MA 105 : Calculus
D1 - T5, Tutorial 05

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Summary

Sheet 4: Problems 5, 6

Sheet 5 (Additional Problems): Problems 1 to 4

5. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0, 2]$.
Then, there exists a unique i_0 such that $x_{i_0} \in [0, 1]$ and $x_{i_0+1} \in (1, 2]$.
For $i = 1, 2, \ldots, i_0$, we have

$$m_i(f) = \inf_{x \in [x_{i-1}, x_i]} f(x) = 1 \text{ and } M_i(f) = \sup_{x \in [x_{i-1}, x_i]} f(x) = 1,$$

for $i = i_0 + 1$, we have

$$m_i(f) = \inf_{x \in [x_{i-1}, x_i]} f(x) = 1 \text{ and } M_i(f) = \sup_{x \in [x_{i-1}, x_i]} f(x) = 2,$$

for $i = i_0 + 2, \ldots, n$, we have

$$m_i(f) = \inf_{x \in [x_{i-1}, x_i]} f(x) = 2 \text{ and } M_i(f) = \sup_{x \in [x_{i-1}, x_i]} f(x) = 2.$$

Thus,

$$L(P, f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = 1(x_{i_0+1} - x_0) + 2(x_n - x_{i_0+1}) = 4 - x_{i_0+1}.$$

Similarly,

$$U(P, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) = 1(x_{i_0} - x_0) + 2(x_n - x_{i_0}) = 4 - x_{i_0}.$$

Note that we have used that $x_0 = 0$ and $x_n = 2$.

Recall the following:
A bounded function $f : [a, b] \to \mathbb{R}$ is (Riemann) integrable if and only if there s a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$. In this case,

$$L(P_n, f) \to \int_a^b f(x)dx \leftarrow U(P_n, f).$$

We shall now construct such a sequence of partitions. For $n \in \mathbb{N}$, let P_n denote the partition of $[0, 2]$ into n equal parts.

Thus, we get $U(P_n, f) - L(P_n, f) = x_{i_0+1} - x_{i_0} = \dfrac{1}{n}$.

Seeing our old friend $1/n$, we can immediately conclude that $U(P_n, f) - L(P_n, f) \to 0$.

Thus, we have now shown that f is (Riemann) integrable on $[0, 2]$.

Now, we must actually compute the integral. This is not tough either.

Using the definition of i_0, show that it must be the (unique!) integer in interval $(n/2 - 1, n/2]$. This gives us that $1 - 2/n \leq x_{i_0} \leq 1$.

Thus, the $L(U_n, f) \to 4 - 1 = 3$, which is the required integral. ∎

A much simpler method is shown in the tutorial.

6. (a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$.
For $i = 1, 2, \ldots, n$, we have

$$M_i(f) = \sup_{x \in [x_{i-1}, x_i]} f(x) \geq 0.$$

We have used that the supremum of a set of non-negative real numbers is nonnegative. (Why?)
Thus, $U(P, f) \geq 0$. As f is given to be Riemann integrable on $[a, b]$, there exists a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) \to \displaystyle\int_a^b f(x)dx$. But $U(P_n, f) \geq 0$ for all n. (Shown above)
Thus, $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty} U(P_n, f) \geq 0$.
Note that here we have used the fact that the limit of a sequence of nonnegative real numbers, if it exists, is nonnegative.

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To prove the next part, let us prove the contrapositive. That is, if $f(x) \neq 0$ for some $x \in [a, b]$, then $\int_a^b f(x)dx \neq 0$.

Suppose $c \in [a, b]$ is the number such that $f(c) \neq 0$. As $f(x) \geq 0$ for all $x \in [a, b]$, we have it that $f(c) > 0$. Let $\epsilon := f(c)$.
As f is continuous, there is a $\delta > 0$ such that if $x \in [a, b]$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon/2$ which implies that $\epsilon/2 < f(x)$.
Now, let us take the partition $P := \{x_0, x_1, x_2, x_3\}$ with $x_0 = a,\ x_1 = c - \delta,\ x_2 = c + \delta$ and $x_3 = b$. If it is the case that $x_1 < x_0$, then discard x_1. If it is the case that $x_2 > x_3$, discard x_2. Relabel if required.
Now, there exists $x_i \in P$ such that $\inf\limits_{x \in [x_{i-1}, x_i]} f(x) \geq \epsilon/2 > 0$.

Thus, $L(P, f) > 0$. As f is Riemann integrable,
$\int_a^b f(x)dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\} > 0$ as we have found a partition that has a strictly positive lower sum.

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(b) Let $a = 0, b = 2$ and $f : [a, b] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & ; x \neq 1 \\ 1 & ; x = 1 \end{cases}$$

Show that f is actually Riemann integrable on $[0, 2]$ with the integral equal to 0.

Sheet 5 (Additional Problems)

1. Brute calculation.

We need the following to happen:

1. $f(1) = 6$,
2. $f(-1) = 10$,
3. $f'(-1) = 0$,
4. $f''(-1) < 0$,
5. $f''(1) = 0$ and
6. f'' changes sign around 1.

It will turn out that condition 1, 2, 3 and 5 are sufficient to find the constants. Verify whether the other conditions are being fulfilled or not.

