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2. (i) Let $S_n := \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \cdots + \dfrac{n}{n^2+n}$.

Define $T_n := \dfrac{n}{n^2+1} + \dfrac{n}{n^2+1} + \cdots + \dfrac{n}{n^2+1}$

and $R_n := \dfrac{n}{n^2+n} + \dfrac{n}{n^2+n} + \cdots + \dfrac{n}{n^2+n}$.

Note that $R_n \leq S_n \leq T_n \quad \forall n \in \mathbb{N}$. (Why?)

Also, $T_n = \dfrac{n^2}{n^2+1}$ and $R_n = \dfrac{n^2}{n^2+n}$.

Observe that $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty} R_n = 1$. (Why?)

Thus, by Sandwich Theorem, $\lim\limits_{n\to\infty} S_n$ exists and is equal to 1.

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2. (ii) To find: $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$.

Observe the following for $n > 2$:

$$a_n = \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n-1}{n} < \frac{1}{n} \cdot 1 \cdots 1 = \frac{1}{n}$$

Thus, $a_n < \dfrac{1}{n}$ for $n > 2$. Moreover, $a_n > 0$ for all $n \in \mathbb{N}$.

$$\therefore 0 < a_n < \frac{1}{n} \quad \forall n > 2.$$

As $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, we have it that $\lim\limits_{n\to\infty} a_n = 0$, by Sandwich Theorem.

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2. (iii) $\lim_{n\to\infty}\left(\dfrac{n^3+3n^2+1}{n^4+8n^2+2}\right)$

Argue from $a_n = \left(\dfrac{n^3+3n^2+1}{n^4+8n^2+2}\right) < \left(\dfrac{n^3+3n^2+1}{n^4}\right) = \dfrac{1}{n} + \dfrac{3}{n^2} + \dfrac{1}{n^4}$ and $a_n > 0$.

2. (iv) $\lim_{n \to \infty} (n)^{1/n}$.

Define $h_n := n^{1/n} - 1$.
Then, $h_n \geq 0 \quad \forall n \in \mathbb{N}$. (Why?)
Observe the following for $n > 2$:

$$n = (1 + h_n)^n > 1 + nh_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2.$$

Thus, $h_n < \sqrt{\dfrac{2}{n-1}} \quad \forall n > 2$.

Using Sandwich Theorem, we get that $\lim_{n \to \infty} h_n = 0$ which gives us that $\lim_{n \to \infty} n^{1/n} = 1$.

Where did we use that $h_n \geq 0$?

2. (v) $\lim_{n\to\infty} \left(\frac{\cos \pi \sqrt{n}}{n^2} \right)$

For all $n \in \mathbb{N}$, we have that $-1 \leq \cos(\pi\sqrt{n}) \leq 1$.

Thus, $\frac{-1}{n^2} \leq \frac{\cos \pi \sqrt{n}}{n^2} \leq \frac{1}{n^2} \quad \forall n \in \mathbb{N}$.

Use Sandwich Theorem to argue that $\lim_{n\to\infty} \frac{\cos \pi \sqrt{n}}{n^2} = 0$.

2. (vi) $\lim_{n\to\infty}(\sqrt{n}(\sqrt{n+1}-\sqrt{n}))$

Observe that

$$a_n = \sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \sqrt{n}(\sqrt{n+1}-\sqrt{n}) \cdot \frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}.$$

Thus, $a_n < \dfrac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \dfrac{1}{2}$.

Also, $a_n > \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+1}} = \dfrac{1}{2}\sqrt{\dfrac{n}{n+1}} = \dfrac{1}{2}\sqrt{1-\dfrac{1}{n+1}} \geq \dfrac{1}{2}\left(1-\dfrac{1}{\sqrt{n+1}}\right)$.

Therefore, we have shown that $\dfrac{1}{2}\left(1-\dfrac{1}{\sqrt{n+1}}\right) < a_n < \dfrac{1}{2}$.

Use Sandwich Theorem to argue that $\lim_{n\to\infty} a_n = \dfrac{1}{2}$.

Sheet 1

4. (i) Determine whether $\left\{ \dfrac{n}{n^2+1} \right\}_{n\geq 1}$ is increasing or decreasing.

Let a_n denote the sequence.

$$a_{n+1} - a_n = \frac{(n+1)}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n\left((n+1)^2-1\right)}{\left((n+1)^2+1\right)\left(n^2+1\right)}$$

$$a_{n+1} - a_n = \frac{-n^2-n+1}{\left((n+1)^2+1\right)\left(n^2+1\right)} < 0.$$

$\therefore a_{n+1} < a_n$, that is, a_n is a decreasing sequence.

