Extra Questions for MA 105

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Notation:
$\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\mathbb{R}$ denotes the set of real numbers.

Week 1

1. Let f be any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0,\ 1]$.
   Define the sequence (a_n) of real numbers as: $a_n := f(n) \quad \forall n \in \mathbb{N}$.
   Prove that (a_n) diverges or find an example of f such that (a_n) converges.

2. Let (a_n) be a sequence of real numbers. We say that (a_n) is *slack-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is slack-convergent.

   (Additional) What happens if we change $n \geq n_0$ to $n > n_0$?

3. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reciprocal-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1/\epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reciprocal-convergent.

4. Let (a_n) be a sequence of real numbers. We say that (a_n) is *natural-convergent* if the following condition holds.
   For every $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} |a_{n+k} - a_n| = 0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is natural-convergent.

5. Let (a_n) be a sequence of real numbers. We say that (a_n) is *weirdly-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for infinitely many $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is weirdly-convergent.

6. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reverse-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $n_0 \in \mathbb{N}$, there is $\epsilon > 0$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reverse-convergent.

7. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Let (a_n) be an increasing sequence in S such that $\lim\limits_{n\to\infty} a_n = L \notin S$.
   Prove or disprove that $L = \sup S$.

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?

Week 2

1. Show that $f : \mathbb{N} \to \mathbb{R}$ is continuous for any f.

2. Let $f : \mathbb{Q} \to \mathbb{R}$ be a continuous function such that the image (range) of f is a subset of $\mathbb{Q}$. Let $a,\ b,\ r \in \mathbb{Q}$ be such that $a < b$ and $f(a) < r < f(b)$. Show (with the help of an example) that it is not necessary that there exists some $c \in \mathbb{Q} \cap [a, b]$ such that $f(c) = r$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *reverse continuous* at c if for all $\delta > 0$, there exists $\epsilon > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
   Is this notion of continuity the same as the normal notion?
   If not, then give an example of a function which is reverse continuous at a point but not continuous or vice-versa.

4. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *upper continuous* at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies f(c) \leq f(x) < f(c) + \epsilon$.
   (a) Prove that a function is continuous at a point if it is upper continuous at that point.
   (b) Show that the converse may not be true.
   (c) Give an example of a function that is upper continuous at only one point.
   (d) Given any $n \in \mathbb{N}$, show that there exists a function that is upper continuous at exactly n points.
   (e) Show that there exists a function that is upper continuous at infinitely many points.
   (f) Give an example of a function f that is upper continuous everywhere.
   (g) Can you give an example of another function g such that g is upper continuous everywhere but $f - g$ is not constant?

5. Let $A, B \subset \mathbb{R}$ and $f : A \to B$ be a bijection. Show with the help of an example that f is continuous $\not\Longrightarrow$ f^{-1} is continuous.

6. Show that there exists a bijection from $(0, 1)$ to $[0, 1]$.

7. Show that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$ or from $[0, 1]$ to $(0, 1)$.

8. Let $f : A \to B$ be a continuous surjective function. Show that it is possible for A to be a bounded open interval and B to be a bounded closed interval.
   Is it possible for A to be a bounded closed interval and B to be a bounded open interval?

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the intermediate value property. Is it necessary that f is continuous *somewhere*?

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that given any $c \in \mathbb{R}$, the limit $\lim\limits_{x \to c} f(x)$ exists. Is it necessary that f is continuous *somewhere*?

The last two questions are just for one to think about. I do not expect solutions for those.