Extra Questions for MA 106

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0 Notation

$\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\mathbb{R}$ denotes the set of real numbers.

The existence of all the above sets will be assumed.
$M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with real entries.
$M_n(\mathbb{C})$ denotes the set of all $n \times n$ matrices with complex entries.
$M_n(\mathbb{F})$ denotes the set of all $n \times n$ matrices with entries from an arbitrary field $\mathbb{F}$. If you're not familiar with fields, you may assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
Whenever either of the above three sets is written, it will be assumed that $n \in \mathbb{N}$.

$A \subset B$ will be written to denote that A is a(n improper) subset of B. In particular, $\{1\} \subset \{1\}$ is a true statement. So is $\{1\} \subset \{1, 2\}$.
$A \subsetneq B$ will be written to denote that A is a proper subset of B. In particular, $\{1\} \subsetneq \{1\}$ is not a true statement. However, $\{1\} \subsetneq \{1, 2\}$ is.
$\mathcal{N}(A)$ denotes the null-space of the matrix A.
If $f : X \to Y$ is a function and $S \subset X$, we define $f(S) := \{y \in Y : \exists s \in S(f(s) = y)\} = \{f(s) : s \in S\}$.

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1 Standard

1. Let V and W be vector spaces over a field $\mathbb{F}$.
   Show that if $T : V \to W$ is a linear transformation, then T is one-one if and only if $\mathcal{N}(T) = \{0\}$.

2. Suppose A is an $m \times n$ matrix and b is an $m \times 1$ column matrix.
   Let $x_0 \in \mathbb{R}^n$ be a particular $n \times 1$ matrix such that $Ax_0 = b$.
   Let $S = \{x \in \mathbb{R}^n : Ax = b\}$, that is the set of all solutions to the equation $Ax = b$.
   Show that $S = x_0 + \mathcal{N}(A)$.

   Notation: $x_0 + \mathcal{N}(A) = \{x_0 + x : x \in \mathcal{N}(A)\}$.

3. Let U be a subspace of a finite dimensional vector space V.
   Show that $\dim U \leq \dim V$.

4. Show that $\mathbb{R}$ and $\mathbb{C}$ are vector spaces over $\mathbb{R}$ when the addition and scalar multiplication have their usual definitions.
   What is the dimension of each?

5. Show that $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ as well as $\mathbb{R}$ for each $n \in \mathbb{N}$. (The addition and multiplication are to be taken as the standard ones.)
   What is the dimension in each case?

6. Show that $\mathbb{R}$ is a vector space over $\mathbb{Q}$ when the addition and scalar multiplication have their usual definitions.
   Show that no finite subset of $\mathbb{R}$ can be a basis of this vector space.

7. Let A be a 7×3 matrix and B be a 9×4 matrix.
   Show that there exists a 3×9 matrix $X \neq O$ such that $AXB = O$.

8. A subset S of $\mathbb{R}^n$ $(n \in \mathbb{N})$ is said to be *convex* if the following is true:

$$\forall \mathbf{x}, \mathbf{y} \in S \quad \forall t \in [0,1] : t\mathbf{x} + (1-t)\mathbf{y} \in S$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $S \subset \mathbb{R}^n$ be convex. Show that $T(S) \subset \mathbb{R}^m$ is also convex.
Note $T(S) := \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in S\} = \{T(\mathbf{x}) : \mathbf{x} \in S\}$.

9. Let A be an $m \times n$ real matrix and let B be an $n \times p$ real matrix.
Show that $\operatorname{rank} AB \le \operatorname{rank} A$. Hence or otherwise, show that $\operatorname{rank} AB \le \operatorname{rank} B$.
Give examples to show that each of these equalities may or may not be achieved.

10. Let A be an $m \times n$ real matrix and let B be an $n \times m$ real matrix.
Show that it is not necessary that $\operatorname{rank} AB = \operatorname{rank} BA$.

11. Let $A, B \in M_n(\mathbb{F})$. Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

12. Let $A, B \in M_n(\mathbb{F})$ with B invertible. Show that $\operatorname{rank} AB = \operatorname{rank} A = \operatorname{rank} BA$.

2 Algebra

1. Let $GL_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : \det M \ne 0\}$, the subset of invertible matrices of $M_n(\mathbb{R})$.
Show the following:

   (a) For every $A, B, C \in GL_n(\mathbb{R})$, the following is true: $A(BC) = (AB)C$.
   (b) $\exists E \in GL_n(\mathbb{R})$ such that $AE = EA = A$ for all $A \in GL_n(\mathbb{R})$.
   (c) For every $A \in GL_n(\mathbb{R})$, there exists $B \in GL_n(\mathbb{R})$ such that $AB = BA = E$.

   (Yes, these questions are easier than they may look at first glance.)

2. Let $E \in M_n(\mathbb{R})$ be such that $EA = A = AE$ for all $A \in M_n(\mathbb{R})$. Show that $E = I_n$.
(That is, there is a unique identity.)

3. Find all $C \in M_2(\mathbb{R})$ such that $MC = CM$ for all $M \in M_2(\mathbb{R})$.
(That is, find all matrices that commute with every other matrix.)

