MA 105 : Calculus
D1 - T5, Tutorial 06

Aryaman Maithani

IIT Bombay

4th September, 2019

Summary

Sheet 4: Problems 7, 8, 9, 10

7. (i) Note that

$$S_n = \frac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2} \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \frac{2}{5}x^{5/2}$. Then, we have that $f'(x) = x^{3/2}$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = S(P_n, f')$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, f') \to \int_0^1 x^{3/2} dx = \int_0^1 f'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x) dx = f(1) - f(0) = \frac{2}{5}.$$

7. (ii) Note that

$$S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2} = \sum_{i=1}^{n} \frac{1}{\left(\frac{i}{n}\right)^2 + 1} \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \tan^{-1} x$. Then, we have that $f'(x) = \frac{1}{x^2+1}$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = S(P_n, f')$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, f') \to \int_0^1 \frac{1}{x^2+1} dx = \int_0^1 f'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n \to \infty} S_n = \int_0^1 f'(x) dx = f(1) - f(0) = \frac{\pi}{4}.$$

7. (iii) Note that

$$S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}} = \sum_{i=1}^{n} \frac{1}{\sqrt{\left(\frac{i}{n}\right) + 1}} \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := 2\sqrt{x+1}$. Then, we have that $f'(x) = \frac{1}{\sqrt{x+1}}$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = S(P_n, f')$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, f') \to \int_0^1 \frac{1}{\sqrt{x+1}} dx = \int_0^1 f'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x) dx = f(1) - f(0) = 2\sqrt{2} - 2.$$

7. (iv) Note that

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right) = \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \frac{1}{\pi}\sin(\pi x)$. Then, we have that $f'(x) = \cos(\pi x)$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = S(P_n, f')$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, f') \to \int_0^1 \cos(\pi x)dx = \int_0^1 f'(x)dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)dx = f(1) - f(0) = 0.$$

7. (v) Note that

$$S_n = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}.$$

We shall find $\lim_{n\to\infty} S_n$ by finding the limits of the individual sums and showing that they all exist.

Sheet 4

Define $A_n := \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right)\right\} = \sum_{i=1}^{n}\left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right)$.
Define $a : [0,1] \to \mathbb{R}$ by $a(x) := \frac{x^2}{2}$. Then, we have that $a'(x) = x$.
As a' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $p_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $A_n = S(P_n, a')$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, a') \to \int_0^1 x dx = \int_0^1 a'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} A_n = \int_0^1 a'(x) dx = a(1) - a(0) = \frac{1}{2}.$$

Sheet 4

Define $B_n := \dfrac{1}{n}\left\{\sum_{i=n+1}^{2n}\left(\dfrac{i}{n}\right)^{3/2}\right\} = \sum_{i=n+1}^{2n}\left(\dfrac{i}{n}\right)^{3/2}\left(\dfrac{i}{n} - \dfrac{i-1}{n}\right).$
Define $b : [1,2] \to \mathbb{R}$ by $b(x) := \frac{2}{5}x^{5/2}$. Then, we have that $b'(x) = x^{3/2}$.
As b' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $R_n := \{1, 1 + 1/n, \ldots, 1 + n/n\}$ and $r_i := (n+i)/n$ for $i = 1, 2, \ldots, n$.
Then, $B_n = S(R_n, b')$. Since $\mu(R_n) = 1/n \to 0$, it follows that

$$S(R_n, b') \to \int_1^2 x^{3/2} dx = \int_0^1 b'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} B_n = \int_1^2 b'(x) dx = b(2) - b(1) = \frac{2}{5}(4\sqrt{2} - 1).$$

Sheet 4

Define $C_n := \frac{1}{n}\left\{\sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^2\right\} = \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^2\left(\frac{i}{n} - \frac{i-1}{n}\right).$
Define $c : [2,3] \to \mathbb{R}$ by $c(x) := \frac{x^3}{3}$. Then, we have that $c'(x) = x^2$.
As c' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $T_n := \{2, 2+1/n, \ldots, 2+n/n\}$ and $t_i := (2n+i)/n$ for $i = 1, 2, \ldots, n$.
Then, $C_n = S(T_n, c')$. Since $\mu(T_n) = 1/n \to 0$, it follows that

$$S(T_n, c') \to \int_2^3 x^2 dx = \int_2^3 c'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} C_n = \int_2^3 c'(x) dx = c(3) - c(2) = \frac{19}{3}.$$

