(Extra)^2 Questions for MA 106

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These are questions that came out of some discussions.
In the following, $\mathbb{F}$ denotes an arbitrary field. You may read this to get an introduction to fields.
Or may assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. (Although your answers then may not work for a general field.)

1. A **nonempty** subset $J \subset \mathbb{F}^{n \times n}$ is said to be a *two-sided ideal* if it has the following properties:

   (a) (Closed under addition) For all $A, B \in J$, we have $A + B \in J$,

   (b) (Absorption) For all $A \in J$ and $C \in \mathbb{F}^{n \times n}$, we have $AC, CA \in J$.

   Show that the (two-sided) ideals of $\mathbb{F}^{n \times n}$ are precisely $\{O\}$ and $\mathbb{F}^{n \times n}$.

2. Let $A \in \mathbb{F}^{n \times n}$ be such that $Ay = y$ for all $y \in \mathbb{F}^{n \times 1}$. Show that $A = I$.
   HIDDEN: Consider $y = e_k$ for $k \in \{1, \ldots, n\}$.

3. Suppose $A \in \mathbb{R}^{2 \times 2}$ is such that $x^\top A x = 0$ for all $x \in \mathbb{R}^{2 \times 1}$. Is it necessary that $A = O$?
   HIDDEN: No. Interpret $x^\top A x$ as $\langle Ax, x \rangle$.

4. Let $P \in \mathbb{R}^{n \times n}$ be invertible and let $A = P^\top P$.
   Show that if $x \in \mathbb{R}^{n \times 1}$, then $x^\top A x = 0 \iff x = 0$.

5. Let $A \in \mathbb{F}^{n \times n}$ be arbitrary. Show that

   (a) A can be written as a sum of two invertible matrices, and

   (b) A can be written as a sum of two non-invertible matrices.