Short Quiz 9: Solution

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Question.
Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \sqrt{x^2+y^2} \text{ and } g(x,y) := |xy| \text{ for } (x,y) \in \mathbb{R}^2.$$

Determine which of the following statements is true. Justify your answer.

(a) Both f and g are differentiable at $(0,0)$.

(b) f is differentiable at $(0,0)$, but g is not differentiable at $(0,0)$.

(c) g is differentiable at $(0,0)$, but f is not differentiable at $(0,0)$.

(d) Neither f nor g is differentiable at $(0,0)$.

[5 marks]

[2 marks for correct alternative (T/F); 3 marks for correct justification]

Answer. The correct alternative is (c). **[2]**
Justification:
Note that for $h \neq 0$, one has

$$\frac{f(0+h,0)-f(0,0)}{h} = \frac{|h|}{h}.$$

Hence, $f_x(0,0)$ does not exist as $\lim\limits_{h\to 0} \dfrac{|h|}{h}$ does not. **[1]**

For $(h,k) \neq (0,0)$ note that $|h| \le \sqrt{h^2+k^2}$ and hence,

$$\frac{g(0+h,0+k)-g(0,0)-0\cdot h-0\cdot k}{\sqrt{h^2+k^2}} = \frac{|hk|}{\sqrt{h^2+k^2}} \le |k|.$$

Hence, we get that

$$\lim_{(h,k)\to(0,0)} \frac{g(0+h,0+k)-g(0,0)-0\cdot h-0\cdot k}{\sqrt{h^2+k^2}} = 0.$$

Thus, g is differentiable at $(0,0)$ with (total) derivative $(0,0)$. **[2]**

Points to be noted -

1. Simply calculating $f_x(x_0,y_0)$ for $(x_0,y_0) \neq (0,0)$ and then concluding that $f_x(0,0)$ does not exist by showing that $\lim\limits_{(x_0,y_0)\to(0,0)} f_x(x_0,y_0)$ does not exist is not correct as limit of derivative not existing does not imply non-existence of derivative of the limit. Hence, no marks for this justification has been given. This had been mentioned in the last quiz as well.

2. Many have incorrectly tried to invoke the sufficient condition for differentiability. Note that $g_x(0,k)$ does not even exist for $k \neq 0$ and hence, g_x isn't continuous at $(0,0)$. Similar argument for g_y as well.

3. Some have concluded the differentiability of g by incorrect arguments such as the following:
   (a) Existence of all directional derivatives.
   (b) Existence of partial derivatives and continuity of g.

(c) $(\mathbf{D_u}g)(0,0) = (\nabla g)(0,0) \cdot \mathbf{u}$ being true for all unit vectors $\mathbf{u} \in \mathbb{R}^2$.

Note that the above are simply necessary conditions but not sufficient. Indeed, consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$h(x,y) := \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

One can verify that the above function satisfies all the condition listed earlier but is not differentiable at $(0,0)$.