Counterexamples in Calculus

Aryaman Maithani
TA of D1-T5

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1. A bounded sequence need not be convergent.
Example: $a_n := (-1)^n$.

2. A continuous function need not have the intermediate value property.
Example: $f : (0,1) \cup (2,3) \to \mathbb{R}$ given by $f(x) := x$.

3. The inverse of a differentiable function need not be continuous.
Example: $f : [0,1] \cup (2,3] \to [0,2]$ given by

$$f(x) := \begin{cases} x & x \in [0,1] \\ x-1 & x \in (2,3] \end{cases}$$

Corollaries: The inverse of a continuous function need not be continuous. The inverse of a differentiable function need not be differentiable.

4. A function defined on an interval with the intermediate value property need not be continuous *anywhere.*
Example: Conway Base 13 function.

5. A Riemann integrable function may have infinitely many discontinuities.
Example: Thomae's functions.

6. A differentiable function with derivative zero everywhere need not be constant.
Example: $f : (0,1) \cup (2,3) \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} 1 & x \in (0,1) \\ 2 & x \in (2,3) \end{cases}$$

7. A differentiable function with strictly negative derivative everywhere need not be monotonically decreasing.
Example: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined as $f(x) := x^{-1}$.

8. Let $f : [0,1] \to \mathbb{R}$ be a bounded function. f need not be integrable on $[0,1]$.
Example: Take f to be the Dirichlet function defined as

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

9. Integrability of $|f|$ does not imply integrability of f.
Example: $f : [-1,1] \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $f_{xy}(0,0)$ and $f_{yx}(0,0)$ exist. It is not necessary that they are equal.
Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) := \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

11. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all directional derivatives of f exist at $(0,0)$. It is not necessary that the function is continuous at $(0,0)$.
Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) := \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

The above function also satisfies $(\mathbf{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \mathbf{u}$ for every unit vector $\mathbf{u} \in \mathbb{R}^2$.

12. Let (x_0, y_0) be an interior point of a subset D of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$. Suppose the following conditions hold:
    (a) Both partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.
    (b) The directional derivative $(\mathbf{D_u} f)(x_0, y_0)$ exists for every unit vector $\mathbf{u} \in \mathbb{R}^2$.
    (c) $(\mathbf{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \mathbf{u}$ for every unit vector $\mathbf{u} \in \mathbb{R}^2$.
    (d) f is continuous at (x_0, y_0).

    It is not necessary that f is differen tiable at (x_0, y_0).
Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) := \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

13. Let $D \subset \mathbb{R}^2$. It is not necessary that ∂D is of content zero or that it is "one dimensional."
Example: Let $D = (\mathbb{Q} \cap [0,1])^2$. Then, $\partial D = [0,1]^2$.

14. Let $R = [a,b] \times [c,d]$ be a closed and bounded rectangle in $\mathbb{R}^2$. Suppose that $f : R \to \mathbb{R}$ is a function such that the iterated integrals $\displaystyle\int_a^b \left(\int_c^d f(x,y)dy \right) dx$ and $\displaystyle\int_c^d \left(\int_a^b f(x,y)dx \right) dy$ both exist.

    It is not necessary that they are equal.
Example: Let $a = c = -1$ and $b = d = 1$ and let $f : R \to \mathbb{R}$ be defined as

$$f(x,y) := \begin{cases} \dfrac{2xy(x^2 - y^2)}{(x^2 + y^2)^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

15. Let $R = [a,b] \times [c,d]$ be a closed and bounded rectangle in $\mathbb{R}^2$. Suppose that $f : R \to \mathbb{R}$ is a function such that the iterated integrals $\displaystyle\int_a^b \left(\int_c^d f(x,y)dy \right) dx$ and $\displaystyle\int_c^d \left(\int_a^b f(x,y)dx \right) dy$ both exist and are equal.

    It is not necessary that f is Riemann integrable on R.
Example: Let $a = c = -1$ and $b = d = 1$ and let $f : R \to \mathbb{R}$ be defined as

$$f(x,y) := \begin{cases} \dfrac{2xy}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

16. Let $R = [a,b] \times [c,d]$ be a closed and bounded rectangle in $\mathbb{R}^2$. Suppose that $f : R \to \mathbb{R}$ is a function such that f is Riemann integrable on R. It is not necessary that both the iterated integrals exist.
Example: Let $a = c = 0$ and $b = d = 1$. Define $f : R \to \mathbb{R}$ as

$$f(x,y) := \begin{cases} n^{-1} & \text{if } (x,y) \in \mathbb{Q}^2 \text{ and } x = m/n \text{ such that } (m,n) \text{ in simplest form} \\ 0 & (x,y) \notin \mathbb{Q}^2 \end{cases}$$

    (Note that $0/1$ is the simplest form for 0.)

    Then, f is integrable on R. (May be tough to show but it's true.)
    However, if you fix $x_0 \in \mathbb{Q} \cap [0,1]$, then $f(x_0, y)$ is not Riemann integrable (as a function of y) on $[0,1]$. It can be seen easily that it behaves like the Dirichlet function mentioned in point 8.
    Thus, the iterated integral $\displaystyle\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx$ does not exist.
    Note the other iterated integral, however, does exist.

17. Let $R = [a,b] \times [c,d]$ be a closed and bounded rectangle in $\mathbb{R}^2$. Suppose $f : R \to \mathbb{R}$ is a function such that one of the iterated integrals exists. It is not necessary that the other does too.
Example: Example 16.