Week 1

11 May 2022 13:30

→ Hi. I'm Aryaman. Almost an alumnus.

→ bit.ly/ma-108 (Recordings of recap and tut - MS Team)

PDFs of whatever I write

MA-108

ODE: A relation involving $x, y, y', \ldots, y^{(n)}$.

$$F(x, y, \ldots, y^{(n)}) = 0.$$

eg. ① $y' + y = 0$

② $(y')^2 + x e^y + \sin(y'') = 0$

NOT

NOT example: $y(y(x)) = y'(x)$.

interested in

$$y^{(n)} = G(x, y, \ldots, y^{(n-1)}) \quad \text{— } (*)$$

EXPLICIT: (explicit)

A solution of $(*)$ is a function ϕ which is defined on some (open) interval I s.t.

$$\phi^{(n)}(x) = G(x, \phi(x), \ldots, \phi^{(n-1)}(x)) \quad \text{for all } x \in I.$$

IVP:

$$\left[\begin{array}{l} y' = f(x,y) \\ y(x_0) = y_0 \end{array}\right.$$

Here, we will try to find an interval I containing x_0 s.t. $\exists\, \phi : I \to \mathbb{R}$ satisfying the above.

Linear ODE: Is an ODE of the form:

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x) y = b(x).$$

The above is said to have order n if $a_n(x) \neq 0$.

EXAMPLE: $(1-x^2) y'' - 2xy' + n(n+1) y = 0 \quad (n \in \mathbb{Z})$

NON-EXAMPLES: ① $y y' + x = 43$

EXAMPLE: $(1-x^2)y'' - 2xy' + \dots y = 0$ (...)

Legendre's equation (MA 207)

NON-EXAMPLES: ① $yy' + x = 4^3$

② $e^y = y'$

EXPLICIT SOLUTION:

IVP: $y' = e^{x^2}$; $y(0) = 0$.

$\phi(x) := \int_0^x e^{t^2}\,dt$ is an EXPLICIT SOLUTION.

IMPLICIT: "Don't have $y = \phi(x)$.
Rather, a relation $F(x, y) = 0$
s.t. on some interval I, we can
get a ϕ s.t. $F(x, \phi(x)) = 0$
$\forall x \in I$."

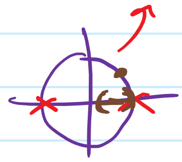
EXAMPLE: ① $yy' + x = 0$.

$y^2 + x^2 = c$

Particular

$y^2 + x^2 = 4 \longrightarrow$ Implicit

Use IFT.



But this $\Rightarrow$

$y = \sqrt{4 - x^2}$ and $y = -\sqrt{4-x^2}$

as two solutions.

② $\left[F(x, y) = y \sin x + x^2 e^y - y\right]$

Let's say $y \sin x + x^2 e^y - y = 0$

From the graph, it is "clear" that we can get a function of x on $(-1/2, 1/2)$.

Formally: IMPLICIT FUNCTION THEOREM

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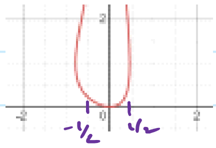


Note: ① $F(0,0) = 0.$

② $$\frac{\partial F}{\partial y}(0,0) = \left(\sin(x) + x^2 e^y - 1\right)\Big|_{(x,y)=(0,0)}$$

$$= -1 \neq 0.$$

The above conditions tell you that you can get y as a function x locally.

$$F(x, y) = c$$

Given (x_0, y_0) s.t. $F(x_0, y_0) = c$.

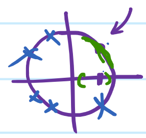
Q: Can you can get ϕ on some interval I around x_0 s.t. $F(x, \phi(x)) = c$ $\forall x \in I$.

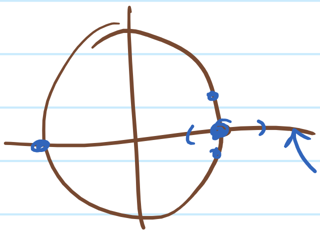
A: (IFT) Yes, if $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$

$x^2 + y^2 = 4$

$\partial/\partial y$

$2y$





Week 2

18 May 2022 12:41

1. Integrating factor
2. Homogeneous ODEs $\rightsquigarrow$ substitute $v = y/x$
3. Orthogonal trajectory $\rightsquigarrow$ example
4. Lipschitz continuity $\rightsquigarrow$

1. Integrating Factors.

$$M\,dx + N\,dy = 0. \qquad (1)$$

functions of x and y

(1) is said to be exact if $M_y = N_x$.

(I will assume domain is nice. For example, convex (or simply-connected, more generally).)

We can then solve the above, i.e., we get a function ϕ s.t. $\phi_x = M$ and $\phi_y = N$.

Then, $\phi = C$ is the gen. solution.

If (1) is NOT exact, we try to find an integrating factor μ s.t.

$$\mu M\,dx + \mu N\,dy = 0$$

is exact.