4. Let S be a set and let $\cdot : S \times S \to S$ be a function.
For the sake of efficient notation, let us write $a \cdot b$ to denote $\cdot(a, b)$.
Suppose that $\cdot$ is associative. That is, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$.
An element $e \in S$ is said to be an identity if $a \cdot e = e \cdot a = a$ for each $a \in S$. (Note that two elements need not commute in general.)

   (a) Show that if an identity exists, then it must be unique.
   (b) Suppose such an identity e exists.
   Let $a, b, c \in S$ be such that $a \cdot b = c \cdot a = e$.
   The element b is called a right inverse of a and the element c is called a left inverse of a. Show that $b = c$.
   (c) Conclude that if an element has both a right and left inverse, then they must unique (and same).

(d) Give an example of S and $\cdot$ to show that existence of a right inverse does not imply the existence of a left inverse.
(e) Give an example to show that an element may have infinitely many distinct left inverses. (Note that in this case, a right inverse cannot exist.)

5. Let $A, B \in M_n(\mathbb{R})$. Note that matrix multiplication is associative and there does exist an identity for this.
Using the above exercise, it would be easy to conclude that if $AB = I$, then $BA = I$. However, we are still left to show that A actually does have a left inverse.
(The above exercise shows that it is not true for general operations.)

Using the interpretation of matrices representing linear transforms, show that this is indeed true.
(Note that there do exist linear transforms between infinite dimensional spaces that have only a one-sided inverse, so this exercise may not be as trivial after all.)

6. A matrix $A \in M_n(\mathbb{F})$ is said to be nilpotent if there exists $m \in \mathbb{N}$ such that $A^m = O$.

(a) Show that if N is nilpotent, then $\det(N) = 0$.

(b) Show that $\det(N) = 0 \;\not\Longrightarrow\; N$ is nilpotent.

(c) Show that if N and M are nilpotent, then $N + M$ need not be.

(d) Show that if N and M are nilpotent and commute with each other, then $N + M$ is nilpotent.

(e) Show that if N is nilpotent, then $I - N$ is invertible.
(Hint: Considering the case that $N^2 = O$ may give an idea for the general case.)

(f) Show that if N is nilpotent, then the only eigenvalue of N is 0.

(g) Show that if N is nilpotent, then $N^n = O$. (Note that n is the size of this matrix.)

7. Let V be a vector space. Let U and W be subspaces of V.

(a) Show that $U \cap W$ is a subspace of V.
(b) Show that $U \cup W$ need not be a subspace of V.
(c) Show that if $U \cup W$ is a subspace of V, then $U \subset W$ or $W \subset U$ (or both).

8. Let V be a vector space. Let U and W be subspaces of V.
Define $U + W := \{v \in V : v = u + w \text{ for some } u \in U,\ w \in W\}$.
Show that $U + W$ is a subspace of V.

9. Let V be a vector space. Let U and W be subspaces of V.
As we saw earlier, it is not necessary that $U \cup W$ is a subspace of V.
Show that

(a) $U + W$ contains $U \cup W$.
(Note that we have already shown that $U + W$ is a subspace.)
(b) If X is a subspace containing $U \cup W$, then X contains $U + W$.

The above results show that $U+W$ is the "smallest" subspace containing $U \cup W$.

3 Linear independence and spanning

1. Let $n \in \mathbb{N} \setminus \{1\}$ and $A \in M_n(\mathbb{R})$.
   Let u and v be non-zero vectors in $\mathbb{R}^n$ such that $Au = u$ and $Av = -v$.
   Prove that u and v are linearly independent.

2. Let $\{v_1, v_2\}$ be a linearly independent subset of a vector space V and let $w \in V$ be such that $w \notin \operatorname{span}\{v_1, v_2\}$.
   Show that $\{v_1 + w, v_2 + w\}$ is linearly independent.

3. Give an example of three vectors x, y, z such that $\{x, y, z\}$ is not linearly independent but the sets formed by taking two of them at a time are.

4. Let V be a vector space over $\mathbb{F}$.
   Let $B \subset V$ be such that every $v \in V$ can be written as a linear combination of elements of B in a unique way.
   Show that B is a basis for V.

5. Let V and W be vector spaces over some field $\mathbb{F}$.
   Let S be a (finite) linearly independent subset of W and let $T : V \to W$ be a linear map.
   Show that if T is injective (one-to-one), then $T(S)$ is a linearly independent subset of W.

6. Let V and W be vector spaces over some field $\mathbb{F}$.
   Let S be a (finite) spanning subset of W and let $T : V \to W$ be a linear map.
   Show that if T is surjective (onto), then $T(S)$ is a spanning subset of W.

7. Let V be a vector space and let $S \subset V$. Show that if W is a subspace of V containing S, then W contains $\operatorname{span} S$.

8. Let V a vector space over $\mathbb{F}$ and let S be a linearly independent subset of V.
   Let $w \in V$. Show that $S \cup \{w\}$ is linearly independent if and only if $w \notin \operatorname{span} S$.

9. Let $n \in \mathbb{N} \setminus \{1\}$ and let $k \in \mathbb{N}$ with $1 \leq k < n$.
   Let $S = \{v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^n$.
   Suppose that S is linearly independent.
   Let $v \in \mathbb{R}^n$ and $\epsilon > 0$ be given.
   Show that there exists $w \in \mathbb{R}^n$ such that

   (a) $S \cup \{w\}$ is linearly independent and
   (b) $\|w - v\| < \epsilon$.
   (Or alternately, each coordinate of w differs from the corresponding coordinate of v by at most ϵ.)

