TSC - QP

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QP

MA 106 Endsem*

Aryaman Maithani

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If nothing is mentioned, assume that similarity and eigenvalues/eigenvectors are being considered over $\mathbb{C}$. The characteristic polynomial of a square matrix A is defined as $p_A(t) = \det(A - tI)$.

1. Let A be a 2×2 real matrix with $\det(A) < 0$. Then,
   (a) A is diagonalisable over $\mathbb{R}$.
   (b) A is not diagonalisable over $\mathbb{R}$.
   (c) The given information is not sufficient to conclude.

complex in conjugate pairs ×

real repeated $\Rightarrow \det(A) \geq 0$ ×

A has 2 real distinct eigen.

wrt alg. mult. { Sum of eigen (in $\mathbb{C}$) = trace; product — " — = det.

2. Let A and B be 2×2 matrices with same eigenvalue(s) with the same geometric and algebraic multiplicities.
   True/False: A and B are similar.

3. Let A be a nonzero square matrix such that $A^k = O$ for some $k \geqslant 2$. Show that A is not diagonalisable.

$\hookrightarrow$ All eigenvalues of A are 0.

If A were diag., then $P^{-1}AP = 0$.

But then $A = POP^{-1} = 0$. Contradiction.

Suppose λ is an eigen. of A.

$Av = \lambda v$ for some $v \neq 0$.

$\vdots$

$A^k v = \lambda^k v$

$A^k = 0$

$\Rightarrow \lambda^k v = 0$ but $v \neq 0$.

Thus, $\lambda^k = 0$.

Thus, $\lambda = 0$.

4. Let A be a 9×7 matrix and B be a 4×3 matrix.
   Show that there exists a nonzero 7×4 matrix X such that $\underbrace{AXB}_{9 \times 3} = O$.

linear system in coordin. of X.

27 homogeneous linear eq's in 28 var

5. Let A and B be $n \times n$ matrices. Consider the following statements.
   (S1) A is similar to B.
   (S2) A and B have the same characteristic polynomial.
   (S3) $\det(A) = \det(B)$.

   Pick the correct options.
   (a) (S1) $\Rightarrow$ (S2) ✓
   (b) (S2) $\Rightarrow$ (S3) ✓
   (c) (S3) $\Rightarrow$ (S1)
   (d) (S1) $\Leftarrow$ (S2)
   (e) (S2) $\Leftarrow$ (S3)
   (f) (S3) $\Leftarrow$ (S1) ✓

$A \sim B$

$\Updownarrow$

$\exists$ inver. P s.t. $A = P^{-1}BP$

(S1) $\Rightarrow$ (S2)

(S2) $\Rightarrow$ (S3)

(S1) $\Rightarrow$ (S3)

$A \sim B \Rightarrow tr(A) = tr(B)$

$\Rightarrow \det(A) = \det(B)$

$A - xI \sim B - xI$

$\Rightarrow$ char poly same.

*Of course, this is not the actual endsem paper.

$A = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} \longrightarrow (x - \lambda)^2$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \not\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Same char poly.

Same det

$\text{rank}(A - \lambda I) = \text{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1$

$g_\lambda = \text{nullity}(A - \lambda I) = 1.$

$m_\lambda = 2.$ A is NOT diagonalisable.

6. Let A and B be square matrices with the same characteristic polynomial. Suppose that for each eigenvalue, the geometric and algebraic multiplicities are the same for A and B.
   True/False: A and B are similar.

7. Let A be a 3×3 matrix with eigenvectors u, v, w corresponding to eigenvalues 0, 1, 2 respectively.
   Show that $Ax = u$ has no solution.

$\{u, v, w\} \longrightarrow$ basis for $\mathbb{R}^3$

Suppose $Ax = u$.

$x = au + bv + cw$.

$Ax = bv + 2cw$

$= u$

$\Rightarrow u = b \cdot v + 2c \cdot w$

Contradiction.

