The Miller-Rabin Primality Test

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Output: isPrime(n).

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Moreover, note that the above algorithm does *more* than what we expected from our algorithm. Namely, it not only tells us that the number is prime but also produces a nontrivial factor in the case that n is composite.

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$$(\mathbb{Z}_n)^* \xrightarrow{\cong} (\mathbb{Z}_{p_1^{e_1}})^* \times \cdots \times (\mathbb{Z}_{p_r^{e_r}})^*.$$

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However, there *are* infinitely many odd composite n for which $L_n = \mathbb{Z}_n^*$

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The first statement is clear. For the second, one recalls that L_n is a subgroup of Z_n^*. Thus, $\frac{|\mathbb{Z}_n^*|}{|L_n|}$ is a positive integer. Thus, if the integer is not 1, it is at least 2. Combine this with the fact that $|Z_n^*| \leqslant n-1$ to get the result. $\square$

However, there *are* infinitely many odd composite n for which $L_n = \mathbb{Z}_n^*$ and thus, they cannot be ignored.

Carmichael numbers

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This algorithm runs in time $O(\text{poly}(\log(n)))$ and thus, satisfies the first criteria.

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Let us now prove the above theorem.

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Proof (continued).

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Proof (continued).

For $j = 0, \ldots, h$, define ρ_j to be the $(t2^j)$-power map on $\mathbb{Z}_n^*$.

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For $j = 0, \ldots, h$, define ρ_j to be the $(t2^j)$-power map on $\mathbb{Z}_n^*$. From the previous claim, and the definition of L'_n, it follows that $\alpha^{t2^{g-1}} = \pm 1$ $\forall \alpha \in L'_n$.

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$$2^r \left|\ker(\rho_{g-1})\right| = \left|\ker(\rho_g)\right| \leqslant \left|\ker(\rho_h)\right|.$$

Does it fit the bill? (Yes)

Proof (continued).

For $j = 0, \ldots, h$, define ρ_j to be the $(t2^j)$-power map on $\mathbb{Z}_n^*$. From the previous claim, and the definition of L'_n, it follows that $\alpha^{t2^{g-1}} = \pm 1$ $\forall \alpha \in L'_n$. Thus, $L'_n \subseteq \rho_{g-1}^{-1}(\{\pm 1\})$ and hence, $|L_n|' \leqslant 2|\ker(\rho_{g-1})|$. Also,

$$|\ker(\rho_j)| = \prod_{i=1}^{r} \gcd(t_i 2^{h_i}, t2^j) \qquad \forall j \in \{0, \ldots, h\}.$$

Since $g \leqslant h$ and $g \leqslant h_i$ for all i, we get

$$2^r |\ker(\rho_{g-1})| = |\ker(\rho_g)| \leqslant |\ker(\rho_h)|.$$

Combining the red expressions, we get

$$|L'_n| \leqslant 2^{-r+1} |\ker(\rho_h)| = \frac{|L_n|}{2^{r-1}}.$$

Does it fit the bill? (Yes)

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$$\frac{|L_n|}{2^{r-1}} = \frac{|L_n|}{2}$$

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Proof (continued).

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If $r \geqslant 3$, then we are done since $|L_n| \leqslant |Z_n^*| \leqslant n-1$, and $2^{r-1} \geqslant 4$.
If $r = 2$, then n is not a Carmichael number and thus,

$$\frac{|L_n|}{2^{r-1}} = \frac{|L_n|}{2} \leqslant \frac{1}{4}(n-1),$$

and we are again done. $\square$