3.6

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3.6 Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$, where $a, b, c \in \mathbb{R}$. Also, prove an analogous formula for a determinant of order n, known as the **Vandermonde determinant**.

$$V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \ddots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}.$$

<u>Claim.</u> $\det(V_n) = \prod_{1 \le i < j \le n} (a_j - a_i)$ for $n \geq 2$.

$\swarrow \binom{n}{2}$ terms

$$= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1)$$
$$(a_3 - a_2) \cdots (a_n - a_2)$$
$$\ddots$$
$$(a_n - a_{n-1})$$

<u>Proof.</u> We prove this via induction on n.

<u>Base case</u>. $n = 2$. $V_2 = \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix}$.

Then, $\det(V_2) = a_2 - a_1$, as desired.

<u>Inductive hypothesis</u>. Assume $n \geq 3$ and the result is true for $n-1$.

<u>Inductive step</u>. We show the result is true for n.

$$\det(V_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & \cdots & & a_n^{n-1} \end{bmatrix}.$$

$R_n \mapsto R_n - a_1 R_{n-1}$

$R_{n-1} \mapsto R_{n-1} - a_1 R_{n-2}$

$\vdots$

$R_2 \mapsto R_2 - a_1 R_1$

Then, $$\det(V_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & \cdots & a_n - a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & & \\ 0 & a_2^{n-1} - a_1 a_2^{n-2} & \cdots & a_n^{n-1} - a_1 a_n^{n-2} \end{bmatrix}$$

Expand along the first column to get

$$\det(V_n) = 1 \cdot \det \begin{pmatrix} a_2 - a_1 & \cdots & a_n - a_1 \\ \vdots & \ddots & \vdots \\ a_2^{n-1} - a_1 a_2^{n-2} & \cdots & a_n^{n-1} - a_1 a_n^{n-2} \end{pmatrix}$$

$$= \det \begin{pmatrix} a_2 - a_1 & \cdots & a_n - a_1 \\ \vdots & \ddots & \vdots \\ a_2^{n-2}(a_2 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{pmatrix}$$

$$= \left[(a_2 - a_1) \cdots (a_n - a_1)\right] \det \begin{bmatrix} 1 & \cdots & 1 \\ a_2 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_2^{n-2} & \cdots & a_n^{n-2} \end{bmatrix}$$

by inductive hypothesis

$$= \left[\prod_{i=2}^{n} (a_i - a_1)\right] \prod_{2 \le i < j \le n} (a_j - a_i)$$

$$= \prod_{1 \le i < j \le n} (a_j - a_i).$$

Thus, by the principle of mathematical induction, we are done.

Let $V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$ for $n \in \mathbb{N}$.

<u>Claim</u>. $\det(V_n) = \prod_{1 \le i < j \le n} (a_j - a_i)$

<u>Proof</u>. Via induction on n. For $n = 1$, it is clearly true since $\det[1] = 1$. Assume $n \geq 2$ and that $\det(V_{n-1}) = \prod_{1 \le i < j \le n-1} (a_j - a_i)$.

Then,

$$\det(V_n) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$$

$R_n \mapsto R_n - a_1 R_{n-1}$

$R_{n-1} \mapsto R_{n-1} - a_1 R_{n-2}$

$\vdots$

$R_2 \mapsto R_2 - a_1 R_1$

$$\Rightarrow \det(V_n) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & \cdots & a_n - a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-1} - a_1 a_2^{n-2} & \cdots & a_n^{n-1} - a_1 a_n^{n-2} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_2 - a_1 & \cdots & a_n - a_1 \\ \vdots & \ddots & \vdots \\ a_2^{n-1} - a_1 a_2^{n-2} & \cdots & a_n^{n-1} - a_1 a_n^{n-2} \end{bmatrix}$$

Note $a_k^{i} - a_1 a_k^{i-1} = (a_k - a_1)(a_k^{i-1})$

Factoring out $a_k - a_1$ from the k^{th} row gives:

$$= (a_2 - a_1) \cdots (a_n - a_1) \det \begin{bmatrix} 1 & \cdots & 1 \\ a_2 & \ddots & a_n \\ \vdots & & \\ a_2^{n-2} & \cdots & a_n^{n-2} \end{bmatrix}$$

$$= \left[\prod_{1 < j \le n} (a_j - a_1)\right] \left[\prod_{2 \le i < j \le n} (a_j - a_i)\right]$$

$$= \prod_{1 \le i < j \le n} (a_j - a_i). \quad \blacksquare$$

3.7

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3.7 For $n \in \mathbb{N}$, prove that

