

Lecture 1 (30-07-2021)

30 July 2021 09:28

Automata on infinite words

Some notations

- Let Σ be a finite nonempty set (called alphabet).
- A finite word (over Σ) is a finite sequence w of letters from Σ .
 $w = a_0 a_1 \dots a_n, \quad a_i \in \Sigma.$
- ϵ is the empty word.
- Σ^* is the set of all finite words (over Σ).
- An infinite word over Σ is an infinite sequence of letters from Σ .
 $\alpha = a_0 a_1 a_2 \dots, \quad a_n \in \Sigma \quad \forall n \in \mathbb{N}_0.$

(Different formality: $\mathbb{N}_0 = \{0, 1, \dots\}$ and $\alpha: \mathbb{N}_0 \rightarrow \Sigma^*$)

Also, $\omega = \mathbb{N}_0$.

$\Sigma^\omega =$ all infinite words (on Σ)

(In general, given sets X and Y , Y^X denotes the set of all functions $x \rightarrow y$)

Examples ① $\Sigma = \{a, b\}$

$\alpha = a b a b a b \dots$

or: $\alpha(n) = \begin{cases} a & ; \quad 2|n \\ b & ; \quad 2 \nmid n \end{cases}$

② $\alpha = a b b a b b a b b$

$\alpha(n) = \begin{cases} a & ; \quad 3|n \\ b & ; \quad 3 \nmid n \end{cases}$

$\alpha = (abb)^\omega$

③ $\gamma: \omega \rightarrow \{a, b\}$

$\gamma(n) = \begin{cases} a & ; \quad n \text{ is prime} \\ b & ; \quad \text{otherwise} \end{cases}$

$\gamma = \underset{0}{b} \underset{1}{b} \underset{2}{a} \underset{3}{a} \underset{4}{b} \underset{5}{a} \underset{6}{b} \underset{7}{a} \underset{8}{b} \underset{9}{b} \underset{10}{a} \dots$

γ has infinitely many 'a's and 'b's.
(Can't write as compactly as before.)

$$\textcircled{1} \quad \Sigma = \{a\}, \quad \Sigma^\omega = \{a^\omega\}. \quad (|\Sigma| = 1)$$

But if $|\Sigma| > 1$, then Σ^ω is not a countable set.
OTOH, Σ^* is always a countable set. ($1 \leq |\Sigma| < \infty$)

Automata:

$$A = (Q, \Sigma, q_0, \Delta \subset Q \times \Sigma \times Q, \text{"Acceptance condition"})$$



→ a finite set of states

→ $q_0 \in Q$ → the initial state (unique)

→ $\Delta \subset Q \times \Sigma \times Q$ → the transition relation

$$(q, a, q') \in \Delta \equiv q \xrightarrow{a} q'$$

(Non-determinism here is fine)

Now, let $\alpha = a_0 a_1 a_2 \dots \in \Sigma^\omega$ be given.

A run β of A on α is an infinite sequence of states

$$\beta = q_0 q_1 q_2 \dots$$

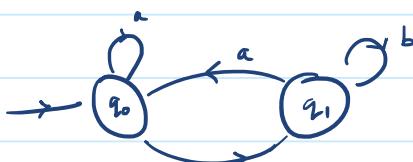
(run)

such that " q_0 " is indeed the initial state and

$$\forall i \in \omega : (q_i, a_i, q_{i+1}) \in \Delta$$

In terms of functions: Given $\alpha : \omega \rightarrow \Sigma$, we have
 $f : \omega \rightarrow \Sigma$ s.t. $f(0) = q_0$ and $(f(n), \alpha(n), f(n+1)) \in \Delta \quad \forall n \in \omega$.

Example.



$$\alpha = (ab)^\omega$$

$$\alpha = a \cdot (a \cdot b)^\omega$$

$$f = q_0 (q_0 q_1)^\omega$$

"Acceptance condition": (Büchi automata)

$\alpha \rightarrow$ input word

$f \rightarrow$ a run of A on α

$\text{Inf}(f) :=$ the set of states which occur infinitely often along f

$$= \{q \in Q : \exists^\infty i \in \omega \text{ s.t. } f(i) = q\}$$

Obs. $\text{Inf}(f) \neq \emptyset$. (There are only finitely many states.)

Büchi automaton (BA): fix $G \subseteq Q$ called the "good state".

A run f is accepted by a BA if $\text{Inf}(f) \cap G \neq \emptyset$.

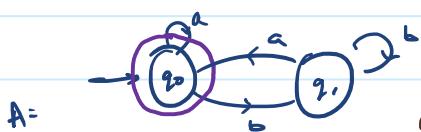
(Thus, some good state appears infinitely often.)

A word $\alpha \in \Sigma^\omega$ is accepted by A if α has an accepting run f on the word α .

$$L(A) := \{ \alpha \in \Sigma^\omega : A \text{ accepts } \alpha \}$$

↳ Language of A

Example



$$G = \{q_0\}, \Sigma = \{a, b\}$$

Claim. $L(A) = \{\alpha \in \Sigma^\omega : \alpha \text{ has inf.}\}$

many 'a' s.

Prof. Let the right side be L.

• $L(A) \subseteq L$:

$$\alpha \in L(A), \quad \alpha = a_0 a_1 a_2 \dots$$

Note that A is deterministic, thus α has a unique run f , which is accepted.

$$f = q_0 q'_1 q'_2 q'_3 \dots$$

Thus, q_0 appears inf. often above. Since it only receives 'a', we see that 'a' appears inf. often.

• $L \subseteq L(A)$:

Let $\alpha \in L$. It has a unique run f .

Then, since α has inf. many 'a's, f will have inf. many ' q'_0 's.

B

Can also write $L = (b^* a)^\omega$ once we have defined what that means.

Question What about $\overline{L} = \Sigma^\omega \setminus L$? Can that be accepted by a Büchi automaton?

Lecture 2 (04-08-2021)

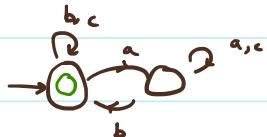
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Note. We do NOT allow ϵ transitions in this course.

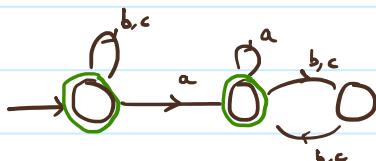
Fact. Even though we insisted on single initial state, the expressive power does not change if we allow more.
(It is simply for convenience.)

Example. (1) L over $\Sigma = \{a, b, c\}$.

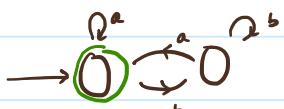
$L = \text{every 'a' is eventually followed by a 'b'}$



(2) $L_2 = \text{any two occurrences of 'a' are separated by even no. of other (b, c) letters}$

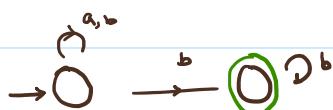


(3) $\Sigma = \{a, b\}$, $L = \text{inf. many 'a's}$

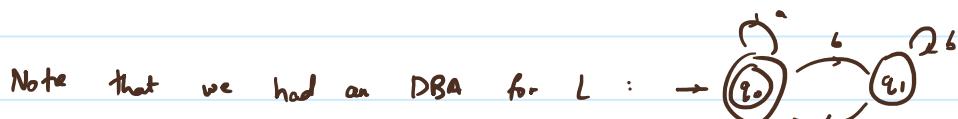


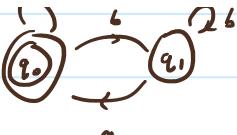
Complement: $\bar{L} = \Sigma^\omega \setminus L = \text{finitely many 'a's}$

Q. What is a BA for \bar{L} ?



Q. Do we have a deterministic Büchi automaton (DBA) for \bar{L} ?



Note that we had an DBA for L : 

$$A : G = \{q_0\}$$

Toggle states . $A' : G = \{q_1\}$

But $L(A') \not\supseteq \bar{L}$.