4. (ii) $a_n = \dfrac{2^n 3^n}{5^{n+1}}$.

$$a_{n+1} = \frac{2^{n+1}3^{n+1}}{5^{n+2}} = \frac{6}{5}a_n.$$

$$\implies a_{n+1} - a_n = \frac{1}{5}a_n > 0.$$

Thus, a_n is an increasing sequence.

4. (iii) $a_n = \dfrac{1-n}{n^2}$ for $n \geq 2$.

$$a_{n+1} - a_n = \frac{1-(n+1)}{(n+1)^2} - \frac{1-n}{n^2} = \frac{n-1}{n^2} - \frac{n}{(n+1)^2}$$

$$a_{n+1} - a_n = \frac{(n^2-n-1)}{n^2(n+1)^2}$$

The numerator factors as $(n-\phi)(n+1/\phi)$ where $1 < \phi < 2$. Thus, for $n \geq 2$, the numerator is positive. Thus, the given sequence is increasing.

5. (i) $a_1 = 1,\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall n \geq 1.$

Claim 1. $a_n > 0 \quad \forall n \in \mathbb{N}$.
Proof. This can be easily seen via induction. Details are left to the reader.

Claim 2. $a_n^2 > 2 \quad \forall n \geq 2$.
Proof. We shall prove this via induction. The base case $n = 2$ is immediate as $a_n = 3/2$.
Assume that it holds for $n = k$.

$$a_{k+1}^2 - 2 = \frac{1}{4}\left(a_k + \frac{2}{a_k}\right)^2 - 2 = \frac{(a_k^2 - 2)^2}{4a_k^2}$$

$(a_k^2 - 2) \neq 0$ by induction hypothesis and thus, $a_{k+1}^2 - 2 > 0$. Therefore, by principle of mathematical induction, we have proven our claim. ■

Claim 3. $a_{n+1} < a_n \quad \forall n \geq 2.$

Proof. Observe that $a_{n+1} - a_n = \dfrac{2 - a_n^2}{2a_n}$.
The quantity on the right is negative, using Claim 1 and Claim 2.
Thus, $a_{n+1} < a_n$. ■

We have shown that the sequence is *eventually* monotonically decreasing. Also, it is bounded below, by Claim 1. Thus, the limit $\lim\limits_{n\to\infty} a_n$ exists. Let $L(\in \mathbb{R})$ denote this limit.

Note: We are assuming that an *eventually* monotonic bounded sequence is also convergent.

Thus, $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$.
We had shown that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n$. Using that and other limit properties, we get that $L^2 = 2$. Thus, L must be $\sqrt{2} > 0$. (Why not $-\sqrt{2}$?)

5. (ii) $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n} \quad \forall n \geq 1.$
Claim 1. $a_n > 0 \quad \forall n \in \mathbb{N}.$
Proof. This can be easily seen via induction. Details are left to the reader.

Claim 2. $a_n < 2 \quad \forall n \in \mathbb{N}.$
Proof. We shall prove this via induction. The base case $n = 1$ is immediate as $\sqrt{2} < 2$.
Assume that it holds for $n = k$.
$a_{k+1}^2 - 4 = \left(\sqrt{a_k + 2}\right)^2 - 4 = a_k + 2 - 4 = a_k - 2.$
But $a_k - 2 < 0$ by induction hypothesis. Thus, $a_{k+1}^2 < 4$ or $a_{k+1} < 2$. By principle of mathematical induction, we have proven the claim. ∎

Claim 3. $a_n < a_{n+1} \quad \forall n \in \mathbb{N}.$
Proof. $a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2 - a_n)(1 + a_n) > 0.$
The last inequality is by the help of Claims 1 and 2.

Thus, we have $a_{n+1}^2 > a_n^2$. Using Claim 1, we can conclude that $a_{n+1} > a_n$. $\blacksquare$

By Claims 1 and 2, we have it that the sequence is bounded. By Claim 3, we have it that the sequence is monotone. Therefore, the sequence must converge. Let the limit be $L (\in \mathbb{R})$.

Taking limit on both sides of the recursive definition gives us $L = \sqrt{2 + L}$. Thus, $L^2 = 2 + L$ or $(L - 2)(L + 1) = 0$.

Note that L cannot be -1. (Why?)

$\therefore L = 2$.

