MA 105 : Calculus (Autumn 2019)

Additional Problems for Tutorial 5

1. For which constants $a, b, c, d \in \mathbb{R}$ does the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = ax^3 + bx^2 + cx + d \quad \text{for } x \in \mathbb{R}$$
have a local maximum at -1, a point of inflection at 1 and satisfy $f(-1) = 10$ and $f(1) = 6$?

2. (**Cauchy Mean Value Theorem**) If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then show that there exists $c \in (a, b)$ such that
$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

3. Find the nth Taylor polynomial of $f : (-1, 1) \to \mathbb{R}$ around $a = 0$ when:
(i) $f(x) = (1 + x)^r$ [where $r \in \mathbb{Q}$], (ii) $f(x) = \cos x$, (iii) $f(x) = \dfrac{1}{1-x}$.

Optional Problems

4. (**L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms**) Let $c \in \mathbb{R}$ and let $D = (c - r, c) \cup (c, c + r)$ for some $r > 0$. Let $f, g : D \to \mathbb{R}$ be differentiable functions such that $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$. Suppose $g'(x) \neq 0$ for all $x \in D$, and $f'(x)/g'(x) \to \ell$ as $x \to c$. Then prove that $f(x)/g(x) \to \ell$ as $x \to c$. Here ℓ can be a real number or ∞ or $-\infty$. (Hint: Use the Cauchy Mean Value Theorem.)

5. (**L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms when $x \to \infty$**) Let $a \in \mathbb{R}$ and let $f, g : (a, \infty) \to \mathbb{R}$ be differentiable functions such that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$. Suppose $g'(x) \neq 0$ for all $x \in (a, \infty)$, and $f'(x)/g'(x) \to \ell$ as $x \to \infty$. Then prove that $f(x)/g(x) \to \ell$ as $x \to \infty$. Here ℓ can be a real number or ∞ or $-\infty$.

6. Evaluate the following limits:
(i) $\displaystyle\lim_{x \to 1} \frac{(2x - x^4)^{1/2} - x^{1/3}}{1 - x^{3/4}}$, (ii) $\displaystyle\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$,
(iii) $\displaystyle\lim_{x \to \infty} \left(x - \sqrt{x + x^2}\right)$, (iv) $\displaystyle\lim_{x \to \infty} \frac{\sqrt{x+2}}{\sqrt{x+1}}$.

7. Show that the Taylor series of the function $f(x) = \dfrac{x}{1 - x - x^2}$ is $\displaystyle\sum_{n=1}^{\infty} f_n x^n$ where f_n is the nth Fibonacci number, that is, $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. By writing $f(x)$ as a sum of partial fractions and thereby obtaining the Taylor series in a different way, find an explicit formula for the nth Fibonacci number.