Short Quiz 1: Solution

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Question. State whether the following statement is true or false. Justify your answer.
If S is a nonempty subset of $\mathbb{R}$ such that S is bounded above and if $c := \sup S$, then there exists a sequence (x_n) of elements of S such that (x_n) is convergent and $x_n \to c$. [5 marks]

[2 marks for correct alternative (T/F); 3 marks for correct justification]

Answer. T
Justification: It is given that c is the supremum of S. Thus, any number strictly less than c cannot be an upper bound. Thus, given any $n \in \mathbb{N}$, $c - 1/n$ is not an upper bound. Thus, there exists $x_n \in S$ such that $c - 1/n < x_n$. As c is an upper bound of S, we have it that

$$c - \frac{1}{n} < x_n \leq c \quad \forall n \in \mathbb{N}.$$

Thus, by Sandwich Theorem, we can conclude that $x_n \to c$.

Common mistakes:

1. Many have said that we can take an increasing sequence (x_n) in S and then $x_n \to c$.
   This is not true. It is true that such a sequence *will* converge as S is bounded above but it is not necessary that it converges to c as $\sup\{a_n | n \in \mathbb{N}\}$ may not be equal to c.
   Example: $S = [0, 2]$ and $x_n := 1 - 1/n$. We have it that $x_n \to 1 \neq 2 = \sup S$.

2. Many have said that we can take the sequence $x_n = c - 1/n$. There is no reason that such a sequence will actually be a sequence in S. For example, $S = \{1\}$.
   Even if we take the case where $c \notin S$, the statement need not be true.
   For example, let $S = (0, 1) \setminus \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$.

3. Many have tried to construct a counterexample but it does not work in the correct manner. For example, if you pick a specific set S and a specific sequence (x_n) of elements of S such that $x_n \not\to c$, you have **not** disproved the question. The question asked for the existence of ***a*** sequence.
   If one did want to give a counterexample, they would have to find a particular set S which satisfies the hypothesis of the question and show that *every* sequence in that set does not converge to c.

4. Similar to the last point, you have **not** proven the statement either if you choose a specific set S and show the existence of a sequence there. It must work for every S.

5. Many have not considered that the constant sequence is a perfectly valid sequence for the case when $c \in S$.