Extra Questions for MA 105

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TA for D1-T5

Semester: Autumn 2019
Latest update: October 7, 2019

Notation:
$\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\mathbb{R}$ denotes the set of real numbers.

Week 1

1. Let f be any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0,\ 1]$.
   Define the sequence (a_n) of real numbers as: $a_n := f(n) \quad \forall n \in \mathbb{N}$.
   Prove that (a_n) diverges or find an example of f such that (a_n) converges.

2. Let (a_n) be a sequence of real numbers. We say that (a_n) is *slack-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is slack-convergent.

   (Additional) What happens if we change $n \geq n_0$ to $n > n_0$?

3. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reciprocal-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1/\epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reciprocal-convergent.

4. Let (a_n) be a sequence of real numbers. We say that (a_n) is *natural-convergent* if the following condition holds.
   For every $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} |a_{n+k} - a_n| = 0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is natural-convergent.

5. Let (a_n) be a sequence of real numbers. We say that (a_n) is *weirdly-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for infinitely many $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is weirdly-convergent.

6. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reverse-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $n_0 \in \mathbb{N}$, there is $\epsilon > 0$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reverse-convergent.

7. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Let (a_n) be an increasing sequence in S such that $\lim\limits_{n\to\infty} a_n = L \notin S$.
   Prove or disprove that $L = \sup S$.

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?

Week 2

1. Show that $f : \mathbb{N} \to \mathbb{R}$ is continuous for any f.

2. Let $f : \mathbb{Q} \to \mathbb{R}$ be a continuous function such that the image (range) of f is a subset of $\mathbb{Q}$. Let $a,\ b,\ r \in \mathbb{Q}$ be such that $a < b$ and $f(a) < r < f(b)$. Show (with the help of an example) that it is not necessary that there exists some $c \in \mathbb{Q} \cap [a, b]$ such that $f(c) = r$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *reverse continuous* at c if for all $\delta > 0$, there exists $\epsilon > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
   Is this notion of continuity the same as the normal notion?
   If not, then give an example of a function which is reverse continuous at a point but not continuous or vice-versa.

4. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *upper continuous* at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies f(c) \le f(x) < f(c) + \epsilon$.
   (a) Prove that a function is continuous at a point if it is upper continuous at that point.
   (b) Show that the converse may not be true.
   (c) Give an example of a function that is upper continuous at only one point.
   (d) Given any $n \in \mathbb{N}$, show that there exists a function that is upper continuous at exactly n points.
   (e) Show that there exists a function that is upper continuous at infinitely many points.
   (f) Give an example of a function f that is upper continuous everywhere.
   (g) Can you give an example of another function g such that g is upper continuous everywhere but $f - g$ is not constant?

5. Let $A, B \subset \mathbb{R}$ and $f : A \to B$ be a bijection. Show with the help of an example that f is continuous $\not\Longrightarrow$ f^{-1} is continuous.

6. Show that there exists a bijection from $(0, 1)$ to $[0, 1]$.

7. Show that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$ or from $[0, 1]$ to $(0, 1)$.

8. Let $f : A \to B$ be a continuous surjective function. Show that it is possible for A to be a bounded open interval and B to be a bounded closed interval.
   Is it possible for A to be a bounded closed interval and B to be a bounded open interval?

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the intermediate value property. Is it necessary that f is continuous *somewhere*?

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that given any $c \in \mathbb{R}$, the limit $\lim\limits_{x \to c} f(x)$ exists. Is it necessary that f is continuous *somewhere*?

The last two questions are just for one to think about. I do not expect solutions for those.

Week 3

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $c \in \mathbb{R}$. Is it necessary that there exist $a,\ b \in \mathbb{R}$ such that $a < c < b$ and $f'(c) = \dfrac{f(b) - f(a)}{b - a}$?

2. Let $k \in \mathbb{N}$. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is k times differentiable everywhere but not $(k+1)$ times differentiable somewhere.

3. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at only one point.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose there is $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f'(x)| \le \alpha < 1$. Let $a_1 \in \mathbb{R}$ and set $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.

5. Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *convex* if
   $$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda) f(y) \quad \forall x,\ y \in D,\ \forall \lambda \in [0, 1].$$
   Prove that if I is an open interval and $f : I \to \mathbb{R}$ is convex, then f is continuous. Where did you use that I is an open interval?
   Give an example to show that if J is not an open interval, then a convex function $f : J \to \mathbb{R}$ need not be continuous.

6. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a differentiable function. Show by example that $f'(x) = 0 \quad \forall x \in D$ does not imply that f is constant.

7. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a differentiable function.
   We say that f is increasing if $\forall x,\ y \in D : x \leq y \implies f(x) \leq f(y)$.
   Show by example that $f'(x) \geq 0 \quad \forall x \in D$ does not imply that f is increasing.

