MA 105 : Calculus
D1 - T5, Tutorial 14

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Summary

Sheet 12: Problems 3, 6, 7, 8
Sheet 13: Problems 2, 3, 5, 6

(3) Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$. There are two pieces of the surface - one below and one above the xy-plane, both having the same surface area. Let S be the upper piece. Then one has

$$\text{Area}(S) = \iint_T \sqrt{1 + z_x^2 + z_y^2}\, d(x, y),$$

where T is the disc

$$\left\{(x, y) \in \mathbb{R}^2 : x^2 + \left(y - \frac{a}{2}\right)^2 \leq \left(\frac{a}{2}\right)^2\right\},$$

and $z(x, y) = \sqrt{a^2 - x^2 - y^2}$ for $(x, y) \in T$.

Now, we calculate z_x and z_y.

$$z_x = -\frac{x}{z} \text{ and } z_y = -\frac{y}{z}.$$

Thus, we get the area integral as

$$\text{Area}(S) = \iint_T \frac{a}{z} d(x, y) = \iint_T \frac{a}{\sqrt{a^2 - x^2 - y^2}} d(x, y)$$

Now, T is described in polar coordinates by

$$x = r\cos\theta, y = r\sin\theta; 0 \leq \theta \leq \pi, 0 \leq r \leq a\sin\theta.$$

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Therefore,

$$\text{Area}(S) = \int_0^{\pi} \left(\int_0^{a\sin\theta} \frac{ar}{\sqrt{a^2 - r^2}} dr \right) d\theta$$

$$= a \int_0^{\pi} [-\sqrt{a^2 - r^2}] \Big|_0^{a\sin\theta} d\theta$$

$$= a \int_0^{\pi} (-a|\cos\theta| + a) d\theta = (\pi - 2)a^2.$$

Thus, the required area is $2(\pi - 2)a^2$.

(6) We shall consider the cylinder (and thus, the sphere) to have radius 1. Moreover, we shall choose our axes such that the center of the sphere is the origin and the axis of the cylinder is the $z-$axis.
Let us find the area of the sphere between the planes $z = 0$ and $z = c$ for some $c \in (0, 1]$. Using this, we can find the area between two planes according to whether or not they are on the same side of $z = 0$ or not.
The surface of interest is parameterised as:

$$\Phi(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \text{ where } -\pi \leq \theta \leq \pi,\ \alpha \leq \varphi \leq \pi/2,$$

where α is the (unique) real number in $[0, \pi/2]$ such that $\cos\alpha = c$.

Now, we have $\Phi_\varphi \times \Phi_\theta = \left(\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, sin\varphi\cos\varphi\right)$.
This gives us $\|\Phi_\varphi \times \Phi_\theta\| = \sin\varphi$.
Thus, the area is given by

$$\int_{-\pi}^{\pi}\int_{\alpha}^{\pi/2} 1\sin\varphi \mathrm{d}\varphi \mathrm{d}\theta$$

$$= 2\pi\cos\alpha = 2\pi c.$$

Moreover, it can be easily seen the surface area of the cylinder between these two planes is also given by $2\pi c$. Thus, we can now conclude the final result by taking two cases.

(7) (i) Let T be the region in the $uv-$plane parameterising the region S as given. Note that $\mathbf{r}_u \times \mathbf{r}_v = -2(\mathbf{i}+\mathbf{j}+\mathbf{k})$ has negative $z-$component. Thus, we get the following, in differential notation:

$$\hat{\mathbf{n}} dS = \hat{\boldsymbol{n}} \left\| \mathbf{r}_u \times \mathbf{r}_v \right\| d(u,v) = 2(\mathbf{i}+\mathbf{j}+\mathbf{k}) d(u,v).$$

Thus, the integral is simply $\displaystyle\iint_T 2dS = 2\,\text{Area}(T)$.

Note that T is the triangle in the $uv-$plane with vertices $(0,0)$ and $\left(\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus, the answer is simply $\frac{1}{2}$.

(ii) The surface satisfies $z = 1 - x - y \geq 0,\ x \geq 0,\ y \geq 0.$

Define $T := \{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\}$. Thus, we then have that S is given by $z = f(x, y) := 1 - x - y$ for $(x, y) \in T$.

Thus, $\hat{\mathbf{n}}dS = (-z_x, -z_y, 1)d(x, y)$, in differential notation.
Moreover, $\mathbf{F} \cdot \mathbf{n}dS = (x, y, z) \cdot (-z_x, -z_y, 1)\, d(x, y) = (x + y + z)d(x, y) = 1d(x, y)$.
Now, one has $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}dS \ = \iint_T 1d(x, y) = \text{Area}(T) = \tfrac{1}{2}$.

(8) Routine calculation is to be done.
Parameterise the sphere as $\Phi(\varphi, \theta) := (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi)$ for $(\varphi, \theta) \in [0, \pi] \times [-\pi, \pi]$.

Then, $\hat{\boldsymbol{n}} dS = (\Phi_\varphi \times \Phi_\theta)\, d(\varphi, \theta) = (a \sin\varphi)\Phi(\varphi, \theta) d(\varphi, \theta)$.

(Note that this is indeed the outwards normal.)
The integrand is now $\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_\varphi) = a^4 \sin^3\varphi \cos\varphi \left(1 + \cos^2\theta\right)$.

