Q2.

20 June 2022 16:19

(ii) $\mathcal{L}^{-1}\left(\dfrac{2as}{(s^2-a^2)^2}\right) = ?$

Sol^n. Note $\dfrac{2as}{(s^2-a^2)^2} = \dfrac{2as}{(s-a)^2(s+a)^2}$

$$= \frac{1}{2}\frac{4as}{(s-a)^2(s+a)^2}$$

$$= \frac{1}{2}\left[\frac{(s+a)^2-(s-a)^2}{(s-a)^2(s+a)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)^2}-\frac{1}{(s+a)^2}\right].$$

$$\boxed{\begin{aligned}\mathcal{L}(e^{at}) &= \frac{1}{s-a}\\ \mathcal{L}(te^{at}) &= -\frac{d}{ds}\left(\frac{1}{s-a}\right)\\ &= \frac{1}{(s-a)^2}\end{aligned}}$$

$$\boxed{\begin{aligned}&\mathcal{L}(t) = \frac{1}{s^2}\\ &\Rightarrow \mathcal{L}(te^{at}) = \frac{1}{(s-a)^2}\end{aligned}}$$

Using linearity, $\mathcal{L}^{-1}\left(\dfrac{2as}{(s^2-a^2)^2}\right) = \dfrac{1}{2}\left[\mathcal{L}^{-1}\left(\dfrac{1}{(s-a)^2}\right) - \mathcal{L}^{-1}\left(\dfrac{1}{(s+a)^2}\right)\right]$

$$= \frac{1}{2}\left(te^{at} - te^{-at}\right)$$

$$= t\sinh(at).$$

Aliter : $\mathcal{L}\left(\dfrac{f(t)}{t}\right)(s) = \displaystyle\int_s^\infty F(\tilde{s})\,d\tilde{s} = \dfrac{a}{s^2-a^2} \Rightarrow \dfrac{f(t)}{t} = \sinh(at).$

(iv) $\mathcal{L}^{-1}\left(\dfrac{s^3}{s^4+4a^4}\right) = ?$

Sol^n. Note : $s^4+4a^4 = s^4+4a^2s^2+4a^4-4a^2s^2$

$$= (s^2+2a^2)^2-(2as)^2$$

$$= (s^2-2as+2a^2)(s^2+2as+2a^2)$$

$$= \left[(s-a)^2+a^2\right]\left[(s+a)^2+a^2\right]$$

$$\frac{s^3}{s^4+4a^4} = \frac{As+B}{(s-a)^2+a^2} + \frac{A's+B'}{(s+a)^2+a^2}$$

Q: What are A, B, A', B'?

Multiply both side with $s^4 + 4a^4$ to get

$$1 s^3 = (As + B)[s^2 + 2as + 2a^2] + (A's + B')[s^2 - 2as + 2a^2]$$
$$= (A + A')s^3 + (B + 2aA + B' - 2aA')s^2$$
$$+ 2(Aa^2 + aB - aB' + A'a^2)s$$
$$+ 2a^2(B + B').$$

$s^4 = -4a^4 \Rightarrow s = \pm a \pm ia$

$$\frac{1}{4}\left(\frac{1}{s-(a+ia)} + \dots\right)$$

Comparing constants: $B = -B'$

Comparing s^2: $A = A'$.

Comparing s^3: $A = A' = 1/2$.

Comparing s: $a^2 + 2aB = 0$

$\Rightarrow B = -\frac{a}{2}$.

$$\frac{s^3}{s^4+4a^4} = \frac{1}{2}\frac{(s-a)}{(s-a)^2+a^2} + \frac{1}{2}\frac{(s+a)}{(s+a)^2+a^2}.$$

$$\boxed{\begin{aligned}\mathcal{L}(\cos \omega t) &= \frac{s}{s^2+\omega^2}\\ \mathcal{L}(e^{at}\cos\omega t) &= \frac{s-a}{(s-a)^2+\omega^2}\\ \mathcal{L}(e^{at}\cos at) &= \frac{s-a}{(s-a)^2+a^2}\end{aligned}}$$

$$= \frac{1}{2}\mathcal{L}\left(e^{at}\cos at + e^{-at}\cos at\right)$$
$$= \mathcal{L}\left(\cosh(at)\cos(at)\right).$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s^3}{s^4+4a^4}\right) = \cos(at)\cosh(at).$$

Aliter: We could work over $\mathbb{C}$:

$$s^4 + 4a^4 = ((s-a)^2 + a^2)((s+a)^2 + a^2)$$
$$= (s-a-ia)(s-a+ia)(s+a-ia)(s+a+ia).$$

$$\frac{s^3}{s^4+4a^4} = \frac{A}{s-(a+ia)} + \frac{B}{s-(a-ia)} + \frac{C}{s-(-a+ia)} + \frac{D}{s-(-a-ia)}.$$

Take conjugates to conclude $A = \overline{B}$ and $C = \overline{D}$.

