MA 105 : Calculus
D1 - T5, Tutorial 11

Aryaman Maithani

IIT Bombay

16th October, 2019

Summary

Sheet 8: Problems 2, 4, 6, 8, 9, 10

(2) (i) We are to compute the following integral:

$$I = \int_0^\pi \left[\int_x^\pi \frac{\sin y}{y} dy \right] dx.$$

(Note that the above is actually problematic as $y = 0$ is in our region. Thus, the question is really asking us to integrate the function f which is 1 when $y = 0$ and $\sin y/y$ otherwise. As the function has a "removable" discontinuity, we aren't bothered.)
Now, note that we are integrating the function f over the following domain:

$$D := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \pi, \ x \leq y \leq \pi\}.$$

By Fubini's theorem, we know that the integral I equals $I' = \iint_D f(x, y) d(x, y)$.

Now, also note that D can also be written as the following:

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \pi,\ 0 \leq x \leq y\}.$$

(It is easy to argue this by looking the region D by drawing its graph and we shall stick with that for this course. Although, how *would* one argue that this does indeed equal D?)

Thus, using our friend Fubini again, we get that I' equals the following integral:

$$I'' = \int_0^\pi \left[\int_0^y \frac{\sin y}{y} dx\right] dy$$

I'' is easy to solve now and we get that $I'' = 2$. (How?)

By our previous observations, we get that $I'' = I' = I$ and hence, $I = 2$.

Sheet 8

(2) (ii) We are to compute the following integral:

$$I = \int_0^1 \left[\int_y^1 x^2 e^{xy} \, dx \right] dy.$$

As before, we shall transform the integral by cleverly switching the order of integrals. We observe that the domain of integration in this case is the following:

$$D := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, \; y \leq x \leq 1\}.$$

Like before, we note that D can also be written as:

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \; 0 \leq y \leq x\}.$$

By Fubini, we will again get that the two integrals are equal and hence,

$$I = \int_0^1 \left[\int_0^x x^2 e^{xy} \, dy \right] dx.$$

This is now easy to integrate and we get that:

$$\implies I = \int_0^1 x(e^{x^2} - 1) dx$$

$$\implies I = \frac{1}{2} \int_0^1 e^u du - \int_0^1 x dx$$

$$\implies I = \frac{1}{2}(e - 2)$$

(The answer given at the back is incorrect.)

(2) (iii) We are to compute the following integral:

$$\int_0^2 \left(\tan^{-1} \pi x - \tan^{-1} x\right) dx.$$

This is a single variable Riemann integral, which is not easy to compute directly. We convert it into a double integral and compute the double integral using Fubini, hoping that that'll be easier. Now, note that:

$$\int_0^2 \left(\tan^{-1} \pi x - \tan^{-1} x\right) dx = \int_0^2 \left(\int_x^{\pi x} \frac{1}{1+y^2} dy\right) dx$$

This is a double integral over the region

$$D := \{(x, y) : 0 \leq x \leq 2, x \leq y \leq \pi x\}.$$

Now, note that this region D is the union of two regions D_1 and D_2 where

$$D_1 = \left\{(x, y) : 0 \leq y \leq 2, \frac{y}{\pi} \leq x \leq y\right\}, \text{ and}$$

$$D_2 = \left\{(x, y) : 2 \leq y \leq 2\pi, \frac{y}{\pi} \leq x \leq 2\right\}.$$

Note that $D_1 \cap D_2$ is of (two-dimensional) content zero. (Verify!)
Therefore, we get that

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f,$$

Where $f : D \to \mathbb{R}$ is defined as $f(x, y) := (1 + y^2)^{-1}$.

