# MA 105 : Calculus
# D1 - T5, Tutorial 13

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(2)

$$\oint_{\partial R} y^2 dx + x dy.$$

In each part, we shall keep our boundary *positively oriented.*
For this question, this would mean that we are always traversing the path in an anti-clockwise direction.
Observe that the vector field $(y^2, x)$ defined on $\mathbb{R}^2$ is smooth.
Each given region $R$ is indeed a subset of $\mathbb{R}^2$.
Also, in each case, $\partial R$ consists of a single piecewise smooth curve.

Thus, by Green's Theorem, we get that:

$$\oint_{\partial R} y^2 dx + x dy = \iint_R \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial y} y^2 \right) d(x, y)$$

$$\oint_{\partial R} y^2 dx + x dy = \iint_R (1 - 2y)\, d(x, y)$$

# Sheet 11

(i) Our region here is simply $R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2,\ 0 \leq y \leq 2\}$.
Our friend Fubini can easily solve this for us now. The desired integral is simply-

$$\int_0^2 \int_0^2 (1 - 2y)dydx$$

$$= \int_0^2 (2 - 4)dx$$

$$= -4$$

(ii) Once again, this is quite straightforward by Fubini. The integral is simply:

$$\int_{-1}^{1}\int_{-1}^{1}(1-2y)dydx = \int_{-1}^{1}(2)dx = 4$$

(iii) Our region now is $R = \{(x,y) \in \mathbb{R}^2 : -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\}$. This being an elementary allows our old friend Fubini to help us again. The given integral can simply be written as:

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(1-2y)dydx &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(1)dydx \\ &= 4\pi\end{aligned}$$

(3) Let the equation of the curve $C$ be given by $r = \rho(t)$ and $\theta = \theta(t)$ for $t \in [a, b]$.
Let $D$ be the closed region enclosed by this curve.
Then, Area($D$) is $\iint_D 1_D d(x, y)$, by definition.
By Green's theorem, we know that $\iint_D 1_D d(x, y) = \frac{1}{2}\oint_{\partial D} xdy - ydx$. In our case, we have $\partial D = C$.
Note: We are assuming that $C$ is positively oriented.

Consider the parameterisation of $C$ given by $(x(t),\ y(t))$, where $x(t) := \rho(t)\cos(\theta(t))$ and $y(t) := \rho(t)\sin(\theta(t))$ for $t \in [a, b]$.

Note that we have $x'(t) = \rho'(t)\cos(\theta(t)) - \rho(t)\sin(\theta(t))\theta'(t)$ and
$y'(t) = \rho'(t)\sin(\theta(t)) + \rho(t)\cos(\theta(t))\theta'(t)$.
Thus, $x(t)y'(t) - y(t)x'(t) = (\rho(t))^2\theta'(t)$.

Hence, the area is given by $\dfrac{1}{2}\displaystyle\oint_{\partial D}(x(\theta)y'(\theta) - y(\theta)x'(\theta))d\theta = \dfrac{1}{2}\int_a^b (\rho(t))^2\theta'(t)dt$.

$= \frac{1}{2}\displaystyle\oint_C r^2 d\theta.$

Was $C$ indeed positively oriented?

# Sheet 11

Using the formula derived makes the questions very simple now as we are given everything explicitly.

(i) The area is given by:

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta$$

$$= \frac{3}{2}\pi a^2$$

(ii) The area is given by:

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2\cos 2\theta d\theta$$

$$= \frac{1}{2}a^2$$

(4) (i) We shall use the derived formula to write the area as:

$$\frac{1}{2}\int_0^{\pi/2} a^2(1-\cos\theta)^2 d\theta = a^2\left(\frac{3\pi}{8}-1\right).$$

(iii) Similarly as before, we get the area to simply be:

$$\frac{1}{2}\int_0^{\pi/2} (1-2\cos\theta)^2 d\theta = \frac{3\pi}{4}-2.$$

Note that in both the above cases, the curve was actually the union of the polar curve given and some part of the axes. However, $\theta$ is constant along the axes and hence, we get the correct answer.