Multiply both sides with $s-(a+ia)$ to get

$$A = \lim_{s \to a+ia} \frac{s^3}{s^4+4a^4} \cdot (s-(a+ia))$$

$$= \lim_{s\to a+ia} \frac{s^3}{(s-a+ia)((s+a)^2+a^2)}$$

$$= \frac{a^3(1+i)^3}{(2ia)\left[(2a+ia)^2+a^2\right]}$$

$$= \frac{a^3}{a^3} \frac{(1+i)^3}{(2i)\left[(2+i)^2+1\right]}$$

$$\left(\begin{aligned} 1+i &= \sqrt{2}e^{i\pi/4} \\ (1+i)^3 &= 2\sqrt{2}\, e^{i3\pi/4} \\ &= 2(-1+i) \end{aligned}\right)$$

$$= \frac{2(-1+i)}{2i\left[4+4i\right]} = \frac{1}{4}\frac{-1+i}{i(1+i)}$$

$$= \frac{1}{4}.$$

Thus, $B = 1/4$.

Similarly, $$C = \lim_{s\to -a+ia} \frac{s^3}{(s^4+4a^4)}(s+a-ia)$$

$$= \lim_{s\to -a+ia} \frac{s^3}{((s-a)^2+a^2)(s+a+ia)}$$

$$= \frac{(-1+i)^3}{\left[(-2+i)^2+1\right]\left[2i\right]}$$

$$\left(\begin{aligned} -1+i &= \sqrt{2}\exp\left(\tfrac{3\pi i}{4}\right) \\ (-1+i)^3 &= 2\sqrt{2}\exp\left(\tfrac{9\pi i}{4}\right) \\ &= 2\sqrt{2}\exp(i\pi/4) \\ &= 2(1+i) \end{aligned}\right)$$

$$= \frac{2(1+i)}{(4-4i)(2i)} = \frac{1}{4}.$$

$= 2\sqrt{2}\exp(i\pi/4)$
$= 2(1+i)$

Now, using $\frac{1}{4}\left[\frac{1}{s-(a+ia)} + \frac{1}{s-(a-ia)}\right]$

$$= \frac{1}{4}\left[\frac{2(s-a)}{(s-a)^2+a^2}\right] = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}\right],$$

we arrive at the earlier form. ☒

Note: it may be tempting to say

$$\mathcal{L}^{-1}\left(\frac{1}{s-(a+ia)}\right) = e^{(a+ia)t}, \text{ etc.}$$

but we have not made sense of Laplace transform of complex functions!

Q3.

20 June 2022 17:37

(i) Solve:

$$y'' + y = \sin(3t), \qquad \text{—— (1)}$$
$$y(0) = y'(0) = 0.$$

Sol^n.

Let $Y := \mathcal{L}(y)$.

Recall: $\mathcal{L}(y') = sY - y(0) = sY.$

$\mathcal{L}(y'') = s^2 Y - s\,y(0) - y'(0) = s^2 Y.$

Taking Laplace transform on both sides of (1) gives

$$(s^2 + 1)\,Y(s) = \frac{3}{s^2 + 9}$$

$$\Rightarrow \quad Y(s) = \frac{3}{(s^2+9)(s^2+1)} = \frac{3}{8}\,\frac{8}{(s^2+9)(s^2+1)}$$
$$= \frac{3}{8}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$\Rightarrow \quad y(t) = \frac{3}{8}\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$
$$= \frac{3}{8}\left(\sin(t) - \frac{1}{3}\sin(3t)\right)$$
$$= \frac{1}{8}\left(3\sin(t) - \sin(3t)\right). \qquad \blacksquare$$

Q4.

20 June 2022 17:54

(v)
$$y_1' + y_2 = 2\cos(t)$$
$$y_1 + y_2' = 0$$

$$y_1(0) = 0, \quad y_2(0) = 1.$$

Sol^n. $Y_1 := \mathcal{L}(y_1)$, $Y_2 := \mathcal{L}(y_2)$.

Then, $\mathcal{L}(y_1') = sY_1 - y_1(0) = sY_1$.

Similarly, $\mathcal{L}(y_2') = sY_2 - y_2(0) = sY_2 - 1$.

