Lecture 1 (30-07-2021)

30 July 2021 09:28

Automata on infinite words

Some notations

- Let Σ be a finite nonempty set (called alphabet).
- A finite word (over Σ) is a finite sequence (possibly empty) of letters from Σ.

$$\omega = a_0 a_1 \cdots a_n, \qquad a_i \in \Sigma.$$

ϵ is the empty word.

Σ^* is the set of all finite words (over Σ).

- An infinite word over Σ is an infinite sequence of letters from Σ.

$$\alpha = a_0 a_1 a_2 \cdots, \qquad a_n \in \Sigma \quad \forall n \in \mathbb{N}_0.$$

(Different formality: $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $\alpha : \mathbb{N}_0 \to \Sigma^*$)

Also, $\omega := \mathbb{N}_0$.

Σ^ω = all infinite words (on Σ)

(In general, given sets X and Y, Y^X denotes the set of all functions $X \to Y$)

Examples ① $\Sigma = \{a, b\}$

$$\alpha = a\ b\ ab\ ab \cdots$$

or: $\alpha(n) = \begin{cases} a & ; 2 \mid n \\ b & ; 2 \nmid n \end{cases}$

② $\alpha = abb\ abb\ abb$

$$\alpha(n) = \begin{cases} a & ; 3 \mid n \\ b & ; 3 \nmid n \end{cases}$$

$$\alpha = (abb)^\omega$$

③ $\gamma : \omega \to \{a, b\}$

$$\gamma(n) = \begin{cases} a & ; n \text{ is prime} \\ b & ; \text{otherwise} \end{cases}$$

$$\gamma = \underset{0}{b}\,\underset{1}{b}\,\underset{2}{a}\,\underset{3}{a}\,\underset{4}{b}\,\underset{5}{a}\,\underset{6}{b}\,\underset{7}{a}\,\underset{8}{b}\,\underset{9}{b}\,\underset{10}{b}\,\underset{11}{a} \cdots$$

γ has infinitely many 'a's and 'b's.
(Can't write as compactly as before.)

④ $\Sigma = \{a\}$, $\Sigma^\omega = \{a^\omega\}$. $\quad (|\Sigma^\omega| = 1.)$

But if $|\Sigma| > 1$, then Σ^ω is not a countable set.

OTOH, Σ^* is always a countable set. $\quad (1 \le |\Sigma| < \infty)$

Automata:

$$A = (Q, \Sigma, q_0, \Delta \subseteq Q \times \Sigma \times Q, \text{"Acceptance condition"})$$

→ a finite set of states

→ $q_0 \in Q$ → the initial state (unique)

→ $\Delta \subseteq Q \times \Sigma \times Q$ → the transition relation

$$(q, a, q') \in \Delta \equiv q \xrightarrow{a} q'$$

(Non-determinism here is fine)

Now, let $\alpha = a_0 a_1 a_2 \cdots \in \Sigma^\omega$ be given.

A run ρ of A on α is an infinite sequence of states

(run)

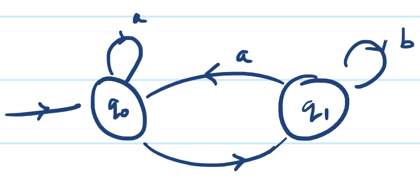
$$\rho = q_0 q_1 q_2 \cdots$$

Such that "q_0" is indeed the intial state and

$$\forall i \in \omega : (q_i, a_i, q_{i+1}) \in \Delta$$

(In terms of functions: Given $\alpha : \omega \to \Sigma$, we have $\rho : \omega \to \Sigma$ s.t. $\rho(0) = q_0$ and $(\rho(n), \alpha(n), \rho(n+1)) \in \Delta \ \forall n \in \omega$.)

Example.



$\alpha = (ab)^\omega$

$\rho = q_0 (q_1 q_0)^\omega$

$\rho = q_0 (q_0 q_1)^\omega$

"Acceptance condition": (Büchi automata)

$\alpha \to$ input word

$\rho \to$ a run of A on α

$\mathrm{Inf}(\rho) :=$ the set of states which occur infinitely often along ρ

$$= \{ q \in Q : \exists^\infty i \in \omega \text{ s.t. } \rho(i) = q \}$$

Obs. $\mathrm{Inf}(\rho) \neq \emptyset$. (There are only finitely many states.)

Büchi automaton (BA): Fix $G \subseteq Q$ called the "good states".

A run ρ is accepted by a BA if $\mathrm{Inf}(\rho) \cap G \neq \emptyset$.

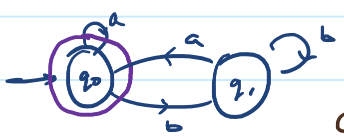
(Thus, some good state appears infinitely often.)

A word $\alpha \in \Sigma^\omega$ is accepted by A if α has an accepting run ρ on the word α.

$L(A) := \{ \alpha \in \Sigma^\omega : A \text{ accepts } \alpha \}$

$\hookrightarrow$ Language of A

Example

$A =$ 

$G = \{q_0\}$, $\Sigma = \{a, b\}$

Claim. $L(A) = \{ \alpha \in \Sigma^\omega :$ α has inf.

many 'a' }.

Proof. Let the right side be L.

- $L(A) \subseteq L$:

$\alpha \in L(A)$, $\alpha \equiv a_0 a_1 a_2 \cdots$

Note that A is deterministic, thus α has a unique run ρ, which is accepted.

$\rho \equiv q_0 q_1' q_2' q_3' \cdots$

Thus, q_0 appear inf. often above. Since it only receives 'a', we see that 'a' appears inf. often.

- $L \subseteq L(A)$:

Let $\alpha \in L$. It has a unique run ρ.

Then, since α has inf. many 'a's, ρ will have inf. many 'q_0's. ∎

Can also write $L = (b^* a)^\omega$ once we have defined what that means.

Question What about $\bar{L} = \Sigma^\omega \setminus L$? Can that be accepted by a Büchi automaton?

Lecture 2 (04-08-2021)

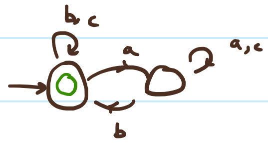
04 August 2021 09:31

Note. We do NOT allow ϵ transitions in this course.

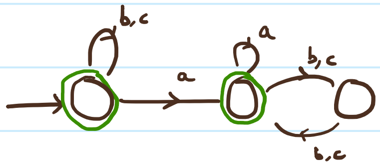
Fact. Even though we insisted on single initial state, the expressive power does not change if we allow more. (It is simply for convenience.)

Examples (1) L_1 over $\Sigma = \{a, b, c\}$.

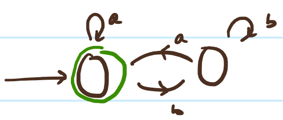
L_1 = every 'a' is eventually followed by a 'b'



(2) L_2 = any two occurrences of 'a' are separated by even no. of other (b, c) letters

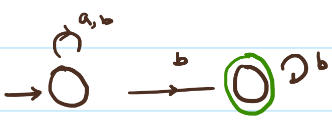


(3) $\Sigma = \{a, b\}$, L = inf. many 'a's



Complement : $\bar{L} = \Sigma^\omega \setminus L$ = finitely many 'a's

Q. What is a BA for $\bar{L}$?



Q. Do we have a deterministic Büchi automaton (DBA) for $\bar{L}$?

Note that we had an DBA for L :

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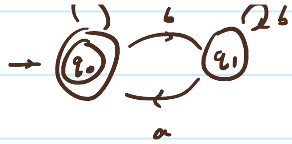


A : $G = \{q_0\}$

Toggle states : A' : $G = \{q_1\}$

But $\underset{\text{infinitely many } b}{L(A')} \overset{?}{\supsetneq} \underset{\text{eventually } b}{\bar{L}}$.

$(ab)^\omega \in L(A')$ but $(ab)^\omega \notin \bar{L}$

Complementing the good state of a DBA does <u>**NOT**</u> complement the accepted language.

Claim. There is No DBA for $\bar{L} = \{\alpha \in \Sigma^\omega :$ α has finitely many 'a'$\}$.

Thus, as opposed to finite languages, non-determinism actually gives us more languages.

Proof. We prove this by contradiction.
For the sake of contradiction, assume that $\exists$ DBA A such that $L(A) = \bar{L}$.
Suppose A has m states.

$\alpha_0 = b^\omega = b\,bb\cdots \in \bar{L}$

$\rho_0 = q_0 q_1 q_2 \cdots$

↳ unique run of α_0

Since ρ_0 is accepting, $\exists\, n_1$ s.t. $q_{n_1+1} \in G$. Pick the smallest such n_1.

$\rho_0 = \overbrace{b\ b\ \cdots\ b}^{n_1} \mid \boxed{\ } b\ b\ \cdots$

↑ first good state

Define $\alpha_1 := b^{n_1}\, a\, b^\omega \in \bar{L}$.

$\alpha_1 = b \cdots b^{n_1} \; a \; b \; b \cdots$

$\rho_1 = \square \cdots \square \; \square \cdots \square$

$\in G$ — again a good state

Then, we can get n_2 s.t. $b^{n_1} a b^{n_2} a$ ends at a good state.

Then, $\alpha_2 = b^{n_1} a b^{n_2} a b^{\omega} \in L$.

Its unique run ρ_2 matches ρ_1 until $b^{n_1} a b^{n_2} a$.

Keep getting $n_1, n_2, n_3, \dots, n_{m+1}$.

$\alpha_m = b^{n_1} a b^{n_2} a \cdots b^{n_{m+1}} a b^{\omega} \in L$.

$\rho_m = \underbrace{\square_{\in G} \;\; \square_{\in G} \;\; \cdots \;\; \square_{\in G}}_{m+1 \text{ states}}$

By PHP, two of these $m+1$ good states are equal. Loop between them to get a word which is accepted but has inf. many 'a's. ∎

Cor. DBA $\subsetneq$ NBA in terms of expressiveness.

Def^n A language $L \subseteq \Sigma^{\omega}$ is said to be *ω-regular* if there exists a (possibly non-deterministic) Büchi automaton A such that $L(A) = L$.

Closure Properties of ω-Regular Languages

1) <u>closure under union</u>:

$L_1 = L(A_1)$, $\quad A_1 = (Q_1, q_0^1, \Sigma, \Delta_1, G_1)$,

$L_2 = L(A_2)$, $\quad A_2 = (Q_2, q_0^2, \Sigma, \Delta_2, G_2)$.

To-do: Construct a BA A s.t. $L(A) = L_1 \cup L_2$.

