MA 105 : Calculus
D1 - T5, Tutorial 09

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(Happy birthday to me!)

An example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,\ y) := \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = (0,\ 0) \end{cases}$$

Then, verify that

$$(\mathbf{D_u} f)(0,\ 0) = \begin{cases} u_1^2/u_2 & u_2 \neq 0 \\ 0 & u_2 = 0 \end{cases}$$

Thus, observe that *all* directional derivatives of f at $(0,\ 0)$ exist even though f is discontinuous at $(0,\ 0)$. Also, note that f is not differentiable, that is, its total derivative does not exist. The last note should not be surprising as f is discontinuous at the origin. Also, note that $(\mathbf{D_u} f)(0,\ 0) = (\nabla f)(0,\ 0) \cdot \mathbf{u}$ is not true in general.

The aim of this example was to drive home the point that even existence of all directional derivatives does not imply differentiability (or even continuity!).

(7) Argue about the continuity of f at $(0,\ 0)$ using the fact that $|f(x,\ y)| \leq |x^2 + y^2|$.
(Recall Tutorial 2, Question 3. (ii))

It can also be easily verified that $f_x(0,\ 0) = f_y(0,\ 0) = 0$. (Write the expression like the previous questions and arrive at the conclusion.)

Now, let us evaluate $f_x(x_0,\ y_0)$ for $(x_0,\ y_0) \neq (0,\ 0)$.
It can be easily evaluated using product and chain rules to be:

$$f_x(x_0,\ y_0) = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,\ 0)$. (How?)

However, $\dfrac{2x}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)$ is not bounded in any such disc. To see this, consider any $r > 0$ and any $M \in \mathbb{R}$. One can find an $n \in \mathbb{N}$ such that $\frac{1}{\sqrt{n\pi}} < r$ and $\sqrt{n\pi} > M$. (How? Archimedean.)

In that case, the point $(x_0, y_0) = (1/\sqrt{2n\pi}, 0)$ will lie in the disc centered at $(0, 0)$ with radius r and $f(x_0, y_0) > M$.

(9) (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given in the question.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f\left(0 + tu_1, 0 + tu_2\right) - f(0, 0)}{t} = u_1 u_2 (u_1^2 - u_2^2) t.$$

Hence, $\left(\mathbf{D_u} f\right)(0, 0)$ exists and equals 0 for all $\mathbf{u}$. Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $(f_x(0, 0), f_y(0, 0)) = (0, 0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f\left(0 + h, 0 + k\right) - f\left(0, 0\right) - 0h - 0k}{\sqrt{h^2 + k^2}} = 0.$$

For $(h, k) \neq (0,0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = hk\frac{(h^2-k^2)}{(h^2+k^2)^{3/2}}.$$

Also, note that

$$\left|hk\frac{(h^2-k^2)}{(h^2+k^2)^{3/2}}\right| \leq \left|h\frac{k}{\sqrt{h^2+k^2}}\right| \leq |h|.$$

Thus, the required limit indeed does exist and equals 0.
Hence, f is differentiable at $(0,0)$ with (total) derivative equal to $(0,0)$.

(9) (ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given in the question.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0 + tu_1, 0 + tu_2) - f(0, 0)}{t} = u_1^3.$$

Hence, $(\mathbf{D_u} f)(0, 0)$ exists and equals u_1 for all $\mathbf{u}$. Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $(f_x(0, 0), f_y(0, 0)) = (1, 0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f(0 + h, 0 + k) - f(0, 0) - 1h - 0k}{\sqrt{h^2 + k^2}} = 0.$$

For $(h,k) \neq (0,0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = -\frac{hk^2}{(h^2+k^2)^{3/2}}.$$

It can be seen that the limit for the above expression as $(h,k) \to (0,0)$ does not exist. Indeed, if one approaches $(0,0)$ along the curve $h = mk$, the limit along that path turns out to be $-m/(1+m^2)^{3/2}$. Thus, taking $m = 1$ and $m = 0$ demonstrates the non-existence of limit.

(9) (iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given in the question.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f\left(0 + tu_1, 0 + tu_2\right) - f(0,0)}{t} = t \sin\left(\frac{1}{t^2}\right).$$

Hence, $\left(\mathbf{D_u} f\right)(0,0)$ exists and equals 0 for all $\mathbf{u}$. (How?)
Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $(f_x(0,0), f_y(0,0)) = (0,0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f\left(0+h, 0+k\right) - f\left(0,0\right) - 0h - 0k}{\sqrt{h^2+k^2}} = 0.$$

Sheet 6

For $(h, k) \neq (0,0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right).$$

Also, note that

$$\left|\sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right)\right| \leq \left|\sqrt{h^2+k^2}\right|.$$

Thus, the required limit indeed does exist and equals 0.
Hence, f is differentiable at $(0,0)$ with (total) derivative equal to $(0,0)$.

