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CS 719

Topics in Mathematical Foundations of Formal Verifications

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Recap of Regular Languages

Different formalisms surprisingly describe the same class of languages. Regular expressions, DFA, NFA, MSO logic.

Notation and Setup (for the rest of course)

Fix a finite alphabet Σ .

A (finite) word over Σ is a finite sequence $a_0 a_1 \dots a_n$ of elements of Σ . $u, v, w \dots$ are used for words.
 $w = a_0 a_1 \dots a_n$ where each $a_i \in \Sigma$.

The empty sequence corresponds to the unique word of length 0 and is denoted by ϵ , the empty word.

Σ^* = the set of all finite words over Σ . ($\epsilon \in \Sigma^*$)

$\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$ = the set of all non-empty words over Σ .

CONCATENATION of words:

$$\cdot : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \\ (u, v) \mapsto u \cdot v$$

defined in the usual manner.

→ The \cdot operation on Σ^* is associative.

That is, $\forall u, v, w \in \Sigma^* : (u \cdot v) \cdot w = u \cdot (v \cdot w)$.

→ ϵ acts as an identity for \cdot .

This is an example of a monoid $(X, *)$

$$\forall w \in \Sigma^*: f \cdot w = w \cdot f = w.$$

(Another example of monoid: $(\mathbb{N}, +)$
 $\mathbb{N} = \{0, 1, \dots\}$ in this course
 (Later we'll look at finite monoids.)

$$l: \Sigma^* \rightarrow \mathbb{N}$$

$$u \mapsto \text{length of } u = l(u)$$

Note $l(u \cdot w) = l(u) + l(w)$
 $l(\epsilon) = l(0)$

Thus, l is a monoid morphism.

Defn: A language L is simply a subset of Σ^* . (Language)

Given languages $L_1, L_2 \subset \Sigma^*$, we define

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}.$$

REGULAR EXPRESSIONS

(Regular expressions)

$$r \equiv \emptyset \mid \epsilon \mid a \mid r_1 + r_2 \mid r_1 \cdot r_2 \mid r^*$$

$r \rightsquigarrow L(r)$ language associated to r

$L(r)$ is defined by structural induction on r .

- $L(\emptyset) = \emptyset$
- $L(\epsilon) = \{\epsilon\}$
- $L(a) = \{a\}$ ($a \in \Sigma$)
- $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
- $L(r_1 \cdot r_2) = L(r_1) \cdot L(r_2)$ (rhs defined earlier)

$$\begin{aligned} L(r^*) &= \{\epsilon\} \cup L(r) \cup L(r) \cdot L(r) \cup L(r) \cdot L(r) \cdot L(r) \cup \dots \\ &= \bigcup_{i=0}^{\infty} L^i \quad (L^0 = \{\epsilon\}, L^1 = L(r), L^{i+1} = L^i \cdot L) \end{aligned}$$

[Example. $(ab)^* = \{\epsilon, ab, abab, \dots\}$.]

Defn. A language $L \subseteq \Sigma^*$ is said to be **regular** if there exists a regular expression r such that $L(r) = L$. (Regular language)

Thm. Regular languages are closed under union, intersection, complementation, concatenation.

(As per our defⁿ using regular expressions, union & concatenation are obvious.)

Some of the above is easier to prove under diff. formalisms. One first shows that two diff. formalisms are actually same.

Defn. (Extended reg. expressions) (Extended regular expressions)

$$r \equiv \phi \mid \epsilon \mid a \mid r_1 + r_2 \mid r_1 \cdot r_2 \mid \neg r \mid r^* \quad (\underbrace{\epsilon}_{\Sigma}) \quad \downarrow \quad \downarrow$$

These we can add, in view of th^m, w/o changing the class of languages

Q: What subclass of language will we get if we restrict ourselves to a subset of the operators?

Defn. (Star-free ^{extended} reg. expressions) Exclude the * operator.