We do the usual product construction.

$$(Q_1 \times Q_2,\ (q_0^1, q_0^2),\ \Sigma,\ \underbrace{\Delta}_{},\ G_1 \times Q_2 \cup Q_1 \times G_2)$$

$$(q_1, q_2) \xrightarrow{a} (q_1', q_2')$$

iff $\quad q_1 \xrightarrow{a} q_1'$ and $q_2 \xrightarrow{a} q_2'$.

$\alpha = a_0\ a_1\ a_2\ \cdots$

$\rho^1 = q_0^1\ q_1^1\ q_2^1\ \cdots$ a run of A_1 on α

$\rho^2 = q_0^2\ q_1^2\ q_2^2\ \cdots$ —— " —— A_2 — " —

"$\rho^1 \times \rho^2$" $= \begin{pmatrix} q_0^1 \\ q_0^2 \end{pmatrix} \begin{pmatrix} q_1^1 \\ q_1^2 \end{pmatrix} \begin{pmatrix} q_2^1 \\ q_2^2 \end{pmatrix} \cdots$ a "product run" on α

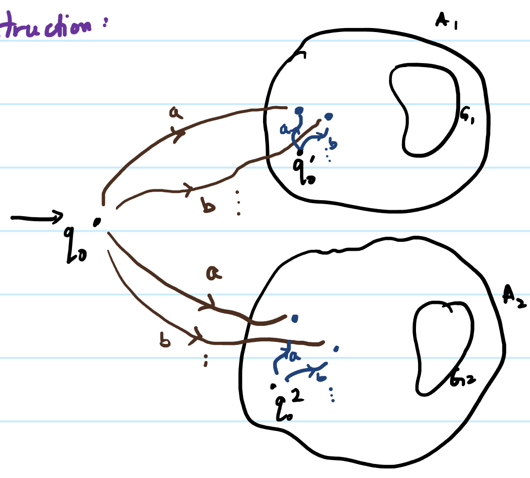
Here we assume that each $\alpha \in \Sigma^\omega$ has at least one run on both A_i. (Can always ensure this by adding a dead state)

With the above assumption,

$$G = (G_1 \times Q_2) \cup (Q_1 \times G_2) \text{ gives}$$

the language as $L_1 \cup L_2$.

A simpler construction:



Closure under intersection.

Do the same product construction as earlier and put
$$G = G_1 \times G_2.$$

$A \equiv A_1 \times A_2.$

<u>Is</u>: $L(A) = L(A_1) \cap L(A_2)$.

($\subseteq$) If $\rho = \rho_1 \times \rho_2$ is an accepting run, so ρ_1 and ρ_2 both are.

($\supseteq$) Let $\alpha \in L(A_1) \cap L(A_2)$.

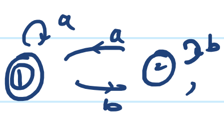
Then there are accepting runs ρ_i on A_i.

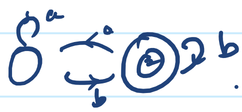
Put $\rho = \rho_1 \times \rho_2$.

But then it is not necessary that ρ is accepting.

For example, ρ_1 has good states at even positions and ρ_2 at odd.

As a concrete example of above:

Take A_1 = ,

A_2 = .

Then $(ab)^\omega \in \big(L(A_1) \cap L(A_2)\big) \setminus L(A_1 \times A_2)$.

Doesn't work! Slightly modified.

$Q = Q_1 \times Q_2 \times \{1, 2\}$, → indicates the component being "searched" for a good state

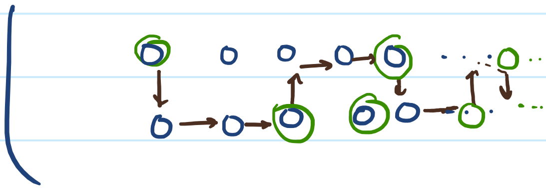
$q_0 = (q_0^1, q_0^2, 1)$

$\Delta = (q_1, q_2, 1) \xrightarrow{a} (q_1', q_2', 1)$ if $q_1 \xrightarrow{a} q_1' \in \Delta_1$, $q_2 \xrightarrow{a} q_2' \in \Delta_2$, $q_1 \notin G_1$

$$(q_1, q_2, 1) \xrightarrow{a} (q_1', q_2', 2) \qquad \begin{matrix} q_1 \xrightarrow{a} q_1' \\ q_2 \xrightarrow{a} q_2' \\ q_1 \in G_1 \end{matrix}$$

similarly for $(, , 2) \to (, , 2)$
$(, , 2) \to (, , 1)$.

$$G = G_1 \times Q_2 \times \{1\}.$$



$$L(A) = L(A_1) \cap L(A_2).$$

Closure under projection:

$\pi : \Sigma_1 \times \Sigma_2 \longrightarrow \Sigma_1$ induces $\pi : (\Sigma_1 \times \Sigma_2)^\omega \longrightarrow \Sigma_1^\omega$

$$\binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots \mapsto a_0 a_1 a_2 \cdots$$

If $L \subseteq (\Sigma_1 \times \Sigma_2)^\omega$ is ω-regular, so is $\pi(L)$.
Let $A = (Q, q_0, \Sigma_1 \times \Sigma_2, \Delta, G)$ be a BA with $L(A) = L$.
Goal: Construct B s.t. $L(B) = \pi(L)$.

Define $B = (Q, q_0, \Sigma_1, \Delta', G)$, where

$$\Delta' = \{ q \xrightarrow{a} q' : \exists b \in \Sigma_2 \text{ s.t. } q \xrightarrow{(a,b)} q' \text{ in } A \}.$$

(Take original automata and erase all the second components.)

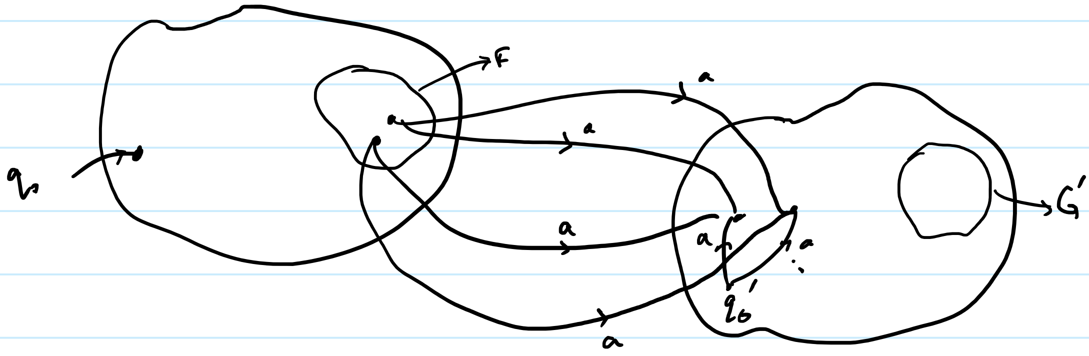
$U \subseteq \Sigma^*$, $L \subseteq \Sigma^\omega$, $U \cdot L \subseteq \Sigma^\omega$.

(The concat. of a finite word followed by an infinite word is defined in the natural way.)

(Nonsense: $\text{Concat} : \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ or $\text{concat} : \Sigma^\omega \times \Sigma^* \to \Sigma^\omega$.)

Closure : $U \subseteq \Sigma^*$ regular $\quad A = (Q_0, q_0, \Sigma, \Delta, F)$, → NFA, $L(A) = U$

$L \subseteq \Sigma^\omega$ ω-regular $\quad B = (Q_0', q_0', \Sigma', \Delta', G')$. ↳ BA, $L(B) = L$



Keep them disjoint and all possible transitions of the form:

$$q_f \xrightarrow{a} q' \quad \text{where } q_f \in F$$
and $q_0' \xrightarrow{a} q'$ in Δ'.

Keep G as G'. #

Give $U \subseteq \Sigma^*$, define

$$U^\omega = \{ \alpha \in \Sigma^\omega : \alpha \text{ has a factorisation of the form}$$
$$\alpha = \alpha_0 \alpha_1 \alpha_2 \alpha_3 \cdots \text{ for } \alpha_i \in U. \}$$

Closure If $U \subseteq \Sigma^*$ is regular, then U^ω is ω-regular.

Let $A = (Q, q_0, \Sigma, \Delta, F)$ recognise U.

Assume that there are no incoming transitions to q_0 and that $q_0 \notin F$.

(Why can we do this?)

(Also note $U^\omega = (U \setminus \{\varepsilon\})^\omega$.)

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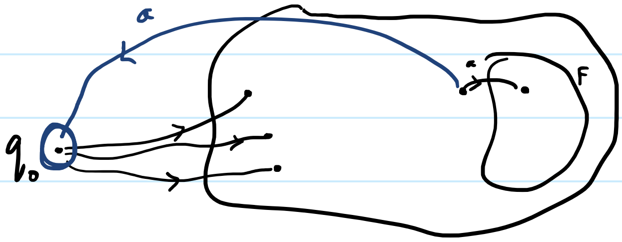


Add all possible transitions of the form:

$$q \xrightarrow{a} q_0 \quad \text{if} \quad \exists q_f \in F \text{ s.t. } q \xrightarrow{a} q_f.$$

Put $G = \{q_0\}$.

Lecture 4 (11-08-2021)

11 August 2021 09:33

To Do : Closure under complementation.

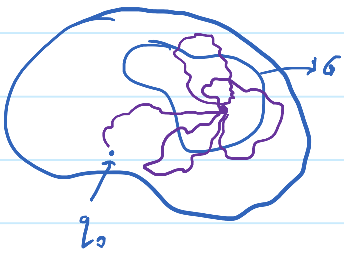
Prop^n. Let L be ω-regular. Then, L can be expressed as

$$L = \bigcup_{i=1}^{n} U_i \cdot V_i^{\omega},$$

where $U_i, V_i \subseteq \Sigma^*$ are regular languages for $i = 1, \ldots, n$.

(By our earlier results, it is clear that any such L is indeed ω-regular.)

Proof. Let A be a BA s.t. $L(A) = L$.
$\overset{\shortparallel}{(Q, \Sigma, q_0, \Delta, G)}$



Given an accepted word $\overset{\in L(A) = L}{\alpha} = a_0 a_1 a_2 \cdots$ with an accepting run $\rho = q_0 q_1 q_2 \cdots$,
$\exists\, g \in G$ which occurs i.o.

$$\underbrace{q_0 q_1 q_2 \cdots}_{u} \; \underbrace{\overset{\uparrow}{g} \cdots}_{v_0} \; \underbrace{\overset{\uparrow}{g} \cdots}_{v_1} \; \underbrace{\overset{\uparrow}{g} \cdots}_{v_2} \cdots$$

For $g \in G$, define

$$U_g := \{ w \in \Sigma^* : \exists \text{ a run } q_0 \xrightarrow{w} g \}$$
$$V_g := \{ w \in \Sigma^* : \exists \text{ a run } g \xrightarrow{w} g \}.$$

regular since $A_g := (Q, \Sigma, q_0, \Delta, \{g\})$ and
$B_g := (Q, \Sigma, \{g\}, \Delta, \{g\})$ accept them.

