TUTORIAL 2
DETERMINANTS AND VECTOR SPACES
MA 106 (LINEAR ALGEBRA)
SPRING 2020

1. Tutorial Problems

(1) Compute the inverse of the matrix

$$\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$$

using Gauss-Jordan Elimination Method and the formula in terms of adjoint separately and compare the results.

(2) Calculate the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \dots & n \end{bmatrix}.$$

(3) (Vandermonde determinant):

(a). Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$

(b). Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column.

(4) Examine whether the following sets of vectors constitute a vector space. If so, find the dimension and a basis of that vector space.

(a). The set of all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ such that
(i) $x_4 = 0$; (ii) $x_1 \leq x_2$; (iii) $x_1^2 - x_2^2 = 0$; (iv) $x_1 = x_2 = x_3 = x_4$; (v) $x_1 x_2 = 0$.

(b). The set of all real functions of the form $a \cos x + b \sin x + c$, where a, b, c vary over all real numbers.

(c). The set of all $n \times n$ real symmetric matrices.

(d). The set of all complex polynomials of degree ≤ 5 with $p(0) = p(1)$ together with the zero polynomial.

(5) Examine whether the set $\{e^x, xe^x, \ldots, x^n e^x\}$ is linearly independent.

(6) Show that the only possible subspaces of $\mathbb{R}^3$ are the zero space $\{0\}$, lines passing through origin, planes passing through origin and the whole space.

2. Practice Problems

(1) Find an algorithm using GEM to calculate the determinant of a square matrix A.

(2) Compute the inverse of the following matrices using JGEM and the formula in terms of adjoint separately and compare the results.

$$\text{(i)} \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

(3) Solve the following systems by Cramer's rule:

$$\text{(i)} \begin{array}{rcr} -x+3y-2z & = & 7 \\ 3x+y+3z & = & -3 \\ 2x+y+2z & = & -1 \end{array} \qquad \text{(ii)} \begin{array}{rcr} 4x+y-z & = & 3 \\ 3x+2y-3z & = & 1 \\ -x+y-2z & = & -2 \end{array}$$

(4) Calculate the determinant of the following matrices :

$$\text{(a)} \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ 1 & & & & & \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ 2 & 2 & 3 & \ldots & n \\ 3 & 3 & 3 & \ldots & n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \ldots & n \end{bmatrix}.$$

$$\text{(c)} \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

(5) (Vandermonde determinant)

(a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

(b) Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column.

(6) Prove that the equation of the line in the plane through the distinct vectors $(a, b), (c, d)$ is given by

$$\det \begin{bmatrix} x & y & 1 \\ a & b & 1 \\ c & d & 1 \end{bmatrix} = 0.$$

(7) Show that the area of the triangle in the plane with vertices $(a, b), (c, d), (e, f)$ is given by

$$\frac{1}{2} \det \begin{bmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{bmatrix}.$$

(8) Show that the volume of the tetrahedron with vertices (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) and (d_1, d_2, d_3) is given by

$$\frac{1}{6} \det \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{bmatrix}$$

(9) Let A be a 2×2 matrix. Show that $\det(A+I) = 1 + \det A$ if and only if trace $(A) = 0$.

(10) Let A be an $n \times n$ matrix having the block form

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_k \end{bmatrix}$$

where A_j is an $r_j \times r_j$ matrix for $j = 1, 2, \dots, k$. Show that $\det A = \det A_1 \det A_2 \dots \det A_k$.

(11) Let $L(S)$ denote the subspace spanned by a subset S of a vector space V. Prove that if $S \subseteq T \subseteq V$ and T is a subspace of V, then $L(S) \subseteq T$. (That is $L(S)$ is the smallest subspace of V which contains S).

(12) Given a set of n linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V, show that for any scalar α, the set $\{\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_{i-1}, \mathbf{v}_i + \alpha \mathbf{v}_j, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$ with $i \neq j$ is linearly independent.

(13) Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices with entries in $\mathbb{C}$. Show that $M_n(\mathbb{C})$ is a vector space over $\mathbb{C}$ and $\{E_{(i,j)}, 1 \leq i, j \leq n\}$, where $E_{(i,j)}$ denote $n \times n$ matrix with 1 at $(i, j)^{\text{th}}$ place and 0 elsewhere is a basis of it.

