Tutorial-4

Inner Product Spaces, Gram-Schmidt Process

MA 106 (Linear Algebra)

Spring 2020

1 Tutorial Problems

1. In the vector space $C[1, e]$, define $\langle f,\, g\rangle = \int_1^e \log x f(x)g(x)\, dx$.

   (a) if $f(x) = \sqrt{x}$, compute $\|f\| = \langle f,\, f\rangle^{1/2}$.

   (b) Find a linear polynomial $g(x) = ax + b$ that is orthogonal to $f(x) = 1$.

2. Let $C_0^1[a, b] = \{f \in C^1[a, b] : f(a) = 0\}$. Show that $\langle f,\, g\rangle = \int_a^b f^{'}(x)g^{'}(x)\, dx$ is an inner product on $C_0^1[a, b]$.

3. Orthonormalize the set $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 1)\}$ of vectors in $\mathbb{R}^4$, using the Gram-Schmidt process.

4. Let A be an $n \times n$ square matrix with real (or complex) entries. Show that the following are equivalent.
   (i) A is orthogonal (unitary).
   (ii) A^t (resp. A^*) is orthogonal (unitary).
   (iii) The column vectors of A form an orthonormal set.
   (iv) The row vectors of A form an orthonormal set.

2 Practice Problems

1. On the vector space $C^1[a, b]$ of continuously differentiable real valued functions, examine whether or not $\langle f,\, g\rangle$, defined below is an inner product in each case. Justify your answer.

   (a) $\langle f,\, g\rangle = \int_a^b f^{'}(t)g^{'}(t)\, dt$

(b) $\langle f,\, g\rangle = \int_a^b (f(t)g(t) + f^{'}(t)g^{'}(t))\, dt.$

(c) $\langle f,\, g\rangle = f(b)g(b)$

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space V. Suppose $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$ and $\mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{v}_i \in V$. Prove that

   (a) $\alpha_i = \langle \mathbf{u},\, \mathbf{v}_i\rangle$ and $\beta_i = \langle \mathbf{v},\, \mathbf{v}_i\rangle$ for $i = 1, 2, \ldots, n$.

   (b) $\langle \mathbf{u},\, \mathbf{v}\rangle = \sum_{i=1}^{n} \alpha_i\beta_i$ and $\|\mathbf{u}\| = \left(\sum_{i=1}^{n} \alpha_i^2\right)^{1/2}$.

3. Prove that in a real inner product space V, the following are equivalent:
   (i) $\langle x,\, y\rangle = 0$;
   (ii) $\|x + y\| = \|x - y\|$;
   (iii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
   Describe this geometrically.

4. In the vector space $\mathcal{P}_n[t]$ of all real polynomials of degree $\leq n$, define

$$\langle f,\, g\rangle = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) g\left(\frac{j}{n}\right).$$

   (a) Prove that $\langle f,\, g\rangle$ is an inner product on P_n.

   (b) Compute $\langle f,\, g\rangle$, when $f(t) = t, g(t) = at + b$.

   (c) If $f(t) = t$, find all linear polynomials g orthogonal to f.

5. Write elements of $\mathbb{R}^n$ as column vectors of length n. Let A be an $n \times n$ real matrix and A^t be its transpose. For usual inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, prove that for any two $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\langle A\mathbf{u},\, \mathbf{v}\rangle = \langle \mathbf{u},\, A^t\mathbf{v}\rangle.$$

   State and prove a similar result about $\mathbb{C}^n$.

6. In a real inner product space V, show that for any $x, y \in V$

$$\langle x + y,\, x - y\rangle = 0 \quad \text{iff } \|x\| = \|y\|.$$

7. Orthonormalize the following set of vectors in $\mathbb{R}^4$, using the Gram-Schmidt process.

   (i) $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 1)\}$;
   (ii) $\{(1, -1, 2, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}$.

8. Let V be an inner product space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal set in V.

(a) Show that for all $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{v} \rangle \geq \sum_{i=1}^{n} |\langle \mathbf{v}, \mathbf{v}_i \rangle|^2$, (**Bessel's inequality**)

(b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is also a basis, show $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} |\langle \mathbf{v}, \mathbf{v}_i \rangle|^2$. (**Parseval's identity**)