It is easy to observe that $S_n = A_n + B_n + C_n$ for all $n \in \mathbb{N}$.
As all the limits individually exist, we can write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n + \lim_{n\to\infty} C_n = \frac{1}{2} + \frac{2}{5}(4\sqrt{2} - 1) + \frac{19}{3}.$$

8. (a) We are given that

$$x = \int_0^y \frac{1}{\sqrt{1+t^2}}\,dt$$

As the integrand is continuous, we have it x is a differentiable function of y. Using Fundamental Theorem of Calculus (Part 1), we can write that

$$\frac{dx}{dy} = \frac{1}{\sqrt{1+y^2}}.$$

As $\dfrac{dx}{dy}$ is positive, we get that x is a strictly increasing function of y. In particular, it is one-one. It is also continuous and its derivative is never zero. Thus, by the inverse function theorem, we get that

$$\frac{dy}{dx} = \sqrt{1+y^2}.$$

Now, we can calculate the double derivative as follows,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sqrt{1+y^2} = \frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} = y.$$

8. (b) Let u and v be differentiable functions defined on appropriate domains. Let g be a continuous function. Define $G(x) := \int_a^x g(t)dt$. Then $G'(x) = g(x)$, by Fundamental Theorem of Calculus (Part 1). Note that

$$\int_{u(x)}^{v(x)} g(t)dt = \int_a^{v(x)} g(t)dt - \int_a^{u(x)} f(t)dt = G(v(x)) - G(u(x)).$$

Thus, by the Chain Rule, one has

$$\frac{d}{dx}\int_{u(x)}^{v(x)} g(t)dt = G'(v(x))v'(x) - G'(u(x))u'(x) = g(v(x))v'(x) - g(u(x))u'(x).$$

We can now easily solve the question.

(i)

Given, $F(x) = \int_1^{2x} \cos(t^2)dt$

$$\begin{aligned}\therefore \frac{dF}{dx} &= \cos\left((2x)^2\right)(2x)' - \cos(1)(1)' \\ &= 2\cos(4x^2).\end{aligned}$$

(ii)

Given, $F(x) = \int_0^{x^2} \cos(t)dt$

$$\therefore \frac{dF}{dx} = \cos\left(x^2\right)\left(x^2\right)' - \cos(0)(0)'$$
$$= 2x\cos(x^2).$$

9. Define $F : \mathbb{R} \to \mathbb{R}$ as

$$F(a) := \int_a^{a+p} f(t)dt.$$

If we show that F is constant, then we are done.
As f is a continuous, Fundamental Theorem of Calculus (Part 1) tells us that F is differentiable everywhere. Using the result we had shown earlier, we have it that $F'(a) = f(a+p) \cdot 1 - f(a) \cdot 1 = 0$.
As F is defined on an interval ($\mathbb{R}$), we have it that F is constant. ■

10.

$$\begin{aligned}
g(x) &= \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt \\
&= \frac{1}{\lambda}\int_0^x f(t)\left(\sin\lambda x\cos\lambda t - \cos\lambda x\sin\lambda t\right)dt \\
&= \frac{1}{\lambda}\sin\lambda x\int_0^x f(t)\cos\lambda t dt - \frac{1}{\lambda}\cos\lambda x\int_0^x f(t)\sin\lambda t dt
\end{aligned}$$

Now, we can differentiate g using product rule and Fundamental Theorem of Calculus (Part 1).

$$\therefore g'(x) = \cos\lambda x\int_0^x f(t)\cos\lambda t dt + \sin\lambda x\int_0^x f(t)\sin\lambda t dt$$

It is easy to verify that both $g(0)$ and $g'(0)$ are 0.
We can differentiate g' in a similar way and get,

$$
\begin{aligned}
g''(x) &= -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t dt + f(x) \cos^2 \lambda x + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t dt \\
&\quad + f(x) \sin^2 \lambda x \\
&= f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t\right) dt \right) \\
&= f(x) - \lambda^2 g(x)
\end{aligned}
$$

$$
\implies g''(x) + \lambda^2 g(x) = f(x) \qquad \blacksquare
$$