MA 105 : Calculus
Simply connected sets

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Simply connected sets

Let $m \in \mathbb{N}$ and $D \subset \mathbb{R}^m$.
A loop in D is simply a continuous function $c : [a, b] \to D$ where $a,\ b \in \mathbb{R}$ with $a < b$ such that $c(a) = c(b)$.
For the purpose of this discussion, we shall assume $a = 0$ and $b = 1$.
We say that c can be continuously shrunk in D to a point $d \in D$ if there is a continuous function $H : [0, 1] \times [0, 1] \to D$ such that

1. $H(s, 0) = c(s)$ for every $s \in [0, 1]$,
2. $H(s, 1) = d$ for every $s \in [0, 1]$, and
3. $H(0, t) = H(1, t)$ for every $t \in [0, 1]$.

This map H is called a homotopy in D between the curve c and the constant curve d.
The domain D is said to be simply-connected if for every simply closed curve c in D, we have a homotopy H between c and some $d \in D$.

Intuitive interpretation

The definition is just how one would formally write what it means to "continuously shrink" a loop.

In (s, t), the t can be thought of as time. The first point is saying that at time $t = 0$, the curve $H(-, t) = H(-, 0)$ is simply the original loop c. Similarly, the second point is saying that at time $t = 1$, it becomes the constant loop d.

The third point is ensuring that during the deformation, at each point of time, we still do have a loop.

Demanding H to be continuous ensures that the shrinking is a "continuous process." Note that the codomain of H being D ensures that the shrinking happens completely within D.

Alternate definition

The previous definition can also be written in a slightly more concise (but equivalent) way.
Let D and c have the same meaning as before. Moreover, let
$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $U^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

We say that D is simply-connected if any loop in D defined by $f : S^1 \to D$ can be contracted to a point: there exists a continuous map $F : U^2 \to D$ such that F restricted to S^1 is f.

Remark

Usually, we also require D to be path-connected as part of our definition but for the purpose of this course, we do not demand that.