Tutorial 2

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Disclaimer

These are **not** complete solutions and should not be regarded as such. The purpose of this is to basically get you started and you must fill in the gaps. To be more explicit, if what you care about is marks, then just the solutions written here won't suffice.

Q.1. All of the ODEs are of the form $Mdx + Ndy = 0$.
We just have to fine the condition where $N_x = M_y$. Note that this implies that the ODE is closed. However, all the functions in the question are defined on $\mathbb{R}^2$, which is convex and thus, exact is the same as closed. (In fact, we only need the domain to be simply-connected.)

(i) $M = f(x) + g(y)$ and $M_y = g'(y)$. Similarly, $N_x = h'(x)$. Thus, we need $g'(y) = h'(x)$. However, note that the LHS is a function purely of y and the RHS purely of x. Thus, $g'(y) = h'(x) = c$ for some $c \in \mathbb{R}$. Thus, $g(y) = cy + c_1$ and $h(x) = cx + c_2$ for some constants c_1 and c_2.

(ii) $M_y = 2xy$ and $N_x = 2axy + 2by^2$. Thus, $a = 1$ and $b = 0$.

(iii) $M_y = 2bx + 2cy$ and $N_x = 2bx + 2cy$. Thus, the ODE is exact for any real values of a, b, c, and g.

Q.2. Given that $Mdx + Ndy = 0$ is exact, we need to find a scalar function $u(x, y)$ such that $u_x = M$ and $u_y = N$. (The existence of such a u is guaranteed by theory.)
Then, the general solution is given by $u(x, y) = c$.
The procedure to find such a u is also not difficult and I illustrate it with one question. The other can similarly be solved.

(i)

$$u_x = 3x^2y - 6x \tag{1}$$

$$u_y = x^3 + 2y \tag{2}$$

$$\tag{3}$$

Integrating the first equation with respect to x gives us $u(x, y) = x^3y - 3x^2 + g(y)$.
Calculating u_y using this gives $u_y = x^3 + g'(y)$.
Substituting in (2) gives us that $g'(y) = 2y$.
Thus, one choice of $u(x, y)$ is $u(x, y) = x^3y - 3x^2 + y^2$.
The general solution is thus, $x^3y - 3x^2 + y^2 = c$.

Q.3. In the following, M and N will have the usual meaning. That is, M and N will be chosen such that the ODE in question is $Mdx + Ndy$.

(i) Already exact.

(ii) Already exact.

(iii) x^2

(iv) Note that $M_y = e^{x/y}(1 - x/y)$ and $N_x = -e^{x/y}(1 + x/y)$.
Thus, $\dfrac{N_x - M_y}{M} = \dfrac{2}{y}$ is a function of y alone.
Thus, we may assume μ to be an IF depending only on y. This sets up the equation

$$\frac{d\mu}{dy} = \frac{2}{y}\mu$$

$$\implies \frac{1}{\mu}d\mu = \frac{2}{y}dy$$

$$\implies \ln|\mu| = 2\ln|y| + C$$

Thus, $\mu = y^2$ is one possible IF.

(v) Already exact.

(vi) Note that $M_y = 2y$ and $N_x = y$.
Then, $\dfrac{M_y - N_x}{N} = \dfrac{y}{xy} = \dfrac{1}{x}$, is a function of x only.
Same idea as before gives $\mu = x$ as a possible integrating factor.

Q.4.

$$
\begin{aligned}
& Mdx + Ndy = 0 \\
\implies & Mx\frac{dx}{x} + Ny\frac{dy}{y} = 0 \\
\implies & 2Mx\frac{dx}{x} + 2Ny\frac{dy}{y} = 0 \\
\implies & Mx\frac{dx}{x} + Mx\frac{dx}{x} + Mx\frac{dy}{y} - Mx\frac{dy}{y} + Ny\frac{dx}{x} - Ny\frac{dx}{x} + Ny\frac{dy}{y} + Ny\frac{dy}{y} = 0 \\
\implies & (Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right) = 0 \\
\implies & (Mx + Ny)d(\ln(xy)) + (Mx - Ny)d\left(\ln\left(\frac{x}{y}\right)\right) = 0 \\
\implies & \frac{1}{2}(Mx + Ny)d(\ln(xy)) + \frac{1}{2}(Mx - Ny)d\left(\ln\left(\frac{x}{y}\right)\right) = 0.
\end{aligned}
$$