10. Let $n \in \mathbb{N} \setminus \{1\}$. Let $\epsilon > 0$ be given.
    Suppose $A = (a_{ij})$ is an $n \times n$ matrix with real entries.
    Show that there exists an *invertible* $n \times n$ matrix $B = (b_{ij})$ such that $|a_{ij} - b_{ij}| < \epsilon$ for each $i, j \in \{1, \ldots, n\}$.

Bonus. Show that you can even find a B which differs from A by no more than n entries. Find a matrix A for which you do require n changes.

11. Let (A_n) be a sequence of $m \times m$ real matrices where $m \in \mathbb{N}$.
    We say that the sequence (A_n) converges to *the* matrix $A \in M_m(\mathbb{R})$, if each coordinate sequence converges to the corresponding coordinate of A in the usual sense. (It is clear that such a limit will be unique.)
    Show that - given any $M \in M_m(\mathbb{R})$, there exists a sequence (M_n) of $m \times m$ real matrices such that M is the limit of (M_n) and M_n is invertible for each $n \in \mathbb{N}$.

    In other words, invertible matrices are *dense* in $M_m(\mathbb{R})$.

12. Let $n \in \mathbb{N}$.
    Show that $\mathbb{R}^n$ (as a vector space over $\mathbb{R}$) has infinitely many distinct bases.

13. Let V be a finite dimensional vector space.
    Let S be any linearly independent subset of V.
    Show that there exists $T \subset V$ such that $S \cap T = \varnothing$ and $S \cup T$ is a basis for V.
    (Make sure you cover all cases, including $S = \varnothing$ and $|S| = \dim V$.)

14. In continuation with the above exercise, show that this T need not be unique.

15. Let V be a vector space. Show that $S = \{v\} \subset V$ is linearly independent if and only if v is not the zero vector.

16. Show that $\varnothing$ is a linearly independent subset of any vector space.

17. Let V be a vector space. Let S be a (finite) subset of V.
    Show that the following two statements are equivalent:

    (a) S is not linearly independent.

    (b) $\exists v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$.

    (Note that if S is infinite, then we say that S is linearly independent if every finite subset of S is linearly independent. Similarly, $\operatorname{span} S$ is defined as the set of all **finite** linear combinations of elements of S.)

18. Let V be a vector space and $S \subset V$.
    Suppose S is not linearly independent.
    By the previous exercise, there exists $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$.
    Show that $\operatorname{span} S = \operatorname{span}(S \setminus \{v\})$.

19. Let V be a finite dimensional vector space.
    Let S be any finite subset of V such that $\operatorname{span} S = V$.
    Show that there exists $T \subset S$ such that T is a basis for V.
    Do this again without the hypothesis that S is finite.

20. In continuation with the above exercise, show that this T need not be unique.

21. Let V be an $n-$dimensional vector space over a field $\mathbb{F}$ for some $n \in \mathbb{N}$.
Let $S \subset V$ with $|S| = n$.
Show that the following are equivalent:
    (a) S is linearly independent.
    (b) $\operatorname{span} S = V$.
    (c) S is a basis for V.
22. Let $I \subset J \subset \mathbb{R}$.
Let f and g be real valued functions defined on J.
Show that if f and g are linearly independent on I, then they are linearly independent on J.
Show that the converse is not true.
23. Let V be a vector space. Let $S \subset V$.
Define $\mathcal{S} = \{W \subset V : S \subset W,\ W \text{ is a subspace of } V\}$, that is, $\mathcal{S}$ is the set of all subspaces of V containing S.
Show that $\mathcal{S} \neq \varnothing$. Show that $\operatorname{span} S = \bigcap_{W \in \mathcal{S}} W$.
Recall that if $\bigcap_{W \in \mathcal{S}} W := \{w : w \in W \text{ for } \textbf{all } W \in \mathcal{S}\}$, if $\mathcal{S} \neq \varnothing$.

4 Linear maps

1. Let V and W be vector spaces over $\mathbb{F}$.
Let $T : V \to W$ be a bijection.
Show that T^{-1} is linear if and only if T is.
2. Let V be a finite dimensional vector space over $\mathbb{F}$ and let W be a subspace of V.
Show that there exists a linear transform $T : V \to V$ such that $\mathcal{N}(T) = W$.
3. Let V be a vector space over $\mathbb{F}$.
Let $T : V \to V$ be a linear map such that $T \circ T = T$.
Show that $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$.
4. Let V and W be vector spaces over $\mathbb{Q}$.
Let $T : V \to W$ be a function satisfying $T(x + y) = T(x) + T(y)$ for all $x, y \in V$.
Show that T is a linear map.
(That is, show that $T(qx) = qT(x)$ for all $x \in V$ and $q \in \mathbb{Q}$.)
5. Let V and W be vector spaces over some field $\mathbb{F}$.
Let $a \in \mathbb{F} \setminus \{1\}$.
Let $T : V \to W$ be a linear map.
Suppose that $T(x) = T(ax)$ for all $x \in V$.
Show that $T(x) = 0$ for all $x \in V$.
6. Let V be a finite dimensional vector space and W_1, W_2 and W be subspaces of V, such that $\dim W_1 = \dim W_2$, $W_1 \cap W = \{0\} = W_2 \cap W$, and $W_1 + W = V = W_2 + W$.

