Short Quiz 4: Solution

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28th August, 2019

Question. Recall that a cubic polynomial is a function $q : \mathbb{R} \to \mathbb{R}$ given by $q(x) = ax^3 + bx^2 + cx + d$ for $a,\ b,\ c,\ d \in \mathbb{R}$ and $a \neq 0$.
State whether the following statement is true or false. Justify your answer.
Every cubic polynomial has an inflection point. [5 marks]

[2 marks for correct alternative (T/F); 3 marks for correct justification]

Answer. T **[2]**
Justification:
We will justify our answer using the third derivative test.
First note that a cubic polynomial is thrice differentiable. **[1]**
We compute the following derivatives: $f''(x) = 6ax + b$ and $f'''(x) = 6a$. **[1]**
As $a \neq 0$, the following quantity is well-defined, $x_0 := -\frac{b}{3a}$. Moreover, $f''(x_0) = 0$ and $f'''(x_0) = 6a \neq 0$. **[1]**
This is a *sufficient* condition for an inflection point. Thus, we have shown that every cubic polynomial has an inflection point.

Points to be noted -

1. We do have an alternate solution which relies only on the function being twice differentiable and checking the sign of f'' around $-\frac{b}{3a}$.
2. Half a mark has been deducted for those who simply wrote that $f''(x) > 0$ for $x > -\frac{b}{3a}$ as this is true only when $a > 0$.
3. Marks have been appropriately deducted when a student has written a wrong statement, some common ones being -
   (a) $f''(c) = 0$ is a sufficient condition for an inflection point.
   Counterexample - $f(x) = x^4$ and $c = 0$.
   (b) $f''(c) = 0$ and $f'''(c) \neq 0$ is a necessary condition for an inflection point.
   Counterexample - $f(x) = x^5$ and $c = 0$.
   This is especially wrong as is not even necessary for a function to be continuous at a point of inflection.