

Extra Questions for MA 105

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TA for D1-T5

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Notation:

$\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers.

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.

\mathbb{Q} denotes the set of rational numbers.

\mathbb{R} denotes the set of real numbers.

WEEK 1

1. Let f be any bijection from \mathbb{N} to $\mathbb{Q} \cap [0, 1]$.
Define the sequence (a_n) of real numbers as: $a_n := f(n) \quad \forall n \in \mathbb{N}$.
Prove that (a_n) diverges or find an example of f such that (a_n) converges.
2. Let (a_n) be a sequence of real numbers. We say that (a_n) is *slack-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$ for all $n \geq n_0$.
Prove or disprove that a sequence is convergent (in the normal sense) \iff it is slack-convergent.

(Additional) What happens if we change $n \geq n_0$ to $n > n_0$?
3. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reciprocal-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1/\epsilon$ for all $n \geq n_0$.
Prove or disprove that a sequence is convergent (in the normal sense) \iff it is reciprocal-convergent.
4. Let (a_n) be a sequence of real numbers. We say that (a_n) is *natural-convergent* if the following condition holds.
For every $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} |a_{n+k} - a_n| = 0$.
Prove or disprove that a sequence is convergent (in the normal sense) \iff it is natural-convergent.
5. Let (a_n) be a sequence of real numbers. We say that (a_n) is *weirdly-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for infinitely many $n \geq n_0$.
Prove or disprove that a sequence is convergent (in the normal sense) \iff it is weirdly-convergent.
6. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reverse-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
For every $n_0 \in \mathbb{N}$, there is $\epsilon > 0$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.
Prove or disprove that a sequence is convergent (in the normal sense) \iff it is reverse-convergent.
7. Let S be a nonempty subset of \mathbb{R} which is bounded above. Let (a_n) be an increasing sequence in S such that $\lim_{n \rightarrow \infty} a_n = L \notin S$.
Prove or disprove that $L = \sup S$.

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?

WEEK 2

1. Show that $f : \mathbb{N} \rightarrow \mathbb{R}$ is continuous for any f .
2. Let $f : \mathbb{Q} \rightarrow \mathbb{R}$ be a continuous function such that the image (range) of f is a subset of \mathbb{Q} . Let $a, b, r \in \mathbb{Q}$ be such that $a < b$ and $f(a) < r < f(b)$. Show (with the help of an example) that it is not necessary that there exists some $c \in \mathbb{Q} \cap [a, b]$ such that $f(c) = r$.
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *reverse continuous* at c if for all $\delta > 0$, there exists $\epsilon > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
Is this notion of continuity the same as the normal notion?
If not, then give an example of a function which is reverse continuous at a point but not continuous or vice-versa.
4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *upper continuous* at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies f(c) \leq f(x) < f(c) + \epsilon$.
(a) Prove that a function is continuous at a point if it is upper continuous at that point.
(b) Show that the converse may not be true.
(c) Give an example of a function that is upper continuous at only one point.
(d) Given any $n \in \mathbb{N}$, show that there exists a function that is upper continuous at exactly n points.
(e) Show that there exists a function that is upper continuous at infinitely many points.
(f) Give an example of a function f that is upper continuous everywhere.
(g) Can you give an example of another function g such that g is upper continuous everywhere but $f - g$ is not constant?
5. Let $A, B \subset \mathbb{R}$ and $f : A \rightarrow B$ be a bijection. Show with the help of an example that f is continuous $\not\Rightarrow f^{-1}$ is continuous.
6. Show that there exists a bijection from $(0, 1)$ to $[0, 1]$.
7. Show that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$ or from $[0, 1]$ to $(0, 1)$.
8. Let $f : A \rightarrow B$ be a continuous surjective function. Show that it is possible for A to be a bounded open interval and B to be a bounded closed interval.
Is it possible for A to be a bounded closed interval and B to be a bounded open interval?
9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function with the intermediate value property. Is it necessary that f is continuous *somewhere*?
10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that given any $c \in \mathbb{R}$, the limit $\lim_{x \rightarrow c} f(x)$ exists. Is it necessary that f is continuous *somewhere*?

