim_{h\to 0} h \sum_{r=1}^{n} (e^h)^r$$

$$= e^a \lim_{h\to 0} h \frac{e^h[(e^{nh}) - 1]}{e^h - 1}$$

$$= e^a \lim_{h\to 0} \frac{e^h(e^{(b-a)} - 1)}{e^h - 1} \cdot h$$

$$= e^a(e^{b-a} - 1) \lim_{h\to 0} \frac{e^h}{e^h - 1} \cdot h$$

$$= (e^b - e^a) \cdot \lim_{h\to 0} \frac{h}{e^h - 1} e^h$$

$$= (e^b - e^a) \cdot 1 \qquad \left[\text{Being } \frac{e^x - 1}{x} = 1\right]$$

$$= e^b - e^a.$$

Thus, $\int_a^b e^x\,dx = e^b - e^a$.

(e) $\int_a^b x^2\,dx$ **[2005, Fall]**

Solution:

Comparing the given integral $\int_a^b x^2\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = x^2$, a = a, b = b. So, nh = b – a.

Also, $f(a + rh) = (a + rh)^2 = r^2h^2$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_a^b x^2\,dx = \lim_{h\to0} h\sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{h\to0} h\sum_{r=1}^{n} (a^2+2arh+r^2h^2)$$

$$= \lim_{h\to0}\left[a^2h\sum_{r=1}^{n}1 + 2ah^2\sum_{r=1}^{n}r + h^3\sum_{r=1}^{n}r^2\right]$$

$$= \lim_{h\to0}\left[a^2h.n + 2ah^2\frac{n(n+1)}{2} + h^3\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to0}\left[a^2nh + a\,nh(nh+h) + \frac{nh(nh+h)(2nh+h)}{6}\right]$$

$$= \lim_{h\to0}\left(a^2(b-a)+a(b-a)(b-a+h)+\frac{(b-a)(b-a+h)(2b-2a+h)}{6}\right)$$

$$= a^2(b-a)+a(b-a)(b-a+0)+(b-a)(b-a+0)\left(\frac{2b-2a+0}{6}\right)$$

$$= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{3}$$

(f) $\int_0^4 (x^2+3x)\,dx$ **[2007, Fall]**

Solution:

Comparing the given integral $\int_0^4 (x^2+3x)\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = x^2+3x$, $a = 0$, $b = 4$. So, $nh = b-a = 4$.

Also, $f(a+rh) = (a+rh)^2 + 3(a+rh) = r^2h^2 + 3rh$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$\int_0^4 (x^2+3x)\,dx = \lim_{h\to0} h\sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{h\to0} h\sum_{r=1}^{n}(r^2h^2+3rh)$$

$$= \lim_{h\to0}\left[h^3.\frac{n(n+1)(2n+1)}{6} + 3h^2\frac{n(n+1)}{2}\right]$$

$$= \lim_{h\to0}\left(\frac{nh(nh+h)(2nh+h)}{6} + \frac{3nh(nh+h)}{2}\right)$$

$$= \lim_{h\to0}\left(\frac{4(4+h)(8+h)}{6} + \frac{3(4)(4+h)}{2}\right)$$

$$= \frac{4(4+0)(8+0)}{6} + \frac{3(4)(4+0)}{2}$$

$$= \frac{64}{3} + 24$$

$$= \frac{136}{3}.$$

(g) $\int_0^1 x^2\,dx$ **[2009, Fall] [2006, Spring]**

Solution:

Comparing the given integral $\int_0^1 x^2\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = x^2$, $a = 0$, $b = 1$. So, $nh = b-a = 1-0 = 1$.

Also, $f(a+rh) = (a+rh)^2 = r^2h^2$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a+rh).$$

Therefore,

$$\int_0^1 x^2\,dx = \lim_{h\to0} h\sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{h\to0} h\sum_{r=1}^{n} r^2h^2$$

$$= \lim_{h\to0} h^3\sum_{r=1}^{n} r^2 = \lim_{h\to0} h^3\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to0}\left(\frac{nh(nh+h)(2nh+h)}{6}\right)$$

$$= \lim_{h\to0}\left(\frac{1(1+h)(2+h)}{6}\right) = \frac{2}{6} = \frac{1}{3}.$$

Thus, $\int_0^1 x^2\,dx = \frac{1}{3}$.

Short Questions

(a) $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$ **[2007, Fall]**

Solution: Let,

$$I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$$

Put, $\sin^{-1} x = y$ then $\dfrac{dx}{\sqrt{1-x^2}} = dy$. Then,

$$I = \int e^y\,dy = e^y + C = e^{\sin^{-1}x} + C.$$

(b) $\displaystyle\int \frac{dx}{x \log x}$ **for $x > 0$** [2006, Fall]

Solution: Let,

$$I = \int \frac{dx}{x \log x} \quad \text{for } x > 0.$$

Put, $\log x = y$ then $\dfrac{dx}{x} = dy$. Then,

$$I = \int \frac{dy}{y} = \log(y) + C = \log(\log x) + C.$$

(c) $\int x^2 e^x\,dx$ **[2011 Spring][2005, Spring]**

Solution: Let,

$$I = \int x^2 e^x\,dx$$
$$= x^2e^x - 2x\,e^x + 2e^x + C \quad [\because \text{applying integration by parts}]$$

(d) $\int x^4 \log x\,dx.$ [2002]

Solution: Let,

$$I = \int x^4 \log(x)\,dx$$
$$= \log x\,\frac{x^5}{5} - \int \frac{1}{x}\frac{x^5}{5}\,dx + C_1 \quad [\because \text{applying integration by parts}]$$
$$= \frac{x^5 \log x}{5} - \frac{1}{5}\int x^4\,dx + C_1$$
$$= \frac{x^5 \log x}{5} - \frac{x^5}{25} + C.$$

(e) $\int x^2 \log x\,dx.$ log x [1999]

Solution: Let

$$I = \int x^2 \log x\,dx$$
$$= \log x\,\frac{x^3}{3} - \int \frac{1}{x}\frac{x^3}{3}\,dx + C_1 \quad [\because \text{applying integration by parts}]$$
$$= \frac{x^3 \log x}{3} - \frac{1}{3}\int x^2\,dx + C_1$$
$$= \frac{x^3 \log x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + C.$$

$\int x^2$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C.$$

Q. Integrate: $\int x \sin^2 x\,dx$ 2012 Fall

Solution. Let,

$$I = \int x \sin^2 x\,dx$$
$$= \int x\left(\frac{1-\cos 2x}{2}\right)dx$$
$$= \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$$
$$= \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[(x)\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-\cos 2x}{4}\right)\right] + C$$
$$= \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} + C$$

Q. Evaluate: $\displaystyle\int_1^e \log(x)\,dx$ 2012 Fall

Solution. Let,

$$I = \int_1^e \log(x)\,dx$$
$$= \int_1^e 1.\log(x)\,dx$$
$$= \left[\log(x)\,(x) - \int \frac{1}{x}(x)\,dx\right]_1^e \quad [\because \text{applying integrating by parts}]$$
$$= e\log(e) - \int_1^e dx \quad [\because \log(1) = 0]$$
$$= e - [x]_1^e \quad [\because \log(e) = 1]$$
$$= e - e + 1$$
$$= 1$$

Q. Evaluate: $\displaystyle\int \frac{x+5}{(x+1)(x+2)^2}\,dx$ 2013 Fall

Solution. Let,

$$I = \int \frac{x+5}{(x+1)(x+2)^2}\,dx$$

Here,

$$\frac{x+5}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}$$

$$\Rightarrow \quad x+5 = A(x^2+2x+a) + B(x^2+3x+2) + C(x+1)$$
$$= (A+B)x^2 + (2A+3B+C)x + (4A+2B+C)$$

Equating the like terms, we get,

$$A+B=0,\ 2A+3B+C=1,\ 4A+2B+C=5$$

Solving we get,

$$A=\frac{4}{3},\ B=\frac{-4}{3} \text{ and } C=\frac{7}{3}$$

Then,

$$I = \frac{4}{3}\int\frac{dx}{x+1} - \frac{4}{3}\int\frac{dx}{x+2} + \frac{7}{3}\int\frac{dx}{(x+2)^2}$$
$$= \frac{4}{3}\log(x+1) - \frac{4}{3}\log(x+2) - \frac{7}{3(x+2)} + C$$

Q. Integrate: $\int\frac{e^x\,dx}{e^x - 3e^{-x} + 2}$ 2014 Fall

Let,

$$I = \int\frac{e^x\,dx}{e^x - 3e^{-x} + 2}$$

Put $e^x = y$ then $e^x\,dx = dy$. So,

$$I = \int\frac{dy}{y - 3y^{-1} + 2}$$
$$= \int\frac{y}{y^2+2y-3}\,dy$$
$$= \int\frac{y}{(y+3)(y-1)}\,dy$$

Here,

$$\frac{y}{(y+3)(y-1)} = \frac{A}{y+3} + \frac{B}{y-1}$$
$$= \frac{A(y-1)+B(y+3)}{(y+3)(y-1)}$$

$$\Rightarrow y = A(y-1) + B(y+3)$$
$$= (A+B)y + (-A+3B)$$

Equating the like terms we get,

$$A+B=0,\ -A+3B=0$$

Solving we get

$$A=\frac{3}{4},\ B=\frac{1}{4}$$

Then,

$$I = \frac{3}{4}\int\frac{dy}{y+3} + \frac{1}{4}\int\frac{dy}{y-1}$$
$$= \frac{3}{4}\log(y+3) + \frac{1}{4}\log(y-1) + C$$
$$= \frac{3}{4}\log(e^x+3) + \frac{1}{4}\log(e^x-1) + C$$

Q. Evaluate: $\int_1^e x\log(x)\,dx$ [2015, Fall]

Solution. Let,

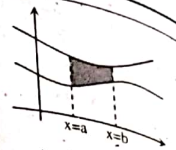
$$I = \int_1^e x\log(x)\,dx$$
$$= \left[\log(x)\left(\frac{x^2}{2}\right) - \int\frac{1}{x}\left(\frac{x^2}{2}\right)\right]_1^e$$
$$= \frac{e^2}{2} - \int_1^e \frac{x}{2}\,dx \qquad [\because \log(e)=1,\ \log(1)=0]$$
$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$
$$= \frac{e^2}{2} - \left(\frac{e^2-1}{4}\right)$$
$$= \frac{1}{4} + \frac{e^2}{4}$$

•••

List of Formulae

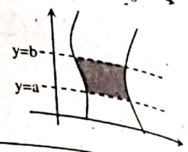
The area of the region bounded by two curves $y_1 = f(x)$, $y_2 = g(x)$ in between $x = a$ and $x = b$ then,

$$A = \int_a^b (y_1 - y_2)\,dx.$$



The area of the region bounded by two curves $x_1 = f(y)$, $x_2 = g(y)$ in between $y = a$ and $y = b$ then,

$$A = \int_a^b (x_1 - x_2)\,dy.$$



Exercise 12.1

A. Find the area bounded by the x–axis and the following curve and ordinates.

1. $y^2 = 4x$, $x = 4$, $x = 9$

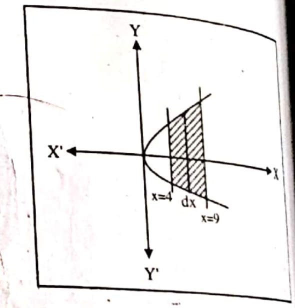
Solution: Given parabola is, $y^2 = 4x$

which is has vertex (0, 0) and a = 1 > 0 and line of symmetry is y = 0.

Also, x = 4 and x = 9.

Graph of given curve and ordinates are shown in figure.

Now,



Required area, $A = \int_{x=4}^{9} y\,dx$.

$$= \int_4^9 2\sqrt{x}\,dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{3/2}\right]_4^9$$

$$= \frac{4}{3}(9^{3/2} - 4^{3/2}) = \frac{4}{3}[27-8] = \frac{4}{3} \times 19 = \frac{76}{3} = 25\tfrac{1}{3} \text{ sq. unit.}$$

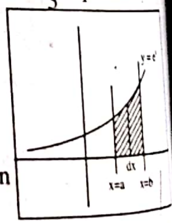
Thus, the area of the area bounded by the given curves is $\frac{76}{3}$ sq. units.

2. $y = e^x$, $x = a$, $x = b$

Solution: Given curves are,

$$y = e^x,\ x = a,\ x = b$$

Curve of given function and ordinate can be drawn as,



The region bounded by the curves $y = e^x$, $x = a$, $x = b$ is the shaded portion. Taking vertical strip in the region of integration and taking limit from x = a to x = b we get the area of the region is,

$$A = \int_a^b e^x\,dx = [e^x]_a^b = (e^b - e^a) \text{ sq. unit}$$

Thus, the area bounded by the given curves $y = e^x$, $x = a$, $x = b$ is $(e^b - e^a)$ sq. unit.

3. $y = x^2 - 4$, $x = 3$, $x = 5$

Solution: Given curves are,

$$y = x^2 - 4$$

$$\Rightarrow x^2 = (y + 4)$$

which is a parabola having vertex (0, –4) and $a = \frac{1}{4} > 0$. So its curve is shown in figure including given ordinates.

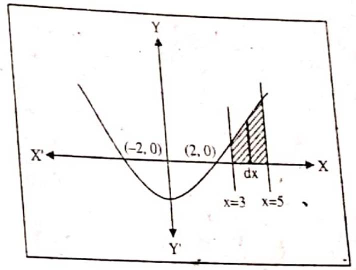
The region bounded by the curves $y = x^2 - 4$, $x = 3$, $x = 5$ is the shaded portion. Taking vertical strip as dx and taking limits from x = 3 to x = 5, we get the area of the region is,

$$A = \int_3^5 (x^2 - 4)\,dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_3^5$$

$$= \left[\frac{125}{3} - 20\right] - \left[\frac{27}{3} - 12\right]$$

$$= \frac{65}{3} + 3 = \frac{74}{3}$$



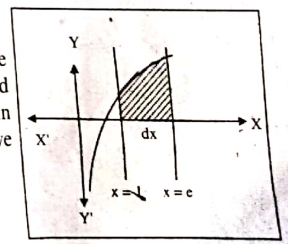
Thus, the area bounded by the given curves $y = x^2 - 4$, $x = 3$, $x = 5$ is $\frac{74}{3}$ sq. unit.

4. $y = \log x$, $x = 1$, $x = e$.

Solution: Given curves are,

$$y = \log x,\ x = 1,\ x = e$$

Tracing the curves the region bounded by the curves $y = \log x$, $x = 1$, $x = e$ is the shaded portion. Taking, vertical strip dx as shown in figure are limits from x = 1 to x = e then we get the area of the region is,



$$A = \int_1^e \log x\,dx = [\log x \cdot x - x]_1^e$$

$$= (e \log e - e) - [(\log 1)(1) - 1]$$

$$= (e - e) - (-1) = 1.$$

Thus, the area bounded by the given curves $y = x^2 - 4$, $x = 3$, $x = 5$ is 1 sq. unit.

B. Find the area bounded by

1. The curve $y = 2 - x^2$ and the line $y = -x$.

Solution: Given curves are,

$$y = 2 - x^2$$

$$\Rightarrow x^2 = -(y-2) \qquad \ldots (i)$$

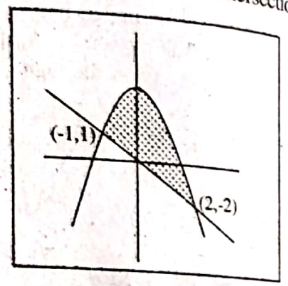
$$\text{and} \quad y = -x \qquad \ldots (ii)$$

Clearly the curve (i) is a parabola having vertex at (0, 2) whose line of symmetry is $x = 0$ i.e. y-axis and having down openward.

Also, the line (ii) passes through (0, 0) and (1, –1).

With this information, the sketch of the curves is shown in figure. Therefore the region bounded by the curves (i) and (ii) is the shaded portion in the figure. Solving the equations (i) and (ii) we get the point of the intersections are (–1, 1) and (2, –2).

Now, area of the region is



$$A = \int_{x=-1}^{2} (y_1 - y_2)\, dx$$

$$= \int_{x=-1}^{2} (2 - x^2 + x)\, dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2} = 4.5$$

Thus, the area bounded by the given curves $y = 2 - x^2$ and the line $y = -x$ is 4.5 sq. units.

2. $x + y^2 = 0$ and $x + 3y^2 = 2$.

Solution: Given curves are,

$$y^2 = -x \qquad \ldots (i)$$

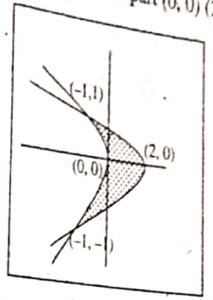
$$\text{and} \quad y^2 = -\frac{1}{3}(x-2) \qquad \ldots (ii)$$

Clearly the curves are parabolas in which (i) has vertex at (0, 0) with line of symmetry is $y = 0$ i.e. x-axis and having left open ward. Also, the parabola (ii) has vertex at (2, 0) with line of symmetry is $y = 0$ i.e. x-axis and having left open ward.

With this information, the sketch of the curves is shown in figure. Therefore the region bounded by the curves (i) and (ii) is the shaded portion in the figure. Solving the equations (i) and (ii) we get the point of the intersections are (–1, 1) and (–1, –1).

From figure it is clear that the required region is 2 times of the part (0, 0) (2, 0) and (–1, 1)

Now, the area of the required region be



$$A = 2\left|\int_{y=0}^{1} (x_1 - x_2)\, dy\right|$$

$$= 2\left|\int_{y=0}^{1} (-y^2 - 2 + 3y^2)dy\right|$$

$$= 2\left|\left[\frac{-y^3}{3} - 2y + y^3\right]_0^1\right|$$

$$= 2\left|-\frac{1}{3} - 2 + 1\right| = \frac{8}{3}$$

Thus, the area bounded by the given curves $x + y^2 = 0$ and $x + 3y^2 = 2$ is $\frac{8}{3}$ sq. units.

3. The curve $y^2 = 4x$ and the line $y = x$.

Solution: Given curves are,

$$y^2 = 4x \qquad \ldots (i)$$

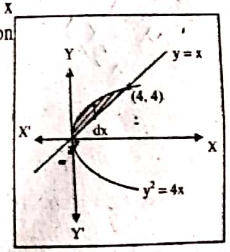
$$\text{and} \quad y = x \qquad \ldots (ii)$$

Clearly the curve (i) is a parabola having vertex at (0, 0) whose line of symmetry is $y = 0$ i.e. x-axis and having right open ward.

Also, the line (ii) passes through (0, 0) and (1, 1).

With this information, the sketch of the curves is shown in figure. Therefore the region bounded by the curves (i) and (ii) is the shaded portion in the figure. Solving the equations (i) and (ii) we get the point of the intersections are (0, 0) and (4, 4). Clearly the region has no symmetrical parts.

Now, Taking vertical strip dx and limit from line x = 0 to x = 4, we get the required area of the region is



$$A = \int_0^4 (y_1 - y_2)\, dx$$

$$= \int_0^4 (2\sqrt{x} - x)\, dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4 = \left[\frac{32}{3} - 8\right] = \frac{8}{3}$$

Thus, the area bounded by the given curves $y^2 = 4x$ and the line $y = x$ is $\frac{8}{3}$ sq. units.

4. The curve $x = y^2$ and $x = -2y^2 + 3$. [2006, Spring]

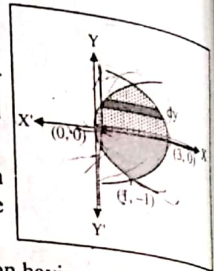
Solution: Given, $x = y^2 \Rightarrow y^2 = x$ ... (i)

which is a parabola with vertex (0, 0) and $a = \frac{1}{4} > 0$, and its line of symmetry is $y = 0$.

Also, $x = -2y^2 + 3$

$$\Rightarrow 2y^2 = -x + 3 \Rightarrow y^2 = -\frac{1}{2}(x-3) \quad \text{... (ii)}$$

which is a of parabola with vertex (3, 0) and $a = -\frac{1}{8} < 0$, and its line of symmetry $y = 0$.



