MA 105 : Calculus
D1 - T5, Tutorial 08

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(1) (i) Given any non-zero real number, it has a multiplicative inverse. Conversely, if a real number has a multiplicative inverse, then the number is non-zero.
Thus, whenever $x^2 = y^2$, we get that the expression is not defined and it is defined otherwise. Thus, the domain is $D = \{(x,\ y) \in \mathbb{R}^2 : x^2 \neq y^2\}$.

(ii) We know that the ln function is defined for positive real numbers. Thus, the expression given is defined whenever $x^2 + y^2 > 0$. It can be seen that the set of all such values of $(x,\ y)$ is precisely the following set $D = \mathbb{R}^2 \setminus \{(0,\ 0)\}$.

Sheet 6

(2) (i) Given any c from the options, the level curve is the line $x - y = c$ in the XY plane, that is, the set of points $\{(x,\ y) \in \mathbb{R}^2 : x - y = c\}$ in $\mathbb{R}^2$.
The contour line for that c is the line in $\mathbb{R}^3$ which consists of the set of points $\{(x,\ y,\ z) \in \mathbb{R}^3 : x - y = c,\ z = c\}$. That is, it is the contour line just shifted parallel-y in the $z-$direction.

(ii) For $c < 0$, the contour lines and level curves are empty sets.
For $c = 0$, the level curve is just the point $(0,\ 0) \in \mathbb{R}^2$ and the counter line is $(0,\ 0,\ 0) \in \mathbb{R}^3$.
For $c > 0$, the level curve L is the circle $\{(x,\ y) \in \mathbb{R}^2 : x^2 + y^2 = c\}$ and the contour line is the "same curve, just shifted c units upwards" in $z-$direction. More precisely, the contour line is the set $L \times \{c\}$.

(iii) You can work this out similarly.

Note

Note: It is technically not correct to say that the contour lines are just the "level curves shifted upwards" because the two curves are not lying in the same space. More precisely, $\mathbb{R}^2 \not\subset \mathbb{R}^3$. However, we do have a natural "embedding" of $\mathbb{R}^2$ into $\mathbb{R}^3$ which is what we were referring to.

P.S.: Thank you, Adway Girish, for pointing out the error in the original slides where I swapped contour lines with level curves.

(3) (i) Claim: the function is not continuous at $(0, 0)$.

Proof. Consider the following sequence $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right)$. It is clear that $(x_n, y_n) \to (0, 0)$.

But $f(x_n, y_n) = \frac{1/n^6}{2/n^6} = \frac{1}{2}$. Thus, $f(x_n, y_n) \to \frac{1}{2} \neq 0$.

Thus, f is not continuous at $(0, 0)$. ∎

(ii) Claim: the given function is continuous at $(0, 0)$.

Proof. Let (x_n, y_n) be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0, 0)$. Then, $x_n \to 0$ and $y_n \to 0$. (1)

Note that if $(x_n, y_n) \neq (0, 0)$, then $\left|\dfrac{x^2 - y^2}{x^2 + y^2}\right| \leq 1$.

Thus, $0 \leq |f(x_n, y_n)| \leq |x_n y_n|$. (This inequality holds even if $(x_n, y_n) = (0, 0)$.)

Note that (1) tells us that $x_n y_n \to 0$.

Now, using our knowledge of limits of real sequences, we get that $\lim\limits_{n\to\infty} |f(x_n, y_n)| = 0$ and we are done. (How?)

(3) (iii) The function is continuous at $(0, 0)$. Similar proof as before will work using the fact that modulus is a continuous function.

(4) (i), (ii), (iii), (iv)
Let $(x_0,\ y_0)$ be any point in $\mathbb{R}^2$. We show that the function is continuous at this point.
Let $(x_n,\ y_n)$ be any sequence in $\mathbb{R}^2$ such that $(x_n,\ y_n) \to (x_0,\ y_0)$. This gives us that $x_n \to x_0$ and $y_n \to y_0$. (Why?)
Hence, $f(x_n) \to f(x_0)$ and $g(y_n) \to g(y_0)$. (Definition of continuity of real functions.)
Now, we can use properties of sum and difference of real sequences to get our answers.

For (iii), use the fact that $\max\{a,\ b\} = \frac{|a+b|+|a-b|}{2}$ and that modulus is a continuous function. Similar considerations apply for (iv).

(5) First we show that the iterated limit $\lim_{x \to 0} \left[\lim_{y \to 0} f(x,\ y) \right]$ exists.
To do this, we must first compute the inner limit. What that means is that we treat x as a constant and let $y \to 0$. The resulting expression must be a function of x alone.
If $x = 0$, then we get that the inner limit is simply 0.
If $x \neq 0$, then we get the function must be continuous at $(x,\ 0)$ as it is quotient of two polynomials such that the denominator is not zero at $(x,\ 0)$. Thus, we can simply substitute $y = 0$ and get our answer as 0, once again.

Thus, the iterated limit now evaluates to $\lim_{x \to 0} [0]$, which is clearly 0. Moreover, observe that $f(x,\ y) = f(y,\ x)$. Thus, it is clear that both the iterated limits exist.

(5) Now we show that the $\lim_{(x,\ y)\to(0,\ 0)} f(x,\ y)$ does not exist.

This is easy as one could take the following sequences:

1. $(x_n,\ y_n) = (0,\ 1/n)$, and
2. $(x_n,\ y_n) = (1/n,\ 1/n)$.

Clearly, in both the cases we have that $(x_n,\ y_n) \to (0,\ 0)$. However, $f(x_n,\ y_n)$ converges to different values in each case.

(6) (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given.
Then,

$$\begin{aligned} f_x(0,\ 0) &= \lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} \\ &= \lim_{h \to 0} \left(h \cdot 0 \cdot \frac{h^2 - 0^2}{h^2 + 0^2} \right) \frac{1}{h} \\ &= 0 \end{aligned}$$

It can be verified that $f_y(0,\ 0)$ also exists and equals 0 in a similar manner.

(6) (ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given.
Then,

$$\begin{aligned} f_x(0,\ 0) &= \lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} \\ &= \lim_{h \to 0} \left(\frac{\sin^2(h)}{h|h|} \right) \end{aligned}$$

The above limit does not exist. (Why?) (Hint: Take a strictly positive sequence and a strictly negative sequence, both of which converge to 0.)
It can be verified that $f_y(0,\ 0)$ also does not exist in a similar manner.

(8) The continuity of f is immediate. It is extremely similar to what we've seen many times by now.
Let us show that the partial derivatives don't exist.
The partial derivative of f at $(0,\ 0)$ with respect to the first variable (x) is given by

$$\lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right),$$

which we know does not exist.
Similar considerations apply for the other partial derivative.