The last two questions are just for one to think about. I do not expect solutions for those.

WEEK 3

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Let $c \in \mathbb{R}$. Is it necessary that there exist $a, b \in \mathbb{R}$ such that $a < c < b$ and $f'(c) = \frac{f(b) - f(a)}{b - a}$?
2. Let $k \in \mathbb{N}$. Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is k times differentiable everywhere but not $(k + 1)$ times differentiable somewhere.
3. Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable at only one point.
4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Suppose there is $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$. Let $a_1 \in \mathbb{R}$ and set $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.
5. Let $D \subset \mathbb{R}$. A function $f : D \rightarrow \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in D, \forall \lambda \in [0, 1].$$

Prove that if I is an open interval and $f : I \rightarrow \mathbb{R}$ is convex, then f is continuous. Where did you use that I is an open interval?

Give an example to show that if J is not an open interval, then a convex function $f : J \rightarrow \mathbb{R}$ need not be continuous.

6. Let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$ be a differentiable function. Show by example that $f'(x) = 0 \quad \forall x \in D$ does not imply that f is constant.
7. Let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$ be a differentiable function.
We say that f is increasing if $\forall x, y \in D : x \leq y \implies f(x) \leq f(y)$.
Show by example that $f'(x) \geq 0 \quad \forall x \in D$ does not imply that f is increasing.
8. Show that the implication in the last two questions would be true if D were an interval.
9. Let A and B be open intervals in \mathbb{R} and $f : A \rightarrow B$ be a bijection such that f is differentiable. Show that it is not necessary that f^{-1} is differentiable.
10. * Construct a function $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties or show that no such function exists:
 1. f_1 is differentiable everywhere except one point x_1 .
 2. Define $f_2 : \mathbb{R} \setminus \{x_1\} \rightarrow \mathbb{R}$ as $f_2(x) :=$ derivative of f_1 at x . This f_2 must be differentiable everywhere in its domain except one point x_2 .
 3. Define $f_3 : \mathbb{R} \setminus \{x_1, x_2\} \rightarrow \mathbb{R}$ as $f_3(x) :=$ derivative of f_2 at x . This f_3 must be differentiable everywhere in its domain except one point x_3 .
 - \vdots
 - n . Define $f_n : \mathbb{R} \setminus \{x_1, \dots, x_{n-1}\} \rightarrow \mathbb{R}$ as $f_n(x) :=$ derivative of f_{n-1} at x . This f_n must be differentiable everywhere in its domain except one point x_n .
 - \vdots
 (Note that we do not stop at any n .)

ANY WEEK

1. Let $D \subset \mathbb{R}$. We say a function $f : D \rightarrow \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
 - (a) Understand how this definition is different from the definition of (usual) continuity.
 - (b) Give an example of a function which is continuous but not uniformly continuous.
 - (c) Show that any uniformly continuous function is also continuous.
2. Let (f_n) be a sequence of real valued functions defined on $[a, b]$ such that each f_n is continuous. Moreover, you are given that for each $x \in [a, b]$, the limit $\lim_{n \rightarrow \infty} f_n(x)$ exists.
Define the function $f : [a, b] \rightarrow \mathbb{R}$ as follows:

$$f(x) := \lim_{n \rightarrow \infty} f_n(x).$$

Show with the help of an example that it is not necessary that f is continuous.

3. Let $f_n : D \rightarrow \mathbb{R}$ be a sequence of functions from the set $D \subset \mathbb{R}$ to \mathbb{R} . We say that the sequence (f_n) *converges uniformly* to the function $f : D \rightarrow \mathbb{R}$ if given $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Prove that if (f_n) is a sequence of continuous functions that converges uniformly to f , then f is continuous. If you have solved the previous question, show that (f_n) didn't uniformly converge to f for that example.

4. Let $f : [a, b] \rightarrow \mathbb{R}$ be any function. Then, we know that if

- (a) f is monotonic, or
- (b) f is bounded and has at most a finite number of discontinuities in $[a, b]$,

then f is (Riemann) integrable.