Solving equation (i) and (ii) we get the points of intersection of equation (i) and (ii) are (1, 1) and (1, –1).

With these information, the sketch of the region is the shaded portion in the figure. Clearly the region has two symmetrical parts.

So the area of the region is the twice of the region having extreme points at (0, 0), (3, 0) and (1, 1).

Therefore, the area of the region is;

$$A = 2\int_{y=0}^{1} (x_1 - x_2)\, dy$$

$$\text{i.e. } A = 2\int_{y=0}^{1} \{(-2y^2 + 3) - y^2\}\, dy$$

$$= 6\int_{y=0}^{1} (-3y^2 + 3)\, dy$$

$$= 6\int_{y=0}^{1} (1 - y^2)\, dy$$

$$= 6\left[y - \frac{y^3}{3}\right]_0^1 = 6\left[\left(1 - \frac{1}{3}\right) - 0\right] = 6 \times \frac{2}{3} = 4.$$

Thus, the area bounded by the given curves $x = y^2$ and $x = -2y^2 + 3$ is 4 sq. units.

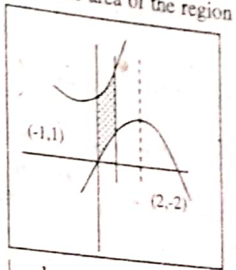
5. $y = \sec^2 x$, $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Solution: The required region is bounded by the curves

$$y = \sec^2 x,\ y = \sin x,\ x = 0 \text{ and } x = \frac{\pi}{4}$$

The sketch of the region bounded by given curves is the shaded portion in the figure.

Clearly the region has no symmetrical part. Therefore the area of the region is,



$$A = \left|\int_{x=0}^{\pi/4} (y_1 - y_2)\, dx\right|$$

$$\text{i.e. } A = \left|\int_{x=0}^{\pi/4} [\sec^2 x - \sin x]\, dx\right|$$

$$= \left|[\tan x + \cos x]_0^{\pi/4}\right|$$

$$= \left|\left(1 - \frac{1}{\sqrt{2}}\right) - (0 + 1)\right| = \left|-\frac{1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}.$$

Thus, the area bounded by the given curves $y = \sec^2 x$, $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$ is $\frac{1}{\sqrt{2}}$ sq. units.

6. the curve $x = y^3$ and $x = y^2$.

Solution: The required region is bounded by the curves $x = y^3$ and $x = y^2$.

Travelling path for $x = y^3$

x	1	–1	8	–8
y	1	–1	2	–2

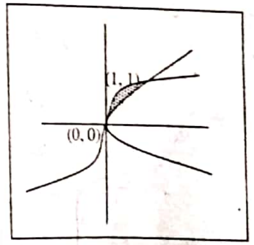
Travelling path for $x = y^2$

x	1	–1	4	4
y	1	–1	2	–2

The region bounded by the given curves is the shaded portion. Clearly, the required region is the shaded portion that has corners (0, 0) and (1, 1).

From figure it is clear that the shaded region has no symmetrical part.

Now, the area of the bounded region is,



$$A = \int_{y=0}^{1} (x_1 - x_2)dy$$

$$= \int_{y=0}^{1} (y^2 - y^3)\, dy$$

$$= \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Thus, the area bounded by the given curves $x = y^3$ and $x = y^2$ is $\frac{1}{12}$ sq. units.

7. The curve $y = 2x - x^2$ and the line $y = -3$.

Solution: Given curves are,

$$y = 2x - x^2$$

$$\Rightarrow -y = x^2 - 2x$$

$$\Rightarrow -y + 1 = x^2 - 2x + 1$$

$$\Rightarrow (x-1)^2 = -(y-1)$$

$2x - x^2 + 3 = 0$

$x^2 - 2x - 3 = 0$

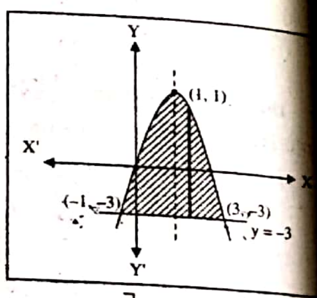
This is a parabola having vertex (1, 1), $a = -\frac{1}{4} < 0$, line of symmetry $x = 1$ and which is passing through origin.

And a given line is,

$$y = -3 \qquad \ldots \text{(ii)}$$

Solving (i) and (ii) we get the point of intersection are (−1, −3) and (3, −3). With these information, the trace of the curves (i) and (ii) is shown in figure. The bounded region by curve (i) and curve (ii) is the shaded portion in the figure.

Clearly the region has no symmetrical part while taking the vertical strip. Now, taking the vertical strip as dx and limit from x = −1 to x = 3, then the area of bounded region by equation (i) and equation (ii) is



$$A = \left|\int_{y=-1}^{3} (y_1 - y_2)\, dx\right|$$

$$\text{i.e. } A = \left|\int_{-1}^{3} \{-3 - (2x - x^2)\}\, dx\right|$$

$$= \left|\int_{-1}^{3} (x^2 - 2x - 3)\, dx\right|$$

$$= \left|\left[\frac{x^3}{3} - \frac{2x^2}{2} - 3x\right]_{-1}^{3}\right|$$

$$= \left[\frac{3^3}{3} - 3^2 - 3\times 3\right] - \left[\frac{(-1)^3}{3} - (-1)^2 - 3\times -1\right]$$

$$= \left|(9 - 9 - 9) - \left(-\frac{1}{3} - 1 + 3\right)\right|$$

$$= \left|-9 + \frac{1}{3} - 2\right|$$

$$= \left|-11 + \frac{1}{3}\right| = \left|\frac{-33+1}{3}\right| = \left|\frac{-32}{3}\right| = \frac{32}{3}.$$

Thus, the area bounded by the given curve $y = 2x - x^2$ and the line $y = -3$ is $\frac{32}{3}$ sq. units.

8. The curve $x + y = 2$, on the left by $y = x^2$ and below by x–axis. [2009 Spring]

Solution: Given line is,

$$x + y = 2 \qquad \ldots \text{(i)}$$

Since the line (i) intersects the axes at (0, 2) to y-axis and (2, 0) at x-axis.

And the given curve is,

$$y = x^2 \qquad \ldots \text{(ii)}$$

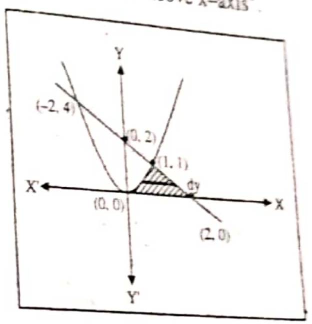
This is parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is x = 0.

Now, solving the equation (i) and (ii) we get the point of intersection of curve (i) and (ii) are (1, 1) and (−2, 4).

Now, tracing curve of equation (i) and (ii) below is given aside. Thus the region bounded by the curves is shown by the shaded in the figure.

Clearly the region has no symmetrical parts. The region bounded on "Left by $y = x^2$" means "along right" "below by x–axis" means "above x–axis".

Now, taking horizontal strip dy in which the strip moves from y = 0 to y = 1, area of shaded region is given by



$$A = \int_{y=0}^{1} (y_1 - y_2)\, dy$$

$$\text{i.e. } A = \int_{y=0}^{1} \{(2 - y) - y^{1/2}\}\, dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{2y^{3/2}}{3}\right]_0^1$$

$$= \left(2 - \frac{1}{2} - \frac{2}{3}\right) - 0 = \frac{12 - 3 - 4}{6} = \frac{5}{6} \text{ sq. unit}$$

Thus, the area bounded by the given curve x + y = 2, on the left by $y = x^2$ and below by x–axis is $\frac{5}{6}$ sq. units.

9. The curve $y = \sin\left(\frac{\pi x}{2}\right)$ and the line y = x.

Solution: Given curve is,

$$y = \sin\left(\frac{\pi x}{2}\right) \qquad \ldots \text{(i)}$$

and the line is,

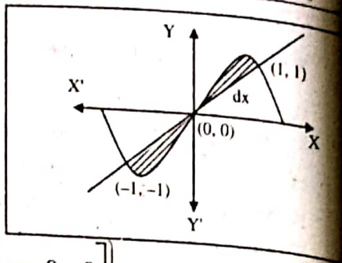
$$y = x \qquad \ldots \text{(ii)}$$

Clearly the curve (i) is the sine curve. And, the line (ii) passes through the point (0, 0) and (1, 1). Solving (i) and (ii) then we get the point of contact of (i) and (ii) are (0, 0), (1, 1) and (−1, −1).

Therefore, the graph of the region bounded by (i) and (ii) is sketch below. The bounded region is the shaded portion in the figure.

Clearly, it has two symmetrical parts. So, taking, vertical strip dx in region that has limits from x = 0 to x = 1.

Now the area of the region bounded by the given curves is,



$$A = \int_{x=0}^{1} (y_1 - y_2)\, dx$$

i.e. $A = 2\int_0^1 \left\{\sin\left(\frac{\pi x}{2}\right) - x\right\} dx$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1$$

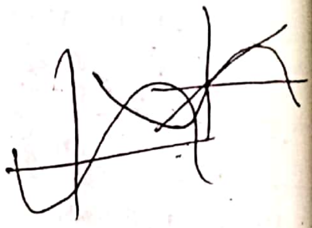
$$= 2\left\{\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{1}{2}\right] - \left[-\frac{2}{\pi}\cos 0 - 0\right]\right\}$$

$$= 2\left\{-\frac{1}{2} + \frac{2}{\pi}\right\}$$

$$= -1 + \frac{4}{\pi} = \frac{4}{\pi} - 1 \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = \sin\left(\frac{\pi x}{2}\right)$ and the line $y = x$ ... $\left(\frac{4}{\pi} - 1\right)$ sq. units.

10. The curves $y = x^2$ and $x = y^2$.



Solution: Given curve is,

$$y = x^2 \qquad \dots \text{(i)}$$

and the curve is,

$$y^2 = x \qquad \dots \text{(ii)}$$

Since the curve (i) is a parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $x = 0$.

And, the curve (ii) is also a parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $y = 0$.

The sketch of the curve is shown in the aside figure. Clearly the region bounded by the curves (i) and (ii) is the shaded portion in the figure, which has extreme points (0, 0) and (1, 1).

Clearly, the region has no symmetrical parts.

Now, the area of the region bounded by the curves (i) and (ii) is,

$$A = \int_{x=0}^{1} (y_1 - y_2)\, dx$$

i.e. $A = \int_0^1 (\sqrt{x} - x^2)\, dx$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3} \text{ sq. unit}$$

Thus, the area bounded by the given curve $y = x^2$ and $x = y^2$ is $\frac{1}{3}$ sq. units.

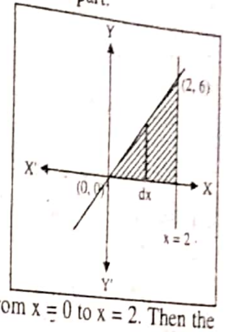
... The curve $y = 3x$, the x-axis and the ordinate $x = 2$.

Solution: Given curve are,

$y = 3x$, x-axis i.e. $y = 0$ and $x = 2$.

Clearly the line $y = 3x$ passes through (0, 0) and (1, 3).

Then the sketch of the bounded region by given curve, is shown in the figure by shaded. Clearly the region has no symmetrical part.



From first and second curves, $y = 0 \Rightarrow x = 0$.

From second and third curves $x = 2 \Rightarrow y = 0$.

From third and first curves $y = 0, x = 0$.

When $x = 2$ we get $y = 6$.

So, intersection point of $y = 3x$ and x-axis is (0, 0) and intersection point of $y = 3x$ and $x = 2$ is (2, 6).

Now, taking vertical region dx that has limit from $x = 0$ to $x = 2$. Then the area of the region is,

$$A = \int_0^2 (y_1 - y_2)\, dx$$

i.e. $A = \int_0^2 3x\, dx = \left[3 \cdot \frac{x^2}{2}\right]_0^2 = 3 \times \frac{2^2}{2} - 0 = 6$ sq. unit

Thus, the area bounded by the given curve $y = 3x$, the x-axis and the ordinate $x = 2$ is 6 sq. units.

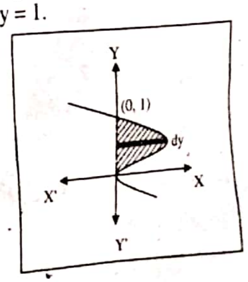
12. The curve y–axis and the curve $x = y^2 - y^3$.

Solution: For y–axis, $x = 0$.

Also, $x = y^2 - y^3$

X	0	0	0.125	2	-4
Y	0	1	0.5	-1	2

The required area bounded by the curve $x = y^2 - y^3$ and y-axis is shown in figure by shaded. Clearly the region has no symmetrical parts. Taking horizontal strip dy that has limits from $y = 0$ to $y = 1$.



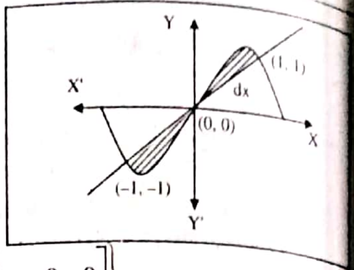
Now, the area of the region bounded by y-axis and the curve $x = y^2 - y^3$ is,

$$A = \int_0^1 (x_1 - x_2)\, dy$$

$y^2 - y^3 = 0$

$y^2(1 - y) = 0$

$$A = \int_{x=0}^{1} (y_1 - y_2)\,dx$$



i.e. $A = 2\int_0^1 \left\{\sin\left(\frac{\pi x}{2}\right) - x\right\} dx$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1$$

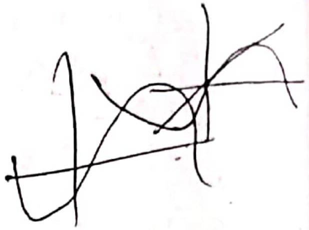
$$= 2\left\{\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{1}{2}\right] - \left[-\frac{2}{\pi}\cos 0 - 0\right]\right\}$$

$$= 2\left\{-\frac{1}{2} + \frac{2}{\pi}\right\}$$

$$= -1 + \frac{4}{\pi} = \frac{4}{\pi} - 1 \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = \sin\left(\frac{\pi x}{2}\right)$ and the line $y = x$ is $\left(\frac{4}{\pi} - 1\right)$ sq. units.

10. The curves $y = x^2$ and $x = y^2$.



Solution: Given curve is,

$$y = x^2 \quad \text{... (i)}$$

and the curve is,

$$y^2 = x \quad \text{... (ii)}$$

Since the curve (i) is a parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $x = 0$.

And, the curve (ii) is also a parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $y = 0$.

The sketch of the curve is shown in the aside figure. Clearly the region bounded by the curves (i) and (ii) is the shaded portion in the figure, which has extreme points (0, 0) and (1, 1).

Clearly, the region has no symmetrical parts.

Now, the area of the region bounded by the curves (i) and (ii) is,

$$A = \int_{x=0}^{1} (y_1 - y_2)\,dx$$

i.e. $A = \int_0^1 (\sqrt{x} - x^2)\,dx$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3} \text{ sq. unit}$$

Thus, the area bounded by the given curve $y = x^2$ and $x = y^2$ is $\frac{1}{3}$ sq. units.

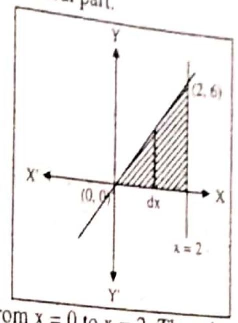
11. The curve $y = 3x$, the x–axis and the ordinate $x = 2$.

Solution: Given curve are,

$$y = 3x, \text{ x–axis i.e. } y = 0 \text{ and } x = 2.$$

Clearly the line $y = 3x$ passes through (0, 0) and (1, 3).

Then the sketch of the bounded region by given curve, is shown in the figure by shaded. Clearly the region has no symmetrical part.



From first and second curves, $y = 0 \Rightarrow x = 0$.

From second and third curves $x = 2 \Rightarrow y = 0$.

From third and first curves $y = 0, x = 0$.

When $x = 2$ we get $y = 6$.

So, intersection point of $y = 3x$ and x–axis is (0, 0) and intersection point of $y = 3x$ and $x = 2$ is (2, 6).

Now, taking vertical region dx that has limit from $x = 0$ to $x = 2$. Then the area of the region is,

$$A = \int_0^2 (y_1 - y_2)\,dx$$

i.e. $A = \int_0^2 3x\,dx = \left[3 \cdot \frac{x^2}{2}\right]_0^2 = 3 \times \frac{2^2}{2} - 0 = 6$ sq. unit

Thus, the area bounded by the given curve $y = 3x$, the x-axis and the ordinate $x = 2$ is 6 sq. units.

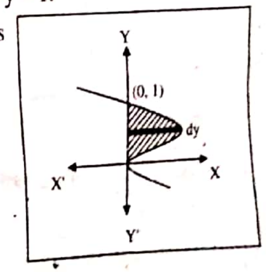
12. The curve y–axis and the curve $x = y^2 - y^3$.

Solution: For y–axis, $x = 0$.

Also, $x = y^2 - y^3$

X	0	0	0.125	2	-4
Y	0	1	0.5	-1	2

The required area bounded by the curve $x = y^2 - y^3$ and y–axis is shown in figure by shaded. Clearly the region has no symmetrical parts. Taking horizontal strip dy that has limits from $y = 0$ to $y = 1$.



Now, the area of the region bounded by y-axis and the curve $x = y^2 - y^3$ is,

$$A = \int_0^1 (x_1 - x_2)\,dy$$

i.e. $A = \int_0^1 (y^2 - y^3)\,dy$

$= \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$

$= \left(\frac{1}{3} - \frac{1}{4}\right) - 0 = \frac{4-3}{12} = \frac{1}{12}$ sq. unit.

Thus, the area bounded by the given curve y–axis and the curve $x = y^2 - y^3$ is $\frac{1}{12}$ sq. units.

13. The curve $y^2 = 12x$ the line $x = 12$.

Solution: Given curve is,

$$y^2 = 12x \qquad \dots (i)$$

which is a parabola having vertex at (0, 0), a = 3 > 0 and the line of symmetry is y = 0.

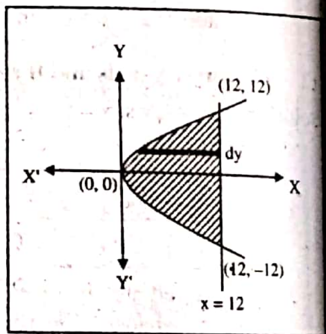
And, the given line is,

$$x = 12 \qquad \dots (ii)$$

Solving the equations (i) and (ii) then we get the points of intersection are (12, –12) and (12, 12).

The sketch of the region bounded by (i) and (ii) is the shaded portion in the figure. Clearly the region has two symmetrical parts from the x-axis. So, the area of the bounded region is the twice of the region having extreme points (0, 0), (12, 0) and (12, 12).

Now the area bounded by given curves is



$$A = 2\int_0^{12} (x_1 - x_2)\,dy$$

i.e. $A = 2\int_0^{12} \left(12 - \frac{y^2}{12}\right) dy$

$= 2\left[12y - \frac{y^3}{36}\right]_0^{12}$

$= 2\left[\left(12 \times 12 - \frac{(12)^3}{36}\right) - 0\right]$

$= 2\,(144 - 48) = 2 \times 96 = 192$ sq. unit.

Thus, the area bounded by the given curve $y^2 = 12x$ the line $x = 12$ is 192 sq. units.