(Star-free regular expressions)

Def: (Star-free ^{or} reg. expressions) Exclude the * operator.

(Star-free regular expressions)

Q. Which regular languages admit a star free representation?

(Non?) Example : $r = (ab)^*$

Can we rewrite this without * ?

The "extended" is important. Else, we get trivial classes.

Lecture 2 (14-01-2021)

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Note that $\neg \phi = \Sigma^*$
can we this freely

Observe: $a^* = \neg(\underbrace{\Sigma^* \cdot b \cdot \Sigma^*}_{\text{words containing at least } b})$

Similarly $(ab)^* \rightarrow \text{words starting with } a, \text{ ending with } b,$
 $\text{no lone active } a \text{ or } b \text{ (or } \epsilon\text{)}$

$$(ab)^* = \epsilon + [\alpha \Sigma^* b \cap \neg(\Sigma^* aa\Sigma^* + \Sigma^* bb\Sigma^*)]$$

It is not even clear a priori whether the question
"Which languages have *-free expression" is even decidable.

Finite state Automata (Finite state automata)

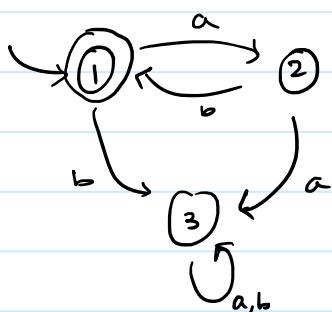
(NFA)

$$A = (Q, \Sigma, Q_0 \subseteq Q, \Delta \subseteq Q \times \Sigma \times Q, F \subseteq Q)$$

finite set initial states transition $(q_i, a, q_j) \in \Delta$ final states

EXAMPLES

①



$$Q = \{1, 2, 3\}$$

$$Q_0 = F = \{1\}$$

$$\Sigma = \{a, b\}$$

Language accepted: $(ab)^*$

Defn Suppose $w = a_0 \dots a_n \in \Sigma^*$.

A run ρ of A on w is a sequence of states

$$\rho = q_0, \dots, q_{n+1}$$

↑ note $n+1$

such that

- $q_0 \in Q_0$
- $(q_i, a_i, q_{i+1}) \in \Delta \quad \forall i = 0, \dots, n$

The run ρ is accepting if $q_{n+1} \in F$.

(Note that a word may have no run or even multiple runs.)

The language $L(A)$ of A is defined as

$$L(A) = \{w \in \Sigma^*: A \text{ has at least one accepting run}\}.$$

A is deterministic if $|Q_0| = 1$ and

$$\forall q \in Q, \forall a \in \Sigma, \exists! q' \in Q \text{ s.t. } (q, a, q') \in \Delta.$$

there exists unique

In other words, $\Delta \subseteq (Q \times \Sigma) \times Q$ is a function
 $Q \times \Sigma \rightarrow Q$.

The example above was actually deterministic. It is called a DFA.

Thm. [TQC] (Kleene's Theorem)

Regular expressions \equiv NFA \equiv DFA.

(That is, all three formalisms talk about the same class of language - regular languages.)

(Recap of Proof.)

Reg. Exp \in NFA

$$\begin{array}{ccc} r & \mapsto & Ar \\ \text{reg exp} & & \uparrow \text{NFA} \end{array}$$

$$L(r) = L(A_r).$$

The way to do this is by induction.

- For ϵ and 'a', easy.
- $r = r_1 + r_2$. We have NFAs for r_1 and r_2 . Then, the NFA $A_r = A_{r_1} \sqcup A_{r_2}$ works.
Allowed since non-determinism → A_{r_1} A_{r_2}
- $r_1 \cdot r_2$. Use ϵ -transitions. Idea is to take union and put ϵ transitions from F of A_{r_1} to Q_0 of A_{r_2} . The final states are now F of A_{r_2} and initial is Q_0 of A_{r_1} .
- r^* . Same sort of idea as above but loop on self.

