COMP3702
Artificial Intelligence

Tutorial 5: Logic

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- A convenient abstraction for dealing with many states.
- Regardless of whether there's a natural notion of "near" or not (i.e. not a metric space), we can use logic to group different states together.
- For example:
  - I have a laptop $\implies$ includes any brand and model.
  - There is a laptop on the table $\implies$ can be at any position on the table.

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- **Model** An interpretation that satisfies the constraints. (Often we don't want to just find a model, but we want to know what is true in all models.
- **Proposition** A statement that is true or false in each interpretation.
- The **formal language representation** and **reasoning system** is made up of:
    - *Syantax* describes what sentences are legal/illegal
    - *Semantics* Specifies the meaning of symbols
    - *Atom* A symbol, starting with a lowercase letter
    - *Definite clause* An atom or rule of the form $atom \impliedby sentence$
    - *Knowledge base* A set of definite clauses. A knowledge base is true iff every definite clause within it is True in every model.

Syntax of Propositional Logic

- Complex propositions (sentences) can be built from simpler propositions using logical connectives, such as:
  - Brackets ()
  - Negation $\neg$
  - And; Conjunction $\wedge$
  - Or; Disjunction $\vee$
  - Implication $\implies$
  - Biconditional/ Equivalence $\iff$

Truth table

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \implies B$	$A \iff B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Lecture example

$$KB = \begin{cases} p \Leftarrow q \\ q \\ r \Leftarrow s \end{cases}$$

	p	q	r	s
I_1	**TRUE**	**TRUE**	**TRUE**	**TRUE**
I_2	**FALSE**	**FALSE**	**FALSE**	**FALSE**
I_3	**TRUE**	**TRUE**	**FALSE**	**FALSE**
I_4	**TRUE**	**TRUE**	**TRUE**	**FALSE**
I_5	**TRUE**	**TRUE**	**FALSE**	**TRUE**

Is a model?

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$$\begin{array}{|c|c|c|}
p \iff q & \text{TRUE} \iff \text{TRUE} & \text{TRUE} \\
q & \text{TRUE} & \text{TRUE} \\
r \iff s & \text{TRUE} \iff \text{TRUE} & \text{TRUE}
\end{array}$$

Since each clause is TRUE, I_1 is a model of the knowledge base KB.

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For I_2 we have p=FALSE, q=FALSE, r=FALSE and s=FALSE

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$p \iff q$	FALSE $\iff$ FALSE	TRUE
q	FALSE	FALSE
$r \iff s$	FALSE $\iff$ FALSE	TRUE

Since not every clause of I_2 is true, this is not a model of the knowledge base KB

Model example

For I_3 we have p=TRUE, q=TRUE, r=FALSE and s=FALSE

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$$\begin{array}{|c|c|c|}
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q & \text{TRUE} & \text{TRUE} \\
r \iff s & \text{FALSE} \iff \text{FALSE} & \text{TRUE}
\end{array}$$

Since each clause is TRUE, I_3 is a model of the knowledge base KB.

Exercise 5.1

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Mr Jones finds three trunks A, B, and C in a cave. Based on studying the history of where these trunks came about, he knows that one trunk contains gold, while two are empty. On the wall of the cave, he found three clues: “A is empty”, “B is empty”, and “gold is in B”. From studying the historical social norm of the villages around the cave, Mr Jones knows that only one of the clues is true, while the other two are false”. Which trunk has the gold? (source: http://disi.unitn.it/ ldkr/ml2014/ExercisesBooklet.pdf)

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- Hint: Break down the problem into:
  - Atoms (symbols, that we need to evaluate the truth)
  - Logical statements

  Then evaluate truth of those statements

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    - Associate an atom with the situation we wish to validate (i.e., gold in trunk)
    - Encode Mr Jones' knowledge of the situation into a set of logical sentences
    - Use these sentences to determine if there is a solution, where one of the chests is guaranteed to contain gold.

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  - Use these sentences to determine if there is a solution, where one of the chests is guaranteed to contain gold.
- We begin by associating a variable for each of the trunks containing gold:
  - A: trunk A contains gold
  - B: trunk B contains gold
  - C: trunk C contains gold

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$$S_1 = (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$$

$(A \wedge \neg B \wedge \neg C)$ Trunk A contains gold, while B and C are empty
$(\neg A \wedge B \wedge \neg C)$ Trunk B contains gold, while A and C are empty
$(\neg A \wedge \neg B \wedge C)$ Trunk C contains gold, while A and B are empty

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Since Mr Jones knows that only one of these are true, and the other two are false, we now need to consider each possibility to form our second logical sentence.

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$$S_2 = (\neg A \wedge \neg(\neg B) \wedge \neg B) \vee (\neg(\neg A) \wedge \neg B \wedge \neg B) \vee (\neg(\neg A) \wedge \neg(\neg(B) \wedge \neg B)$$

Exercise 5.1 - Solution

We now use both sentences to determine if there is a situation where one of the chests is guaranteed to contain gold.

$$S_1 = (A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C) \lor (\neg A \land \neg B \land C)$$

$$S_2 = (\neg A \land \neg(\neg B) \land \neg B) \lor (\neg(\neg A) \land \neg B \land \neg B) \lor (\neg(\neg A) \land \neg(\neg(B) \land \neg B)$$