14. The x–axis and the curve $y = 5x - x^2 - 6$.

y=0

Solution: For x–axis, $y = 0 \quad \dots (i)$

10-4-6

And the given curve is,

$y = 5x - x^2 - 6$

$\Rightarrow -y = x^2 - 2.x.\frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6$

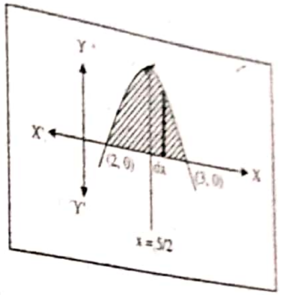
$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -y + \frac{25}{4} - 6$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{-4y + 25 - 24}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{-4y + 1}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -y + \frac{1}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -1\left(y - \frac{1}{4}\right) \qquad \dots (ii)$



This is a parabola having vertex at $\left(\frac{5}{2}, \frac{1}{4}\right)$, $a = -\frac{1}{4} < 0$ and the line of symmetry is $x = \frac{5}{2}$.

Solving the equations (i) and (ii) then we get the point of intersection of x–axis and parabola are (2, 0) and (3, 0).

With this information the trace of the region is given in the aside figure. Clearly the region has no symmetrical parts.

Now, taking vertical strip dx that has the limits from x = 2 to x = 3, required area bounded by x–axis and given parabola is

$$A = \int_{x=2}^{1} (y_1 - y_2)\,dx$$

i.e. $A = \int_{x=2}^{1} (5x - x^2 - 6)\,dx$

$= \left[5\frac{x^2}{2} - \frac{x^3}{3} - 6x\right]_2^3$

$= \left(\frac{45}{2} - 9 - 18\right) - \left(10 - \frac{8}{3} - 12\right)$

$= \frac{45}{2} - 27 + 2 + \frac{8}{3}$

$= \frac{45}{2} + \frac{8}{3} - 25 = \frac{135 + 16 - 150}{6} = \frac{151 - 150}{6} = \frac{1}{6}$ sq. unit

Thus, the area bounded by the given curve x–axis and the curve $y = 5x - x^2 - 6$ is $\frac{1}{6}$ sq. units.

15. The curves $y = x^4 - 2x^2$ and $y = 2x^2$

Solution: Given curve is,

$$y = x^4 - 2x^2 \quad \dots (i)$$

x	0	$\sqrt{2}$	$-\sqrt{2}$	1	−1	2	−2
y	0	0	0	−1	−1	8	8

And the given curve is,

$$y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y \quad \dots (ii)$$

This is a parabola having vertex (0, 0), $a = \frac{1}{8} > 0$ and the line of symmet

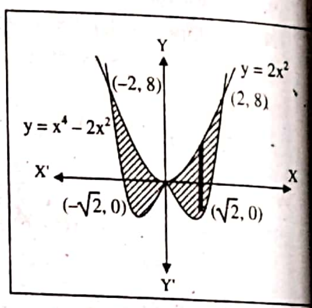
is x = 0.

Solving the equations (i) and (ii) then we get the points of intersection o the curves (i) and (ii) are (0, 0), (2, 8) and (−2, 8).

With this information the trace of the graph of the given curves are show in the figure. Clearly the figure has two symmetrical parts. So, the area o the bounded region is the twice of the region having extremity at (0, 0) an (2, 8).

Now, taking vertical strip dx over the limits from x = 0 to x = 2. Then th area of region bounded by given curves is,

$$A = 2\int_0^2 (y_1 - y_2)\,dx$$



$$\text{i.e. } A = 2\int_0^2 \{2x^2 - (x^4 - 2x^2)\}\,dx$$

$$= 2\int_0^2 (4x^2 - x^4)\,dx$$

$$= 2\left[4\cdot\frac{x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= 2\left\{\left(\frac{32}{3} - \frac{32}{5}\right) - 0\right\}$$

$$= 2\left(\frac{32\times5 - 32\times3}{15}\right) = 2\times\frac{2\times32}{15} = \frac{128}{15} \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = x^4 - 2x^2$ and $y = 2x^2$ is $\frac{128}{15}$ sq. units.

16. The curves $y = \sqrt{4-x}$, $x = 0$, $y = 0$ in the first quadrant.

Solution: Given curve is,

$$y = \sqrt{4-x} \Rightarrow y^2 = 4 - x \Rightarrow y^2 = -1\,(x-4).$$

This is a parabola having vertex (4, 0), $a = -\frac{1}{4} < 0$, line of symmetry is y = 0.

Also given that, x = 0, y = 0.

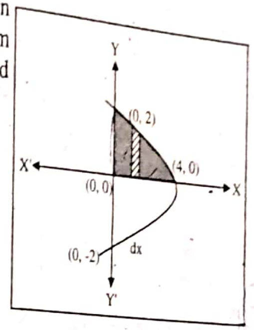
And, $x = 0 \Rightarrow y = \sqrt{4} = \pm 2$ and $y = 0 \Rightarrow x = 4$

Thus, the points of intersection of coordinates and given parabola are (0, 2), (0, −2) and (4, 0).

Therefore, the bounded region by the given curves $y = \sqrt{4-x}$, x = 0, y = 0 in the first quadrant is shaded portion that has extremity at (0, 0), (0, −2) and (4, 0).

Clearly, the bounded region has no symmetrical parts.

Now, taking vertical strip dx in region bounded by curves and taking limits from x = 0 to x = 4, then the area of bounded region is,



$$A = \int_0^4 (y_1 - y_2)\,dx$$

$$\text{i.e. } A = \int_0^4 [\sqrt{4-x} - 0]\,dx$$

$$= \left[\frac{-(4-x)^{3/2}}{3/2}\right]_0^4 = 0 - \left(-\frac{2\times4^{3/2}}{3}\right) = \frac{2\times8}{3} = \frac{16}{3} \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = \sqrt{4-x}$, x = 0, y = 0 in the first quadrant is $\frac{16}{3}$ sq. units.

7. The curve $y = x^2 + 1$ and the line $y = -x + 3$. [2011 Spring][2002]

lution: Given curve is,

$$y = x^2 + 1$$

$$\Rightarrow x^2 = \pm(y - 1) \quad \dots (i)$$

This is a parabola having vertex (0, 1), $a = \frac{1}{4} > 0$ and the line of symmetry is x = 0.

Also the given line is,

$$y = -x + 3 \quad \dots (ii)$$

which cuts x–axis at (3, 0) and y–axis at (0, 3).

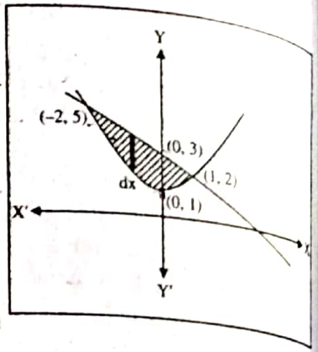
Solving given curves (i) and (ii) then we get the point of intersection of the curves are (1, 2) and (−2, 5).

With this information, the sketch of the region bounded by the curves (i) and (ii) is shown in the figure by shaded.

Clearly the region has no symmetrical parts.

Now, taking vertical strip dx and limits from $x = -2$, to $x = 1$ in bounded region, then the area of the region is,

$$A = 2\int_0^2 (y_1 - y_2)\,dx$$



$$\text{i.e. } A = \int_{-2}^{1} [(-x+3)-(x^2+1)]\,dx$$

$$= \int_{-2}^{1} (2 - x - x^2)\,dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= \frac{3}{2} - \frac{1}{3} + 6 - \frac{8}{3} = \frac{45-18}{6} = \frac{9}{2} = 4.5 \text{ sq. unit}$$

Thus, the area bounded by the given curve $y = x^2 + 1$ and the line $y = -x + 3$ is 4.5 sq. units.

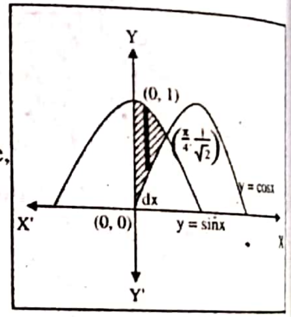
18. The curve $y = \sin x$, $y = \cos x$ and the line y–axis in the first quadrant.

Solution: Given curves are,

$y = \sin x$ ... (i)

$y = \cos x$ ... (ii)

and, y–axis i.e. $x = 0$



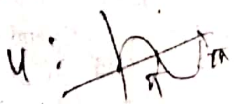
Tracing graph of given curves shown aside, Solving the curves (i) and (ii) then we get the points of intersection of curves is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Therefore, the region bounded by the curves (i), (ii) and y-axis is the shaded portion in the figure that has extreme points at (0, 1), (0, 0) and $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Clearly, the bounded portion has no symmetrical part.

Now, the area of the bounded region is,

$$A = \int_0^{\pi/4} (y_1 - y_2)\,dx$$



$$\text{i.e. } A = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= [\sin x + \cos x]_0^{\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1) = \frac{2}{\sqrt{2}} - 1 = (\sqrt{2} - 1) \text{ sq. unit}$$

Thus, the area bounded by the given curve $y = \sin x$, $y = \cos x$ and the line y–axis in the first quadrant is $(\sqrt{2} - 1)$ sq. units.

19. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Given curve is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \frac{b}{a}\sqrt{(a^2 - x^2)} \quad \text{... (i)}$$

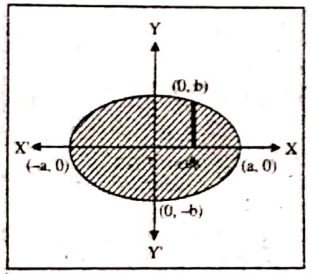
This is an ellipse having centre at (0, 0) and suppose $a > b$.

The sketch of the ellipse is shown in the figure aside with $a > b$.

Clearly it has four symmetrical parts.

Taking vertical strip dx in 1[st] quadrant and limits from $x = 0$ to $x = a$, then total area of ellipse is,

A = 4 × Area of ellipse in 1[st] quadrant



$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left\{\left(0 + \frac{a^2}{2}.\sin^{-1}(1)\right) - \left(0 + \frac{a^2}{2}\sin^{-1}0\right)\right\}$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2} - 0\right] = \frac{4b}{a}.\frac{a^2\pi}{4} = \pi ab \text{ sq. unit}$$

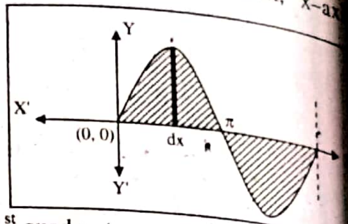
Thus, the area bounded by the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq. units.

20. The curve $y = \sin x$ and x–axis between $x = 0$ and $x = 2\pi$.

Solution: Given curve is,

$y = \sin x$

Given that the region is bounded by the curve $y = \sin x$, x–ax[is] (i.e. $y = 0$), $x = 0$ and $x = 2\pi$.



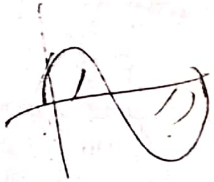
The region is the shaded portion in the aside figure.

Clearly the portion has two symmetrical parts. So, the area of the portion is the twice of the portion bounded from $x = 0$ to $x = 2\pi$ is,

$A = 2 \times$ Area bounded in 1st quadrant

$$\text{i.e. } A = 2\int_0^{\pi} (y_1 - y_2)\,dx$$

$$\text{i.e. } A = 2\int_0^{\pi} (\sin x - 0)\,dx$$



$$= 2\int_0^{\pi} \sin x\,dx$$

$$= 2[-\cos x]_0^{\pi} = 2[\cos\pi + \cos 0] = 2 \text{ sq. units}$$

Thus, the area bounded by the given curve $y = \sin x$ and x–axis betwee[n] $x = 0$ and $x = 2\pi$ is 2 sq. units.

C 1. Find the area of the region of the circle $x^2 + y^2 = 4$ cut off by the lin[e] $x-2y = -2$ in the first two quadrants. [2016 Fall][2013 Fall][2006, Fall]

Solution: Given curve is,

$$x^2 + y^2 = 4 \quad \dots (i)$$

which is circle having centre at (0, 0) and radius = 2. Clearly, it i[s] symmetry about both axis

$$x - 2y = -2 \quad \dots (ii)$$

$$\Rightarrow \frac{x}{(-2)} + \frac{y}{1} = 1$$

$x = 2y - 2$

$(0, 1)$

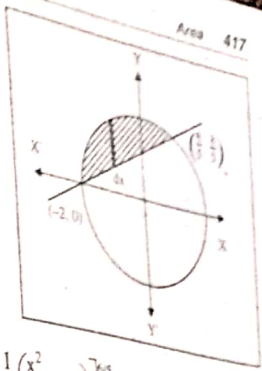
which is a line, cuts x and y–axis on (–2, 0) and (0, 1).

Also, solving the equations (i) and (ii) then we get the point of intersectio[n] are (–2, 0) and $\left(\frac{6}{5}, \frac{8}{5}\right)$.

Now, trace of given curve and line is shown in the figure.

Clearly the region bounded by the given curve and the line (i.e. the shade[d] part) has no symmetrical parts.

Now, taking any vertical strip dx and limits from $x = -2$ to $x = \frac{6}{5}$. Then the area bounded by the curves is,



$$A = \int_{-2}^{6/5} (y_1 - y_2)\,dx$$

$$\text{i.e. } A = \int_{-2}^{6/5} \left\{\sqrt{4 - x^2} - \frac{x+2}{2}\right\} dx$$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\left(\frac{x^2}{2} + 2x\right)\right]_{-2}^{6/5}$$

$$= \left\{\frac{6}{10}\sqrt{4 - \frac{36}{25}} + 2\sin^{-1}\left(\frac{6}{10}\right) - \frac{1}{2}\left(\frac{36}{50} + \frac{12}{5}\right)\right\}$$

$$- \left\{0 + 2\sin^{-1}(-1) - \frac{1}{2}(2 - 4)\right\}$$

$$= (0.96 + 2\sin^{-1}(0.6) - 3.12) - \left(2 \times \frac{\pi}{2} + 1\right)$$

$$= 0.96 + 2\sin^{-1}(0.6) - 3.12 + \pi - 1$$

$$= 2\sin^{-1}(0.6) - 0.02 \text{ sq. unit}$$

$$= 73.72 \text{ sq. units.}$$

$\sqrt{a^2 - x^2}$

Thus, the area of the region of the circle $x^2 + y^2 = 4$ cut off by the line $x - 2y = -2$ in the first two quadrants is 73.72 sq. units.

2. Find the area bounded between the parabola $x^2 = 4y$ and the curve $y = |x|$. [2016 Spring][2014 Fall][2008, Spring][2005, Spring]

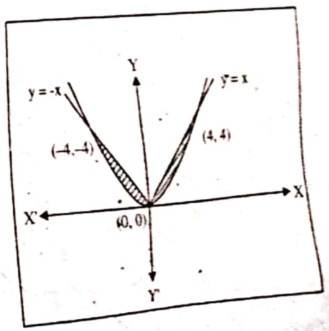
Solution: Given curve is,

$$x^2 = 4y \quad \dots (i)$$

which is a parabola having vertex at (0, 0), $a = 1 > 0$ and the line of symmetry is $x = 0$.

Also given line is,

$$y = |x| = \begin{cases} x & x > 0 \\ -x & x < 0 \\ 0 & x = 0 \end{cases} \quad \dots (ii)$$



It passes through origin.

Now trace the curve of above curve is shown in the figure.

Solving the equations (i) and $y = x$, then we get the point of intersections are (0, 0) and (4, 4).

And, solving the equations (i) and $y = -x$, then we get the point intersections are (0, 0) and (−4, −4).

Therefore, the point of intersection of curve (i) and (ii) are (0, 0), (4, 4) (−4, −4).

The bounded region by (i) is the shaded portion that has two symmetric parts. Therefore, the area of the bounded region is the twice of the region having extreme points (0, 0) and (4, 4).

Now, taking the vertical strip of width dx that has limits from $x = 0$ to $x = 4$. Therefore, the area of the bounded region is,

$$A = 2\int_0^4 (y_1 - y_2)\,dx$$

$$\text{i.e. } A = 2\int_0^4 \left\{|x| - \frac{x^2}{4}\right\} dx$$

$$= 2\int_0^4 \left(x - \frac{x^2}{4}\right) dx = 2\left[\frac{x^2}{2} - \frac{x^3}{12}\right]_0^4$$

$$= 2\left(8 - \frac{16}{3}\right) = 2 \times \frac{(24-16)}{3} = 2 \times \frac{8}{3} = \frac{16}{3} \text{ sq. unit}$$

Thus, the area bounded between the parabola $x^2 = 4y$ and the curve $y = |x|$ is $\frac{16}{3}$ sq. units.

3. Find the area bounded by the parabola $y = 16(x-1)(4-x)$ and the x–axis.

Solution: Given parabola is,

$$y = 16(x-1)(4-x)$$

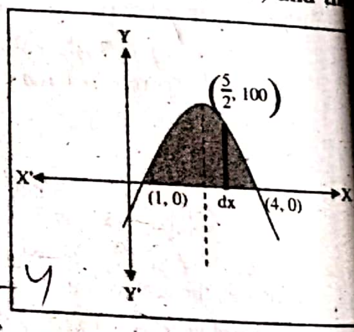
$$\Rightarrow \frac{y}{16} = 4x - x^2 - 4 + x$$

$$\Rightarrow \frac{-y}{16} = -x^2 + 5x - 4$$

$$\Rightarrow -\frac{y}{16} = x^2 - 2 \cdot x \cdot \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \frac{25}{4} - \frac{y}{16} = \left(x - \frac{5}{2}\right)^2 \Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{100-y}{16}$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -\frac{1}{16}(y - 100) \quad \dots (i)$$



which is a parabola having vertex at $\left(\frac{5}{2}, 100\right)$, $a = -\frac{1}{64} < 0$ and line of symmetry is $x = \frac{5}{2}$.

And the given line is x–axis i.e. $y = 0$ ... (ii)

Solving (i) and (ii) then we get the points of intersection are (1, 0) and (4, 0).

Now, the trace of the curve is shown in the figure aside. In the figure the region bounded by the curves (i) and (ii) is shaded portion.

Clearly the bounded region has no symmetrical parts.

Now, taking vertical strip of width dx that has limits from $x = 1$ to $x = 4$. Therefore, the area bounded by curves is given by

$$A = \int_1^4 (y_1 - y_2)\,dx$$

$$\text{i.e. } A = \int_1^4 (-16x^2 + 80x - 64)\,dx$$

$$= \left[-16 \cdot \frac{x^3}{4} + 40x^2 - 64x\right]_1^4$$

$$= \left(-\frac{1024}{3} + 640 - 256\right) - \left(-\frac{16}{3} + 40 - 64\right)$$

$$= \frac{128}{3} - \left(-\frac{88}{3}\right) = \frac{128}{3} + \frac{88}{3} = \frac{216}{3} = 72 \text{ sq. unit}$$

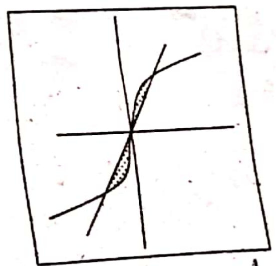
Thus, the area bounded by the parabola $y = 16(x-1)(4-x)$ and the x–axis is 72 sq. units.

4. Find the area of the propeller shaped region enclosed by the curve $x = y^3 = 0$ and the line $x - y = 0$.

Solution: The required region is bounded by the curves $x = y^3$ and $x = y$. That is shaded and have corners (1, 1), (0, 0) and (−1, −1).

Clearly the region has 2-symmetrical parts that are 2–times of the region between (0, 0) and (1, 1).

Now, the area of the bounded region is,



$$A = 2\left|\int_{y=0}^{1} (y_1 - y_2)\,dx\right|$$

$$\text{i.e. } A = 2\left|\int_{y=0}^{1} (y^3 - y)\,dy\right|$$

$$= 2\left|\left[\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1\right| = 2\left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{2} = 0.5$$

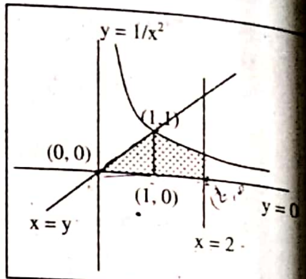
Thus, the area of the propeller shaped region enclosed by the curve $x = y^3 = 0$ and the line $x - y = 0$ is 0.5 sq. units.