5. (iii) $a_1 = \sqrt{2},\ a_{n+1} = 3 + \dfrac{a_n}{2} \quad \forall n \geq 1.$

Claim 1. $a_n < 6 \quad n \in \mathbb{N}.$

Proof. We shall prove this via induction. The base case $n = 1$ is immediate as $2 < 6$. Assume that it holds for $n = k$.

$a_{k+1} = 3 + \dfrac{a_n}{2} < 3 + \dfrac{6}{2} = 6.$

By principle of mathematical induction, we have proven the claim. ■

Claim 2. $a_n < a_{n+1} \quad \forall n \in \mathbb{N}.$

Proof. $a_{n+1} - a_n = 3 - \dfrac{a_n}{2} = \dfrac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n.$ ■

Thus, (a_n) is a monotonically increasing sequence that is bounded above. Therefore, it must converge. Using the same method as earlier gives this limit to be 6.

10. To show:
$\{a_n\}_{n\geq 1}$ is convergent $\iff$ $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to the same limit.

Proof. ($\implies$) Let $b_n := a_{2n}$ and $c_n := a_{2n+1}$. We are given that $\lim\limits_{n\to\infty} a_n = L$. We must show that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$.
Let $\epsilon > 0$ be given. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for $n \geq n_0$.
Note that $2n > n$ and $2n + 1 > n$ for all $n \in \mathbb{N}$. Thus, we have that $|b_n - L| < \epsilon$ and $|c_n - L| < \epsilon$ for all $n \geq n_0$.
Thus, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = L$. ∎

$(\Longleftarrow)$ Let (b_n) and (c_n) be as defined before. We are given that
$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$. We must show that (a_n) converges.
Let $\epsilon > 0$ be given. By hypothesis, there exists $n_1,\ n_2 \in \mathbb{N}$ such that

$$|b_n - L| < \epsilon \text{ for all } n \geq n_1 \tag{1}$$

$$\text{and } |c_n - L| < \epsilon \text{ for all } n \geq n_2. \tag{2}$$

Choose $n_0 = \max\{2n_1,\ 2n_2 + 1\}$.
Let $n \geq n_0$ be even. Then, $n \geq 2n_1$ or $n/2 \geq n_1$ and $a_n = b_{n/2}$. By (1), we have it that $|a_n - L| < \epsilon$.
Similarly, let $n \geq n_0$ be odd. Then, $n \geq 2n_2 + 1$ or $(n-1)/2 \geq n_2$ and $a_n = c_{(n-1)/2}$. By (2), we have it that $|a_n - L| < \epsilon$.
Thus, we have shown that $|a_n - L| < \epsilon$ whenever $n \geq n_0$. This is precisely what it means for (a_n) to converge to L. ■

Sheet 2

1. (i) We shall show that the statement is false with the help of a counterexample.
Let $a = -1,\ b = 1,\ c = 0$. Define f and g as follows:

$$f(x) = x \text{ and } g(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}.$$

It can be seen that $\lim_{x \to 0} f(x) = 0$ but $\lim_{x \to c} [f(x)g(x)] = \lim_{x \to 0} 1 = 1$.

(ii) We shall prove that the given statement is true.
We are given that g is bounded. Thus, $\exists M \in \mathbb{R}^+$ such that $|g(x)| \leq M \quad \forall x \in (a,\ b)$.
Let $\epsilon > 0$ be given. We want to show that there exists $\delta > 0$ such that
$|f(x)g(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.
Let $\epsilon_1 = \epsilon / M$. As $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that
$0 < |x - c| < \delta \implies |f(x)| < \epsilon_1$.
Thus, whenever $0 < |x - c| < \delta$, we have it that
$|f(x)g(x) - 0| = |f(x)||g(x)| \leq |f(x)| \cdot M < \epsilon_1 \cdot M = \epsilon$. ■

(iii) We shall prove that the given statement is true.
Let $\epsilon > 0$ be given.
Let $l := \lim_{x \to c} g(x)$.
Let $\epsilon_1 = \epsilon/(|l| + \epsilon)$.