\downarrow
infinitely
many b

\downarrow
eventually
 a

$$(ab)^\omega \in L(A') \text{ but } (ab)^\omega \notin \bar{L}$$

Complementing the good state of a DBA does

NOT complement the accepted language.

Claim: There is no DBA for $\bar{L} = \{\alpha \in \Sigma^\omega : \alpha \text{ has finitely many } 'a's\}$.

Thus, as opposed to finite languages, non-determinism actually gives us more languages.

Proof. We prove this by contradiction.

For the sake of contradiction, assume that \exists DBA A such that $L(A) = \bar{L}$.

Suppose A has m states.

$$\alpha_0 = b^\omega = b\ b b \dots \in \bar{L}$$

$$f_0 = q_0\ q_1\ q_2 \dots$$

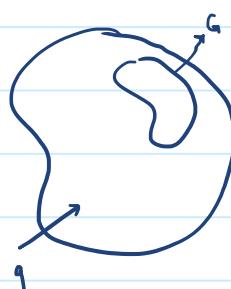
↳ unique run of α_0 .

Since f_0 is accepting, $\exists n_1 \text{ s.t. } q_{n_1, n_1} \in G$.

$\underbrace{b\ b \dots b}_{n_1} \mid b\ b \dots$ Pick the smallest such n_1 .

$$f_0 = \boxed{\quad} \uparrow \text{first good state}$$

$$\text{Define } a_1 := b^{n_1} a b^\omega \in \bar{L}.$$



$\alpha_1 = b \cdots b^{n_1} a b b \cdots$
 $\beta_1 = \square \cdots \square \xrightarrow{\text{EG}} \cdots \square$
 \nwarrow_{EG}
again a
good state

Then, we can get n_2 s.t. $b^{n_1} a b^{n_2} a$ ends at a good state.

Then, $\alpha_2 = b^{n_1} a b^{n_2} a b^\omega \in L$.

Its unique run β_2 matches β_1 until $b^{n_1} a b^{n_2} a$.

Keep getting $n_1, n_2, n_3, \dots, n_{m+1}$.
 $\alpha_m = b^{n_1} a b^{n_2} a \cdots b^{n_{m+1}} a b^\omega \in L$.

$\beta_m = \underbrace{\square_{\text{EG}} \square_{\text{EG}}}_{m+1 \text{ states}} \cdots \xrightarrow{\text{EG}}$

By PMP, two of these $m+1$ good states are equal. Loop between them to get a word which is accepted but has inf. many 'a's. \square

Cor. DBA \subsetneq NBA in terms of expressiveness.

Defn A language $L \subseteq \Sigma^\omega$ is said to be ω -regular if there exists a (possibly non-deterministic) Büchi automaton A such that $L(A) = L$.

CLOSURE PROPERTIES OF ω -REGULAR LANGUAGES

i) closure under union:

$$L_1 = L(A_1), \quad A_1 = (Q_1, q_0^1, \Sigma, \Delta_1, \delta_1),$$

$$L_2 = L(A_2), \quad A_2 = (Q_2, q_0^2, \Sigma, \Delta_2, \delta_2).$$

To-do: Construct a BA A s.t. $L(A) = L_1 \cup L_2$.

We do the usual product construction.

$$(Q_1 \times Q_2, (q_1^1, q_2^1), \Sigma, \Delta, \underbrace{G_1 \times G_2 \cup Q_1 \times b_2}_{\delta})$$

$$(q_1, q_2) \xrightarrow{a} (q_1', q_2')$$

If $q_1 \xrightarrow{a} q_1'$ and $q_2 \xrightarrow{a} q_2'$.

$$\alpha = c_0 a_1 a_2 \dots$$

$$s^1 = q_0' q_1' q_2' \dots \quad \text{a run of } A_1 \text{ on } \alpha$$

$$s^2 = q_0^2 q_1^2 q_2^2 \dots \quad \overbrace{\quad \quad \quad}^n \quad \overbrace{A_2 \quad \quad}^n$$

$$"s^1 s^2" = \begin{pmatrix} q_0' \\ q_0^2 \end{pmatrix} \begin{pmatrix} q_1' \\ q_1^2 \end{pmatrix} \begin{pmatrix} q_2' \\ q_2^2 \end{pmatrix} \dots \quad \text{a "product run" on } \alpha$$

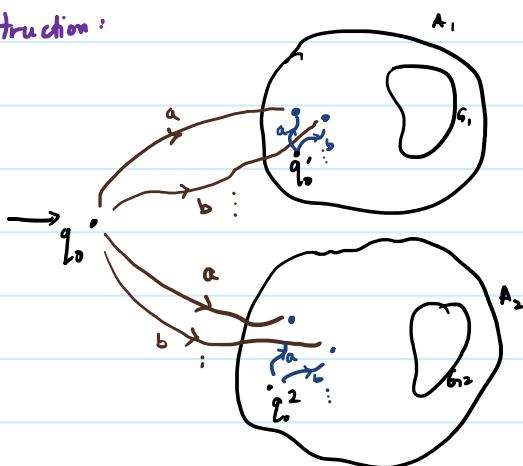
Here we assume that each $\alpha \in \Sigma^\omega$ has at least one

run on both A_i . (Can always ensure this by adding a
dead state)

With the above assumption,

$G = (G_1 \times Q_2) \cup (Q_1 \times G_2)$ give
the language as $L_1 \cup L_2$.

A simpler construction:



Lecture 3 (06-08-2021)

06 August 2021 09:39

Closure under intersection.

Do the same product construction as earlier and put

$$G = G_1 \times G_2.$$

$$A = A_1 \times A_2.$$

Is: $L(A) = L(A_1) \cap L(A_2).$

(\Leftarrow) If $p = p_1 \times p_2$ is an accepting run, so p_1 and p_2 both are.

(\Rightarrow) Let $\alpha \in L(A_1) \cap L(A_2).$

Then there are accepting runs p_i on A_i .

$$\text{But } p = p_1 \times p_2.$$

But then it is not necessary that p is accepting.

For example, p_1 has good states at even positions and p_2 at odd.

As a concrete example of above:



$$\text{Then } (ab)^\omega \in (L(A_1) \cap L(A_2)) \setminus L(A_1 \times A_2).$$

Doesn't work! Slightly modified.

$$Q = Q_1 \times Q_2 \times \{1, 2\}, \quad \xrightarrow{\text{indicates}} \text{the component being "searched" for a good state}$$

$$q_0 = (q_0^1, q_0^2, 1)$$

$$\Delta = (q_1, q_2, 1) \xrightarrow{a} (q_1', q_2', 1) \text{ if } \begin{cases} q_1 \xrightarrow{a} q_1' & \text{if } q_1 \in G_1 \\ q_2 \xrightarrow{a} q_2' & \text{if } q_2 \in G_2 \end{cases}$$

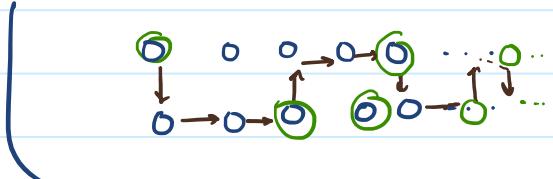
$$q_1 \notin G_1,$$

$$(q_1, q_2, 1) \xrightarrow{a} (q'_1, q'_2, 2) \quad \begin{matrix} q_1 \xrightarrow{a} q'_1 \\ q_2 \xrightarrow{a} q'_2 \\ q_1 \in G_1 \end{matrix}$$

similarly for $(\cdot, \cdot, 2) \rightarrow (\cdot, \cdot, 2)$
 $(\cdot, \cdot, 2) \rightarrow (\cdot, \cdot, 1)$.

$$G = G_1 \times Q_2 \times \{1\}.$$

$$L(A) = L(A_1) \cap L(A_2).$$



Closure under projection:

$$\pi: \Sigma_1 \times \Sigma_2 \rightarrow \Sigma_1 \quad \text{induces} \quad \pi: (\Sigma_1 \times \Sigma_2)^\omega \rightarrow \Sigma_1^\omega$$

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \dots \mapsto a_0 a_1 a_2 \dots$$

If $L \subseteq (\Sigma_1 \times \Sigma_2)^\omega$ is ω -regular, so is $\pi(L)$.

