

MA 105 : Calculus D1 - T5, Tutorial 10

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5. (i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7$, $(x_0, y_0) = (-1, 2)$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, the given point is an interior point of D . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

Note that $(\nabla f)(x, y) = (4x^3 + 4, 4y^3 - 32)$. Hence, $(\nabla f)(x_0, y_0) = 0$.

Thus, we can appeal to the determinant test.

$$(\Delta f)(x, y) = (12x^2)(12y^2) - (0)^2 = 144x^2y^2.$$

Thus, $(\Delta f)(x_0, y_0) > 0$.

Also, $f_{xx}(x_0, y_0) = 12x_0^2 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0) .

5. (ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$, $(x_0, y_0) = (0, 0)$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, the given point is an interior point of D . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

Note that $(\nabla f)(x, y) = (3x^2 + 6x - 2y, -2x + 10y - 12y^2)$. Hence, $(\nabla f)(x_0, y_0) = 0$.

Thus, we can appeal to the determinant test.

$$(\Delta f)(x, y) = (6x + 6)(10 - 24y) - (-2)^2.$$

$$\text{Thus, } (\Delta f)(x_0, y_0) = (6)(10) - 4 = 56 > 0.$$

$$\text{Also, } f_{xx}(x_0, y_0) = 6 > 0.$$

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0) .

6. (i) $f(x, y) = (x^2 - y^2) e^{-(x^2+y^2)/2}$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, every point is an interior point of D . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)

For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = x e^{1/2(-x^2-y^2)} (-x^2 + y^2 + 2)$.

Also, $f_y(x, y) = y e^{1/2(-x^2-y^2)} (-x^2 + y^2 - 2)$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that

$$(x_0, y_0) \in \{(0, 0), (0, \sqrt{2}), (0, -\sqrt{2}), (-\sqrt{2}, 0), (\sqrt{2}, 0)\}.$$

Now, we determine the exact nature using the determinant test.

Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.

Hence, in our case,

$$(\Delta f)(x, y) = -e^{-x^2-y^2} (x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4).$$

Moreover, $f_{xx}(x, y) = e^{-(x^2+y^2)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$

For $(x_0, y_0) = (0, 0)$, it is clear that it is a saddle point for f as discriminant is $-4 < 0$.

Note that if $x = 0$, the discriminant reduces to $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$.

Substituting $y = \pm\sqrt{2}$ gives us that the discriminant is positive with f_{xx} positive and hence, the points are points of local minima.

Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima as they have discriminant positive and f_{xx} negative.

6. (ii) $f(x, y) = x^3 - 3xy^2$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, every point is an interior point of D . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)

For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = 3x^2 - 3y^2$.

Also, $f_y(x, y) = -6xy$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that $(x_0, y_0) = (0, 0)$.

Now, we determine the exact nature using the determinant test.

Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.

Hence, in our case,

$$(\Delta f)(x_0, y_0) = -36(x_0^2 + y_0^2).$$

Thus, for $(x_0, y_0) = (0, 0)$, we get the discriminant is 0.

Hence, we get that the discriminant test is **inconclusive!**

This means that we must turn to some other analytic methods of determining the nature.

Now, we note that $f(\delta, 0) = \delta^3$ for all $\delta \in \mathbb{R}$.

Thus, given any $\epsilon > 0$, choose $\delta = \pm\epsilon/2$.

This gives us that $(0, 0)$ is saddle point.

(How?)

7. To find: Absolute maxima and minima of

$$f(x, y) = (x^2 - 4x) \cos y \text{ for } 1 \leq x \leq 3, -\pi/4 \leq y \leq \pi/4.$$

Note that $f_x(x, y) = (2x - 4) \cos y$ and $f_y(x, y) = -(x^2 - 4x) \sin y$ for interior points (x, y) .

Thus, the only critical point is $p_1 = (2, 0)$.

Now we restrict ourselves to the boundaries to find the local extrema.

“Right boundary:” This is the line segment $x = 3, -\pi/4 \leq y \leq \pi/4$.

The function now reduces to $-3 \cos y$ on this segment.

Using our theory from one-variable calculus, we get that we need to check the points $(3, 0), (3, \pi/4), (3, -\pi/4)$. (How?)