Now, by our earlier observation, it is easy to argue that

$$L = \bigcup_{g \in G} U_g \cdot V_g^{\omega} .$$

Obs. The following problem is decidable: (Non emptiness problem)

- Input :– $A \to$ a BA
- Output :– YES if $L(A) \neq \emptyset$,

  NO if $L(A) = \emptyset$.

$\left(L(A) \neq \emptyset \Leftrightarrow \exists g \in G \text{ s.t. } \exists\, q_0 \xrightarrow{w} g \text{ and } \exists\, g \xrightarrow{w} g . \right)$

reachable from initial state — check if part of cycle

both efficient

Obs. If $L(A) \neq \emptyset$, then there exist finite words u and v s.t. $|u|, |v| \leq |Q|$ and $u \cdot v^{\omega} \in L(A)$.

ultimately periodic

Let $A = (Q, \Sigma, q_0, \Delta, G)$ be a BA accepting L.

Goal: To show that $\overline{L} = \Sigma^{\omega} \setminus L$ is also ω-regular.

For $u, v \in \Sigma^*$, define

$u \sim_A v \Leftrightarrow \forall q, q' \in Q$, $q \xrightarrow{u} q'$ iff $q \xrightarrow{v} q'$ and $q \xrightarrow{u}_G q'$ iff $q \xrightarrow{v}_G q'$.

[Notation: $s \xrightarrow{x}_G s'$ means $\exists$ a run on x from s to s' with an intermediate visit to G.]

Observations:

(i) $\sim_A$ is an equivalence relation on Σ^*

(i) $\sim_A$ is an equivalence relation on Σ^*.

(ii) $\sim_A$ is of finite index, i.e., it has finitely many equivalence classes.

Proof. Fix $q, q' \in Q$.

$$U_{q,q'} = \{ w \in \Sigma^* : q \xrightarrow{w} q' \}.$$

$$V_{q,q'} = \{ w \in \Sigma^* : q \xrightarrow[G]{w} q' \}$$

$2n^2$ such sets. $(n := |Q|)$

For each $u, v \in \Sigma^*$, we can ask $2n^2$ questions about set membership. $u \sim_A v \iff$ they have same answers.

Thus, there are $\leq 2^{2n^2}$ classes.

$$[u]_{\sim_A} = \left(\bigcap_{\substack{q,q' \in Q \\ u \in U_{q,q'}}} U_{q,q'} \right) \cap \left(\bigcap_{\substack{q,q' \in Q \\ u \in V_{q,q'}}} V_{q,q'} \right) \cap \left(\bigcap_{\substack{q,q' \in Q \\ u \notin U_{q,q'}}} \overline{U_{q,q'}} \right) \cap \left(\bigcap_{\substack{q,q' \in Q \\ u \notin V_{q,q'}}} \overline{V_{q,q'}} \right).$$

The above discussion also shows that each equivalence class is a regular language.

($U_{q,q'}$ is clearly regular. Some argument shows the same for $V_{q,q'}$.)

Let $U_1, \ldots, U_m$ be the equivalence classes of $\sim_A$.

Lemma Suppose $L \cap \left(U_i \cdot U_j^{\omega} \right) \neq \emptyset$ for some i, j, then $U_i \cdot U_j^{\omega} \subseteq L$.

Proof. Let $\alpha \in L \cap (U_i \cdot U_j^{\omega})$.

$$\alpha = a_0 a_1 a_2 a_3 \cdots \in L.$$

Let $\rho = q_0 q_1 q_2 q_3 \cdots$ be an accepting run of A

Let $\rho = q_0\, q_1\, q_2\, q_3 \cdots$ be an accepting run of A on α.

We can also write $\alpha = u \cdot v_0 \cdot v_1 \cdot v_2 \cdots$ s.t. $u \in U_i$ and $v_0, v_1, v_2, \ldots \in U_j$.

$$\rho_\alpha = q_0 \xrightarrow{u} q_1' \xrightarrow{v_0} q_2' \xrightarrow{v_1} q_3' \rightarrow \cdots$$

Now, let $\beta \in U_i\, U_j^\omega$. Then, $\beta = u'\ v_0'\ v_1' \ldots$.

Then, we have a run

$\rho_B = q_0\ q_1'\ q_2'\ q_3' \cdots$ by def^n of $\sim_A$.

Moreover if ρ_A saw a good state $q_i' \rightarrow q_{i+1}'$, so does ρ_B.

$\therefore \beta \in L(A)$. ∎

Lecture 5 (13-08-2021)

13 August 2021 09:36

Ramsey's Theorem

$\mathbb{N} = \{0, 1, 2, \dots\}$

$E = \{(i,j) \in \mathbb{N} \times \mathbb{N} : i < j\}$. } Complete graph on $\mathbb{N}$

$\mathcal{C}$ — a finite set of colours.

$\chi : E \to \mathcal{C}$ is called an edge-colouring of the complete graph on $\mathbb{N}$.

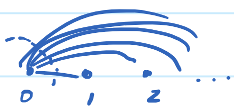
Thm. Given an arbitrary χ, $\exists$ an infinite monochromatic clique in χ.
That is,

$\exists S \subseteq \mathbb{N}$, $|S| = \infty$, $\exists c \in \mathcal{C}$ such that every edge within S is coloured 'c'.

$$(\forall i,j \in S : i < j \Rightarrow \chi((i,j)) = c.)$$

Proof. Fix $\chi : E \to \mathcal{C}$.

$X_0 := \mathbb{N}$, $m_0 := \min(X_0)$ $(= 0)$.



$\exists c_0 \in \mathcal{C}$ s.t. $\exists$ infinitely many x s.t. $\chi((m_0, x)) = c_0$.

Let $X_1 := \{$neighbours of m_0 in $X_0\} \subseteq X_0 \setminus \{m_0\}$.

Note: $X_1 \subseteq \mathbb{N}$ is infinite.

Let $m_1 := \min(X_1)$ and proceed similarly to pick c_1 and $X_2 \subseteq X_1 \setminus \{m_1\} \dots$

In general, we have an infinite subset X_{k+1} and colour c_k s.t. every element of X_{k+1} is connected to $\min(X_k)$ by c_k.

Define $X_\infty := \{m_0, m_1, m_2, \dots\}$. $(m_0 < m_1 < m_2 < \dots)$

Then, X_∞ is an infinite set s.t. $\forall i,j : \chi((m_i, m_j)) = c_i \ \forall i < j$.

As usual, $\exists\, c \in \mathcal{C}$ which occurs infinitely many often.
Simply restrict graph to those vertices.

Continuing from last lecture: $U_1, \ldots, U_m$ are equiv. classes of $\sim_A$.
We saw: U_i are regular.

<u>Claim</u>. $\Sigma^\omega = \bigcup_{i,j} U_i \cdot U_j^\omega$.

<u>Proof</u>. Only $(\subseteq)$ is to be shown.
Let $\alpha \in \Sigma^\omega$ be arbitrary.
<u>TS</u>: $\exists i,j$ $\alpha = u_0 v_0 v_1 v_2 \cdots$ for $u_0 \in U_i$ and $v_k \in U_j \ \forall k$.

Write $\alpha = a_0 a_1 a_2 a_3 \cdots \in \Sigma^\omega$ for $a_i \in \Sigma$.

Define the colouring χ_α on $(\mathbb{N}, \in)$ as: $\mathcal{C} = \{U_1, \ldots, U_m\}$

$$\chi_\alpha((i,j)) = [a_i a_{i+1} \cdots a_{j-1}]_{\sim_A}.$$

equiv class of $a_i a_{i+1} \cdots a_{j-1}$

By Ramsey's theorem, $\exists U_j$ with a clique, i.e., $\exists m_1 < m_2 < m_3 < \cdots$
s.t. $\chi_\alpha((x_k, x_{k+1})) = U_j \quad \forall j$.

Defining $u_0 = a_0 \cdots a_{m_1 - 1}$, $v_0 = a_{m_1} \cdots a_{m_2 - 1}$,
$v_1 = a_{m_2} \cdots a_{m_3 - 1}, \ldots$
does the job. ∎

$$\Sigma^* = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_m,$$
$$\Sigma^\omega = \bigcup_{i,j} U_i \cdot U_j^\omega.$$

Note that U_i are regular. Moreover, we have

$$L \cap (U_i \cdot U_j^\omega) \neq \phi \Rightarrow U_i \cdot U_j^\omega \subseteq L.$$

Thus, $L = \bigcup_{\text{some } i,j} U_i \cdot U_j^\omega.$

Thus, $\Sigma^\omega \setminus L = \bigcup_{i,j : U_i \cdot U_j^\omega \not\subseteq L} U_i \cdot U_j^\omega.$

Thus, it is again ω-regular. ∎

→ Effective construction of BA for $\overline{L}$.

- → Construct automaton for U_i.
- → Construct BA for $U_i \cdot U_j^\omega$.
- → Take union of those not in L.

(Can effectively check if $L \cap (U_i \cdot U_j^\omega) = \phi$.)

Lecture 6 (18-01-2021)

18 August 2021 09:31

Büchi's Theorem:

Want to talk about properties of words (finite or infinite).

First-order Logic (over words)

Fix $\Sigma \to$ alphabet.

First-order variables — $x, y, z, x_1, x_2, x_3, \ldots$ range over positions in the word

Atomic-predicate — $a(x), b(x), \ldots$ $\to$ unary predicate

for $a, b, \ldots \in \Sigma$ and x is a FO variable.