Let $A = (Q, q_0, \Sigma_1 \times \Sigma_2, \Delta, \delta)$ be a BA with $L(A) = L$.

Goal: Construct B s.t. $L(B) = \pi(L)$.

Define $B = (\emptyset, q_0, \Sigma, \Delta', G)$, where

$$\Delta' = \left\{ q \xrightarrow{a} q' : \exists b \in \Sigma \cdot q \xrightarrow{(a,b)} q' \right\}_{\text{in } A}.$$

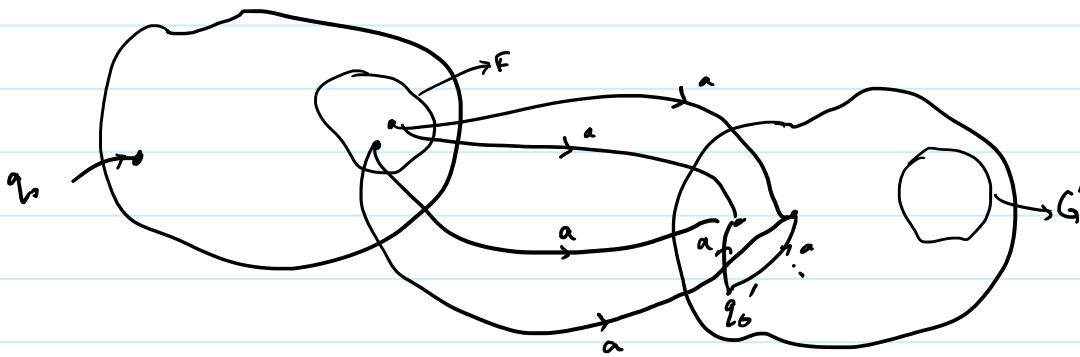
(Take original automata and erase all the second components.)

$$U \subseteq \Sigma^*, \quad L \subseteq \Sigma^{\omega}, \quad U \cdot L \subseteq \Sigma^{\omega}.$$

(The concat. of a finite word followed by an infinite word is defined in the natural way.)

(Nonsense : Concat : $\Sigma^\omega \times \Sigma^* \rightarrow \Sigma^*$ or Concat : $\Sigma^* \times \Sigma^* \rightarrow \Sigma^\omega$)

Closure : $U \subseteq \Sigma^*$ regular $A = (Q_0, q_0, \Sigma, \Delta, F)$, $L(A) = U$
 $L \subseteq \Sigma^\omega$ ω -regular $B = (Q'_0, q'_0, \Sigma', \Delta', G)$, $L(B) = L$



Keep them disjoint and all possible transitions of the form:

$$q_f \xrightarrow{a} q'_f \quad \text{where } q_f \in F \text{ and } q'_f \xrightarrow{a} q'_f \text{ in } \Delta'$$

Keep G as G' .

Given $U \subseteq \Sigma^*$, define

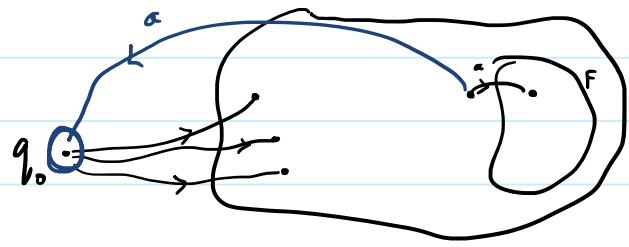
$U^\omega = \{ \alpha \in \Sigma^\omega : \alpha \text{ has a factorisation of the form}$

$$\alpha = \alpha_0 \alpha_1 \alpha_2 \alpha_3 \dots \text{ for } \alpha_i \in U.$$

Closure If $U \subseteq \Sigma^*$ is regular, then U^ω is ω -regular.

Let $A = (Q, q_0, \Sigma, \Delta, F)$ recognise U .

Assume that there are no incoming transitions to q_0 .
and that $q_0 \notin F$.
(Why can we do this?)
(Also note $U^\omega = (U \setminus \{\epsilon\})^\omega$)



(Also note $U^\omega = (U \setminus \{\epsilon\})^\omega$)

Add all possible transitions of the form:

$$q \xrightarrow{a} q_0 \quad \text{if} \quad \exists q_f \in F \text{ s.t. } q \xrightarrow{*} q_f.$$

Put $b_1 = \{q_0\}$.

Lecture 4 (11-08-2021)

11 August 2021 09:33

To Do: Closure under complementation.

Prop: Let L be ω -regular. Then, L can be expressed as

$$L = \bigcup_{i=1}^n U_i \cdot V_i^\omega,$$

where $U_i, V_i \subseteq \Sigma^*$ are regular languages for $i = 1, \dots, n$.

(By our earlier results, it is clear that any such L is indeed ω -regular.)

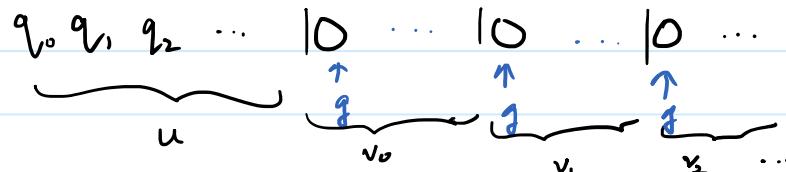
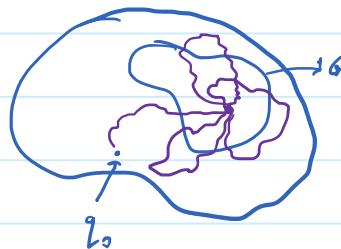
Proof: Let A be a BA s.t. $L(A) = L$.

$$(Q, \Sigma, q_0, \Delta, G)$$

Given an accepted word $w = a_0 a_1 a_2 \dots$

with an accepting run $\rho = q_0 q_1 q_2 \dots$,

$\exists g \in G$ which occurs i.o.



For $g \in G$, define

$$\begin{cases} U_g := \{w \in \Sigma^* : \exists \text{ a run } q_0 \xrightarrow{\omega} g\} \\ V_g := \{w \in \Sigma^* : \exists \text{ a run } g \xrightarrow{\omega} g\}. \end{cases}$$

regular since $A_g := (Q, \Sigma, q_0, \Delta, \{g\})$ and

$B_g := ((Q, \Sigma, \{g\}), \Delta, \{g\})$ accept them

Now, by our earlier observation, it is easy to argue that

$$L = \bigcup_{g \in G} U_g \cdot V_g^\omega.$$

Obs. The following problem is decidable: (Non emptiness problem)

- Input :— $A \rightarrow a \text{ BA}$
- Output :— YES if $L(A) \neq \emptyset$,
NO if $L(A) = \emptyset$.

$(L(A) \neq \emptyset \Leftrightarrow \exists g \in G \text{ st. } \exists q_0 \xrightarrow{\omega} g \text{ and } \exists g \xrightarrow{\omega} g)$

reachable from initial state
 both
 check if part of cycle
 efficient ✓

Obs. If $L(A) \neq \emptyset$, then there exist finite words u and v s.t. $|u|, |v| \leq |Q|$ and $u \cdot v^\omega \in L(A)$.