By hypothesis, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1 \implies |g(x) - l| < \epsilon$.
Also, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2 \implies |f(x)| < \epsilon_1$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x - c| < \delta$, we have that:
$|f(x)g(x)| = |f(x)g(x) - lf(x) + lf(x)| \leq |f(x)||(g(x) - l)| + |l||f(x)| <$
$|f(x)|\epsilon + |l||f(x)| = |f(x)|(\epsilon + |l|) < \epsilon_1(\epsilon + |l|) = \epsilon.$
Thus, we have it that $0 < |x - c| < \delta \implies |f(x)g(x) - 0| < \epsilon.$ ■

2. We are given that $\lim_{x \to \alpha} f(x)$ exists. Let it be $c(\in \mathbb{R})$. Note that it's not necessary that $c = f(\alpha)$.
Let us evaluate $\lim_{h \to 0} f(\alpha + h)$. Let (h_n) be an arbitrary sequence of real numbers such that $h_n \neq 0$ and $h_n \to 0$. We need to find $\lim_{n \to \infty} f(\alpha + h_n)$.
Consider the sequence (x_n) of real numbers defined as $x_n := \alpha + h_n$. Thus, $x_n \neq \alpha$ and $x_n \to \alpha$. By hypothesis, we must have that $\lim_{n \to \infty} f(x_n) = c$.
Thus, by definition of x_n, we must have that $\lim_{n \to \infty} f(\alpha + h_n) = c$. This gives us that $\lim_{h \to 0} f(\alpha + h_n) = c$.
Similar consideration will give $\lim_{h \to 0} f(\alpha - h_n) = c$ as well.
Using the limit theorems for functions, we have that:

$$\lim_{h \to 0} [f(\alpha + h) - f(\alpha - h)] = \lim_{h \to 0} f(\alpha + h) - \lim_{h \to 0} f(\alpha - h) = c - c = 0.$$

Sheet 2

Converse of 2.

The converse of 2 does not hold. That is, given $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $\lim_{h \to 0} [f(a+h) - f(a-h)] = 0$, it is not necessary that $\lim_{x \to \alpha} f(x)$ exists.

We shall demonstrate this will the help of a counterexample.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It can be easily observed that $f(x) = f(-x) \quad \forall x \in \mathbb{R}$.

Let $\alpha = 0$.

Then, $\lim_{h \to 0} [f(\alpha + h) - f(\alpha - h)] = \lim_{h \to 0} [f(h) - f(-h)] = \lim_{h \to 0} [0] = 0$.

However, $\lim_{x \to 0} f(x)$ does **not** exist.

3. (i) The function is continuous everywhere except at $x = 0$.
Proof. For $x \neq 0$, we have it that f is a composition of continuous functions. Thus, it is continuous.

To show that f is discontinuous at $x = 0$:
Consider the sequence (x_n) where $x_n = \dfrac{2}{(4n+1)\pi}$.
Then, $x_n \to 0$ but $f(x_n) = 1 \quad \forall n \in \mathbb{N}$ and thus, $f(x_n) \to 1 \neq f(0)$.
Thus, f is discontinuous at $x = 0$, by definition.

3. (ii) The function is continuous everywhere.

Proof. For $x \neq 0$, it simply follows from the fact that product and composition of continuous functions is continuous.

To show continuity at $x = 0$:

Let (x_n) be any sequence of real numbers such that $x_n \to 0$. We must show that $f(x_n) \to 0$.

Let $\epsilon > 0$ be given.

Observe that $|f(x_n) - 0| = \left| x_n \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|$.

Now, we shall use the fact $x_n \to 0$. By this hypothesis, there must exist $n_1 \in \mathbb{N}$ such that $|x_n| = |x_n - 0| < \epsilon \quad \forall n \geq n_1$.

Choosing $n_0 = n_1$, we have it that $|f(x_n) - 0| \leq |x_n| < \epsilon \quad \forall n \geq n_0$. ■

3. (iii) The function can be rewritten as: $f(x) = \begin{cases} x & \text{if} \quad 1 \leq x < 2 \\ 1 & \text{if} \quad x = 2 \\ \sqrt{6-x} & \text{if} \quad 2 < x \leq 3 \end{cases}$

We claim that the function is continuous on $[1, 2) \cup (2, 3]$ and discontinuous at 2.
Given $x \in [1, 2)$ and any sequence (x_n) in the domain such that $x_n \to x$, there must exist $n \in n_0$ such that $x_n \in [1, 2) \quad \forall n \geq n_0$. Thus, $f(x_n) = x_n \quad \forall n \geq n_0$. It can now be easily shown that $f(x_n) \to x = f(x)$. (We have essentially used the continuity of the function $x \mapsto x$.) Thus, f is continuous on $[1, 2)$.
Similarly, we can argue that f is continuous on $(2, 3]$. Again, this will follow from the fact that the function $x \mapsto \sqrt{6-x}$ is continuous on its domain.

