Multivariable Calculus

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1 Vectors

- **Parametric equations** are a way to rewrite all the coordinates in terms of one parameter. The parametric equation of a **line** passing through p_0 and parallel to vector $\mathbf{v}$ is $\overrightarrow{OP_0} + t\mathbf{v}$.
- The **magnitude** of a vector is $\sqrt{\mathbf{v} \cdot \mathbf{v}}$.
- One thing to note is that dot product is equal to $\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$.
- **Projection**:
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{b}}\mathbf{a}$$
- **Cross product**: Set up the determinant matrix, remember $ad - bc$.
  - Length of cross product is equal to area of spanned parallelogram, which is $\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$.
  - Area of triangle is $\frac{1}{2}\|a \times b\|$.
- **Plane** that passes through fixed point p_0 and perpendicular to **normal vector v** has equation $\overrightarrow{p_0 p} \cdot n = 0$.
  - $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
  - Distance between two planes is just $\|\text{proj}_{\mathbf{n}}\overrightarrow{p_1 p_2}\|$, where those are two points on different planes and $\mathbf{n}$ is one of the normal vectors.
- **Coordinate systems** don't have to be our normal dimensional thing.
  - **Polar**: $\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{cases}$, **Spherical**: $\begin{cases} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{cases}$

2 Multivariable differentiation

- A **function** is **onto** if its range is the whole codomain, and **one-to-one** if no two inputs map to one output. Write as $f : \mathbb{R} \to \mathbb{R}^2$.
- **Image** of f, use set notation $\{\text{codomain} \mid \text{condition}\} = \text{range} \subseteq \text{codomain}$, **graph** is $(\text{input}, \text{output})$ together.
  - $z = x^2 + y^2$ (elliptic paraboloid), $z = x^2 - y^2$ (hyperbolic paraboloid), $z^2 = x^2 + y^2$ (elliptic cone)
- **Limits**: you remember these.
  - How do we do limits in $\mathbb{R}^3$? Have to consider approaching from every angle around the point. Use polar coordinates, **squeeze theorem** ($g(x) \leq f(x) \leq h(x)$, $\lim_{x\to a} g(x) = \lim_{x\to a} h(x)$).
  - Function is not **differentiable** if discontinuous, undefined, or sharp turn.
- **Partial derivatives**: treat all except one dimension as constants, differentiate as normal.
  - **Tangent plane**:

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-a) + f(a,b)$$

- **Directional derivative**: Given $\nabla f = (f_x, f_y, \ldots)$ (which is max directional derivative bc orthogonal), derivative in direction of $\mathbf{v}$ is $\overrightarrow{\nabla f} \cdot \mathbf{v}$ (must be unit vector!!).
- **Chain rule**: construct dependency graph

3 Vector-Valued Functions

- **Vector-valued functions**: parametric function of a path in $\mathbb{R}^n$
  - **Arclength reparametrisation**: $L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\| dt$. Note that $\mathbf{t}'$ is orthogonal to $\mathbf{t}$.
  - **Unit tangent vector**: $\mathbf{T} = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$
  - **Binormal**: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, $\mathbf{N} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$
  - **Curvature**: $\kappa = \frac{\|d\mathbf{T}/dt\|}{ds/dt}$
- **Vector field**: $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$
  - A **gradient vector field** is the gradient of a function: $\nabla f(\mathbf{x})$
  - A **flow line** of a vector field is such that $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$
  - $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$, $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$. If curl is 0 everywhere, then **irrotational**.

4 Maxima and Minima

- **Taylor polynomial**:

5 Multiple Integration

- **Fubini's theorem** says you can flip the order of the integrals.
- **Triple**:

$$\iiint_W f\ dV = \int_a^b \int_{\gamma(x)}^{\delta(x)} \int_{\psi(x,y)}^{\phi(x,y)} f(x, y, z)\ dz\ dy\ dx$$

- **Jacobian**:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1}$$

- **Change of variable**:

$$\iint_D f(x, y) dx\ dy = \iint_{D*} f(u(x, y), v(x, y)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\ dv$$

6 Line Integrals

- **Scalar line integral**:

$$\int_{\mathbf{x}} f\ ds = \int_a^b f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| dt$$

- **Vector line integral**:

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt$$

  - For a closed path, this is the **circulation**.

- **Green's theorem**: D is a region with a differentiable boundary C, such that D is to its left. Given $\mathbf{F}(x, y) = (M(x, y), N(x, y))$:

$$\oint_C M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy$$

  Note you can break up the curve as long as all of it combined is closed.

- **Path-independent** vector field means that all line integrals are only dependent on the endpoints, not the path itself. If $\mathbf{F} = (M(x, y), N(x, y))$'s domain is simply-connected and $\nabla \times F = 0$ then it is conservative. Or you flip partials for 2d: $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. Or, $\operatorname{curl} \mathbf{F} = 0$.

- **Finding scalar potential**: $\nabla f = \mathbf{F}$