Short Quiz 2: Solution

Aryaman Maithani

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Question. Using the substitutions $u = y - x$ and $v = x + y$, express the integral $\iint_D \exp\left(\frac{y-x}{y+x}\right) d(x, y)$, where D is the triangular region with vertices $(0, 0),\ (1, 0)$ and $(0, 1)$ in the following form:

$$\int_a^b \int_{c(v)}^{d(v)} f(u, v) du dv.$$

[5 marks]

Answer.
The given substitution can be rearranged to give $y = \frac{1}{2}(u + v)$ and $x = \frac{1}{2}(v - u)$.
Let $\Phi : \Omega \to \mathbb{R}^2$ be defined as $\Phi(u, v) := \left(\frac{1}{2}(v - u),\ \frac{1}{2}(v + u)\right)$, where $\Omega = \mathbb{R}^2$.
Note that Φ is a one-to-one transformation. Moreover, if we write $\Phi = (\phi_1,\ \phi_2)$, then ϕ_1 and ϕ_2 have continuous partial derivatives in Ω.
Also,

$$\begin{aligned} J(\Phi)(u_0, v_0) &= \det \begin{bmatrix} \frac{\partial x}{\partial u}(u_0, v_0) & \frac{\partial x}{\partial v}(u_0, v_0) \\ \frac{\partial y}{\partial u}(u_0, v_0) & \frac{\partial y}{\partial v}(u_0, v_0) \end{bmatrix} \\ &= \det \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = -1/2 \neq 0 \end{aligned}$$

Let $E := \{(u, v) \in \Omega : 0 \le v \le 1,\ -v \le u \le v\}$.
Then, $\Phi(E) = D$. (A linear transformation with nonzero Jacobian as above sends vertices of a polygon to the vertices. Alternately, one can show that if $(u, v) \in E$, then $\Phi(u, v) \in D$ and if $(x, y) \in D$, then there exists $(u, v) \in E$ such that $\Phi(u, v) = (x, y)$.)

Thus, the given integral is the same as

$$\iint_E e^{u/v} |J(\Phi)(u, v)| d(u, v) = \iint_E e^{u/v} \left|-\frac{1}{2}\right| d(u, v).$$

Using Fubini's theorem, we can write it in the desired form as -

$$\int_0^1 \int_{-v}^{v} \frac{1}{2} e^{u/v} du dv.$$

That is, $a = 0,\ b = 1,\ c(v) = -v$, and $d(v) = v$ where c and d are defined on $[0, 1]$. Moreover, $f(u, v) = \frac{1}{2}e^{u/v}$ for $(u, v) \in E$.

Broad marking scheme - 2 marks for correct Jacobian, 2 marks for correct E, and 1 mark for correct final form using Fubini.

Points to be noted -

1. Note that dx, dy, du, dv are not real numbers. It makes no sense to write $dy = \frac{1}{2}(du + dv)$ or any other such algebraic expression.
2. It also makes not much sense to write $d\left(\frac{v-u}{2},\ \frac{u+v}{2}\right)$.
3. Marks have not been deducted for lack of mentioning Fubini's theorem this time but that must be kept in mind from next time onwards.

4. Some have written the Jacobian in the opposite manner. Make note of what is to be differentiated with respect to what.

5. Remember that the Jacobian is the determinant itself, not the matrix.

6. When writing the integral as an integral over E, the modulus of the Jacobian must be multiplied, not the Jacobian itself.