NFA \subseteq Reg. Expr.



$$Q = \{1, \dots, n\}$$

r_{ij} = a reg. exp. which captures the words which allow to go from i to j .

Then $r := \bigcup_{\substack{i \in Q_0 \\ j \in F}} r_{ij}$ works.

Thus, only need to figure out r_{ij} .

'Dynamic Programming'

Introduce a third parameter k .

r_{ij}^k = reg. expression words w which have a

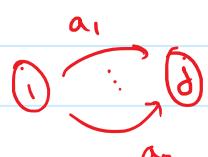
- run ρ of A s.t.
- ρ starts at i
 - ρ ends at j
 - all intermediate states of ρ are in $\{1, \dots, k\}$.
(include $k = 0$)

Start building r_{ij}^k going from $k = 0$ to $k = n$.
Note that $r_{ij}^k = r_{ij}^n$.

r_{ij}^0 = start at i , end at j , no intermediate state
= all those letters which allow transition from i to j .
(if any)

$$\begin{aligned} r_{ij} &= a_1 + a_2 + \dots + a_p && (i \neq j) \\ &= a_1 + \dots + a_p + \epsilon && (i = j) \end{aligned}$$

a_1, \dots, a_p
 ϵ



$$r_{ij}^k = r_{ij}^{k-1} + r_{ik}^{k-1} \cdot (r_{kk}^{k-1})^* \cdot r_{kj}^{k-1}$$

(We build for lower k first for all (i, j))

Thus, Reg f.g = NFA.

NFA = DFA. DFA \subseteq NFA is obvious.

Converse:

$$A = (Q, \Sigma, Q_0, \Delta, F).$$

The idea to get an equivalent DFA is the powerset construction.

$$B = (2^Q, \Sigma, \{Q_0\}, \delta: 2^Q \times \Sigma \rightarrow 2^Q, F')$$

Idea is to keep track of all the states that you can reach from given state.

$$\delta(x, a) = \{q \in Q : \exists q' \in X, \quad q' \xrightarrow{a} q\}.$$

Lecture 3 (18-01-2021)

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Today, we see another formalism to describe regular languages.
A natural way to describe a language is to give a "property" of words.

Examples:

1) Every (occurrence of an) 'a' is eventually followed by a 'b'.

a a b a a b ✓ b a b a b a c ✗

2) There is exactly one 'a' in the word.

3) The first position is labelled 'a'.

4) There are even number of 'a's.

We need a formal language to do so.

Formal Language : Should allow us to do "Boolean" properties like "and", "or", et cetera.

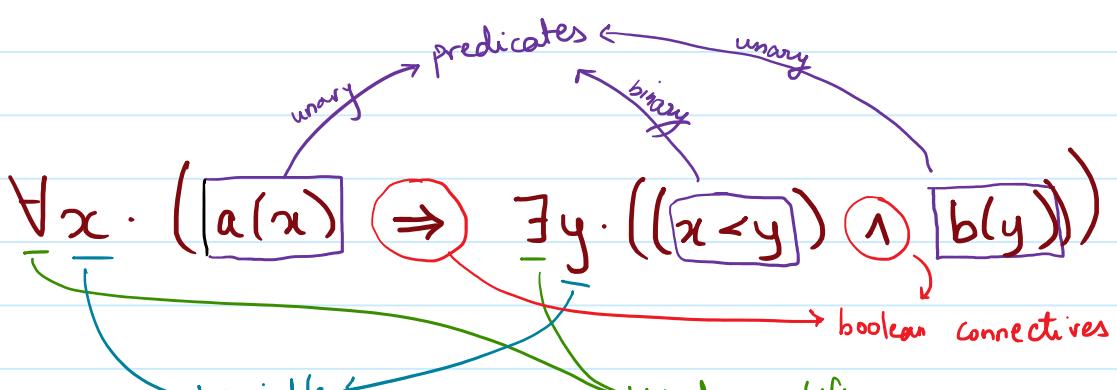
Going to use a Mathematical Logic for doing so.

