

# MA 105 : Calculus D1 - T5, Tutorial 10

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Sheet 7: Problems 5, 6, 7, 8, 9, 10

5. (i)  $f(x, y) = x^4 + y^4 + 4x - 32y - 7$ ,  $(x_0, y_0) = (-1, 2)$ .

Note that the above function is defined on  $D = \mathbb{R}^2$ .

Thus, the given point is an interior point of  $D$ . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

Note that  $(\nabla f)(x, y) = (4x^3 + 4, 4y^3 - 32)$ . Hence,  $(\nabla f)(x_0, y_0) = 0$ .

Thus, we can appeal to the determinant test.

$$(\Delta f)(x, y) = (12x^2)(12y^2) - (0)^2 = 144x^2y^2.$$

Thus,  $(\Delta f)(x_0, y_0) > 0$ .

Also,  $f_{xx}(x_0, y_0) = 12x_0^2 > 0$ .

Thus, by the determinant test, we get that  $f$  has a local minimum at  $(x_0, y_0)$ .

5. (ii)  $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$ ,  $(x_0, y_0) = (0, 0)$ .

Note that the above function is defined on  $D = \mathbb{R}^2$ .

Thus, the given point is an interior point of  $D$ . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

Note that  $(\nabla f)(x, y) = (3x^2 + 6x - 2y, -2x + 10y - 12y^2)$ . Hence,  $(\nabla f)(x_0, y_0) = 0$ .

Thus, we can appeal to the determinant test.

$$(\Delta f)(x, y) = (6x + 6)(10 - 24y) - (-2)^2.$$

$$\text{Thus, } (\Delta f)(x_0, y_0) = (6)(10) - 4 = 56 > 0.$$

$$\text{Also, } f_{xx}(x_0, y_0) = 6 > 0.$$

Thus, by the determinant test, we get that  $f$  has a local minimum at  $(x_0, y_0)$ .

6. (i)  $f(x, y) = (x^2 - y^2) e^{-(x^2+y^2)/2}$ .

Note that the above function is defined on  $D = \mathbb{R}^2$ .

Thus, every point is an interior point of  $D$ . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)

For  $(x_0, y_0)$  to be a point of extrema or a saddle point, it must be the case that  $(\nabla f)(x_0, y_0) = (0, 0)$ .

Note that  $f_x(x, y) = x e^{1/2(-x^2-y^2)} (-x^2 + y^2 + 2)$ .

Also,  $f_y(x, y) = y e^{1/2(-x^2-y^2)} (-x^2 + y^2 - 2)$ .

Thus, solving  $(\nabla f)(x_0, y_0) = (0, 0)$  gives us that

$$(x_0, y_0) \in \{(0, 0), (0, \sqrt{2}), (0, -\sqrt{2}), (-\sqrt{2}, 0), (\sqrt{2}, 0)\}.$$

Now, we determine the exact nature using the determinant test.

Recall that  $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$ .

Hence, in our case,

$$(\Delta f)(x, y) = -e^{-x^2-y^2} (x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4).$$

Moreover,  $f_{xx}(x, y) = e^{-(x^2+y^2)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$

For  $(x_0, y_0) = (0, 0)$ , it is clear that it is a saddle point for  $f$  as discriminant is  $-4 < 0$ .

Note that if  $x = 0$ , the discriminant reduces to  $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$ .

Substituting  $y = \pm\sqrt{2}$  gives us that the discriminant is positive with  $f_{xx}$  positive and hence, the points are points of local minima.

Similarly, we get that the points  $(\pm\sqrt{2}, 0)$  are points of local maxima as they have discriminant positive and  $f_{xx}$  negative.

6. (ii)  $f(x, y) = x^3 - 3xy^2$ .

Note that the above function is defined on  $D = \mathbb{R}^2$ .

Thus, every point is an interior point of  $D$ . Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)

For  $(x_0, y_0)$  to be a point of extrema or a saddle point, it must be the case that  $(\nabla f)(x_0, y_0) = (0, 0)$ .

Note that  $f_x(x, y) = 3x^2 - 3y^2$ .

Also,  $f_y(x, y) = -6xy$ .

Thus, solving  $(\nabla f)(x_0, y_0) = (0, 0)$  gives us that  $(x_0, y_0) = (0, 0)$ .

Now, we determine the exact nature using the determinant test.

Recall that  $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$ .

Hence, in our case,

$$(\Delta f)(x_0, y_0) = -36(x_0^2 + y_0^2).$$

Thus, for  $(x_0, y_0) = (0, 0)$ , we get the discriminant is 0.

Hence, we get that the discriminant test is **inconclusive!**

This means that we must turn to some other analytic methods of determining the nature.

Now, we note that  $f(\delta, 0) = \delta^3$  for all  $\delta \in \mathbb{R}$ .

Thus, given any  $\epsilon > 0$ , choose  $\delta = \pm\epsilon/2$ .

This gives us that  $(0, 0)$  is saddle point.

(How?)



7. To find: Absolute maxima and minima of

$$f(x, y) = (x^2 - 4x) \cos y \text{ for } 1 \leq x \leq 3, -\pi/4 \leq y \leq \pi/4.$$

Note that the domain is a closed and bounded set. As  $f$  is continuous on the domain,  $f$  does achieve a maximum and a minimum. Note that  $f_x(x, y) = (2x - 4) \cos y$  and  $f_y(x, y) = -(x^2 - 4x) \sin y$  for interior points  $(x, y)$ .

Thus, the only critical point is  $p_1 = (2, 0)$ .

Now we restrict ourselves to the boundaries to find the local extrema.

“Right boundary:” This is the line segment  $x = 3, -\pi/4 \leq y \leq \pi/4$ .

The function now reduces to  $-3 \cos y$  on this segment.