Prove that there exists a linear map $f : W_1 \to W$ such that $W_2 = \{w_1 + f(w_1) : w_1 \in W_1\}$.

7. Let $n \in \mathbb{N}$.
   Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be linear maps such that given any $x \in \mathbb{R}^n$, $f(x) = 0 \implies g(x) = 0$.
   Show that there exists $\lambda \in \mathbb{R}$ such that $g = \lambda f$.

5 Eigenvalues and diagonalisation

1. Let $M \in M_n(\mathbb{C})$.
   Suppose M has eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ with algebraic multiplicities $m_1, \ldots, m_k$.
   Prove that $\lambda_1^{m_1} \cdots \lambda^{m_k} = \det M$.

   Is the result true if we replace $\mathbb{C}$ by $\mathbb{R}$?

2. Let $A \in M_n(\mathbb{C})$ have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| > |\lambda_i|$ for each $i \in \{2, \ldots, n\}$.
   Prove that for "most" vectors $v \in \mathbb{C}^n$, the sequence $v_k := \lambda_1^{-k} A^k v$ converges to a vector w which is an eigenvector of A with eigenvalue λ_1.
   Determine precisely the values of v for which this happens.

3. Suppose $A \in M_n(\mathbb{R})$ with eigenvalues $0, 1, \ldots, n-1$ corresponding to eigenvectors $v_0, v_1, \ldots, v_{n-1}$, respectively.
   Show that $Ax = v_0$ has no solution for $x \in \mathbb{R}^n$.

4. Let M be the matrix corresponding to a (standard 9×9) solved Sudoku.
   Show that 45 is an eigenvalue of M.

5. Let $D \in M_n(\mathbb{R})$.
   Let $f(t) = \det(D - tI_n)$.
   Note that for a given polynomial $p(t) = a_0 + a_1 t + \cdots + a_m t^m$, we define $p(A) = a_0 I_n + a_1 A + \cdots + a_m A^m$ for every matrix $A \in M_n(\mathbb{R})$.

   (a) Show that $f(D) = O$, if D is a diagonal matrix.

   (b) Show that $f(D) = O$, if D is similar to a diagonal matrix.

   (No, letting $t = D$ is not a correct proof.)

   Remark. It is true that $f(D) = O$ for any $D \in M_n(\mathbb{R})$. That is, every (square) matrix satisfies its own characteristic matrix. This is known as the Cayley-Hamilton theorem.

6. Let $A \in M_n(\mathbb{R})$. Show that A and A^T have the same eigenvalues (including algebraic and geometric multiplicities).
   Show that they need not have the same eigenvectors.

7. Let $M \in M_2(\mathbb{C})$.
   Then, M has 2 (possibly same) eigenvalues λ and μ. (Why?)
   Note that if $\lambda \neq \mu$, then M is diagonalisable. (How?)

Suppose $\mu = \lambda$.

Now, it not necessary that M is diagonalisable. For example, show that $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalisable.

Now, we show that even if M is not diagonalisable, not all hope is lost, there is still a canonical similar matrix.

More precisely, we will show that if M is not diagonalisable, them M is similar to $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

(Note that the converse is also true, that is, if M is similar to such a matrix, then M is not diagonalisable.)

Show the following:

(a) Show that $M - \lambda I \neq O$.

(b) Show that $(M - \lambda I)^2 = O$.
(Hint: The only root of the characteristic polynomial is λ.)

(c) Show that there exists $v \in \mathbb{C}^2$ such that $(M - \lambda I)v \neq 0$.

(d) Let $w = (M - \lambda I)v$. Show that w is an eigenvector of M.

(e) Show that v and w are linearly independent. Conclude that $\{w, v\}$ is a(n ordered) basis for $\mathbb{C}^2$.

(f) Represent M using this new basis. Observe that this is J. Thus, $M \sim J$.

Thus, we can conclude by saying that a 2×2 complex matrix with eigenvalues μ and λ is similar to exactly one of the following matrices:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \qquad \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}$$

Where the latter situation is possible only if $\lambda = \mu$.

In general, we can do this for higher order matrices as well. The interested reader can look up Jordan canonical form to read more about it.

Note that the assumption of being in $\mathbb{C}$ was necessary to ensure that 2 roots do exist. For example, the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ cannot be put into any of the above two forms using only real matrices for the similarity transform. However, if you start with a matrix whose characteristic polynomial can be factored into linear factors, then the above result follows.

8. Let $A \in M_n(\mathbb{C})$.
Show that A is invertible if and only if 0 is not an eigenvalue of A.
In this case, show that if λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}.

9. Show that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

10. Consider the Fibonacci sequence $(F_n)_{n \in \mathbb{N} \cup \{0\}}$ which is defined as follows:
$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Find an appropriate sized matrix A such that:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

Show that

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}.$$

Find the eigenvalues of A. Note that these are two distinct eigenvalues and thus, A is diagonalisable. Find an invertible P such that $P^{-1}AP = D$ where D is a diagonal matrix.
Hence, find an explicit formula for F_n.