Similar consideration of the “left boundary” gives us the points

$(1, 0), (1, \pi/4), (1, -\pi/4)$.

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Now, we look at the “top boundary.”

The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.

Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4)$, $(2, \pi/4)$, $(3, \pi/4)$.

Similarly, checking the “bottom boundary” gives us the points

$(1, -\pi/4)$, $(2, -\pi/4)$, $(3, -\pi/4)$.

We now tabulate our results as follows:

(x_0, y_0)	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$
(x_0, y_0)	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\min} = -4$ at $(2, 0)$ and $f_{\max} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$.

8. Let $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined as $g(x, y, z) := x^2 + y^2 + z^2 - 1$. We need to maximise the function $T(x, y, z) = 400xyz$ subject to the constraint $g = 0$. Note that the set $S^2 = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$ is nonempty, closed and bounded, and T is continuous on it.

Now, $(\nabla T)(x, y, z) = \lambda(\nabla g)(x, y, z)$ and $g(x, y, z) = 0$ means

$$400yz = 2\lambda x, \quad 400xz = 2\lambda y, \quad 400xy = 2\lambda z, \quad x^2 + y^2 + z^2 = 1.$$

The above gives us that $400xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$.

Also, $(\nabla g)(x, y, z) \neq (0, 0, 0)$ whenever $g(x, y, z) = 0$.

Thus, the hypotheses of the Lagrange Multiplier Theorem are satisfied.

Now, we solve the equations to get the points of maxima.

If $\lambda \neq 0$, then $x^2 = y^2 = z^2$ and hence, we get the 8 points $\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$.

If $\lambda = 0$, then $yz = zx = xy = 0$. This, combined with $g = 0$ gives us the 6 points $(0, 0, \pm 1)$, $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$.

Now, we check the value of T at these 14 points.

The first 8 points give either $T = \frac{400}{3\sqrt{3}}$ or $T = -\frac{400}{3\sqrt{3}}$. The last 6 points give $T = 0$.

Thus, the highest value of T is $\frac{400}{3\sqrt{3}}$.

9. We wish to maximise $f(x, y, z) = xyz$ subject to the constraints $g(x, y, z) = x + y + z - 40 = 0$ and $h(x, y, z) = x + y - z = 0$. We solve $\nabla f = \lambda \nabla g + \mu \nabla h$ along with $g = h = 0$ for λ, μ, x, y, z .

We see that $\nabla f = (yz, zx, xy)$, $\nabla g = (1, 1, 1)$, $\nabla h = (1, 1, -1)$.

Thus, we get $yz = \lambda + \mu$, $zx = \lambda + \mu$, $xy = \lambda - \mu$. (*)

Note that $g = h = 0$ gives us that $z = 20$.

Also, the first two equalities of (*) now give us that $x = y$.

Combining that with $h = 0$ gives us $x = y = 10$.

Thus, we get $(x, y, z) = (10, 10, 20)$ as the only solution along with $(\lambda, \mu) = (150, 50)$.

Now, we calculate f at the above point to get 2000 as an extremal value. To see that it is indeed the maximum and not the minimum, we substitute some other point like $(0, 20, 20)$ satisfying the constraints and compare.

10. We wish to maximise $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = x + 2y + 3z - 6 = 0$ and $h(x, y, z) = x + 3y + 4z - 9 = 0$. We solve $\nabla f = \lambda \nabla g + \mu \nabla h$ along with $g = h = 0$ for λ, μ, x, y, z .

We see that $\nabla f = (2x, 2y, 2z)$, $\nabla g = (1, 2, 3)$, $\nabla h = (1, 3, 4)$. Thus, it is clear that ∇g and ∇h are always non-zero. Moreover, they are non-parallel at all points.

Thus, we get $2x = \lambda + \mu$, $2y = 2\lambda + 3\mu$, $2z = 3\lambda + 4\mu$. (*)

Note that $2g = 2h = 0$ along with (*) gives us that $\lambda = -10$, $\mu = 8$.

Now, the equalities of (*) give us that $x = -1$, $y = 2$, $z = 1$.

Now, we calculate f at the above point to get 6 as an extremal value. To see that it is indeed the minimum and not the maximum, we substitute some other point like $(0, 3, 0)$ satisfying the constraints and compare.