(i) In the case that $Mx + Ny = 0$, the given ODE transforms to $\frac{1}{2}(Mx - Ny)d\left(\ln\left(\frac{x}{y}\right)\right) = 0$.
Multiplying with the given factor gives $\frac{1}{2}d\left(\ln\left(\frac{x}{y}\right)\right) = 0$, which is clearly exact. Thus, the given factor is indeed an IF.

(ii) Same as above.

Q.5. If μ is an IF of the given ODE, we then have

$$
\begin{aligned}
& \mu_y M + \mu M_y = \mu_x N + \mu N_x \\
\implies & M_y - N_x = N\frac{\mu_x}{\mu} - M\frac{\mu_y}{\mu}.
\end{aligned}
$$

Noting that $\dfrac{\partial}{\partial x}\ln|\mu| = \dfrac{\mu_x}{\mu}$ and $\dfrac{\partial}{\partial y}\ln|\mu| = \dfrac{\mu_y}{\mu}$ gives the result.

(a) $M_y - N_x = Nf(x)$.
In this case, the above equation transforms to

$$M\left(\frac{\mu_y}{\mu}\right) = N\left(\frac{\mu_x}{\mu} - f(x)\right).$$

If $\mu = \exp\left(\int_a^x f(x')\mathrm{d}x'\right)$, then the bracketed term on both sides is 0.

(b) Same as above.

(c) $M_y - N_x = Nf(x) - Mg(y)$.
In this case, the above equation transforms to

$$M\left(g(y) - \frac{\mu_y}{\mu}\right) = N\left(\frac{\mu_x}{\mu} - f(x)\right).$$

If $\mu = \exp\left(\int_a^x f(x')\mathrm{d}x' + \int_a^{y'} g(y')\mathrm{d}y'\right)$, then the bracketed term on both sides is 0.

(i) In this case, we have $M_y - N_x = 4(x - 3y) = \dfrac{2}{x}[2x(x - 3y)]$.
Thus, this is case (a) with $f(x) = 2x^{-1}$. Solve it.

(ii) In this case, we have $M_y - N_x = (-1)[3(x^2 + y^2)]$.
Thus, this is case (b) with $g(y) = -1$. Solve it. (Note that the $-$ will get canceled.)

(iii) In this case, we have $M_y - N_x = 2x + 4y = \frac{2}{x}[x(x + 2y)]$.
Thus, this is case (a) with $f(x) = 2x^{-1}$. Solve it.

Q.6. (i) Can be rearranged to give

$$\frac{1}{x-a} = \frac{1}{y + ay^2}y'.$$

Solve using partial fractions.

(ii)

(iii) We have $M_y - N_x = 4y\sqrt{x^2+y^2} = \left(-\frac{4}{x}\right)N$.
Conclude.

(iv) Substitute $Y = x + y$. Note that $y' = Y' - 1$.

(v) Rearrange to get

$$\frac{1}{y+y^2}y' = \frac{1}{x}.$$

Solve using partial fractions.

(vi)

$$\begin{aligned} x^2y' + 2xy &= \sinh 3x \\ \implies \frac{d}{dx}(x^2y) &= \sinh 3x \\ \implies x^2y &= \frac{1}{3}\cosh 3x + c \end{aligned}$$

(vii) Multiplying both sides with $\sec x$ gives an exact ODE. Solve it.

(viii) Use Q. 10. from the previous sheet.

Q.7. Substitute $y = vx$ in each and solve. Note that $y' = v + v'x$.

Q.8. All of the ODEs are first order linear ODEs of the form $y' + p(x)y = g(x)$.
From the theory done in class, we know that $\exp\left(\int_{x_0}^{x} p(t)\mathrm{d}t\right)$ is an IF.
Moreover, we know that the final solution is given as

$$y = \exp\left(-\int_{x_0}^{x} p(t)\mathrm{d}t\right)\left(\int_{x_1}^{x}\exp\left(-\int_{x_0}^{t}p(u)\mathrm{d}u\right)g(t)\mathrm{d}t + c\right).$$

Note that the constants x_0 and x_1 are arbitrary and one may choose them according to convenience. (This is effectively doing the same as indefinite integration without keeping the constant.)
The first has been done for illustration. The others are done similarly. Only the IFs have been written.

