Short Quiz 2: Solution

Aryaman Maithani

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Question. State whether the following statement is true or false. Justify your answer.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = x^3 + \sin x + \frac{x}{1+x^2} \quad \text{for } x \in \mathbb{R}.$$

Then there exists a real number c such that $f(c) = 2019$. [5 marks]

[2 marks for correct alternative (T/F); 3 marks for correct justification]

Answer. T **[2]**
Justification: The function f is continuous as it is the sum of x^3, $\sin x$, and $\frac{x}{1+x^2}$, each of which is a continuous function. The function $\frac{x}{1+x^2}$ is continuous as it the quotient of two continuous functions where the denominator is never 0. **[1]**

There exist $a,\ b \in \mathbb{R}$ such that $f(a) < 2019$ and $f(b) > 2019$. For example, one can take $a = 0$ and $b = 2019$. **[1]**

The conclusion follows by applying the Intermediate value theorem on the *interval* $[a, b] \subset \mathbb{R}$. **[1]**

Points to be noted -

1. For those who have argued only using the justification that $\lim\limits_{x\to\infty} f(x) = \infty$ or $\lim\limits_{x\to-\infty} = -\infty$, half a mark has been deducted. The reason for this is simply - We have talked only about real numbers when talking about limits of functions. $\pm\infty$ are not real numbers.
2. For those who have not mentioned the crucial point that IVT can be used since the function is continuous *on an interval*, half a mark has been deducted. This is crucial as functions which are continuous on sets which are not intervals need not have the Intermediate value property.
3. The choice of a and b is obviously not unique and marks have been awarded as long as the bounds are correct, whether it be as narrow as $[12, 13]$ or as wide as $[0, 10^6]$.
4. No marks have been cut even if the person has not explicitly mentioned the reason for continuity of $\frac{x}{1+x^2}$.