Lecture 1 (30-07-2021)

30 July 2021 09:28

Automata on infinite words

Some notations

- Let Σ be a finite nonempty set (called alphabet).
- A finite word (over Σ) is a finite (possibly empty) sequence of letters from Σ.

$$\omega = a_0 a_1 \cdots a_n, \qquad a_i \in \Sigma.$$

ϵ is the empty word.

Σ^* is the set of all finite words (over Σ).

- An infinite word over Σ is an infinite sequence of letters from Σ.

$$\alpha = a_0 a_1 a_2 \cdots, \qquad a_n \in \Sigma \quad \forall n \in \mathbb{N}_0.$$

(Different formality: $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $\alpha : \mathbb{N}_0 \to \Sigma^*$)

Also, $\omega := \mathbb{N}_0$.

Σ^ω = all infinite words (on Σ)

(In general, given sets X and Y, Y^X denotes the set of all functions $X \to Y$)

Examples ① $\Sigma = \{a, b\}$

$\alpha = a\ b\ ab\ ab \cdots$

or: $\alpha(n) = \begin{cases} a & ; \ 2 \mid n \\ b & ; \ 2 \nmid n \end{cases}$

② $\alpha = abb\ abb\ abb$

$\alpha(n) = \begin{cases} a & ; \ 3 \mid n \\ b & ; \ 3 \nmid n \end{cases}$

$\alpha = (abb)^\omega$

③ $\gamma : \omega \to \{a, b\}$

$\gamma(n) = \begin{cases} a & ; \ n \text{ is prime} \\ b & ; \ \text{otherwise} \end{cases}$

$$\gamma = \underset{0}{b}\,\underset{1}{b}\,\underset{2}{a}\,\underset{3}{a}\,\underset{4}{b}\,\underset{5}{a}\,\underset{6}{b}\,\underset{7}{a}\,\underset{8}{b}\,\underset{9}{b}\,\underset{10}{b}\,\underset{11}{a} \cdots$$

y has infinitely many 'a's and 'b's.
(Can't write as compactly as before.)

④ $\Sigma = \{a\}$, $\Sigma^\omega = \{a^\omega\}$. $(|\Sigma^\omega| = 1.)$
But if $|\Sigma| > 1$, then Σ^ω is not a countable set.
OTOH, Σ^* is always a countable set. $(1 \leq |\Sigma| < \infty)$

Automata:

$A = (Q, \Sigma, q_0, \Delta \subseteq Q \times \Sigma \times Q,$ "Acceptance condition")

→ a finite set of states
→ $q_0 \in Q$ → the initial state (unique)
→ $\Delta \subseteq Q \times \Sigma \times Q$ → the transition relation
$(q, a, q') \in \Delta \equiv q \xrightarrow{a} q'$
(Non-determinism here is fine)

Now, let $\alpha = a_0 a_1 a_2 \cdots \in \Sigma^\omega$ be given.
A run ρ of A on α is an infinite sequence of states

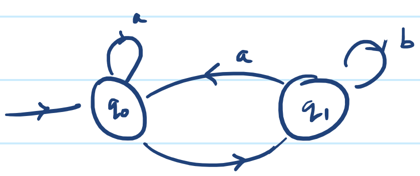
(run)

$$\rho = q_0 q_1 q_2 \cdots$$

Such that "q_0" is indeed the initial state and
$$\forall i \in \omega : (q_i, a_i, q_{i+1}) \in \Delta$$

(In terms of functions: Given $\alpha : \omega \to \Sigma$, we have
$\rho : \omega \to \Sigma$ s.t. $\rho(0) = q_0$ and $(\rho(n), \alpha(n), \rho(n+1)) \in \Delta \ \forall n \in \omega$.)

Example.



$\alpha = (ab)^\omega$
$\rho = q_0 (q_0 q_1)^\omega$

$$\rho = q_0 (q_0 q_1)^\omega$$

"Acceptance condition": (Büchi automata)

$\alpha \to$ input word

$\rho \to$ a run of A on α

$\mathrm{Inf}(\rho) :=$ the set of states which occur infinitely often along ρ

$$= \{ q \in Q : \exists^\infty i \in \omega \text{ s.t. } \rho(i) = q \}$$

Obs. $\mathrm{Inf}(\rho) \neq \emptyset$. (There are only finitely many states.)