↑
 ultimately periodic

Let $A = (Q, \Sigma, q_0, \Delta, G)$ be a BA accepting L .

Goal: To show that $\bar{L} = \Sigma^\omega \setminus L$ is also ω -regular.

For $u, v \in \Sigma^*$, define

$$u \sim_p v \Leftrightarrow \forall q, q' \in Q, q \xrightarrow{\omega} q' \text{ iff } q \xrightarrow{v} q' \text{ and } q \xrightarrow{\omega_G} q' \text{ iff } q \xrightarrow{\omega} q'.$$

Notation: $s \xrightarrow{\pi} s'$ means
 $\exists \text{ a run on } \pi \text{ from } s \text{ to } s'$
 with an intermediate visit to G .

Observations:

(i) \sim_p is an equivalence relation on Σ^*

(i) \sim_A is an equivalence relation on Σ^* .

(ii) \sim_A is of finite index, i.e., it has finitely many equivalence classes.

Proof. Fix $q, q' \in Q$.

$$U_{q,q'} = \{w \in \Sigma^*: q \xrightarrow{w} q'\}$$

$$V_{q,q'} = \{w \in \Sigma^*: q \xrightarrow{w} q'\}$$

2^{n^2} such sets. ($n := |Q|$)

For each $u, v \in \Sigma^*$, we can ask 2^{n^2} questions about set membership. $u \sim_A v \Leftrightarrow$ they have same answers.

Thus, there are $\leq 2^{n^2}$ classes.

$$[u]_{\sim_A} = \left(\bigcap_{\substack{q, q' \in Q \\ u \in U_{q,q'}}} U_{q,q'} \right) \cap \left(\bigcap_{\substack{q, q' \in Q \\ u \in V_{q,q'}}} V_{q,q'} \right) \cap \left(\bigcap_{\substack{q, q' \in Q \\ u \notin U_{q,q'}}} \bar{U}_{q,q'} \right) \cap \left(\bigcap_{\substack{q, q' \in Q \\ u \notin V_{q,q'}}} \bar{V}_{q,q'} \right).$$

The above discussion also shows that each equivalence class is a regular language.

($U_{q,q'}$ is clearly regular. Some argument shows the same for $V_{q,q'}$.)

Let U_1, \dots, U_m be the equivalence classes of \sim_A .

Lemma: Suppose $L \cap (U_i \cdot U_j^\omega) \neq \emptyset$ for some i, j , then $U_i \cdot U_j^\omega \subseteq L$.

Proof. Let $\alpha \in L \cap (U_i \cdot U_j^\omega)$.

$$\alpha = \alpha_0 \alpha_1 \alpha_2 \alpha_3 \dots \in L.$$

Let $\rho = q_0 q_1 q_2 q_3 \dots$ be an accepting run of A

Let $\beta = q_0 q_1 q_2 q_3 \dots$ be an accepting run of A
on α .

We can also write $\alpha = u \cdot v_0 \cdot v_1 \cdot v_2 \dots$ s.t. $u \in U_i$ and
 $v_0, v_1, v_2, \dots \in U_j$.

$$\beta_\alpha = \underbrace{q_0}_{u}, \underbrace{q'_1}_{v_0}, \underbrace{q'_2}_{v_1}, \underbrace{q'_3}_{v_2} \dots$$

Now, let $\beta \in U_i \cup U_j^\omega$. Then, $\beta = u' v'_0 v'_1 \dots$

Then, we have a run

$$\beta_B = q_0 q'_1 q'_2 q'_3 \dots \text{ by def" of } \beta_A.$$

Moreover if β_A saw a good state $q'_i \xrightarrow{\alpha} q'_{i+1}$,

so does β_B .

$\therefore \beta \in L(A)$. B

Lecture 5 (13-08-2021)

13 August 2021 09:36

Ramsey's Theorem

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

$$E = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i < j\}$$

} Complete graph on \mathbb{N}

\mathcal{C} - a finite set of colours.

$x: E \rightarrow \mathcal{C}$ is called an edge-colouring of the complete graph on \mathbb{N} .

Thm. Given an arbitrary x , \exists an infinite monochromatic clique in x .
That is,

$\exists S \subseteq \mathbb{N}, |S| = \infty, \exists c \in \mathcal{C}$ such that every edge with S is coloured ' c '.

$$(\forall i, j \in S : i < j \Rightarrow x((i, j)) = c)$$

Proof. Fix $x: E \rightarrow \mathcal{C}$.

$$x_0 := \mathbb{N}, m_0 := \min(x_0) (= 0).$$



$\exists c \in \mathcal{C}$ s.t. \exists infinitely many x s.t.
 $x((m_0, x)) = c$.

Let $x_1 := \{\text{neighbours of } m_0 \text{ in } x_0\} \subseteq x_0 \setminus \{m_0\}$.

Note: $x_1 \subseteq \mathbb{N}$ is infinite.

Let $m_1 := \min(x_1)$ and proceed similarly to pick
 c and $x_2 \subseteq x_1 \setminus \{m_1\} \dots$

In general, we have an infinite subset x_{k+1} and colour c_k
s.t. every element of x_{k+1} is connected to $\min(x_k)$ by c_k .

Define $x_\infty := \{m_0, m_1, m_2, \dots\}$. $(m_0 < m_1 < m_2 < \dots)$

Then, x_∞ is an infinite set s.t. $\forall i, j : x((m_i, m_j)) = c_i \quad \forall i < j$.

As usual, $\exists c \in \mathcal{C}$ which occurs infinitely many often.

Simply restrict graph to these vertices.



Continuing from last lecture: U_1, \dots, U_m are equiv. classes of \sim_A .

We know: U_i are regular.

Claim. $\sum^\omega = \bigcup_{i,j} U_i \cdot U_j^\omega$.

Proof. Only (\subseteq) is to be shown.

Let $\alpha \in \sum^\omega$ be arbitrary.

IS: $\stackrel{\exists i, j}{\alpha} = u_0 v_0 v_1 v_2 \dots$ for $u_0 \in U_i$ and $v_k \in U_j \forall k$.

Write $\alpha = a_0 a_1 a_2 a_3 \dots \in \sum^\omega$ for $a_i \in \Sigma$.

Define the coloring x_α on (\mathbb{N}, \leq) as:

$$\mathcal{C} = \{U_1, \dots, U_m\}$$

$$x_\alpha(i, j) = [a_i a_{i+1} \dots a_{j-1}]_{\sim_A}.$$

\hookrightarrow equiv class of $a_i a_{i+1} \dots a_{j-1}$

By Ramsey's theorem, $\exists U_j$ with a clique, i.e., $\exists m_1 < m_2 < m_3 < \dots$
s.t. $x_\alpha((n_k, n_{k+1})) = U_j \quad \forall j$.

Defining

$$u_0 = a_0 \dots a_{m_1-1}, \quad v_0 = a_{m_1} \dots a_{m_2-1}, \\ v_1 = a_{m_2} \dots a_{m_3-1}, \dots$$

does the job. \square

$$\sum^* = U_1 \sqcup U_2 \sqcup \dots \sqcup U_m,$$

$$\sum^\omega = \bigcup_{i,j} U_i \cdot U_j^\omega.$$

Note that U_i are regular. Moreover, we have

$$L \cap (U_i \cdot U_j^\omega) \neq \emptyset \Rightarrow U_i \cdot U_j^\omega \subseteq L.$$

Thus, $L = \bigcup_{\text{some } i, j} U_i \cdot U_j^\omega.$

Thus, $\sum^\omega \setminus L = \bigcup_{i, j : U_i \cdot U_j^\omega \not\subseteq L} U_i \cdot U_j^\omega.$

Thus, it is again ω -regular. \square

→ Effective construction of BA for \bar{L} .

