MA 105 : Calculus
D1 - T5, Tutorial 11

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(3) Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$. There are two pieces of the surface - one below and one above the xy-plane, both having the same surface area. Let S be the upper piece. Then one has

$$\text{Area}(S) = \iint_T \sqrt{1 + z_x^2 + z_y^2}\, d(x, y),$$

where T is the disc

$$(x, y) \in \mathbb{R}^2 : x^2 + \left(y - \frac{a}{2}\right)^2 \leq \left(\frac{a}{2}\right)^2,$$

and $z(x, y) = \sqrt{a^2 - x^2 - y^2}$ for $(x, y) \in T$.

Now, we calculate z_x and z_y.

$$z_x = -\frac{x}{z} \text{ and } z_y = -\frac{y}{z}.$$

Thus, we get the area integral as

$$\text{Area}(S) = \iint_T \frac{a dx dy}{z} = \iint_T \frac{a dx dy}{\sqrt{a^2 - x^2 - y^2}}$$

Now, T is described in polar coordinates by

$$x = r\cos\theta, y = r\sin\theta; 0 \le \theta \le \pi, 0 \le r \le a\sin\theta.$$

Therefore,

$$\text{Area}(S) = \int_0^{\pi} \left(\int_0^{a \sin\theta} \frac{ar}{\sqrt{a^2 - r^2}} dr \right) d\theta$$

$$= a \int_0^{\pi} [-\sqrt{a^2 - r^2}]\Big|_0^{a\sin\theta} d\theta$$

$$= a \int_0^{\pi} (-a|\cos\theta| + a) d\theta = (\pi - 2)a^2.$$

Thus, the required area is $2(\pi - 2)a^2$.

(6) We shall consider the cylinder (and thus, the sphere) to have radius 1. Moreover, we shall choose our axes such that the center of the sphere is the origin and the axis of the cylinder is the $z-$axis.
Let us find the area of the sphere between the planes $z = 0$ and $z = h$ for some $h \in (0, 1]$. Using this, we can find the area between two planes according to whether or not they are on the same side of $z = 0$ or not.
The surface of interest is parameterised as:

$$\Phi(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \text{ where } -\pi \leq \theta \leq \pi,\ \alpha \leq \varphi \leq \pi/2,$$

where α is the (unique) real number in $[0, \pi/2]$ such that $\cos\alpha = c$.

Now, we have $\Phi_\varphi \times \Phi_\theta = \left(\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, sin\varphi\cos\varphi\right)$.
This gives us $\|\Phi_\varphi \times \Phi_\theta\| = \sin\varphi$.
Thus, the area is given by

$$\int_{-\pi}^{\pi}\int_{\alpha}^{\pi/2} 1\sin\varphi d\varphi d\theta$$

$$= 2\pi\cos\alpha = 2\pi c.$$

Moreover, it can be easily seen the surface area of the cylinder between these two planes is also given by $2\pi c$. Thus, we can now conclude the final result by taking two cases.

(7) (i) Let T be the region in the $uv-$plane parameterising the region S as given. Note that $\mathbf{r}_u \times \mathbf{r}_v = -2(\mathbf{i} + \mathbf{j} + \mathbf{k})$ has negative $z-$component. Thus, we get the following, in differential notation:

$$\hat{\mathbf{n}} dS = \hat{\boldsymbol{n}} \left\| \mathbf{r}_u \times \mathbf{r}_v \right\| d(u, v) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) d(u, v).$$

Thus, the integral is simply $\displaystyle\iint_T 2 dS = 2 \operatorname{Area}(T)$.

Note that T is the triangle in the $uv-$plane with vertices $(0, 0)$ and $\left(\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus, the answer is simply $\frac{1}{2}$.

(ii) The surface satisfies $z = 1 - x - y \geq 0,\ x \geq 0,\ y \geq 0$.

Define $T := \{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\}$. Thus, we then have that S is given by $z = f(x, y) := 1 - x - y$ for $(x, y) \in T$.

Thus, $\hat{\mathbf{n}} dS = (-z_x, -z_y, 1) d(x, y)$, in differential notation.
Moreover, $\mathbf{F} \cdot \mathbf{n} dS = (x, y, z) \cdot (-z_x, -z_y, 1)\, d(x, y) = (x + y + z) d(x, y) = 1 d(x, y)$.
Now, one has $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n} dS \ = \iint_T 1 d(x, y) = \text{Area}(T) = \tfrac{1}{2}$.

(8) Routine calculation is to be done.
Parameterise the sphere as $\Phi(\varphi, \theta) := (a \sin \varphi \cos \theta, a \sin \varphi \sin \theta, a \cos \varphi)$ for $(\varphi, \theta) \in [0, \pi] \times [-\pi, \pi]$.