Büchi automaton (BA): Fix $G \subseteq Q$ called the "good states".

A run ρ is accepted by a BA if $\mathrm{Inf}(\rho) \cap G \neq \emptyset$.

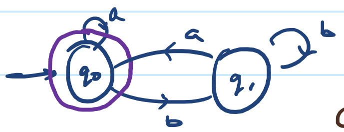
(Thus, some good state appears infinitely often.)

A word $\alpha \in \Sigma^\omega$ is accepted by A if α has an accepting run ρ on the word α.

$L(A) := \{ \alpha \in \Sigma^\omega : A \text{ accepts } \alpha \}$

$\hookrightarrow$ Language of A

Example

$A =$ 

$G = \{q_0\}$, $\Sigma = \{a, b\}$

Claim. $L(A) = \{ \alpha \in \Sigma^\omega : \alpha \text{ has inf.}$

many 'a' }.

Proof. Let the right side be L.

- $L(A) \subseteq L$:

  $\alpha \in L(A)$, $\alpha \equiv a_0 a_1 a_2 \cdots$

  Note that A is deterministic, thus α has a unique run ρ, which is accepted.

  $\rho \equiv q_0 q_1' q_2' q_3' \cdots$

  Thus, q_0 appear inf. often above. Since it only receives 'a', we see that 'a' appears inf. often.

- $L \subseteq L(A)$:

  Let $\alpha \in L$. It has a unique run ρ.

  Then, since α has inf. many 'a's, ρ will have inf. many 'q_0's. ∎

Can also write $L = (b^* a)^\omega$ once we have defined what that means.

Question: What about $\overline{L} = \Sigma^\omega \setminus L$? Can that be accepted by a Büchi automaton?

Lecture 2 (04-08-2021)

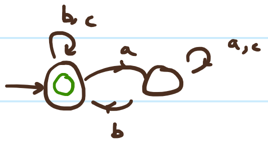
04 August 2021 09:31

Note. We do NOT allow ϵ transitions in this course.

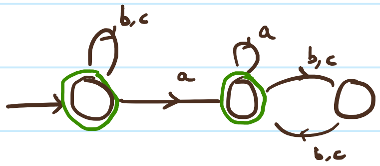
Fact. Even though we insisted on single initial state, the expressive power does not change if we allow more. (It is simply for convenience.)

Examples (1) L_1 over $\Sigma = \{a, b, c\}$.

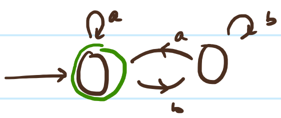
L_1 = every 'a' is eventually followed by a 'b'



(2) L_2 = any two occurences of 'a' are separated by even no. of other (b, c) letters

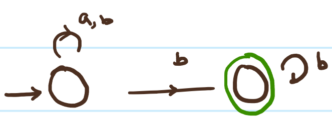


(3) $\Sigma = \{a, b\}$, L = inf. many 'a's



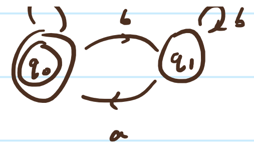
Complement : $\bar{L} = \Sigma^\omega \setminus L$ = finitely many 'a's

Q. What is a BA for $\bar{L}$?



Q. Do we have a deterministic Büchi automaton (DBA) for $\bar{L}$?

Note that we had an DBA for L :

Note that we had an DBA for L : 

A : $G = \{q_0\}$

Toggle states, A' : $G = \{q_1\}$

But $L(A') \overset{?}{\neq} \bar{L}$.

$L(A')$ ↓ infinitely many b

$\bar{L}$ ↓ eventually b

$(ab)^\omega \in L(A')$ but $(ab)^\omega \notin \bar{L}$

Complementing the good state of a DBA does <u>**NOT**</u> complement the accepted language.