5. Find the area of the region in the first quadrant bounded by the line $y = x$, $x = 2$ and the curve $y = \frac{1}{x^2}$ and x–axis.

Solution: The required region is bounded by

$y = x$, $x = 2$, $y = \frac{1}{x^2}$ and x–axis i.e. $y = 0$.

The region is shown by shaded in the figure.



Now, the area of the bounded region is,

$$A = \int_{x=0}^{2} (y_1 - y_2)\, dx$$

$$\text{i.e. } A = \int_{x=0}^{1} (y_1 - y_2)\, dx + \int_{x=1}^{2} (y_1 - y_2)\, dx$$

$$= \int_{x=0}^{1} x\, dx + \int_{x=1}^{2} x^{-2}\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^{-1}}{-1}\right]_1^2 = \frac{1}{2} - \frac{1}{2} + 1 = 1.$$

Thus, the area of the region in the first quadrant bounded by the line $y = x$, $x = 2$ and the curve $y = \frac{1}{x^2}$ and x–axis is 1 sq. units.

D. Show that the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{2a^2\pi}{8}$.

Solution: Given that the asteroid is,

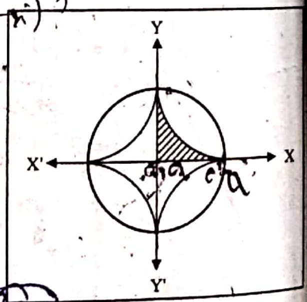
$$x^{2/3} + y^{2/3} = a^{2/3}$$

Comparing it with $(x - h)^{2/3} + (y - k)^{2/3} = r^{2/3}$ then we get, centre $(h, k) = (0, 0)$ and $r = a$.

Since the asteroid has symmetrical as in the figure. So, area of the asteroid is equal to the four time area of shaded portion. Therefore,

$$A = 4\int_{x=0}^{a} (y_1 - y_2)\, dx$$

$$\text{i.e. } A = 4\int_{x=0}^{a} [(a^{2/3} - x^{2/3})^{3/2} - 0]\, dx$$



$$= 4\int_{x=0}^{a} (a^{2/3} - x^{2/3})^{3/2}\, dx$$

Put, $x = a\sin^3\theta$ then $dx = 3a\sin^2\theta\cos\theta\, d\theta$.

Also, $x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$A = 4\int_{\theta=0}^{\pi/2} a\,.\,(1 - \sin^2\theta)^{3/2}\,.\,3a\sin^2\theta\cos\theta\, d\theta$$

$$= 12a^2 \int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta$$

$$= 12a^2 \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)}$$

$$= 12a^2 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)}$$

$$= 12a^2 \,.\, \frac{\frac{1}{2}\,.\,\Gamma\left(\frac{1}{2}\right)\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,.\,\Gamma\left(\frac{1}{2}\right)}{2\,.\,3\,.\,2\,.\,1}$$

$$= \frac{12a^2}{48}\sqrt{\pi}\,.\,\sqrt{\pi} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{a^2\pi}{4}.$$

Thus, the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{2a^2\pi}{8}$ i.e. $\frac{a^2\pi}{4}$ sq. units.

E. Show that the area bounded by the circle $x^2 + y^2 = a^2$ is πa^2. [2011 Fall]

Solution: Given circle is,

$$x^2 + y^2 = a^2.$$

Compare it with $(x - h)^2 + (y - k)^2 = r^2$ then we get

$(h, k) = (0, 0)$ and $r = a$.

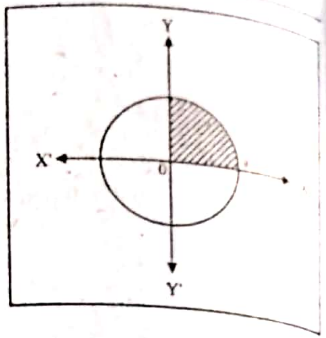
Since the circle has symmetrical figure that is divided into four equal parts. So, the area of circle is equal to 4 times of the area of the shaded portion.

Now, the area of the circle is,

$$A = 4\int_0^a (y_1 - y_2)\,dx$$

$$= 4\int_{x=0}^{a} [(a^2 - x^2)^{1/2} - 0]\,dx$$

$$= 4\int_{x=0}^{a} (a^2 - x^2)^{1/2}\,dx$$



Put $x = a\sin\theta$ then $dx = a\cos\theta\,d\theta$.

Also, when $x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$A = 4\int_0^{\pi/2} a(1 - \sin^2\theta)^{1/2} \cdot a\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{4a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{4a^2}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2}\right] = \frac{4a^2\pi}{4} = a^2\pi.$$

Thus, the area bounded by the circle $x^2 + y^2 = a^2$ is πa^2.

$y^2 = 1 - x$
0 1

F. Show that the area common to the circle $x^2 + y^2 = 1$ and the parabola $y^2 = 1 - x$ is $\frac{4}{3} + \frac{\pi}{2}$. **[2014 Spring]**

Solution: Given region be the common part of $x^2 + y^2 = 1$ and $y^2 = 1 - x$. Clearly, the common part is half circle and twice of the shaded portion. Therefore, area of common region be,

$$A = \frac{1}{2}\text{area of circle} + 2\text{ Area of shaded portion} \quad \dots (i)$$

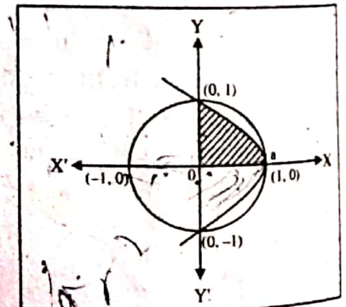
Here,

A_1 = Area of circle

$\Rightarrow A_1 = \pi$ [a = 1] [by (E)]

A_2 = Area of shaded portion

$$= \int_{y=0}^{1} (x_1 - x_2)\,dy$$



$$= \int_{y=0}^{1} (1 - y^2 - 0)\,dy = \left[y - \frac{y^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence, (i) becomes,

$$A = \frac{1}{2}\pi + 2 \cdot \frac{2}{3} = \frac{4}{3} + \frac{\pi}{2}$$

Thus, the area common to the circle $x^2 + y^2 = 1$ and the parabola $y^2 = 1 - x$ is $\frac{4}{3} + \frac{\pi}{2}$ sq. units.

OTHER IMPORTANT QUESTIONS FROM FINAL EXAM

1. Find the area bounded by the curve on the left by the parabola $x = y^2$, on the right by the line $y = x - 2$ and below by the x-axis. **[2009, Fall]**

OR

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$. **[2005, Fall]** **[2015 Spring]**

Solution: Given that the required region is bounded by the curves on the left by $y^2 = x$, on the right by $y = x - 2$ and below by x-axis.

Then the required region is the shaded portion on the adjoining figure that has corners (0, 0), (2, 0) and (4, 2).

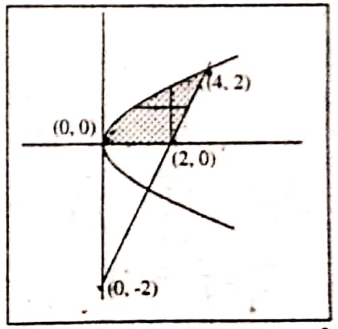
Now, the area of shaded portion be

$$A = \int_{y=0}^{2} (x_1 - x_2)\,dy$$

$$= \int_{y=0}^{2} (y + 2 - y^2)\,dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2$$

$$= \left(2 + 4 - \frac{8}{3} - 0\right) = \frac{10}{3}$$



(0, 0)
(0, -2)
$y = \sqrt{x}$

$y = x - 2$
$0 = x - 2$
$x = 0$
$y = -2$

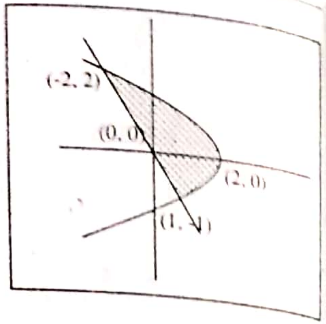
Thus, area of the shaded portion is $\frac{10}{3}$ square units.

2. Find the area of the region enclosed by the parabola $y^2 = 2 - x$ and the line $y = -x$. **[2001] [1999]**

Solution: Given that the required region is enclosed by the curves $y^2 = 2 - x$ and the line $y = -x$.

Then the required region is the shaded portion that has corners (1, −1) and (−2, 2).

Now, area bounded by the given curves be



$$A = \left|\int_{y=-1}^{2} (x_1 - x_2)\, dy\right|$$

$$= \left|\int_{y=-1}^{2} (-y - 2 + y^2) dy\right|$$

$$= \left|\left[\frac{-y^2}{2} - 2y + \frac{y^3}{3}\right]_{y=-1}^{2}\right|$$

$$= \left(-\frac{4}{2} - 4 + \frac{8}{3}\right) - \left(-\frac{1}{2} + 2 + \frac{1}{3}\right)$$

$$= \left|-2 - 4 + \frac{8}{3} + \frac{1}{2} - 2 + \frac{1}{3}\right| = \left|-\frac{9}{2}\right| = \frac{9}{2} = 4.5$$

Thus, the area of shaded portion is A = 4.5 sq. units.

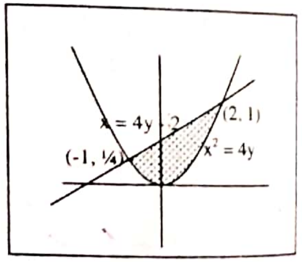
3. **Find the area bounded by the parabola $x^2 = 4y$ and the line $x = 4y - 2$.** **[2012 Fall] [2002]**

Solution: Given that the required region is bounded by the curves $x^2 = 4y$ and $x = 4y - 2$.

Clearly the required region is the shaded portion that has corners (−1, ¼) and (2, 1).

Clearly the region has no symmetrical parts (from figure).

Now, area of the shaded portion be,



$$A = \left|\int_{x=-1}^{2} (y_1 - y_2)\, dx\right|$$

$$= \left|\int_{x=-1}^{2} \left(\frac{x^2}{4} - \frac{x+2}{4}\right) dx\right|$$

$$= \frac{1}{4}\left|\int_{x=-1}^{2} (x^2 - x - 2)\, dx\right|$$

$$= \frac{1}{4}\left|\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}\right|$$

$$= \frac{1}{4}\left|\left[\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)\right]\right|$$

$$= \frac{1}{4}\left|-\frac{10}{3} - \frac{7}{6}\right| = \left|\frac{-27}{24}\right| = \left|-\frac{9}{8}\right| = \frac{9}{8}$$

So, the required area is $\frac{9}{8}$ sq. unit.

4. **Find the area bounded by curve $x = y^2$ and the line $x = y + 2$.** **[2003, Fall]**

Solution: Given that the required region is enclosed by the curves $y^2 = x$ and the line $x = y + 2$.

Then the required region is the shaded portion that has corners (1, −1) and (4, 2).

Now, area bounded by the given curves be

$$A = \left|\int_{y=-1}^{2} (x_1 - x_2)\, dy\right|$$

$$= \left|\int_{y=-1}^{2} (y^2 - y - 2) dy\right|$$

$$= \left|\left[\frac{y^3}{3} - \frac{y^2}{2} - 2y\right]_{y=-1}^{2}\right|$$

$$= \left(-\frac{4}{2} - 4 + \frac{8}{3}\right) - \left(-\frac{1}{2} + 2 + \frac{1}{3}\right)$$

$$= \left|\frac{8}{3} - 2 - 4 + \frac{1}{3} + \frac{1}{2} - 2\right| = \left|-\frac{9}{2}\right| = \frac{9}{2} = 4.5$$

Thus, the area of shaded portion is A = 4.5 sq. units.

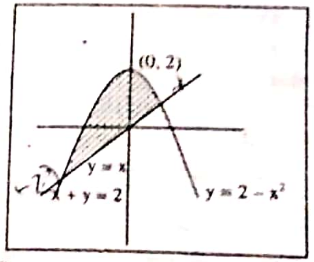
5. **Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y = x$.** **[2003, Spring]**

Solution: Given that the required region is bounded by the curves,

$$y = 2 - x^2 \Rightarrow x^2 = 2 - y \text{ and } y = x.$$

The region is the shaded portion that has corners (−2, −2) and (1, 1).

Now, area of the region be



$$A = \int_{x=-2}^{1} (y_1 - y_2)\, dx$$

$$= \int_{x=-2}^{1} (2 - x^2 - x)\, dx$$

$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - \frac{4}{2}\right) = \frac{7}{6} + \frac{10}{3} = 4.5$$

Thus, the area of the shaded portion be 4.5 sq. units.

6. **Find the area enclosed by $y^2 - 4x = 4$ and $4x - y = 16$.** **[2007, Spring] [2004, Fall]**

Solution: Given curve is,

$$y^2 = 4(x + 1) \qquad \ldots \text{(i)}$$

which is a parabola having vertex at (−1, 0), a = 1 > 0 and the line of symmetry is y = 0.

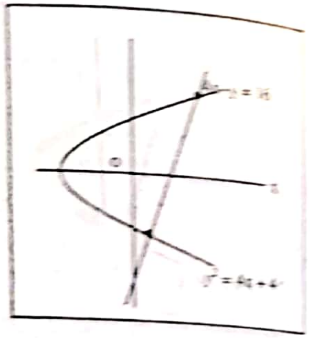
And, the given line is,

$$4x - y = 16 \qquad \text{...(ii)}$$

Solving the equations (i) and (ii) then we get the points of intersection are $(3, -4)$ and $\left(\frac{21}{4}, 5\right)$.

The sketch of the region bounded by (i) and (ii) is the shaded portion in the figure. Clearly the region has no symmetrical parts.

Now the area bounded by given curves is



$$A = \left|\int_{-4}^{5} (x_1 - x_2)\, dy\right|$$

$$\text{i.e. } A = \left|\int_{-4}^{5} \left(\frac{y}{4} - \frac{y^2}{4}\right) dy\right|$$

$$= \left[\frac{y^2}{8} - \frac{y^3}{12}\right]_{-4}^{5}$$

$$= \left|\left(\frac{25}{8} - \frac{125}{12}\right) - 0\right|$$

$$= \left|\frac{150 - 500}{48}\right| = \left|\frac{-350}{48}\right| = \frac{175}{24} \text{ sq. unit.}$$

Thus, the area enclosed by $y^2 - 4x = 4$ and $4x - y = 16$ is $\frac{175}{24}$ sq. units.

7. Find the area inside the circle $x^2 + y^2 = 1$ and the outside the parabola $y^2 = 1 - x$. Also sketch the bounded region. [2008, Fall]

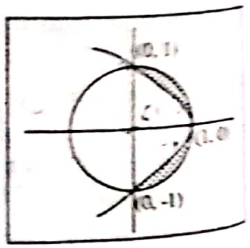
Solution: Given that the required region is bounded by the curves $x^2 + y^2 = 1$ and the outside of the parabola $y^2 = 1 - x$.

Clearly, the required region is the shaded part in the figure that has corners $(0, 1)$ and $(1, 0)$.

Clearly the region has two symmetrical parts. So, the area of the region is twice of the region having extreme point $(0, 1)$ and $(1, 0)$.

Now, the area of the region is,

$$A = 2\int_{y=0}^{1} (x_1 - x_2)\, dy$$



$$= 2\int_{y=0}^{1} [\sqrt{1-y^2} - (1 - y^2)]\, dy$$

$$= 2\left[\frac{y\sqrt{1-y^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{y}{1}\right) - y + \frac{y^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2}[\sin^{-1}(1) - \sin^{-1}(0)] - 1 + \frac{1}{3}\right]$$

$\frac{\pi}{4} - (1 - \frac{1}{3})$

$$= \frac{\pi}{2} - \frac{4}{3}$$

Thus, the area of the region be, $A = \frac{\pi}{2} - \frac{4}{3}$.

8. Find the area enclosed by the curve $y = 4x^3 - 3x^2$, x-axis and the ordinates $x = 1$ and $x = 2$. [2004, Spring]

Solution: Given, $y = 4x^3 - 3x^2$

For x-axis, i.e. $y = 0$

Ordinates are $x = 1, x = 2$

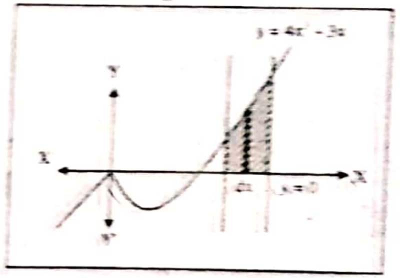
$y = 4x^3 - 3x^2$

x	1	1.5	2
y	1	6.75	20

Now, the trace of the curves is as shown in figure.

Clearly the bounded region has no symmetrical parts.

Now, taking vertical strip of width dx that has limit from $x = 1$ to $x = 2$ in the region. Then the area of the bounded region is,



$$A = \int_1^2 (y_1 - y_2)\, dx$$

$$\text{i.e. } A = \int_1^2 (4x^3 - 3x^2)\, dx$$

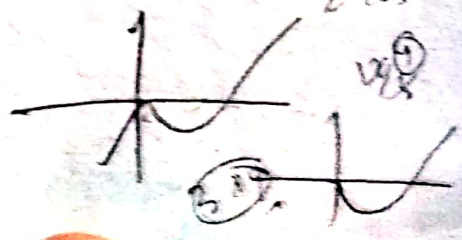
$$= \left[4 \cdot \frac{x^4}{4} - 3 \cdot \frac{x^3}{3}\right]_{x=1}^{x=2}$$

$$= [x^4 - x^3]_1^2 = (16 - 8) - (1 - 1) = 8 - 0 = 8 \text{ sq. units.}$$

Thus, the area of the region be, $A = 8$ sq. units.

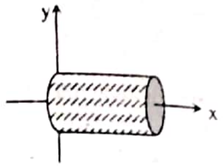
9. Find the area bounded between the curve $y = x^2 + 1$ and the line $x - y + 3 = 0$. [2017 Spring]

•••



List of Formulae

The volume of the region bounded by two curves $y_1 = f(x)$, $y_2 = g(x)$ and revolving about x–axis then the limit should be as $x = a$ and $x = b$ then,



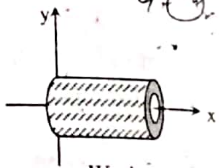
$$V = \pi\int_a^b [(y_1)^2 - (y_2)^2]\, dx.$$

The volume of the region bounded by two curves $x_1 = f(y)$, $x_2 = g(y)$ and revolving about y–axis then the limit should be as $y = a$ and $y = b$ then,

$$V = \pi\int_a^b [(x_1)^2 - (x_2)^2]\, dy.$$

For Washer method

The volume of the region bounded by two curves $y_1 = f(x)$, $y_2 = g(x)$ and revolving about the line $y = m$ then the limit should be as $x = a$ and $x = b$ then,



Washer

$$V = \pi\int_a^b [(y_1 - y)^2 - (y_2 - y)^2]\, dx.$$

The volume of the region bounded by two curves $x_1 = f(y)$, $x_2 = g(y)$ and revolving about the line $x = n$ then the limit should be as $y = a$ and $y = b$ then,

$$V = \pi\int_a^b [(x_1 - x)^2 - (x_2 - x)^2]\, dy.$$

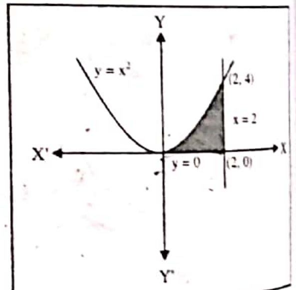
Exercise 12.2

1. **Find the volume of the solids generated by revolving the regions bounded by the lines and the curves about the x–axis.**

(i) $y = x^2, y = 0, x = 2$

Solution: First we trace the curves from the figure, $R(x) = y = x^2$.

Since the curve $y = x^2$ is a parabola having vertex at (0, 0) and having up-openward. Therefore, the region bounded by the curve $y = x^2$ and the line $y = 0$, $x = 2$, is the shaded portion in the figure.



