

# MA 105 : Calculus D1 - T5, Tutorial 01

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This is an extremely interesting course. You will be introduced to mathematical rigour.

Do not fear this, try to appreciate the elegance behind it.

The learning curve will be quite steep compared to any other course but do not fret; with sincere and regular efforts from your side, you should be able to understand the course quite well.



# About the course policy

Here is the course policy relevant to the tutorials -

There will be a quiz in almost all tutorials.

There will be total 12 quizzes in all.

The best 10 out of this 12 will be counted.

Due to this reason, there will be no re-quiz under any circumstances.

The quiz will begin sharp at 2:00 PM and end at 2:05 PM. For those who come later than 2:00 PM but before 2:05 PM, they can still take the quiz but will have limited time.

This quiz will also serve as your attendance. If you have taken the quiz, you are not allowed to leave the tutorial until it has ended.

# Expectations

What we expect from you, before you come to the tutorial is the following:

- ① You have read the lecture slides that have been uploaded up to that tutorial.
- ② You have *attempted* the questions that are to be discussed in the tutorials.

# Some elementary concepts

## Definition (Interval)

*An interval  $I$  is any subset of  $\mathbb{R}$  with the following property:*

$$x, y \in I, x < y \implies [x, y] \subset I.$$

## Definition (Arbitrary intersection)

*Given a collection of sets  $\{A_i\}_{i \in I}$  where  $I$  is any arbitrary nonempty set, we define the following intersection:*

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \quad \forall i \in I\}.$$

*What this means is that  $x$  will belong to the intersection if and only if it belongs to each set  $A_i$ .*

- ①  $+\infty$  and  $-\infty$  are real numbers.
- ② The set of all even natural numbers is bounded.
- ③ The set  $\{x\}$  is an open interval for every  $x \in \mathbb{R}$ .
- ④ The set  $\{2/m : m \in \mathbb{N}\}$  is bounded above.

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Note: any number greater than 2 is also an upper bound.

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Thus,  $y \in N$ . ■

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Note: In the previous proof, we had talked about arbitrary intersections. That is, the argument works for infinite intersections as well.

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To show the above equality, one must show that each side is a subset of the other.



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The reverse containment is left as an exercise.

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Note that  $\{a_i\}_{i \in I}$  is a nonempty subset of  $\mathbb{R}$  that is bounded above. (By what?) Similarly,  $\{b_i\}_{i \in I}$  is a nonempty subset of  $\mathbb{R}$  that is bounded below.

Thus, we can define  $a = \sup\{a_i | i \in I\}$  and  $b = \sup\{b_i | i \in I\}$ . Like before, we must have  $a \leq b$ .

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Let  $p, q \in \mathbb{Q}$  such that  $p < q$ .

Define  $r := a + \frac{b-a}{\sqrt{2}}$ . Show that  $r \in \mathbb{R} \setminus \mathbb{Q}$  and that  $p < r < q$ .

# Recap - Convergence of a sequence

## Definition (Convergence of a sequence)

*Let  $(a_n)$  be a sequence of real numbers. We say that  $(a_n)$  is convergent if there is a  $a \in \mathbb{R}$  such that the following condition holds.*

*For every  $\epsilon > 0$ , there is  $n_0 \in \mathbb{N}$  such that  $|a_n - a| < \epsilon$  for all  $n \geq n_0$ .*

In this case, we say that  $(a_n)$  **converges** to  $a$ , or that  $a$  is a limit of  $(a_n)$ , and we write

$$\lim_{n \rightarrow \infty} a_n = a \text{ or } a_n \longrightarrow a \text{ (as } n \longrightarrow \infty \text{)}.$$

If a sequence doesn't converge, we say that the sequence **diverges** or that it is **divergent**.

# Sheet 1

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Let  $n_0 = \left\lfloor \frac{10}{\epsilon} \right\rfloor + 1$ . It is clear that  $n_0 > \frac{10}{\epsilon}.$

Moreover, for any  $n \geq n_0$ , we will have  $n > \frac{10}{\epsilon}.$

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Let  $n_0 = \left\lfloor \frac{10}{\epsilon} \right\rfloor + 1$ . It is clear that  $n_0 > \frac{10}{\epsilon}.$

Moreover, for any  $n \geq n_0$ , we will have  $n > \frac{10}{\epsilon}.$

Thus, we have shown that for every  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $\left| \frac{10}{n} \right| < \epsilon$  for

all  $n \geq n_0$ .  $\therefore \lim_{n \rightarrow \infty} \frac{10}{n} = 0.$

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One such choice is  $n_0 = \max \left\{ 1, \left\lfloor \frac{1}{3} \left( \frac{5}{\epsilon} - 1 \right) \right\rfloor \right\} + 1$ .

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One such choice is  $n_0 = \max \left\{ 1, \left\lfloor \frac{1}{3} \left( \frac{5}{\epsilon} - 1 \right) \right\rfloor \right\} + 1$ .

Note: The choice of  $n_0$  is not unique. Our choice of  $n_0$  might not be the smallest but that is okay.

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Note the direction of implication of the red arrow. We have used the fact that  $|\sin x| < 1$  for all real  $x$ .

$$\left| \frac{n^{2/3}}{n+1} \right| < \epsilon \iff \left| \frac{n^{2/3}}{n} \right| < \epsilon$$



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By our arrows of implication, it can be seen that for  $n \geq n_0$ , the desired inequality holds.

1. (iv)  $\lim_{n \rightarrow \infty} \left( \frac{n}{n+1} - \frac{n+1}{n} \right) = 0$

Let  $\epsilon > 0$  be given. We must show that there exists  $n_0 \in \mathbb{N}$  such that

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Observe the following:

$$\left| \frac{n}{n+1} - \frac{n+1}{n} \right| = \left| 1 - \frac{1}{n+1} - 1 - \frac{1}{n} \right| = \left| -\frac{1}{n+1} - \frac{1}{n} \right|$$

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Thus, the sequence given is bounded below by  $n-1$ , but by Archimedean property, we know that  $n-1$  is not bounded above. Thus, our sequence is not bounded (above). As a result, it is not convergent. ■

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Now, for the sake of contradiction, let us assume that  $(a_n)$  converges. Then, by (a), we have it that  $c_n := a_n + b_n = \frac{(-1)^n}{2}$  must be convergent.

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We will use the following two results: (a) Sum of convergent sequences is convergent.

(b) The sequence  $\{(-1)^n\}_{n \geq 1}$  is not convergent.

We now proceed as follows:

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However, by (b), we know that the above is false. Thus, we have arrived at a contradiction.

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Hint for **optional**: Use the inequality  $\left| \sqrt[n]{a} - \sqrt[n]{b} \right| \leq \sqrt[n]{|a - b|}$  for  $n \in \mathbb{N}$ .

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Both are **false**.

The sequences,  $a_n := 1 \quad \forall n \in \mathbb{N}$  and  $b_n := (-1)^n \quad \forall n \in \mathbb{N}$  act as a counterexample for both the statements.