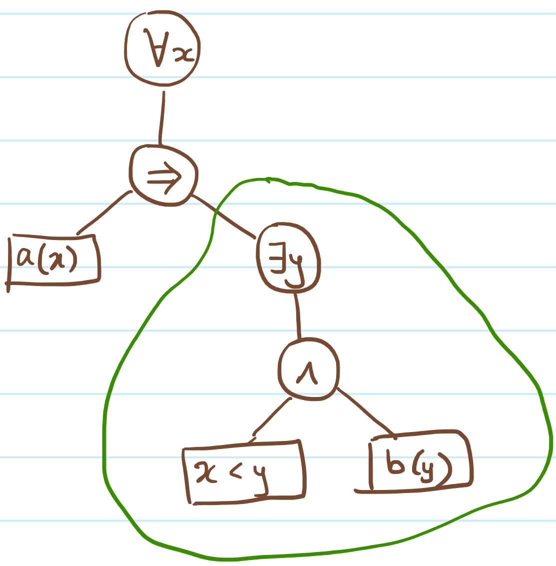
$x < y$ $\to$ binary predicate

Syntax:

$$\varphi \equiv a(x) \mid x < y \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

derived: $\varphi_1 \Rightarrow \varphi_2 \equiv \neg\varphi_1 \vee \varphi_2$

Example: $\varphi_1 \equiv \forall x. \left[a(x) \Rightarrow \exists y \, (x < y) \wedge b(y) \right]$



Semantics:

$\varphi(x_1, \ldots, x_m)$ — φ is a formula with free variables

$\varphi(x_1, \ldots, x_m)$ — φ is a formula with free variables $x_1, \ldots, x_m$

$\varphi' \equiv \exists y[(x<y) \wedge b(y)]$ → x free, y bound

$\varphi(x_1, \ldots, x_m)$, w → word

$w, x_1 \leftarrow p_1, \ldots, x_m \leftarrow p_m \models \varphi(x_1, \ldots, x_m)$

defined by structural induction

1. "$w, x_1 \leftarrow p_1 \models a(x_1)$" iff the letter in w at position p_1 is a

2. "$w, x_1 \leftarrow p_1, x_2 \leftarrow p_2 \models x_1 < x_2$" $\Leftrightarrow$ $p_1 < p_2$

$\vdots$

Example. ① $\varphi_2 \equiv \exists x \forall y [(x<y) \Rightarrow \neg a(y)]$.

If w $(\neq \varepsilon)$ is finite, then it will satisfy φ_2.

But if w is infinite, then $w \models \varphi_2 \Leftrightarrow w$ has finitely many 'a's.

② $\varphi_3 \equiv \forall x \exists y (x<y)$

if w is a (nonempty) finite word, then $w \not\models \varphi_3$.
OTOH, all infinite words satisfy this property

Büchi - Elgot Theorem → a logical characterisation of (finite) regular languages

Büchi Theorem → a logical characterisation of ω-regular languages

---

Monadic Second-Order Logic over words

extends FO - over words

position variables — $x, y, x_1, x_2, x_3, \dots$
sets-of-positions variables — $X, Y, X_1, X_2, X_3, \dots$
atomic-predicate — $a(x), x < y, X(x)$.

Syntax

$$\varphi \equiv \text{atomic-predicates} \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x \cdot \varphi \mid \forall x \cdot \varphi \mid \exists X \cdot \varphi \mid \forall X \cdot \varphi$$

$S(x,y) \equiv$ position y is successor of position x
$\equiv (x < y) \wedge \neg(\exists z \cdot (x < z) \wedge (z < y))$.

first $(x) \equiv x$ is the first position
$\equiv \forall y\, (x = y \vee x < y)$.

last $(x) \equiv \dots$

Remark. In FO, the '<' predicate cannot be expressed using 'S' predicate.

$x < y \iff$ $x \neq y$ and every successor-closed set of positions which contains x, also contains y

↳ can define in MSO

Thus, we can write '<' in terms of 'S' in MSO.

Def^n. Let φ be a MSO sentence.

$$L_\varphi = \{\alpha \in \Sigma^\omega : \alpha \models \varphi\}.$$

$L \subseteq \Sigma^\omega$ is called MSO-definable if $\exists$ MSO φ s.t. $L = L_\varphi$.

Theorem (Büchi's Theorem)

Let $L \subseteq \Sigma^\omega$.

L is MSO-definable $\iff$ L is ω-regular.

Lecture 7 (20-08-2021)

20 August 2021 09:37

Thm. (Buchi) Let $L \in \Sigma^\omega$.
L is ω-regular $\iff$ L is MSO-definable.

Proof. $(\Rightarrow)$ Suppose L is ω-regular, say $L = \mathcal{L}(A)$, where
$A = (Q, q_0, \Sigma, \Delta \subseteq Q \times \Sigma \times Q, G)$ is a BA.

Goal: Construct MSO sentence φ_A s.t.
$\forall \alpha \in \Sigma^\omega$: $\alpha \models \varphi_A \iff A$ accepts α.

$\alpha = a_0 a_1 a_2 a_3 a_4 \ldots$

Suppose A accepts α via an accepting run ρ.
$\rho = q_0 q_1 q_2 q_3 q_4 \ldots$

$\forall q \in Q$, $X_q \equiv$ the set of positions in α when this run ρ is in the state 'q'
$= \{i \in \mathbb{N} : q_i \in q\}$. $(0 \in \mathbb{N})$

Note that $\{X_q\}_{q \in Q}$ is a partition of $\mathbb{N}$. (Allowing $\emptyset$ in partition.)

1) $0 \in X_{q_0}$
2) for any two consecutive positions x and y, if $x \in X_q$, $y \in X_{q'}$, then $(q, a, q') \in \Delta$, where a is the letter at position x.
3) for any position x, there is a position y to the right of x such that $y \in X_q$ for some $q \in G$.

Conversely, given a partition with above 3 properties, we can build an accepting run.

For convenience, write $Q = \{0, 1, \ldots, m\}$.
$\uparrow$ initial

$$\varphi_A \equiv \exists X_0 \, \exists X_1 \ldots \exists X_m \, . \, \text{partition}(X_0, \ldots, X_m) \wedge$$
$$[\forall x \, . \, \text{first}(x) \Rightarrow X_0(x)] \wedge$$
$$\left[\forall x \forall y \; S(x,y) \Rightarrow \left(\bigvee_{(i,a,j) \in \Delta} X_i(x) \wedge X_j(y) \wedge a(x) \right)\right] \wedge$$
$$\left[\forall x \; \exists y \; (x < y) \wedge \left(\bigvee_{i \in G} X_i(y) \right)\right].$$

$$\text{partition}(X_0, X_1, \ldots, X_m)$$
$$\equiv \forall x \left(\bigvee_{i \in Q} X_i(x) \wedge \bigwedge_{i \neq j} \neg(X_i(x) \wedge X_j(x)) \right).$$

$\text{Length}(\varphi_A) = O(|A|).$

($\Leftarrow$) Given: MSO sentence φ.
Goal: Construct BA A s.t.
$$L(A) = \{\alpha \in \Sigma^\omega \mid \alpha \models \varphi\}.$$

As in the finite case, we use MSO_0-logic
- $\hookrightarrow$ substitute position variables by singleton set vars.
- $\hookrightarrow$ more atomic predicates: $\text{Sing}(X)$, $a(X)$, $S(X,Y)$, $X \subseteq Y$

$\text{Sing}(X)$: singleton

$S(X,Y)$: X, Y are Sing and the single positions are related by S

these can be defined in MSO.

The converse is true too. Thus, we use them interchangeably.
$\hookrightarrow$ That is, they have same expressive power.

Lecture 8 (25-08-2021)

25 August 2021 09:39

Goal: Given a MSO_0 formula, construct a BA A s.t. $L(A) = L(\varphi)$.
The construction of A proceeds by structural induction φ.

In fact: Let $\varphi(X_1, \dots, X_n)$ be an MSO_0-formula with free (set-) variables $X_1, \dots, X_n$

$$\alpha, \; X_1 \leftarrow P_1, \dots, X_n \leftarrow P_n \overset{?}{\models} \varphi$$

$$\begin{array}{llllllll} \alpha & = & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ P_1 & = & 1 & 0 & 0 & 1 & 1 & \dots \\ P_2 & = & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & & & & & & & \\ P_n & = & 1 & 0 & 1 & 0 & 1 & \dots \end{array}$$

$P_1 = \{0, 3, 4, \dots\}$

$P_2 = \phi$

$P_n = 2\mathbb{N} = \{0, 2, 4, \dots\}$

characteristic vectors

$$\begin{pmatrix} a_0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\begin{pmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} a_2 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\begin{pmatrix} a_3 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} a_4 \\ 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\dots$$

The model of φ (an inf. word Σ + n sets) can be seen as an inf. word over $\Sigma \times \{0,1\}^n$.

$$\mathrm{Free}(\varphi) = \{X_1, \dots, X_n\}.$$

$$L(\varphi) = \{ \alpha' \in (\Sigma \times \{0,1\}^n)^\omega : \alpha' \models \varphi \}.$$

Claim. $L(\varphi)$ is ω-regular over $\Sigma_n = \Sigma \times \{0,1\}^n$.

Proof. $\varphi \rightsquigarrow A_\varphi$ by structural induction.

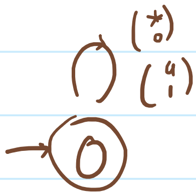
Base. φ – atomic predicate

- $\varphi = \mathrm{Sing}(X_1)$ $\qquad \alpha' = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\dots \qquad a_i \in \Sigma, \; b_i \in \{0,1\}$

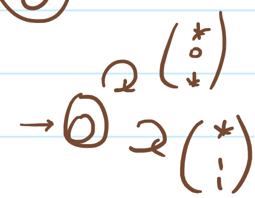
- $\varphi = \mathrm{Sing}(X_1)$ $\alpha' = \binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots$ $a_i \in \Sigma$, $b_i \in \{0,1\}$

- $\varphi = S(X_1, X_2)$; $\Sigma_2 = \Sigma \times \{0,1\} \times \{0,1\}$

- $\varphi = a(X_1)$



- $\varphi = X_1 \subseteq X_2$



<u>Inductive step.</u>

- $\varphi = \varphi_1 \vee \varphi_2$. $\underbrace{\mathrm{Free}(\varphi_1)}_{} \subseteq \{x_1, \dots, x_n\}$. ($= \mathrm{Free}(\varphi)$)

WLOG, we may assume $\mathrm{Free}(\varphi_1) = \{x_1, \dots, x_n\}$. ($= \mathrm{Free}(\varphi_2)$)

By induction, we have appropriate automata A_{φ_i} for φ_i.
But the alphabet for both is same. Can take union of DFAs.

- $\varphi = \varphi_1 \wedge \varphi_2$, $\varphi = \neg \varphi_1$ similarly done.

- $\varphi = \exists x_n \; \varphi'(x_1, \dots, x_n)$ $\mathrm{Free}(\varphi) = \{x_1, \dots, x_{n-1}\}$

Note: $\alpha' \models \varphi \iff \exists$ a set P_n s.t.
$\alpha'; x_n \leftarrow P_n \models \varphi'$

Consider the projection map

$$\pi : \Sigma \times \{0,1\}^n \longrightarrow \Sigma \times \{0,1\}^{n-1},$$
$$(a, b_1, \dots, b_n) \longmapsto (a, b_1, \dots, b_{n-1}).$$

This induces a map $\pi : (\Sigma_n)^\omega \to (\Sigma_{n-1})^\omega$.

$\alpha' \in (\Sigma_{n-1})^\omega$, $\alpha' \models \varphi \iff \exists\, \alpha'' \in (\Sigma_n)^\omega$ s.t.
$\pi(\alpha'') = \alpha'$ and $\alpha'' \models \varphi'$.