(i) In standard form, we have

$$y' + -\frac{2}{x}y = x^3.$$

Thus, the IF is $\exp\left(\int_1^x \left(-\frac{2}{t}\right)\mathrm{d}t\right) = \frac{1}{x^2}$.

Thus, the solution is given by

$$y = x^2\int x\mathrm{d}x = x^3 + Cx^2.$$

(ii) IF $= e^{2x}$.

(iii) IF $= \cos^3 x$.

(iv) IF $= \operatorname{cosec} x$.

(v) IF $= \sin x$.

(vi) IF $= e^{-mx}$.

Q.9. Under the transformation $y^{1-\alpha} = u$, we have that $(1-\alpha)y^{-\alpha}y' = u'$.
Also, diving the original equation by y^α and multiplying with $1-\alpha$ gives us

$$y^{-\alpha}(1-\alpha)y' + f(x)(1-\alpha)y^{1-\alpha} = (1-\alpha)g(x).$$

The above is equivalent to

$$u' + (1-\alpha)f(x)u = (1-\alpha)g(x).$$

The above is a first order linear ODE.

(i) Not sure why this is given here as this is not a Bernoulli equation. However, the spirit of derivation is the same.
Make the substitution $e^y = u$ to arrive at

$$u' - u = 2x - x^2.$$

This is a first order linear ODE. Solve this and substitute back. (IF $= e^{-x}$.)

(ii) First make the substitution $y + 1 = Y$ to get $2YY' - \dfrac{2}{x}Y^2 = x^4$.
This is (almost) a Bernoulli equation. Substitute $Y^2 = u$ to get a first order linear ODE. Solve that then substitute back to get things in terms of y.

(iii) Divide by x to get

$$y' + \left(\frac{1}{x} + 1\right) y = 1.$$

This is clearly a first order linear ODE. Solve.

(iv)

(v) Clearly a Bernoulli equation after subtracting xy from both sides. Do the substitution mentioned at the beginning with $\alpha = 3$. Solve.

(vi) Divide by x to get a Bernoulli equation with $\alpha = 4$. Solve.

(vii) Rearrange to get

$$\frac{dx}{dy} + \left(-\frac{1}{6y}\right) x = \left(\frac{1}{3y^2}\right) x^4.$$

This is a Bernoulli equation (in the other way). Substitute $x^{-3} = v$.

Q.10. (i) Rearrange to get

$$\frac{dy}{dx} + \left(-\frac{3}{2x}\right) y = \frac{x}{4y}.$$

This is clearly a Bernoulli equation with $\alpha = -1$. Substitute $y^2 = u$ and solve.

(ii) With separation of variables, we run into a problem as we divide by y. We had seen this in the last tutorial's Q.8.(iii).
As a Bernoulli equation, we again have the problem of substituting $y^{-1} = u$.
Now, let us make the substitution given to obtain $-u' = (1-u)u$ or $u' + u = u^2$.
Substituting $u^{-1} = v$ gives us $-v' + v = 1$ or $(e^{-x}v)' = -e^x$.
Thus, the solution is $v = 1 + Ce^x$.
Substituting back gives $u = (1 + Ce^x)^{-1}$. The initial condition gives $1 = (1 + C)^{-1}$ or $C = 0$. Thus, the solution is $u(x) = 1$. This tells us that the solution of the original ODE was $y(x) = 0$.
No wonder dividing by y was a problem.

(iii) Rearrange the given equation as

$$\frac{dx}{dy} + \left(-\frac{1}{2y}\right) x = \left(-\frac{\ln y}{2y}\right) x^3.$$

This is clearly a Bernoulli equation. Substitute $x^{-2} = v$.