(14) Examine whether the following sets of vectors constitute a vector space. If so, find the dimension and a basis of that vector space.

(a) The set of all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ such that

(i) $x_4 = 0$; (ii) $x_1 \leq x_2$; (iii) $x_1^2 - x_2^2 = 0$; (iv) $x_1 = x_2 = x_3 = x_4$; (v) $x_1 x_2 = 0$.

(b) The set of all real functions of the form $a \cos x + b \sin x + c$, where a, b, c vary over all real numbers.

(c) Homogeneous polynomials in two variables of degree 3 together with the zero polynomial.

(d) The set of all $n \times n$ real matrices $((a_{ij}))$ which are:

(i) diagonal; (ii) upper triangular; (iii) having zero trace; (iv) symmetric; (v) anti-symmetric (i.e., those satisfying $A^t = -A$;) (vi) invertible.

(e) The set of all real polynomials of degree 5 together with the zero polynomial.

(f) The set of all complex polynomials of degree ≤ 5 with $p(0) = p(1)$ together with the zero polynomial.

(g) The real functions of the form $(ax + b)e^x, \quad a, b \in \mathbb{R}$.

(15) Consider the following subsets of the space $M_n(\mathbb{C})$ of $n \times n$ matrices over complex numbers:

(a) The space $Sym_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = A^T\}$ of **symmetric matrices**.

(b) The space $Herm_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = A^*\}$ of **Hermitian matrices**.

(c) The space $Skew_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = -A^*\}$ of **skew-Hermitian Matrices**.

Show that each of them is a $\mathbb{R}$ vector subspace of $M_n(\mathbb{C})$ and compute their dimension by explicitly writing down a basis for each of them. Is any one of them a complex vector subspace?

(16) Given $A \in M_n(\mathbb{C})$, show that the mappings

$$\alpha_A(B) = \frac{1}{2}(AB + BA^*); \ \beta_A(B) = \frac{1}{2\imath}(AB - BA^*)$$

define $\mathbb{R}$-linear maps $\mathrm{HERM}_n(\mathbb{C}) \to \mathrm{HERM}_n(\mathbb{C})$.

Also show that α_A, β_A commute with each other.

(17) Let $P_n[x]$ denote the vector space consisting of the zero polynomial and all real polynomials of degree $\leq n$, where n is fixed. Let S be a subset of all polynomials $p(x)$ in $P_n[x]$ satisfying the following conditions. Check whether S is a subspace; if so, compute dimension of S.

(i) $p(0) = 0$; (ii) p is an odd function; (iii) $p(0) = p''(0) = 0$.

(18) Examine whether the following subsets of the set of real valued functions on $\mathbb{R}$ are linearly dependent or independent. Compute the dimension of the subspace spanned by each set

(a) $\{1 + t, (1 + t)^2\}$; (b) $\{x, |x|\}$.

(19) Examine whether the following sets are linearly independent.

(a). $\{(a, b), (c, d)\} \subset \mathbb{R}^2$, with $ad - bc \neq 0$.

(b). $\{(1+i, 2i, 2), (1, 1+i, 1-i)\}$ in $\mathbb{C}^3$.

(c). For $\alpha_1, \ldots, \alpha_k$ distinct real numbers, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ where $\mathbf{v}_i = (1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{k-1})$.

(d). $\{e^{\alpha_1 x}, e^{\alpha_2 x}, \cdots, e^{\alpha_n x}\}$ for distinct real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(e). $\{1, \cos x, \cos 2x, \ldots, \cos nx\}$.

(f). $\{1, \sin x, \sin 2x, \ldots, \sin nx\}$.

(g). $\{e^x, xe^x, \ldots, x^n e^x\}$.

(20) Let $\alpha_1, \alpha_2, \alpha_3$ be fixed real numbers. Show that the vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_4 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ forms a subspace, which is spanned by $(1, 0, 0, \alpha_1)$, $(0, 1, 0, \alpha_2)$ and $(0, 0, 1, \alpha_3)$. Find the dimension of this subspace.

(21) Show that the only possible subspaces of $\mathbb{R}^3$ are the zero space $\{0\}$, lines passing through origin, planes passing through origin and the whole space.