Is the converse true?

That is, if f is (Riemann) integrable, then is it necessary that one of (a) or (b) should be true? Prove or disprove via counterexample. (Credit: Amit)

5. Show that any function $f : \mathbb{N} \rightarrow \mathbb{R}$ is uniformly continuous.

6. Let $a \in \mathbb{R}$ and (a_n) be a sequence of real numbers with the following property: Given any subsequence (a_{n_k}) of (a_n) , there exists a subsequence $(a_{n_{k_l}})$ of (a_{n_k}) with the property that $\lim_{l \rightarrow \infty} a_{n_{k_l}} = a$.
Prove that $\lim_{n \rightarrow \infty} a_n = a$.
7. Let E be a bounded subset of \mathbb{R} with the following property:
There exists $x_0 \in \mathbb{R} \setminus E$ such that there exists a sequence (x_n) in E which converges to x_0 . (For those familiar with the lingo, E is not a closed set.)
Show that there exists:
- (a) A function $g : E \rightarrow \mathbb{R}$ which is continuous but not bounded.
 - (b) A function $f : E \rightarrow \mathbb{R}$ such that $f(E)$ is bounded but does not have a maximum.
 - (c) A function $h : E \rightarrow \mathbb{R}$ such that h is continuous but not uniformly continuous.
8. Let $f : (a, b) \rightarrow \mathbb{R}$ be a monotonically increasing function, that is, $a < x < y < b \implies f(x) \leq f(y)$.
Show that for any $x \in (a, b)$, both $\lim_{t \rightarrow x^-} f(t)$ and $\lim_{t \rightarrow x^+} f(t)$ exist. Moreover, show that $\lim_{t \rightarrow x^-} f(t) \leq f(x) \leq \lim_{t \rightarrow x^+} f(t)$.
Also show that if $x < y$, then $\lim_{t \rightarrow x^+} f(t) \leq \lim_{t \rightarrow y^-} f(t)$.
(Hint: Try relating $\lim_{t \rightarrow x^-} f(t)$ with $\sup_{a < t < x} f(t)$.)
9. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Show that given any $x \in \mathbb{R}$, there exists a sequence (s_n) in S that converges to x .
Bonus 1: Generalise the argument by replacing $\sqrt{2}$ by any irrational square root of a natural number.
Bonus 2: Generalise the argument by replacing $\sqrt{2}$ by any irrational number.
10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a periodic function with period $p > 0$. That is, $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Moreover, assume that f is Riemann integrable on $[x, x + p]$ for any $x \in \mathbb{R}$. Is it necessary that $\int_x^{x+p} f(x) dx$ is independent of x ? (Note that f is not necessarily continuous.)
11. Let $A \subset \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ be a continuous and periodic function.
- (a) Show that if $A = \mathbb{R}$, then f is bounded.
 - (b) Show that there exists some A and some f for which the hypothesis holds but f is not bounded.
12. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that it is differentiable at 0. Is it necessary that there exist $a < 0 < b$ such that f is continuous at every point in (a, b) ?
13. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotonically increasing function. Show that the set of discontinuities of f is countable. (A set E is said to be countable if there exists a one-to-one function from E to \mathbb{N} . Examples - \emptyset , $\{1, 5, 6\}$, \mathbb{Q})
14. Show with the help of an example that there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous and bounded but not uniformly continuous.
15. Suppose $E \subset \mathbb{R}$. Let $f : E \rightarrow \mathbb{R}$ be a uniformly continuous function. Show that if (x_n) is a convergent sequence in E , then the sequence $(f(x_n))$ converges in \mathbb{R} . (Hint: Cauchy)
Show with the help of an example that the result need not hold if the function is just “continuous.”
16. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$.
Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
Is the result true if we drop the continuity hypothesis?
Can you think of a more general result? More simply, what sort of sets can we replace \mathbb{Q} with?
17. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.
18. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function. Suppose f has the property that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Is it necessary that there exists $\epsilon > 0$ such that f is constant in the interval $(-\epsilon, \epsilon)$?
19. Does there exist $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x)$ is rational iff x is irrational?

