Week 1 (10-03-2021)

10 March 2021 13:30

- Matrices $\rightarrow$ Multiply them.

(1) $\rightarrow$ $\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$=$

$a_1 b_1 + \cdots + a_n b_n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} \\ \\ \end{bmatrix}_{m\times 1}$$

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$A_1, \ldots, A_m \in \mathbb{R}^{1\times n}$

$$Ab = \begin{bmatrix} A_1 b \\ \vdots \\ A_m b \end{bmatrix}$$

$A \in \mathbb{R}^{m\times n}$ $\qquad$ $B \in \mathbb{R}^{n\times p}$

$B = \begin{bmatrix} b_1 & \cdots & b_p \end{bmatrix}$; $b_1, \ldots, b_p \in \mathbb{R}^{n\times 1}$

$$AB = \begin{bmatrix} Ab_1 & \cdots & Ab_p \end{bmatrix} \in \mathbb{R}^{m\times p}$$

$\in \mathbb{R}^{m\times 1}$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$

We say that B is an inverse of A if
$AB = I = BA$.

Fact. (Will see later) $AB = I \Rightarrow BA = I$
(this was not clear, a priori)

→ Functions $f, g : X \longrightarrow X$. ($X \neq \emptyset$ is some set.)
If $(f \circ g)(x) = x \quad \forall x \in X$,
is it necessary that $(g \circ f)(x) = x \quad \forall x \in X$?

No. Find example.

$Ax = b$. —— $(*)$ $\quad A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^{n\times 1}$, $b \in \mathbb{R}^{m\times 1}$
If A is upper triangular, it is easy by back-substitution. (Whether consistent or not is also clear.)

Idea: Do operations on both A and b to get something as above.

→ If $Ax_0 = b$, i.e., x_0 is a particular sol^n,
and $S = \{x \in \mathbb{R}^{n\times 1} : Ax = 0\}$.
Then, all solutions of $(*)$ are precisely of the form $x_0 + s$ for some $s \in S$.

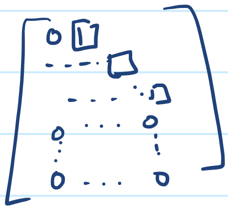
Idea: Row echelon form (REF)

(i) All zero rows at bottom. (Possibly none.)

(No zero row can be above a nonzero row.)

first $\neq 0$ element from left

(2) Pivots should be strictly from left to right as you go from top to bottom.



Week 2

17 March 2021 09:42

Outline:

1. Recall REF. n variables, r pivots ⇒ (n - r) free variables
2. **Ax** = **0** has ***only*** the zero solution ⇔ n = r ← every column has a pivot
3. EROs
4. GEM
5. **Ax** = **0** has ***only*** the zero solution ⇔ any REF of **A** has n non-zero rows
6. Inverse
7. **Ax** = **0** has ***only*** the zero solution ⇔ A is invertible
8. Let **A**, **B** ∈ ℝ^{n × n}. **AB** = **I** ⇔ **BA** = **I**
9. RCF. REF + pivots are 1 + the entries above the pivots are 0s
10. **A** can be transformed to **I** via EROs ⇔ **A** is invertible
11. GJM
12. Linear (in)dependence
13. Row rank
14. Given n column vectors, make a matrix with those as columns and find its row rank r. We know r ≤ n. The vectors are linearly independent ⇔ r = n.
15. EROs don't change row rank. Thus, **A** and REF(**A**) have the same row rank.
16. If **A'** is in REF, then row-rank(**A'**) = number-of-non-zero-rows(**A'**).

$A \in \mathbb{R}^{m \times n}$ (items 1–5)

$A \in \mathbb{R}^{n \times n}$ (items 6–7)

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3. EROs → Elementary <u>Row</u> operations

Type I : Interchange two rows

Type II : Add a scalar multiple of R_i to R_j where $i \neq j$.

Type III : Multiplying a row with a non-zero scalar

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4. GEM → Gauss Elimination Method

↳ Algo to convert a matrix into an REF using EROs.

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5. # non-zero rows of A' = # pivots of A' (A' is in REF)

5 follows from 2.

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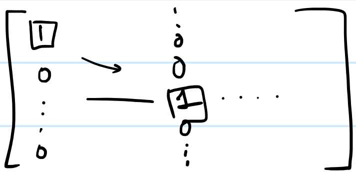
6. If $A \in \mathbb{R}^{n \times n}$, then $B \in \mathbb{R}^{n \times n}$ is ~~an~~ the inverse of A if $AB = I = BA$.

9. RCF if (i) it is REF

(ii) it has all pivots as 1

(iii) everything above pivot are also 0



RCF is unique. (REF need not be.)

10. A is invertible $\iff$ RCF of A is I

$\iff$ A can be transformed to I via EROs

11. Take $A \in \mathbb{R}^{n \times n}$.

Make the augmented matrix

$$[A \mid I]$$

performs EROs to make A into its RCF (so same operations on I as well)

$$[A' \mid B]$$

If A is inv., then $A' = I$ and $B = A^{-1}$.

If A is not inv., then $A' \neq I$.

11. Linear dependence

- $S \subset \mathbb{R}^{n \times 1}$ (or $\mathbb{R}^{1 \times n}$)

(possibly infinite)

- S is linearly dependent if there exist (distinct) $v_1, \dots, v_s \in S$ and $\alpha_1, \dots, \alpha_s \in \mathbb{R}$, not all zero such that

$$\alpha_1 v_1 + \cdots + \alpha_s v_s = \mathbf{0}$$

in $\mathbb{R}^{n \times 1}$ (or $\mathbb{R}^{1 \times n}$)

- For example, if $\alpha_1 \neq 0$ and $n \geq 2$, then

$$v_1 = -\frac{1}{\alpha_1}(\alpha_2 v_2 + \cdots + a_s v_s).$$

- if $\mathbf{0} \in S$, then S is lin. dep.
  Take $n=1$, $v_1 = \mathbf{0}$, $\alpha_1 = 1 \neq 0$.
  Then, $1 \cdot \mathbf{0} = \mathbf{0}$.

- If $S = \{v\}$ and $v \neq \mathbf{0}$. Then, S is not lin. dep.

- if $S = \emptyset$, then S is not lin. dep.

- S is linearly independent if S is not linearly dependent.

- $\emptyset$ is lin. indep. $\{v\}$ is lin indep iff $v \neq \mathbf{0}$.

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13. row-rank (A) = maximum no. of lin. indep rows of A.

if $A = \mathbf{0}$, then row-rank$(A) = 0$.

$$\text{row-rank} \begin{bmatrix} [1 & 1] \\ 2 & 2 \end{bmatrix} = 1$$

this is lin indep

$\{[1 \quad 1], [2 \quad 2]\}$ is lin. dep.

15. In general, row-rank (A) = row-rank (A')
where A' is an REF of A.