8. Let A be a solved Sudoku interpreted as a 9×9 real matrix. Let $p(t)$ be the characteristic polynomial of A. Show that $p(45) = 0$.

$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an evec with eval 45.

9. Let A be an $n \times n$ matrix with characteristic polynomial $(-1)^n(t-1)(t-2)\cdots(t-n)$. Show that
   $$Ax = \begin{bmatrix} 1 \\ 4 \\ \vdots \\ n^2 \end{bmatrix}$$
   has a solution.

0 is not an eigenval

$\Leftrightarrow A$ is invert.

Thus, $Ax = b$ always has a sol^n.

10. Let A and B be 3×3 polynomials with characteristic polynomial $-t^3 + 6t^2 - 11t + 6$. Are A and B necessarily similar? Yes. Both A & B are diag.

$-(t-1)(t-2)(t-3)$

11. Let A and B be 3×3 matrices with characteristic polynomial $-t(t-1)^2$. Are A and B necessarily similar? No.

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

12. Let A be an $m \times n$ real matrix. Show that $\mathcal{N}(A^{\mathsf{T}}A) = \mathcal{N}(A)$.

$A^TAx = 0$

$\Downarrow$

$x^TA^TAx = 0$

$\Downarrow$

$Ax = 0$

13. Given $A = \begin{bmatrix} 6.5 & -2.5 & 2.5 \\ -2.5 & 6.5 & -2.5 \\ 0 & 0 & 4 \end{bmatrix}$, find a matrix B such that $B^2 = A$.

Check that A is diagon...

Find P s.t.

$P^{-1}AP = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Take $B = P \begin{bmatrix} \sqrt{a} & & \\ & \sqrt{b} & \\ & & \ddots \end{bmatrix} P^{-1}.$

14. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Prove that
   $$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} = \prod_{1 \leqslant i < j \leqslant n} (\lambda_j - \lambda_i).$$

Check Tut 3 from 2021 (bit.ly/ma-106)

15. Let A be an $n \times n$ matrix satisfying $A^2 = A$. Suppose that A is neither the zero matrix nor the identity matrix.
    Choose the correct option(s).
    (a) A must be invertible. ×
    (b) A cannot be invertible. ✓
    (c) The only possible eigenvalues of A are 0 and 1. ✓

Suppose $v \in N(A) \cap C(A)$.

$Av = 0$.

$v = Aw$

$0 = Av = A(Aw) = \ldots$

If A is inv, then $A^2 = A \Rightarrow A = I$.

A contradiction.

Choose the correct option(s).

- ~~(a)~~ A must be invertible. ×
- ✓(b) A cannot be invertible.
- ✓(c) The only possible eigenvalues of A are 0 and 1.
- ~~(d)~~ The null space and column space of A have a nonzero vector in common.

Sup. $v \in N(A) \cap C(A)$; $Av = 0$. $v = Aw$. $0 = Av = A(Aw) = A^2 w = Aw = v$ ✓

If ... then $A^2 = A \Rightarrow A = I$. A contradiction.

Let λ — eval, v → evec.

$Av = \lambda v$, $A^2 v = \lambda^2 v$ $\Rightarrow \lambda v = \lambda^2 v$, $v \neq 0$ $\Rightarrow \lambda = \lambda^2$ $\Rightarrow \lambda \in \{0, 1\}$.

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Take $B = P \begin{bmatrix} \sqrt{a} & & \\ & \sqrt{b} & \\ & & \sqrt{c} \end{bmatrix} P^{-1}$.

$B^2 = P \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} P^{-1} = A$. ✓

Let $x \in \mathbb{R}^n$. $x = (x - Ax) + Ax$.
(↳ 0-eigenspace) (↳ 1-eigenspace)

$A(x - Ax) = 0$

$A(Ax) = Ax$.