Thus, the required integral is given by

$$I = \int_0^2 \left(\int_{y/\pi}^{y} \frac{1}{1+y^2} dx \right) dy + \int_2^{2\pi} \left(\int_{y/\pi}^{2} \frac{1}{1+y^2} dx \right) dy$$

$$= \left(1 - \frac{1}{\pi}\right) \int_0^2 \frac{y}{1+y^2} dy + \int_2^{2\pi} \frac{2}{1+y^2} dy - \frac{1}{\pi} \int_2^{2\pi} \frac{y}{1+y^2} dy$$

$$= \frac{1}{2}\left(1 - \frac{1}{\pi}\right) \left[\ln\left(1+y^2\right)\right]_0^2 + 2\left[\tan^{-1} y\right]_2^{2\pi} - \frac{1}{2\pi} \left[\ln\left(1+y^2\right)\right]_2^{2\pi}$$

$$= \frac{\ln 5}{2}\left(1 - \frac{1}{\pi}\right) + 2\left[\tan^{-1} 2\pi - \tan^{-1} 2\right] - \frac{1}{2\pi} \ln \frac{1+4\pi^2}{5}$$

Was this indeed easier? I'm not sure.

(4) We are to compute the following integral:

$$I = \iint_D (x-y)^2 \sin^2(x+y) d(x,y),$$

where D is the square with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.
Let $u := x + y$ and $v := x - y$, that is, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$.

Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ as $\Phi(u,\ v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$.

Then, Φ is one-one and if $\Phi = (\phi_1, \phi_2)$, then the partial derivatives of ϕ_1 and ϕ_2 exist and are continuous.

Now, we get that

$$J(\Phi)(u,v) = \det \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = -1/2 \neq 0 \quad \text{for all } (u,v) \in \mathbb{R}^2.$$

Now, we must see how our domain of integration changes under this transform.
Note that if $E := [\pi, 3\pi] \times [-\pi, \pi]$, then $\Phi(E) = D$. (How?)
Since the integrand is continuous on D, change of variable formula gives us

$$I = \iint_E v^2 \sin^2 u |-1/2| d(u,v)$$

$$\implies I = \frac{1}{2} \left(\int_{\pi}^{3\pi} \sin^2 u du \right) \left(\int_{-\pi}^{\pi} v^2 dv \right)$$

$$\implies I = \frac{1}{3}\pi^4$$

(6) (i) Let $r > 0$ be given, we first compute the following integral:

$$I(r) := \iint_{D(r)} e^{-(x^2+y^2)} d(x,y).$$

The nature of the integrand and $D(r)$ makes it very natural to consider the polar coordinates substitution.
We define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ as $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$. If $\Phi = (\phi_1, \phi_2)$, then ϕ_1 and ϕ_2 have continuous partial derivatives in $\mathbb{R}^2$ and we have that $J(\Phi)(r,\theta) = r \geq 0$ for all $(r,\theta) \in \mathbb{R}^2$.
Note that if we consider $E(r) := [0,r] \times [-\pi,\pi]$, then we get $\Phi(E(r)) = D(r)$.

Sheet 8

Hence, our integral transforms to the following:

$$
\begin{aligned}
I(r) &= \iint_{E(r)} e^{-r^2} r d(r, \theta) \\
&= \left(\int_0^r r e^{-r^2} dr \right) \left(\int_{-\pi}^{\pi} 1 d\theta \right) \\
&= \pi \left(1 - e^{-r^2} \right).
\end{aligned}
$$

Now, we can easily calculate $\lim_{r \to \infty} I(r)$.
It is simply π.

(At this point, one stops being surprised by seeing a wild appearance of π.)

(6) (ii) There are two ways to proceed with this part.

Way 1. Do the whole process like last time and change the limits of θ from 0 to $\pi/2$.

Way 2. Write $D(r)$ as the union of the four quarters of the circle.
$D(r) = D_1(r) \cup D_2(r) \cup D_3(r) \cup D_4(r)$.
Note that with this "partitioning," we get that the intersection of any two sections is a set of (two-dimensional) content zero.
Thus, the integral is simply the sum of integrals over these regions.
Appropriate change of variables will give that all the four integrals are actually the same, giving us the answer as $\pi/4$.
(This is essentially arguing via "symmetry".)

(6) (iii) Given $D(r)$ as defined for part (iii), we define two more regions:

$$D_1(r) := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\right\}, \text{ and}$$

$$D_2(r) := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2r^2\right\}.$$

Observe that $D_1(r) \subset D(r) \subset D_2(r)$.

Now, note that $f(x, y) := \exp(-(x^2 + y^2)) > 0$ for all $(x, y) \in \mathbb{R}^2$. Thus, we get that:

$$\iint_{D_1(r)} f \leq \iint_{D(r)} f \leq \iint_{D_2(r)} f.$$

(How? Note that this is *not* one of the order properties seen in class.)