6 Similarity

1. Let A and B be similar matrices. Show that $\operatorname{rank} A = \operatorname{rank} B$.
2. Let $A, B \in M_n(\mathbb{F})$ be similar matrices.
   Show that A is invertible if and only if B is invertible.
   Do this with and without appealing to an argument using determinants.
3. Prove that if $A, B \in M_n(\mathbb{F})$ and if A is nonsingular, then AB is similar to BA.
4. Let $A, P \in M_n(\mathbb{R})$ with P invertible. Show that $\operatorname{trace}(A) = \operatorname{trace}(PAP^{-1})$.
5. Show that similar matrices have the same characteristic polynomial. Show that non-similar matrices can also have the same characteristic polynomial.

7 Some other vector spaces

1. Let $\mathcal{C}[0,1] := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous.}\}$
   Given two functions $f, g \in \mathcal{C}[0,1]$, define $f + g : [0,1] \to \mathbb{R}$ as $(f+g)(x) = f(x) + g(x)$.
   Given $f \in \mathcal{C}[0,1]$ and $r \in \mathbb{R}$, define $r \cdot f : [0,1] \to \mathbb{R}$ as $(r \cdot f)(x) = rf(x)$.

   Show that if $r \in \mathbb{R}$ and $f, g \in \mathcal{C}[0,1]$, then $f + g \in \mathcal{C}[0,1]$ and $r \cdot f \in \mathcal{C}[0,1]$.
   Show that $\mathcal{C}[0,1]$ is a vector space over $\mathbb{R}$ with the above operations.
2. Given $f, g \in \mathcal{C}[0,1]$, define $\langle f, g \rangle$ as:

   $$\langle f, g \rangle := \int_0^1 f(x)g(x)\mathrm{d}x.$$

   Show that the above defines an inner product.
   (Note: It is crucial that $\mathcal{C}[0,1]$ contains continuous functions. Why?)
3. Let $\mathbb{R}[x]$ denote the set of polynomials with real coefficients.
   Show that:

   (a) $\mathbb{R}[x]$ is vector space over $\mathbb{R}$. (Addition and scalar multiplication is defined in the usual sense.)

(b) No finite set S can be a basis for $\mathbb{R}[x]$.

(c) The set $S = \{1, x, x^2, \ldots\}$ is a basis for $\mathbb{R}[x]$.

(d) The set $S' = \{1, 1+x, 1+x+x^2, \ldots\}$ is a basis for $\mathbb{R}[x]$.

(e) The set $S'' = \{x, x^2, x^3, \ldots\}$ is linearly independent but not a basis for $\mathbb{R}[x]$.

(f) The set $S''' = \{1, 2, x, x^2, x^3, \ldots\}$ is spanning but not a basis for $\mathbb{R}[x]$.

What difference (between finite dimensional and infinite dimensional spaces) is reflected by the last two parts?

4. Define $f, g : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := \begin{cases} x^3 & x > 0 \\ 0 & x \le 0 \end{cases} \Bigg| \quad g(x) := \begin{cases} x^3 & x < 0 \\ 0 & x \ge 0 \end{cases}$$

Show that f and g are linearly independent. (Think of them being elements of the vector space consisting of all functions from $\mathbb{R}$ to $\mathbb{R}$. The zero-vector would be the function which is identically zero on all of $\mathbb{R}$.)
Prove that f and g are differentiable everywhere.

Define $W : \mathbb{R} \to \mathbb{R}$ as:

$$W(x) := \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Show that $W(x) = 0$ for all $x \in \mathbb{R}$.

Thus, the Wronskian of two functions can be zero everywhere even if the two functions are linearly independent.

5. Let $\mathbb{R}^\infty$ denote the set of all those real sequences that are eventually 0. Show that it is a vector space over $\mathbb{R}$ where addition and scalar multiplication have their usual meanings.
Show that this vector space has a countable basis by exhibiting one.

6. Let $\mathbb{R}^\mathbb{N}$ denote the vector space of all real spaces. (Vector space over $\mathbb{R}$.)
Define a sequence $(x_n) \in \mathbb{R}^\mathbb{N}$ to be Fibonacci-like if $x_{n+2} = x_{n+1} + x_n$ for all $n \in \mathbb{N}$.
Let $F \subset \mathbb{R}^\mathbb{N}$ be the set of all Fibonacci-like sequences. Show that F is a finite dimensional subspace of $\mathbb{R}^\mathbb{N}$.
Find a basis for F.

8 Inner product

1. Let V be an inner product space. Let $v \in V$.
Show that if $\langle w, v \rangle = 0$ for all $w \in V$, then $v = 0$.
Show that the converse is true too.

2. Let $A \in M_n(\mathbb{C})$. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then $\bar{\lambda}$ is an eigenvalue of A^*.

3. Let $A \in M_n(\mathbb{C})$. We say that A is normal if $AA^* = A^*A$.
   Let A be normal.
   Show the following:

   (a) $\langle Av, Av \rangle = \langle A^*v, A^*v \rangle$ for all $v \in \mathbb{C}^n$.

   (b) $A - \lambda I$ is normal for any $\lambda \in \mathbb{C}$.

   (c) $Av = \lambda v \implies A^*v = \bar{\lambda} v$ where $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$.
   Thus, adjoint of a matrix has the same eigenvectors corresponding to the eigenvalue's conjugate.