The limit of integration is $x = 0$ to $x = 2$. Then the volume of the solid thus generated by revolving the region bounded by the given curves is,

$$V = \int_0^2 \pi\{R(x)\}^2 dx = \int_0^2 \pi x^4\, dx = \pi\left|\frac{x^5}{5}\right|_0^2 = \frac{2^5\pi}{5} = \frac{32\pi}{5}.$$

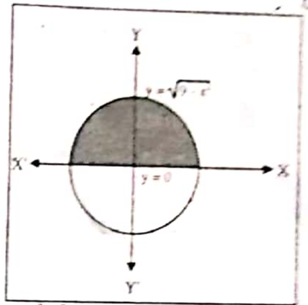
Thus, the volume of the circle is $\frac{32\pi}{5}$ cubic units.

(ii) $y = \sqrt{9 - x^2}, y = 0.$

Solution: Here, by the curve $y = \sqrt{9 - x^2}$.

Clearly the given curve is a half circle having centre at (0, 0) and radius 3. But y takes only the non-negative value being $y = \sqrt{9 - x^2}$.

First we trace the curves from the figure.



$R(x) = y = \sqrt{9 - x^2}$

The limit of integration is $x = 0$ to $x = 3$.

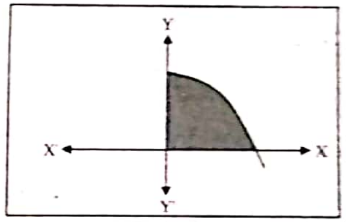
Then the volume of the solid thus generated by revolving the region bounded by the given curves is, $V = 2\int_0^3 \pi\{R(x)\}^2 dx$

$$= 2\pi\int_0^3 (9 - x^2)\, dx = 2\pi\left|9x - \frac{x^3}{3}\right|_0^3 = 2\pi\left[27 - \frac{27}{3}\right] = 36\pi$$

Thus, the volume of the circle is 36π cubic units.

(iii) $y = \sqrt{\cos x}, 0 \le x \le \frac{\pi}{2}, y = 0, x = 0.$

Solution: Here,



$$R(x) = y = \sqrt{\cos x}$$

The limit of integration is $x = 0$ to $x = \frac{\pi}{2}$.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is,

$$V = \int_0^{\pi/2} \pi\{R(x)\}^2 dx = \pi\int_0^{\pi/2} \cos x\, dx = \pi\left|\sin x\right|_0^{\pi/2} = \pi.$$

Thus, the volume of the circle is π cubic units.

2. **Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines below by $y = 1$, $x = 4$ about the line $y = 1$.** **[2018 Spring][2004, Fall] [2009, Fall]**

OR **Use the Washer to find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, and the lines below by $y = 1$, $x = 4$ about the line $y = 1$.** **[2004, Spring]**

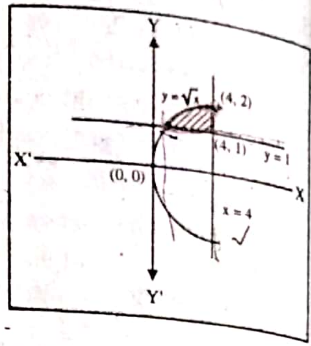
Solution: Here, given curve is $y = \sqrt{x}$ and bounded below by $y = 1$ and $x = 4$.

First we trace the curves from figure,

$R(x) = \sqrt{x}$ and $(x) = 1$

The limit of integration is $x = 1$ to $x = 4$.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is,



$$V = \int_1^4 \pi[\{R(x)-1\}^2 - \{r(x)-1\}^2]\,dx$$

$$= \pi\int_1^4 [(\sqrt{x}-1)^2-(1-1)^2]\,dx \quad = \pi\int_1^4 (x-2\sqrt{x}+1)\,dx$$

$$\Rightarrow V = \pi\left|\frac{x^2}{2}-\frac{4}{3}x^{3/2}+x\right|_1^4 \quad = \pi\left|8-\frac{32}{3}+4-\frac{1}{2}+\frac{4}{3}-1\right|$$

$$= \pi\left|11-\frac{32}{3}-\frac{1}{2}+\frac{4}{3}\right|$$

$$= \pi\left|\frac{66-64-3+8}{6}\right| = \frac{7\pi}{6}.$$

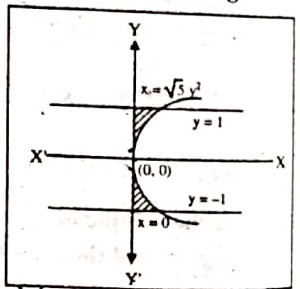
Thus, the volume of the circle is $\frac{7\pi}{6}$ cubic units.

3. Find the volumes of the solids generated by revolving the regions bounded by the lines and the curves about y–axis of the following.

(i) $x = \sqrt{5}\,y^2, x = 0, y = -1, y = 1$

Solution: Here, the region bounded by the curves $x = \sqrt{5}\,y^2, x = 0, y = -1, y = 1$.

First we traces the figure with the help of given curves,



$R(y) = x = \sqrt{5}\,y^2$

The limit of the integration is $y = 0$ to $y = 1$.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is,

$$V = 2\int_0^1 \pi\{R(y)\}^2\,dy \qquad \text{[because the region has 2 symmetrical parts]}$$

$$= 2\pi\int_0^1 5y^4\,dx = 2\pi\left|y^5\right|_0^1 = 2\pi.$$

The volume of the circle is 2π cubic units.

(ii) $x = \sqrt{2\sin 2y}, 0 \le y \le \frac{\pi}{2}, x = 0$

Solution: Here, the region bounded by the curves

$$x = \sqrt{2\sin 2y}, 0 \le y \le \frac{\pi}{2}, x = 0.$$

The limit of the integration is $y = 0$ to $\frac{\pi}{2}$. Then the volume bounded by the given curves be,

$$V = \int_0^{\pi/2} \pi(x)^2\,dy = \pi\int_0^{\pi/2} 2\sin 2y\,dy$$

$$= 2\pi\left|\frac{-\cos 2y}{2}\right|_0^{\pi/2} = \pi(-1)-(-1) = 2\pi.$$

Thus, the volume of the circle is 2π cubic units.

$x = \sqrt{5}y^2$

$x = 0, y = -1, y = 1$

(iii) $x = \frac{2}{y+1}, x = 0, y = 0, y = 3.$

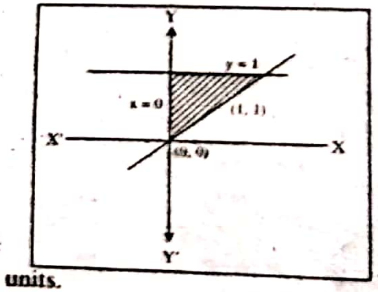
Similarly to Q.No. 2 (ii)

4. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x–axis of the following:

(i) $y = x, y = 1, x = 0$

Solution: The region bounded by the curves $y = x, y = 1, x = 0$.

Here, first we traces the lines from figure, the limit of the integration is $x = 0$ to $x = 1$. Then the volume bounded by the given curves be,



$$V = \int_0^1 (\pi[-R(x)] + [r(x)]^2)\,dx$$

$$= \pi\int_0^1 (1-x^2)\,dx$$

$$= \pi\left|x-\frac{x^3}{3}\right|_0^1 = \frac{3-1}{3}\pi = \frac{2\pi}{3}.$$

Thus, the volume of the circle is $\frac{2\pi}{3}$ cubic units.

(ii) $y = 2\sqrt{x}, y = 2, x = 0$

Solution: *Similarly Q. No. 4(i).*

(iii) $y = x^2 + 1, y = x + 3$

Solution: *Similarly Q. No. 4(i).*

(iv) $y = \sec x, y = \sqrt{2}, -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

Solution: Here, $R(x) = y = \sqrt{2}$ and $r(x) = y = \sec x$

The limit of the function $x = -\pi/4$ to $x = \pi/3$. Then the volume bounded by the given curves be,

$$V = \int_{-\pi/4}^{\pi/4} \pi[\{R(x)\}^2 - \{r(x)\}^2]\,dx$$

$$= \pi \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\,dx = \pi \int_{-\pi/4}^{\pi/4} (2x - \sec^2 x)\,dx$$

$$= \pi \left|2x - \tan x\right|_{-\pi/4}^{\pi/4}$$

$$= \pi(2(\pi/4) - 1 + 2(\pi/4) - 1)$$

$$= \pi(\pi - 2).$$

Thus, the volume of the circle is $\pi(\pi - 2)$ cubic units.

5. Find the volume of the solid generated by revolving the region enclosed by the triangle with vertices (1, 0), (2, 1) and (1, 1) about the y-axis.

Solution: Here, the equation of line having co–ordinates (1, 0) and (2, 1) is,

$$y - 0 = \frac{1-0}{2-1}(x - 1)$$

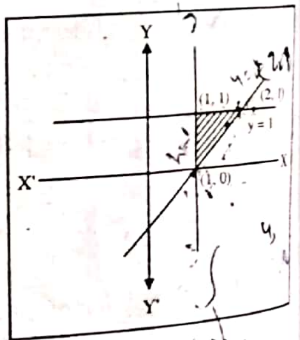
or $y = x - 1 \quad \Rightarrow x = y + 1.$

The equation line having co–ordinate (1, 1) and (1, 0) is x = 1.

The equation of line having co–ordinate (1, 1) and (2, 1) is y = 1.

From figure, the limit of the integration is y = 0 to y = 1.

Then the volume bounded by the given curves be,



$$V = \int_{2}^{1} [x_1^2 - x_2^2]\,dy$$

$$= \pi \int_{2}^{1} [(y + 1)^2 - 1^2]\,dy$$

$$= \pi \left|\frac{(y+3)^3}{3} - y\right|_0^1 = \pi \left|\frac{8}{3} - 1 - \frac{1}{3}\right|$$

$$= \frac{4\pi}{3}.$$

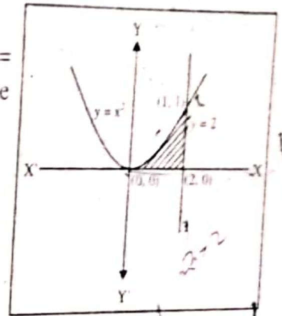
Thus, the volume of the circle is $\frac{4\pi}{3}$ cubic units.

6. Find the volume of the solid generated by revolving the region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis and on the right by the line x = 2 about y-axis. [2017 Fall][2005, Spring]

Solution: The region bounded by the given curves in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis and on the right by the line x = 2.

Here, first we trace the curve,

$R(y) = x = 2$ and $r(y) = x = \sqrt{y}$



From figure, the limit of the integration is y = 0 to y = 4. Then the volume bounded by the given curves be,

$$V = \int_0^4 \pi[\{R(y)\}^2 - \{r(y)\}^2]\,dy$$

$$= \pi \int_0^4 (4 - y)\,dy = \pi \left|4y - \frac{y^2}{2}\right|_0^4$$

$$= \pi(16 - 8) = 8\pi.$$

Thus, the volume of the circle is 8π cubic units.

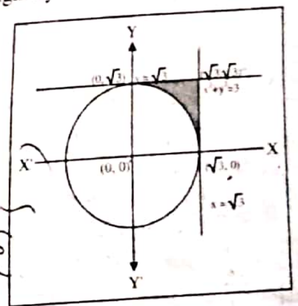
7. Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$ about y-axis.

Solution: The region bounded by the given curves in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$.

Here, first we trace the circle and lines

$R(y) = x = \sqrt{3 - y^2}$

$r(y) = x = \sqrt{3}$



From figure, the limit of the integration is $y = 0$ to $y = \sqrt{3}$. Then the volume bounded by the given curves be,

$$V = \int_0^{\sqrt{3}} \pi[\{R(y)\}^2 - \{r(y)^2\}]\,dy$$

$$= \pi \int_0^{\sqrt{3}} [3 + 3 + y^2]\,dy = \pi \int_0^{\sqrt{3}} [y^2]\,dy = \pi \left|\frac{y^3}{3}\right|_0^{\sqrt{3}} = \sqrt{3}\,\pi.$$

Thus, the volume of the circle is $\sqrt{3}\,\pi$ cubic units.

8. Find the volume of the solid in the region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ revolved about the x-axis. [2003, Fall]

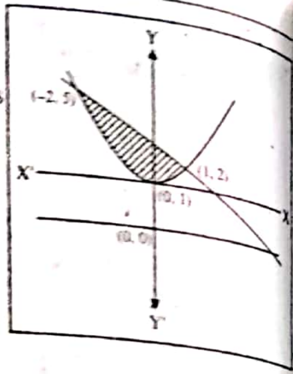
Solution: The region bounded by the given curves $y = x^2 + 1$ and the line $y = -x + 3$.

Here, first we trace the curve,

$R(x) = y = x^2 + 1$

$r(x) = y = -x + 3$

From figure, the limit is $x = -2$ to $x = 1$.
Then the volume bounded by the given curves be,



$$V = \int_{-2}^{1} \pi [\{R(x) - 0\}^2 - \{r(x) - 0\}^2]\, dx$$

$$= \pi \int_{-2}^{1} (x^2 + 1)^2 - (-x + 3)^2 \}$$

$$= \pi \int_{-2}^{1} (x^4 + 2x^2 + 1 - x^2 + 6x - 9)$$

$$= \pi \left| \frac{x^5}{5} + \frac{x^3}{3} + \frac{6x^2}{2} - 8x \right|_{-2}^{1}$$

$$= \pi \left| \frac{1}{5} + \frac{1}{3} + 3 - 8 + \frac{32}{5} + \frac{8}{3} - 12 - 16 \right|$$

$$= \pi \left| \frac{3 + 5 + 96 - 495 + 40}{15} \right| = \left| -\frac{351}{15} \right| = \frac{351}{15}\pi = \frac{117\pi}{5}.$$

Thus, the volume of the circle is $\frac{117\pi}{5}$ cubic units.

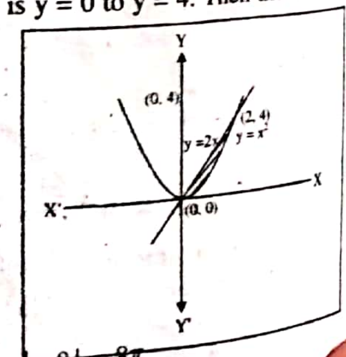
9. Find the volume of the solid in the region in the region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant about y-axis.
[2016 Fall][2006, Fall][2007, Spring]

Solution: The region bounded by the given curves $y = x^2$ and $y = 2x$.

Here, first we trace the curves,

$R(y) = x = \frac{y}{2}$ and $r(y) = x = \sqrt{y}$

From figure, the limit of the integration is $y = 0$ to $y = 4$. Then the volume bounded the curves be,



$$V = \int_{0}^{4} [\{R(y)\}^2 - \{r(y)\}^2]\, dy$$

$$= \pi \int_{0}^{4} \left\{ \left(\frac{y}{2}\right)^2 - (\sqrt{y})^2 \right\} dy$$

$$= \pi \left| \frac{y^3}{12} - \frac{y^2}{2} \right|_0^4$$

$$= \pi \left| \frac{64}{12} - \frac{16}{2} \right| = \pi \left| \frac{16}{3} - 8 \right| = \pi \left| -\frac{8}{3} \right| = \frac{8\pi}{3}.$$

Thus, the volume of the circle is $\frac{8\pi}{3}$ cubic units.

10. Find the volume of the solid in the region in the first quadrant evolved by the parabola $y = x^2$ the y-axis and the line $y = 1$ revolved about the line $x = \frac{3}{2}$. [2014 Fall]

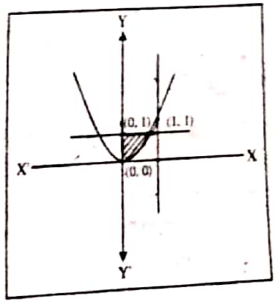
Solution: The region bounded by the given curves $y = x^2$ the y-axis and the line $y = 1$ in the first quadrant.

Here, first we trace the curves,

$R(y) = x = \sqrt{y}$ and $r(y) = x = 0$.

From figure, the limit of the integration is $y = 0$ to $y = 1$. Then the volume bounded the curves be,

$$V = \left| \int_0^1 \pi [(R(y) - 3/2)]^2 - (r(y) - 3/2)^2]\, dy \right|$$



$$= \pi \left| \int_0^1 \left\{ \left(0 - \frac{3}{2}\right)^2 - \left(\sqrt{y} - \frac{3}{2}\right)^2 \right\} dy \right|$$

$$= \pi \left| \int_0^1 \left(\frac{9}{4} - y + 3\sqrt{y} - \frac{9}{4} \right) dy \right|$$

$$= \pi \left| \int_0^1 \left\{ -\sqrt{y^2 + 3\sqrt{y}} \right\} dy \right|$$

$$= \pi \left| \frac{y^2}{2} - 2y^{3/2} \right|_0^1 = \pi \left| -\frac{1}{2} + 2 \right| = \pi \left| \frac{3}{2} \right| = \frac{3\pi}{2}.$$

Thus, the volume of the circle is $\frac{3\pi}{2}$ cubic units..

11. Find the volume of the solid in the region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis and on the right by the line $x = 1$ about the line $x = -1$.
[2014 Spring][2008, Spring][2009 Spring]

OR Find the area bounded by $y = x^2$, below by the x-axis and on the right by the line $x = 1$ about the line $x = -1$. Find the volume thus generated. [2013 Spring]

Solution: Since the given region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis and on the right by the line $x = 1$.

Here, first we trace the curves,

$R(y) = x = 1$ and $r(y) = x = \sqrt{y}$

From figure, the limit of the integration is $y = 0$ to $y = 1$. Then the volume [illegible] by the curves be,

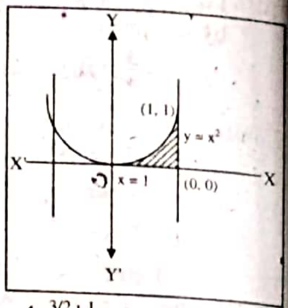
$$V = \int_0^1 \pi[\{R(y)-(-1)\}^2 - \{r(y)-(-1)\}^2]\,dy$$

$$= \pi\int_0^1 [(2)^2 - (\sqrt{y}+1)^2]$$

$$= \pi\int_0^1 [4 - y - 2\sqrt{y} - 1]\,dy$$

$$= \pi\int_0^1 [3 - y - 2\sqrt{y})\,dy \quad = \pi\left|-\frac{y^2}{2} + 3y - \frac{4y^{3/2}}{3}\right|_0^1$$

$$= \pi\left|3 - \frac{1}{2} - \frac{4}{3}\right| \quad = \frac{7\pi}{6}.$$



Thus, the volume of the circle is $\frac{7\pi}{6}$ cubic units.

12. Show that the volume of the paraboloid formed by revolving the parabola $y^2 = 4ax$ and the line $x = a$, about x-axis is $2\pi a^3$.

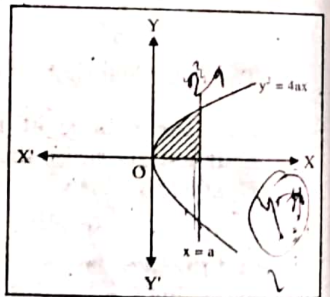
Solution: The region bounded by the given curves are $y^2 = 4ax$ and the line $x = a$. The sketch of the curves is trace below.

Clearly, the half-part region (i.e. shaded region) has same form as given by the rest region. So, we take the shaded region as required region and revolve to it about x-axis i.e. $y = 0$ with $x = 0$ to $x = a$. Then the required volume be,

$$V = \pi\int_{x=a}^{b} [(y_1 - y)^2 - (y_2 - y)^2]\,dx$$

$$= \pi\int_{x=0}^{a} [(\sqrt{4ax} - 0)^2 - (0 - 0)^2]\,dx$$

$$= \pi(\sqrt{4a})^2\left[\frac{x^2}{2}\right]_0^a \quad = \frac{4a^3\pi}{2} \quad = 2a^3\pi.$$

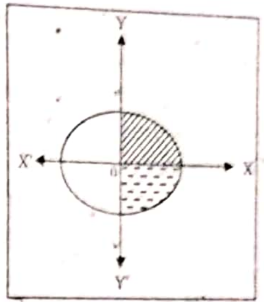


Thus, the volume of the circle is $2a^3\pi$ cubic units.