- Construct automaton for U_i .
- Construct BA for $U_i \cdot U_j^\omega$.
- Take union of those not in L .

(Can effectively check if $L \cap (U_i \cdot U_j^\omega) = \emptyset$.)

Lecture 6 (18-01-2021)

18 August 2021 09:31

Büchi's Theorem:

Want to talk about properties of words (finite or infinite).

First-order Logic (over words)

Fix $\Sigma \rightarrow \text{alphabet}$.

First-order variables - $x, y, z, x_1, x_2, x_3, \dots$

Range over positions
in the word

Atomic-predicate - $a(x), b(x), \dots$

→ unary predicate

for $a, b, \dots \in \Sigma$

and x is a Fo variable.

$x < y$

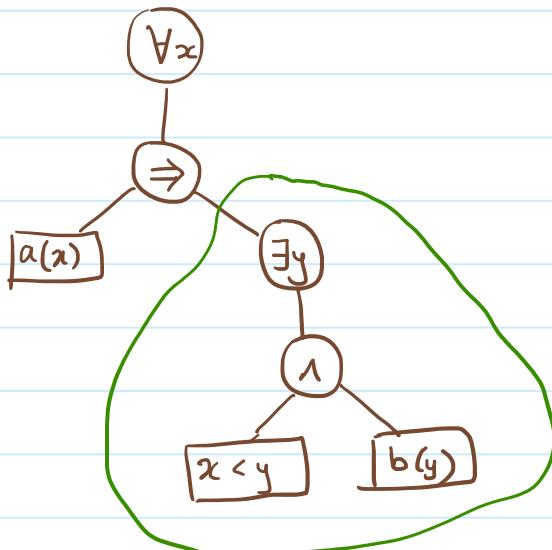
→ binary predicate

Syntax :

$\varphi \equiv a(x) \mid x < y \mid \neg \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x. \varphi \mid \forall x. \varphi$

derived : $\varphi_1 \Rightarrow \varphi_2 \equiv \neg \varphi_1 \vee \varphi_2$

Example : $\varphi_1 \equiv \forall x. [a(x) \Rightarrow \exists y (x < y) \wedge b(y)]$



Semantics :

$\varphi(x_1, \dots, x_m) \equiv \varphi$ is a formula with free variables

$\varphi(x_1, \dots, x_m) — \varphi$ is a formula with free variables

x_1, \dots, x_m

$$\varphi' = \exists y[(x < y) \wedge b(y)] \rightarrow x \text{ free, } y \text{ bound}$$

$\varphi(x_1, \dots, x_m), w \rightarrow \text{word}$

$$w, x_1 \leftarrow p_1, \dots, x_m \leftarrow p_m \models \varphi(x_1, \dots, x_m)$$

defined by structural induction

1. " $w, x_i \leftarrow p_i \models a(x_i)$ " iff the letter in w at position p_i is a

2. " $w, x_1 \leftarrow p_1, x_2 \leftarrow p_2 \models x_1 < x_2$ " $\Leftrightarrow p_1 < p_2$

:

Example. ① $\varphi_2 \equiv \exists x \forall y [(x < y) \Rightarrow \neg a(y)].$

If w is finite, then it will satisfy φ_2 .

But if w is infinite, then $w \models \varphi_2 \Leftrightarrow w$ has finitely many 'a's

② $\varphi_3 \equiv \forall x \exists y (x < y)$

if w is a (nonempty) finite word, then $w \not\models \varphi_3$.

OTOH, all infinite words satisfy this property

Büchi - Elgot Theorem \rightarrow a logical characterisation of (finite) regular languages

Büchi Theorem \rightarrow a logical characterisation of ω -regular languages

Monadic Second-Order Logic over words

Extends F0 - over words

position variables - $x, y, x_1, x_2, x_3, \dots$

sets-of-positions variables - $X, Y, X_1, X_2, X_3, \dots$

atomic-predicate - $a(x), x < y, X(x)$.

Syntax

$\varphi \equiv \text{atomic-predicates} \mid \neg \varphi \mid (\varphi \vee \varphi) \mid \varphi \wedge \varphi \mid \exists x \cdot \varphi \mid \forall x \cdot \varphi \mid \exists X \cdot \varphi \mid \forall X \cdot \varphi$

$S(x, y) \equiv$ position y is successor of position x
 $\equiv (x < y) \wedge \neg (\exists z \cdot (z < x) \wedge (z < y))$.

$\text{first}(x) \equiv x$ is the first position
 $\equiv \forall y \cdot (x = y \vee x < y)$.

$\text{last}(x) \equiv \dots$

Remark. In F_0 , the ' $<$ ' predicate cannot be expressed using ' S ' predicate.

$x < y \Leftrightarrow$ $x \neq y$ and
 $x < y \Rightarrow$ every successor-closed set of positions which
contains x , also contains y
 \hookrightarrow can define in $M\sigma$

Thus, we can write ' $<$ ' in terms of ' S ' in $M\sigma$.

Defn. Let φ be a $M\sigma$ sentence.

$$L_\varphi = \{\alpha \in \Sigma^\omega : \alpha \models \varphi\}.$$

$L \subseteq \Sigma^\omega$ is called $M\sigma$ -definable if $\exists M\sigma \varphi$ s.t. $L = L_\varphi$.

Theorem (Büchi's Theorem)

Let $L \subseteq \Sigma^\omega$.

L is $M\sigma$ -definable $\Leftrightarrow L$ is ω -regular.

Lecture 7 (20-08-2021)

20 August 2021 09:37

Thm. (Buchi) Let $L \subseteq \Sigma^\omega$.

L is ω -regular $\Leftrightarrow L$ is MSO-definable.

Proof. (\Rightarrow) Suppose L is ω -regular, say $L = L(A)$, where $A = (Q, q_0, \Sigma, \Delta \subseteq Q \times \Sigma \times Q, G)$ is a BA.

Goal: Construct MSO sentence φ_A s.t.

$$\forall \alpha \in \Sigma^\omega : \alpha \models \varphi_A \Leftrightarrow A \text{ accepts } \alpha.$$

$$\alpha = a_0 a_1 a_2 a_3 a_4 \dots$$

Suppose A accepts α via an accepting run ρ .

$$\rho = q_0 q_1 q_2 q_3 q_4 \dots$$

$\forall q \in Q, X_q \equiv$ The set of positions in α when
the run ρ is in the state ' q '
 $= \{i \in \mathbb{N} : q_i \in q\}$. $(0 \in \mathbb{N})$

Note that $\{X_q\}_{q \in Q}$ is a partition of \mathbb{N} .
(Allowing \emptyset in partition.)

- 1) $0 \in X_{q_0}$
- 2) for any two consecutive positions x and y ,
if $x \in X_q$, $y \in X_{q'}$, then $(q, a, q') \in \Delta$,
where a is the letter at position x .
- 3) for any position x , there is a position y
to the right of x such that $y \in X_q$ for
some $q \in G$.

Conversely, given a partition with above 3 properties, we
can build an accepting run.

For convenience, write $Q = \{0, 1, \dots, m\}$.

$$\varphi_A = \exists x_0 \exists x_1 \dots \exists x_m \cdot \text{partition}(x_0, \dots, x_m) \wedge$$

$$[\forall x \cdot \text{first}(x) \Rightarrow x_0(x)] \wedge$$

$$[\forall x \forall y \ S(x, y) \Rightarrow \left(\bigvee_{(i, j) \in A} x_i(x) \wedge x_j(y) \wedge s(i, j) \right)] \wedge$$

$$[\forall x \exists y \ (x < y) \wedge \left(\bigvee_{i \in A} x_i(y) \right)].$$

$$\text{partition}(x_0, x_1, \dots, x_m)$$

$$\equiv \forall x \left(\bigvee_{i \in A} x_i(x) \wedge \bigwedge_{i \neq j} \neg(x_i(x) \wedge x_j(x)) \right).$$

$$\text{length}(\varphi_A) = O(|A|).$$

(\Leftarrow) Given: MSO sentence φ .

