TUTORIAL 1
MATRIX OPERATIONS AND GAUSS ELIMINATION
MA 106 (LINEAR ALGEBRA)
SPRING 2020

1. Tutorial Problems

(1) A matrix is called *symmetric* if $A^t = A$ and *skew-symmetric* if $A^t = -A$. Let A and B be symmetric matrices of same size. Show that AB is a symmetric matrix iff $AB = BA$. Show also that any square matrix can be written as sum of symmetric and skew symmetric matrices in a unique way.

(2) A square matrix A is said to be *nilpotent* if $A^n = 0$ for some $n \geq 1$. Let A, B be nilpotent matrices of the same order.
(i) Show by an example that $A + B, AB$ need not be nilpotent.
(ii) However, prove that this is the case if A and B commute with each other (i.e., if $AB = BA$). (Hint: In this case, show that the binomial theorem holds for expansion of $(A + B)^n$.)

(3) If A and B are square matrices, show that $I - AB$ is invertible iff $I - BA$ is invertible. [Hint: Start from $B(I - AB) = (I - BA)B$.]

(4) Solve the following system of linear equations in the unknowns $x_1, \dots, x_5$ by GEM

$$\begin{array}{ccccl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(5) Find all solutions of the equation $x + y + 2z - u = 3$.

2. Practice Problems

(1) Let A, B be matrices of type $n \times n$. Show that $A = B$ iff $\mathbf{e}_i^t A \mathbf{e}_j = \mathbf{e}_i^t B \mathbf{e}_j$ for all i, j where $\mathbf{e}_i$ are standard column vectors $= (0, \dots, 0, 1, 0, \dots, 0)^t$.

(2) Given square matrices $A_1, \dots, A_n$ of the same size, prove that:
(i) $(A_1 \cdots A_n)^t = A_n^t \cdots A_1^t$.
(ii) $(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1}$.
(iii) $(A^t)^{-1} = (A^{-1})^t$.
(iv) $((A_1 \cdots A_n)^{-1})^t = (A_1^{-1})^t \cdots (A_n^{-1})^t$.

(v) If $A_1A_2 = A_2A_1$ then $A_1^kA_2^l = A_2^lA_1^k$ for positive integers k, l.

(3) For any θ, let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Prove that for any two real numbers α, β, we have, $R_{\alpha+\beta} = R_\alpha R_\beta$.

(4) The (complex) *conjugate* $\overline{A}$, of a $m \times n$ matrix $A = (a_{ij})$ with complex entries is defined as the matrix $((\overline{a_{ij}}))$. Prove that $\overline{A+B} = \overline{A} + \overline{B}$, $\overline{\alpha A} = \overline{\alpha}\overline{A}$, and $\overline{AB} = \overline{A}\,\overline{B}$, where the complex matrices A, B are of appropriate sizes and α is a complex number.

(5) The *trace* of a square matrix $A = (a_{ij})$ is defined as the sum of its diagonal elements, i.e., $tr\,A := \sum_i a_{ii}$. Prove that if A, B are square matrices of the same order and α, β are scalars then
(i) $tr\,(\alpha A + \beta B) = \alpha\,tr\,(A) + \beta\,tr\,(B)$; (ii) $tr(AB) = tr(BA)$;
(iii) If A is invertible, then $tr\,(ABA^{-1}) = tr\,(B)$.

(6) A square matrix A is called *nilpotent* if $A^m = 0$ for some positive integer m. Show that a $n \times n$ matrix $A = (a_{ij})$ in which $a_{ij} = 0$ if $i \geq j$ is nilpotent.

(7) Show that the sum and the product of any two diagonal matrices of the same order are diagonal.

(8) The *conjugate transpose* or (*Hermitian*) *adjoint* A^* of a complex $m \times n$ matrix A is defined as the transpose of its conjugate (or equivalently, the conjugate of its transpose). Prove that the properties of the adjoint are analogous to that of the transpose e.g. $(A+B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$. Note, however, that $(\alpha A)^* = \overline{\alpha}A^*$.

(9) A square matrix A is called *Hermitian* if $A^* = A$ and *skew-Hermitian* if $A^* = -A$. Show that every square matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.

(10) If A and B are $n \times n$ Hermitian matrices and α and β are any real numbers, show that $C = \alpha A + \beta B$ is a Hermitian matrix.

(11) If A and B are skew Hermitian matrices and α, β are real numbers, show that $\alpha A + \beta B$ is skew Hermitian. What happens if α and β are allowed to be complex numbers?