First-Order Logic (over words)

(First Order Logic)

before formal defn & syntax:

An example of a formula in this logic:





$\text{FO } [\Sigma]$ - variables :- x, y, z, \dots | range over
 x_0, x_1, x_2, \dots positions

- predicates :-
- letter predicates
 $a \in \Sigma, a(x)$ says the letter ' a ' at position ' x ' (true/false)
- binary predicates, $y \in z$
- equality $x = y$

$$\varphi \equiv a(x) \mid x < y \mid x = y \mid \varphi \vee \varphi \mid \neg \varphi \mid \exists x \cdot \varphi$$

can get $\varphi \wedge \varphi, \varphi \Rightarrow \varphi, \varphi \Leftrightarrow \varphi, \forall x \cdot \varphi$
using these

The above was a sentence, there was no free variable.

$$\text{first}(x) \equiv \forall y \cdot [(x=y) \vee (x < y)] \quad \leftarrow \text{here } x \text{ is free}$$

Given this formula, if we wish to find truth of $\text{first}(x)$ on some word w , we need to give x .

$$w = \begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ abaaab \end{smallmatrix} \quad w, x \leftarrow ? \models \text{first}(x) ?$$

if true, we write: $w, x \leftarrow 2 \models \text{first}(x)$
else : $w, x \leftarrow 2 \not\models \text{first}(x)$

Easy to see $w, x \leftarrow 2 \not\models \text{first}(x)$. Why?

We need to check if for all positions 'p' in w :

$$w, x \leftarrow z, y \leftarrow p \models (x = y) \vee (x < y)$$

If $P=4$, then we check $(2=4) \vee (2 < 4)$

<u>false</u>	<u>true</u>
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true!

However, if $P = 1$, then

$$(2=1) \vee (2 < 1)$$

<u>false</u>	<u>false</u>
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false

Then, $w, x \leftarrow 2 \not\models \text{first}(x)$. (We had the universal quantifier.)

Easy to see $\text{first}(x)$ is true iff x is the first position.

Now, we can use $\text{first}(x)$.

Defn.: A sentence is a formula without free variables.

(Sentence)

Example • $\varphi = \exists x \cdot [\text{first}(x) \wedge b(x)]$ is a sentence.
Now makes sense to ask " $\text{abaab} \models \varphi$ " without any assignment. ($\text{abaab} \not\models \varphi$)

the first position is labelled 'b'

• Exactly one 'a':

$$\{\forall x \forall y [(a(x) \wedge a(y)) \Rightarrow x=y]\} \wedge \{\exists x \cdot a(x)\}$$

• $(ab)^*$ ← can you write a first order sentence which gives this regex?

$$\{\forall x [\text{first}(x) \Rightarrow a(x)]\} \wedge \{\forall x [a(x) \Rightarrow \exists y \cdot (s(x,y) \wedge b(y))]\}$$

$s(x,y) = (x < y) \wedge \forall z (z < x \vee y < z).$

• There are even numbers of 'a's → is regular, can

come up with an automata

The other three examples were also regular. (Also expressible by FOL)
However, FOL cannot describe this logic!
But every language definable by FOL WILL be regular!

FOL → FOL-definable languages

REG → collection of reg. languages

$\boxed{\text{FOL} \subsetneq \text{REG}}$

We shall extend FOL to MSO → Monadic Second Order Logic.