Now, we show that f is discontinuous at 2. Consider the sequence $x_n := 2 - 1/n$. It is clear that $x_n \to 2$.
Observe that $1 \leq x_n < 2$. Thus, $f(x_n) = 2 - 1/n$.
This gives us that $f(x_n) \to 2 \neq f(2)$. $\blacksquare$

Sheet 2

4. We are given that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Thus, we can let $x = y = 0$. This gives us that:
$f(0+0) = f(0) + f(0) \implies f(0) = 2f(0) \implies f(0) = 0.$
As f is continuous at 0, we have it that $\lim_{h \to 0} f(h) = f(0) = 0$.

Now, we will show that f is continuous at every $c \in \mathbb{R}$.
Substituting $x = c$ in the original equation gives us: $f(c+y) = f(c) + f(y)$. As this is true for every $y \in \mathbb{R}$, we have that: $\lim_{y \to 0} f(c+y) = \lim_{y \to 0} [f(c) + f(y)]$.

We know that $\lim_{y \to 0} f(c) = f(c)$ (constant sequence) and $\lim_{y \to 0} f(y) = 0$ (shown above).
Thus, we can write:
$\lim_{y \to 0} f(c+y) = \lim_{y \to 0} f(c) + \lim_{y \to 0} f(y) = f(c)$. This is precisely what it means for f to be continuous at c.

4. **(Optional)** Here's a sketch of how one can show that f satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.
Step 1. Use induction and show that $f(nx) = nf(x) \quad \forall n \in \mathbb{N}$.
Step 2. Show that $f(nx) = nf(x) \quad \forall n \in \mathbb{N}$.
Step 3. Show that $f(qx) = qf(x) \quad \forall q \in \mathbb{Q}$.
Step 4. Use density of rationals and continuity of f to argue that $f(kx) = kf(x) \quad \forall k \in \mathbb{R}$.

Note that we didn't require continuity of f in the first 3 steps.

12. Let $c \in \mathbb{R}$.
Recall that given any $a,\ b \in \mathbb{R}$ such that $a < b$, we can construct a rational number $r(a, b)$ such that $a < r(a, b) < b$. Similarly, we can construct $i(a, b) \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < i(a, b) < b$. (Note that we have explicit constructions of these.)

Define the two sequences (r_n) and (i_n) as follows:
$r_n := r(c, c + 1/n)$ and $i_n := i(c, c + 1/n)$.
Thus, we have it that $r_n \to c$ and $i_n \to c$ and also that $r_n \neq c \neq i_n$.
However, observe that $f(r_n) = 1 \quad \forall n \in \mathbb{N}$ and $f(i_n) = 0 \quad \forall n \in \mathbb{N}$. This gives us that $f(r_n) \to 1$ and $f(i_n) \to 0$.
As $\lim\limits_{n\to\infty} f(r_n) = 1 \neq 0 = \lim\limits_{n\to\infty} f(i_n)$, f cannot be continuous at c.

13. Let $c \in \mathbb{R}$ be such that f is continuous at c. Define sequences (r_n) and (i_n) as before.
Thus, $f(r_n) = r_n$. As $r_n \to c$, we have it that $f(r_n) \to c$.
Similarly, $f(i_n) \to 1 - c$.
For f to be continuous at c, we must have it that $c = 1 - c = f(c)$. Solving this gives us that $c = 1/2$.
Thus, what we have shown so far is that: f continuous at $c \implies c = 1/2$.

However, we must now show that f actually *is* continuous at $1/2$.
This is done as follows: Let (x_n) be any sequence of real numbers such that $x_n \to 1/2$.
We claim that $f(x_n) \to 1/2$. If we can prove this claim, then we are done as $f(1/2) = 1/2$.
Note that if $x_n \in \mathbb{Q}$, then $|f(x_n) - 1/2| = |x_n - 1/2|$ and if $x \notin \mathbb{Q}$, then $|f(x_n) - 1/2| = |1/2 - x_n| = |x_n - 1/2|$.

Thus, we have it that $|f(x_n) - 1/2| = |x_n - 1/2| \quad \forall n \in \mathbb{N}$.
Let $\epsilon > 0$ be given. As $x_n \to 1/2$, there exists $n_1 \in \mathbb{N}$ such that $|x_n - 1/2| < \epsilon$ for all $n \geq n_1$.
Choose $n_0 = n_1$. Thus, $|f(x_n) - 1/2| < \epsilon$ for all $n \geq n_0$.
$\therefore \lim_{n\to\infty} f(x_n) = 1/2$. $\blacksquare$