Goal: Construct BA A s.t.

$$L(A) = \{\alpha \in \Sigma^\omega \mid \alpha \models \varphi\}.$$

As in the finite case, we use MSO₀-logic

\hookrightarrow substitute position variables by singleton set vars.
 \hookrightarrow more atomic predicates: $\text{Sing}(x)$, $a(x)$, $S(x, y)$, $x \leq y$

\downarrow singleton
 \downarrow x, y are sing and the single positions are related by S
 these can be defined
 in MSO.

\hookrightarrow The converse is true too. Thus, we use them interchangeably.
 That is, they have some expressive power.

Lecture 8 (25-08-2021)

25 August 2021 09:39

Goal: Given a MSO₀ formula, construct a BA A s.t. $L(A) = L(\varphi)$.

The construction of A proceeds by structural induction 4.

In fact: Let $\varphi(x_1, \dots, x_n)$ be an MSO_0 -formula with free
(set-) variables x_1, \dots, x_n

$$\alpha, x_1 \leftarrow p_1, \dots, x_n \leftarrow p_n \quad \Pi^? \quad \varphi$$

$$\alpha = \alpha_0 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \dots$$

$\beta_i = \{0, 1, 3, 4, \dots\}$

$$\beta_i = 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad \dots$$

$$p_2 = \begin{matrix} 0 & 0 & 0 & 0 & 0 & \dots \end{matrix}$$

$$P_n = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \dots \end{pmatrix}$$

Characteristic vectors

$$\begin{pmatrix} a_0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} a_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} a_3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} a_4 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots$$

The model of Ψ (an inf. word Σ + n sets)
 can be seen as an inf. word over $\Sigma \times \{0, 1\}^n$.

$$\text{Free}(\varphi) = \{x_1, \dots, x_n\}.$$

$$L(\varphi) = \{ \alpha' \in (\Sigma \times \{0, 1\})^\omega : \alpha' \models \varphi \}.$$

Claim. $L(\ell)$ is (ϵ) -regular over $\Sigma_n = \Sigma \times \{0, 1\}^n$.

Proof. $\varphi \leadsto A_\varphi$ by structural induction.

base. Ψ - atomic predicate

$$\bullet \varrho = \text{Sing}(X)$$

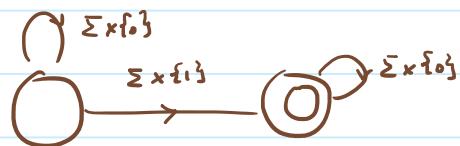
$$\alpha' = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \dots$$

$$a_i \in \Sigma$$

$$\varphi = \text{Sing}(x_1)$$

$$\alpha' = \left(\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix} \right) \left(\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix} \right) \left(\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix} \right) \dots$$

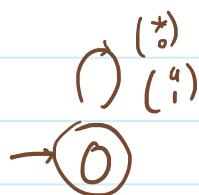
$a_i \in \Sigma$
 $b_i \in \{0, 1\}$



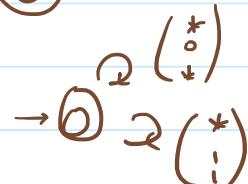
$$\varphi = S(x_1, x_2) \quad ; \quad \Sigma_2 = \Sigma_x \cup \{f_0, 1\} \cup \{0, 1\}$$



$$\varphi = a(x_1)$$



$$\varphi = x_1 \leq x_2$$



Inductive step.

$$\varphi = \varphi_1 \vee \varphi_2.$$

$$\text{Free}(\varphi) \subseteq \{x_1, \dots, x_n\}.$$

wlog, we may assume $\text{Free}(\varphi_1) = \{x_1, \dots, x_n\}$.
 $\text{Free}(\varphi_2)$

By induction, we have appropriate automata A_{φ_i} for φ_i .

But the alphabet for both is same. Can take union of $S\Delta\kappa$.

$$\varphi = \varphi_1 \wedge \varphi_2, \quad \varphi = \neg \varphi_1, \quad \text{similarly done.}$$

$$\varphi = \exists x_n \varphi'(x_1, \dots, x_n)$$

$$\text{Free}(\varphi) = \{x_1, \dots, x_{n-1}\}$$

Note : $\alpha' \models \varphi \Leftrightarrow \exists \text{ a set } P_n \text{ st. }$

$$\alpha', x_n \leftarrow P_n \models \varphi'$$

Consider the projection map

$$\pi : \Sigma \times \{0, 1\}^n \rightarrow \Sigma \times \{0, 1\}^{n-1},$$

$$(a, b_1, \dots, b_n) \mapsto (a, b_1, \dots, b_{n-1}).$$

This induces a map $\pi : (\Sigma^n)^\omega \rightarrow (\Sigma^{n-1})^\omega$.

$$\alpha' \in (\Sigma^{n-1})^\omega, \quad \alpha' \models \varphi \Leftrightarrow \exists \alpha'' \in (\Sigma^n)^\omega \text{ s.t.}$$

$$\pi(\alpha'') = \alpha' \text{ and } \alpha'' \models \varphi'.$$

The question is reduced to asking if projection of an ω -regular language is ω -regular.

But this is simple to see. Take an automaton for φ' and erase the last coordinate on all transitions.

$$\bullet \varphi = \forall X_n. \varphi'.$$

$$\text{Some } a_0 \exists X_n \rightarrow \varphi'.$$

B

Thus, we are done.

Thm.

(Büchi's Theorem) let $L \subseteq \Sigma^\omega$. Then,

$$L \text{ is } \omega\text{-regular} \Leftrightarrow L \text{ is MSO-definable.}$$

Moreover, the translations are effective.

The above theorem was proven a few years after the Büchi - Elgot theorem (the analogous theorem about (finite) regular languages).

It is easy to see how MSO-definability translates to ω -words but was not so clear how to extend regularity.

Thus, $\text{MSO}(\Sigma)$ is decidable.

Given an MSO sentence φ , we can check if there

exists an inf. word $\alpha \in \Sigma^\omega$ s.t. $\alpha \models \varphi$.

$\left[\varphi \rightsquigarrow A_\varphi$ is effective and we can check $L(A_\varphi) \neq \emptyset \right]$

In fact, if $L(A_\varphi) \neq \emptyset$, then $\exists u, v$ s.t. $uv^\omega \in L(A_\varphi)$

and we can produce the above u, v .

Note: $\varphi \rightsquigarrow A_\varphi$ is non-elementary.

We cannot bound $|A_\varphi|$ in terms of any

(fixed) k -ary exponential of $|\varphi|$.

$(n = |\varphi|, 2^{P(n)}, 2^{2^{P(n)}}, \dots \leftarrow \text{elementary})$

singly exp

doubly exp

$P \leftarrow \text{polynomial}$

The tower (we get) will have length in terms of n .

↙ can we do better for satisfiability?

FACT. There is a non-elementary type lower bound for MSO-satisfiability problem.

Note: $\varphi \rightsquigarrow A_\varphi \rightsquigarrow \varphi_{A_\varphi}$

↳ This has a nice form

$\exists x_1 \dots \exists x_n$ ("first-order type").

Lecture 9 (27-08-2021)

27 August 2021 09:34

"First-order theory" of arithmetic

$$(\mathbb{N}, +, \cdot, 0, 1)$$

- $\text{add}(x, y, z) \rightarrow \text{asserts } x + y = z$
- $\text{mult}(x, y, z) \rightarrow \text{asserts } xy = z$

Usual Fo syntax.

