MA 105 : Calculus
D1 - T5, Tutorial 12

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This is desired arc length parameterisation.

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by FTC for Line Integrals (Part II)

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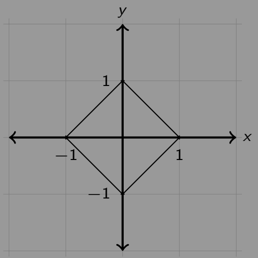

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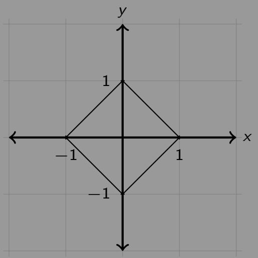

This is precisely the square given by $|x| + |y| = 1$ for $(x, y) \in \mathbb{R}^2$.

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(Note that C isn't smooth itself but rather, it is piecewise smooth. Thus, we can't directly appeal to the fundamental theorem but rather, we can break it up into its smooth components and then apply the theorem piecewise.)

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We shall show that **F** is not a gradient of a scalar field on S by showing that the third condition is not true.

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Thus, we have shown that $\mathbf{F}$ cannot be the gradient of a scalar field on S.

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Can you think of examples of when geometry of domain affected the behaviour of functions in the case of one-variable calculus?