Sheet 6

(10) The continuity of f at $(0, 0)$ is easy to show using the $\epsilon - \delta$ condition.
Indeed, observe that $|f(x, y) - f(0, 0)| = \left|\sqrt{x^2 + y^2}\right|$ for $y \neq 0$ and
$|f(x, y) - f(0, 0)| = 0$ for $y = 0$.
Thus, in general, we have that $|f(x, y) - f(0, 0)| \leq \left|\sqrt{x^2 + y^2}\right|$.
Let $\delta := \epsilon$ and call it a day.

For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0 + tu_1, 0 + tu_2) - f(0, 0)}{t} = \begin{cases} 0 & u_2 = 0 \\ \frac{u_2}{|u_2|} t & u_2 \neq 0 \end{cases}$$

Hence, $(\mathbf{D_u} f)(0, 0)$ exists and equals 0 for all $\mathbf{u}$. Thus, all directional derivatives exist.

Sheet 6

If f is differentiable, then the total derivative *must* be $(f_x(0,0), f_y(0,0)) = (0,0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = 0.$$

For $(h,k) \neq (0,0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = \frac{k}{|k|}.$$

It is clear that the limit of the above expression as $(h,k) \to (0,0)$ does not exist. Hence, f is not differentiable at $(0,0)$.

(1) Note that the partial derivatives of F do exist at $(1, -1, 3)$. Indeed, given any (x_0, y_0, z_0), we have it that,

$$F_x(x_0, y_0, z_0) = 2x_0 + 2y_0,$$

$$F_y(x_0, y_0, z_0) = 2x_0 - 2y_0,$$

$$F_z(x_0, y_0, z_0) = 2z_0.$$

Thus, $(\nabla F)(1, -1, 3) = (0, 4, 6)$.
Moreover, the direction of the normal at to the surface $F(x, y, z) = c$ at the point (x_0, y_0, z_0) is given by $(\nabla F)(x_0, y_0, z_0)$. (How?)
Thus, the required normal line is $(1, -1, 3) + t(0, 4, 6)$ as t varies over $\mathbb{R}$.

Also, the corresponding tangent plane is given by

$$0 \cdot (x - 1) + 4(y + 1) + 6(z - 3) = 0.$$

(2) It is not too tough to show that the direction of the normal to a sphere at a point on the sphere is the same as the direction of the vector joining the center to that point. Indeed, we get that $(\nabla S)(x_0, y_0, z_0) = 2(x_0, y_0, z_0)$, where $S(x, y, z) := x^2 + y^2 + z^2$ for $(x, y, z) \in \mathbb{R}^3$.
Thus, the required $\mathbf{u}$ is $\frac{1}{3}(2, 2, 1)$.
Hence,

$$(\mathbf{D_u} F)(2, 2, 1) = \lim_{t \to 0} \frac{3(2t/3) - 5(2t/3) + 2(t/3)}{t} = -\frac{2}{3}.$$

(3) We shall assume that z is a "sufficiently smooth" function of x and y.
We are given that $\sin(x+y)+\sin(y+z)=1$ and $\cos(y+z)\neq 0$.
Differentiating with respect to x while keeping y constant gives us

$$\cos(x+y)+\cos(y+z)\frac{\partial z}{\partial x}=0. \tag{$*$}$$

Similarly, differentiating with respect to y while keeping x constant gives us

$$\cos(x+y)+\cos(y+z)\left(1+\frac{\partial z}{\partial y}\right)=0. \tag{$**$}$$

Differentiating $(*)$ with respect to y gives us

$$-\sin(x+y)-\sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}+\cos(y+z)\frac{\partial^2 z}{\partial x \partial y}=0.$$

Thus, using $(*)$ and $(**)$, we get

Sheet 7

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)} \left[\sin(x+y) + \sin(y+z) \cdot \left(1 + \frac{\partial z}{\partial y}\right) \frac{\partial z}{\partial x} \right]$$

$$= \frac{1}{\cos(y+z)} \left[\sin(x+y) + \sin(y+z) \left(-\frac{\cos(x+y)}{\cos(y+z)}\right) \left(-\frac{\cos(x+y)}{\cos(y+z)}\right) \right]$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z) \frac{\cos^2(x+y)}{\cos^2(y+z)}$$

(4) We have that

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}.$$

For $k \neq 0$, we know that

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = -k.$$

We also know that

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

Thus, we get that

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1.$$

By similar calculations, we get that $f_{yx}(0,0) = 1$.
Thus, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

For $(x,\ y) \neq (0,\ 0)$, one can calculate the second derivatives and see that they turn out to be discontinuous.

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}, f_{yx}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$