MA 105 : Calculus
D1 - T5, Tutorial 01

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IIT Bombay

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Introduction

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I will be your TA for the course MA 105.
This is an extremely interesting course. You will be introduced to mathematical rigour.
Do not fear this, try to appreciate the elegance behind it.
The learning curve will be quite steep compared to any other course but do not fret; with sincere and regular efforts from your side, you should be able to understand the course quite well.

About the course policy

Here is the course policy relevant to the tutorials -

There will be a quiz is almost all tutorials.

There will be total 12 quizzes in all.

The best 10 out of this 12 will be counted.

Due to this reason, there will be no re-quiz under any circumstances.

The quiz will begin sharp at 2:00 PM and end at 2:05 PM. For those who come later than 2:00 PM but before 2:05 PM, they can still take the quiz but will have limited time.

This quiz will also serve as your attendance. If you have taken the quiz, you are not allowed to leave the tutorial until it has ended.

Expectations

What we expect from you, before you come to the tutorial is the following:

1. You have read the lecture slides that have been uploaded up to that tutorial.
2. You have *attempted* the questions that are to be discussed in the tutorials.

Some elementary concepts

Definition (Interval)

An interval I is any subset of $\mathbb{R}$ with the following property:

$$x,\ y \in I,\ x < y \implies [x, y] \subset I.$$

Definition (Arbitrary intersection)

Given a collection of sets $\{A_i\}_{i \in I}$ where I is any arbitrary nonempty set, we define the following intersection:

$$\bigcap_{i \in I} A_i = \{x | x \in A_i \quad \forall i \in I\}.$$

What this means is that x will belong to the intersection if and only if it belongs to each set A_i.

Sheet 0

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2. The set of all even natural numbers is bounded.
3. The set $\{x\}$ is an open interval for every $x \in \mathbb{R}$.
4. The set $\{2/m \ : \ m \in \mathbb{N}\}$ is bounded above.

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   Note: any number greater than 2 is also an upper bound.

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   As $x,\ z \in N$, this means that $x,\ z \in A_i$ for each $i \in I$.
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Thus, $y \in N$. $\blacksquare$

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Note: In the previous proof, we had talked about arbitrary intersections. That is, the argument works for infinite intersections as well.

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    Let $\{A_i\}_{i \in I}$ be a collection of closed intervals, where $I = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
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To show the above equality, one must show that each side is a subset of the other.

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The reverse containment is left as an exercise.

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Note that $\{a_i\}_{i \in I}$ is a nonempty subset of $\mathbb{R}$ that is bounded above. (By what?)
Similarly, $\{b_i\}_{i \in I}$ is a nonempty subset of $\mathbb{R}$ that is bounded below.

Thus, we can define $a = \sup\{a_i | i \in I\}$ and $b = \inf\{b_i | i \in I\}$. Like before, we must have $a \leq b$.
Once again, we claim that $\bigcap_{i \in I} A_i = [a, b]$, which is a closed interval.

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    True. Follows from the Archimedean property of real numbers and that $\mathbb{N} \subset \mathbb{Q}$.
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12. Between any two rational numbers there lies an irrational number.
    Let $p,\ q \in \mathbb{Q}$ such that $p < q$.
    Define $r := a + \dfrac{b-a}{\sqrt{2}}$. Show that $r \in \mathbb{R} \setminus \mathbb{Q}$ and that $p < r < q$.

Recap - Convergence of a sequence

Definition (Convergence of a sequence)

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is a $a \in \mathbb{R}$ such that the following condition holds.
For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.

In this case, we say that (a_n) **converges** to a, or that a is a limit of (a_n), and we write

$$\lim_{n \to \infty} a_n = a \text{ or } a_n \longrightarrow a \text{ (as } n \longrightarrow \infty).$$

If a sequence doesn't converge, we say that the sequence **diverges** or that is is **divergent**.

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Let $n_0 = \left\lfloor \frac{10}{\epsilon} \right\rfloor + 1$. It is clear that $n_0 > \frac{10}{\epsilon}$.

Moreover, for any $n \geq n_0$, we will have $n > \frac{10}{\epsilon}$.

