MA 105 : Calculus
D1 - T5, Tutorial 10

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Summary

Sheet 7: Problems 5, 6, 7, 8, 9, 10

5. (i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, the given point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.
Note that $(\nabla f)(x, y) = (4x^3 + 4, 4y^3 - 32)$. Hence, $(\nabla f)(x_0, y_0) = 0$.
Thus, we can appeal to the determinant test.

$(\Delta f)(x, y) = (12x^2)(12y^2) - (0)^2 = 144x^2y^2$.
Thus, $(\Delta f)(x_0, y_0) > 0$.
Also, $f_{xx}(x_0, y_0) = 12x_0^2 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0).

5. (ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0,0).$
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, the given point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.
Note that $(\nabla f)(x,y) = (3x^2 + 6x - 2y, -2x + 10y - 12y^2)$. Hence, $(\nabla f)(x_0, y_0) = 0$.
Thus, we can appeal to the determinant test.

$(\Delta f)(x,y) = (6x + 6)(10 - 24y) - (-2)^2$.
Thus, $(\Delta f)(x_0, y_0) = (6)(10) - 4 = 56 > 0$.
Also, $f_{xx}(x_0, y_0) = 6 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0).

6. (i) $f(x, y) = \left(x^2 - y^2\right) e^{-\left(x^2+y^2\right)/2}$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, every point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)
For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = xe^{1/2\left(-x^2-y^2\right)} \left(-x^2 + y^2 + 2\right)$.
Also, $f_y(x, y) = ye^{1/2\left(-x^2-y^2\right)} \left(-x^2 + y^2 - 2\right)$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that
$(x_0, y_0) \in \{(0, 0),\ (0, \sqrt{2}),\ (0, -\sqrt{2}),\ (-\sqrt{2}, 0),\ (\sqrt{2}, 0)\}$.
Now, we determine the exact nature using the determinant test.

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Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.
Hence, in our case,

$$(\Delta f)(x, y) = -e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4\right).$$

Moreover, $f_{xx}(x, y) = e^{-\left(x^2+y^2\right)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$
For $(x_0, y_0) = (0, 0)$, it is clear that it is a saddle point for f as discriminant is $-4 < 0$.

Note that if $x = 0$, the discriminant reduces to $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$.
Substituting $y = \pm\sqrt{2}$ gives us that the discriminant is positive with f_{xx} positive and hence, the points are points of local minima.

Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima as they have discriminant positive and f_{xx} negative.

6. (ii) $f(x, y) = f(x, y) = x^3 - 3xy^2$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, every point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)
For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = 3x^2 - 3y^2$.
Also, $f_y(x, y) = -6xy$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that $(x_0, y_0) = (0, 0)$.
Now, we determine the exact nature using the determinant test.

Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.
Hence, in our case,

$$(\Delta f)(x_0, y_0) = -36(x_0^2 + y_0^2).$$

Thus, for $(x_0, y_0) = (0, 0)$, we get the discriminant is 0.
Hence, we get that the discriminant test is inconclusive!
This means that we must turn to some other analytic methods of determining the nature.

Now, we note that $f(\delta, 0) = \delta^3$ for all $\delta \in \mathbb{R}$.
Thus, given any $\epsilon > 0$, choose $\delta = \pm\epsilon/2$.
This gives us that $(0, 0)$ is saddle point. (How?)

7. To find: Absolute maxima and minima of
$f(x,y) = (x^2 - 4x)\cos y$ for $1 \le x \le 3, -\pi/4 \le y \le \pi/4$.
Note that the domain is a closed and bounded set. As f is continuous on the domain, f does achieve a maximum and a minimum. Note that $f_x(x,y) = (2x-4)\cos y$ and $f_y(x,y) = -(x^2-4x)\sin y$ for interior points (x,y).
Thus, the only critical point is $p_1 = (2,0)$.

Now we restrict ourselves to the boundaries to find the local extrema.
"Right boundary:" This is the line segment $x = 3, -\pi/4 \le y \le \pi/4$.
The function now reduces to $-3\cos y$ on this segment.
Using our theory from one-variable calculus, we get that we need to check the points $(3,0)$, $(3,\pi/4)$, $(3,-\pi/4)$. (How?)

Similar consideration of the "left boundary" gives us the points
$(1,0)$, $(1,\pi/4)$, $(1,-\pi/4)$.

Now, we look at the "top boundary."
The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.
Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4)$, $(2, \pi/4)$, $(3, \pi/4)$.