13. Show that the volume of the solid generated by revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis is $\frac{4}{3}\pi ab^2$.

…ion: Here, in the corresponding ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the shaded portion by …nes, has same volume as given by the shaded portion shaded by dots. Thus …he volume of the solid ellipse is equal to twice of line–shaded region.

Clearly, the shaded portion by lines, is bounded by the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = 0$. Therefore volume of ellipse revolving about x-axis i.e. $y = 0$ be,



$$V = 2\pi\int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\,dx$$

$$= 2\pi\int_{x=0}^{a}\left[\left(\sqrt{\frac{a^2b^2 - b^2x^2}{a^2}} - 0\right)^2 - (0-0)^2\right]dx$$

$$= 2\pi\int_{x=0}^{a}\left(\frac{a^2b^2 - b^2x^2}{a^2}\right)dx \quad = \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx$$

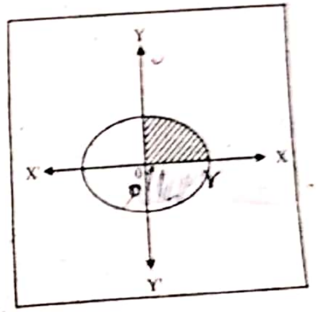
$$\Rightarrow V = \frac{2\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{2\pi b^2}{a^2}\cdot\frac{2a^3}{3} = \frac{4\pi ab^2}{3}$$

Thus, the volume of the circle is $\frac{4\pi ab^2}{3}$ cubic units.

14. Show that the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution: Since the volume of the sphere is obtain by twice of revolving the shaded portion about x-axis or y-axis. Let the region has boundary $x^2 + y^2 = r^2$ and $y = 0$ and it revolve about x-axis i.e. $y = 0$ with limits $x = 0$ to $x = r$.

Therefore, volume of sphere be,



$$V = 2\pi\int_{x=0}^{r} [(y_1 - y)^2 - (y_2 - y)^2]\,dx$$

$$= 2\pi\int_{x=0}^{r} [(y_1 - 0)^2 - (y_2 - 0)^2]\,dx$$

$$= 2\pi\int_{x=0}^{r} [(\sqrt{r^2 - x^2} - 0^2) - (0-0)^2]\,dx$$

$$= 2\pi\int_{x=0}^{r} (r^2 - x^2)^2\,dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4\pi r^3}{3}$$

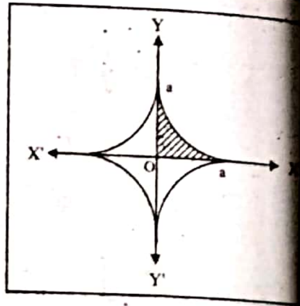
Thus, the volume of the circle is $\frac{4\pi r^3}{3}$ cubic units.

15. Show that the volume of the solid generated by revolving the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis is $\frac{32}{105}\pi a^3$.

Solution: Given asteroid be

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Clearly, if the half part of asteroid is revolved about x–axis then it gives the whole asteroid. Therefore, the volume is twice of revalued form of shaded portion.



Since the shaded portion has limits $x = 0$ to $x = a$ and is bounded by the curves $y = (a^{2/3} - x^{2/3})^{3/2}$ and $y = 0$.

Now, the volume of asteroid revolving about x–axis i.e. $y = 0$ be,

$$V = 2\pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= 2\pi \int_{x=0}^{a} \{(a^{2/3} - x^{2/3})^{3/2} - 0\}^2 - (0-0)^2\}\, dx$$

$$= 2\pi \int_{x=0}^{a} (a^{2/3} - x^{2/3})^3\, dx$$

$$= 2\pi \int_{x=0}^{a} [(a^{2/3})^3 - 3(a^{2/3})^2 . x^{2/3} + 3a^{2/3}(x^{2/3})^2 - (x^{2/3})^3]\, dx$$

$$[\because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3]$$

$$= 2\pi \int_{x=0}^{a} (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)\, dx$$

$$= 2\pi \left[a^2x - 3a^{4/3}\frac{x^{5/3}}{5/3} + 3a^{2/3}\frac{x^{7/3}}{7/3} - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^{4/3}.a^{5/3}}{5} + \frac{a^{2/3}.a^{7/3}}{7} - \frac{a^3}{3}\right]$$

$$= 2\pi \left(a^3 - \frac{a^3}{5} + \frac{a^3}{7} - \frac{a^3}{3}\right) = \frac{2\pi}{105}a^3(105 - 21 + 15 - 35)$$

$$= \frac{2\pi a^3}{105} . 64 = \frac{128\,\pi a^3}{105}.$$

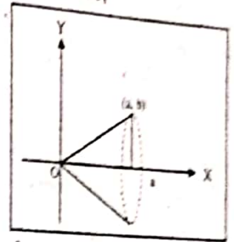
Thus, the volume of the circle is $\frac{128\,\pi a^3}{105}$ cubic units.

16. **Show that the volume of the solid general by revolving the line joining origin and the points (a, b) about x–axis is $\frac{1}{3}\pi ab^2$.**

Solution: Let (a, b) be a point. Then the equation of line joining (0, 0) and (a, b) be, $y = \frac{b}{a}x$.

Thus, the region is bounded by the curves $y = \frac{b}{a}x$ and x–axis i.e. $y = 0$, the solid has limits $x = 0$ and $x = a$.

Then, the volume of the solid by revolving about x–axis be,



$$V = \pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= \pi \int_{x=0}^{a} \left[\left(\frac{b}{a}x - 0\right)^2 - (0-0)^2\right] dx$$

$$= \frac{\pi b^2}{a^2}\int_{x=0}^{a} x^2\, dx = \frac{\pi b^2}{a^2}\left(\frac{x^3}{3}\right)_0^a = \frac{\pi b^2}{a^2}\cdot\frac{a^3}{3} = \frac{\pi ab^2}{3}.$$

Thus, the volume of the circle is $\frac{\pi ab^2}{3}$ cubic units.

17. **Show that the volume of the solid generated by revolving the catenary $y = c\cosh\left(\frac{x}{c}\right)$, ordinates $x = 0$, $x = a$, about x–axis, $\frac{\pi}{2}c^2\left(a + c\sinh\frac{a}{c}\cosh\frac{a}{c}\right)$**

Solution: Given curves are $y = c\cosh\left(\frac{x}{c}\right)$, $y = 0$ with $x = 0$, $x = a$.

Now, the volume of solid bounded by the curves about x–axis be,

$$V = \pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= \pi \int_{x=0}^{a} \left[\left(c\cosh\frac{x}{c} - 0\right)^2 - (0-0)^2\right] dx$$

$$= \pi c^2 \int_{x=0}^{a} \cosh^2\frac{x}{c}\, dx, = \frac{\pi c^2}{2}\int_{x=0}^{a}\left(1 + \cosh\frac{2x}{c}\right) dx$$

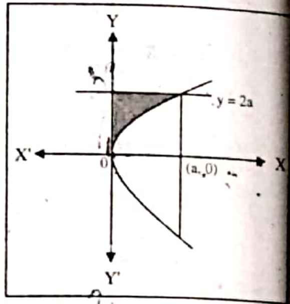
$$= \frac{\pi c^2}{2}\left[x + \sinh\frac{2x}{c}\cdot\frac{c}{2}\right]_0^a$$

$$= \frac{\pi c^2}{2}\left[a + \frac{c}{2}\sinh\frac{2a}{c}\right]$$

$$= \frac{\pi c^2}{2}\left[a + c\sin\left(\frac{a}{c}\right) . \cos h\left(\frac{a}{c}\right)\right]$$

Thus, the volume of the circle is $\frac{\pi c^2}{2}\left[a + c\sin\left(\frac{a}{c}\right) . \cos h\left(\frac{a}{c}\right)\right]$ cubic units.

18. Show that the volume of the solid generated by revolving the area bounded by parabola $y^2 = 4ax$, and y-axis, $y = 2a$ about y-axis is $\left(\frac{2}{5}\pi a^3\right)$. [2002]



Solution: By hypothesis the revolved area is the bounded region by $y^2 = 4ax$ and its latus rectum. Since, the region revolved about y-axis. So, the volume of solid is twice of volume of double shaded portion. Since the solide revolve about y-axis, so we need limits in y.

By solving $y^2 = 4ax$ and its latus rectum then we get $y = 0$ and $y = 2abc$ the limits of double shaded portion.

Now, volume of solid be,

$$V = \pi \int_{y=0}^{2a} [(x_1 - x)^2 - (x_2 - x)^2]\, dy$$

$$= \pi \int_{y=0}^{2a} \left[\left(\frac{y^2}{4a} - 0\right)^2 - (0-0)^2\right] dy$$

$$= \frac{\pi}{16a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{16a^2 \cdot 5}(2a)^5 = \frac{\pi \cdot 32a^5}{16a^2 . 5} = \frac{2\pi a^3}{5}.$$

Thus, the volume of the circle is $\frac{2\pi a^3}{5}$ cubic units.

Additional Questions

1. Find the volume generated in each case bounded by curve $2y = x^2$ and the line $y = 2$ about:

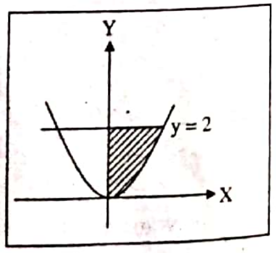
(i) the y-axis (ii) the line $y = 2$ (iii) the line $y = 4$

(iv) the line $y = -1$ (v) the line $x = 2$ (vi) the x-axis

Solution: Given curves are $2y = x^2$ and $y = 2$.

Solving we get $x = \pm 2$ and $y = 0$ and $y = 2$

(i) The solid revolved about y-axis. Then the volume of the solid be,



$$V = 2\pi \int_{y=0}^{2} [(x_1 - x)^2 - (x_2 - x)^2]\, dy$$

$$= 2\pi \int_{y=0}^{2} [(\sqrt{2y} - 0)^2 - (0-0)^2]\, dy$$

$$= 2\pi \int_{y=0}^{2} 2y\, dy = 2\pi . \left[2\frac{y^2}{2}\right]_0^2 = 8\pi.$$

Thus, the volume of the circle is 8π cubic units.

(ii) The solid revolved about the line $y = 2$. Then the volume of the solid be,

$$V = 2\pi \int_{x=0}^{2} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$\Rightarrow V = 2\pi \int_{x=0}^{2} \left[\left(\frac{x^2}{2} - 2\right)^2 - (0-2)^2\right] dx$$

$$\Rightarrow V = 2\pi \int_{x=0}^{2} \left[\frac{x^4}{4} + 4 - 2x^2 - 4\right] dx = 2\pi \int_{x=0}^{2} \left(\frac{x^4}{4} - 2x^2\right) dx$$

$$\Rightarrow V = 2\pi\left[\frac{1}{4} \cdot \frac{x^5}{5} - 2 \cdot \frac{x^3}{3}\right]_0^2 = 2\pi\left[\frac{1}{4} \cdot \frac{32}{5} - 2 \cdot \frac{8}{3}\right]$$

$$\Rightarrow V = 2\pi\left(\frac{8}{5} - \frac{16}{3}\right) = 2\pi\left(\frac{24-80}{15}\right) = -2\pi\frac{56}{15}$$

Thus, $V = -\frac{112\pi}{15}$.

Since V is always measure in positive form.

Thus, the volume of the circle is $\frac{112}{15}\pi$ cubic units.

(iii) - (iv) Process as **(ii)** with $y = 4$ and $y = -1$.

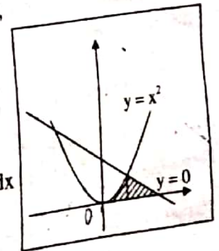
(v) Process as **(i)** with $x = 2$.

(vi) Process as **(ii)** with $y = 0$ i.e. x-axis.

2. Find the volume of solid revolution of the region in the first quadrant bounded by $y = x^2$, $x + y = 3$ and the x-axis, about the x-axis.

Solution: The required region bounded by the given curves $y = x^2$, $x + y = 3$ and $y = 0$ is the shaded portion. Here the required portion is bounded by three curves, so we partitioned the region into two parts from (0, 0) to (1, 0) and (1, 0) to (2, 0).

Now, volume of the solid revolved about x-axis be,



$$V = \pi \int_{x=0}^{1} (y_1^2 - y_2^2)\, dx + \pi \int_{x=1}^{2} (y_1^2 - y_2^2)\, dx$$

$$= \pi \int_{x=0}^{1} (x^4 - 0)\, dx + \pi \int_{x=1}^{2} \{(3-x)^2 - 0\}\, dx$$

$$= \pi \int_{x=0}^{1} x^4\, dx + \pi \int_{x=1}^{2} (9 - 6x + x^2)\, dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^1 + \pi \left[9x - \frac{6x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \frac{\pi}{5} + \pi\left(18 - \frac{24}{2} + \frac{8}{3} - 9 + \frac{6}{2} - \frac{1}{3}\right) = \pi\left[\frac{1}{5} - \frac{1}{3}\right] = -\frac{2\pi}{15}.$$

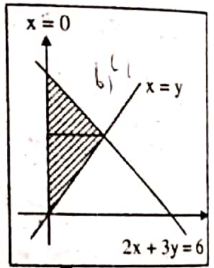
Since, the volume measures only in positive value.

Thus, the volume of the circle is $\frac{2\pi}{15}$ cubic units.

3. Find the volume of solid revolution of the triangular region bounded by $2x + 3y = 6$, $y = x$ and $x = 0$ about the y–axis.

Solution: Clearly, the line $2x + 3y = 6$ passes through (3, 0) and (0, 2). Then the region bounded by the curves $2x + 3y = 6$, $x = y$ and $x = 0$ be the shaded portion. The region is the added part of two different regions moves from (0, 0) to (0, 6/5) and (0, 6/5) to (0, 2).

Now, the volume by revolving the region about y–axis be,



$$V = \pi \int_{x=0}^{6/5} (x_1^2 - x_2^2)\, dy + \pi \int_{x=6/5}^{2} (x_1^2 - x_2^2)\, dy$$

$$= \pi \int_{x=0}^{6/5} (y^2 - 0)\, dy + \pi \int_{x=6/5}^{2} \left[\left(\frac{6-3y}{2}\right)^2 - 0\right] dy$$

$$= \pi\left[\frac{y^3}{3}\right]_0^{6/5} + \frac{\pi}{4}\left[36y + \frac{9y^3}{3} - \frac{36y^2}{2}\right]_{6/5}^{2}$$

$$= \pi\left[\frac{216}{3 \times 125} + \frac{72}{4} + \frac{24}{4} - \frac{72}{4} - \frac{216}{4 \times 5} - \frac{3 \times 216}{125} 3 \times \frac{216}{25}\right]$$

$$= \pi\left[\frac{216}{25}\left(\frac{1}{15} - \frac{5}{4} - \frac{3}{5} - 3\right) + \frac{24}{4}\right]$$

$$= -\frac{4416}{125}\pi$$

Thus, the volume of the circle is $\frac{4416}{125}\pi$ cubic units.

4. Find the volume of solid revolution of the region bounded between $x^2 = 4y$ and $y = |x|$ about $y = -2$. **[2008, Fall]**

Solution: Similar to Q 3.

OTHER IMPORTANT QUESTION FROM FINAL EXAM

1. Find the volume of the solid of revolution of the region bounded by the curve $y = \frac{3}{4}\sqrt{16 - x^2}$ and the line by $y = 0$ about x–axis. **[2001] [1999]**

Solution: Given that the required region is bounded by the curve,

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$\Rightarrow 16y^2 = 9(16 - x^2)$$

$$\Rightarrow \frac{9x^2 + 16y^2}{9(16)} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Then see Q. 13 with a = 4, b = 3.

2. The region in the first quadrant bounded by the parabola $y = x^2$ and the line $y = x$ is revolved about y–axis. Find the volume of the solid thus generated. **[2002]**

Solution: The given curves are $y = x^2$ and $y = x$. Here, $y = x^2$, is the equation of parabola with vertex (0, 0) and concave up.

For other points:

x	±1	±2	±3
y	1	4	9

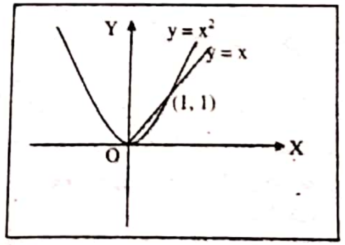
Here $y = x$, is the equation of straight line passing through (0, 0) and (1, 1).

We have to calculate the volume of solid generated by the bounded region revolved about y–axis. Thus, the volume of the solid be,

$$V = \pi \int_0^1 (y - y^2)\, dy$$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right).$$



Thus, the volume of the circle is $\frac{\pi}{6}$ cubic units.

Q. The region in the first quadrant bounded by the parabola $\sqrt{y} = x$ and the line $y = x$ is revolved about y–axis. Find the volume of the solid thus generated. **[2018 Fall]**

Solution: Similar to Q 2.

3. Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$. [2015 Fall][2005, Fall]

Solution: Given that the required region is bounded by the curves

$$x = y^2 + 1 \Rightarrow y^2 = x - 1 \text{ and } x = 3.$$

Then the region has two symmetrical parts in which the corners of a part are (1, 0), (3, 0) and (3, $\sqrt{2}$).

Now, we have to revolve the region about x = 3. Then the volume of the region revolving about x = 3 be,

$$V = 2\pi \int_{y=0}^{\sqrt{2}} [(x_1 - x)^2 - (x_2 - x)^2]\, dy$$

$$= 2\pi \int_{y=0}^{\sqrt{2}} [(3 - y^2 - 1)^2 - (3 - 3)^2] dy$$

$$= 2\pi \int_{y=0}^{\sqrt{2}} (y^4 - 4y^2 + 4)\, dy$$

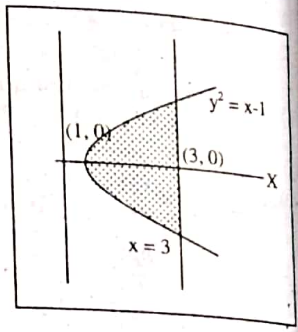
$$= 2\pi \left[\frac{y^5}{5} - \frac{4y^3}{3} + 4y\right]_0^{\sqrt{2}}$$

$$= 2\pi \left[\frac{(\sqrt{2})^5}{5} - \frac{4(\sqrt{2})^3}{3} + 4\sqrt{2}\right]$$

$$= 2\pi \left(\frac{4\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} + 4\sqrt{2}\right)$$

$$= 8\sqrt{2}\,\pi \left(\frac{1}{5} - \frac{2}{3} + 1\right) \qquad = \pi 8\sqrt{2}\left(\frac{3 - 10 + 15}{15}\right)$$

$$= \pi 8\sqrt{2}\,\frac{8}{15} = \frac{\pi\, 64\sqrt{2}}{15}$$



Thus, the volume of the solid by revolving about $x = 3$ $\frac{\pi\, 64\sqrt{2}}{15}$ cubic units.

4. Find the volume of the solid in the region bounded by the curve $x = y^2$ and the line $y = x - 2$ revolved about y-axis. **[2006, Spring]**

Solution: The required region is bounded by the curves $y^2 = x$ and $y = x - 2$.

Then the required region is the shaded portion that has corners $(1, -1)$ and $(4, 2)$.