MULTI-VARIABLE CALCULUS

Notation: For $x \in \mathbb{R}^n$ and $\epsilon > 0$, we define $B_\epsilon(x) := \{y \in \mathbb{R}^n : \|y - x\| < \epsilon\}$.

- Are the following subsets of \mathbb{R}^2 closed? Identify ∂D in each case (except the last four).
 - \mathbb{R}^2
 - \mathbb{Q}^2
 - $(\mathbb{R} \setminus \mathbb{Q})^2$
 - \mathbb{N}^2
 - $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
 - $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
 - Any finite set of points.
 - $\{(x, 1/x) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \{0\}\}$.
 - $\left\{ \left(x, \sin \left(\frac{1}{x} \right) \right) \in \mathbb{R}^2 : x \in (0, 1] \right\} \cup \{0\} \times [-1, 1]$.
 - $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \times \{0\}$.
 - $C_1 \cup C_2$, where C_1 and C_2 are closed subsets of \mathbb{R}^2 .
 - $C_1 \cap C_2$, where C_1 and C_2 are closed subsets of \mathbb{R}^2 .
 - $\bigcup_{i \in \mathbb{N}} C_i$, where each C_i is a closed subset of \mathbb{R}^2 . (Not always, give a counterexample)
 - $\bigcap_{i \in \mathbb{N}} C_i$, where each C_i is a closed subset of \mathbb{R}^2 . (Yes)
- Let D be a subset of \mathbb{R}^2 . Let's call $x \in \mathbb{R}^2$ a limit point of D if for every $\epsilon > 0$, there exists $y \in B_\epsilon(x)$ such that $y \in D$ and $y \neq x$.
 Prove that D is closed if and only if it contains all of its limit points.
 For each of the examples above (except for the last four), find the set of its limit points.
- Let $n \in \mathbb{N}$. Show that there exists a countable subset E of \mathbb{R}^n such that $\mathbb{R}^n = E \cup \partial E$.
 (Hint: \mathbb{Q} is countable.)
- Let D be a subset of \mathbb{R}^2 such that every point of D is an interior point. Let $f : D \rightarrow \mathbb{R}$ be a differentiable function such that $\nabla f = 0$ on D .
 Show that it is not necessary that f is constant on D .
- Let $D \subset \mathbb{R}^2$ be defined as $D := \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$. Suppose $f : D \rightarrow \mathbb{R}$ is a differentiable function such that $\nabla f = 0$ on D .
 Prove that f is constant on D .
 (Note that you can't directly use bivariate MVT.) (Why not?)
- Let $D \subset \mathbb{R}^2$ be defined as $D = \{(x, y) \in \mathbb{R}^2 : (x, y) \in (\mathbb{Q} \cap [0, 1])^2\}$. That is, the set of all points in the rectangle $[0, 1] \times [0, 1]$ with both coordinates rational.
 Show that $\partial D = [0, 1] \times [0, 1]$.
 Show that the function $f : D \rightarrow \mathbb{R}$ defined as $f(x, y) := 0$ for $(x, y) \in D$ is integrable over D .
 (This is an example of a function that is integrable over a domain D even though ∂D is not of content 0.)
- Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as $f(x, y) := |xy|$.
 Show that f is differentiable at $(0, 0)$.
 Show that the partial derivative $f_x(0, k)$ does not exist whenever $k \neq 0$. Show the analogous result for f_y .
 Conclude that the function is differentiable at $(0, 0)$ even though the partial derivatives aren't continuous at $(0, 0)$. (They don't even exist in a neighbourhood!)
- Let $f : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}$ be defined as follows:

$$f(x, y) := \begin{cases} \frac{2xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Show that for any fixed $x \in [-1, 1]$ the function is Riemann integrable on $[-1, 1]$ with respect to y . Compute $I_x = \int_{-1}^1 f(x, y) dy$.

Show that this function I_x is Riemann integrable on $[-1, 1]$ with respect to x . Compute $\int_0^1 I_x dx$.

