MA 105 : Calculus
D1 - T5, Tutorial 03

Aryaman Maithani

IIT Bombay

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Summary

Sheet 2: Problems 6 to 10

Sheet 3: Problems 2, 3, 5 to 7

6. Given: $|f(x+h) - f(x)| \leq C|h|^{\alpha}$ for all $x, x+h \in (a,b)$.
Assuming $h \neq 0$, we can write:

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq C|h|^{\alpha-1}$$

$$\implies -C|h|^{\alpha-1} \leq \frac{f(x+h) - f(x)}{h} \leq C|h|^{\alpha-1}$$

As $\alpha > 1$, we have it that $\lim\limits_{h \to 0} C|h|^{\alpha-1} = 0$. Thus, by Sandwich Theorem, we have it that the limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists and is equal to 0. Thus, the function is differentiable, by definition.

By the definition of $f'(x)$, we also have it that $f'(x) = 0$.

7.
$$\lim_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)+f(c)-f(c-h)}{2h}$$

Now, it is given that f is differentiable at c. This means that $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ exists. Moreover, it is equal to $f'(c)$.

Similarly, the limit $\lim_{h\to 0^+} \frac{f(c)-f(c-h)}{h}$ exists and equals $f'(c)$. Now that we know the existence of these limits, we can split the sum above.

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)+f(c)-f(c-h)}{2h}$$
$$= \frac{1}{2}\left(\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} + \lim_{h\to 0^+} \frac{f(c)-f(c-h)}{h}\right)$$
$$= \frac{1}{2}\left(f'(c)+f'(c)\right) = f'(c). \qquad \blacksquare$$

(Converse.)
The converse need not be true. That is,

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

may exist but f could still be non-differentiable at c.
Show this explicitly using $f(x) := |x|$ as an example.

8. Given: $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. (1)
Let $x = y = 0$. This gives us that $f(0) = (f(0))^2$.
Thus, $f(0) = 0$ or $f(0) = 1$.

Case 1. $f(0) = 0$.
Substitute $y = 0$ in (1). Thus, $f(x) = f(0)f(x) = 0$.
Therefore, f is identically 0 which means it's differentiable everywhere with derivative 0.
Verify that $f'(c) = f'(0)f(c)$ does hold for all $x \in \mathbb{R}$. (We did not need to use the fact that f is differentiable at 0, it followed from definition.)

Case 2. $f(0) = 1$.
As f is differentiable at 0, we know that:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0) \implies \lim_{h \to 0} \frac{f(h) - 1}{h} = f'(0). \tag{2}$$

Now, let us show that f is differentiable everywhere.
Let $c \in \mathbb{R}$. We must show that the following limit exists:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Using (1), we can write the above expression as:

$$\lim_{h \to 0} \frac{f(c)f(h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c)(f(h) - 1)}{h} = f(c) \cdot \lim_{h \to 0} \frac{f(h) - 1}{h}.$$

By (2), we know that the above limit exists. Thus, we have it that f is differentiable at c for every $c \in \mathbb{R}$. Moreover, $f'(c) = f'(0)f(c)$.

(Optional) We have gotten that the derivative of f is a scalar multiple of f. Use this to conclude.

9. (i) Let $f(x) := \cos x$ for $x \in (0, \pi)$. Then f is one-one and continuous. Consider $c \in (0, \pi)$. Now $f'(c) = -\sin c \neq 0$.
Further, $f((0, \pi)) = (-1, 1)$. If $d \in (-1, 1)$ and $f(c) = \cos c = d$, then

$$(f^{-1})'(d) = \frac{1}{f'(c)} = -\frac{1}{\sin c} = -\frac{1}{\sqrt{1 - \cos^2 c}} = -\frac{1}{\sqrt{1 - d^2}}.$$

(ii) Let $f(x) := \operatorname{cosec} x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$. Then f is one-one and continuous. Consider $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$. Now $f'(c) = -\operatorname{cosec} c \cot c = -\operatorname{cosec}^2 c \cos c \neq 0$.
Further, $f\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}\right) = (-\infty, -1) \cup (1, \infty)$. If $|d| > 1$ and $f(c) = \operatorname{cosec} c = d$, then

$$(f^{-1})'(d) = \frac{1}{f'(c)} = -\frac{1}{\operatorname{cosec}^2 c \cos c} = -\frac{1}{\operatorname{cosec}^2 c \sqrt{1 - \frac{1}{\operatorname{cosec}^2 c}}} = -\frac{1}{|d|\sqrt{d^2 - 1}}.$$

10. Define $g(x) := \dfrac{2x-1}{x+1}$ for $x \in \mathbb{R} \setminus \{1\}$.

Given, $y = (f \circ g)(x)$. As g is differentiable in its domain and so is f, we know that $f \circ g$ is differentiable wherever defined and its derivative is given by:

$$\frac{dy}{dx} = (f \circ g)'(x) = f'(g(x))g'(x) = \sin\left((g(x))^2\right) g'(x).$$

Let us compute $g'(x)$. $g(x) = \dfrac{2x-1}{x+1} = \dfrac{2x+2-3}{x+1} = 2 - \dfrac{3}{x+1}$.