The question is reduced to asking if projection of an ω-regular language is ω-regular.
But this is simple to see. Take an automaton for φ' and erase the last coordinate on all transitions.

- $\varphi = \forall X_n.\, \varphi'$.

  Same as $\neg \exists X_n\, \neg \varphi'$. ∎

Thus, we are done.

Thm (Büchi's Theorem) Let $L \in \Sigma^\omega$. Then,
L is ω-regular $\iff$ L is MSO-definable.

Moreover, the translations are effective.

The above theorem was proven a few years after the Büchi-Elgot theorem (the analogous theorem about (finite) regular languages).
It is easy to see how MSO-definability translates to ω-words but was not so clear how to extend regularity.

Thus, MSO(Σ) is decidable.
Given an MSO sentence φ, we can check if there

exists an inf. word $\alpha \in \Sigma^\omega$ s.t. $\alpha \models \varphi$.

[$\varphi \rightsquigarrow A_\varphi$ is effective and we can check $L(A_\varphi) \neq \emptyset$.]

In fact, if $L(A_\varphi) \neq \emptyset$, then $\exists u, v$ s.t. $uv^\omega \in L(A_\varphi)$

and we can produce the above u, v.

Note: $\varphi \rightsquigarrow A_\varphi$ is NON-elementary.

We cannot bound $|A_\varphi|$ in terms of any (fixed) k-ary exponential of $|\varphi|$.

$\left(n = |\varphi|, \quad 2^{p(n)}, \quad 2^{2^{p(n)}}, \ldots \leftarrow \text{elementary} \right)$

$p \leftarrow$ polynomial

singly exp, doubly exp, ..., k-ary exp, ..

The tower (we get) will have length in terms of n.

can we do better for satisfiability?

FACT. There is a non-elementary type lower bound for MSO-satisfiability problem.

Note: $\varphi \rightsquigarrow A_\varphi \rightsquigarrow \varphi_{A_\varphi}$

This has a nice form

$\exists X_1 \cdots \exists X_n$ ("first-order type").

Lecture 9 (27-08-2021)

27 August 2021 09:34

"First-order theory" of arithmetic

$$(\mathbb{N}, +, \cdot, 0, 1)$$

- $add(x, y, z) \longrightarrow$ asserts $x + y = z$
- $mult(x, y, z) \longrightarrow$ asserts $xy = z$

Usual FO syntax.

- $zero(x) \equiv add(x, x, x)$
- $x < y \equiv \exists z \; add(x, z, y)$
- $S(x, y) \equiv x < y \land \neg \exists z \, (x < z \land z < y)$
- $one(x) \equiv \exists y \; (zero(y) \land S(y, x))$
- $prime(x) \equiv \neg one(x) \land \forall y \forall z \, (mult(y, z, x) \Rightarrow one(y) \lor one(z))$
- $even(x) \equiv \exists y \; add(y, y, x)$.

Goldbach's conjecture:

$$\varphi \equiv \forall z \; even(z) \Rightarrow \exists y \exists x \; prime(x) \land prime(y) \land add(x, y, z).$$

Given a sentence φ, we would like to known if φ is true.

Hilbert's belief: Perhaps, we can mechanically figure out the truth/falsity of φ.

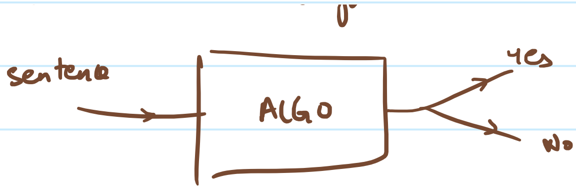
? what does this mean?

Church/Gödel/Turing: "Computability" ← defined

Moreover, it was shown that:

The first-order theory of arithmetic is undecidable/ non-computable

That is, there is no algorithm s.t.



Now, let us look at $(\mathbb{N}, +)$ → Presburger arithmetic.
First order and only add(x, y, z).

This IS decidable!

$(\mathbb{N}, +, <)$ → first-order theory

↳ Büchi showed that Presburger arithmetic is decidable using automata theory

$\varphi(x_1, \dots, x_n)$ → first-order formula

↑ encode $x_1, \dots, x_n$ in reverse binary order
↳ finite words over $\{0,1\}^n$.

→ **S1S** : $(\mathbb{N}, <)$ or $(\mathbb{N}, S)$

second order theory of 1 successor

Both same if monadic second order

(In fact, MSO + add gives mult, which we know is undecidable)

1960s:

$\boxed{W}$S1S : Subsets are only allowed to be finite subsets

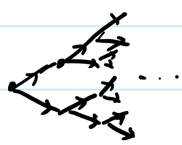
weak

Büchi showed that S1S is decidable

1970s: (Rabin's theorem)

S2S is decidable

two successors



MONA : Logic $\longrightarrow$ Automata

New modes of acceptance for automata on infinite words

$$A : (Q, \Sigma, q_0, \Delta \subseteq Q \times \Sigma \times Q, Acc)$$

<u>Büchi</u> : $G \subseteq Q$, ρ is accepting if $Inf(\rho) \cap G \neq \phi$.

<u>Muller</u> : $\mathcal{F} = \{F_1, \ldots, F_k\}$ is a collection of subsets of Q.
ρ is accepting if $\underbrace{Inf(\rho)}_{\text{"states at } \infty\text{"}} = F_i$ for some i.

<u>Rabin</u> : $\Omega = \{(E_1, F_1), \ldots, (E_k, F_k)\}$, each $E_i, F_i \subseteq Q$.
ρ is accepting if
$\exists i : Inf(\rho) \cap E_i = \phi$ and $Inf(\rho) \cap F_i \neq \phi$.

$$\bigvee_{i=1}^{k} \left[(Inf(\rho) \cap E_i = \phi) \wedge (Inf(\rho) \cap F_i \neq \phi) \right].$$

<u>Streett</u> : Dual to Rabin.
$\Omega = \{(E_1, F_1), \ldots, (E_k, F_k)\}$
ρ is accepting if

$$\bigwedge_{i=1}^{k} \left[(Inf(\rho) \cap E_i \neq \phi) \vee (Inf(\rho) \cap F_i = \phi) \right]$$

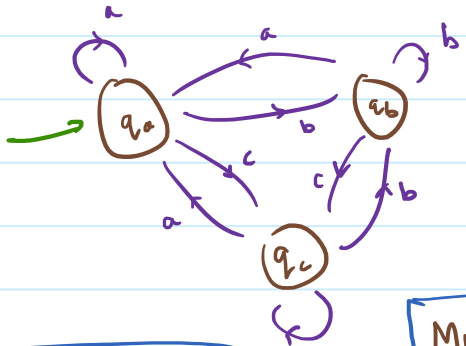
equivalently

$$= \bigwedge_{i=1}^{k} \left([Inf(\rho) \cap F_i \neq \phi] \Rightarrow [Inf(\rho) \cap E_i \neq \phi] \right).$$

If F_i is visited infinitely often, then so is E_i.

$\Sigma = \{a, b, c\}$

$L = \{\alpha \in \Sigma^\omega \mid$ if 'a' occurs inf. often in α, then so does 'b' $\}$.



Streett-condition:

$$\Omega = \{(\{q_b\}, \{q_a\})\}.$$

Muller-condition

$$\mathcal{F} = \{\{q_a, q_b\}, \{q_a, q_b, q_c\}, \{q_c\}, \{q_b\}, \{q_b, q_c\}\}$$

$$= \{X \subseteq Q : q_a \in X \Rightarrow q_b \in X\} \setminus \{\emptyset\}.$$

Note that putting $\emptyset$ in $\mathcal{F}$ makes no difference since $\mathrm{Inf}(\rho) \neq \emptyset$ $\forall \rho$.

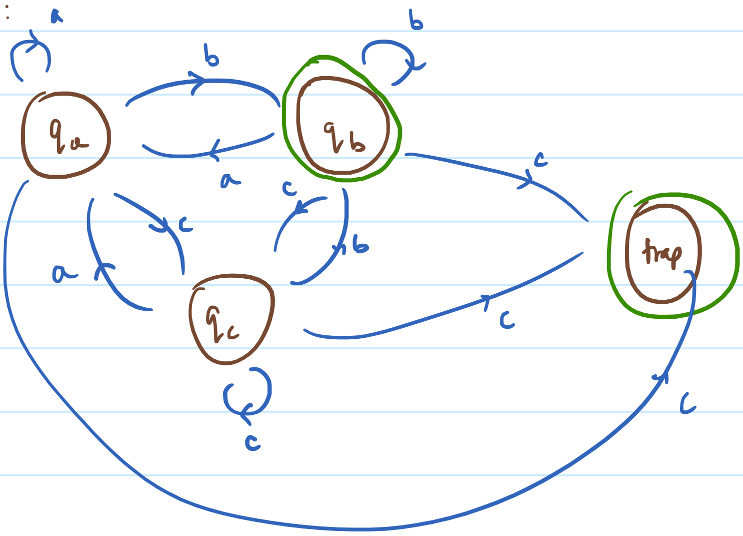
Rabin-condition

Note:

$L = \{\alpha \mid b$ occurs inf. often$\}$
$\cup \{\alpha \mid$ both a & b fin. often$\}$

$$\Omega = \{(\emptyset, \{q_b\}), (\{q_a, q_b\}, Q)\}$$

Büchi-condition:



Thm (McNaughton) DMA = DRA = DSA = NBA = ω-regular.

D = deterministic

N = non-D

M = Muller, R = Rabin, S = Streett, B = Büchi

A = Automaton

Lecture 10 (03-09-2021)

01 September 2021 09:58

Thm.

$$\begin{tikzcd}
 & NMA \arrow[dd] & \\
NRA \arrow[ur, hook] & & NSA \arrow[ul, hook] \\
 & NBA \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Thus, all are equivalent in terms of expressive power.

($X \hookrightarrow Y$: any lang. acc. by X can also be acc. by Y)

Proof.

- $NBA \hookrightarrow NRA$

  Let $A = (Q, \Sigma, q_0, \Delta, G) \leftarrow NBA$.

  For Rabin: Put $\Omega = \{(\phi, G)\}$.

- $NBA \hookrightarrow NSA$.

  For A as above, put $\Omega = \{(G, Q)\}$.

  For Muller: $\mathcal{F} = \{X \subseteq Q : X \cap G \neq \phi\}$.

- $NRA \hookrightarrow NMA$.

  Let $A = (Q, \Sigma, q_0, \Delta, \Omega)$ be a NRA.