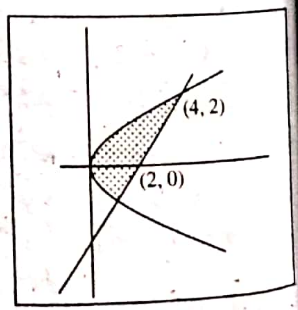
Now, volume of the solid obtained by revolving the region about y-axis be

$$V = \pi \int_{y=-1}^{2} [(x_1 - x)^2 - (x_2 - x)^2]\, dxy$$

$$= \pi \int_{y=-1}^{2} [(x_1 - 0)^2 - (x_2 - 0)^2]\, dy$$

$$= \pi \int_{y=-1}^{2} [y^4 - (y + 2)^2]\, dy$$

$$= \pi \int_{y=-1}^{2} (y^4 - y^2 - 4y - 4) dy$$



$$= \pi \left[\frac{y^5}{5} - \frac{y^3}{3} - \frac{4y^2}{2} - 4y\right]_{y=-1}^{2}$$

$$= \pi \left[\left(\frac{32}{5} - \frac{8}{3} - 8 - 8\right) - \left(-\frac{1}{5} + \frac{1}{3} - 2 + 4\right)\right]$$

$$= \pi \left[\left(\frac{96 - 40 - 240}{15}\right) - \left(\frac{-3 + 5 + 30}{15}\right)\right]$$

$$= \pi \left[\frac{-184}{15} - \frac{32}{15}\right] \quad = -\pi \frac{216}{15} \quad = -\frac{72}{5}\pi.$$

Thus, the volume of the solid obtained by revolving about y-axis is. $\frac{72}{5}\pi$ cubic units.

5. Obtain the volume of the solid in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis and on the right by the line $x = 1$ about the line $x = -2$. **[2011 Fall]**

Solution: Since the given region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis and on the right by the line $x = 1$.

Here, first we trace the curves,

$R(y) = x = 1$ and $r(y) = x = \sqrt{y}$

From figure, the limit of the integration is $y = 0$ to $y = 1$. Then the volume bounded by the curves be,

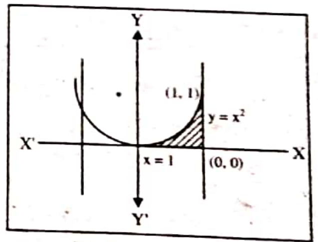
$$V = \int_0^1 \pi\, [\{R(y) - (-2)\}^2 - \{r(y) - (-2)\}^2]\, dy$$

$$= \pi \int_0^1 [(3)^2 - (\sqrt{y} + 2)^2]$$

$$= \pi \int_0^1 [9 - y - 4\sqrt{y} - 4]\, dy$$

$$= \pi \int_0^1 [5 - y - 4\sqrt{y})\, dy \quad = \pi \left| -\frac{y^2}{2} + 5y - \frac{8y^{3/2}}{3} \right|_0^1$$

$$= \pi \left| 5 - \frac{1}{2} - \frac{8}{3} \right| \quad = \frac{11\pi}{6}.$$



Thus, the volume of the circle is $\frac{11\pi}{6}$ cubic units.

6. Find the volume of the solid in the region bounded by the curves $x = y^2$, $x = 0$, $y = -1$, $y = 1$ revolved about y-axis. **[2013 Fall]**

Solution: Similar to Q. 3(i) exercise 12.2.

●●●

Chapter 12 Application of Integration
ARC LENGTH

List of Formulae

(i) If the curve is given in variables x and y then

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \text{OR} \quad L=\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

(ii) If the curve is given in parametric form then such as if the curve is in x and y in the form of θ, independently. Then,

$$L=\int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

If the curve is in polar form i.e. in (r, θ) then

(i) the length of the arc from $\theta=\theta_1$ to $\theta=\theta_2$ is

$$L=\int_{\theta_1}^{\theta_2}\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta$$

(ii) the length of the arc from $r=r_1$ to $r=r_2$ is

$$L=\int_{r_1}^{r_2}\left(1+\left(r\frac{d\theta}{dr}\right)^2\right)dr$$

Exercise 12.3

1. Find the arc length of the curves.

(i) $y=x^2$, $-1\le x\le 2$.

[2017 Spring short]
[2011 Fall (Short)][2003 Spring(Short)][2007 Fall(Short)]

Solution: Here, $y=x^2$ for $-1\le x\le 2$

Different w. r. t. x, then

$$\frac{dy}{dx}=2x.$$

Now, arc length of given curve for $-1\le x\le 2$ be

$$L=\int_{-1}^{2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_{-1}^{2}\sqrt{1+4x^2}\,dx$$

Put $2x=t$ then $2\,dx=dt$. Also, $x=-1\Rightarrow t=-2$, $x=2\Rightarrow t=4$. Then,

$$L=\frac{1}{2}\int_{-2}^{4}\left(\sqrt{1+t^2}\right)dx$$
$$=\frac{1}{2}\left[\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right]_{-2}^{4}$$
$$=\frac{1}{2}\left[\left(2\sqrt{17}+\frac{1}{2}\log\left(4+\sqrt{17}\right)\right)-\left(-\sqrt{5}+\frac{1}{2}\log\left(-2+\sqrt{5}\right)\right)\right]$$
$$=\frac{1}{2}\left[2\sqrt{17}+\sqrt{5}+\frac{1}{2}\log\left(\frac{4+\sqrt{17}}{-2+\sqrt{5}}\right)\right]$$

Thus, the length of the curve $y=x^2$ for $-1\le x\le 2$ is

$$\frac{1}{2}\left[2\sqrt{17}+\sqrt{5}+\frac{1}{2}\log\left(\frac{4+\sqrt{17}}{-2+\sqrt{5}}\right)\right]\text{ unit.}$$

Q. Find the arc length of the curve $y=x^2+1$, from $x=1$ to $x=2$.
[2016 Spring Short]

(ii). $y^2+2y=2x+1$ from (–1, –1) to (7, 3).

Solution: Here, $y^2+2y=2x+1$ from (–1, –1) to (7, 3).

So, differentiating w. r. t. x then,

$$2y\frac{dy}{dx}+2\frac{dy}{dx}=2 \Rightarrow \frac{dx}{dy}=y+1.$$

Now, arc length of given curve from (–1, –1) to (7, 3) is,

$$L=\int_{-1}^{3}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_{-1}^{3}\sqrt{1+(y+1)^2}\,dy.$$

Put $y+1=t$ then $dy=dt$. And $y=-1\Rightarrow t=0$, $y=3\Rightarrow t=4$.

Then,

$$L=\int_0^4\sqrt{1+t^2}\,dt$$
$$=\left[\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right]_0^4$$
$$=2\sqrt{17}+\frac{1}{2}\log\left(4+\sqrt{17}\right) \qquad [\log(1)=0]$$
$$=9.29.$$

Thus, the length of the curve $y^2+2y=2x+1$ from (–1, –1) to (7, 3) is 9.29 unit.

(iii) $x=\dfrac{y^3}{3}+\dfrac{1}{4y}$ from $y=1$ to $y=3$

Solution: Here, $x=\dfrac{y^3}{3}+\dfrac{1}{4y}$ from $y=1$ to $y=3$.

So, differentiating w. r. t. y then,

$$\frac{dx}{dy}=\frac{3y^2}{3}+\frac{1}{4}\left(\frac{-1}{y^2}\right) \Rightarrow \frac{dx}{dy}=y^2-\frac{1}{4y^2}=\frac{4y^4-1}{4y^2}.$$

rc length of given curve from $y=1$ to $y=3$ be

$$L = \int_1^3 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_1^3 \sqrt{1+\left(\frac{4y^4-1}{4y^2}\right)^2}\,dy$$

$$\Rightarrow L = \int_1^3 \sqrt{\frac{16y^4+16y^8-8y^4+1}{16y^4}}\,dy$$

$$\Rightarrow L = \int_1^3 \sqrt{\frac{16y^8+8y^4+1}{18y^4}}\,dy$$

$$\Rightarrow L = \int_1^3 \sqrt{\left(\frac{4y^4+1}{4y^2}\right)^2}\,dy = \int_1^3 \frac{4y^4+1}{4y^2}\,dy$$

$$\Rightarrow L = \int_1^3 \left(y^2+\frac{1}{4y^2}\right)dy = \left[\frac{y^3}{3}+\frac{1}{4}\frac{y^{-1}}{-1}\right]_1^3$$

$$\Rightarrow L = 9-\frac{1}{12}-\frac{1}{3}+\frac{1}{4} = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}.$$

Thus, the length of the curve $x = \frac{y^3}{3}+\frac{1}{4y}$ from $y = 1$ to $y = 3$ is $\frac{53}{6}$ unit.

(iv) $x = \frac{y^4}{4}+\frac{1}{8y^2}$**, from y = 1 to y = 2**

Solution: Here, $x = \frac{y^4}{4}+\frac{1}{8y^2}$ from $y = 1$ to $y = 2$.

So, differentiating w. r. t. y then,

$$\frac{dx}{dy} = y^3-\frac{1}{4y^3} = \frac{4y^6-1}{4y^3}$$

Now, arc length of the given curve from y = 1 to y = 2 be,

$$L = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_1^2 \sqrt{1+\left(\frac{4y^6-1}{4y^3}\right)^2}\,dy$$

$$= \int_1^2 \sqrt{\frac{16y^6+16y^{12}-8y^6+1}{16y^6}}\,dy$$

$$= \int_1^2 \sqrt{\left(\frac{4y^6+1}{4y^3}\right)^2}\,dy = \int_1^2 \left(y^3+\frac{1}{4}y^{-3}\right)dy$$

$$= \left[\frac{y^4}{4}-\frac{1}{8y^2}\right]_1^2$$

$$= 4-\frac{1}{32}-\frac{1}{4}+\frac{1}{8}$$

$$= \frac{128-1-8+4}{32} = \frac{123}{32}.$$

Thus, the length of the curve $x = \frac{y^4}{4}+\frac{1}{8y^2}$ from $y = 1$ to $y = 2$ is $\frac{123}{32}$ unit.

(v) $y = \frac{3}{4}x^{2/3}-\frac{3}{8}x^{2/3}+5$**, for** $-1 \le x \le 8$.

Solution: Here, $y = \frac{3x^{2/3}}{4}-\frac{3x^{2/3}}{8}+5$ for $-1 \le x \le 8$

$$\Rightarrow y = \frac{3x^{2/3}}{8}+5$$

So, differentiating w. r. t. x then,

$$\frac{dy}{dx} = \frac{3}{8}\cdot\frac{2}{3}x^{-1/3} = \frac{x^{-1/3}}{4}$$

Now, arc length of given curve from x = –1 to x = 8 be.

$$L = \int_{-1}^8 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{-1}^8 \sqrt{1+\frac{x^{-2/3}}{16}}\,dx$$

$$= \int_{-1}^8 \sqrt{\frac{x^{2/3}+\frac{1}{16}}{x^{2/3}}}\,dx$$

$$= \int_{-1}^8 x^{-1/3}\sqrt{x^{2/3}+\frac{1}{16}}\,dx.$$

Put $x^{2/3}+\frac{1}{16} = t^2$ then $\frac{2}{3}x^{-1/3}dx = dt \Rightarrow x^{-1/3}dx = \frac{3}{2}dt$.

Also, $x = -1 \Rightarrow t = \frac{17}{16}$ and $x = 8 \Rightarrow t = \frac{65}{16}$. Then,

$$L = \int_{17/16}^{65/16} \frac{3}{2}t\,dt$$

$$= \frac{3}{2}[t^2]_{17/16}^{65/16} = \frac{3}{2}\left[\frac{1}{256}(65^2-17^2)\right] = \frac{369}{16} = 23.0625$$

Thus, the length of the curve $y = \frac{3x^{2/3}}{4}-\frac{3x^{2/3}}{8}+5$ for $-1 \le x \le 8$ is 23.0625 unit.

2. **Find the distance traveled between t = 0 and t = π by the particle p(x, y) whose position at time t is x = cos t, y = t + sin t.**

... $x = \cos t$, $y = t+\sin t$ for $t = 0$ to $t = \pi$.

...ntiating w. r. t. t then,

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = 1 + \cos t$$

Now, arc length of the particle p(x, y) from $t = 0$ to $t = \pi$ is,

$$L = \int_{t=0}^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{i.e. } L = \int_{t=0}^{\pi} \sqrt{\sin^2 t + (1+\cos t)^2}\, dt$$

$$= \int_{t=0}^{\pi} \sqrt{\sin^2 t + 1 + \cos^2 t + 2\cos t}\, dt$$

$$= \sqrt{2}\int_{t=0}^{\pi} \sqrt{1+\cos t}\, dt$$

$$= \sqrt{2}\int_{t=0}^{\pi} \sqrt{2\cos^2 \frac{t}{2}}\, dt$$

$$= 2\int_{t=0}^{\pi} \cos\frac{t}{2}\, dt = 2\left[\frac{\sin t/2}{1/2}\right]_0^{\pi} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4.$$

Thus, the distance traveled between $t = 0$ and $t = \pi$ by the particle p(x, y) whose position at time t is $x = \cos t$, $y = t + \sin t$. is 4 unit.

3. **Find the length of the arc of the parabola $y^2 = 4x$ cut off by the line $y = 2x$.**

Solution: Given curves is,

$$y^2 = 4x \qquad \dots (i)$$

So, $\dfrac{dx}{dy} = \dfrac{y}{2}$

And the curve (i) is cut by the line $y = 2x$. Therefore, solving (i) and $y = 2x$ then we get, $x = 0, 1$ and so $y = 0, 2$.

Now, the arc length of $y^2 = 4x$ from $y = 0$ to 2 is,

$$L = \int_{y=0}^{2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_{y=0}^{2} \sqrt{1 + \frac{y^2}{4}}\, dy$$

$$= \frac{1}{2}\int_{y=0}^{2} \sqrt{4+y^2}\, dy$$

$$= \frac{1}{2}\left[\frac{y}{2}\sqrt{4+y^2} + \frac{4}{2}\log\left(y + \sqrt{4+y^2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[\sqrt{8} + 2\log\left(2+\sqrt{8}\right) - 2\log 2\right]$$

$$= \frac{1}{2}\left[2\sqrt{2} + 2\log\left(\frac{2+2\sqrt{2}}{2}\right)\right]$$

$$= \sqrt{2} + \log(1+\sqrt{2}).$$

Thus, the length of the arc of the parabola $y^2 = 4x$ cut off by the line $y = 2x$ is $\sqrt{2} + \log(1+\sqrt{2})$.

4. **Show that the length of perimeter of the circle $x^2 + y^2 = a^2$ is $2\pi a$.**

Solution: Here, $x^2 + y^2 = a^2$

Put, $x = a\cos\theta$, $y = a\sin\theta$

So, differentiating w. r. t. θ then,

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta.$$

Clearly for a circle, θ varies from 0 to 2π.

Now, arc length of the perimeter of $x^2 + y^2 = a^2$ then

$$L = \int_{\theta=0}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\, d\theta = a\int_0^{2\pi} d\theta = 2\pi a.$$

Thus, the length of perimeter of the circle $x^2 + y^2 = a^2$ is $2\pi a$.

5. **Show that the entire length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is 6a.**

Solution: Here, $x^{2/3} + y^{2/3} = a^{2/3}$.

Then, differentiating w. r. t. x then,

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

Clearly this asteroid has radius a with having 4-symmetrical parts.

Now, the arc length of whole asteroid is,

$$L = 4\int_0^a \sqrt{1 + \left(-\frac{y^{1/3}}{x^{1/3}}\right)^2}\, dx$$

$$= 4\int_0^a \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx = 4a^{1/3}\left[\frac{x^{-1/3+1}}{-(1/3)+1}\right]_0^a = 4a^{1/3}\cdot\frac{a^{2/3}}{2/3} = 6a.$$

Thus, the entire length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is 6a.

6. **Show that the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a) is $\frac{a}{27}(13\sqrt{13} - 8)$.**

Solution: Given that $ay^2 = x^3$.

Then, differentiating w.r.t. x then,

$$2ay\frac{dy}{dx} = 3x^2. \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2a\sqrt{x^3/a}} = \frac{3\sqrt{x}}{2\sqrt{a}}.$$

So, $$\left(\frac{dy}{dx}\right)^2 = \frac{9x}{4a}$$

Clearly the semi–cubical parabola (i) has vertex (0, 0) and radius from (0, 0) to (a, a).

Now, are length from (0, 0) to (a, a) is,

$$L = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1+\frac{9x}{4a}}\, dx$$

$$= \frac{3}{2\sqrt{a}}\int_0^a \sqrt{x+\frac{4a}{9}}\, dx.$$

Put, $x + \frac{4a}{9} = t^2$ then $dx = 2t\,dt$. Also, $x = 0 \Rightarrow t = \sqrt{4a/9}$ and $x = a \Rightarrow t = \sqrt{13a/9}$. Then,

$$L = \frac{3}{2\sqrt{a}}\int_{\sqrt{\frac{4a}{9}}}^{\sqrt{\frac{13a}{9}}} 2t^2\, dt = \frac{3}{\sqrt{a}}\left[\frac{t^3}{3}\right]_{\sqrt{\frac{4a}{9}}}^{\sqrt{\frac{13a}{9}}}$$

$$= \frac{1}{\sqrt{a}}\left[\left(\sqrt{\frac{13a}{9}}\right)^3 - \left(\sqrt{\frac{4a}{9}}\right)^3\right]$$

$$= \frac{1}{27\sqrt{a}}a\sqrt{a}\,[13\sqrt{13} - 4\sqrt{4}]$$

$$= \frac{a}{27}[13\sqrt{13} - 8].$$

Thus, the length of the arc of the semi–cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a) is $\frac{a}{27}(13\sqrt{13} - 8)$.

7. Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $\left[a\left(\log 2 + \frac{15}{16}\right)\right]$.

Solution: Given curve is $y^2 = 4ax$ ... (1)

and the line is, $3y = 8x$ ... (2)

Since we have to find the length of curve segment of (1) that is cut off by the line (2).

Here, solving (1) and (2) then the points of contact, are (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$.

Here, differentiating (1) w. r. t. y then

$$2y = 4a\frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{y}{2a}.$$

Now, arc length of (1) from (0, 0) to $\left(\frac{9a}{16}, \frac{3a}{2}\right)$ is

$$L = \int_0^{3a/2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^{3a/2}\sqrt{1+\frac{y^2}{4a^2}}\, dy = \frac{1}{2a}\int_0^{3a/2}\sqrt{y^2+4a^2}\, dy$$

$$\Rightarrow L = \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2}\left\{\log\left(y+\sqrt{y^2+4a^2}\right)\right\}\right]_0^{3a/2}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\cdot\frac{5a}{2} + \frac{4a^2}{2}\log\left(\frac{3a}{2}+\frac{5a}{2}\right) - \frac{4a^2}{2}\log(2a)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + \frac{4a^2}{2}\left\{\log\left(\frac{8a}{2}\right) - \log(2a)\right\}\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + \frac{4a^2}{2}\log\left(\frac{8a}{2.2a}\right)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + \frac{4a^2}{2}\log(2)\right]$$

$$= \frac{15a}{16} + a\log(2).$$

Thus, the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $\left[a\left(\log 2 + \frac{15}{16}\right)\right]$.

8. Show that the length of the arc of the parabola $x^2 = 4ay$, the vertex to an extremity of the latus rectum $a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$.

Solution: Given parabola is, $x^2 = 4ay$ ... (1)

Clearly the parabola has vertex (0, 0) and the extremities of the latus rectum are $(\pm 2a, a)$.

Here, differentiating (1) w. r. t. x, $2x = 4a\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

Now, arc length of the parabola (1) from vertex (0, 0) to the extremity (2a, 0) be,

$$L = \int_0^{2a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{2a}\sqrt{1+\frac{x^2}{4a^2}}\, dx$$

$$= \frac{1}{2a}\int_0^{2a}\sqrt{x^2+4a^2}\, dx$$

$$= \frac{1}{2a}\left[\frac{x}{2}\sqrt{x^2+4a^2} + \frac{4a^2}{2}\log\left(x+\sqrt{x^2+4a^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a}\left[a.2a.\sqrt{2} + 2a^2\log(2a+2a\sqrt{2}) - 2a^2\log(2a)\right]$$

$$= a\sqrt{2} + a\log(1+\sqrt{2})$$

$$= a\left[\sqrt{2} + \log(1+\sqrt{2})\right].$$

Thus, the length of the arc of the parabola $x^2 = 4ay$, the vertex to extremity of the latus rectum $a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$.