<u>Claim.</u> There is No DBA for $\bar{L} = \{\alpha \in \Sigma^\omega : \alpha$ has finitely many 'a' $\}$.

Thus, as opposed to finite languages, non-determinism actually gives us more languages.

<u>Proof.</u> We prove this by contradiction.

For the sake of contradiction, assume that $\exists$ DBA A such that $L(A) = \bar{L}$.

Suppose A has m states.

$\alpha_0 = b^\omega = b\,bb\cdots \in \bar{L}$

$\rho_0 = q_0\, q_1 q_2 \cdots$

↳ unique run of α_0

Since ρ_0 is accepting, $\exists\, n_1$ s.t. $q_{n_1+1} \in G$.

Pick the smallest such n_1.

$$\overbrace{b\ b\ \cdots\ b}^{n_1} \ \Big|\ b\ \ b\ \ \cdots$$

$\rho_0 =$ $\boxed{}$ ↑ first good state

Define $a_1 := b^{n_1}\, a\, b^\omega \in \bar{L}$.

$\alpha_1 = b \cdots b^{n_1} \; a \; b \; b \cdots$

$\rho_1 = \square \cdots \square \; \underset{\in G}{\square} \cdots \underset{\text{again a good state}}{\square}$

Then, we can get n_2 s.t. $b^{n_1} a b^{n_2} a$ ends at a good state.

Then, $\alpha_2 \cdot b^{n_1} a b^{n_2} a b^{\omega} \in L$.

Its unique run ρ_2 matches ρ_1 until $b^{n_1} a b^{n_2} a$.

Keep getting $n_1, n_2, n_3, \ldots, n_{m+1}$.

$\alpha_m = b^{n_1} a b^{n_2} a \cdots b^{n_{m+1}} a b^{\omega} \in \bar{L}$.

$\rho_m = \underbrace{\underset{\in G}{\square} \; \underset{\in G}{\square} \; \cdots \; \underset{\in G}{\square}}_{m+1 \text{ states}}$

By PHP, two of these $m+1$ good states are equal. Loop between them to get a word which is accepted but has inf. many 'a's. ☒

Cor. DBA $\subsetneq$ NBA in terms of expressiveness.

Def^n A language $L \subseteq \Sigma^{\omega}$ is said to be ω-regular if there exists a (possibly non-deterministic) Büchi automaton A such that $L(A) = L$.

Closure Properties of ω-Regular Languages

1) closure under union:

$L_1 = L(A_1)$, $\quad A_1 = (Q_1, q_0^1, \Sigma, \Delta_1, G_1)$,

$L_2 = L(A_2)$, $\quad A_2 = (Q_2, q_0^2, \Sigma, \Delta_2, G_2)$.

To-do: Construct a BA A s.t. $L(A) = L_1 \cup L_2$.

We do the usual product construction.

$$(Q_1 \times Q_2,\ (q_0^1, q_0^2),\ \Sigma,\ \Delta,\ G_1 \times Q_2 \cup Q_1 \times G_2)$$

$$(q_1, q_2) \xrightarrow{a} (q_1', q_2')$$

iff $\quad q_1 \xrightarrow{a} q_1'$ and $q_2 \xrightarrow{a} q_2'$.

$\alpha = a_0\ a_1\ a_2 \cdots$

$\rho^1 = q_0^1\ q_1^1\ q_2^1 \cdots$ a run of A_1 on α

$\rho^2 = q_0^2\ q_1^2\ q_2^2 \cdots$ —"— A_2 —"—

"$\rho^1 \times \rho^2$" $= \begin{pmatrix} q_0^1 \\ q_0^2 \end{pmatrix} \begin{pmatrix} q_1^1 \\ q_1^2 \end{pmatrix} \begin{pmatrix} q_2^1 \\ q_2^2 \end{pmatrix} \cdots$ a "product run" on α

