MA 105 : Calculus
D1 - T5, Tutorial 10

Aryaman Maithani

IIT Bombay

9th October, 2019

5. (i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, the given point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

Note that $(\nabla f)(x, y) = (4x^3 + 4, 4y^3 - 32)$. Hence, $(\nabla f)(x_0, y_0) = 0$.

Thus, we can appeal to the determinant test.

$(\Delta f)(x, y) = (12x^2)(12y^2) - (0)^2 = 144x^2 y^2$.

Thus, $(\Delta f)(x_0, y_0) > 0$.

Also, $f_{xx}(x_0, y_0) = 12x_0^2 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0).

5. (ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0, 0)$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, the given point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

Note that $(\nabla f)(x, y) = (3x^2 + 6x - 2y, -2x + 10y - 12y^2)$. Hence, $(\nabla f)(x_0, y_0) = 0$.

Thus, we can appeal to the determinant test.

$(\Delta f)(x, y) = (6x + 6)(10 - 24y) - (-2)^2$.

Thus, $(\Delta f)(x_0, y_0) = (6)(10) - 4 = 56 > 0$.

Also, $f_{xx}(x_0, y_0) = 6 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0).

6. (i) $f(x, y) = (x^2 - y^2) e^{-(x^2+y^2)/2}$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, every point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.   (How?)

For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = xe^{1/2(-x^2-y^2)} (-x^2 + y^2 + 2)$.

Also, $f_y(x, y) = ye^{1/2(-x^2-y^2)} (-x^2 + y^2 - 2)$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that
$(x_0, y_0) \in \{(0, 0),\ (0, \sqrt{2}),\ (0, -\sqrt{2}),\ (-\sqrt{2}, 0),\ (\sqrt{2}, 0)\}$.

Now, we determine the exact nature using the determinant test.

Sheet 7

Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.
Hence, in our case,

$$(\Delta f)(x, y) = -e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4\right).$$

Thus, for $(x_0, y_0) = (0, 0)$, it is clear that it is a saddle point for f as discriminant is $-4 < 0$.

Note that if $x = 0$, the discriminant reduces to $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$.
Substituting $y = \pm\sqrt{2}$ gives us that the discriminant is positive and hence, the points are points of local minima.

Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima.

6. (ii) $f(x, y) = f(x, y) = x^3 - 3xy^2$.

Note that the above function is defined on $D = \mathbb{R}^2$.

Thus, every point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.   (How?)

For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = 3x^2 - 3y^2$.

Also, $f_y(x, y) = -6xy$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that $(x_0, y_0) = (0, 0)$.

Now, we determine the exact nature using the determinant test.

Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.

Hence, in our case,

$$(\Delta f)(x_0, y_0) = -36(x_0^2 + y_0^2).$$

Thus, for $(x_0, y_0) = (0, 0)$, we get the discriminant is 0.

Hence, we get that the discriminant test is inconclusive!

This means that we must turn to some other analytic methods of determining the nature.

Now, we note that $f(\delta, 0) = \delta^3$ for all $\delta \in \mathbb{R}$.

Thus, given any $\epsilon > 0$, choose $\delta = \pm \epsilon/2$.

This gives us that $(0, 0)$ is saddle point. (How?)

7. To find: Absolute maxima and minima of
$f(x, y) = (x^2 - 4x) \cos y$ for $1 \le x \le 3, -\pi/4 \le y \le \pi/4$.
Note that $f_x(x, y) = (2x - 4) \cos y$ and $f_y(x, y) = -(x^2 - 4x) \sin y$ for interior points
(x, y).
Thus, the only critical point is $p_1 = (2, 0)$.

Now we restrict ourselves to the boundaries to find the local extrema.
"Right boundary:" This is the line segment $x = 3, -\pi/4 \le y \le \pi/4$.
The function now reduces to $-3 \cos y$ on this segment.
Using our theory from one-variable calculus, we get that we need to check the points
$(3, 0), (3, \pi/4), (3, -\pi/4)$.   (How?)

Similar consideration of the "left boundary" gives us the points
$(1, 0), (1, \pi/4), (1, -\pi/4)$.

Now, we look at the "top boundary."

The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.

Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4)$, $(2, \pi/4)$, $(3, \pi/4)$.