(12) A square matrix A over $\mathbb{C}$ is called **unitary** if $AA^* = Id = A^*A$. In addition, if A has real entries then it is called **orthogonal.** (This is the same as saying $AA^T = Id = A^TA$. Show that A, B are $n \times n$ unitary (orthogonal) matrices then AB is unitary (orthogonal).

(13) A square matrix A with complex entries is called **normal** if $AA^* = A^*A$. Determine all 2×2 normal matrices. In particular, show that there are normal matrices which are neither unitary, Hermitian, skew-Hermitian, symmetric, nor skew-symmetric.

(14) Given a polynomial $p(x)$ with real or complex coefficients, and a square matrix A, let p(A) denote the matrix obtained by 'substituting' A for the variable x. Thus if $p(x) = a_0 + a_1 x \cdots + a_k x^k$ then $p(A) = a_0 I_n + a_1 A + \cdots A_k A^k$. Given two polynomials p, q, show that (i) $(p+q)(A) = p(A) + q(A)$ and (ii) $(pq)(A) = p(A)q(A)$.

(15) Show that any diagonal matrix D whose only possible entries are 0 and 1 is idempotent, i.e., $D^2 = D$.

(16) If $A^2 = A$, prove that $(A+I)^n = I + (2^n - 1)A$.

(17) The matrix $A = \begin{bmatrix} a & i \\ i & b \end{bmatrix}$, where $i^2 = -1, a = \frac{1}{2}(1+\sqrt{5})$, and $b = \frac{1}{2}(1-\sqrt{5})$, has the property $A^2 = A$. Describe completely all 2×2 matrices A with complex entries such that $A^2 = A$.

(18) A square matrix $A = ((a_{ij}))$ is called *upper triangular* if $a_{ij} = 0$ for all $j < i$. A *lower triangular* matrix is defined similarly (or equivalently, as the transpose of an upper triangular matrix). Prove that the sum as well as the product of two upper triangular matrices (of equal orders) is upper triangular.

(19) Show that $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$, for all $\lambda \in \mathbb{R}, n \geq 1$.

(20) An $m \times n$ matrix, all whose entries are 1 is often denoted by $J_{m\times n}$ or simply by J if m, n are understood. Let $A = J_{n\times n}$ and $B = J_{n\times 1}$. Prove that $AB = nB, A^2B = n^2B, \ldots$ and, in general, for any polynomial $p(x) = a_0 + a_1 x + \ldots + a_r x^r, p(A)B = p(n)B$.

(21) If $I_n + A$ is invertible show that $(I_n + A)^{-1}$ and $I_n - A$ commute.

(22) A *Markov* (or *stochastic*) matrix is a square matrix $((a_{ij}))$ such that $0 \leq a_{ij} \leq 1$, and $\sum_{j=1}^n a_{ij} = 1$, for all $i = 1, 2 \ldots, n$. Prove that the product of two Markov matrices is a Markov matrix.

(23) Let $N_{n\times n}$ be an upper triangular matrix with diagonal entries zero. Show that $(I_n + N)^{-1} = I - N + N^2 - + \cdots + (-1)^{n-1} N^{n-1}$.

(24) Show that if N is any nilpotent matrix then $I - N$ is invertible.

(25) Find the inverses of the permutation matrices:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(26) Let $N = \{1, 2, \ldots, n\}$. By a permutation on n letters we mean a bijective mapping $\sigma : N \longrightarrow N$. Given a permutation $\sigma : N \longrightarrow N$ define the permutation matrix P_σ to be the $n \times n$ matrix $((p_{ij}))$ where

$$p_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $P_{\sigma \circ \tau} = P_\tau P_\sigma$. Deduce that all permutation matrices are invertible and

$$P_\sigma^{-1} = P_{\sigma^{-1}} = P_\sigma^T.$$

(27) Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM

$$\text{(i)} \quad \begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = 8 \end{array} \qquad \text{(ii)} \quad \begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

$$\text{(iii)} \quad \begin{array}{rrrrl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array} \qquad \text{(iv)} \quad \begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

(28) Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

(29) Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, E_2, \ldots, E_n$ such that $E_n \ldots E_1 A$ is either the identity matrix or its bottom row is zero.

(30) The n^{th} **Hilbert matrix** H_n is defined as the $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is $\dfrac{1}{i + j - 1}$. Obtain H_3^{-1} by Gauss-Jordan elimination Method (GJEM).

(31) Find the inverse of the following matrix using elementary row-operations:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}.$$

(32) Compute the last row of the inverse of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -17 & 8 & -5 \\ 0 & -5 & -2 & 1 \end{bmatrix}.$$