- $\text{zero}(x) \equiv \text{add}(x, x, x)$
- $x < y \equiv \exists z \text{ add}(x, z, y)$
- $S(x, y) \equiv x < y \wedge \exists z (x < z \wedge z < y)$
- $\text{one}(x) \equiv \exists y (\text{zero}(y) \wedge S(y, x))$
- $\text{prime}(x) \equiv \neg \text{one}(x) \wedge \forall y \forall z (\text{mult}(y, z, x) \Rightarrow \text{one}(y) \vee \text{one}(z))$
- $\text{even}(x) \equiv \exists y \text{ add}(y, y, x)$.

Goldbach's conjecture:

$$\psi \equiv \forall z \text{ even}(z) \Rightarrow \exists y \exists x \text{ prime}(x) \wedge \text{prime}(y) \wedge \text{add}(x, y, z).$$

Given a sentence ψ , we would like to know if ψ is true.

Hilbert's belief: Perhaps, we can mechanically figure out the truth/falsity of ψ .

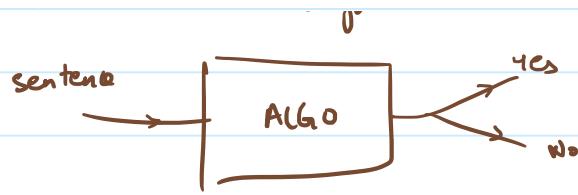
? what does this mean?

Church/ Gödel/ Turing : "Computability" ← defined

Moreover, it was shown that ·

The first-order theory of arithmetic is undecidable/
non-computable

That is, there is no algorithm s.t.



Now, let us look at $(\mathbb{N}, +) \rightarrow$ Presburger arithmetic.

First order and only add (x, y, z).

This IS decidable!

$(\mathbb{N}, +, <) \rightarrow$ first-order theory

Büchi showed that Presburger arithmetic is decidable using automata theory

$\varphi(x_1, \dots, x_n) \rightarrow$ first-order formula

encode x_1, \dots, x_n in reverse binary order
finite words over $\{0, 1\}^n$.

$\rightarrow \underline{\text{S1S}}$: $(\mathbb{N}, <) \text{ or } (\mathbb{N}, S)$

Second order theory of 1 successor

Both same if monadic second order

(In fact, MSO + add gives mult, which we know is undecidable)

1960s:

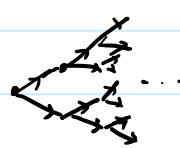
WS1S : Subsets are only allowed to be finite subsets
weak

Büchi showed that S1S is decidable

S2S is decidable

(Rabin's theorem)

two successors



MONA : Logic \rightarrow Automata

New modes of acceptance for automata on infinite words

$A : (Q, \Sigma, q_0, \Delta \subseteq Q \times \Sigma \times Q, A \alpha)$

Büchi : $G \subseteq Q$, ρ is accepting if $\text{Inf}(\rho) \cap G \neq \emptyset$.

Muller : $\mathcal{F} = \{F_1, \dots, F_k\}$ is a collection of subsets of Q .

ρ is accepting if $\text{Inf}(\rho) = F_i$ for some i .

\downarrow
"states at ∞ "

Rabin : $\Omega = \{(E_1, F_1), \dots, (E_k, F_k)\}$, each $E_i, F_i \subseteq Q$.

ρ is accepting if

$\exists i : \text{Inf}(\rho) \cap E_i = \emptyset$ and $\text{Inf}(\rho) \cap F_i \neq \emptyset$.

$$\bigvee_{i=1}^k [(\text{Inf}(\rho) \cap E_i = \emptyset) \wedge (\text{Inf}(\rho) \cap F_i \neq \emptyset)].$$

Streett : Dual to Rabin.

$$\Omega = \{(E_1, F_1), \dots, (E_k, F_k)\}$$

ρ is accepting if

$$\bigwedge_{i=1}^k [(\text{Inf}(\rho) \cap E_i \neq \emptyset) \vee (\text{Inf}(\rho) \cap F_i = \emptyset)]$$

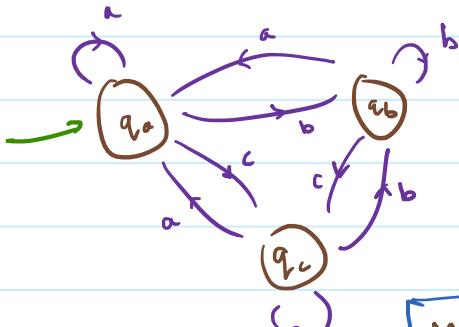
$\underbrace{\quad}_{\text{equivalently}}$

$$= \bigwedge_{i=1}^k \left([\text{Inf}(\rho) \cap F_i \neq \emptyset] \Rightarrow [\text{Inf}(\rho) \cap E_i \neq \emptyset] \right).$$

\hookrightarrow If F_i is visited infinitely often, then
 $\therefore E_i$.

$$\Sigma = \{a, b, c\}$$

$L = \{\alpha \in \Sigma^\omega \mid \text{if 'a' occurs inf. often in } \alpha, \text{ then 'b' does inf. often}\}$



Street-condition:

$$\Omega = \{(\{q_a\}, \{q_a\})\}.$$

Rabin-condition

Note:

$$L = \{\alpha \mid b \text{ occurs inf. often}\} \cup \{\alpha \mid \text{both } a \text{ & } b \text{ fin. often}\}$$

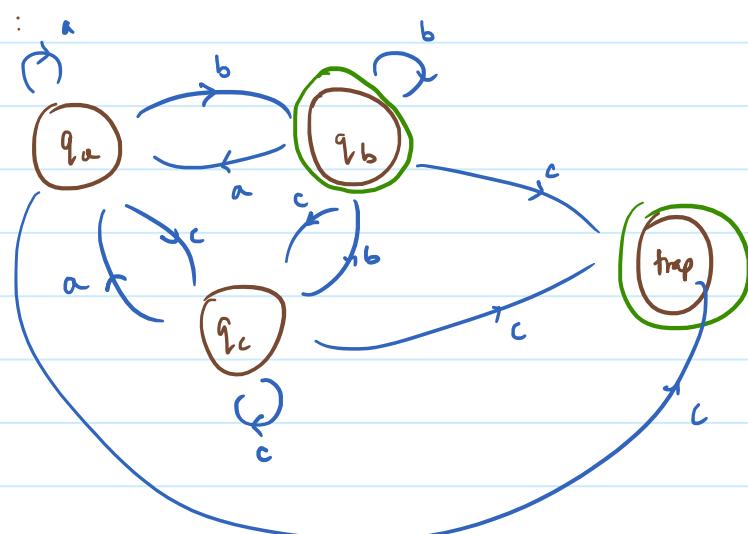
$$\Omega = \{(\emptyset, \{q_b\}), (\{q_a, q_b\}, \emptyset)\}$$

Muller-condition

$$\begin{aligned} F &= \{\{q_a, q_b\}, \{q_a, q_b, q_c\}, \\ &\quad \{q_b\}, \{q_b\}, \{q_b, q_c\}\} \\ &= \{X \subseteq Q : q_{fa} \in X \Rightarrow q_b \in X\} \setminus \{\emptyset\}. \end{aligned}$$

Note that putting \emptyset in F makes no difference since $\inf(\emptyset) \neq \emptyset$ vs.

Büchi-condition:



Thm (McNaughton)

DMA = DRA = DSA = NBA = ω -regular.

D = deterministic

N = non-D

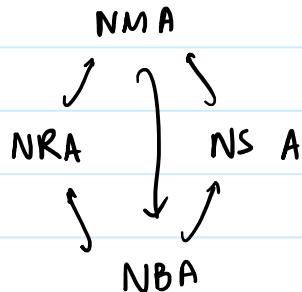
M = Muller, R = Rabin, S = Streetter, B = Büchi

A = Automaton

Lecture 10 (03-09-2021)

01 September 2021 09:58

Thm.



