Sheet 10 Question 11

$\mathbf{F}(x, y, z) = f(r)\mathbf{r} = f(r)x\mathbf{i} + f(r)y\mathbf{j} + f(r)z\mathbf{k}$.
As $r = (x^2 + y^2 + z^2)^{1/2}$, we get that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

If $\mathbf{F}$ is to be $\nabla\varphi$ for some scalar field φ, then we must have $\varphi_x = f(r)x$, $\varphi_y = f(r)y$, $\varphi_z = f(r)z$; that is,

$$\varphi_x = xf(x) = \frac{x}{r}rf(r) = \frac{\partial r}{\partial x}rf(r),$$

$$\varphi_y = yf(y) = \frac{y}{r}rf(r) = \frac{\partial r}{\partial y}rf(r),$$

$$\varphi_z = zf(z) = \frac{z}{r}rf(r) = \frac{\partial r}{\partial z}rf(r).$$

Conversely, if φ satisfies the above properties, then $\nabla\varphi = \mathbf{F}$.

Now, it can be seen that if we define $\varphi(x, y, z) := \int_0^r tf(t)dt$, then φ satisfies the above properties. Note that we use the fact that $t \mapsto tf(t)$ is a continuous function and hence, φ is differentiable, by FTC (part I).