MA 105 : Calculus
D1 - T5, Tutorial 04

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4. (i) $p \leq 0$ follows from question 2 of sheet 3. (Contrapositive)
If $p = 0$, then $f(x) = (x + q^{1/3})(x^2 - q^{1/3}x + q^{2/3})$. The latter has no real roots if $q \neq 0$. In the case that $q = 0$, we still have only one (distinct) root.
Thus, $p < 0$. ■

(ii) Let us look at $f'(x)$.
We have it that $f'(x) = 3x^2 + p$.
As $p > 0$, we have it that $f'(x)$ has two distinct roots which are $x = \pm\sqrt{-p/3}$.
By the second derivative test, it can be verified that we have maximum and minimum at $-\sqrt{-p/3}$ and $\sqrt{-p/3}$, respectively. ■

(iii) Let $\lambda := \sqrt{-p/3}$. We have to show that $f(\lambda)$ and $f(-\lambda)$ are of different signs. We are given that $f(x)$ has three distinct roots. Let these roots be α, β, γ such that $\alpha < \beta < \gamma$.
By Rolle's Theorem, we know that $f'(x)$ has a root in (α, β) and one in (β, γ). Recall that we have already found all the roots of $f'(x)$. It must be the case that $\alpha < -\lambda < \beta < \lambda < \gamma$.
Moreover, by looking at the sign of $f'(x)$, we know that it is strictly decreasing in $[-\lambda, \lambda]$.
As we have it that $\beta \in [-\lambda, \lambda]$ and $f(\beta) = 0$, we know that that $f(\lambda) < 0$ and $f(-\lambda) > 0$. ∎

Thus, we now know that $f(\lambda)f(-\lambda) < 0$.

Sheet 3

$$
\begin{aligned}
f(\lambda)f(-\lambda) &< 0 \\
(\lambda^3 + p\lambda + q)(-\lambda^3 - p\lambda + q) &< 0 \\
\implies q^2 - (\lambda^3 + p\lambda)^2 &< 0 \\
\implies q^2 - \lambda^2(\lambda^2 + p)^2 &< 0 \\
\implies q^2 + (p/3)(-p/3 + p)^2 &< 0 \\
\implies q^2 + 4p^3/27 &< 0 \\
\implies 4p^3 + 27q^2 &< 0 \qquad \blacksquare
\end{aligned}
$$

8. (i) Not possible.
Assume not. Then, we are given that f'' exists. Thus, f' must be continuous and differentiable everywhere.
We have that $f'(0) = f'(1)$. Thus, by Rolle's Theorem, $f''(c) = 0$ for some $c \in (0, 1)$.
This contradicts the hypothesis. $\blacksquare$

(ii) Possible. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := x + \dfrac{x^2}{2}$ is one such function.

(iv) Possible. $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) := \begin{cases} \dfrac{1}{1-x} & \text{if } x \leq 0 \\ 1 + x + x^2 & \text{if } x > 0 \end{cases}$$

is one such function.

(iii) Not possible.
Assume not. Then, we are given that f'' exists. Thus, f' must be continuous and differentiable everywhere.
As f'' is nonnegative, f' must be increasing everywhere. We are given that $f'(0) = 1$.
Thus, given any $c > 0$, we know that $f'(c) \geq 1$. (1)
Let $x \in (0, \infty)$. By MVT, we know that there exists $c \in (0, x)$ such that
$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$
Thus, by (1), we have it that $f(x) - f(0) \geq x$ for all positive x.
This contradicts that $f(x) \leq 100$ for all positive x. (How?)

9. As $f : [-2, 5] \to \mathbb{R}$ is continuous, we have it that the absolute extrema of f on $[a, b]$ is attained either at a critical point of f or at an end-point of $[a, b]$.
Recall that an interior point c of the domain is called a critical point of f if either f is not differentiable at c, or if f is differentiable at c and $f'(c) = 0$.
It is clear that 0 is a critical point as f is not differentiable at 0. Moreover, 0 is the only point at which f is not differentiable.
Rewriting $f(x)$, we get

$$f(x) = \begin{cases} 1 - 12x - 3x^2 & \text{if } x \leq 0 \\ 1 + 12x - 3x^2 & \text{if } x > 0 \end{cases}$$

For $x < 0$, we get the derivative of f as $f'(x) = -12 - 6x = -6(x + 2)$. Thus, no negative number in the domain is a critical point. (Note that -2 is **not** an interior point of the domain.)

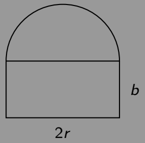
For $x > 0$, we get the derivative as $f'(x) = 12 - 6x = 6(2 - x)$. Thus, 2 is a critical point of f. (Note that 2 **is** an interior point of the domain.)

