MA 105 : Calculus
D1 - T5, Tutorial 07

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1. (i)
Writing y in terms of x gives us $y = 1 - 2\sqrt{x} + x$.
The desired area is

$$\int_0^1 y dx = \int_0^1 (1 - 2\sqrt{x} + x) dx = \left(x - 2 \cdot \frac{2}{3} x^{3/2} + \frac{1}{2} x^2 \right) \Big|_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}.$$

(ii)
Solving for the intersection point of the two curves gives us

$$x^4 - 2x^2 = 2x^2 \text{ or } x = 0, \pm 2.$$

It can be verified that for $x \in [-2, 2]$, we have $x^4 - 2x^2 \leq 2x^2$.
Thus, the desired area is

$$\int_{-2}^2 |x^4 - 2x^2 - 2x^2| dx = \int_{-2}^2 4x^2 - x^4 dx = \left(\frac{4}{3} x^3 - \frac{1}{5} x^5 \right) \Big|_{-2}^2 = \frac{128}{15}.$$

1. (iii)
Solving for the intersection point gives us

$$3y - y^2 = 3 - y \text{ or } y = 1, 3.$$

It can be verified that for $y \in [1, 3]$, we have $3y - y^2 \geq 3 - y$.
Thus, the desired area is

$$\int_1^3 -y^2 + 4y - 3dy = \left(-\frac{1}{3}y^3 + 2y^2 - 3y\right)\Bigg|_1^3 = \frac{4}{3}.$$

Sheet 5

2. It is easy to see that the curves $y = f(x)$ and $y = g(x)$ intersect at $(0, 0)$ and $(1 - a, a - a^2)$.
Let us assume that $a < 1$ and find the area A.

$$A = \int_0^{1-a} (x - x^2 - ax)dx = \frac{(1-a)^3}{6}.$$

As $A = 4.5$, we get that $(1 - a)^3 = 27$. Thus, we have it that $a = -2$.
Now, if we assume that $a > 1$, we get that $A = \frac{(a-1)^3}{6} = 4.5$ which gives us that $a = 4$.

3. Solving for the intersection point of the two curves gives us

$$6a\cos\theta = 2a(1+\cos\theta) \text{ or } \theta = \pm\frac{\pi}{3}.$$

It is for $\theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ that the circle is outside the cardioid. Thus, the desired area is

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (6a\cos\theta)^2 - (2a(1+\cos\theta))^2 d\theta = 2a^2 \int_{-\pi/3}^{\pi/3} 8\cos^2\theta - 1 - 2\cos\theta d\theta$$

$$= 2a^2 \int_{-\pi/3}^{\pi/3} 4\cos(2\theta) - 2\cos\theta + 3d\theta = 4\pi a^2$$

Sheet 5

6. The region in the XY plane between the two curves is given by
$\{(x, y, 0) \in \mathbb{R}^3 : -2 < x < 2, x^2 < y < 8 - x^2\}$.
It can be seen that the plane $y = 4$ is a plane of symmetry for the region. Thus, the centers of the circles mentioned must lie on this plane.
Given a plane $x = x_0$, the area of the circle of cross-section is given by $\pi(4 - x_0^2)^2$.
Thus, the required volume is

$$V = \int_{-2}^{2} \pi(4 - x^2)^2 dx = \frac{512\pi}{15}.$$

8. We can fix the line to be along $z-$axis, $0 \leq z \leq h$.
For any fixed z, the area of the cross-section of the solid is r^2.
Thus, the required volume is $\displaystyle\int_0^h r^2 dz = hr^2$.

9. Washer Method.
In the washer method, the slices are taken **perpendicular** to the axis of revolution. In this case, the axis of revolution is the line $y = -1$.
Let $f_1(x) := 3 - x^2$ and $f_2(x) := -1$ where both the functions are defined from $[-2, 2]$ to $\mathbb{R}$.
The volume will given by

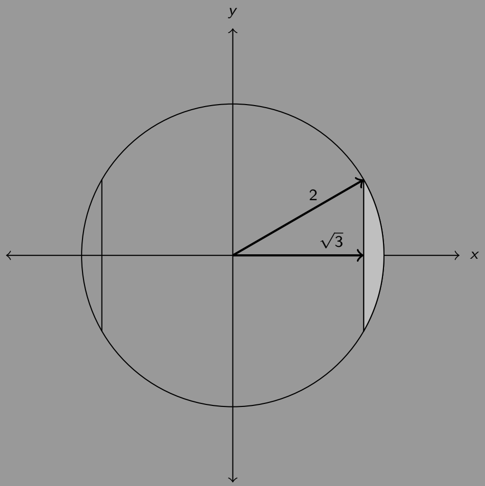
$$V = \pi \int_{-2}^{2} (f_1(x) - (-1))^2 - (f_2(x) - (-1))^2 dx = \pi \int_{-2}^{2} (4 - x^2)^2 dx = \frac{512\pi}{15}.$$

9. Shell Method.
In the shell method, the slivers (which look like cylindrical shells) are taken **parallel** to the axis of revolution.
For $y_0 \in [-1, 3]$, the line $y = y_0$ cuts the curve at the points $(-\sqrt{3 - y_0}, y_0)$ and $(\sqrt{3 - y_0}, y_0)$. Thus, the height of the sliver is $2\sqrt{3 - y_0}$.
Moreover, the radius, that is, the distance of this sliver from the axis of rotation is: $y_0 + 1$.
Thus, by shell method, we get that the volume of the solid is

$$V = 2\pi \int_{-1}^{3} (y + 1)(2\sqrt{3 - y})dy = 4\pi \int_{0}^{4} y\sqrt{4 - y}dy = \frac{512\pi}{15}.$$

The last integral can be calculated easily using Integration by parts.

10.



10. We can compute the volume of the portion cut out by finding the volume of revolution (about the $y-$axis) of the shaded part. This can be done easily using washer method.

$$V' = \pi \int_{-1}^{1} \left(\sqrt{4-y^2}\right)^2 - \left(\sqrt{3}\right)^2 dy = \frac{4\pi}{3}.$$

The total volume of the ball is $V = \frac{4\pi}{3}(2)^2$.

Thus, the volume cut out is $V - V' = \frac{28\pi}{3}$.