Applying Laplace transform to the ODEs gives

$$sY_1 + Y_2 = 2s/s^2+1$$
$$Y_1 + sY_2 = 1$$

Thus,
$$\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2s/s^2+1 \\ 1 \end{bmatrix}$$

or
$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{s^2-1}\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}\begin{bmatrix} 2s/s^2+1 \\ 1 \end{bmatrix}.$$

Thus,
$$Y_1 = \frac{1}{s^2-1} \cdot \left[\frac{2s^2}{s^2+1} - 1\right] = \frac{1}{s^2-1}\left[\frac{s^2-1}{s^2+1}\right] = \frac{1}{s^2+1},$$
$$Y_2 = \frac{1}{s^2-1}\left[-\frac{2s}{s^2+1} + s\right] = \frac{1}{s^2-1}\left[\frac{-2s+s^3+s}{s^2+1}\right]$$
$$= \frac{1}{s^2-1}\left[\frac{s^3-s}{s^2+1}\right]$$
$$= \frac{s}{s^2+1}.$$

Thus, $y_1(t) = \mathcal{L}^{-1}(Y_1(s)) = \sin(t)$,

$$y_2(t) = \mathcal{L}^{-1}(Y_2(s)) = \cos(t).$$ ☑

Q6.

20 June 2022 18:00

Suppose $f(t) = f(t+p)$ (here $p>0$)

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-ts} f(t)\, dt$$

$$= \sum_{n=0}^\infty \int_{np}^{(n+1)p} e^{-ts} f(t) dt$$

$t = u + np$

$$= \sum_{n=0}^\infty \int_0^p e^{-(u+np)s} f(u)\, du$$

$$= \left(\sum_{n=0}^\infty e^{-nps} \right) \int_0^p e^{-us} f(u)\, du$$

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-us} f(u)\, du.$$

(ii) $f(t) = |\sin(\omega t)|$

Sol^n: Can take $p = \frac{\pi}{\omega}$. (Assume $\omega > 0$.)

Then,

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-\frac{\pi}{\omega}s}} \int_0^{\pi/\omega} |\sin(\omega t)|\, e^{-st}\, dt$$

sin ≥ 0 on $[0,\pi]$

$$= \frac{1}{1-e^{-\pi/\omega s}} \int_0^{\pi/\omega} \sin(\omega t)\, e^{-st} dt$$

$\int e^{at} \sin(bt)\, dt$

$\int (u+iv) = \int u + i\int v$

$$\boxed{J(u+iv) = Ju + iJv}$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \, \mathrm{Im} \int_0^{\pi/\omega} e^{(-s+i\omega)t} \, dt$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \, \mathrm{Im} \left[\frac{e^{-(s+i\omega)\pi/\omega} - 1}{-s+i\omega} \right]$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \, \mathrm{Im} \left[\frac{e^{-\frac{s\pi}{\omega}} \cdot \cancel{e^{-i\pi}}^{\,-1} - 1}{-s+i\omega} \right]$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \, \mathrm{Im} \left[\frac{e^{-s\pi/\omega} + 1}{s-i\omega} \right]$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \left(e^{-s\pi/\omega} + 1 \right) \left(\frac{\omega}{s^2+\omega^2} \right)$$

$$= \frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}} \left(\frac{\omega}{s^2+\omega^2} \right)$$

$$= \frac{e^{\frac{\pi s}{2\omega}} + e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}} - e^{-\frac{\pi s}{2\omega}}} \cdot \frac{\omega}{s^2+\omega^2}$$

$$= \coth\left(\frac{\pi s}{2\omega}\right) \frac{\omega}{s^2+\omega^2}.$$

Q10.

20 June 2022 18:10

$$\log\left(\frac{s+a}{s+b}\right)$$
$$= \log\left(\frac{1+a/s}{1+b/s}\right)$$
$$= \log\left(1+a/s\right) - \log\left(1+b/s\right)$$

$$\mathcal{L}^{-1}\left(\log\left(\frac{s^2+4s+5}{s^2+2s+5}\right)\right).$$

Sol^n Let f be s.t. $\mathcal{L}(f)(s) = \underbrace{\log\left(s^2+4s+5\right)}-\log\left(s^2+2s+5\right).$

Then, $\mathcal{L}(tf(t))(s) = -\dfrac{2s+4}{s^2+4s+5} + \dfrac{2s+2}{s^2+2s+5}.$

$\mathcal{L}(tf(t)) = -F'(s)$

But we can easily take inverse laplace transforms of the above!

$$\mathcal{L}^{-1}\left(\frac{2s+4}{s^2+4s+5}\right) = 2\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right)$$
$$= 2e^{-2t}\cos(t).$$

Similarly, $\mathcal{L}^{-1}\left(\dfrac{2s+2}{s^2+2s+5}\right) = 2e^{-t}\cos(2t).$

Thus, $f(t) = \dfrac{2}{t}\left(-e^{-2t}\cos(t) + e^{-t}\cos(2t)\right).$ ∎

$\uparrow$ lim exists as $t \to 0^+$

Q14.

20 June 2022 18:13

$f \to$ exponential type.

$$\mathcal{L}f = \frac{1}{\sqrt{s^2+1}}.$$

$f * f = ?$

Sol

Solⁿ. By convolution theorem,

$$\mathcal{L}(f*f) = \mathcal{L}(f)\,\mathcal{L}(f)$$
$$= \frac{1}{s^2+1}.$$

$$\therefore f*f = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t).$$

Q15.

