MA 105 : Calculus
D1 - T5, Tutorial 09

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(Happy birthday to me!)

(7) Argue about the continuity of f at $(0,\ 0)$ using the fact that $|f(x,\ y)| \leq |x^2 + y^2|$.
(Recall Tutorial 2, Question 3. (ii))

It can also be easily verified that $f_x(0,\ 0) = f_y(0,\ 0) = 0$. (Write the expression like the previous questions and arrive at the conclusion.)

Now, let us evaluate $f_x(x_0,\ y_0)$ for $(x_0,\ y_0) \neq (0,\ 0)$.
It can be easily evaluated using product and chain rules to be:

$$f_x(x_0,\ y_0) = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,\ 0)$. (How?)

However, $\dfrac{2x}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)$ is not bounded in any such disc. To see this, consider any $r > 0$ and any $M \in \mathbb{R}$. One can find an $n \in \mathbb{N}$ such that $\frac{1}{\sqrt{n\pi}} < r$ and $\sqrt{n\pi} > M$. (How? Archimedean.)

In that case, the point $(x_0, y_0) = (1/\sqrt{2n\pi}, 0)$ will lie in the disc centered at $(0, 0)$ with radius r and $f(x_0, y_0) > M$.

(3) We shall assume that z is a "sufficiently smooth" function of x and y.
We are given that $\sin(x+y)+\sin(y+z)=1$ and $\cos(y+z)\neq 0$.
Differentiating with respect to x while keeping y constant gives us

$$\cos(x+y)+\cos(y+z)\frac{\partial z}{\partial x}=0. \tag{$*$}$$

Similarly, differentiating with respect to y while keeping x constant gives us

$$\cos(x+y)+\cos(y+z)\left(1+\frac{\partial z}{\partial y}\right)=0. \tag{$**$}$$

Differentiating $(*)$ with respect to y gives us

$$-\sin(x+y)-\sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}+\cos(y+z)\frac{\partial^2 z}{\partial x\partial y}=0.$$

Thus, using $(*)$ and $(**)$, we get

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)} \left[\sin(x+y) + \sin(y+z) \cdot \left(1 + \frac{\partial z}{\partial y}\right) \frac{\partial z}{\partial x} \right]$$

$$= \frac{1}{\cos(y+z)} \left[\sin(x+y) + \sin(y+z) \left(-\frac{\cos(x+y)}{\cos(y+z)}\right) \left(-\frac{\cos(x+y)}{\cos(y+z)}\right) \right]$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z) \frac{\cos^2(x+y)}{\cos^2(y+z)}$$

(4) We have that

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}.$$

For $k \neq 0$, we know that

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = -k.$$

We also know that

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

Thus, we get that

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1.$$

By similar calculations, we get that $f_{yx}(0,0) = 1$.
Thus, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

For $(x,\ y) \neq (0,\ 0)$, one can calculate the second derivatives and see that they turn out to be discontinuous.

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}, f_{yx}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$