4. Let V and W be vector spaces over $\mathbb{C}$. Let $T : V \to W$ be a linear map.
   Let $U \subset V$.
   Show that if $T(U) \subset U$, then $T^*(U^\perp) \subset U^\perp$.

5. Let $A \in M_n(\mathbb{C})$ be a normal matrix.
   Let $\lambda, \mu \in \mathbb{C}$. Suppose $x, y \in \mathbb{C}^n$ are such that $Ax = \lambda x$ and $Ay = \mu y$.
   Show that $\lambda \neq \mu \implies \langle x, y \rangle = 0$.
   Thus, if x and y are eigenvectors corresponding to distinct eigenvalues, then not only are they linearly independent but also orthogonal.
   (Note that if x and y are eigenvectors, then $x \neq 0 \neq y$.)

6. Let $A \in M_n(\mathbb{C})$ be a normal matrix.
   Let λ be an eigenvalue of A.
   Show that $\mathcal{N}(A - \lambda I) = \mathcal{N}(A - \lambda I)^2$.

7. Let V be an inner product space. Let S be a subset of V.
   Define $W := \{w \in V : \langle w, s \rangle = 0 \text{ for every } s \in S\}$.
   Show that W is a subspace of V.

8. Let V be an inner product space of dimension $n \in \mathbb{N}$. Let S be a subspace of V.
   Define $W := \{w \in V : \langle w, s \rangle = 0 \text{ for every } s \in S\}$.
   Show that $\dim S + \dim W = n$.

9. Suppose A is an $m \times n$ real matrix.
   Show that $\mathcal{N}(A^T A) = \mathcal{N}(A)$.
   (Hint: Recall the natural inner product on $\mathbb{R}^n$.)

9 Miscellaneous

1. Find a matrix $M \in M_2(\mathbb{C})$ such that $X^2 = M$ has no solution for $X \in M_2(\mathbb{C})$.

2. Suppose you are given that a function $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree of at most n.
   Show that f cannot be uniquely determined by specifying its value at n distinct real numbers.
   Show that f can be uniquely determined by specifying its value at $n+1$ distinct real numbers.
   Find this polynomial.

3. Let A be an $m \times n$ real matrix and B be an $m \times 1$ real matrix.
Show that:
   (a) If $Ax = B$ has two distinct solutions for $x \in \mathbb{R}^n$, then there are infinitely many such solutions.
   (b) If $Ax = B$ has a solution for $x \in \mathbb{C}^n \setminus \mathbb{R}^n$, then there are infinitely many solutions for $x \in \mathbb{R}^n$.
4. Let A be an $m \times n$ real matrix and b be an $m \times 1$ real matrix. Suppose that $Ax = 0$ has infinitely many solutions.
Prove or disprove: $Ax = b$ has infinitely many solutions.
5. Let A be an $m \times n$ real matrix and let B be an $n \times m$ real matrix.
Show that $I_m - AB$ is invertible if and only if $I_n - BA$ is.
6. Let V be a vector space over $\mathbb{R}$ containing two distinct elements. Show that V has infinitely many distinct elements.
7. Suppose $A \in M_n(\mathbb{R})$. If $Ax = 0 \iff x = 0$, find the value of $\operatorname{rank} AA^T + \operatorname{rank} A^T A$.
(Hint: Recall the natural inner product on $\mathbb{R}^n$.)
8. Let $u = (u_1, \ldots, u_n)^T$ and $v = (v_1, \ldots, v_n)^T$ be nonzero vectors in $\mathbb{R}^n$, where $n > 1$.
Define $(b_{ij}) = B \in M_n(\mathbb{R})$ according to the rule $b_{ij} = u_i v_j$ for $i, j \in \{1, \ldots, n\}$.
Find all the eigenvalues of B along with the geometric and algebraic multiplicities.
Find the rank of B.
Show that u is an eigenvector of B.
9. Let $A, B \in M_n(\mathbb{R})$.
Show that $\operatorname{rank} A + \operatorname{rank} B \leq \operatorname{rank} AB + n$.
10. Suppose $A \in M_2(\mathbb{R}) \setminus \{I, -I\}$ satisfies $A^2 = I$.
Show that the characteristic polynomial of A factors into linear factors.
Show that $\operatorname{trace} A = 0$ and $\det A = -1$.
11. Consider the following subset S of $\mathbb{R}^5$:

$$S := \left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \right\}.$$

Note that S is not orthogonal with respect to the standard inner product of $\mathbb{R}^5$.
Find a subset $G \subset \mathbb{R}^5$ such that:
   (a) G is orthogonal with respect to the standard inner product of $\mathbb{R}^5$.
   (b) $\operatorname{span} G = \operatorname{span} S$.
   (c) G has 6 elements.

Or show that no such G exists.

12. Let $A \in M_n(\mathbb{C})$.
    Show that if A is unitary, then $\det A \in \{1, -1\}$.
    Show that the converse is not true.

13. Give an example of $A \in M_2(\mathbb{C})$ such that $X^2 \neq A$ for any $X \in M_2(\mathbb{C})$.

14. Let $A \in M_3(\mathbb{R})$ and $0 \neq y \in \mathbb{R}^3$.
    Suppose the equation $Ax = y$ has the following three solutions for x :

    (a) $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}$ (c) $\begin{pmatrix} 7 \\ 24 \\ 25 \end{pmatrix}$

    Find the complete set of solutions for $Ax = y$.