(ii) The required area is

$$\frac{1}{2}\oint_C xdy - ydx$$

Where $C$ is the line segment $[0, 2\pi] \times \{0\}$ traversed from left to right along with the cycloid traversed in the "opposite direction."
However, the integrand is zero on the $x$-axis as $y = 0$ and is constant.
Thus, the required area is:

$$-\frac{a^2}{2}\int_0^{2\pi}[(t - \sin t)(\sin t) - (1 - \cos t)(1 - \cos t)]dt \ = 3\pi a^2.$$

(The answer given at the back is wrong.)

# Sheet 11

(5)

$$\oint_C xe^{-y^2} dx + \left[ -x^2 ye^{-y^2} + \frac{1}{x^2 + y^2} \right] dy.$$

Observe that the above can be re-written as

$$\left( \oint_C xe^{-y^2} dx + -x^2 ye^{-y^2} dy \right) + \oint_C \frac{1}{x^2 + y^2} dy.$$

Note that the first integral can be written as:

$$\frac{1}{2} \oint_C \nabla(x^2 e^{-y^2}) \cdot d\mathbf{r}$$

Now, we turn to the second integral. We need to integrate $\frac{1}{x^2+y^2}$ around the square with boundaries given by $|x| = a$ and $|y| = a$ in the counter-clockwise direction.
It is clear that the integral along the "top" and "bottom" curves will be zero as $y$ is constant there.
Moreover, the integral along the "right" curve will be equal in magnitude but opposite in sign to the integral along the "left" curve.
Thus, the second integral along $C$ is zero and so is the first as it's the integral of a gradient field along a closed loop.
Using the famous identity $0 + 0 = 0$ gives us the answer to be 0.

(10) (i)

$$\oint_C \frac{xdy - ydx}{x^2 + y^2}.$$

Consider the open set $S := \mathbb{R}^2 \setminus \{0, 0\}$.
Define the scalar fields $Q(x, y) := \frac{x}{x^2+y^2}$ and $P(x, y) := -\frac{y}{x^2+y^2}$ for $(x, y) \in S$.
It can be seen that they are smooth and moreover, $P_y = Q_x$ on $S$.
Now, let us consider the following two scenarios:
(a) $C$ does not enclose the origin. (b) $C$ encloses the origin.
Let $D$ denote the closed and bounded subset of $\mathbb{R}^2$ enclosed by $C$.

(a) $C$ does not enclose the origin. In this case, $(P, Q)$ is defined everywhere on $D$ and hence, we simply get that the integral is 0.

(b) $C$ encloses the origin.
As $C$ does not pass through the origin, there exists $\epsilon > 0$ such that the disc of radius $\epsilon$ centered at $(0, 0)$ lies completely inside $D$. Let this disc be $D_\epsilon$. (How?)
Now, the field $(P, Q)$ is defined everywhere on the region outside $D_\epsilon$ and within $D$.

# Sheet 11

Thus, using the so-called deformation principle gives us that

$$\oint_C Pdx + Qdy = \oint_{\partial D_\epsilon} Pdx + Qdy,$$

where the boundary $\partial D_\epsilon$ is oriented positively.
The integral on the left can be easily calculated using the parameterisation $(\epsilon \cos\theta, \epsilon \sin\theta)$ for $\theta \in [-\pi, \pi]$.
The answer comes out to be $-2\pi$.
This completes the question.

(10) (ii) Once again, note that $\dfrac{\partial}{\partial y}\dfrac{x^2 y}{(x^2+y^2)^2} = \dfrac{x^2(x^2-3y^2)}{(x^2+y^2)^3} = \dfrac{\partial}{\partial x}\left(-\dfrac{x^3}{(x^2+y^2)^2}\right)$.

Thus, the same argument as before shows that we can integrate the integrand over any other curve containing the origin that lies inside the curve given, along the same direction.

We can simply choose the curve to be the unit circle centered at origin.

Hence, we need to compute the following:

$$\int_0^{2\pi} [(\cos\theta)^2(\sin\theta)(-\sin\theta) - (\cos\theta)^3(\cos\theta)]d\theta = -\pi.$$

(The answer given at the back is wrong.)