(iv) Following the hint given, we get

$$\cos y dz + (\cos^2 y - z)dy = 0.$$

Rearrange to get

$$\frac{dz}{dy} + (-\sec y)\, z = -\cos y.$$

This is a linear first order ODE. Solve.

Q.11. The idea is to find the ODE of the given family of curves by eliminating the arbitrary constant. Then, we replace y' with $-(y')^{-1}$ and then solve the ODE obtained to find the orthogonal family of curves. I shall do the first example.

(i) $x^2 - y^2 = c^2$. Differentiating wrt x gives:
$x - yy' = 0$.
$y \mapsto -\dfrac{1}{y'}$ gives us

$xy' + y = 0$. Dividing with xy separates the variables to give
$\dfrac{1}{y}y' + \dfrac{1}{x} = 0$.

Integrate to obtain $\ln|xy| = C$ or $xy = K$.

Q.12. First, let us find the ODE of the given family. Note that only λ is to be eliminated. a and b are fixed. λ is the parameter that varies to give us the family.

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

Differentiating wrt x

$$\begin{aligned}
\frac{x}{a^2+\lambda} + \frac{yy'}{b^2+\lambda} &= 0\\
\frac{b^2+\lambda}{a^2+\lambda} &= -\frac{yy'}{x}\\
1 + \frac{b^2-a^2}{a^2+\lambda} &= -\frac{yy'}{x}\\
1 + \frac{yy'}{x} &= \frac{a^2-b^2}{a^2+\lambda}\\
a^2+\lambda &= \frac{(a^2-b^2)x}{x+yy'}\\
b^2+\lambda = b^2 - a^2 + \frac{(a^2-b^2)x}{x+yy'} &= \frac{(b^2-a^2)yy'}{x+yy'}
\end{aligned}$$

Now, we substitute the values of $a^2+\lambda$ and $b^2+\lambda$ from the last two equations in the original curve equation to get:

$$\begin{aligned}
\frac{x^2(x+yy')}{(a^2-b^2)x} - \frac{y^2(x+yy')}{(a^2-b^2)yy'} &= 1\\
x(x+yy') - \frac{y(x+yy')}{y'} &= a^2-b^2\\
\frac{(xy'-y)(x+yy')}{y'} &= a^2-b^2
\end{aligned}$$

The above is the ODE of the given family of curves. Now, we substitute $y' \mapsto -\dfrac{1}{y'}$.

However, before doing so, we take a good look at the ODE obtained.
Hopefully, you've taken a good look.
Now, without further adieu, we proceed to obtain the ODE of the orthogonal family of curves as:

$$\begin{aligned}
\frac{(-x/y'-y)(x-y/y')}{-1/y'} &= a^2-b^2\\
\iff \frac{[(x+yy')/y'][(xy'-y)/y']}{1/y'} &= a^2-b^2\\
\iff \frac{(x+yy')(xy'-y)}{y'} &= a^2-b^2
\end{aligned}$$

Et viola! If you had indeed taken a good look earlier, you may have noticed that we have obtained the same ODE. Thus, the family of orthogonal curves is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1.$$

A paradox?

Q.13. (i) We have $y' = 1 + (-x^3)y + x^2y^2$. Thus, $P(x) = 1, Q(x) = -x^3$, and $R(x) = x^2$.
The Bernoulli equation then reads

$$u' - x^3u = x^2u^2.$$

To solve this, we make the substitution $u^{-1} = v$ to obtain

$$v' + x^3v = -x^2.$$

We get the IF as $\exp\left(\frac{1}{4}x^4\right)$.

Thus, the solution is given by

$$v = \exp\left(-\frac{1}{4}x^4\right)\int x^2\exp\left(\frac{x^4}{4}\right)\mathrm{d}x.$$

The last integral cannot be explicitly solved in terms of elementary functions, so we leave it as it is. Using $uv = 1$ gives us u. Thus, the general solution $y(x) = y_1(x) + u(x)$ is given as

$$y(x) = x + \frac{\exp\left(\frac{1}{4}x^4\right)}{\int x^2\exp\left(\frac{x^4}{4}\right)\mathrm{d}x}.$$

(ii) Same idea.