2. If $g(a) = g(b)$, then it will directly follow from LMVT. (How?)
Let us assume that $g(a) \neq g(b)$.
Define $h : [a, b] \to \mathbb{R}$ as $h(x) := f(x) - c_0 g(x)$ for a particular constant c_0 which we shall choose such that $h(a) = h(b)$.
Thus, $f(a) - c_0 g(a) = f(b) - c_0 g(b) \iff f(a) - f(b) = c_0(g(a) - g(b))$. As $g(a) - g(b) \neq 0$, the desired c_0 exists.
By hypothesis, h is also continuous on $[a, b]$ and differentiable on (a, b). (Why?)
As $h(a) = h(b)$, by Rolle's Theorem, we know that there exists $c \in (a, b)$ such that $h'(c) = 0$.

$$\begin{aligned}
& h'(c) = 0 \\
\implies & f'(c) - c_0 g'(c) = 0 \\
\implies & f'(c)(g(b) - g(a)) - c_0(g(b) - g(a))g'(c) = 0 \\
\implies & f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))
\end{aligned}$$

Sheet 5 (Additional Problems)

3. (i) Let us assume that we know the derivative of x^n when $n \in \mathbb{N} \cup \{-1\}$ and derive it for the case when n is any rational number.
Given $n \in \mathbb{N}$, let us first find the derivative of the following function,
$g : (0,2) \to (0, 2^{1/n})$ defined as $g(x) := x^{1/n}$. We shall do so using the inverse function theorem.
Define $f : (0, 2^{1/n}) \to (0,2)$ as $f(x) := x^n$, then f is 1-1 and continuous in the domain.
Moreover, $f'(c) = nc^{n-1} \neq 0$.
Also, $f\left((0, 2^{1/n})\right) = (0,2)$. Thus, for $d \in (0,2)$, $d = f(c) = c^n$ for some $c \in (0, 2^{1/n})$ and

$$(f^{-1})'(d) = \frac{1}{f'(c)} = \frac{1}{nc^{n-1}} = \frac{1}{n} d^{\frac{1}{n}-1}.$$

As $f^{-1} = g$, we have it that $g'(x) = \dfrac{1}{n} x^{\frac{1}{n}-1}$.

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Given $m \in \mathbb{Z}^-$, that is, the set of negative integers, we can find the derivative of $f : (0, 2) \to (0, 2^m)$ defined as $f(x) := x^m$.
This can be done using the chain rule as follows:

$$f'(x) = \left(\left(\frac{1}{x}\right)^{-m}\right)' = (-m)\left(\frac{1}{x}\right)^{-m-1}\left(-\frac{1}{x^2}\right) = mx^{m-1}.$$

For $m = 0$, one derive $(x^m)'$ easily.
Thus, we can now derive the derivative of the following function:

$$f : (-1, 1) \to \mathbb{R}$$

$$f(x) := (1 + x)^r$$

for $r \in \mathbb{Q}$.

Sheet 5 (Additional Problems)

Given $r \in \mathbb{Q}$, we can write r as m/n for some $m \in \mathbb{Z}$ and some $n \in \mathbb{N}$. Using our results from earlier, we have differentiate the function using chain rule as follows:

$$((1+x)^r)' = \left(((1+x)^m)^{1/n}\right)' = \frac{1}{n}\left[((1+x)^m)^{1/n-1}\right]\left[m(1+x)^{m-1}\right] = r(1+x)^{r-1}.$$

Now, one can inductively show that $f^{(n)}(x) = r(r-1)\cdots(r-(n-1))(1+x)^{r-n}$. The n^{th} Taylor polynomial around $a = 0$ is given by

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Thus, in this case, it is

$$1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots \frac{r(r-1)\cdots(r-n+1)}{n!}x^n.$$

Note the *Taylor polynomial* does not include the remainder term.

3. (ii) Left as an exercise. Be careful with the signs.
(iii) Exercise.

4. Let us extend f and g by defining $f(c) := 0 =: g(c)$. By hypothesis, f, g are now continuous on $(c-r, c+r)$. (Why?)
As $g'(x)$ is never zero, $g(x) \neq g(y)$ whenever $x \neq y$. (Why?)
We will show that that if $\lim\limits_{x \to c^+} \dfrac{f'(x)}{g'(x)} = l$, then $\lim\limits_{x \to c^+} \dfrac{f(x)}{g(x)} = l$. (Given the remaining hypothesis.

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Suppose (x_n) in D is a sequence such that $x_n \to c^+$. Given any $n \in \mathbb{N}$, f and g are continuous on $[c, x_n]$ and differentiable on (c, x_n). Then, by CMVT, there exists $c_n \in (c, x_n)$ such that

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f(x_n)}{g(x_n)}$$

$$\implies \lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)}$$

As $c < c_n < x_n$ and $x_n \to c$, we have it that $c_n \to c$ along with $c_n > c$.

As we are given that $\lim_{x\to c} f'(x)/g'(x)$ exists, we can write the LHS as $\lim_{x\to c^+} \frac{f'(c_n)}{g'(c_n)}$. As (x_n) was arbitrarily chosen and now that we know that $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)}$ exists, it must be equal to $\lim_{x\to c^+} \frac{f(x_n)}{g(x_n)}$, by definition.

Thus, we have shown that

$$\lim_{x\to c^+} \frac{f'(c_n)}{g'(c_n)} = \lim_{x\to c^+} \frac{f(x_n)}{g(x_n)}.$$

Similarly, one can show a similar result for $x \to c^-$. As we are give that $f'(x)/g'(x) \to l$ as $x \to c$, we know that both the limits must coincide. Thus, we have proven the theorem.

Note that we didn't require l to be a real number.