MA 105 : Calculus
Simply connected sets

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Simply connected sets

Let $m \in \mathbb{N}$ and $D \subset \mathbb{R}^m$.
We know that a curve in D is simply a $\mathcal{C}^1$ function $c : [a, b] \to D$ where $a,\ b \in \mathbb{R}$ with $a < b$.
For the purpose of this discussion, we shall assume $a = 0$ and $b = 1$.
We say that c can be *continuously* shrunk to a point $d \in D$ if there is a continuous function $H : [0, 1] \times [0, 1] \to D$ such that

1. $H(0, t) = c(t)$ for every $t \in [0, 1]$,
2. $H(1, t) = d$ for every $t \in [0, 1]$, and
3. $H(s, 1) = H(s, 0)$ for every $s \in [0, 1]$.

This map H is called a homotopy in D between the curve c and the constant curve d.
The domain D is said to be simply-connected if D is path-connected and if for every simply closed curve c in D, we have a homotopy H between c and *some* $d \in D$.

Alternate definition

The previous definition can also be written in a slightly more concise (but equivalent) way.
Let D and c have the same meaning as before. Moreover, let
$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $U^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

We say that D is simply-connected if D is path-connected and any loop in D defined by $f : S^1 \to D$ can be contracted to a point: there exists a continuous map $F : U^2 \to D$ such that F restricted to S^1 is f.