Using our theory from one-variable calculus, we get that we need to check the points  $(3, 0), (3, \pi/4), (3, -\pi/4)$ . (How?)

Similar consideration of the “left boundary” gives us the points

$(1, 0), (1, \pi/4), (1, -\pi/4)$ .

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Now, we look at the “top boundary.”

The function there reduces to  $\frac{x^2-4x}{\sqrt{2}}$ .

Once again, using our theory from one-variable calculus, we get that we need to check the points  $(1, \pi/4)$ ,  $(2, \pi/4)$ ,  $(3, \pi/4)$ .

Similarly, checking the “bottom boundary” gives us the points

$(1, -\pi/4)$ ,  $(2, -\pi/4)$ ,  $(3, -\pi/4)$ .

We now tabulate our results as follows:

$(x_0, y_0)$	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	$-4$	$-3$	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$
$(x_0, y_0)$	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	$-3$	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

Thus, we get that  $f_{\min} = -4$  at  $(2, 0)$  and  $f_{\max} = -\frac{3}{\sqrt{2}}$  at  $(1, \pm\pi/4)$  and  $(3, \pm\pi/4)$ .

8. Let  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined as  $g(x, y, z) := x^2 + y^2 + z^2 - 1$ . We need to maximise the function  $T(x, y, z) = 400xyz$  subject to the constraint  $g = 0$ . Note that the set  $S^2 = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$  is nonempty, closed and bounded, and  $T$  is continuous on it. Thus,  $f$  will attain its maximum on  $S^2$ .

Now,  $(\nabla T)(x, y, z) = \lambda(\nabla g)(x, y, z)$  and  $g(x, y, z) = 0$  means

$$400yz = 2\lambda x, \quad 400xz = 2\lambda y, \quad 400xy = 2\lambda z, \quad x^2 + y^2 + z^2 = 1.$$

The above gives us that  $400xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$ .

Also,  $(\nabla g)(x, y, z) \neq (0, 0, 0)$  whenever  $g(x, y, z) = 0$ .

Thus, the hypotheses of the Lagrange Multiplier Theorem are satisfied.

Now, we solve the equations to get the points of maxima.

If  $\lambda \neq 0$ , then  $x^2 = y^2 = z^2$  and hence, we get the 8 points  $\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$ .

If  $\lambda = 0$ , then  $yz = zx = xy = 0$ . This, combined with  $g = 0$  gives us the 6 points  $(0, 0, \pm 1)$ ,  $(0, \pm 1, 0)$ ,  $(\pm 1, 0, 0)$ .

Now, we check the value of  $T$  at these 14 points.

The first 8 points give either  $T = \frac{400}{3\sqrt{3}}$  or  $T = -\frac{400}{3\sqrt{3}}$ . The last 6 points give  $T = 0$ .

Thus, the highest value of  $T$  is  $\frac{400}{3\sqrt{3}}$ .

9. We wish to maximise  $f(x, y, z) = xyz$  subject to the constraints

$g(x, y, z) = x + y + z - 40 = 0$  and  $h(x, y, z) = x + y - z = 0$ .

$g = h = 0$  clearly gives us that  $z = 20$ .

Using this and  $h$ , we get that  $x + y = 20$ .

Thus,  $f(x, y, z) = 20x(20 - x) = -20((x - 10)^2 - 100)$ .

Note that  $(x - 10)^2 \geq 0$  and hence,  $f(x, y, z) \leq 2000$ .

Thus, we get that  $f$  is bounded above by 2000 under the constraints  $g = h = 0$ .

Moreover,  $f$  does attain this value at  $(10, 10, 20)$  which is a point satisfying the constraints.

Thus, we get that the maximum value attained by  $f$  is 2000, given the constraints.

Note that this time, our constraint set was not bounded. Thus, we had no reason to assume a priori that the maximum is attained.

10. We wish to minimise  $f(x, y, z) = x^2 + y^2 + z^2$  subject to the constraints  $g(x, y, z) = x + 2y + 3z - 6 = 0$  and  $h(x, y, z) = x + 3y + 4z - 9 = 0$

Note that the set  $E := \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = h(x, y, z) = 0\}$  is **not** bounded.

Thus, we can't straight away say that  $f$  does indeed attain a minimum on  $E$ .

However, observe that  $(0, 3, 0) \in E$  and  $f(0, 3, 0) = 9$ . Thus, if there were to exist a global minimum at some point  $(x, y, z)$ , then it would have to be the case that  $f(x, y, z) \leq 9$ . This motivates us to consider the new set

$$E' = \{(x, y, z) \in E : f(x, y, z) \leq 9\}.$$

This set is clearly bounded. Moreover, it is closed and non-empty as well. Thus,  $f$  attains a minimum on  $E'$  which would in turn be a minimum on  $E$  as well. Now, we turn back to Lagrange.

We solve  $\nabla f = \lambda \nabla g + \mu \nabla h$  along with  $g = h = 0$  for  $\lambda, \mu, x, y, z$ .

We see that  $\nabla f = (2x, 2y, 2z)$ ,  $\nabla g = (1, 2, 3)$ ,  $\nabla h = (1, 3, 4)$ .

Thus, it is clear that  $\nabla g$  and  $\nabla h$  are always non-zero. Moreover, they are non-parallel at all points.

Thus, we get  $2x = \lambda + \mu$ ,  $2y = 2\lambda + 3\mu$ ,  $2z = 3\lambda + 4\mu$ . (\*)

Note that  $2g = 2h = 0$  along with (\*) gives us that  $\lambda = -10$ ,  $\mu = 8$ .

Now, the equalities of (\*) give us that  $x = -1$ ,  $y = 2$ ,  $z = 1$ . It is clear that this is indeed a point of minimum.