Similarly, checking the "bottom boundary" gives us the points

$(1, -\pi/4)$, $(2, -\pi/4)$, $(3, -\pi/4)$.

We now tabulate our results as follows:

(x_0, y_0)	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\dfrac{-3}{\sqrt{2}}$	$\dfrac{-4}{\sqrt{2}}$	$\dfrac{-3}{\sqrt{2}}$

(x_0, y_0)	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\dfrac{-3}{\sqrt{2}}$	$\dfrac{-4}{\sqrt{2}}$	$\dfrac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\min} = -4$ at $(2, 0)$ and $f_{max} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$.

8. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined as $g(x, y, z) := x^2 + y^2 + z^2 - 1$. We need to maximise the function $T(x, y, z) = 400xyz$ subject to the constraint $g = 0$. Note that the set $S^2 = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$ is nonempty, closed and bounded, and T is continuous on it.

Now, $(\nabla T)(x, y, z) = \lambda(\nabla g)(x, y, z)$ and $g(x, y, z) = 0$ means

$$400yz = 2\lambda x, \quad 400xz = 2\lambda y, \quad 400xy = 2\lambda z, \quad x^2 + y^2 + z^2 = 1.$$

The above gives us that $400xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$.

Also, $(\nabla g)(x, y, z) \neq (0, 0, 0)$ whenever $g(x, y, z) = 0$.
Thus, the hypotheses of the Lagrange Multiplier Theorem are satisfied.

Now, we solve the equations to get the points of maxima.

If $\lambda \neq 0$, then $x^2 = y^2 = z^2$ and hence, we get the 8 points $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.

If $\lambda = 0$, then $yz = zx = xy = 0$. This, combined with $g = 0$ gives us the 6 points $(0, 0, \pm1), (0, \pm1, 0), (\pm1, 0, 0)$.

Now, we check the value of T at these 14 points.

The first 8 points give either $T = \frac{400}{3\sqrt{3}}$ or $T = -\frac{400}{3\sqrt{3}}$. The last 6 points give $T = 0$.

Thus, the highest value of T is $\frac{400}{3\sqrt{3}}$.

Sheet 7

9. We wish to maximise $f(x, y, z) = xyz$ subject to the constraints
$g(x, y, z) = x + y + z - 40 = 0$ and $h(x, y, z) = x + y - z = 0$.
We solve $\nabla f = \lambda \nabla g + \mu \nabla h$ along with $g = h = 0$ for λ, μ, x, y, z.

We see that $\nabla f = (yz, zx, xy)$, $\nabla g = (1, 1, 1)$, $\nabla h = (1, 1, -1)$.
Thus, we get $yz = \lambda + \mu$, $zx = \lambda + \mu$, $xy = \lambda - \mu$. $\hspace{2cm}$ $(*)$

Note that $g = h = 0$ gives us that $z = 20$.
Also, the first two equalities of $(*)$ now give us that $x = y$.
Combining that with $h = 0$ gives us $x = y = 10$.

Thus, we get $(x, y, z) = (10, 10, 20)$ as the only solution along with $(\lambda, \mu) = (150, 50)$.
Now, we calculate f at the above point to get 2000 as an extremal value. To see that
it is indeed the maximum and not the minimum, we substitute some other point like
$(0, 20, 20)$ satisfying the constraints and compare.

10. We wish to maximise $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints
$g(x, y, z) = x + 2y + 3z - 6 = 0$ and $h(x, y, z) = x + 3y + 4z - 9 = 0$
We solve $\nabla f = \lambda \nabla g + \mu \nabla h$ along with $g = h = 0$ for λ, μ, x, y, z.

We see that $\nabla f = (2x, 2y, 2z)$, $\nabla g = (1, 2, 3)$, $\nabla h = (1, 3, 4)$.
Thus, we get $2x = \lambda + \mu$, $2y = 2\lambda + 3\mu$, $2z = 3\lambda + 4\mu$.   (*)

Note that $2g = 2h = 0$ along with (*) gives us that $\lambda = -10$, $\mu = 8$.
Now, the equalities of (*) give us that $x = -1$, $y = 2$, $z = 1$.

Now, we calculate f at the above point to get 6 as an extremal value. To see that it is indeed the minimum and not the maximum, we substitute some other point like $(0, 3, 0)$ satisfying the constraints and compare.