Lecture 4 (19-01-2021)

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MSO (Monadic Second Order Logic - Over Words)

(MSO Monadic Second Order Logic)

Here, we have position variables : $x, y, z, \dots, x_0, x_1, \dots$
Set of position variables : $X, Y, Z, \dots, X_0, X_1, \dots$

Predicates : $a(x) - a \in \Sigma$ (Unary)

$x = y$ (Binary)

$S(x, y)$ - successor : 'y' is a successor of 'x'
(membership predicate) $X(x) - 'x' \text{ belongs to } X [x \in X]$

$$\varphi = a(x) | x = y | S(x, y) | X(x) | \varphi \vee \varphi | \neg \varphi | \exists x. \varphi | \forall x. \varphi$$

Eg of formula: $\forall X \exists x. x \in X$

Convention (notation) : $\varphi(x_1, \dots, x_m, X_1, \dots, X_n)$ - φ is an MSO formula
 x_1, \dots, x_m are free pos. var
 X_1, \dots, X_n ——— set var.

Semantics (Semantics) "truth" / "models" relation.

$w \in \Sigma^*$ - a finite word

p_1, \dots, p_m - m positions in w ,

Q_1, \dots, Q_n - n sets of positions in w .

$w, p_1, \dots, p_m, Q_1, \dots, Q_n \models \varphi$

(p_1, \dots, p_m are "concrete" positions)
(Q_1, \dots, Q_n ——— sets)

want to define when this happens.
 $(x \leftarrow p_1, \dots, x_m \leftarrow p_m, t_n \leftarrow Q_n \text{ is understood})$

Defined by structural induction on φ

- $w, p_i \models a(x_i)$ if the letter in w at position p_i is a a
- $w, p_i, Q_i \models x_i(a_i)$ if $p_i \in Q_i$
- $w, p_1, \dots, p_m, Q_1, \dots, Q_n \models \psi(x_1, \dots, x_m, x_1, \dots, x_n) = \psi_1 \vee \psi_2$
iff $w, p_1, \dots, p_m, Q_1, \dots, Q_n \models \psi_1$ or $w, \dots \models \psi_2$
- $w, \dots \models \neg \psi$ iff $w, \dots \not\models \psi$
- $w, p_1, \dots, p_m, Q_1, \dots, Q_n \models \psi(x_1, \dots, x_m, x_1, \dots, x_n) = \exists x_{m+1} \psi'(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$
iff there exists a position p_{m+1} in w s.t.
 $w, p_1, \dots, p_m, p_{m+1}, Q_1, \dots, Q_n \models \psi'(x_1, \dots, x_{m+1}, x_1, \dots, x_n)$.

Example $\psi \equiv \forall x \exists z . X(z) \wedge a(z) \equiv \forall x . \psi'(x)$

"
 $\exists z . X(z) \wedge a(z)$

$aa \stackrel{?}{\models} \psi$; $aa \models \psi$ if for all subsets Q of positions in aa ,
 $aa, Q \models \psi'(x)$
 $aa, \{1, 2\} \stackrel{?}{\models} \psi' = \exists z . X(z) \wedge a(z)$

Yes! $x=1$ works

$aa, \phi \not\models \psi'$ since $\phi(x)$ is never true.

Thus, $aa \not\models \psi$.

Fo: $a(x), x < y, x = y$, boolean, $\exists x, \forall x$

MSO: $a(x), S(x, y), \underline{\quad u \quad}$, $\exists x, \forall x$

Is $Fo \subseteq MSO$? If we had ' $<$ ' in MSO, would be obvious.

As it turns out, we can write ' $<$ ' in MSO, since we have set variables.

Lecture 5 (21-01-2021)

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$$\text{MSO } [S] : \varphi \equiv a(x) \mid x = y \mid \leq(x, y) \mid x(x) \mid \varphi \vee \varphi \mid \neg \varphi \mid \exists x \cdot \varphi \mid \exists x \cdot \varphi$$

$$\text{FO } [<] : \varphi \equiv a(x) \mid x = y \mid x < y \mid \varphi \vee \varphi \mid \neg \varphi \mid \exists x \cdot \varphi$$

$$\text{FO } [S] : \varphi \equiv a(x) \mid x = y \mid \leq(x, y) \mid \dots$$

(Obvious semantics for all three above.)