Then, $\hat{\boldsymbol{n}} dS = = (\Phi_\varphi \times \Phi_\theta)\, d(\varphi, \theta) = (a \sin \varphi) d(\varphi, \theta)$.

(Note that this is indeed the outwards normal.)
The integrand is now $\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_\varphi) = a^4 \sin^3 \varphi \cos \varphi \left(1 + \cos^2 \theta\right)$.

(Check! I might have made a sign mistake.)
Thus, the required integral is

$$\int_0^{2\pi} \left(\int_0^{\pi} a^4 \sin^3 \varphi \cos \varphi \left(1 + \cos^2 \theta\right) d\theta \right) d\varphi$$

$$= a^4 \left(\int_0^{\pi} \sin^3 \varphi \cos \varphi d\varphi \right) \left(\int_0^{2\pi} \left(1 + \cos^2 \theta\right) d\theta \right) = 0.$$

(2) For $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$,

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = \mathbf{0}.$$

Let S be any *good enough* (geometric) surface in $\mathbb{R}^3$ such that $\eth S = C$. Moreover, assume that S is oriented such that orientation it induces on C is the desired orientation. Then, we have that the line integral is given by

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS = 0.$$

(3) By Stokes' theorem, we have

$$\iint_S \operatorname{curl}(\mathbf{v}) \cdot \mathbf{n} dS = \oint_{C_1} \mathbf{v} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{v} \cdot d\mathbf{s}.$$

where C_1 is the circle $x^2 + y^2 = 4, z = -3$ with the counterclockwise orientation when viewed from "high above," and C_2 is the circle $x^2 + y^2 = 4, z = 0$, with the opposite orientation.

(How did we decide the orientation?)

Now, we can write $\oint_{C_i} \mathbf{v} \cdot d\mathbf{s}$ as $\oint_{C_i} ydx + xz^3 dy - zy^3 dz$.

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For $i = 1$, we have:

$$\oint_{C_1} ydx + xz^3dy - zy^3dz = \oint_{C_1} ydx - 27xdy = \oint_{C_1} \nabla(xy) \cdot d\mathbf{s} - \oint_{C_1} 28y^2dy.$$

The latter integral can be easily evaluated by a suitable parameterisation of C_1 to give us $-28\displaystyle\int_{-\pi}^{\pi} 4\cos^2\theta d\theta = -112\pi$.

Similarly, for $i = 2$, we have

$$\oint_{C_2} ydx = -\int_{\pi}^{\pi} (-4\sin^2\theta)d\theta = 4\pi.$$

Hence, the required integral is -108π.

(5) Note the following:

$$\mathbf{F} = \left(y^2 - z^2\right)\mathbf{i} + \left(z^2 - x^2\right)\mathbf{j} + \left(x^2 - y^2\right)\mathbf{k},$$

$$\nabla \times \mathbf{F} = (-2y - 2z)\mathbf{i} + (-2z - 2x)\mathbf{j} + (-2x - 2y)\mathbf{k},$$

$$\text{and } \hat{\mathbf{n}} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}.$$

Now, along the surface S which is a part of the plane $x + y + z = \frac{3a}{2}$ and which is bounded by C, we have

$$\nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} = -\frac{4}{\sqrt{3}}\frac{3a}{2}.$$

Hence,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS = -2\sqrt{3}a \iint_S dS = (-2\sqrt{3}a)(\text{Area of } S).$$

The surface S is a regular hexagon with vertices
$(a/2, 0, a), (a, 0, a/2), (a, a/2, 0), (a/2, a, 0), (0, a, a/2), (0, a/2, a)$.
Hence, its area is $\frac{3\sqrt{3}}{4}a^2$.
Using Stokes' theorem, we get that the above integral is equal to our desired integral which comes out to be $-\frac{9a^3}{2}$.

(6) Consider the following:

$$\mathbf{F} = (y, z, x),$$
$$\nabla \times \mathbf{F} = -(1, 1, 1).$$

We are to compute $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{s}$.

Let S be the surface lying on the hyperboloid bounded by C. We shall now describe S.

Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$.

Let $f(x, y) := \dfrac{xy}{b}$ for $(x, y) \in D$.

Then, the surface S is given by $z = f(x, y)$. Following differential notation, we get $\hat{\mathbf{n}} dS = (-z_x, -z_y, 1) d(x, y) = (-\frac{y}{b}, -\frac{x}{b}, 1) d(x, y)$.

By Stokes' Theorem, the required integral is simply equal to $\dfrac{1}{b} \displaystyle\iint_D (y + x - b) d(x, y)$.

This can be easily solved using polar coordinates.

Farewell