TUTORIAL SHEET 5 : EIGENVALUES AND EIGENVECTORS

Tutorial Problems

(1) Let u be a unit vector in $\mathbb{R}^n$. Define $H = I - 2uu^t$. Find all the eigenvalues and eigenvectors of H. Find a geometric interpretation of $T_H : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_H(v) = Hv$ for all $v \in \mathbb{R}^n$.

(2) If $A, A' \in \mathbb{F}^{n\times n}$ are **similar** (i.e. $A' = P^{-1}AP$ for some invertible $n \times n$ matrix $P \in \mathbb{F}^{n\times n}$) then show that (a) A and A' have same eigenvalues (b) if $\mathbf{v}$ is an eigenvector of A then $P^{-1}\mathbf{v}$ is an eigenvector of A'.

(3) Let A be $n\times n$ matrix. Prove that (i) 0 is an *eigenvalue* of A if and only if A is singular. (ii) if λ is an *eigenvalue* of A then it is also an *eigenvalue* of A^t (where A^t denotes the transpose of A). (iii) If x is an *eigenvector* of A corresponding to λ then x need not be an *eigenvector* of A^t corresponding to λ.

(4) Show that the map $T : C^\infty[0,1] \to C^\infty[0,1]$ given by $T(f)(x) = \int_0^x f(t)dt$ has no eigenvalue while every real number is an eigenvalue of $T(f)(x) = \frac{df(x)}{dx}$.

(5) Transform the following quadratic equations to principal axis form and find out what conics they represent. (a) $41x_1^2 - 24x_1x_2 + 34x_2^2 = 0$ (b) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$; (c) $91x^2 - 24xy + 84y^2 = 25$. (d) $4xy + 3y^2 = 10$.

Practice Problems

(6) Examine whether the following matrices can be diagonalised. If yes, find P such that $P^{-1}AP$ is diagonal.

$$\text{(i)} \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

(7) Prove that (a) the *trace* of $A \in \mathbb{C}^{n\times n}$ is equal to the sum of its *eigenvalues.* (b) the determinant of A is equal to the product of its *eigenvalues.*

(8) Let A be a square complex matrix. Show that (i) the eigenvalues of A are real if A is Hermitian or real symmetric. (ii) the eigenvalues of A are either 0 or purely imaginary if A is skew Hermitian. (iii) the eigenvalues of A are of modulus equal to 1, if A is unitary. (v) A^tA has only non negative eigenvalues, if A is real.

(9) Let $A \in \mathbb{C}^{n\times n}$ and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a complex polynomial. Suppose that λ is an eigenvalue of A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$. Find all the eigenvalues of $f(A)$.

(10) Let $V = \mathbb{R}^{2\times 2}$. Let $T : V \to V$ be defined by $T(A) = A^t$. Find the eigenvalues and eigenvectors of T.

(11) Let A be a 2×2 real matrix and $p_A(x)$ be its characteristic polynomial. Show that $p_A(A) = 0$. This is called the Cayley-Hamilton Theorem. It is valid for all square matrices.

(12) Find a nonzero matrix so that $N^3 = 0$. Find all the eigenvalues of N. Show that N cannot be symmetric.

(13) Let an $n \times n$ matrix B have n distinct *eigenvalues.* Show that every $n \times n$ matrix A such that $AB = BA$, is diagonalizable.