Hence, we get that $\pi\left(1 - e^{-r^2}\right) \leq \iint_{D(r)} f \leq \pi\left(1 - e^{-2r^2}\right)$.

A simple use of Sandwich Theorem gives us our desired answer as π.

Sheet 8

(6) (iv) Similar argument as in part (ii). Way two is easier.

(8) First, observe that ξ_1 and ξ_2 are already given to us. Indeed, $\xi_1(x,y) = \sqrt{x^2+y^2}$ and $\xi_2(x,y) = 1$.
Let us now find bounds for the other variables in the appropriate manner.

Note that $|x| \leq \sqrt{x^2+y^2} \leq 1$.
Thus, $-1 \leq x \leq 1$. Also, one can note that x indeed does take these extreme values as the points $(1,0,1)$ and $(-1,0,1)$ belong to D.

Now, given a fixed x, we have that $y^2 \leq 1 - x^2$.
Thus, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$.
Once again, it can be confirmed that y does attain the extreme values.

To summarise, we get
$a = -1,\ b = 1,\ \phi_1(x) = -\sqrt{1-x^2},\ \phi_2(x) = \sqrt{1-x^2},\ \xi_1(x,y) = \sqrt{x^2+y^2}$ and $\xi_2(x,y) = 1$.

(9) Note that- by Fubini- the integral can be written as

$$\iiint_D x d(x, y, z),$$

where $D := \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq \sqrt{2},\ 0 \leq y \leq \sqrt{2 - x^2},\ x^2 + y^2 \leq z \leq 2\}$.

Now, observe that D can also be written as:

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 2,\ 0 \leq y \leq \sqrt{z},\ 0 \leq x \leq \sqrt{z - y^2}\right\}.$$

Using Fubini's Theorem again, we get that the integral equals:

$$\int_0^2 \left(\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} x dx\right) dy\right) dz$$

$$= \frac{1}{2}\int_0^2 \left(\int_0^{\sqrt{z}} \left(z - y^2\right) dy\right) dz$$

$$= \frac{1}{2}\int_0^2 \frac{2}{3} z^{3/2} dz$$

$$= \frac{2^{7/2}}{15}.$$

(10) (i) Given the nature of the integrand and the region of integration, it is natural to go for a cylindrical substitution.
Let $f : D \to \mathbb{R}$ be defined as $f(x, y, z) := z^2x^2 + z^2y^2$.
Now, let $E := \{(r, \theta, z) \in \mathbb{R}^3 : 0 \le r \le 1,\ -\pi \le \theta \le \pi,\ -1 \le z \le 1\}$.
Thus, $(r, \theta, z) \in E \iff (r\cos\theta, r\sin\theta, z) \in D$.
As f is continuous and E has a volume, we get that the required integral of f over D equals

$$\iiint_E f(r\cos\theta, r\sin\theta, z) r d(r, \theta, z)$$

$$= \left(\int_{-\pi}^{\pi} 1 d\theta\right)\left(\int_0^1 r^3 dr\right)\left(\int_{-1}^1 z^2 dz\right)$$

$$= \frac{\pi}{3}.$$

(10) (ii) Given the nature of the integrand and the region of integration, it is natural to go for a spherical substitution in this case.
Let $f : D \to \mathbb{R}$ be defined as $f(x, y, z) := \exp(x^2 + y^2 + z^2)^{3/2}$.
Now, let $E := \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : 0 \leq \rho \leq 1,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi\}$.
Thus, $(\rho, \varphi, \theta) \in E \iff (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \in D$.
As f is continuous and E has a volume, we get that the required integral of f over D equals

$$\iiint_E f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\rho^2 \sin\varphi d(\rho, \varphi, \theta)$$

$$= \iiint_E \exp(\rho^3)\rho^2 \sin\varphi d(\rho, \varphi, \theta) = \left(\int_0^\pi \sin\varphi d\varphi\right)\left(\int_{-\pi}^\pi 1 d\theta\right)\left(\int_0^1 \rho^2 e^{\rho^3} d\rho\right)$$

$$= \frac{4\pi}{3}(e - 1).$$