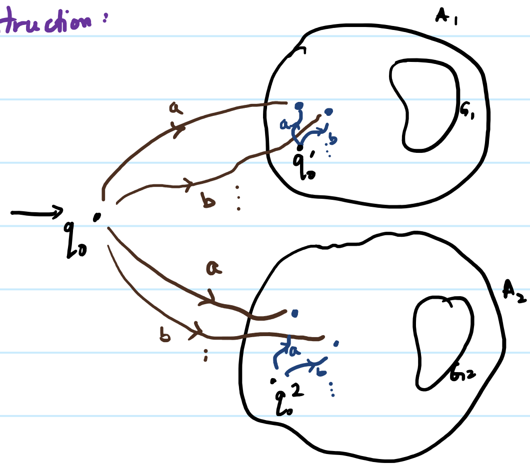
Here we assume that each $\alpha \in \Sigma^\omega$ has at least one run on both A_i. (Can always ensure this by adding a dead state)

With the above assumption,

$$G = (G_1 \times Q_2) \cup (Q_1 \times G_2) \text{ gives}$$

the language as $L_1 \cup L_2$.

A simpler construction:



Lecture 3 (06-08-2021)

06 August 2021 09:39

Closure under intersection.

Do the same product construction as earlier and put
$$G = G_1 \times G_2.$$

$A = A_1 \times A_2$.

Is: $L(A) = L(A_1) \cap L(A_2)$.

($\subseteq$) If $\rho = \rho_1 \times \rho_2$ is an accepting run, so ρ_1 and ρ_2 both are.

($\supseteq$) Let $\alpha \in L(A_1) \cap L(A_2)$.

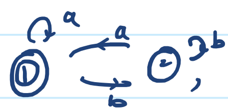
Then there are accepting runs ρ_i on A_i.

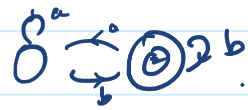
Put $\rho = \rho_1 \times \rho_2$.

But then it is not necessary that ρ is accepting.

For example, ρ_1 has good states at even positions and ρ_2 at odd.

As a concrete example of above:

Take A_1 = 

A_2 = 

Then $(ab)^\omega \in (L(A_1) \cap L(A_2)) \setminus L(A_1 \times A_2)$.

Doesn't work! Slightly modified.

$Q = Q_1 \times Q_2 \times \{1, 2\}$, → indicates the component being "searched" for a good state

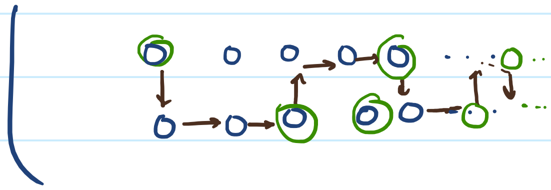
$q_0 = (q_0^1, q_0^2, 1)$

$\Delta = (q_1, q_2, 1) \xrightarrow{a} (q_1', q_2', 1)$ if $q_1 \xrightarrow{a} q_1' \in \Delta_1$, $q_2 \xrightarrow{a} q_2' \in \Delta_2$, $q_1 \notin G_1$

$$(q_1, q_2, 1) \xrightarrow{a} (q_1', q_2', 2) \qquad \begin{array}{l} q_1 \xrightarrow{a} q_1' \\ q_2 \xrightarrow{a} q_2' \\ q_1 \in G_1 \end{array}$$

similarly for $(\ , \ , 2) \to (\ , \ , 2)$
$(\ , \ , 2) \to (\ , \ , 1)$.

$G = G_1 \times Q_2 \times \{1\}$.



$L(A) = L(A_1) \cap L(A_2)$.

Closure under projection:

$\pi : \Sigma_1 \times \Sigma_2 \to \Sigma_1$ induces $\pi : (\Sigma_1 \times \Sigma_2)^\omega \to \Sigma_1^\omega$

$$\binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots \mapsto a_0 a_1 a_2 \cdots$$

If $L \subseteq (\Sigma_1 \times \Sigma_2)^\omega$ is ω-regular, so is $\pi(L)$.

Let $A = (Q, q_0, \Sigma_1 \times \Sigma_2, \Delta, G)$ be a BA with $L(A) = L$.

Goal: Construct B s.t. $L(B) = \pi(L)$.