EXERCISE FOR PRACTICE FROM FINAL EXAM

SHORT QUESTIONS

a. Find the arc length of the curve $y = \frac{1}{3}(x^2+2)^{3/2}$. [2009, Fa

b. Find the expression for the arc length of the curve
$y = \tan x, 0 \le x \le \frac{3}{4}$. [200

c. Find the arc length of curve $y = \frac{4\sqrt{2}}{3}x^{3/2}$ for $x = 0$ to 1. [2003, Fa

d. Find the arc length of curve $x = \sin y$ in $0 \le y \le \pi$. [2004, Sprin

e. Find the length of the arc of the parabola $y^2 = 4x$ from (0, 0) to (1, 2 [2008, Fa

f. Find the arc length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 2$. [2014 Spring (Short

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...pter 12

APPLICATION OF INTEGRATION
Trapezoidal and Simpson Rule

...t of Formulae

(i) For approximate area by Simpson's rule is obtained by using the formula,

$$S = \frac{b-a}{3n}[(y_0+y_n)+4(y_1+y_3+\dots+y_{n-1})+2(y_2+y_4+\dots+y_{n-2})]$$

$$= \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\dots+y_{n-1})+2(y_2+y_4+\dots+y_{n-2})]$$

When $n = 4$,

$$S = \frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2y_2]$$

Note: If n is odd then Simpson's process can not be applied. So n should be even.

(ii) For approximate area by Trapezoidal's rule is obtained by using the formula,

$$T = \frac{b-a}{2n}[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})] \text{ (n is odd or even)}$$

$$= \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})] \text{ (n is odd or even)}$$

When $n = 4$,

$$T = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]$$

(iii) For exact value, the integral, is solve by the process of definite integral.

...or Error Estimation

(i) With the area determined by Simpson's rule,

$$\text{Error}_s\ \% = \frac{|S-E|}{E}\times 100\%$$

(ii) With the area determined by Trapezoidal's rule,

$$\text{Error}_T\ \% = \frac{|T-E|}{E}\times 100\%$$

Where E denotes the exact value of the integral.

Exercise 12.4

1. (i) $\int_0^2 x\,dx$

Solution: Let

$$I = \int_0^2 x\,dx \qquad \dots (i)$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$f(x) = x$, $a = 0$, $b = 2$, $n = 4$.

Then

and $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$.

Here,

$x_0 = a = 0$ and $y_0 = f(x_0) = f(0) = 0$

$x_1 = a + h = \frac{1}{2}$ and $y_1 = f(x_1) = f\left(\frac{1}{2}\right) = \frac{1}{2}$

$x_2 = a + 2h = 1$ and $y_2 = f(x_2) = f(1) = 1$

$x_3 = a + 3h = \frac{3}{2}$ and $y_3 = f(x_3) = f\left(\frac{3}{2}\right) = \frac{3}{2}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = f(2) = 2$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2] \quad \dots (i)$$

$$\Rightarrow S \approx \frac{1/2}{3}\left[(0+2) + 4\left(\frac{1}{2} + \frac{3}{2}\right) + 2 \cdot 1\right]$$

$$= \frac{1}{6}(2 + 8 + 2) = \frac{1}{6}(12) = 2 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1/2}{2}\left[(0+2) + 2\left(\frac{1}{2} + 1 + \frac{3}{2}\right)\right] = \frac{1}{4}[8] = 2 \text{ sq. units.}$$

(c) Exact value:

$$E = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_2^0 = \left[\frac{2^2}{2} - 0\right] = 2 \text{ sq. units.}$$

Error percentage with the value given by Simpson rule is,

$$\text{Error}_s\,\% = \frac{|S-E|}{E} \times 100\% = \frac{|2-2|}{2} \times 100\% = 0\%.$$

Error percentage with the value given by trapezoidal rule is,

$$\text{Error}_T\,\% = \frac{|T-E|}{E} \times 100\% = \frac{|2-2|}{2} \times 100\% = 0\%.$$

Hence, Simpson's approximate value and trapezoidal approximate value is equal with exact area bounded by given curve.

(ii) $\int_0^2 x^2\,dx$

Solution: Let

$$I = \int_0^2 x^2\,dx \quad \dots (i)$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$f(x) = x^2$, $a = 0$, $b = 2$, $n = 4$

and $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$.

$y_0 = f(x_0)$

$f(x)$

$x^2 = 0$

Then,

$x_0 = a = 0$ and $y_0 = f(x_0) = x_0^2 = 0$

$x_1 = a + h = \frac{1}{2}$ and $y_1 = f(x_1) = x_1^2 = \frac{1}{4}$

$x_2 = a + 2h = 1$ and $y_2 = f(x_2) = x_2^2 = 1$

$x_3 = a + 3h = \frac{3}{2}$ and $y_3 = f(x_3) = x_3^2 = \frac{9}{4}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = x_4^2 = 4$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1/2}{3}\left[(0+4) + 4\left(\frac{1}{4} + \frac{9}{4}\right) + 2 \cdot 1\right]$$

$$= \frac{1}{6}[4 + 1 + 9 + 2] = \frac{1}{6} \times 16 = \frac{8}{3} \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1/2}{2}\left[(0+4) + 2\left(\frac{1}{4} + 1 + \frac{9}{4}\right)\right]$$

$$= \frac{1}{4}\left[4 + \frac{1}{2} + 2 + \frac{9}{2}\right]$$

$$= \frac{1}{4}\left[\frac{8+1+4+9}{2}\right] = \frac{1}{4} \times 11 = 2.75 \text{ sq. units.}$$

(c) Also the exact value is,

$$E = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \left[\frac{2^3}{3} - 0\right] = \frac{8}{3}$$

Error percentage with the value given by Simpson's rule is,

$$\text{Error}_s\,\% = \frac{|S-E|}{E} \times 100\% = \frac{\left|\frac{8}{3} - \frac{8}{3}\right|}{8/3} \times 100\% = 0\%.$$

Error percentage with the value given by Trapezoidal ruleis

$$\text{Error}_T\,\% = \frac{|S-E|}{E} \times 100\% = \frac{\left|\frac{11}{4} - \frac{8}{3}\right|}{8/3} \times 100\% = 3.125\%.$$

Thus, Simpson's approximate is more consistence than trapezoidal approximation with exact area bounded by given curves.

(iii) $\int_0^2 x^3\,dx$ **[2003, Spring]**

OR Find the approximate area by using Simpson's and trapezoidal rule for the region bounded by the curve $y = x^3$, the x–axis, $x = 0$ and $x = 2$, with $n = 4$ and compare with exact value. **[2008, Fall]**

Solution: Let

$$I = \int_0^2 x^3\,dx \qquad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$$f(x) = x^3,\ a = 0,\ b = 2,\ n = 4$$

and, $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$.

Here,

$x_0 = a = 0$ and $y_0 = f(x_0) = f(0) = 0$

$x_1 = a + h = \frac{1}{2}$ and $y_1 = f(x_1) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$x_2 = a + 2h = 1$ and $y_2 = f(x_2) = f(1) = (1)^3 = 1$

$x_3 = a + 3h = \frac{3}{2}$ and $y_3 = f(x_3) = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = f(2) = (2)^3 = 8$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1/2}{3}\left[(0+8) + 4\left(\frac{1}{8} + \frac{27}{8}\right) + 2.1\right]$$

$$= \frac{1}{6}\left[8 + \frac{1}{2} + \frac{27}{2} + 2\right]$$

$$= \frac{1}{6}\left[\frac{16 + 1 + 27 + 4}{2}\right] = \frac{1}{6} \times \frac{48}{2} = 4 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{\frac{1}{2}}{2}\left[(0+8) + 2\left(\frac{1}{8} + 1 + \frac{27}{8}\right)\right]$$

$$= \frac{1}{4}\left[8 + \frac{1}{4} + 2 + \frac{27}{4}\right] = \frac{17}{4} \text{ sq. unit.}$$

(c) Also, the exact value is,

$$E = \int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = \left[\frac{2^4}{4} - 0\right] = \frac{16}{4} = 4 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule

$$\text{Error}_s\,\% = \frac{|S-E|}{E} \times 100\% = \frac{|4-4|}{4} \times 100\% = 0\%$$

Error percentage with the value given by trapezoidal rule:

$$\text{Error}_T\,\% = \frac{|T-E|}{E} \times 100\% = \frac{\left|\frac{17}{4} - 4\right|}{4} \times 100\% = \frac{1/4}{4} \times 100\% = 6.25\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

(iv) $\int_1^2 \left(\frac{1}{x^2}\right) dx$ **[2011 Fall]**

Solution: Let

$$I = \int_1^2 \left(\frac{1}{x^2}\right) dx \qquad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$$f(x) = \frac{1}{x^2},\ a = 1,\ b = 2,\ n = 4$$

$\frac{1}{4}$

and, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$ $1 + \frac{1}{4}$ $\frac{5}{4}$

Here,

$x_0 = a = 1$ and $y_0 = f(x_0) = \frac{1}{x_0^2} = 1$

$x_1 = a + h = \frac{5}{4}$ and $y_1 = f(x_1) = \frac{1}{x_1^2} = \frac{16}{25}$

$x_2 = a + 2h = \frac{3}{2}$ and $y_2 = f(x_2) = \frac{1}{x_2^2} = \frac{4}{9}$

$x_3 = a + 3h = \frac{7}{4}$ and $y_3 = f(x_3) = \frac{1}{x_3^2} = \frac{16}{49}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = \frac{1}{x_4^2} = \frac{1}{4}$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$
$$= \frac{1/4}{3}\left[\left(1 + \frac{1}{4}\right) + 4\left(\frac{16}{25} + \frac{16}{49}\right) + 2\left(\frac{4}{9}\right)\right]$$
$$= \frac{1}{12}[(1 + 0.25) + 4(0.64 + 0.33) + 2(0.44)]$$
$$= 0.5008 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$
$$= \left(\frac{1/4}{2}\right)\left[\left(1 + \frac{1}{4}\right) + 2\left(\frac{16}{25} + \frac{4}{9} + \frac{16}{49}\right)\right]$$
$$= \frac{1}{8}[(1 + 0.25) + 2(0.64 + 0.44 + 0.33)]$$
$$= 0.5088 \text{ sq. unit.}$$

(c) Also, the exact value is,

$$E = \int_1^2 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^2 = -\left[\frac{1}{2} - 1\right] = 0.5 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule is

$$\text{Error, \%} = \left|\frac{S - E}{E}\right| \times 100\%$$
$$= \left|\frac{0.5008 - 0.5}{0.5}\right| \times 100\% = 0.16\ \%$$

Error percentage with the value given by trapezoidal rule:

$$\text{Error}_T\ \% = \frac{|T - E|}{E} \times 100\% = \frac{|0.5088 - 0.5|}{0.5} \times 100\% = 1.76\ \%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

(v) $\int_1^4 \sqrt{x}\,dx$

OR Find approximate area bounded by given curves $y = \sqrt{x}$ from $x = 1$ to $x = 4$, by using Simpson's and Trapezoidal rule with $n = 4$. Compare these values with exact value. [2009 Spring]

OR Evaluate $\int_1^4 \sqrt{x}\,dx$ with $n = 4$ by Simpson's and Trapezoidal rule and compare this with the exact value of the integral. [2004, Spring]

Solution: Let

$$I = \int_1^4 \sqrt{x}\,dx \quad \dots \text{(i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$f(x) = \sqrt{x}$, $a = 1$, $b = 4$, $n = 4$

and, $h = \frac{b - a}{n} = \frac{4 - 1}{4} = \frac{3}{4}$.

Here,

$x_0 = a = 1$ and $y_0 = f(x_0) = 1$

$x_1 = a + h = \frac{7}{4}$ and $y_1 = f(x_1) = 1.32$

$x_2 = a + 2h = \frac{5}{2}$ and $y_2 = f(x_2) = 1.58$

$x_3 = a + 3h = \frac{13}{4}$ and $y_3 = f(x_3) = 1.80$

$x_4 = a + 4h = 4$ and $y_4 = f(x_4) = 2$

(a) Approximate area by using Simpson's rule.

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$
$$= \frac{3/4}{3}[(1 + 2) + 4(1.32 + 1.80) + 2(1.58)]$$
$$= (0.25)[3 + 12.48 + 3.16]$$
$$= 4.66 \text{ sq. units.}$$

(b) Approximate area by using trapezoidal rule

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$
$$= \frac{3/4}{2}[(1+2) + 2(1.32 + 1.58 + 1.80)]$$
$$= (0.375)[3 + 9.4] \quad = 3.525 \text{ sq. unit.}$$

(c) Exact value:

$$E = \int_1^4 \sqrt{x}\,dx$$
$$= \left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2\left[\frac{8-1}{3}\right] = 4.66 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule

$$\text{Error}_s\,\% = \frac{|S-E|}{E} \times 100\% = \frac{|4.66 - 4.66|}{4.66} \times 100\% = 0\%$$

Error percentage with the value given by trapezoidal rule:

$$\text{Error}_T\,\% = \frac{|T-E|}{E} \times 100\% = \frac{|3.525 - 4.66|}{4.66} \times 100\% = 24.35\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

(vi) $\int_0^{\pi} \sin x\,dx$ **[2011 Spring]**

Solution: Let

$$I = \int_0^{\pi} \sin x\,dx \quad \ldots \text{(i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$f(x) = \sin x$, $a = 0$, $b = \pi$, $n = 4$

Then, $h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$

Here,

$x_0 = a = 0$ and $y_0 = f(x_0) = \sin 0 = 0$

$x_1 = a + h = \frac{\pi}{4}$ and $y_1 = f(x_1) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$x_2 = a + 2h = \frac{\pi}{2}$ and $y_2 = f(x_2) = \sin\frac{\pi}{2} = 1$

$x_3 = a + 3h = \frac{3\pi}{4}$ and $y_3 = f(x_3) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$x_4 = a + 4h = \pi$ and $y_4 = f(x_4) = \sin\pi = 0.$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$
$$= \frac{\pi/4}{3}\left[(0+0) + 4\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2(1)\right]$$
$$= \frac{\pi}{12}\left[0 + 4(\sqrt{2}) + 2\right]$$
$$= 0.6381\,\pi \text{ sq. units.} = 2.0047 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$
$$= \left(\frac{\pi/4}{2}\right)\left[(0+0) + 2\left(\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right)\right]$$
$$= 0.6036\,\pi \text{ sq. unit.} = 1.8963 \text{ sq units.}$$

(c) Also, the exact value is,

$$E = \int_0^{\pi} \sin x\,dx$$
$$= [-\cos x]_0^{\pi} = [-\cos\pi + \cos 0] = 1 + 1 = 2 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule is,

$$\text{Error}_s\,\% = \left|\frac{S-E}{E}\right| \times 100\%$$
$$= \left|\frac{2.0047 - 2}{2}\right| \times 100\% = 0.235\,\%$$

Error percentage with the value given by trapezoidal rule is

$$\text{Error}_T\,\% = \frac{|T-E|}{E} \times 100\% = \frac{|1.8963 - 2|}{2} \times 100\% = 5.185\,\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

OTHER QUESTION AS AN EXERCISE FROM FINAL EXAM

1. Evaluate $\int_0^4 \left(\frac{1}{x^2+1}\right) dx$ by using Trapezoid Rule, Simpson's Rule and compare the result with the exact value taking $n = 4$. **[2009, Fall]**

2. Use Simpson's rule with $n = 4$ to approximate the area between the curve $y = (2x+1)^2$, ordinates at $x = 1$, $x = 3$ and x-axis. Also evaluate the same by using trapezoidal rule. **[2007, Spring] [1999] [2001]**

$y = (2x+1)^2$

3. Evaluate $\int_0^2 (x^2 + 1)\,dx$ by Trapezoidal and Simpson's rule taking $n = 4$. Compare your result with exact value. [2002]

4. Evaluate: $\int_0^2 8x^3\,dx$ by Trapezoidal and Simpson's rule with $n = 4$. Compare your results with exact value. [2002]

5. Find the approximate area using Simpson's and Trapezoidal Rule for the region bounded by $y = x^2 + 1$, x–axis, ordinates at $x = 1$ and $x = 5$ taking number of subintervals 4(i.e., $n = 4$) and compare those with exact area using define integral. [2004, Fall]

6. Find the approximate area using Simpson's and Trapezoidal rule for the area bounded by the curve $y = 2x^2 + 1$, the x–axis and the lines $x = 1$ and $x = 5$ (using $n = 4$) and compare these results with exact value. [2005, Fall]

7. Use Simpson's rule with $n = 4$ to approximate the area between the curve $y = 2x - 1$, ordinates at $x = 1$, $x = 3$ and x–axis. Also evaluate the same by using trapezoidal rule and hence compare these results with exact values. [2005, Spring]

8. Evaluate: $\int_0^{2\pi} \sin x\,dx$ using Simpson's rule with $n = 4$ and compare it with the exact value. [2006, Fall]

9. Find the approximate area using Simpson's and Trapezoidal rules for the area bounded by curve $y = x^2 + 3$, the x–axis and the lines $x = 1$ and $x = 5$ (using $n = 4$) and compare these results with exact values. [2006, Spring]

10. Evaluate: $\int_0^2 \frac{1}{x^3 + 1}\,dx$ by Trapezoidal rule and Simpson's rule taking $n = 4$. [2008, Spring]

11. Find the approximate area using Simpson's and Trapezoidal rule for the area bounded by the curve $y = x^2 + 2$, the x–axis and lines $x = 1$ and $x = 5$(using $n = 4$) and compared with exact value. [2003, Fall]

12. Approximate the integral $\int_1^4 \frac{dx}{x + 1}$ with $n = 4$, using Trapezoidal and Simpson's rule. [2015 Spring][2012 Fall]

13. Evaluate: $\int_1^5 (2x^2 + 1)\,dx$ using Simpson's and Trapezoidal rule with $n = 4$ and compare it with the exact value. [2013 Fall]

14. Find approximate value of $\int_1^3 (2x + 1)^2\,dx$, using Simpson's and Trapezoidal rule with $n = 4$. Compare these values with exact value. [2013 Spring]

15. Using Trapezoidal rule and Simpson's rule, estimate the integral $\int_0^4 \frac{1}{x^2 + 4}\,dx$ with $n = 4$ subintervals. [2008, Spring]

16. Find approximate value of $\int_1^2 \frac{1}{x}\,dx$ using Trapezoidal and Simpson's rule with $n = 10$ and then compare the results with the exact value of the integral. [2014, Spring]

17. Use Trapezoidal and Simpson's rule with $n = 6$ to approximate the area between the curve $y = (2x + 1)^2$ ordinates $x = 1$, $x = 4$ and x-axis. Compare the result with exact value. [2014, Fall]

18. Evaluate: $\int_0^{\pi} \sin x\,dx$ using Trapezoidal rule, Simpson's rule and compare it with the exact value taking $n = 6$. [2017 Spring]

19. Find the approximate value of $\int_2^5 (x^2 + 1)\,dx$ using Trapezoidal and Simpson's rule taking with $n = 6$. Also, compare the results with exact value. [2017 Fall]

20. Find the approximate area using Simpson's and Trapezoidal rule for the area bounded by the curve $y = x^2 + 4$, the x-axis and lines $x = 1$ and $x = 4$(using $n = 6$) and compared with exact value. [2016, Fall]

21. Use Trapezoid Rule and Simpson's Rule, estimate the integral $\int_0^4 \left(\frac{1}{x^2 + 1}\right) dx$ with $n = 6$. [2016 Spring]

22. Find the approximate value of $\int_0^3 (x^2 + 1)\,dx$ by Trapezoidal and Simpson's rule taking with $n = 6$. Compare the results with exact value. [2018 Fall]

23. Approximate the integral $\int_1^4 \frac{dx}{x + 1}$ with $n = 6$, using Trapezoidal and Simpson's rule. Also, compare with exact. [2018 Spring]

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