$$\det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = (-1)^{n(n-1)/2}.$$

$(-1)^{n/2}$ if n even

$\shortparallel$

$\left((-1)^{n-1}\right)^{n/2}$

__Proof.__ Let $M_n = \begin{bmatrix} & & 1 \\ & 0 & 1 & \\ & & \ddots & \\ & 1 & & 0 \\ 1 & & & \end{bmatrix}$.

__Claim.__ $\det(M_n) = (-1)^{\frac{n(n-1)}{2}}$ for $n \geq 1$.

__Proof.__ Via induction on n.

__$n=1$.__ $\det(M_1) = \det[1] = 1 = (-1)^0$, as desired.

Assume $\det(M_{n-1}) = (-1)^{\frac{(n-1)(n-2)}{2}}$ for some $n \geq 2$.

Then, $\det(M_n) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & & \\ \vdots & \vdots & & \cdot^{\cdot^{\cdot}} & \\ \vdots & 1 & & & \\ 1 & 0 & 0 & & 0 \end{bmatrix}$.

n

Expand along first column to get

$$\det(M_n) = (-1)^{n+1} \cdot \det(M_{n-1})$$

$$= (-1)^{n+1} \cdot (-1)^{\frac{(n-1)(n-2)}{2}}$$

$$= 1 \cdot (-1)^{n-1} \cdot (-1)^{\frac{(n-1)(n-2)}{2}}$$

$$= (-1)^{\frac{n(n-1)}{2}}.$$

3.8

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$D_n = (-1)^{n+1} n$

$= -(-1)^n \cdot (n-1) \frac{n}{n-1}$

$= -\frac{n}{n-1} D_{n-1}$

3.8 For $n \in \mathbb{N}$, prove that

$$D_n = \det \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 2 & 3 & \dots & n-1 & n \\ 3 & 3 & 3 & \dots & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & n-1 & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{bmatrix} = (-1)^{n+1} n.$$

$R_n - \frac{n}{n-1} R_{n-1}$

Take n common from last row to get

$$D_n = n \det \begin{bmatrix} 1 & 2 & 3 \dots & n-1 & n \\ 2 & 2 & 3 \dots & n-1 & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & n-1 \dots & n-1 & n \\ 1 & 1 & 1 \dots & 1 & 1 \end{bmatrix}$$

$R_{n-1} - (n-1) R_n$; $R_{n-2} - (n-2) R_n$; ..., $R_1 - R_n$

$$= n \det \begin{bmatrix} 0 & 1 & 2 & \dots & n-2 & n-1 \\ 0 & 0 & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

Expand along first column

$$= n (-1)^{n+1} \det \begin{bmatrix} 1 & 2 & \dots & n-1 \\ & 1 & \dots & n-2 \\ & 0 & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

upper triangular

$$= (-1)^{n+1} \cdot n \cdot 1^{n-1}$$

$$= (-1)^{n+1} n.$$

3.9

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3.9 Find rank **A** using determinants, where **A** is

$$\text{(i)} \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}.$$

Verify by transforming **A** to a REF.

$A \in \mathbb{R}^{m \times n}$

$\operatorname{rank}(A) = \max\{ r : \exists \text{ an } r \times r \text{ submatrix with } \det \neq 0 \}.$

If $\operatorname{rank}(A) = 2$, then $\exists$ a 2×2 submatrix with det $\neq 0$
but every 3×3 submatrix has det $= 0$.

(i) We check for 3×3 submatrices.
There is only one such. We have

$$\det \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} = (-2)(15) + (-3)(10)$$
$$= -60 \neq 0.$$

Thus, rank $= 3$.

Verification: REF
$$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} -3 & 5 & 0 \\ 2 & 0 & 5 \\ 0 & 2 & -3 \end{bmatrix}$$

$R_2 + \frac{2}{3} R_1$

$$\begin{bmatrix} -3 & 5 & 0 \\ 0 & 10/3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$$

$R_3 - \frac{3}{5} R_2$

$$\begin{bmatrix} -3 & 5 & 0 \\ 0 & 10/3 & 5 \\ 0 & 0 & -6 \end{bmatrix} \rightarrow \text{REF}$$

rank $= 3$

2. $$\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}$$

The largest possible square submatrix is of size 2×2. There are three such.