20 June 2022 18:15

Evaluate

(ii) $$f(t) = \int_0^\infty \frac{\cos(tx)}{x^2+a^2}\, dx.$$

Sol^n. Take Laplace to get

$$F(s) := \mathcal{L}(f)(s)$$

$$= \int_0^\infty \mathcal{L}\left(\frac{\cos(tx)}{x^2+a^2}\right) dx$$

$$= \int_0^\infty \frac{1}{x^2+a^2} \cdot \frac{s}{s^2+x^2}\, dx$$

$$= \frac{s}{s^2-a^2}\int_0^\infty \left[\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right] dx$$

$$= \frac{s}{s^2-a^2}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^\infty - \frac{1}{s}\tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty\right]$$

$$= \frac{\pi}{2}\ \frac{s}{s^2-a^2} \cdot \left(\frac{1}{a} - \frac{1}{s}\right)$$

$$= \frac{\pi}{2}\ \frac{s}{s^2-a^2} \cdot \frac{s-a}{sa}$$

$$= \frac{\pi}{2a}\ \frac{1}{s+a}.$$

$$\therefore\ f(t) = \mathcal{L}^{-1}(F)(t) = \frac{\pi}{2a} e^{-at}.$$

$\blacksquare$

Q16.

20 June 2022 18:18

(iii) $$\frac{dy}{dt} = 1 - \int_0^t y(t-\tau)\,d\tau.$$

$$y(0) = 0.$$

Sol^n. Let $\mathbb{1} : (0, \infty) \longrightarrow \mathbb{R}$ be the constant function $\mathbb{1}(t) = 1 \ \forall t$.

Then, $$\int_0^t y(t-\tau)\,d\tau = \int_0^t \mathbb{1}(\tau)\, y(t-\tau)\,d\tau$$

$$= (\mathbb{1} * y)(t).$$

Thus, we have $\boxed{y' = 1 - \mathbb{1} * y.}$

Taking $\mathcal{L}$ gives

$$\mathcal{L}(y') = \mathcal{L}(\mathbb{1}) - \mathcal{L}(\mathbb{1} * y)$$

$\mathcal{L}(f')$ convolution theorem

$$\Rightarrow sY - y(0) = \frac{1}{s} - \mathcal{L}(\mathbb{1})\,\mathcal{L}(y)$$

$$\Rightarrow sY = \frac{1}{s} - \frac{Y}{s}$$

$$\Rightarrow s^2 Y + Y = 1$$

$$\Rightarrow Y = \frac{1}{s^2+1}$$

$$\Rightarrow y(t) = \sin(t).$$

Q17.

20 June 2022 18:29

(v) $$y_1' = y_2 - 5\sin t \quad ; \quad y_1(0) = 5$$
$$y_2' = -4y_1 + 17\cos t \quad ; \quad y_2(0) = 2$$

Sol^n. As before, take Laplace to get

$$sY_1 - 5 = Y_2 - \frac{5}{s^2+1} \Rightarrow sY_1 - Y_2 = 5 + \frac{5}{s^2+1}$$

$$sY_2 - 2 = -4Y_1 + \frac{17s}{s^2+1} \Rightarrow 4Y_1 + sY_2 = 2 + \frac{17s}{s^2+1}$$

This gives $$(s^2+4)Y_1 = 5s\left(1 + \frac{1}{s^2+1}\right) + 2 + \frac{17s}{s^2+1}$$

$$(s^2+4)Y_1 = 5s+2 + \frac{22s}{s^2+1}$$

$$\Rightarrow Y_1 = \frac{5s}{s^2+4} + \frac{2}{s^2+4} + \frac{22s}{(s^2+1)(s^2+4)}$$

$$= \frac{5s}{s^2+4} + \frac{2}{s^2+4} + \frac{22}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

Thus, $$y_1(t) = 5\cos(2t) + \sin(2t) + \frac{22}{3}\cos(t) - \frac{22}{3}\cos(2t)$$

$$\boxed{y_1(t) = -\frac{7}{3}\cos(2t) + \sin(2t) + \frac{22}{3}\cos(t).}$$

Also, $$y_2 = y_1' + 5\sin(t)$$

$$= \frac{14}{3}\sin(2t) + 2\cos(2t) - \frac{22}{3}\sin(t) + 5\sin(t)$$

$$\boxed{y_2(t) = \frac{14}{3}\sin(2t) + 2\cos(2t) - \frac{7}{3}\sin(t).}$$

$$\boxed{y_2(t) = \frac{14}{3}\sin(2t) + 2\cos(2t) - \frac{7}{3}\sin(t).}$$