15. Let $M \in M_n(\mathbb{R})$ such that $M^2 = O$. Show that $\operatorname{rank} M \leq n/2$.
    Let n be an even natural number. Show that the equality can be achieved.
    (Obviously it can't be achieved if n is odd.)

16. Let V be a finite dimensional vector space. Let U and W be subspaces of V.
    Define $U + W := \{v \in V : v = u + w \text{ for some } u \in U,\ w \in W\}$.
    Prove the following result:

    $$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

    (Hint: Consider a basis B of $U \cap W$ and extend that to bases B_1 and B_2 of U and W and use that to obtain a basis for $U + W$.)

17. Show that $AB - BA \neq I_n$ for every $A, B \in M_n(\mathbb{R})$.

18. Let V and W be finite dimensional vector spaces over some field $\mathbb{F}$.
    Show that V and W are isomorphic if and only if they have the same dimension.

19. Show the the following sets of functions are linearly independent on $(0, \infty)$.

    (a) $\{\sin x, \sin 2x, \ldots, \sin nx\}$ for any $n \in \mathbb{N}$.
    (b) $\{e^{a_1 x}, e^{a_2 x}, \ldots, e^{a_n x}\}$, where $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ are distinct real numbers.
    (c) $\{1, x, \ldots, x^n\}$ for any $n \in \mathbb{N}$.
    (d) $\{e^{mx}, xe^{mx}, \ldots, x^n e^{mx}\}$ for any $n \in \mathbb{N}$ and $m \in \mathbb{R}$.
    (e) $\{x^k, x^k(\log x), \ldots, x^k(\log x)^n\}$ for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

    These sets of functions will show up in your next course, MA 108.

20. Consider the following $n \times n$ complex matrix C given as

    $$C := \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

That is, the first column of C is the column vector $c = [c_0\ c_1\ \ldots\ c_{n-1}]^T$ and the remaining columns are cyclic permutations of the vector c with offset equal to the column index.
Find all the eigenvalues and eigenvectors of C.
Is C always diagonalisable? (That is, for any choice of the first vector c.)
What is the determinant of C?
You may give your answer in terms of the following polynomial: $f(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$.

(Hint: The *root* of the problem lies in *unity*.)

21. Show that the equation $X^2 = -I_n$ has a solution for $X \in M_n(\mathbb{R})$ if and only if n is even.
(Hint: For the even case, think in terms of linear functions.)

22. Let $A \in M_2(\mathbb{R})$ be an orthogonal matrix.
Is it true that there exists $\theta \in [0, 2\pi)$ such that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$?

10 Big Boi Stuff

1. Let $\mathbb{F}$ be a finite field with q elements.
Find the cardinality of $GL_n(\mathbb{F}) = \{M \in M_n(\mathbb{F}) : \det M \neq 0\}$.

2. Let $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
Define $+ : S^1 \times S^1 \to S^1$ and $\cdot : \mathbb{R} \times S^1 \to S^1$ in a way such that $(S, +, \cdot)$ is a vector field over $\mathbb{R}$.

3. Let V be a vector space over some field $\mathbb{F}$. Assume that V is not the zero vector space.
Let $\mathcal{I}$ be the set of all linearly independent subsets of V. (Note that the elements of $\mathcal{I}$ are not vectors but rather, sets of vectors. In other words, $\mathcal{I}$ is not a subset of V but rather a subset of the power set of V.)

We shall call a subset $\mathcal{C}$ of $\mathcal{I}$ to be a *chain* if it has the following property:

- Given $A,\ B \in \mathcal{C}$, either $A \subset B$ or $B \subset A$.

We shall call an element $M \in \mathcal{I}$ to be *maximal* if there exists no $M' \in \mathcal{I}$ such that $M \subsetneq M'$. (That is, there is no proper superset of M in $\mathcal{I}$.)
Show that the following two statements are true:

(a) $\mathcal{I}$ is nonempty.

(b) Given any chain $\mathcal{C}$ of $\mathcal{I}$, there exists some $U \in \mathcal{I}$ such that $C \subset U$ for every $C \in \mathcal{C}$.

(c) If there exists a maximal element M, then M must be a basis for V.

(Note that fancy capital letters ($\mathcal{I}$ and $\mathcal{C}$) are used for set of sets of vectors and normal capital letters (A, B, C, M) are used for sets of vectors, they are subsets of V.)

Remark. Zorn's Lemma states that (a) and (b) imply the existence of a maximal element

M. Thus, what you have shown is that every non-zero vector space has a basis. By convention, we say that $\varnothing$ is a basis for the zero space and thus, every vector space has a basis.
In fact, Zorn's lemma is a bit more general but that would require the introduction of some more concepts like that of a *partially ordered set*.
The reader is encouraged to read up more about *Zorn's Lemma* or its equivalent, the infamous *Axiom of Choice*.

4. Show that no countable subset of $\mathbb{R}$ can be a basis of $\mathbb{R}$, considered as a vector space over $\mathbb{Q}$.

5. Given an (infinite) basis B of $\mathbb{R}$ over $\mathbb{Q}$, construct a basis B' over $\mathbb{C}$ over $\mathbb{Q}$.
Show that there exists a bijection $\varphi : B \to B'$.
Using this, conclude that $\mathbb{C}$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{Q}$.
(Using the above, also show that $\mathbb{R}$ and $\mathbb{C}$ are isomorphic as additive groups.)