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(Check! I might have made a sign mistake.)
Thus, the required integral is

$$\int_0^{2\pi}\left(\int_0^{\pi} a^4 \sin^3\varphi\cos\varphi\left(1+\cos^2\theta\right)d\theta\right)d\varphi$$

$$= a^4\left(\int_0^{\pi}\sin^3\varphi\cos\varphi d\varphi\right)\left(\int_0^{2\pi}\left(1+\cos^2\theta\right)d\theta\right) = 0.$$

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(2) For $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$,

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = \mathbf{0}.$$

Let S be any *good enough* (geometric) surface in $\mathbb{R}^3$ such that $\eth S = C$. Moreover, assume that S is oriented such that orientation it induces on C is the desired orientation. Then, we have that the line integral is given by

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS = 0.$$

(3) By Stokes' theorem, we have

$$\iint_S \mathrm{curl}(\mathbf{v}) \cdot \mathbf{n} dS = \oint_{C_1} \mathbf{v} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{v} \cdot d\mathbf{s}.$$

where C_1 is the circle $x^2 + y^2 = 4, z = -3$ with the counterclockwise orientation when viewed from "high above," and C_2 is the circle $x^2 + y^2 = 4, z = 0$, with the opposite orientation.

(How did we decide the orientation?)

Now, we can write $\oint_{C_i} \mathbf{v} \cdot d\mathbf{s}$ as $\oint_{C_i} ydx + xz^3dy - zy^3dz$.

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For $i = 1$, we have:

$$\oint_{C_1} ydx + xz^3dy - zy^3dz = \oint_{C_1} ydx - 27xdy = \oint_{C_1} \nabla(xy) \cdot d\mathbf{s} - \oint_{C_1} 28y^2dy.$$

The latter integral can be easily evaluated by a suitable parameterisation of C_1 to give us $-28\displaystyle\int_{-\pi}^{\pi} 4\cos^2\theta d\theta = -112\pi$.

Similarly, for $i = 2$, we have

$$\oint_{C_2} ydx = -\int_{\pi}^{\pi}(-4\sin^2\theta)d\theta = 4\pi.$$

Hence, the required integral is -108π.

(5) Note the following:

$$\text{Let } \mathbf{F} = \left(y^2 - z^2\right)\mathbf{i} + \left(z^2 - x^2\right)\mathbf{j} + \left(x^2 - y^2\right)\mathbf{k},$$

$$\text{then } \nabla \times \mathbf{F} = (-2y - 2z)\mathbf{i} + (-2z - 2x)\mathbf{j} + (-2x - 2y)\mathbf{k},$$

$$\text{and } \hat{\mathbf{n}}_S = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}.$$

Now, along the surface S, which is a part of the plane $x + y + z = \frac{3a}{2}$ and is bounded by C, we have

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = -\frac{4}{\sqrt{3}}\frac{3a}{2}.$$

Hence,
$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS = -2\sqrt{3}a \iint_S dS = (-2\sqrt{3}a)(\text{Area of } S).$$
The surface S is a regular hexagon with vertices
$(a/2, 0, a), (a, 0, a/2), (a, a/2, 0), (a/2, a, 0), (0, a, a/2), (0, a/2, a)$.
Hence, its area is $\frac{3\sqrt{3}}{4}a^2$.
Using Stokes' theorem, we get that the above integral is equal to our desired integral which comes out to be $-\frac{9a^3}{2}$, assuming that the curve was oriented in the orientation induced by the surface.
(Which has to be mentioned. I leave that to you.)

(Note that this isn't the way to write this answer as I've actually mentioned $\hat{\mathbf{n}}$ even before defining S. Hopefully, you will be able to write it properly.)

(6) Consider the following:

$$\mathbf{F} = (y, z, x),$$
$$\nabla \times \mathbf{F} = -(1, 1, 1).$$

We are to compute $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

Let S be the surface lying on the hyperboloid bounded by C. We shall now describe S.

Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$.

Let $f(x, y) := \dfrac{xy}{b}$ for $(x, y) \in D$.

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Then, the surface S is given by $z = f(x, y)$. Following differential notation, we get $\hat{\mathbf{n}}dS = (-z_x, -z_y, 1)d(x, y) = (-\frac{y}{b}, -\frac{x}{b}, 1)d(x, y)$.

By Stokes' Theorem, the required integral is simply equal to $\dfrac{1}{b}\displaystyle\iint_D (y + x - b)d(x, y)$.

This can be easily solved using polar coordinates.

Farewell

So, with that, we end this tutorial.
There are a few concluding remarks that I would like to make.

Farewell

Things I want you to take away from this course:

1. Appreciation of how theory is built.
2. Elegance of the way one can capture the notions of things being "arbitrarily small" or "big" in a formal manner.

I would mention more specific things but I think that they'll just fall in the above two categories anyway.

Also, as promised earlier, here's the link to the memes -

https://github.com/aryamanmaithani/ma-105-tut/blob/master/Memes/Memes.pdf

Advice no one really asked for

To summarise what I said in class -
Life from second year onwards will get hectic. Courses will be tougher. You'll need to be at war with the courses.

Try to appreciate the courses. If you're not academically oriented, do something to make your stay here worthwhile.

Lastly, be happy.

And here's the link to the reddit comment I had mentioned -

https://www.reddit.com/r/AskReddit/comments/dp56ya/

Emotional note

To end on an emotional note, here is a picture that I feel describes the situation -

Emotional note

Fin.