8. Show that the implication in the last two questions would be true if D were an interval.

9. Let A and B be open intervals in $\mathbb{R}$ and $f : A \to B$ be a bijection such that f is differentiable. Show that it is not necessary that f^{-1} is differentiable.

10. * Construct a function $f_1 : \mathbb{R} \to \mathbb{R}$ with the following properties or show that no such function exists:
    1. f_1 is differentiable everywhere except one point x_1.
    2. Define $f_2 : \mathbb{R} \setminus \{x_1\} \to \mathbb{R}$ as $f_2(x) :=$ derivative of f_1 at x. This f_2 must be differentiable everywhere in its domain except one point x_2.
    3. Define $f_3 : \mathbb{R} \setminus \{x_1,\ x_2\} \to \mathbb{R}$ as $f_3(x) :=$ derivative of f_2 at x. This f_3 must be differentiable everywhere in its domain except one point x_3.
    $\vdots$
    n. Define $f_n : \mathbb{R} \setminus \{x_1, \cdots, x_{n-1}\} \to \mathbb{R}$ as $f_n(x) :=$ derivative of f_{n-1} at x. This f_n must be differentiable everywhere in its domain except one point x_n.
    $\vdots$
    (Note that we do not stop at any n.)

Any Week

1. Let $D \subset \mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x,\ y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
   (a) Understand how this definition is different from the definition of (usual) continuity.
   (b) Give an example of a function which is continuous but not uniformly continuous.
   (c) Show that any uniformly continuous function is also continuous.

2. Let (f_n) be a sequence of real valued functions defined on $[a, b]$ such that each f_n is continuous. Moreover, you are given that for each $x \in [a, b]$, the limit $\lim\limits_{n\to\infty} f_n(x)$ exists.
   Define the function $f : [a, b] \to \mathbb{R}$ as follows:
   $$f(x) := \lim_{n\to\infty} f_n(x).$$
   Show with the help of an example that it is not necessary that f is continuous.

3. Let $f_n : D \to \mathbb{R}$ be a sequence of functions from the set $D \subset \mathbb{R}$ to $\mathbb{R}$. We say that the sequence (f_n) *converges uniformly* to the function $f : D \to \mathbb{R}$ if given $\epsilon > 0$, there exists an integer N such that
   $$|f_n(x) - f(x)| < \epsilon$$
   for all $n > N$ and all $x \in D$.
   Prove that if (f_n) is a sequence of continuous functions that converges uniformly to f, then f is continuous.
   If you have solved the previous question, show that (f_n) didn't uniformly converge to f for that example.

4. Let $f : [a, b] \to \mathbb{R}$ be any function. Then, we know that if
   (a) f is monotonic, or
   (b) f is bounded and has at most a finite number of discontinuities in $[a, b]$,
   then f is (Riemann) integrable.
   Is the converse true?
   That is, if f is (Riemann) integrable, then is it necessary that one of (a) or (b) should be true? Prove or disprove via counterexample. (Credit: Amit)

5. Show that any function $f : \mathbb{N} \to \mathbb{R}$ is uniformly continuous.

6. Let $a \in \mathbb{R}$ and (a_n) be a sequence of real numbers with the following property: Given any subsequence (a_{n_k}) of (a_n), there exists a subsequence $\left(a_{n_{k_l}}\right)$ of (a_{n_k}) with the property that $\lim_{l \to \infty} a_{n_{k_l}} = a$.
Prove that $\lim_{n \to \infty} a_n = a$.

7. Let E be a bounded subset of $\mathbb{R}$ with the following property:
There exists $x_0 \in \mathbb{R} \setminus E$ such that there exists a sequence (x_n) in E which converges to x_0. (For those familiar with the lingo, E is not a closed set.)
Show that there exists:
   (a) A function $g : E \to \mathbb{R}$ which is continuous but not bounded.
   (b) A function $f : E \to \mathbb{R}$ such that $f(E)$ is bounded but does not have a maximum.
   (c) A function $h : E \to \mathbb{R}$ such that h is continuous but not uniformly continuous.

8. Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function, that is, $a < x < y < b \implies f(x) \leq f(y)$.
Show that for any $x \in (a, b)$, both $\lim_{t \to x^-} f(t)$ and $\lim_{t \to x^+} f(t)$ exist. Moreover, show that $\lim_{t \to x^-} f(t) \leq f(x) \leq \lim_{t \to x^+} f(t)$.

   Also show that if $x < y$, then $\lim_{t \to x^+} f(t) \leq \lim_{t \to y^-} f(t)$.

   (Hint: Try relating $\lim_{t \to x^-} f(t)$ with $\sup_{a<t<x} f(t)$.)

9. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Show that given any $x \in \mathbb{R}$, there exists a sequence (s_n) in S that converges to x.
Bonus 1: Generalise the argument by replacing $\sqrt{2}$ by any irrational square root of a natural number.
Bonus 2: Generalise the argument by replacing $\sqrt{2}$ by any irrational number.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $p > 0$. That is, $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Moreover, assume that f is Riemann integrable on $[x, x + p]$ for any $x \in \mathbb{R}$. Is it necessary that $\displaystyle\int_x^{x+p} f(x)dx$ is independent of x? (Note that f is not necessarily continuous.)

11. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous and periodic function.
   (a) Show that if $A = \mathbb{R}$, then f is bounded.
   (b) Show that there exists some A and some f for which the hypothesis holds but f is not bounded.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that it is differentiable at 0. Is it necessary that there exist $a < 0 < b$ such that f is continuous at every point in (a, b)?

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Show that the set of discontinuities of f is countable.
(A set E is said to be countable if there exists a one-to-one function from E to $\mathbb{N}$. Examples - $\varnothing$, $\{1, 5, 6\}$, $\mathbb{Q}$)

14. Show with the help of an example that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous and bounded but not uniformly continuous.

15. Suppose $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that if (x_n) is a convergent sequence in E, then the sequence $(f(x_n))$ converges in $\mathbb{R}$. (Hint: Cauchy)
Show with the help of an example that the result need not hold if the function is just "continuous."

16. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$.
Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
Is the result true if we drop the continuity hypothesis.
Can you think of a more general result? More simply, what sort of sets can we replace $\mathbb{Q}$ with?

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Suppose f has the property that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Is it necessary that there exists $\epsilon > 0$ such that f is constant in the interval $(-\epsilon, \epsilon)$?

Multi-variable Calculus

Notation: For $x \in \mathbb{R}^n$ and $\epsilon > 0$, we define $B_\epsilon(x) := \{y \in \mathbb{R}^n : \|y - x\| < \epsilon\}$.

1. Are the following subsets of $\mathbb{R}^2$ closed? Identify ∂D in each case (except the last four).

(a) $\mathbb{R}^2$
(b) $\mathbb{Q}^2$
(c) $(\mathbb{R} \setminus \mathbb{Q})^2$
(d) $\mathbb{N}^2$
(e) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
(f) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
(g) Any finite set of points.
(h) $\{(x, 1/x) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \{0\}\}$.
(i) $\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \in \mathbb{R}^2 : x \in (0, 1]\right\} \cup \{0\} \times [-1, 1]$.
(j) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \times \{0\}$.
(k) $C_1 \cup C_2$, where C_1 and C_2 are closed subsets of $\mathbb{R}^2$.
(l) $C_1 \cap C_2$, where C_1 and C_2 are closed subsets of $\mathbb{R}^2$.
(m) $\bigcup_{i\in\mathbb{N}} C_i$, where each C_i is a closed subset of $\mathbb{R}^2$. (Not always, give a counterexample)
(n) $\bigcap_{i\in\mathbb{N}} C_i$, where each C_i is a closed subset of $\mathbb{R}^2$. (Yes)

2. Let D be a subset of $\mathbb{R}^2$. Let's call $x \in \mathbb{R}^2$ a limit point of D if for every $\epsilon > 0$, there exists $y \in B_\epsilon(x)$ such that $y \in D$ and $y \neq x$.
Prove that D is closed if and only if it contains all of its limits points.
For each of the examples above (except for the last four), find the set of its limit points.

3. Let $n \in \mathbb{N}$. Show that there exists a countable subset E of $\mathbb{R}^n$ such that $\mathbb{R}^n = E \cup \partial E$.
(Hint: $\mathbb{Q}$ is countable.)

4. Let D be a subset of $\mathbb{R}^2$ such that every point of D is an interior point. Let $f : D \to \mathbb{R}$ be a continuously differentiable function such that $\nabla f = 0$ on D.
Show that it is not necessary that f is constant on D.

5. Let $D \subset \mathbb{R}^2$ be defined as $D := \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$. Suppose $f : D \to \mathbb{R}$ is a continuously differentiable function such that $\nabla f = 0$ on D.
Prove that f is constant on D.
(Note that you can't directly use bivariate MVT.) (Why not?)

6. Let $D \subset \mathbb{R}^2$ be defined as $D = \{(x, y) \in \mathbb{R}^2 : (x, y) \in (\mathbb{Q} \cap [0, 1])^2\}$. That is, the set of all points in the rectangle $[0, 1] \times [0, 1]$ with both coordinates rational.
Show that $\partial D = [0, 1] \times [0, 1]$.
Show that the function $f : D \to \mathbb{R}$ defined as $f(x, y) := 0$ for $(x, y) \in D$ is integrable over D.
(This is an example of a function that is integrable over a domain D even though ∂D is not of content 0.)

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) := |xy|$.
Show that f is differentiable at $(0, 0)$.
Show that the partial derivative $f_x(0, k)$ does not exist whenever $k \neq 0$. Show the analogous result for f_y.
Conclude that the function is differentiable at $(0, 0)$ even though the partial derivatives aren't continuous at $(0, 0)$. (They don't even exist in a neighbourhood!)