Short Quiz 7: Solution

Aryaman Maithani

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Question. Let $a, b \in \mathbb{R}$ with $a > b > 0$. Set up the surface area of the oblate ellipsoid formed by rotating the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ around the $y-$axis as a Riemann integral, that is, express the area in the form

$$\int_c^d \varphi(y)dy.$$

[5]

Answer. Since we interested in revolving the figure about $y-$axis, it suffices to revolve the curve given by $x = \dfrac{a}{b}\sqrt{b^2 - y^2}$. Thus, we have $\dfrac{dx}{dy} = -\dfrac{ay}{b\sqrt{b^2 - y^2}}$. **[2]**

Given, a curve of the form $x = f(y)$ for $y \in [c, d]$ such that $f(y)$ is always nonnegative, we know that the surface area of revolution of such a curve about the $y-$axis is given by

$$\int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

[2]

Thus, in our case we have it that the surface area S is given as:

$$S = \int_{-b}^{b} 2a\pi \sqrt{1 + \left(\frac{a^2}{b^2} - 1\right)\frac{y^2}{b^2}} dy.$$

Which gives us that $c = -b$, $d = b$, and $\varphi(y) = 2a\pi\sqrt{1 + \left(\dfrac{a^2}{b^2} - 1\right)\dfrac{y^2}{b^2}}$. **[1]**

Points to be noted -

1. Even if the integrand hasn't been simplified to this extent, it is still accepted as long as it's correct.
2. One may simplify further and write the integral from 0 to b and multiplying with a 2, that is accepted as well.
3. Forgetting to mention 2 or π has been penalised.
4. Many have made a mistake in writing the expression for the surface area. The required surface area is *not* given by $\displaystyle\int_{-b}^{b} 2\pi x dy$ or $\displaystyle\int_{-b}^{b} \pi x^2 dy$.
5. Some have incorrectly calculated the volume instead of surface area. Marks have not been awarded in that case, unless there are some common computations such as that of $\frac{dx}{dy}$.
6. Some have started by writing the following formula:

   $$S = \int_{-b}^{b} 2\pi x\sqrt{(dx)^2 + (dy)^2}.$$

   This does not make sense by the way we have defined things as dx and dy are not real numbers. Thus, operations such as squaring and adding them are not defined. Half a mark has been cut for this.

7. On similar lines as above, implicitly differentiating in the following manner has also been penalised:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \frac{2xdx}{a^2} + \frac{2ydy}{b^2} = 0 \implies \frac{dx}{dy} = -\frac{ay}{b\sqrt{b^2 - y^2}}.$$

The reason for doing so is hopefully clear. Note that the following has *not* been penalised:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} \cdot \frac{dx}{dy} + \frac{2y}{b^2} = 0 \implies \frac{dx}{dy} = -\frac{ay}{b\sqrt{b^2 - y^2}}.$$

8. Many have found $\frac{dy}{dx}$ and simply substituted its reciprocal as $\frac{dx}{dy}$. Note that is not correct as you are appealing to the inverse function theorem most likely. However, when considering x as a function of y, we are looking at the curve in the first and fourth quadrants. In this region, y is *not* a function of x.
Most importantly, this calculation is *not* valid at the point $(0, b)$.

9. Unexplained parametrisations in terms of θ are not awarded marks.

10. Lastly, few have misread the question as surface of revolution about $x-$axis. They have been awarded up to 4 marks on the basis of their steps.