To summarise,
Critical points of f : $0, 2$. End-points of $[-2, 5]$: $-2, 5$.

x	0	2	-2	5
f(x)	1	13	13	-14

$\therefore$ f attains its global maximum 13 at 2 as well as -2, and its global minimum -14 at 5.

10.



Let $l(r, b)$ denote the amount of light that enters through the window for r and b specified in the diagram.
We know that $l(r, b) = k(2rb) + \left(\frac{k}{2}\right)\left(\frac{\pi r^2}{2}\right)$. Where k is some positive constant.
We are given that $p = 2b + 2r + \pi r = 2b + (2+\pi)r$ is fixed. Thus, we can rewrite the amount of light just in terms of the radius as follows:

$$L(r) = l\left(r, \frac{1}{2}(p - (2+\pi)r)\right) = k\left[r(p - (2+\pi)r) + \frac{\pi r^2}{4}\right] = k\left[pr - 2r^2 - \frac{3\pi r^2}{4}\right].$$

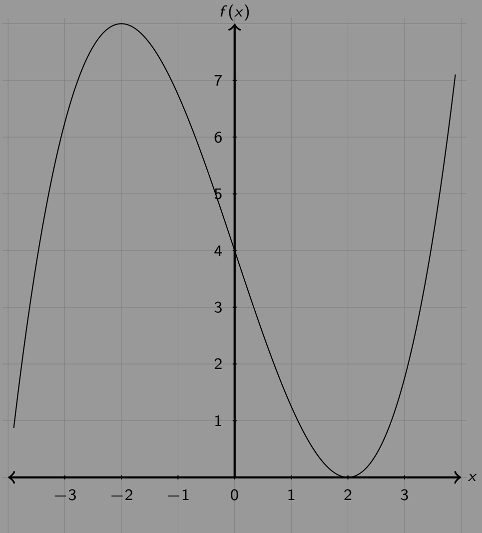
As the above is defined for $(0, \infty)$ and is differentiable everywhere, we only need to check where the derivative is zero.

$$L'(r) = k\left[p - 4r - \tfrac{3\pi r}{2}\right] = k\left[p - \left(\tfrac{8+3\pi}{2}\right) r\right].$$
$$\therefore L'(r) = 0 \implies r = \frac{2p}{8+3\pi}.$$

It can be easily verified that for this value of r, $L''(r)$ is indeed negative. (In fact, it is always negative.)

b can now be calculated as we know the relation between b, p and r.
It comes out to be $\dfrac{1}{2}\left(\dfrac{4+\pi}{8+3\pi}\right) p$.

2.



Sheet 4

2. (contd.)
I have actually graphed a polynomial that satisfies the given properties.
Can you come up with it?
Is there a unique such polynomial?
What's the minimum degree of such a polynomial?
Is there a unique polynomial with that degree?
Suppose you have two distinct polynomials f and g that satisfy the given conditions.
Can you come up with a distinct third polynomial such that it satisfies the conditions as well?

3. (i) $f(x) = x^2$.
(ii) $f(x) = \sqrt{x}$.
(iii) $f(x) = -\sqrt{x}$.
(iv) $f(x) = -x^2$.

Note that these functions are strictly convex (or concave) as well.

4. (i) True.

Note that we **cannot** use the derivative test to prove this as we do not know whether f and/or g are differentiable. (They may not even be continuous!)

By definition, we know that there exists $\delta_1 > 0$ such that $f(c) \geq f(x)$ whenever $x \in (c - \delta_1, c + \delta_1)$.
Similarly, $\exists \delta_2 > 0$ such that $|x - c| < \delta_2 \implies g(x) \leq g(c)$.
(Note how the same statement can be written in different styles.)
Let $\delta := \min\{\delta_1, \delta_2\}$. Thus, for $x \in (c - \delta, c + \delta)$, we know that $f(x) \leq f(c)$ *and* $g(x) \leq g(c)$.
As $f(x)$ and $g(x)$ are always nonnegative, we know that

$$h(c) = f(c)g(c) \geq f(c)g(x) \geq f(x)g(x) = h(x) \quad \forall x \in (c - \delta, c + \delta).$$

Thus, h has a local maximum at c. ∎

4. (ii) False.
Take $f(x) = g(x) = x^4 - x^3 + 1$.
Both f and g have a point of inflection at 0.5 but h does not. ($h''(0.5)$ turns out to be 0.125)

A "simpler" counterexample would have probably been $f(x) = g(x) = 1 + \sin x$. The reason I avoid this is because I wouldn't consider sin to be a simple function!
It is for a similar reason that I did not give e^x as an example for Question 8 part (iv).

Here's an even weirder example which I claim is a counterexample -

$$f(x) = g(x) = |x|, \quad c = 0$$

How's that for simplicity?
Can you argue that 0 indeed is a point of inflection of f and g? Can one even be so bold as to claim that *all* real numbers are points of inflection of f and g both?