16. Show that if A is an $n \times n$ matrix satisfying $A^2 = A$, then A is diagonalisable. ✓
Conclude that if $A^2 = cA$ for some $c \neq 0$, then too A is diagonalisable. → $(A/c)^2 = (A/c)$ → A/c is diagonal. ⇓ A is ...

17. Let A be a matrix such that $A^k = O$ for some $k \geqslant 1$. Show that $I - A$ is invertible.
↳ $I + A + \cdots + A^{k-1}$

18. Let A and B be 4×4 matrices defined by

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Mark the correct option(s).

- ✓(a) Both A and B have the same characteristic polynomial. → $(x-2)^4$
- ✓(b) Both A and B have the same eigenvalues and their geometric multiplicities are also the same.
  $A - 2I = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → REF rank $= 2$, nullity $= 2$. Similarly for B.
- ✓(c) Both A and B have the same eigenvalues and their algebraic multiplicities are also the same.
- ~~(d)~~ A and B are similar.

$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$

$A \sim B$
$\Rightarrow A - 2I \sim B - 2I$
$\Rightarrow (A - 2I)^2 \sim (B - 2I)^2$
$\Rightarrow O \sim (B - 2I)^2 \neq 0$.
Thus, $A \not\sim B$.

19. Consider

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Choose the correct option(s):

- (a) $A\mathbf{x} = \mathbf{u}$ has a solution.
- (b) $\mathbf{v}$ is in the column space of A. ⟺ $Ax = v$ has a sol^n.
- (c) None of the above.

$[A \mid u \mid v]$ → REF. Check!

20. Find the value(s) of k for which the system

$$\begin{aligned} y + 3kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2ky + 12z &= -4 \end{aligned}$$

has no solution.

$\left[\begin{array}{ccc|c} 0 & 1 & 3k & 0 \\ 1 & 2 & 6 & 2 \\ k & 2k & 12 & -4 \end{array}\right]$ → REF check.

21. Let A be an $m \times n$ matrix. Let $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\mathcal{C}(A)$ denote the null space, row space, and column space of A, respectively. Pick the correct option(s).

- ~~(a)~~ $\dim(\mathcal{N}(A)) = \dim(\mathcal{R}(A))$.
- ✓(b) $\dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = n$.
- ✓(c) $\dim(\mathcal{N}(A)) + \dim(\mathcal{C}(A)) = n$.

$\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A)) = \text{rank}(A)$

3

$\mathcal{N}(A) \subseteq \mathbb{R}^n$
$\mathcal{C}(A) \subseteq \mathbb{R}^m$
$\mathcal{R}(A) \subseteq \mathbb{R}^n$

$\begin{bmatrix} — a_1 — \\ \vdots \\ — a_m — \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \cdot x \\ \vdots \\ a_m \cdot x \end{bmatrix} = 0$

- ✓(d) $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are orthogonal.
- ~~(e)~~ $\mathcal{N}(A)$ and $\mathcal{C}(A)$ are orthogonal.

Recall that subspaces $V, W \subseteq \mathbb{R}^n$ are said to be orthogonal if $\langle v, w \rangle = 0$ for all $v \in V$ and all $w \in W$.

22. Let A be a self-adjoint matrix. Show that if $\langle A\mathbf{x}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{C}^n$, then $A = O$. → let λ be an eval of A. v — evec. to λ. $\langle Av, v \rangle = 0$, $\lambda \|v\|^2$ $\Rightarrow \lambda = 0$.
A is diago... since A is Herm. But all evals are 0. $A \sim O$ $\Rightarrow A = O$.
23. Show that if $\|A\mathbf{x}\| = \|A^*\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is a normal matrix.
24. Show that if $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is a unitary matrix. → $B = AA^* - I$. ∴
25. Which of the following matrices are diagonalisable?

$B := AA^* - A^*A$. TS: $B = 0$. But $B^* = B$. Check: $\langle Bx, x \rangle = 0$. Use #22.

