Tutorial-3
Rank of Matrices and Linear Transformations
MA 106 (Linear Algebra)
Spring 2020

1 Tutorial Problems

1. Consider the linear transformations $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$. If T_2 is one-one then show that $rank(T_2 \circ T_1) = rank(T_1)$.

2. Obtain the REF of the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & 3 & 2 \end{bmatrix}.$$

   Use this to find rank and nullity of the matrix. Also write down a basis for the range. Finally obtain the RCF and use it to write down a basis for the null space.

3. Define $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by $f((x_1, x_2, x_3, x_4, x_5)^t)$

   $= (2x_3 - 2x_4 + x_5, 2x_2 - 8x_3 + 14x_4 - 5x_5, x_2 + 3x_3 + x_5)^t$.

   Find bases for the null-space and the range of f, using GJEM.

4. Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Using linear transformations show that $rk(AB) \leq Min\{rk(A), rk(B)\}$.

5. Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors satisfying $4x_1 - 3x_2 + x_3 = 0$ is – (i) the null-space of T, (ii) the range of T.

6. Examine whether the transformation $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined as $T(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2)$ is linear or not. Is it linear over $\mathbb{R}$?

2 Practice Problems

1. Let A be a $m \times n$ matrix and let f_A be the linear transformation from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ induced by A. Show that column-rank $(A) =$ dim (Range (f_A))=rk f_A.

2. Let A, B be $n \times n$ matrices. If A is invertible, then show that $rk\,(AB) = rk\,B = rk\,(BA)$.

3. Two $n \times n$ matrices A, B are said to be **similar** if there exists a non-singular matrix C (which will, in general, depend on both A and B) such than $B = C^{-1}AC$. Prove that:
   (i) Similarity is an equivalence relation.
   (ii) Similar matrices have equal traces. (A more fancy way to express this is to say that the trace is a similarity invariant.)
   (iii) Similar matrices have equal ranks.

4. Give an example of two square matrices A, B (of equal orders) such that $rk(A) = rk(B)$ but $rk(A^2) \neq rk(B^2)$.

5. Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Using linear transformations show that $rk(AB) \leq Min\{rk(A), rk(B)\}$.

6. Let A, B be $n \times n$ matrices. We say A is a left inverse of B and B is a right inverse of A if $AB = I_n$. Show that
   (i) If A has a left inverse then rk $(A) = n$ and hence A is invertible.
   (ii) If B has a right inverse then $rk\,(B) = n$ and hence B is invertible.

7. Let f be a bijective linear map. Show that the inverse is also linear.

8. Let $f : V \longrightarrow W$ be a linear transformation. Put

$$\mathcal{R}(f) := f(V) := \{f(v) \in W \;:\; v \in V\},\; \mathcal{N}(f) := \{v \in V \;:\; f(v) = 0\}.$$

   (i) Show that $\mathcal{R}(f)$ and $\mathcal{N}(f)$ are both vector subspace of W and V respectively.
   (They are respectively called the *range* and the *null space* of f. The dimensions of these two spaces are respectively called the rank and nullity of f.)
   (ii) $dim\,\mathcal{R}(f) \leq dim\,V$ where V is the domain of f.
   (iii) Rank of $f+$ nullity of f is equal to $dim\,V$.

9. Let $T : U \longrightarrow V$ be a linear transformation. Show that
   (i) T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$.
   (ii) If $L(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = U$ then $\mathcal{R}(T) = L(\{T(\mathbf{u}_1), T(\mathbf{u}_2), \cdots, T(\mathbf{u}_n)\})$.

10. Let $f : V \longrightarrow W$ be a linear transformation.
   (a) Suppose f is injective and $S \subset V$ is linearly independent. Then show that $f(S)$ is linearly independent.

(b) Suppose f is onto and S spans V. Then show that $f(S)$ spans W.

(c) Suppose S is a basis for V and f is an isomorphism then show that $f(S)$ is a basis for W.

11. Let V be a finite dimensional vector space and $f : V \longrightarrow V$ be a linear map. Prove that the following are equivalent:
    (i) f is an isomorphism.
    (ii) f is injective.
    (iii) f is surjective.
    (iv) there exist $g : V \longrightarrow V$ such that $g \circ f = Id_V$.
    (v) there exists $h : V \longrightarrow V$ such that $f \circ h = Id_V$.

12. Prove that linear independence is preserved under one-to-one linear transformations.

13. Let U , V be finite dimensional vector spaces having equal dimension. Prove that a linear transformation $T : U \longrightarrow V$ is onto only if it is one-to-one.

14. Consider the linear transformations $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$. If T_2 is one-one then show that $rank(T_2 \circ T_1) = rank(T_1)$.

15. Let A and B be any two $n \times n$ matrices and $AB = I_n$. Show that both A and B are invertible and they are inverses of each other.

16. Obtain the REF of the following matrices. Use them to find rank and nullity of the matrix. Also write down a basis for the range. Finally obtain the RCF and use to write down a basis for the null space.

$$\text{(i)} \begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & 3 & 2 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 0 \\ 2 & -3 & 1 \\ 5 & 1 & 1 \end{bmatrix}.$$

17. Define $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by $f((x_1, x_2, x_3, x_4, x_5)^t)$
    $= (2x_3 - 2x_4 + x_5, 2x_2 - 8x_3 + 14x_4 - 5x_5, x_2 + 3x_3 + x_5)^t$.
    Find bases for the null-space and the range of f, using GJEM.

18. Find the range and null-space of the following linear transformations. Also find the rank and nullity wherever applicable.
    (a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2)^t = (x_1 + x_2, x_1)^t$.
    (b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2)^t = (x_1, x_1 + x_2, x_2)^t$.
    (c) $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^t = (x_1 - x_4, x_2 + x_3, x_3 - x_4)^t$.
    (d) $T : C(0, 1) \longrightarrow C(0, 1)$ defined by $T(f)(x) = f(x) \sin x$.

(e) $T : C^1(0,1) \longrightarrow C(0,1)$ defined by $T(f)(x) = f'(x)e^x$.

19. Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors satisfying $4x_1 - 3x_2 + x_3 = 0$ is – (i) the null-space of T, (ii) the range of T.

20. Show that each of the following linear transformations is nonsingular and find its inverse.
(i) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (5x_1, 3x_2)$
(ii) $T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_3 + x_2, x_3)$.

21. Examine whether the following transformations are linear or not. In case of linear transformations, write down their matrix representation with respect to the standard bases.
(i) $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that $T(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2)$.
(ii) $T :\ \mathbf{P}_3 \longrightarrow \mathbf{P}_3$ such that $T(a_0 + a_1x + a_2x^2 + a_3x^3) = -a_0 + 2a_1x + 3(a_0 - a_1)x^2$.
(iii) $T :\ \mathbf{P}_3 \longrightarrow \mathbf{P}_4$ such that $T(p(x)) = xp(x) + \int_0^x p(t)\,dt$.
(iv) $T :\ \mathbf{P}_3 \longrightarrow \mathbf{P}_3$ such that $T(p) = p'$.

22. Let $f, g : V \to V$ be two linear maps which commute with each other, i.e., $f \circ g = g \circ f$. Show that $f(\mathcal{R}(g)) \subset \mathcal{R}(g)$, and $f(\mathcal{N}(g)) \subset \mathcal{N}(g)$.

23. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ of dimension $n > 2$. Show that if U_j are some finitely many subspaces of V each of dimension $< n$ then V cannot be written as union of U_j's.