Thus, all are equivalent
in terms of expressive power.

($X \hookrightarrow Y$: any lang. acc. by X
can also be acc. by Y)

Proof.

• $NBA \hookrightarrow NRA$

Let $A = (Q, \Sigma, q_0, \Delta, \delta)$ $\hookleftarrow NBA$.

For Robin: Put $S_2 = \{(q_0, \delta)\}$.

• $NBA \hookrightarrow NSA$.

For A as above, put $S_2 = \{(q_0, Q)\}$.

For Muller: $F = \{x \in Q : x \cap G \neq \emptyset\}$.

• $NRA \hookrightarrow NMA$.

Let $A = (Q, \Sigma, q_0, \Delta, \delta)$ be a NRA.

$\delta = \{(E_1, F_1), \dots, (E_k, F_k)\}$.

$F = \{x \in Q : \exists i \text{ s.t. } x \cap E_i = \emptyset \text{ and } x \cap F_i = \emptyset\}$.

• $NSA \hookrightarrow NMA$: similar.

• $NMA \hookrightarrow NBA$.

Let $A = (Q, \Sigma, q_0, \Delta, F)$ be a NMA.

$F = \{x_1, \dots, x_m\}$.

For $i \in [m]$, define $A_i := (Q, \Sigma, q_0, \Delta, \{x_i\})$.

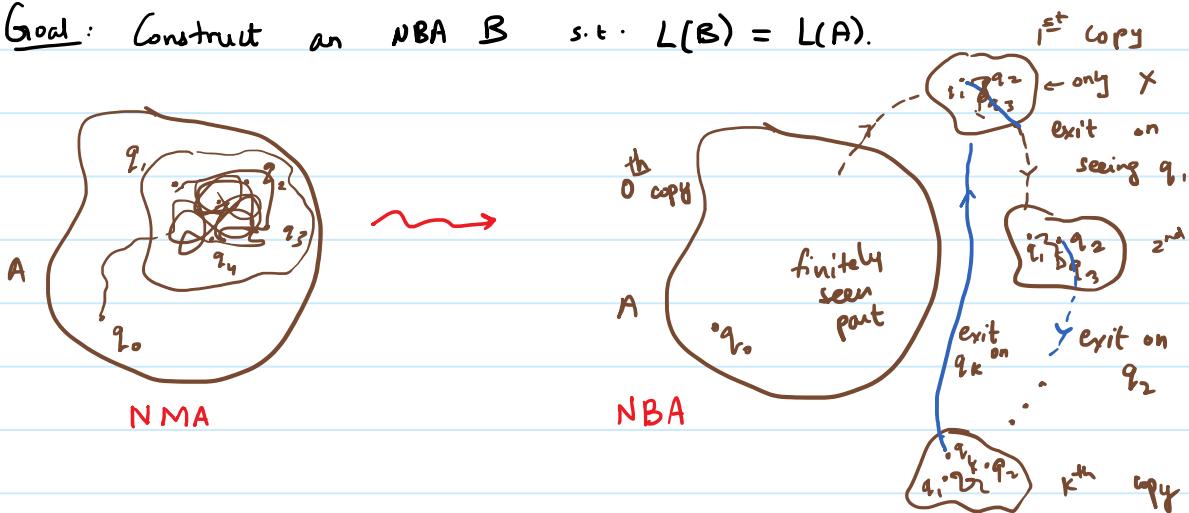
Note: $L(A) = \bigcup_{i=1}^k L(A_i)$.

Since ω -reg. languages are closed under union, suffice to

translate each A_i to an NBA. Equivalently, we may assume $m = 1$.

$$A = (Q, \Sigma, q_0, \Delta, \{x\}), \text{ and write } x = \{q_1, \dots, q_k\}.$$

Goal: Construct an NBA B s.t. $L(B) = L(A)$.



$$B = (Q', (q_0, 0), \Sigma, \Delta', \{(q_1, 1)\})$$

$$Q' = (Q \times \{0\}) \cup (\times \times \{1, 2, \dots, k\}).$$

$$\begin{aligned} \Delta' : & (q_1, 0) \xrightarrow{a} (q'_1, 0) \quad \text{if } q \xrightarrow{a} q'_1 \\ & (q_1, 0) \xrightarrow{a} (q'_1, 1) \quad \text{if } q \xrightarrow{a} q'_1 \text{ and } q'_1 \in x \\ & q, q' \in x \rightarrow (q, i) \xrightarrow{a} (q', i) \quad \text{if } q \neq q_i \text{ and } q \xrightarrow{a} q' \\ & (q, i) \xrightarrow{a} (q', i+1) \quad \text{if } q = q_i \text{ and } q \xrightarrow{a} q' \end{aligned}$$

$\sum_{k=1}^k$

↑
the arrows here
indicate those in
original Δ

Thus, all these models have same expressive power.

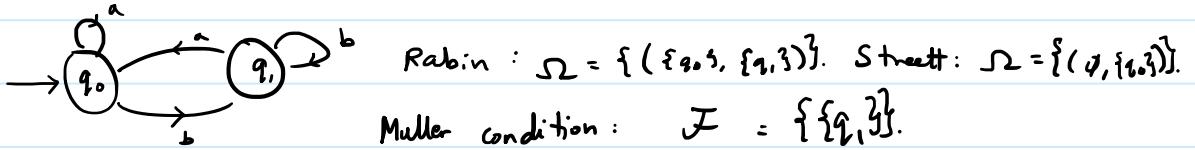
$$NRA = NFA = NSA = NBA$$

But

DBA

We had seen $L = \{\alpha \mid \alpha \text{ has finitely many 'a's}\} \in NBA \setminus DBA$.
(on $\Sigma = \{a, b\}$)

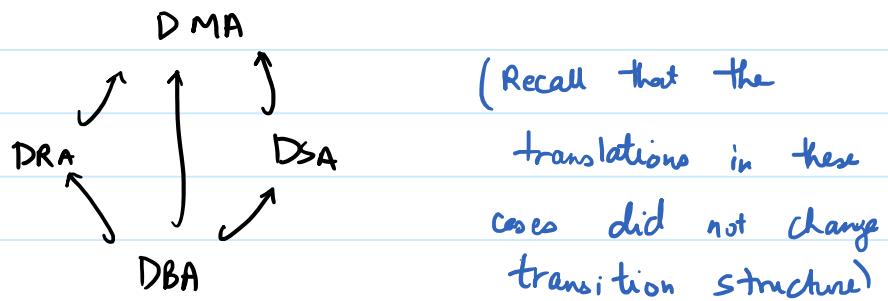
We had seen $L = \{\alpha \mid \alpha \text{ has finitely many 'a's}\} \in \text{NSA} \setminus \text{DBA}$.
 (on $\Sigma = \{a, b\}$)



Rabin: $\mathcal{R} = \{\{\{q_0\}, \{q_1\}\}\}$. Streett: $\mathcal{S} = \{\langle q, \{q_0\} \rangle\}$.
 Muller condition: $\mathcal{F} = \{\{q_1\}\}$.

Then, L is accepted by this DFA.

In fact, from the proof of the last theorem, we get



The L above shows that $\text{DMA} \not\hookrightarrow \text{DBA}$.

The following is true:

$$\text{NMA} \equiv \text{NRA} \equiv \text{NSA} \equiv \text{NBA}$$

$$\text{|||} \quad \text{|||} \quad \text{|||} \quad \text{JHR}$$

$$\text{DMA} \equiv \text{DRA} \equiv \text{DSA} \not\equiv \text{DBA}$$