Linear Algebra TSC

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Matrices

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Corresponding to each of the operations above, there are obvious row operations that can be performed on A^+, called the elementary row operations (EROs).

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It is fairly straightforward to perform EROs to turn A into an REF (and further an RREF).

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Note: The above algorithm does not require prior knowledge that A is invertible. If you perform it on a non-invertible matrix, you'll end up finding out that it is not invertible.

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We write $V = \operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and say that V is spanned (or generated) by $\mathbf{w}_1, \ldots, \mathbf{w}_k$.

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$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbf{0}.$$

Else, they are said to be linearly independent.

Rephrasing slightly, linear independence means that

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbf{0} \Rightarrow a_1 = \cdots = a_k = 0.$$

Basis

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Then, V has a basis B. If B' is any other basis of V, then B and B' have the same size.
The size of B is called the dimension of V, denoted $\dim(V)$.

Linear independence, spanning, basis

Linear algebra is nice, it works like you would intuitively want it to. Let V be a vector subspace of dimension d, and let $S \subseteq V$. We have the following.

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The above was the *First Fundamental Lemma in linear algebra*.

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Theorem 9

Row rank = Column rank = number of pivots in any REF.

The common quantity above is called the rank of A.

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On the other hand, if you are asked to find a basis for the row space, then you pick the nonzero rows from any REF of A.

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Note that n is the number of *columns*.

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Let $A \in \mathbb{K}^{m \times n}$ and $\mathbf{b} \in \mathbb{K}^m$. Consider the system

$$A\mathbf{x} = \mathbf{b}. \tag{$*$}$$

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What if "infinitely many" is replaced with "unique"?

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True/False: Let A be a square matrix such that $A\mathbf{x} = \mathbf{0}$ has a unique solution, and fix $\mathbf{b} \in \mathbb{K}^n$. Does $A\mathbf{x} = \mathbf{b}$ also have a unique solution?

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Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ be vectors.

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Recall that the adjugate of a square matrix A is the transpose of the cofactor matrix, and is denoted by $\mathrm{adj}(A)$. We have $\mathrm{adj}(A)A = \det(A)\mathbf{I}$. If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$.

Applications of determinants

Theorem 13

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ be vectors. Construct the matrix G whose (i, j)-th entry is the dot product $\mathbf{v}_i^\top \mathbf{v}_j$. Then, the k vectors are linearly independent iff $\det(G) \neq 0$.

Recall that the adjugate of a square matrix A is the transpose of the cofactor matrix, and is denoted by $\operatorname{adj}(A)$. We have $\operatorname{adj}(A)A = \det(A)\mathbf{I}$. If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$. In this case, the unique solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = \frac{(\operatorname{adj}(A))\mathbf{b}}{\det(A)}.$$

The above is essentially Cramer's rule.

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Similarly, one has Pythagoras theorem and Cauchy Schwarz: $|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle| \leqslant \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|$.

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Any orthogonal set of nonzero vectors is linearly independent.
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λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

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The geometric multiplicity of λ is defined as nullity$(A - \lambda I)$.

Theorem 20

Let λ be an eigenvalue of an $n \times n$ matrix A. Let g and m denote the geometric and algebraic multiplicities of λ respectively. Then,

$$1 \leqslant g \leqslant m \leqslant n.$$

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If A satisfies either (and hence, both) condition, then A is said to be diagonalisable over $\mathbb{K}$.

Criteria for diagonalisability

Let $A \in \mathbb{K}^{n \times n}$ be a square matrix, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ be the distinct eigenvalues of A.

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A is diagonalisable <u>over $\mathbb{K}$</u> iff $\boxed{\sum g_i = n}$.

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Example: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalisable over $\mathbb{C}$ but not over $\mathbb{R}$.

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Also note that a diagonal matrix is Hermitian iff it is real.
Similarly, a diagonal matrix is skew-Hermitian iff it is purely imaginary.

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The converse of the above is true as well: if A is unitarily diagonalisable (or has an orthonormal eigenbasis), then A is normal.

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Theorem 25

Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Then, there exists an orthogonal matrix $O \in \mathbb{R}^{n \times n}$ such that $O^\top AO$ is diagonal.