ODEs TSC

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IIT Bombay

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The ODE is said to be linear if it of the form

$$a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = b(x)$$

for some $n \geqslant 0$ and functions $a_0, \ldots, a_n, b$ of x.

Solutions

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Example: $x^2 + y^2 = \lambda^2$. Differentiating gives $x + yy' = 0$. Replacing y with $-1/y'$ gives

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Solving it gives $y = cx$ ($c \in \mathbb{R}$) as the family of orthogonal trajectories.

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$$H_1(x) + H_2(y) = c$$

for $c \in \mathbb{R}$.

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To solve: put $y = xv$ and things "magically" fall in place by becoming a separable ODE in v.

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Just integrate the above to get $k(y)$ and in turn, get $u(x, y)$.

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$$\mu = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

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$$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|$$

for all $x_1, x_2 \in I$.

Now, if f is a function of two variables defined on some $D \subseteq \mathbb{R}^2$, then we say that f is Lipschitz continuous with respect to y if there exists some $L \geqslant 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

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An non-example of Lipschitz function (in y) is: $f(x, y) = \sqrt{|y|}$ defined on $[-1, 1] \times [-1, 1]$. Similarly, $f(x, y) = y^2$ is not Lipschitz w.r.t. y on $\mathbb{R}^2$ but is so on bounded domains.

Back to uniqueness

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Suppose that we have the IVP

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As before, there may a solution on a larger interval. Moreover, there may still be a larger interval where the solution is unique.

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Under the assumptions of the existence-uniqueness theorem, the above converges to the solution y of the IVP defined by $y(x) := \lim_{n\to\infty} y_n(x)$.

Table of Contents

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Disclaimer

Our results will always assume that the ODE is in standard form. This is crucial.

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The standard ODE is said to be homogeneous if $b(x) = 0$, i.e., it is of the form

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From now on, "homogeneous" will refer to the above, not the one we had defined earlier.

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(Bernoulli) If the ODE was instead $y' + P(x)y = Q(x)y^n$ for some $n \neq 0, 1$, then substitute $v = y^{1-n}$ and it will "magically" get reduced to the above.

Consider the following second order homogeneous linear ODE:

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Theorem 7 (Dimension result)

The solution space of (2) is a two-dimensional real vector space.

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Note that the Wronskian is defined for any two functions, without any mention of any DE.

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Consequently, if $x_0 \in I$, then

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$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int p(x)\, dx\right)}{y_1(x)^2}\, dx.$$

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Note that basis being $\{y_1, y_2\}$ means that the general solution is given by $c_1 y_1 + c_2 y_2$ for $c_1, c_2 \in \mathbb{R}$.

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