Counterexamples in Calculus

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Semester: Autumn 2019
Latest update: September 10, 2019

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1. A bounded sequence need not be convergent.
   Example: $a_n := (-1)^n$.

2. A continuous function need not have the intermediate value property.
   Example: $f : (0,1) \cup (2,3) \to \mathbb{R}$ given by $f(x) := x$.

3. The inverse of a differentiable function need not be continuous.
   Example: $f : [0,1] \cup (2,3] \to [0,2]$ given by

   $$f(x) := \begin{cases} x & x \in [0,1] \\ x-1 & x \in (2,3] \end{cases}$$

   Corollaries: The inverse of a continuous function need not be continuous. The inverse of a differentiable function need not be differentiable.

4. A function defined on an interval with the intermediate value property need not be continuous *anywhere.*
   Example: Conway Base 13 function.

5. A Riemann integrable function may have infinitely many discontinuities.
   Example: Thomae's functions.

6. A differentiable function with derivative zero everywhere need not be constant.
   Example: $f : (0,1) \cup (2,3) \to \mathbb{R}$ defined as

   $$f(x) := \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (2,3) \end{cases}$$