6. Give an example of a vector space V (over $\mathbb{R}$) and a linear function $T : V \to V$ such that null space of T is non-zero but the image is still V.
(Hint: Show that V cannot be finite dimensional.)

7. Let $M \in M_n(\mathbb{R})$ be a diagonalisable matrix with k eigenvalues. Show that M cannot satisfy a polynomial with degree strictly less than k.
Show that it does satisfy a polynomial of degree k.
(Hint: Show that every eigenvalue *must* be a root of the polynomial.)

8. Let $(G_1, +, \cdot)$ and $(G_2, +, \cdot)$ be vector spaces over $\mathbb{Q}$.
Show that G_1 and G_2 are isomorphic as groups if and only if they are isomorphic as vector spaces (over $\mathbb{Q}$).

9. Let G_1 and G_2 be additive groups which are also vector spaces over some field $\mathbb{F}$.
Show that G_1 and G_2 are isomorphic as groups if they are isomorphic as vector spaces over $\mathbb{F}$.
Show that the converse need not be true.
(Converse does hold in the special case that $\mathbb{F} = \mathbb{Q}$.)

10. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$.
Assume that there exists a basis B for this vector space.
Using this basis, construct a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x + y) = f(x) + f(y)$ but $f(x)$ cannot be written as rx for any $r \in \mathbb{R}$.
In fact, one can come up with extremely bizarre, wildly discontinuous and unbounded everywhere functions.

    Remark. Recall the question from MA105 where you showed that requiring continuity at just 0 is enough to prove that the function is of the form rx. This shows that any other function must be discontinuous.
The assumption that there does exist a basis can be proven via the so-called *Axiom of Choice*, or its equivalent, Zorn's Lemma. Axiom of Choice is a seemingly innocent looking statement but it has quite unintuitive results such as the existence of these extremely badly behaved functions. The reader is encouraged to read more about this.

11. Let B be a basis for the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Let $a \in \mathbb{R} \setminus \{1\}$.

Show that there exists $x \in B$ such that $ax \notin B$.

12. Let $\mathbb{R}^{\mathbb{N}}$ denote the set of real sequences. Show that it is a vector space over $\mathbb{R}$ where addition and scalar multiplication have their usual meanings.
Show that the following set S is linearly independent:
$$S = \{(1, x, x^2, x^3, \dots) \in \mathbb{R}^{\mathbb{N}} : x \in \mathbb{R}\}.$$
Note that S has the same cardinality as that of $\mathbb{R}$.
Conclude that this vector space cannot have a countable basis.
Note that $\mathbb{R}^{\mathbb{N}}$ has the same cardinality as that of $\mathbb{R}$. (How?)
Assuming Axiom of Choice, $\mathbb{R}^{\mathbb{N}}$ has a basis and by the above exercise, its cardinality must also be that of $\mathbb{R}$. (How?)

13. Suppose that you are given a recursively defined sequence $(s_n)_{n\in\mathbb{N}}$ of the following form.
$s_n = a_1 s_{n-1} + a_2 s_{n-2} + a_3 s_{n-3}$ for $n \geq 4$ along with initial values $s_1,\ s_2$, and s_3. a_1, a_2, a_3 are some fixed constants.
Represent the above in matrix form like question 10 of Section 5.
Suppose that you are asked to evaluate the last 8 digits of $s_{10^{18}}$. How would you write a program which could that for you? Can you write an algorithm that performs around $\lfloor \log_2(10^{18}) \rfloor$ matrix multiplications?

Remark. There was nothing special about the fact that the recursion went only till 3 terms. This can be extended easily for higher lengths. However, matrix multiplication itself will become more time consuming.

14. Let V be a vector space over $\mathbb{Q}$ of dimension 3. Let $u, v, w \in V$ be vectors such that $u \neq 0$ and there exists a linear map $T : V \to V$ such that $T(u) = v, T(v) = w$, and $T(w) = u + v$.
Show that $\{u, v, w\}$ is a basis of V.
Show that if we replace $\mathbb{Q}$ with $\mathbb{R}$, then the result need not hold.

11 Matrix Exponentiation

For $A \in M_n(\mathbb{C})$, define e^A as:
$$e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots.$$
It can be shown that the series on the right converges for every $A \in M_n(\mathbb{C})$.
In general, it isn't easy to calculate the entries of e^A. In particular, they are not the entries individually exponentiated. This should be no surprise given the nature of matrix multiplication.

1. Show that if D is a diagonal matrix, then e^D is also diagonal. Moreover, compute the diagonal entries explicitly in terms of the original entries.

2. Let P be an invertible $n \times n$ complex matrix.
Show that $Pe^A P^{-1} = e^{PAP^{-1}}$.

3. Calculate e^A for $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

4. Note that, in general, it is not true that $e^{A+B} = e^A e^B$.
   However, it is true in the case that $AB = BA$.
   Using this or otherwise, calculate e^A for $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

5. Find matrices A and B such that $e^{A+B} \neq e^A e^B$.

6. Show that $e^{\operatorname{trace} A} = \det(e^A)$.
   You may use the following fact - for any A, there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that PAP^{-1} is a triangular matrix.