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1. Through the elimination of double-negatives, we get:
   $S_2 = (\neg A \land B \land \neg B) \lor (A \land \neg B \land \neg B) \lor (A \land B \land B)$
2. We know that $X \land \neg X = FALSE$ (as we can't have the same variable / atom be true and false at the same time)
   $S_2 = (\neg A \land FALSE) \lor (A \land \neg B \land \neg B) \lor (A \land B \land B)$
3. We additionally know that $A \land A = A$ and can use this fact to simplify our expression:
   $S_2 = FALSE \lor (A \land \neg B) \lor (A \land B)$
4. Additionally, we know that $FALSE \lor A = A$.
   $S_2 = (A \land \neg B) \lor (A \land B)$
5. By the distributivity law, we know that, the above statement is equivalent to:
   $S_2 = A \land (\neg B \land B)$
6. We know that $(\neg X \land X) = T$

   $S_2 = A$ Therefore, to satisfy S_2, A must be true (i.e. the gold is in trunk A)

Exercise 5.1 - Solution

A	B	C	S_1	S_2
F	F	F	F	F
F	F	T	T	F
F	T	F	T	F
F	T	T	F	F
T	F	F	T	T
T	F	T	F	T
T	T	F	F	T
T	T	T	F	T

Exercise 5.2

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Exercise 5.2. Are the following entailments correct? Please provide the proof.

a) $(A \land B) \models (A \Leftrightarrow B)$

b) $(A \Leftrightarrow B) \models (A \land B)$

Hint: When we are asked to determine if an entailment is correct (or holds, or is true) we can convert the entailment into an implication, and check if the implication is valid. Checking whether or not the implication is valid means solving a validity problem. Remember that a sentence is valid when every combination of variable assignments in the sentence causes it to be true.

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We can convert the entailment $(A \wedge B) \models (A \iff B)$ to an implication of the form $(A \wedge B) \implies (A \iff B)$

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A	B	A∧B	$A \iff B$	$(A \wedge B) \implies (A \iff B)$
F	F	F	T	T
T	F	F	F	T
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F	F	F	T	T
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Since all of the rows of the implication column of the truth table is true, the implication is valid, and thus the original entailment is correct.

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F	T	F	F	T
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Since not every row of the implication column is true, the implication is not valid, and thus the original entailment is not correct.

Resolution Refutation

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- Convert all sentences to Conjugative Normal Form (CNF)
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- Apply the resolution rule, until we either:
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  - Can't apply the rule anymore.

The resolution refutation rule is sound and complete (for propositional logic):

- If we derive a contradiction, the conclusion follows from the axioms (proof by contradiction)
- If we can't apply any more, the conclusion cannot be proven from the axioms (no entailment; invalid statement)

Conjugative Normal Form

Conjunctions of Disjunctions

Statements of the form $(A \vee B) \wedge (C \vee D)$

In CNF:

- Clause A disjunction of literals $(A \vee B)$
- Literals Variables, or the negation of variables e.g. $A, \neg A, B$

Conjugative Normal Form

Suppose we have the statement $(A \vee B) \implies (C \implies D)$ hat we want to convert to CNF – to do this, we need to follow three key steps

- Eliminate arrows, using the rule $A \implies B = \neg A \vee B$

$$\neg(A \vee B) \vee (\neg C \vee D)$$

- Drive in negations

$$(\neg A \wedge \neg B) \vee (\neg C \vee D)$$

- Distribute OR over AND

$$(\neg A \vee \neg C \vee \neg D) \wedge (\neg B \vee \neg C \vee D)$$

Exercise 5.3

Exercise 5.3. Please use resolution refutation to show

a) $(P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R)$ (same as the lecture)

b) $(P \wedge \neg P) \models R$

(Note: negation can be represented by "$\neg$" or "$\sim$" or "!" or just plain "not")

Refutation is like a proof by contradiction – assume that the statement that we want to prove is false, and then derive a contradiction (show that the knowledge base cannot be true, for the negated statement we want to prove)

Exercise 5.4

Exercise 5.4. UQPark is a theme park with 5 rides: Bumper cars, carousel, haunted class, roller coaster, and ferris wheel, where each ride can be turned on and off independently of the other rides. After performing a cost-benefit analysis, UQPark Management decided that only 3 rides should be open at any given day, and the set of rides that are open/closed must satisfy the following constraints:

1. Either bumper cars or carousel must be open.
2. If bumper cars is closed, then roller coaster must open.
3. If carousel is open, then either bumper cars or haunted class must be open too.
4. If haunted class is open, then ferris wheel must be open too.
5. Bumper cars and ferris wheel cannot both be open at the same day.
6. If roller coaster is open, then ferris wheel must be open too.
7. If roller coaster is closed, then either haunted class or ferris wheel must be open.

UQPark Facilities thinks there is no combination of the rides that can satisfy all of Management's constraints.

(a) Please frame this problem as a satisfiability problem with propositional logic representation.

(b) Please solve the problem in (a) using DPLL. If both Management and Facilities are correct or both are incorrect, please also explain why using DPLL.

2021 Exam question

Question 2. **(10 marks)**

Michael went exploring in a cave and found himself trapped in a room with three doors labelled (A), (B) and (C), from left to right. Behind one of the doors is a path to the exit. Behind the other two doors, however, is a dragon. Opening a door to a dragon results in almost certain death. On each door there is some text:

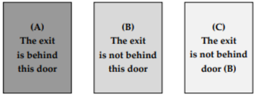

Figure 1: Three doors (A, B, and C respectively)

We know that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false. Please use the following language in solving the problem:

- a: "the exit is behind door A"
- b: "the exit is behind door B"
- c: "the exit is behind door C"

- Convert the knowledge base into propositional logic statements
- Determine which door would lead Michael to safety