Q.14. (i) We have $y' = 2\sqrt{y} =: f(t, y)$.
Thus, the corresponding integral equation is

$$\phi(t) = \int_0^t 2\sqrt{\phi(s)}\mathrm{d}s$$

$$\begin{aligned}
\phi_0(t) &= 0 \\
\phi_1(t) &= \int_0^t f(s, \phi_0(s))\mathrm{d}s \\
&= \int_0^t 0\mathrm{d}s \\
&= 0
\end{aligned}$$

... oh. Clearly, we shall keep getting $\phi_n(x) = 0$ for all $n \in \mathbb{N}$.
One may verify that $y(x) = 0$ for all $x \in \mathbb{R}$ is indeed a solution to the given IVP.
However, one may note that $\frac{\partial f}{\partial y}$ is not continuous on any rectangle containing the point $(1, 0)$. (It is not even defined at $(1, 0)$.) Thus, the hypothesis of Picard's theorem is not satisfied. It is not surprising that we miss out on the solution $y(x) = x^2$ for $x \geq 0$.

(ii) $y' = xy + 1$ and $y(0) = 1$.
We try to solve the integral equation

$$\phi(t) = 1 + \int_0^t (s\phi(s) + 1)\mathrm{d}s.$$

We define the Picard iterates as

$$\begin{aligned}
\phi_0(t) &= 1 \\
\phi_1(t) &= 1 + \int_0^t (s\phi_0(s) + 1)\mathrm{d}s \\
&= 1 + \int_0^t (s + 1)\mathrm{d}s \\
&= 1 + t + \frac{t^2}{2}
\end{aligned}$$

$$\begin{aligned}
\phi_2(t) &= 1 + \int_0^t \left(s\left(1 + s + \frac{s^2}{2}\right) + 1\right)\mathrm{d}s \\
&= 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}
\end{aligned}$$

$$\begin{aligned}
\phi_3(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8} + \frac{t^5}{15} + \frac{t^6}{48} \\
&\vdots \\
\phi_n(t) &= \sum_{k=0}^{2n} \frac{t^k}{k!!}
\end{aligned}$$

Where $n!!$ is the double factorial of n. (Not the factorial of the factorial of n.)
The double factorial is recursively defined as $n!! = n \cdot (n-2)!!$ with base cases $0!! = 1!! = 1$.

To compare with the exact solution, notice that $y(x)$ defined as

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!!}$$

satisfies the ODE given. To check, simply plug it in the equation and verify. Assume that the series does converge for all real x and that term-by-term differentiation is valid.

(iii)

Q.15. First we prove the following claim:
Claim. $|\sin y_2 - \sin y_1| \le |y_2 - y_1|$ for all $y_1, y_2 \in \mathbb{R}$.

Proof. If $y_1 = y_2$, then the claim is certainly true.
Assume $y_1 \ne y_2$. Then, by LMVT, we have

$$\frac{\sin y_2 - \sin y_1}{y_2 - y_1} = \sin'(y_3) = \cos(y_3),$$

for some y_3 between y_1 and y_2.
(Note that we're not assuming $y_1 < y_2$.)
As $|\cos y_3| \le 1$, the claim follows. □

Armed with this claim, we may proceed as follows:

$$\begin{aligned}
|f(x, y_1) - f(x, y_2)| &= ||\sin y_1| - |\sin y_2|| \\
&\le |\sin y_1 - \sin y_2| \\
&\le |y_1 - y_2|.
\end{aligned}$$

This proves the claim about being Lipschitz with $M = 1$.

Now, we show that f_y does not exist at $(x_0, 0)$ for any $x_0 \in \mathbb{R}$.
Recall, from calculus, the definition of f_y as

$$f_y(x_0, 0) = \lim_{k \to 0} \frac{f(x_0, 0 + k) - f(x_0, 0)}{k},$$

if the limit exists. We show that it does not. For $k \neq 0$, we may note that

$$\begin{aligned}\frac{f(x_0, 0+k) - f(x_0, 0)}{k} &= \frac{|\sin k| + x_0 - 0 - x_0}{k} \\ &= \frac{|\sin k|}{k}\end{aligned}$$

It is clear that the limit of the above expression as $k \to 0$ does not exist.