  $\Omega = \{(E_1, F_1), \ldots, (E_k, F_k)\}$.

  $$\mathcal{F} = \{X \in Q : \exists i \text{ s.t. } X \cap E_i = \phi \text{ and } X \cap F_i = \phi\}.$$

- $NSA \hookrightarrow NMA$: Similar.

- $NMA \hookrightarrow NBA$.

  Let $A = (Q, \Sigma, q_0, \Delta, \mathcal{F})$ be a NMA.

  $\mathcal{F} = \{X_1, \ldots, X_m\}$.

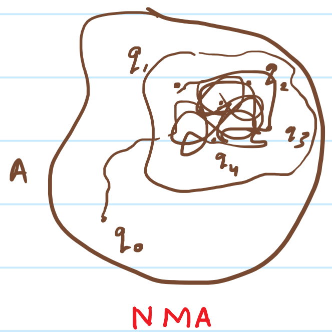
  For $i \in [m]$, define $A_i := (Q, \Sigma, q_0, \Delta, \{X_i\})$.

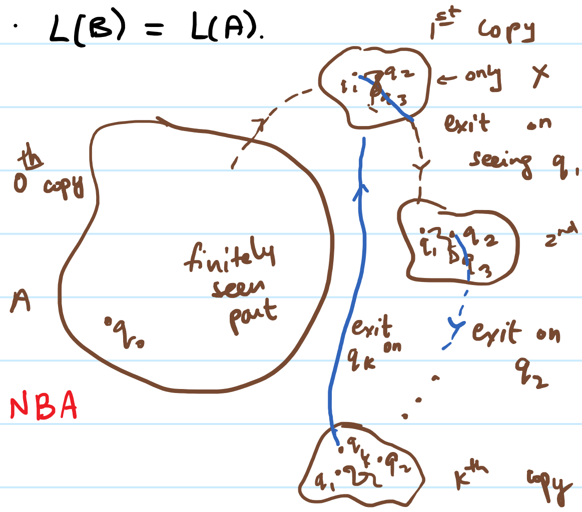
  Note: $L(A) = \bigcup_{i=1}^{k} L(A_i)$.

Since ω-reg. languages are closed under union, suffices to translate each A_i to an NBA. Equivalently we may assume $m = 1$.

$A = (Q, \Sigma, q_0, \Delta, \{X\})$, and write $X = \{q_1, \dots, q_k\}$.

Goal: Construct an NBA B s.t. $L(B) = L(A)$.





$$B = (Q', (q_0, 0), \Sigma, \Delta', \{(q_1, 1)\})$$

$$Q' = (Q \times \{0\}) \cup (X \times \{1, 2, \dots, k\}).$$

Δ' :

$(q, 0) \xrightarrow{a} (q', 0)$ if $q \xrightarrow{a} q'$

$(q, 0) \xrightarrow{a} (q', 1)$ if $q \xrightarrow{a} q'$ & $q' \in X$

$q, q' \in X \to$ $(q, i) \xrightarrow{a} (q', i)$ if $q \neq q_i$ & $q \xrightarrow{a} q'$

$(q, i) \xrightarrow{a} (q', \underbrace{i+1}_{k+1 \equiv 1})$ if $q = q_i$ & $q \xrightarrow{a} q'$

the arrows here indicate those in original Δ

Thus, all these models have same expressive power.

$$NRA \equiv NFA \equiv NSA \equiv NBA$$
$$\cup\!\!\!\!\!\!\; +$$
$$DBA$$

We had seen $L = \{\alpha \mid \alpha$ has finitely many 'a's$\} \in NBA \setminus DBA$.
(on $\Sigma = \{a, b\}$)

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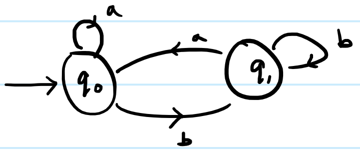


Rabin : $\Omega = \{(\{q_0\}, \{q_1\})\}$. Streett: $\Omega = \{(\emptyset, \{q_0\})\}$.

Muller condition : $\mathcal{F} = \{\{q_1\}\}$.

Then, L is accepted by this DMA.

In fact, from the proof of the last theorem, we get

$$\begin{tikzcd}
 & DMA & \\
DRA \arrow[ur] & & DSA \arrow[ul] \\
 & DBA \arrow[ul] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

(Recall that the translations in these cases did not change transition structure)

The L above shows that DMA $\not\hookrightarrow$ DBA.

The following is true:

$$\begin{array}{ccccccc}
NMA & \equiv & NRA & \equiv & NSA & \equiv & NBA \\
\parallel\!| & & \parallel\!| & & \parallel\!| & & \uparrow \not\equiv \\
DMA & \equiv & DRA & \equiv & DSA & \overset{\leftarrow}{\not\equiv} & DBA
\end{array}$$

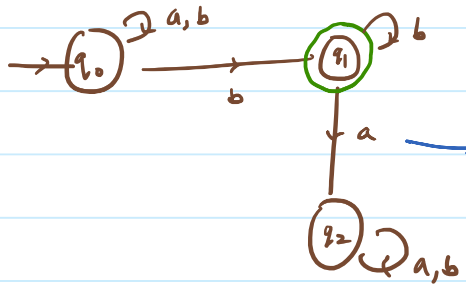
Lecture 11 (03-09-2021)

03 September 2021 09:28

Determinisation of Büchi automata:

Let $A = (Q, \Sigma, q_0, \Delta, G)$ be a NBA.

Finitely many 'a's :

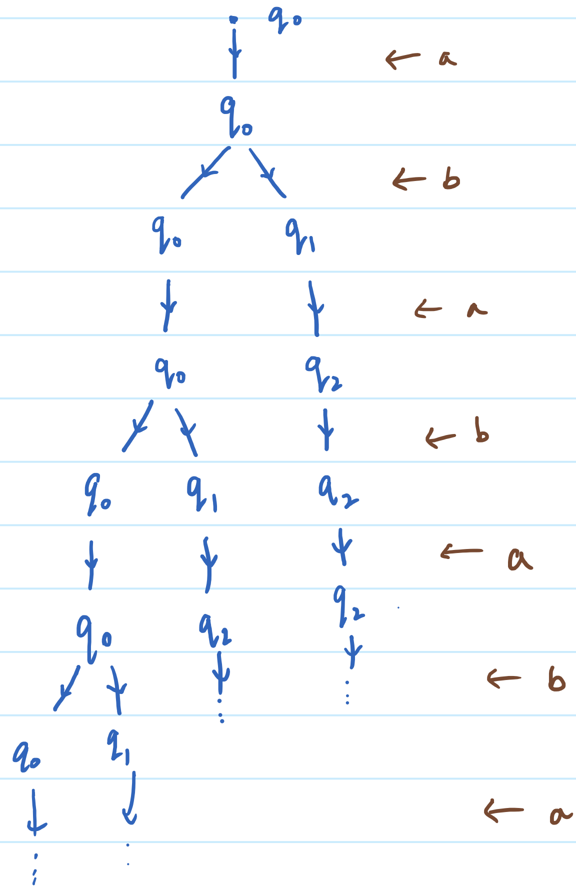


this ensures that every word has some run

Want : deterministic automaton accepting the above.

All runs of A on $\alpha \in \Sigma^\omega$ can be seen as a "run-tree" of A on α.

For example, take $\alpha = (ab)^\omega$.



A run ρ of A on α corresponds to an inf path in this run-tree.

Büchi acceptance ≡ ∃ an inf path where a good state appears infinitely often.

The subset / powerset construction:

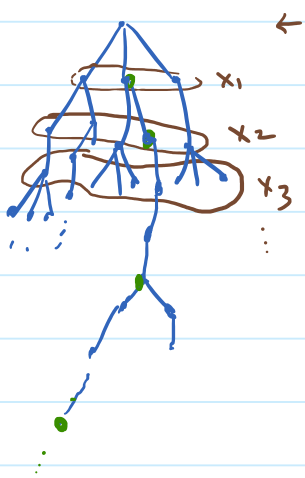
$$A_S := (2^Q, \Sigma, \{q_0\}, \delta_S, \text{acceptance}).$$

$$\delta_S : 2^Q \times \Sigma \to 2^Q \text{ is defined as}$$

$$\delta_S(X, a) := \{ q \in Q : \exists q' \in X, (q', a, q) \in \Delta \}.$$

(Look at X and take all states q s.t. $q' \xrightarrow{a} q$ for some $q' \in X$)

Suppose A accepts the word $\alpha = a_0 a_1 a_2 a_3 \cdots$



The run of α on A_S looks like: $X_0 \xrightarrow{a_0} X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3 \to \cdots$

There are infinitely many indices i s.t. X_i contains a good state.

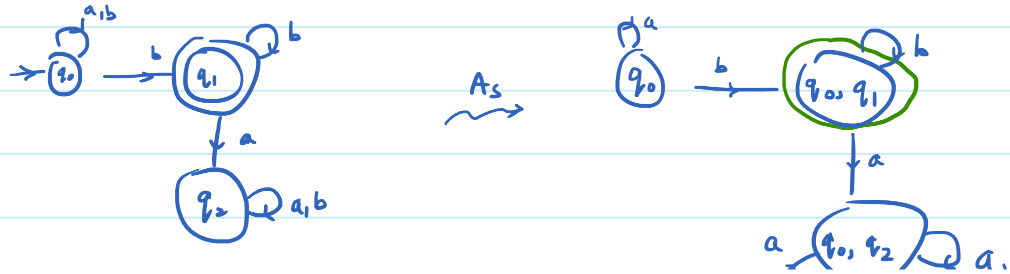
Define
$$G' := \{ X \subseteq Q : X \cap G \neq \emptyset \} \subseteq 2^Q.$$

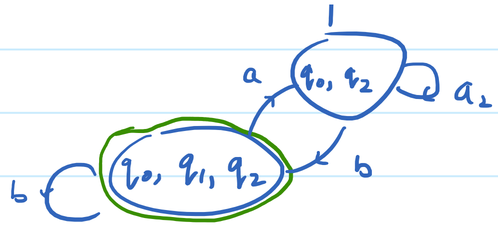
G good states

Observation: $L(A) \subseteq L(A_S, G')$.

But in general, $L(A) \subsetneq L(A_S, G')$ as we've already seen DBA ≢ NBA.

Recall





For $(ab)^\omega$, good states appear infitely many often.

(Along the original tree, there were infinitely many levels with a good state but no single path with infitely many good states.)

Key property : If $X \xrightarrow{u} Y$ in A_s, then
$\forall q \in Y \; \exists q' \in X$ s.t. $q' \xrightarrow{u} q$.

A more refined "acceptance" condition on the subset automaton:

Defn $X \xrightarrow[G]{u} Y$ if $\forall q \in Y \; \exists q' \in X$ s.t.
$q' \xrightarrow[G]{u} q$. (There is a run of A on u from q' to q which visits a good state.)