① $\det\begin{bmatrix} 4 & 3 \\ -8 & -6 \end{bmatrix} = -24 + 24 = 0$

② $\det\begin{bmatrix} 4 & 3 \\ 16 & 12 \end{bmatrix} = 48 - 48 = 0$

③ $\det\begin{bmatrix} -8 & -6 \\ 16 & 12 \end{bmatrix} = -144 + 144 = 0.$

Thus, rank < 2. — (I)

On the other hand, consider the 1×1 submatrix $[4]$. $\det[4] = 4 \neq 0$. Thus, rank ≥ 1. — (II)

By (I) and (II), rank $= 1$. ☒

Verify via REF :

$$\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}$$

$R_2 + 2R_1$

$R_3 - 4R_2$

$$\begin{bmatrix} 4 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{REF}$$

rank $= 1$

4.1

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> 4.1 Find the value(s) of α for which Cramer's rule is applicable. For the remaining value(s) of α, find the number of solutions, if any.
>
> $$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 20 \\ x & + & 3y & + & z & = & 13 \\ x & + & 6y & + & \alpha z & = & \alpha. \end{array}$$

Consider the co-efficient matrix $C_\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \alpha \end{bmatrix}$.

Recall: Cramer's rule is applicable iff co-efficient matrix is invertible.

Let us find the values of α for which C_α is invertible.

Recall that: let $A \in \mathbb{R}^{n \times n}$.

Then, A is invertible $\Leftrightarrow \det(A) \neq 0$.

Here, $\det(C_\alpha) = \det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \alpha \end{bmatrix}$

$R_2 - R_1$
$R_3 - R_1$

$= \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 4 & \alpha - 3 \end{bmatrix}$

$= \det \begin{bmatrix} 1 & -2 \\ 4 & \alpha - 3 \end{bmatrix}$

$= (\alpha - 3) + 8 = \alpha + 5$

Thus, Cramer's rule is applicable iff $\alpha \neq -5$.

For $\alpha = -5$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right]$$

$R_2 - R_1, \quad R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & -8 & -25 \end{array}\right]$$

$R_3 - 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Thus, the system is inconsistent, i.e., no solutions. ∎

4.2

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4.2 Find the cofactor matrix $\mathbf{C}$ of the matrix $\mathbf{A}$, and verify $\mathbf{C}^{\mathsf{T}}\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^{\mathsf{T}}$. If $\det \mathbf{A} \neq 0$, find $\mathbf{A}^{-1}$, where $\mathbf{A}$ is

$$\text{(i)} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{(ii)} \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \text{(iii)} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

(i) $M_{11} = d$; $M_{12} = c$; $M_{21} = b$; $M_{22} = a$

Thus, $C = \begin{bmatrix} (-1)^2 d & (-1)^3 c \\ (-1)^3 b & (-1)^4 a \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now, $AC^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$

$= (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $C^TA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da-bc & 0 \\ 0 & -cb+ad \end{bmatrix}$

$= (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $\det(A) = ad-bc$, we are done.

Moreover, $\det(A) \neq 0$ iff $ad \neq bc$ and then we have

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(ii) $\begin{bmatrix} 0 & 9 & 5 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 & 1.4 \end{bmatrix}$

(ii)

$$A = \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \cdot 4 \\ (-1)(-10) & 0 & 0 \\ 0 & (-1)(-10) & 1 \cdot (-18) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 4 \\ 10 & 0 & 0 \\ 0 & 10 & -18 \end{bmatrix}$$

$$\det(A) = (-2) \cdot (-10) = 20$$

$$AC^T = \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= 20 \cdot I_3.$$

work it out! (verify.)

Thus, $\frac{C^T}{20}$ is a right inverse of A. Thus, we must also have $\frac{C^T}{20} A = I$ or $C^T A = 20 I_3$.

(iii) $$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \rightarrow H_3 \quad \text{(Hilbert matrix)}$$

In general, H_n is defined similarly. Moreover, H_n is invertible $\forall n \in \mathbb{N}$ and $H_n^{-1} \in \mathbb{Z}^{n \times n}$.

$$\det(H_n) = \frac{C_n^4}{C_{2n}}, \text{ where}$$

$$C_n = \prod_{i=1}^{n-1} i!.$$

$$\det(H_3) = \frac{C_3^4}{C_6} = \frac{1}{2160}$$

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$\frac{1}{15} - \frac{1}{16} = \frac{1}{15 \times 16}$$

$$\frac{1}{10} - \frac{1}{12} = \frac{1}{60}$$

$$\frac{1}{4} - \frac{1}{6} = \frac{2}{24} = \frac{1}{12}$$

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$C = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$$

$$C^T = C.$$

$$AC^T = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2160 & 0 & 0 \\ 0 & 1/2160 & 0 \\ 0 & 0 & 1/2160 \end{bmatrix}$$

$$C^T A = \frac{1}{2160} I_3.$$

Thus, $A^{-1} = 2160\, C^T = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$. ☒