§1. Introduction

The topic of study will be Representation Theory of Finite Groups. We shall be using the following notes to study it: https://aryamanmaithani.github.io/math/rep-theory/Notes.pdf. The aim is to cover Sections 1 and 2. Section 0 can be skipped more or less completely if you are able to solve the questions below.

The prereqs are Group Theory and Linear Algebra. For the former, if you've done MA 419 or an SoS on Group Theory, you should be okay. For the latter, if you've done MA 401, then very good. Having only done MA 106 is also okay but you should be comfortable with abstract vector spaces.
If you're able to solve the questions below within an hour, then you should be comfortable with the content.

I expect you to make a report of your progress in LaTeX as the month goes on. (You should not be starting a report on the last day.)

You **must** type your solutions in LaTeX and send me a PDF file via email to aryaman@iitb.ac.in.
If you have any doubt, you may ask it via WhatsApp if you have my number. Otherwise, you can email it to me. The first email that you send to me regarding this should be with the following subject (excluding quotes): "WoP - Representation Theory".
After that, we can just use that thread for future correspondence, whenever required.
I am not going to be too particular about the tiniest of details, your solutions should essentially convince me that you know what you are doing. If you look something up, then mention the source (and only write the solution if *you* understand it!).

There are many tutorials online for learning LaTeX. One starter is on my website: https://aryamanmaithani.github.io/latex. You can also find the source files for the earlier linked notes at https://github.com/aryamanmaithani/math/tree/master/rep-theory.

§2. Questions

If you wish to use a result from an earlier question, you can do so even if you have not solved the earlier question.

You may assume that the vector spaces are finite-dimensional, if you want. Just mention that.

1. Let G be a group and $N \trianglelefteq G$ be a normal subgroup. Let $\varphi : G \to H$ be a group homomorphism. We say that φ factors through G/N if there exists a homomorphism $\widetilde{\varphi} : G/N \to H$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ru, dashed, "\widetilde{\varphi}"'] & \end{tikzcd} \quad . \tag{2.1}$$

   Show that φ factors through G/N iff $N \subset \ker(\varphi)$.

   Terminology. By (2.1) commuting, we mean that $\widetilde{\varphi} \circ \pi = \varphi$. As usual, $\pi : G \to G/N$ is the natural map.

2. Let G be a group. For the elements $g, h \in G$, define the commutator $[g, h]$ by

$$[g, h] := ghg^{-1}h^{-1}.$$

   Define $C \subset G$ to be the set of all commutators in G, i.e., $C := \{[g, h] : g, h \in G\}$. The subgroup generated by C is denoted by $[G, G]$.

   (a) Show that $[g, h] = 1$ iff g and h commute.

   (b) Show that $[G, G] = \{1\}$ iff G is abelian.

   (c) Show that $[G, G] \trianglelefteq G$.

   (d) Show that $G/[G, G]$ is abelian.

   (e) Let $N \trianglelefteq G$ be such that G/N is abelian. Show that $[G, G] \subset N$. Thus, $[G, G]$ is the smallest normal subgroup we must quotient by, to get an abelian quotient.

   (f) Let $\varphi : G \to H$ be a group homomorphism. Show that if H is abelian, then φ factors through $G/[G, G]$.

3. Let V be a $\mathbb{C}$-vector space. Let $T : V \to V$ be linear. A subspace $W \leq V$ is said to be T-invariant if $T(W) \subset W$, i.e., $T(w) \in W$ for all $w \in W$.

   (a) Show that $\{0\}$, V, $\ker(T)$, and $\operatorname{im}(T)$ are T-invariant. (ker is the kernel, which you may know as null-space. im is the image.)

   (b) Let $S : V \to V$ and $W \leq V$ be such that W is both S and T-invariant. Show that W is also $S \circ T$-invariant.

(c) Let $S : V \to V$ be such that $T \circ S = S \circ T$. Show that $\ker(S)$ and $\operatorname{im}(S)$ are T-invariant.

(d) Assume that T is invertible. Assume that W is finite-dimensional. Show that W is T-invariant iff W is T^{-1}-invariant.
Optional. Is the problem still true if W is not assumed to be finite-dimensional?

(e) Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space. By T^*, we denote the adjoint of T. Suppose that T is unitary and W is T-invariant. Show that $W^\perp$ is also T-invariant.

4. Given $\mathbb{C}$-vector spaces V and W, define $\operatorname{Hom}_{\mathbb{C}}(V, W)$ to be the set of all $\mathbb{C}$-linear maps from V to W.

   (a) Describe how $\operatorname{Hom}_{\mathbb{C}}(V, W)$ has a natural $\mathbb{C}$-vector space structure.

   (b) Assuming V and W to be finite-dimensional, what is the dimension of $\operatorname{Hom}_{\mathbb{C}}(V, W)$?

5. From Section 0.1.3. of my notes here: https://aryamanmaithani.github.io/math/rep-theory/Notes.pdf#page=11, look at the concept of linearisation. Prove the following:

   (a) Let X be a finite set, and V a $\mathbb{C}$-vector space. Let $f : X \to V$ be a function (note that it does not make sense to talk about f being linear). Show that there exists a unique linear function $F : \mathbb{C}X \to V$ such that $F|_X = f$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}X \arrow[r, dashed, "F"] & V \\ X \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}.$$

   (What is the vertical map here?)

   (b) Let X and Y be finite sets, and $f : X \to Y$ be a function. Show that there exists a unique linear function $F : \mathbb{C}X \to \mathbb{C}Y$ such that $F|_X = f$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}X \arrow[r, dashed, "F"] & \mathbb{C}Y \\ X \arrow[u, hook] \arrow[r, "f"'] & Y \arrow[u, hook] \end{tikzcd}.$$

   For these questions, it is okay to be brief. Try following the following outline:

   (i) Define what F should be, in terms of an equation/formula.

   (ii) Justify why that F is indeed well-defined.

   (iii) Mention why that F works.

(iv) Mention why that is the only F that works.

6. How many conjugacy classes are there in S_5? Write down exactly one element from each class.
   If you're not aware of how conjugacy classes in S_n look, you should check Section 0.2.3. of my notes here: `https://aryamanmaithani.github.io/math/rep-theory/Notes.pdf#page=17`. Even if you don't want to go through the proof, Theorem 0.39 is something you should know. (The proof is actually simple but notation is daunting. Check out the computations for simple cases to see what's really going on.)