Extra Questions for MA 105

Aryaman Maithani

Semester: Autumn 2019

Notation:
$\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\mathbb{R}$ denotes the set of real numbers.

Week 1

1. Let f be any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0,\ 1]$.
   Define the sequence (a_n) of real numbers as: $a_n := f(n) \quad \forall n \in \mathbb{N}$.
   Prove that (a_n) diverges or find an example of f such that (a_n) converges.

2. Let (a_n) be a sequence of real numbers. We say that (a_n) is *slack-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is slack-convergent.

   (Additional) What happens if we change $n \geq n_0$ to $n > n_0$?

3. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reciprocal-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1/\epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reciprocal-convergent.

4. Let (a_n) be a sequence of real numbers. We say that (a_n) is *natural-convergent* if the following condition holds.
   For every $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} |a_{n+k} - a_n| = 0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is natural-convergent.

5. Let (a_n) be a sequence of real numbers. We say that (a_n) is *weirdly-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for infinitely many $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is weirdly-convergent.

6. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reverse-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $n_0 \in \mathbb{N}$, there is $\epsilon > 0$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reverse-convergent.

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?