Using quotient rule, we get that $g'(x) = \dfrac{3}{(x+1)^2}$.

$$\therefore \frac{dy}{dx} = \sin\left(\left(\frac{2x-1}{x+1}\right)^2\right) \frac{3}{x+1}$$

2. Assume that the cubic (denote it by $f(x)$) has two roots, a and b. We may assume that $a < b$. Then, we know the following:
(i) f is continuous on $[a, b]$,
(ii) f is differentiable on (a, b), and
(iii) $f(a) = f(b)$.
Thus, by Rolle's Theorem, there exists $c \in (a, b)$ such that $f'(c) = 0$.
However, $f'(c) = 3c^2 + p$ cannot be 0 as $3c^2$ is always non-negative and p is strictly positive.
Note: We have shown that the cubic has **at most** 1 root. We haven't actually shown that f has *a* root. This can be shown using IVT. (How?)

3. Part 1. We will first show the existence of such an $x_0 \in (a, b)$.

Proof. $I := [a, b]$ is an interval and f is continuous. Thus, f has the intermediate value property on I. Thus, the range $J := f(I)$ must be an interval. As $f(a)$ and $f(b)$ are of different signs, 0 lies between them. As $f(a), f(b) \in J$ and J is an interval, we have it that $0 \in J = f(I)$. Thus, $0 = f(x_0)$ for some $x_0 \in I = (a, b)$. ∎

Part 2. Now we will show the uniqueness of x_0. Assume that there exists $x_1 \in (a, b)$ such that $f(x_1) = 0$. We may assume that $x_0 < x_1$.

Now, we know the following:

(i) f is continuous on $[x_0, x_1]$,

(ii) f is differentiable on (x_0, x_1), and

(iii) $f(x_0) = f(x_1)$.

Thus, by Rolle's Theorem, there exists $x_2 \in (x_0, x_1)$ such that $f'(x_2) = 0$. But this contradicts the hypothesis that $f'(x) \neq 0$ for all $x \in (a, b)$. ∎

Lagrange's Mean Value Theorem (MVT)

Theorem (MVT)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that
(i) f is continuous on $[a, b]$, and
(ii) f is differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

5. To prove that $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
Case 1. $a = b$. Trivial.
Case 2. $a \neq b$. Without loss of generality, we can assume that $a < b$.
As $f := \sin$ is continuous and differentiable on $\mathbb{R}$, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \text{(By MVT)}$$

Also, we know that $|f'(c)| = |\cos c| \leq 1.$

Thus, we have it that $\left|\dfrac{f(b) - f(a)}{b - a}\right| \leq 1.$

This is equivalent to what we wanted to prove. $\blacksquare$

6. Let $c := \dfrac{a+b}{2}$. It is clear that $a < c < b$. Moreover, we have it that $2(c-a) = 2(b-c) = b-a$.
By MVT, there exists $c_1 \in (a,c)$ such that $f'(c_1) = \dfrac{f(c)-f(a)}{c-a}$ and there exists $c_2 \in (c,b)$ such that $f'(c_2) = \dfrac{f(b)-f(c)}{b-c}$. As c_1 and c_2 belong to disjoint intervals, it is clear that $c_1 \neq c_2$.

Observe that

$$f'(c_1) + f'(c_2) = \frac{f(c)-f(a)}{c-a} + \frac{f(b)-f(c)}{b-c} = 2\left(\frac{f(c)-f(a)+f(b)-f(c)}{b-a}\right) = 2. \blacksquare$$

7. Assume not. That is, $f(0) \neq 0$. Then, there are two possibilities.
Case 1. $f(0) > 0$.
The function f satisfies the hypothesis of MVT, thus there must exist $c \in (-a, 0)$ such that $f'(c) = \dfrac{f(0) - f(-a)}{0 - (-a)} = \dfrac{f(0)}{a} + 1$.
As $f(0) > 0$ and $a > 0$, we get that $f'(c) > 1$ which contradicts the hypothesis.

Case 2. $f(0) < 0$.
The function f satisfies the hypothesis of MVT, thus there must exist $d \in (0, a)$ such that $f'(d) = \dfrac{f(a) - f(0)}{a - 0} = 1 - \dfrac{f(0)}{a}$.
As $f(0) < 0$ and $a > 0$, we get that $f'(d) > 1$ which contradicts the hypothesis.

(Optional) Note the following:

$$\frac{f(x)-f(-a)}{x-(-a)} = \frac{f(x)-x+x+a}{x+a} = \frac{f(x)-x}{x+a} + 1$$

and

$$\frac{f(a)-f(x)}{a-x} = \frac{a-x+x-f(x)}{a-x} = 1 + \frac{x-f(x)}{a-x}$$

Choose $x \in (-a, a)$ and use MVT appropriately to get contradictions for $f(x) > x$ and $f(x) < x$.