Similarly, checking the "bottom boundary" gives us the points $(1, -\pi/4)$, $(2, -\pi/4)$, $(3, -\pi/4)$.
We now tabulate our results as follows:

(x_0, y_0)	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$
(x_0, y_0)	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\min} = -4$ at $(2, 0)$ and $f_{max} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$.

8. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined as $g(x, y, z) := x^2 + y^2 + z^2 - 1$. We need to maximise the function $T(x, y, z) = 400xyz$ subject to the constraint $g = 0$. Note that the set $S^2 = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$ is nonempty, closed and bounded, and T is continuous on it. Thus, f will attain its maximum on S^2.
Now, $(\nabla T)(x, y, z) = \lambda(\nabla g)(x, y, z)$ and $g(x, y, z) = 0$ means

$$400yz = 2\lambda x, \quad 400xz = 2\lambda y, \quad 400xy = 2\lambda z, \quad x^2 + y^2 + z^2 = 1.$$

The above gives us that $400xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$.

Also, $(\nabla g)(x, y, z) \neq (0, 0, 0)$ whenever $g(x, y, z) = 0$.
Thus, the hypotheses of the Lagrange Multiplier Theorem are satisfied.

Now, we solve the equations to get the points of maxima.
If $\lambda \neq 0$, then $x^2 = y^2 = z^2$ and hence, we get the 8 points $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.
If $\lambda = 0$, then $yz = zx = xy = 0$. This, combined with $g = 0$ gives us the 6 points $(0, 0, \pm 1)$, $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$.

Now, we check the value of T at these 14 points.
The first 8 points give either $T = \frac{400}{3\sqrt{3}}$ or $T = -\frac{400}{3\sqrt{3}}$. The last 6 points give $T = 0$.

Thus, the highest value of T is $\frac{400}{3\sqrt{3}}$.

9. We wish to maximise $f(x, y, z) = xyz$ subject to the constraints
$g(x, y, z) = x + y + z - 40 = 0$ and $h(x, y, z) = x + y - z = 0$.
$g = h = 0$ clearly gives us that $z = 20$.
Using this and h, we get that $x + y = 20$.
Thus, $f(x, y, z) = 20x(20 - x) = -20\left((x - 10)^2 - 100\right)$.
Note that $(x - 10)^2 \geq 0$ and hence, $f(x, y, z) \leq 2000$.
Thus, we get that f is bounded above by 2000 under the constraints $g = h = 0$.
Moreover, f does attain this value at $(10, 10, 20)$ which is a point satisfying the constraints.
Thus, we get that the maximum value attained by f is 2000, given the constraints.

Note that this time, our constraint set was not bounded. Thus, we had no reason to assume a priori that the maximum is attained.

10. We wish to maximise $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = x + 2y + 3z - 6 = 0$ and $h(x, y, z) = x + 3y + 4z - 9 = 0$
Note that the set $E := \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = h(x, y, z) = 0\}$ is **not** bounded. Thus, we can't straight away say that f does indeed attain a minimum on E. However, observe that $(0, 3, 0) \in E$ and $f(0, 3, 0) = 9$. Thus, if there were to exist a global minimum at some point (x, y, z), then it would have to be the case that $f(x, y, z) \leq 9$. This motivates us to consider the new set

$$E' = \{(x, y, z) \in E : f(x, y, z) \leq 9\}.$$

This set is clearly bounded. Moreover, it is closed and non-empty as well. Thus, f attains a minimum on E' which would in turn be a minimum on E as well. Now, we turn back to Lagrange.

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We solve $\nabla f = \lambda \nabla g + \mu \nabla h$ along with $g = h = 0$ for λ, μ, x, y, z.

We see that $\nabla f = (2x, 2y, 2z)$, $\nabla g = (1, 2, 3)$, $\nabla h = (1, 3, 4)$.
Thus, it is clear that ∇g and ∇h are always non-zero. Moreover, they are non-parallel at all points.
Thus, we get $2x = \lambda + \mu$, $2y = 2\lambda + 3\mu$, $2z = 3\lambda + 4\mu$. $(*)$

Note that $2g = 2h = 0$ along with $(*)$ gives us that $\lambda = -10$, $\mu = 8$.
Now, the equalities of $(*)$ give us that $x = -1$, $y = 2$, $z = 1$. It is clear that this is indeed a point of minimum.