Q. How do $\text{FO} [<]$ and $\text{FO} [S]$ compare?

Can a property in one logic be written in the other?

- If ' S ' can be expressed in $\text{FO} [<]$, then $\text{FO} [S] \subseteq \text{FO} [<]$.

$$S(x, y) \equiv (x < y) \wedge \neg(\exists z ((x < z) \wedge (z < y)))$$

- Can ' $<$ ' be expressed in $\text{FO} [S]$?

No.

Thus, $\text{FO} [S] \subsetneq \text{FO} [<]$.

-
- $\text{FO} [S] \subseteq \text{MSO} [S]$. Clear.

However, we also have $\text{FO} [<] \subseteq \text{MSO} [S]$.

Suffices to show ' $<$ ' can be expressed in $\text{MSO} [S]$

$$x < y \equiv (\neg(x = y)) \wedge (\forall x [(x(x) \wedge S(x)) \Rightarrow x(y)])$$

$$\text{succ}(x) \equiv \forall z \forall w \left\{ [x(z) \wedge s(z, w)] \Rightarrow x(w) \right\}$$

successor closed

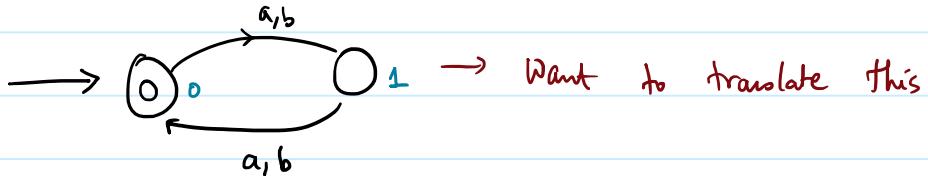
iff x is not equal to y and every subset which contains x and closed under successor also contains y .

$$\text{Thus, } \text{FO}[S] \subsetneq \text{FO}[\langle] \subseteq \text{MSO}[S] = \text{MSO}[S, \langle].$$

In fact,

$$\text{FO}[\langle] \subsetneq \text{MSO}[S].$$

"Words of even length" can be expressed in $\text{MSO}[S]$
but not in $\text{FO}[\langle]$. (Proof. Later. B)



$\epsilon \# w$ has even length $\Leftrightarrow \exists$ a subset X of positions in ω s.t.
 first(x) \leftarrow 1) X contains the first position
 note this used \nwarrow 2) X contains every alternate position
 ' \wedge ' but can use that now 3) X does not contain the last position

$$\exists X \left[\left[\exists x. [\text{first}(x) \wedge x(x)] \right] \wedge \left[\forall y \forall z. [s(y, z) \Rightarrow [x(y) \Leftrightarrow \neg x(z)]] \right] \wedge \left[\exists x. [\text{last}(x) \wedge \neg x(x)] \right] \right]$$

[non empty words]

$$\text{Note } \epsilon \models \forall x. \neg(x = x)$$

$$\left[\text{Recall: } \omega = \epsilon \not\models \exists x. \varphi \right]$$

\rightarrow can or with this

$$\omega = \epsilon \models \forall x. \varphi$$

for convenience, we may switch to Σ' and forget about ϵ
since we can always take care of it separately.

since we can always take care of it separately.

Defⁿ. Let $L \subseteq \Sigma^*$. We say L is **MSO[s]-definable** if \exists a MSO[s] sentence φ s.t.

$$L = \{w \mid w \models \varphi\} = L(\varphi).$$

(We will drop the "[s]" and just say "MSO".)

Thm. [Büchi- Elgot] Let $L \subseteq \Sigma^*$.
 L is regular iff L is MSO-definable.

More importantly, the proof (transitions b/w automata & MSO)
is effective.

↳ Can write a program which does this conversion.