Tutorial 5

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19th February 2020

1. Recall the definition of an eigenvalue.
   We want to find a real number λ such that $T_H(v) = \lambda v$ has a non-zero solution for v.
   Using the definition, note that $Hv = (I - 2uu^t)v = v - 2u(u^t v) = v - 2\langle u, v\rangle u$.
   (Recall that $\langle u, v\rangle = u^t v$.)
   Now, we make the following two observations for $v \neq 0$:

   (a) If $\langle v, u\rangle = 0$, then v is an eigenvector with eigenvalue $\lambda = 1$.
   (b) If $v \in \text{span}\{u\}$ then v is an eigenvector with eigenvalue $\lambda = -1$.

   Note that the dimension of the vector space $W = \{v \in \mathbb{R}^n : \langle v, u\rangle = 0\}$ is at least $n - 1$.
   (Since $\{u\}$ is a linearly independent set, you may extend it to a basis $\{u, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$. Then, use Gram Schmidt to orthogonalise the set and get $n - 1$ linearly independent vectors which are orthogonal to u. Note that $\{u\}$ is linearly independent since $\|u\| = 1$ and hence, $u \neq 0$.)
   Moreover, note that $\dim \text{span}\{u\} = 1$.   (Why?)
   Thus, we already have n linearly independent eigenvectors corresponding to these two eigenvalues. Clearly, there can't be more eigenvalues. Conclude.

2. (a) Note that $P^{-1}(\lambda I - A)P = \lambda I - A'$ and thus, $\det(\lambda I - A) = \det(\lambda I - A')$. Conclude.

   (b) First we note that if $\mathbf{v} \neq \mathbf{0}$, then $P^{-1}\mathbf{v} \neq \mathbf{0}$. (Making this note is important as eigenvectors are nonzero by definition.)
   Observe that $A\mathbf{v} = \lambda\mathbf{v} \implies PA'P^{-1}\mathbf{v} = \lambda v \implies A'(P^{-1}\mathbf{v}) = \lambda P^{-1}\mathbf{v}$.
   Conclude.

3. (a)
$$
\begin{aligned}
& 0 \text{ is an eigenvalue of } A \\
\iff & \det(0I - A) = 0 \\
\iff & \det(A) = 0 \\
\iff & A \text{ is not invertible} \\
\iff & A \text{ is singular}
\end{aligned}
$$

   (b) Note that $(\lambda I - A)^t = \lambda I - A^t$. Use the fact that $\det M = \det M^t$ and conclude.

   (c) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $x = [1\ 0]^t$. Conclude.

4. First part.
   Suppose that T did have an eigenvalue. Let $\lambda \in \mathbb{R}$ be an eigenvalue. Then, by hypothesis, there exists a nonzero $f \in C^\infty[0, 1]$ such that $T(f) = \lambda f$.
   Thus, $\lambda f(x) = \int_0^x f(t)dt$ for all $x \in [0, 1]$. Note that we may use Fundamental Theorem of Calculus (Part I) and conclude $\lambda f'(x) = f(x)$ for $x \in (0, 1)$. (Note the open interval.)
   However, since both sides are continuous functions, we have that $\lambda f'(x) = f(x)$ for all $x \in [0, 1]$.
   At this point, we note that λ cannot be zero, since f is not identically zero.

Thus, $f'(x) = \frac{1}{\lambda} f(x)$.

We may now multiply with $e^{-x/\lambda}$ to write:

$e^{x/\lambda} f'(x) - e^{-x/\lambda} \frac{1}{\lambda} f(x) = 0$ or $(e^{-x/\lambda} f(x))' = 0$.

Thus, $e^{-x/\lambda} f(x) = c$ for some $c \in \mathbb{R}$. (In fact, $c = f(0)$.)

However, now one may check that $f(x) = ce^{-x/\lambda}$ does *not* actually satisfy $T(f) = \lambda f$.

Second part. Let $\lambda \in \mathbb{R}$. We show that there exists some nonzero f such that $T(f) = \lambda f$.

Note that $f(x) = e^{\lambda x}$ works.

(Note that I went via that long route of multiplying with $\exp\left(-\frac{x}{\lambda}\right)$ since I didn't want to divide both sides by f.)

5. (a) The given conic can be rewritten as

$$X^T A X = 0, \qquad (*)$$

where $X = [x_1 x_2]^T$ and $A = \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}$.

One may calculate the eigenvalues of A to be 25 and 50 with corresponding unit vectors as $\frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\frac{1}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, respectively. (You may also take the negative of these.)

Thus, setting $U = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ gives us that $U^* A U = \begin{bmatrix} 25 & 0 \\ 0 & 50 \end{bmatrix}$.

Set $X = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $(*)$ to get the transformed equation as $25y_1^2 + 50y_2^2 = 0$.

Thus, the conic represented is a point.

(b) The given conic can be rewritten as

$$X^T A X = 40, \qquad (*)$$

where $X = [x_1 x_2]^T$ and $A = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$.

One may calculate the eigenvalues of A to be 0 and 10 with corresponding unit vectors as $\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, respectively. (You may also take the negative of these.)

Thus, setting $U = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$ gives us that $U^* A U = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}$.

Set $X = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $(*)$ to get the transformed equation as $0y_1^2 + 10y_2^2 = 40$.

Thus, the conic represented is a pair of parallel lines.

(c) The given conic can be rewritten as

$$X^T A X = 25, \qquad (*)$$

where $X = [x_1 x_2]^T$ and $A = \begin{bmatrix} 91 & -12 \\ -12 & 84 \end{bmatrix}$.

One may calculate the eigenvalues of A to be 75 and 100 with corresponding unit vectors as $\frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\frac{1}{5} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, respectively. (You may also take the negative of these.)

Thus, setting $U = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ gives us that $U^* A U = \begin{bmatrix} 75 & 0 \\ 0 & 100 \end{bmatrix}$.

Set $X = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $(*)$ to get the transformed equation as $75y_1^2 + 100y_2^2 = 25$.

Thus, the conic represented is an ellipse.

(d) The given conic can be rewritten as

$$X^T A X = 10, \qquad (*)$$

where $X = [x_1 x_2]^T$ and $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$.

One may calculate the eigenvalues of A to be -1 and 4 with corresponding unit vectors as $\dfrac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively. (You may also take the negative of these.)

Thus, setting $U = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ gives us that $U^* A U = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

Set $X = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $(*)$ to get the transformed equation as $-y_1^2 + 4y_2^2 = 10$.

Thus, the conic represented is a hyperbola.

That brings us to the end of this tutorial and in turn, this course. Hope you had fun. See you next course.