$(t-3)(t-1)(t+1)(t-1/2)$

$$\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

× ✓ ×

$(t-4)^2$ but $g_4 = 1$.

26. Find necessary and sufficient conditions on a, b, c for the following matrix to be diagonalisable:

$$\begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}. \rightarrow -(t-2)^2(t-1).$$

$b + ac = 0$

Only need to care about 2.
Need: nullity$(A - 2I) = 2$ ⟺ rank$(A - 2I) = 1$

$A - 2I = \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$ ⇝ $\begin{bmatrix} 0 & -1 & c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$

27. Let $\lambda \in \mathbb{C}$. Show that λ is an eigenvalue of A iff $\bar{\lambda}$ is an eigenvalue of A^*.

$(A - \lambda I)x = 0$; check $\|(A^* - \bar{\lambda} I)x\| = 0$.

28. Use Gram-Schmidt to orthonormalise the ordered subset

$$\left(\begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 3 & 0 & 0 & 1 \end{bmatrix}^T \right)$$

and obtain an ordered orthonormal set $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Also, find $\mathbf{v}_4$ such that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is an orthonormal basis for $\mathbb{R}^4$. → Tut 5 '21 (bit.ly/ma-106)
Express $\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$ as a linear combination of these four basis vectors.

29. Write down the symmetric matrix A such that the quadric

$$7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy = 36$$

can be expressed as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 36.$$

$A = \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$

$$7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy = 36$$

can be expressed as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 36.$$

Find an orthogonal matrix Q such that $Q^T AQ$ is diagonal.

30. Let A be an $n \times n$ normal matrix and $\lambda \in \mathbb{C}$.
Show that $A - \lambda I$ is a normal matrix.
Show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^*\mathbf{x} = \bar{\lambda}\mathbf{x}$.

(27 is a special case.)

4

$$A = \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$$

$\rightsquigarrow$

$P_A(t) \to \lambda_1, \lambda_2, \lambda_3$ (possibly same)

$A - \lambda_1 I \to$ find orthonormal basis

Construct P.

$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightsquigarrow$

$$\begin{bmatrix} 0 & -1 & c \\ 0 & 0 & b+ac \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 1 \iff \boxed{b + ac = 0}$.

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

31. Give an example of a square matrix over $\mathbb{C}$ that is not diagonalisable.

32. Suppose $A \in \mathbb{R}^{3\times3}$ satisfies $A^3 - 2A^2 = A - 2I$ and has the property that $\det(A) < 0$ and $\operatorname{trace}(A) > 2$.
Find the characteristic polynomial $p(t) = \det(A - tI)$.

It is not $-(t^3 - 2t^2 - t + 2)$.

Possible evals of A are $\{1, -1, 2\}$.

Write $p(t) = -(t - \lambda_1)(t - \lambda_2)(t - \lambda_3)$.

$\lambda_1, \lambda_2, \lambda_3 \in \{1, -1, 2\}$.

$\det(A) < 0 \Rightarrow$ odd # of λ_i are -1.

If all 3, the tr $= -3$. ✗

Only one is -1.

tr $> 2 \Rightarrow$ others are 2 & 2.

$\therefore p(t) = -(t+1)(t-2)^2$. ∎

5

λ is an eval of A.

v ⟼ — - - -

$Av = \lambda v$
$A^2 v = \lambda^2 v$
$A^3 v = \lambda^3 v$

$A^2 v = A(Av)$
$= A(\lambda v)$
$= \lambda(Av)$
$= \lambda(\lambda v) = \lambda^2 v.$

$(A^3 - 2A^2)v = (A - 2I)v$

$\Rightarrow (\lambda^3 - 2\lambda^2)v = (\lambda - 2)v$

$\Rightarrow \lambda^3 - 2\lambda^2 = \lambda - 2$ ($v \neq 0$)

$\Rightarrow \lambda \in \{1, -1, 2\}$.