A run of A_s on α

$$X_0 \xrightarrow{a_0} X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3 \rightarrow \cdots$$

is said to be strongly accepting if $\exists i_0 < i_1 < i_3 < \cdots$ s.t.

$$X_{i_0} \xrightarrow[G]{u_1} X_{i_1} \xrightarrow[G]{u_2} X_{i_2} \xrightarrow[G]{u_3} \cdots$$

and $a_0 a_1 a_2 \cdots = u_0 u_1 u_2 \cdots$.

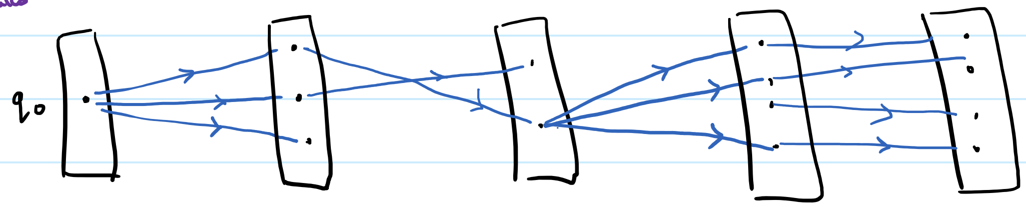
Lemma If A_s strongly accepts $\alpha \in \Sigma^\omega$, then A accepts α.

Proof. $\alpha = u_0 u_1 u_2 u_3 \cdots$

$$\vee \xrightarrow{u_0} \vee . \xrightarrow{u_1} \vee . \xrightarrow{u_2} \vee . \xrightarrow{u_3} \vee . \xrightarrow{u_4} \cdots$$

$$X_0 \xrightarrow{u_0} X_{i_0} \xrightarrow{u_1}_G X_{i_1} \xrightarrow{u_2}_G X_{i_2} \xrightarrow{u_3}_G X_{i_3} \xrightarrow{u_4}_G \cdots$$

write in terms of states

$\mathcal{G}$: 

All the arrows here ~~n~~ pass through a good state. (except at first level)

To show : $\exists$ an infinite path in the directed graph $\mathcal{G}$.

Obs. 1 : Every vertex of $\mathcal{G}$ is reachable from initial vertex.

Question. Let T be an infinite tree s.t. every vertex has finite deg. Does T contain an infinite path?
That is, does $\exists$ an inf seq $v_0, v_1, v_2, \ldots$
s.t. v_i is a child of v_{i-1}?

Ans. Yes. Keep picking a child s.t. the subtree below it is infinite. (Such a child exists since tree is infinite and # children is finite)

König's Lemma (A finitely branching infinite tree must have an infinite path.)

Applying the lemma to our graph yields the result. $\boxtimes$

Q. How to implement the "stronger acceptance"?

Marked-subset automaton:

$X \rightarrow$ the set of reachable states
$Y \rightarrow$ the set of states which can be

reached via good states

$$A_m = \left(2^Q \times 2^Q,\ \begin{matrix} (\{q_0\}, \phi) & \text{if } q_0 \notin G \\ (\{q_0\}, \{q_0\}) & \text{if } q_0 \in G \end{matrix},\ \Sigma,\ \delta_m\right).$$

$(Y \subseteq X)$

$$\delta_m((X, Y), a) = \begin{cases} \big(\delta_s(X, a),\ \delta_s(Y, a) \cup (\delta_s(X, a) \cap G)\big) & \text{if } X \neq Y \\ \big(\delta_s(X, a),\ \delta_s(X, a) \cap G\big) & \text{if } X = Y \end{cases}$$

$$G_m = \{ (X, X) : X \subseteq Q \}.$$

$$(X_1, X_1) \xrightarrow{u} (X_2, X_2) \Rightarrow \forall q \in X_2\ \exists q' \in X_1 \text{ s.t. } q' \xrightarrow{u}_G q.$$

Thus, $L(A_m, G_m) \subseteq L(A) \subseteq L(A_s, G')$.

Lecture 12 (08-09-2021)

08 September 2021 09:47

Determinisation

$A = (Q, \Sigma, q_0, \Delta, G) \longrightarrow$ NBA

Runtree : Given a word α, we have the runtree of A on α which computes all runs of A on α.

$A_s = (2^Q, \Sigma, \{q_0\}, \delta_s : 2^Q \times \Sigma \to 2^Q, G' = \{X : X \cap G \neq \emptyset\})$

The current state keeps track of the set of reachable states of A.

$L(A) \subseteq L(A_s)$

$A_m = (\{(X, Y) \in 2^Q \times 2^Q : Y \subseteq X\}, \Sigma, \text{initial}, \delta_m, \{(X, X) : X \in Q\})$

$$\delta_m((X,Y), a) = \begin{cases} (\delta_s(X, a), \delta_s(Y,a) \cup (\delta_s(X, a) \cap G)) & \text{if } X \neq Y \\ (\delta_s(X, a), \delta_s(X, a) \cap G) & \text{else} \end{cases}$$

Then, $L(A_m) \subseteq L(A)$.

Recall proof.

let $\alpha = a_0 a_1 a_2 a_3 \dots \in L(A_m)$

$$\rho_m = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \xrightarrow{a_0} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \xrightarrow{a_1} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \xrightarrow{a_2} \dots$$

ρ_m is acc. $\Rightarrow \exists$ inf many i s.t. $X_i = Y_i$.

$$\begin{pmatrix} X_i \\ \shortparallel \\ Y_i \end{pmatrix} \xrightarrow{u} \begin{pmatrix} X_j \\ \shortparallel \\ Y_j \end{pmatrix}$$

$$\begin{pmatrix} X_{i+1} \\ X_{i+1} \cap G \end{pmatrix} \to \dots$$

Y_j

$\shortparallel$

$\forall q \in X_j \; \exists q' \in X_i$ s.t. $q' \xrightarrow[G]{u} q$

$\forall q \in X_j \; \exists q' \in X_i$ s.t. $q' \xrightarrow[G]{u} q$

$X_0 \xrightarrow{u_0} X_{i_1} \xrightarrow[G]{u_1} X_{i_2} \xrightarrow[G]{u_2} X_{i_3} \xrightarrow[G]{u_3} \cdots$

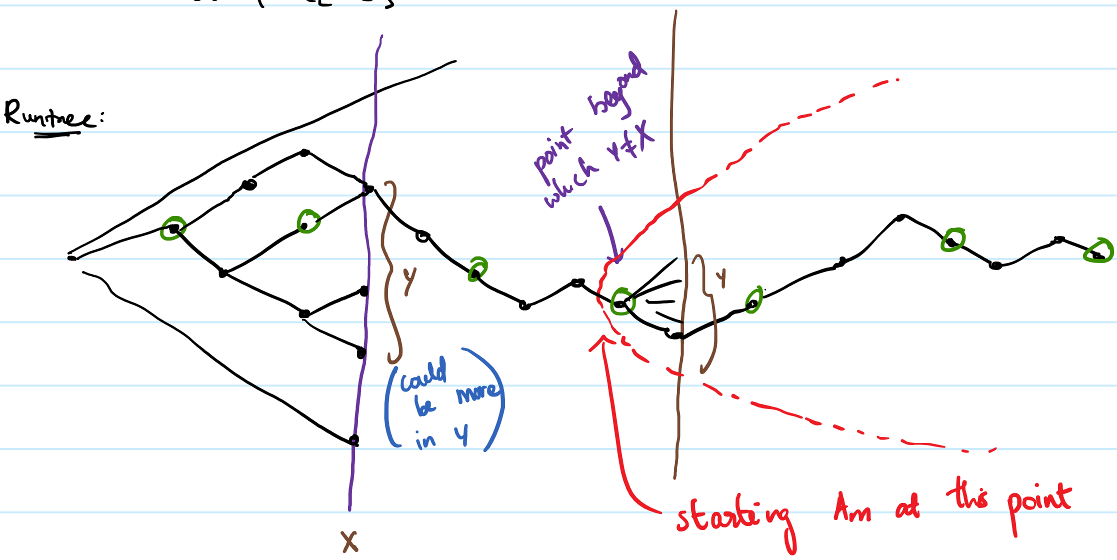
At this point, we applied Königs lemma to see that A has an acc. run on α. Thus, $\alpha \in L(A)$.

- $L(A_m) \subseteq L(A) \subseteq L(A_S)$.

Pick $\alpha \in L(A)$. Suppose $\alpha \notin L(A_m)$.

$\alpha = a_0 a_1 a_2 a_3 \cdots$

Runtree:



α not acc. by $A_m \Rightarrow$ beyond a point, ρ_m does not encounter (X, X).

Trying to show: $\forall$ word $\in L(A)$ $\exists$ some state s.t. starting A_m at that state results in accepting α.

Factor α as $u \cdot \beta$ s.t. 'u' allows A to go to a good state g, and the A_m started at 'g' accepts β.

Lecture 13 (29-09-2021)

29 September 2021 09:39

Determinisation of BA continued

A — NBA $\longrightarrow A = (Q, q_0, \Sigma, \Delta, G)$

A_s — subset automaton $\longrightarrow G_m = \{X : X \cap G \neq \phi\}$

A_m — marked subset automaton $\longrightarrow G_m = \{(X, X) : X \in 2^Q\}$

$L(A_m) \subseteq L(A) \subseteq L(A_s)$.

If $\alpha \in L(A)$, then $\exists$ a factorisation of α as
$\alpha = u \cdot \beta$
and a good state g of A s.t.

$$q_0 \xrightarrow[A]{u} g \quad \text{and} \quad (\{g\}, \{g\}) \xrightarrow[A_m]{\beta} \quad \text{accepts } \beta,$$

i.e., A_m accepts when started at 'g'.

Prop^n. Let $L = L(A)$ be an ω-regular language. Then, L can be written as

$$L = \bigcup_{i=1}^{n} U_i \cdot \overrightarrow{V_i}, \quad \text{where } U_i, V_i \text{ are regular languages.}$$

Recall: $\overrightarrow{V} := \lim V := \{\alpha \in \Sigma^\omega : \text{inf many prefixes of } \alpha \text{ are in } V\}$.

Also, if V is accepted by a DFA, then $\overrightarrow{V}$ is accepted by the same automata interpreted as a DBA.

Earlier, we had seen: Every ω-regular L can be written as
$$\underbrace{\bigcup}_{\text{finite union}} U V^\omega$$

for regular languages $\mathcal{U}$ and V.

$L = L(A)$, $\quad A = (Q, \Sigma, q_0, \Delta, G)$.