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Thus, we have shown that for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{10}{n}\right| < \epsilon$ for all $n \geq n_0$. $\therefore \lim_{n\to\infty} \frac{10}{n} = 0$.

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One such choice is $n_0 = \max\left\{1, \left\lfloor \frac{1}{3}\left(\frac{5}{\epsilon} - 1\right)\right\rfloor\right\} + 1$.

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One such choice is $n_0 = \max\left\{1, \left\lfloor \frac{1}{3}\left(\frac{5}{\epsilon} - 1\right)\right\rfloor\right\} + 1$.
Note: The choice of n_0 is not unique. Our choice of n_0 might not be the smallest but that is okay.

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$$\left|\frac{n^{2/3}\sin(n!)}{n+1} - 0\right| < \epsilon \iff \left|\frac{n^{2/3}\sin(n!)}{n+1}\right| < \epsilon \Longleftarrow \left|\frac{n^{2/3}}{n+1}\right| < \epsilon$$

Note the direction of implication of the red arrow. We have used the fact that $|\sin x| < 1$ for all real x.

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$$\left|\frac{n^{2/3}}{n+1}\right| < \epsilon \impliedby \left|\frac{n^{2/3}}{n}\right| < \epsilon$$

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By our arrows of implication, it can be seen that for $n \geq n_0$, the desired inequality holds.

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Let $\epsilon > 0$ be given. We must show that there exists $n_0 \in \mathbb{N}$ such that

$\left| \frac{n^{2/3} \sin(n!)}{n+1} - 0 \right| < \epsilon$ for all $n \geq n_0$.

Thus, if we choose $n_0 = \left\lfloor \frac{2}{\epsilon} \right\rfloor + 1$, we have it that the desired inequality holds.

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Observe the following:

$$\left|\frac{n}{n+1}-\frac{n+1}{n}\right|=\left|1-\frac{1}{n+1}-1-\frac{1}{n}\right|=\left|-\frac{1}{n+1}-\frac{1}{n}\right|$$

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$$= \frac{1}{n+1}+\frac{1}{n} < \frac{2}{n}$$

Thus, if we choose $n_0 = \left\lfloor \frac{2}{\epsilon} \right\rfloor + 1$, we have it that the desired inequality holds.

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Thus, the sequence given is bounded below by $n-1$, but by Archimedean property, we know that $n-1$ is not bounded above. Thus, our sequence is not bounded (above). As a result, it is not convergent. ■

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Now, for the sake of contradiction, let us assume that (a_n) converges. Then, by (a), we have it that $c_n := a_n + b_n = \frac{(-1)^n}{2}$ must be convergent.

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However, (c_n) converging is equivalent to $\{(-1)^n\}_{n\geq 1}$ converging.

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Now, for the sake of contradiction, let us assume that (a_n) converges. Then, by (a), we have it that $c_n := a_n + b_n = \dfrac{(-1)^n}{2}$ must be convergent.
However, (c_n) converging is equivalent to $\{(-1)^n\}_{n\ge1}$ converging. (Why?)
However, by (b), we know that the above is false. Thus, we have arrived at a contradiction.

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$n + 1 > n \geq n_0 = n_1$ and using (1).

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Choose $n_0 = n_1$, then, for any $n \geq n_0$, we have that $|b_n - L| = |a_{n+1} - L| < \epsilon$. (2)
The last inequality is due to the following:
$n + 1 > n \geq n_0 = n_1$ and using (1).
Thus, by (2), we have shown that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_{n+1} = L$.

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8. If $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.

Let $\epsilon > 0$ be given. This means that $\epsilon^2 > 0$.

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At what place(s) did we use that $a_n \geq 0$?

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By definition of limit, we have shown that $\lim\limits_{n\to\infty} a_n^{1/2} = 0$. $\blacksquare$

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Hint for **optional:** Use the inequality $\left|\sqrt[n]{a} - \sqrt[n]{b}\right| \leq \sqrt[n]{|a-b|}$ for $n \in \mathbb{N}$.

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Both are false.
The sequences, $a_n := 1 \quad \forall n \in \mathbb{N}$ and $b_n := (-1)^n \quad \forall n \in \mathbb{N}$ act as a counterexample for both the statements.