Define $B = (Q, q_0, \Sigma_1, \Delta', G)$, where

$$\Delta' = \left\{ q \xrightarrow{a} q' \ : \ \begin{array}{l} \exists b \in \Sigma_2 \text{ s.t.} \\ q \xrightarrow{(a,b)} q' \text{ in } A \end{array} \right\}.$$

(Take original automata and erase all the second components.)

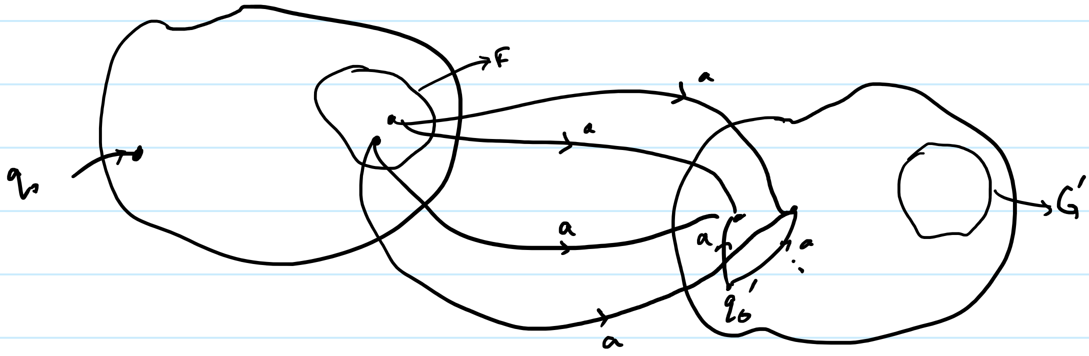
$U \subseteq \Sigma^*$, $L \subseteq \Sigma^\omega$, $U \cdot L \subseteq \Sigma^\omega$.

(The concat. of a finite word followed by an infinite word is defined in the natural way.)

(Nonsense : Concat : $\Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ or concat : $\Sigma^\omega \times \Sigma^* \to \Sigma^\omega$.)

Closure: $U \subseteq \Sigma^*$ regular $\quad A = (Q_0, q_0, \Sigma, \Delta, F)$, $\to$ NFA, $L(A) = U$

$L \subseteq \Sigma^\omega$ ω-regular $\quad B = (Q_0', q_0', \Sigma', \Delta', G')$. $\hookrightarrow$ BA, $L(B) = L$



Keep them disjoint and all possible transitions of the form:

$$q_f \xrightarrow{a} q' \quad \text{where } q_f \in F$$

and $q_0' \xrightarrow{a} q'$ in Δ'.

Keep G as G'. #

Give $U \subseteq \Sigma^*$, define

$$U^\omega = \{ \alpha \in \Sigma^\omega : \alpha \text{ has a factorisation of the form}$$

$$\alpha = \alpha_0 \alpha_1 \alpha_2 \alpha_3 \cdots \text{ for } \alpha_i \in U. \}$$

Closure If $U \subseteq \Sigma^*$ is regular, then U^ω is ω-regular.

Let $A = (Q, q_0, \Sigma, \Delta, F)$ recognise U.

Assume that there are no incoming transitions to q_0 and that $q_0 \notin F$. (Why can we do this?)

(Also note $U^\omega = (U \setminus \{\varepsilon\})^\omega$.)

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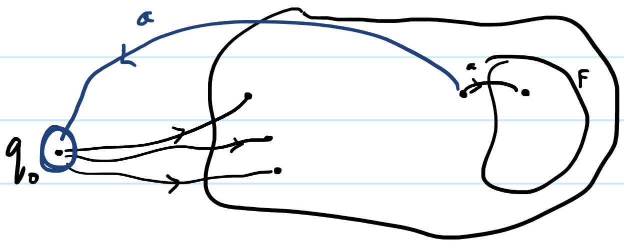

Add all possible transitions of the form:

$$q \xrightarrow{a} q_0 \quad \text{if} \quad \exists q_f \in F \text{ s.t. } q \xrightarrow{a} q_f.$$

Put $G = \{q_0\}$.