$\alpha \in L(A)$, then $\alpha = u \cdot \beta$ where $q_0 \xrightarrow[A]{u} g$

and β is accepted by the Am started at g.

$$U_g := \{ u \in \Sigma^* : q_0 \xrightarrow[A]{u} g \}.$$

$$V_g = \{ v \in \Sigma^* : v \text{ is acc. by Am starting at } (\{g\}, \{g\}) \}.$$

interpreting it as a DFA

Then, $\alpha \in U_g \cdot \overrightarrow{V_g}$.

In fact,

$$L = \bigcup_{g \in G} U_g \cdot \overrightarrow{V_g}.$$

Next class: Construct det Rabin automaton to accept $U \cdot \overrightarrow{V}$.

<u>Exercise</u>: Show that det Rabin —"— are closed under union.

Thus, we conclude that every ω-reg. language can be accepted by a det. Rabin automaton. (In turn, Muller as well.)

Lecture 14 (01-10-2021)

01 October 2021 09:30

Suppose U and V are regular languages with corresponding DFAs given by

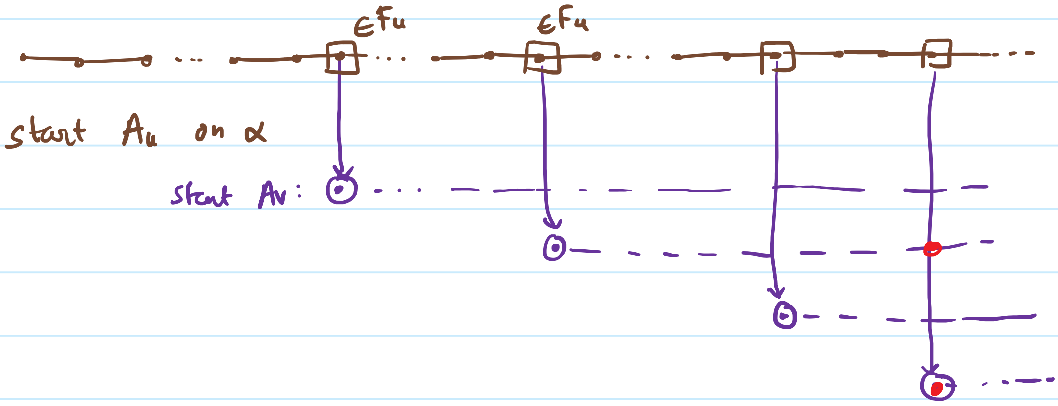
$$A_U = (Q_U, q_0^U, \Sigma, \delta_U, F_U),$$
$$A_V = (Q_V, q_0^V, \Sigma, \delta_V, F_V).$$

Goal: To construct a deterministic ω-automaton (of appropriate kind) which accepts $U \cdot \vec{V}$ and is closed under finite unions.

Let $\alpha \in U \cdot \vec{V}$.

Then, we can write $\alpha = u \cdot \beta$, where $u \in U$ and $\beta \in \vec{V}$. i.e., ∃ infinitely many prefixes of β in V.

$\alpha = a_0 a_1 a_2 a_3 a_4 a_5 \cdots$



If we made the guess to jump from A_U at the correct stage, we are done.

Also note: if we are on the red dot on the two A_Vs, then the run from that point is identical for both. (Since deterministic)

Thus, we need to run at most $O(|Q_V|)$ copies of A_V at a time.

Here is the automaton:

States: $S = Q_U \times (Q_V \times \{\perp\})^{N+1}$

$= \{(q_U, q_V^1, \dots, q_V^{N+1}) : q_U \in Q_U,\ q_V^i \in Q_V \cup \{\perp\}\}$

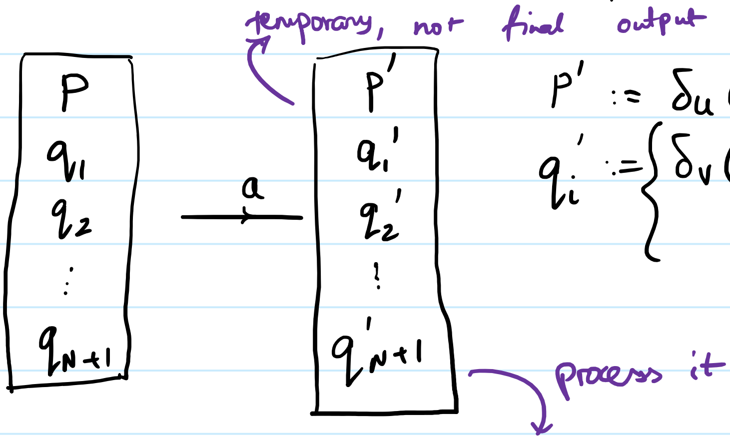
$q_V^j = \perp$ indicates that no copy of A is running in slot j.

Initial state:

$$S = \begin{cases} (q_0^U, \perp, \perp, \dots, \perp) & \text{if } q_0^U \notin F_U \\ (q_0^U, q_0^V, \perp, \perp, \dots, \perp) & \text{if } q_0^U \in F_U \end{cases}$$

We will maintain invariant that $q_i^V \neq q_j^V$ for $i \neq j$ (unless both are $\perp$).

In general, $\delta : S \times \Sigma \to S$ is defined as follows



$p' := \delta_U(p, a)$

$$q_i' := \begin{cases} \delta_V(q_i, a) & ; q_i \neq \perp \\ \perp & ; q_i = \perp \end{cases}$$

Note that by distinctness, $\exists\, i$ s.t. $q_i = \perp$.

In that case $q_i' = \perp$.

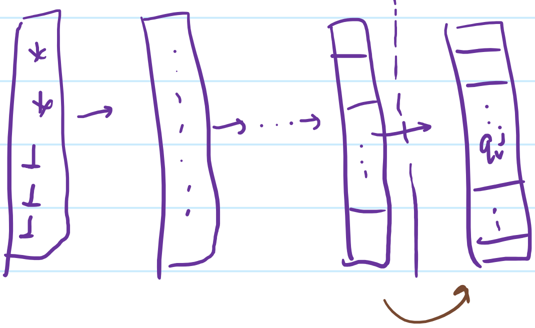
- If $p' \in F_U$, then we redefine $q_i' := q_0^V$.
- Now, we do the "merge", if there are multiple indices which have ended in same state, change all but lowest index one to $\perp$.

Final: After this processing, whatever we are left with, is the new state.

Suppose $\alpha \in U \cdot \vec{V}$. Consider the run of $B = (S, S_0, \Sigma, \delta, *)$ on α.

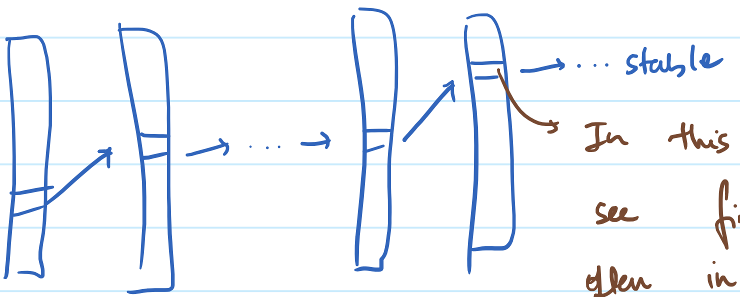
$B = (S, s_0, \Sigma, \delta, \ldots)$ on α.

$$\alpha = \underbrace{a_0\ a_1\ a_2\ a_3 \cdots a_k}_{\in \mathcal{U}} \;\Big|\; \underbrace{a_{k+1}\ \ a_{k+2}\ \cdots}_{\in \vec{V}}$$



A copy of A_V is started in slot j

may remain in slot j forever or it may merge with another copy running at a lower indexed slot. And this may happen repeatedly. This jumping cannot happen inf. often since the index strictly decreases. Thus, it will eventually settle at a fixed slot.



In this stable slot, we will see final states of A_V infinitely often in that slot.

Thus, the unique run ρ of B on α has the property:

$\exists$ a slot i such that

1) the state at slot i will be in $F_{\mathcal{U}}$ inf. often,
2) the state at slot i will be $\perp$ finitely often.

Define the sets:

$$E_i = \{ (p, q_1, \ldots, q_{i-1}, \perp, q_{i+1}, \ldots, q_{N+1}) \in S \}$$
$$= \{ s \in S : \pi_i(s) = \perp \}.$$

$$F_i = \{ s \in S : \pi_i(s) \in F_v \}.$$ i^{th} coordinate

If $\alpha \in U \cdot \vec{V}$, then the unique run ρ has the property that $\exists i$ s.t.

$$\mathrm{Inf}(\rho) \cap E_i = \phi \neq \mathrm{Inf}(\rho) \cap F_i.$$

Now, put $\Omega = \{ (E_1, F_1), \ldots, (E_{N+1}, F_{N+1}) \}$ as the Rabin condition on B.

Our discussion so far has shown: $U \cdot \vec{V} \subseteq L(B)$.

Need to argue the reverse containment.

Let $\alpha \in L(B)$.

$\alpha = a_0\ a_1\ a_2 \cdots$

$\rho = s\ s_1\ s_2 \cdots$

Since ρ satisfies the Rabin condition, $\exists i$ s.t. $\mathrm{Inf}(\rho) \cap E_i = \phi$ & $\mathrm{Inf}(\rho) \cap F_i \neq \phi$.

Since $\mathrm{Inf}(\rho) \cap E_i = \phi$, $\exists$ a point at which i is started and never stopped. The prefix until that would've been in U and the suffix from there is in $\vec{V}$ since $\mathrm{Inf}(\rho) \cap F_i \neq \phi$.

McNaughton's Theorem:

Every ω-reg lang. can be accepted by a Muller automaton.

We showed for Rabin. Can do from Rabin to Muller without changing states and transitions.

Thm. Every ω-regular language is a boolean combination of languages accepted by DBA (or equivalently, languages of the form $\vec{U}$).

$\wedge, \vee, \neg$

Proof. Let $A = (Q, q_0, \Sigma, \delta, \mathcal{F} = \{F_1, \dots, F_s\})$ be a det Muller automaton.

$$L(A) = \bigcup_{i=1}^{s} L((Q, q_0, \Sigma, \delta, \{F_i\})).$$

Thus, wlog assume $|\mathcal{F}| = 1$. Write $\mathcal{F} = \{F\}$.

$\forall q \in Q$, define $L_q = \{\alpha :$ unique run of A on α visits q inf. often$\}$

$= \overrightarrow{U_q}$, where U_q is the lang. accepted with $\{q\}$ as final set.

Since we are using Muller condition, we have

$$L(A) = \left(\bigcap_{q \in F} \overrightarrow{U_q} \right) \cap \left(\bigcap_{q \notin F} \neg \left(\overrightarrow{U_q} \right) \right). \quad \blacksquare$$