The Miller-Rabin Primality Test

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Output: isPrime(n).

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Moreover, note that the above algorithm does *more* than what we expected from our algorithm. Namely, it not only tells us that the number is prime but also produces a nontrivial factor in the case that n is composite.

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However, there *are* infinitely many odd composite n for which $L_n = \mathbb{Z}_n^*$

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The first statement is clear. For the second, one recalls that L_n is a subgroup of Z_n^*. Thus, $\frac{|\mathbb{Z}_n^*|}{|L_n|}$ is a positive integer. Thus, if the integer is not 1, it is at least 2. Combine this with the fact that $|Z_n^*| \leqslant n-1$ to get the result. $\square$

However, there *are* infinitely many odd composite n for which $L_n = \mathbb{Z}_n^*$ and thus, they cannot be ignored.

Carmichael numbers

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The above is not hard to prove, it only requires some basic facts about cyclic groups. However, we skip the proof.

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This algorithm runs in time $O(\text{len}(n)^3)$ and thus, satisfies the first criteria.

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For $j = 0, \ldots, h$, define ρ_j to be the $(t2^j)$-power map on $\mathbb{Z}_n^*$.

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$$|\ker(\rho_j)| = \prod_{i=1}^{r} \gcd(t_i 2^{h_i}, t2^j) \qquad \forall j \in \{0, \ldots, h\}.$$

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$$2^r \left|\ker(\rho_{g-1})\right| = |\ker(\rho_g)| \leqslant |\ker(\rho_h)| \,.$$

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$$2^r \left|\ker(\rho_{g-1})\right| = \left|\ker(\rho_g)\right| \leqslant \left|\ker(\rho_h)\right|.$$

Combining the red expressions, we get

$$\left|L_n'\right| \leqslant 2^{-r+1} \left|\ker(\rho_h)\right| = \frac{|L_n|}{2^{r-1}}.$$

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Proof (continued).

$$\left|L'_n\right| \leqslant \frac{\left|L_n\right|}{2^{r-1}}$$

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$$\frac{|L_n|}{2^{r-1}} = \frac{|L_n|}{2} \leqslant \frac{1}{4}(n-1),$$

and we are again done. □