Thus, we want

$$(\mu M)_y = (\mu N)_x$$

or

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

For simplicity, we assume μ is a function of (say) x alone. Then, $\mu_y \equiv 0$ and we get

$$\mu_x N = \mu(M_y - N_x) \quad \text{or}$$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}.$$

μ $\sim$

4\. Let $D \subseteq \mathbb{R}^2$ (say a rectangle).
$f : D \to \mathbb{R}$.

f is said to be Lipschitz (in y) if
$\exists M \geq 0$ s.t.

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$
for all $(x, y_1), (x, y_2) \in D$.

Let $I \subseteq \mathbb{R}$. $g : I \to \mathbb{R}$ is said to be Lipschitz if $\exists L \geq 0$ s.t.

$$|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$$
for all $x_1, x_2 \in I$.

Example: • $I = \mathbb{R}$, $g(x) = x$. ($L = 1$ works.)

• $I = [1, 2]$, $g(x) = x^2$.

$(x_1 \neq x_2)$ $$\frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|} = |x_1 + x_2| \leq 4.$$

$L = 4$ works.

• $I = [0, 1]$, $g(x) = \sqrt{x}$.
Not Lipschitz.
Suppose not. Then, $\exists L \geq 0$ s.t.
$$|g(x) - g(0)| \leq L|x - 0| \quad \forall x \in I.$$
$$\Leftrightarrow \sqrt{x} \leq Lx \quad \forall x \in I$$
$$\Rightarrow \frac{1}{\sqrt{x}} \leq L \quad \text{for all } x \in I \setminus \{0\}.$$

But this is a contradiction, choose
$$x = \frac{1}{L^2 + 1} \in I \setminus \{0\}.$$

Digression:

$f: X \to X$ is conti (X → complete metric space.)

s.t.

$$|f(x) - f(y)| \leq \alpha |x-y|$$

for some $0 < \alpha < 1$.

Then, f has a unique fixed pt.,

$x \in X$ s.t. $f(x) = x$.

DE : $y' = f(x,y)$

$\Updownarrow$

$$\phi(x) = \int_{x_0}^{x} f(s, \phi(s))\, ds$$

$$\|f-g\| = \sup_{x\in[0,1]} |f(x)-g(x)|.$$

X = appropriate metric space

$f: X \to X$

$$f(\phi) := \int_{x_0}^{x} f(s, \phi(s))\, ds.$$

Show f is a contraction.

Done.

MA 108
D1 – T3

- Constant coefficient (second order) linear ODE
- Cauchy Euler (second order) ODE

① $1\, y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$
↳ looking to solve this on $\mathbb{R}$.

② $\underline{\underline{x^n}}\, y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y^{(1)} + a_0 y = 0$
↳ looking to solve this on $(0, \infty)$.

In both cases: $a_0, \ldots, a_{n-1} \in \mathbb{R}$.

To solve ①, we substitute $y = e^{mx}$.
② $y = x^m$.

General theory tells that ① and ② have an n-dimensional solution space.

Our "algorithm" actually gives us n distinct linearly independent solutions. Thus, we have found all.

ALGORITHM : ① Put $y = e^{mx}$. We end up with an n-degree polynomial in 'm', given as:

$$m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

In $\mathbb{C}$, the above has n solutions (counted with multiplicity). If m_0 is a solution with multiplicity $k+1$ $(k \geq 0)$, then we have the solutions

(space) $P(D)$ $(D - m_0)^{k+1} y = 0$

$(D - m_0)^{k+1} y = 0$

$$\underbrace{e^{m_0 x},\ x e^{m_0 x},\ \ldots,\ x^k e^{m_0 x}}_{k+1 \text{ linearly indep.}}.$$

If m_0 is non-real complex, then $\overline{m_0}$ is a root with same multiplicity

$m_0 = a + ib$
$a, b \in \mathbb{R}$

$e^{ax}\cos(bx)$
$e^{ax}\sin(bx)$

$$\begin{cases} e^{m_0 x}, \\ e^{\overline{m_0} x}, \end{cases} \quad \begin{cases} x e^{m_0 x}, & \ldots, & x^k e^{m_0 x} \\ x e^{\overline{m_0} x}, & \ldots, & x^k e^{\overline{m_0} x} \end{cases}$$

$x e^{ax} \cos bx,\ x e^{ax} \sin bx,\ \ldots$

② Here the n-degree polynomial is:

$$m(m-1)\cdots(m-(n-1)) + a_{n-1} m(m-1)\cdots(m-(n-2)) + \cdots + a_1 m + a_0 = 0.$$

Again : n solutions in $\mathbb{C}$ ...

m_0 has mult. $k+1$, then the functions:

$$x^{m_0},\ x^{m_0} \cdot \log x,\ \ldots,\ x^{m_0} (\log x)^k.$$

If $m_0 \in \mathbb{C} \setminus \mathbb{R}$, then write $m_0 = a + ib$

$$\begin{aligned} x^{m_0} &= \exp(m_0 \cdot \log x) \\ &= \exp(a \log x + i b \log x) \\ &= x^a \cdot [\cos(b \log x) + i \sin(b \log x)] \end{aligned}$$

As before $\overline{m_0}$ is a root ...

$$x^a \cos(b\log x), \quad x^a \cos(b\log x)\log x, \quad \ldots$$
$$x^a \sin(b\log x), \quad x^a \sin(b\log x)\log x, \quad \ldots$$