Equivalence of the two definitions of continuity

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Definition 1. *Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in D$. We say that f is continuous at c if*

$$(x_n) \text{ is a sequence in } D,\ x_n \to c \implies f(x_n) \to f(c).$$

Definition 2. *Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in D$. We say that f is continuous at c if:*
For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

We shall now prove the equivalence of the two definitions.

Proof. Definition 2 $\implies$ Definition 1:
Let $\epsilon > 0$ be given. Let (x_n) be any sequence in D such that $x_n \to c$. We must show that $f(x_n) \to f(c)$.
By hypothesis, there exists $\delta > 0$ such that $x \in D$ and $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$. (1)
As $x_n \to c$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$.
Therefore, by (1), we have it that $|f(x_n) - f(c)| < \epsilon$ for all $n \geq n_0$. This is precisely what it means for $f(x_n) \to f(c)$.

Definition 1 $\implies$ Definition 2:
We shall prove this by proving its contrapositive. That is, we assume that the $\epsilon - \delta$ condition does not hold and show that the sequential criterion does not hold either.
By assumption, there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, we have it that there exists $x \in D$ such that $|x-c| < \delta$ but $|f(x) - f(c)| \geq \epsilon_0$. (2)
Using this, we shall now construct a sequence (x_n) in D such that $x_n \to c$ but $f(x_n) \not\to f(c)$.
Let $n \in \mathbb{N}$. As (2) is true for all $\delta > 0$, we can choose $\delta = 1/n$. Thus, there exists $x_n \in D$ such that $|x_n - c| < 1/n$ and $|f(x_n) - f(c)| \geq \epsilon_0$.
Thus, we now have a sequence (x_n) in D such that $x_n \to c$. However, if choose $\epsilon = \epsilon_0 > 0$, we have it that $|f(x_n) - f(c)| \geq \epsilon$ for all $n \in \mathbb{N}$. Thus, $f(x_n)$ can not converge to $f(c)$. $\square$

Remark. We require Axiom of Choice (or one of its weaker version) for this argument.