MA 108 : ODE
Closed, exact, and simply connected

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Simply connected sets

Let $m \in \mathbb{N}$ and $D \subset \mathbb{R}^m$.
We know that a curve in D is simply a $\mathcal{C}^1$ function $c : [a, b] \to D$ where $a, \ b \in \mathbb{R}$ with $a < b$.
For the purpose of this discussion, we shall assume $a = 0$ and $b = 1$.
We say that c can be continuously shrunk to a point $d \in D$ if there is a continuous function $H : [0, 1] \times [0, 1] \to D$ such that

1. $H(0, s) = c(s)$ for every $s \in [0, 1]$,
2. $H(1, s) = d$ for every $s \in [0, 1]$, and
3. $H(t, 1) = H(t, 0)$ for every $t \in [0, 1]$.

This map H is called a homotopy in D between the curve c and the constant curve d.
The domain D is said to be simply-connected if for every simply closed curve c in D, we have a homotopy H between c and some $d \in D$.

Interpretation of the last slide

The map H can be interpreted as follows:
It is a function of two variables. The first function may be thought of as "time".
For every fixed instant of time $t_0 \in [0, 1]$, we get a curve $H(t_0, s)$ as s varies from 0 to 1. This is capturing the "continuous shrink."
Let us look what the three points are saying:

1. At time $t_0 = 0$, the curve drawn is the initial curve c that we started with.
2. At time $t_0 = 1$, the curve is the final point d.
3. At any given time $t_0 \in [0, 1]$, the curve is still a loop, that is, we aren't opening it up.

Alternate definition

The previous definition can also be written in a slightly more concise (but equivalent) way.
Let D and c have the same meaning as before. Moreover, let $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $U^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

We say that D is simply-connected if any loop in D defined by $f : S^1 \to D$ can be contracted to a point: there exists a continuous map $F : U^2 \to D$ such that F restricted to S^1 is f.

This is admittedly a terser definition. It's okay if you don't understand it right away.

A more intuitive idea

For the purpose of MA 108, we will be concerned about the case that $m = 2$. That is, our domains are (open) subsets of $\mathbb{R}^2$. To recall, a subset $\Omega \subset \mathbb{R}^2$ is said to be open if for every $p \in \Omega$, there exists some $r > 0$ such that the open disc of radius r centered at p is a subset of Ω. (You can draw a small enough circle about every point within Ω.)

Loosely speaking, a subset $D \subset \mathbb{R}^2$ will be simply connected if for every (closed) loop $C \subset D$, we have that the points "inside" C are also points of D.

That gives us that any closed loop in the domain can be continuously shrunk (without opening the loop) to a point in the domain.

Examples

The following subsets of $\mathbb{R}^2$ **are** simply connected:

1. $\mathbb{R}^2$,
2. $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$ (the plane after removing the $x-$axis),
3. $\{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x+2)^2 + y^2 < 1\}$ (union of two disjoint open discs).

The following subsets of $\mathbb{R}^2$ are **not** simply connected:

1. $\mathbb{R}^2 \setminus \{(0, 0)\}$ (the plane after removing the origin),
2. the plane after removing any finite set of points,
3. $\mathbb{R}^2 \setminus S^1$, where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle.

Verify that all the examples given above are open subsets of $\mathbb{R}^2$.

Closed and Exact Forms

Definition

A first order ODE $M(x,y) + N(x,y)y^1 = 0$ *is called exact if there is a function* $u(x,y)$ *such that*

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N.$$

Definition

The differential form

$$M(x,y)dx + N(x,y)dy$$

is called closed if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Their connection

Recall that if M and N are "nice enough", then exact $\implies$ closed.
More precisely, if $M, N : D \to \mathbb{R}$ are such that their first partial derivatives exist and are continuous, then $M + Ny^1 = 0$ being exact implies that $M_y = N_x$.

The idea behind the proof was to note that we have $M = u_x$ and $N = u_y$. By hypothesis, we have that u has all of its second partial derivatives continuous. The mixed partials theorem from MA 105 told us that this implies $u_{xy} = u_{yx}$ which gives us the desired equality.
(If you have forgotten the theorem, it is there on the last slide.)

Their connection

Let M, N, D be as before. Now, we additionally assume that D is simply connected. Then we have the following:

$$M + Ny^1 = 0 \text{ is exact} \iff Mdx + Ndy \text{ is closed.}$$

Note that we already had exact $\implies$ closed. Thus, we only need to prove that closed $\implies$ exact.

The idea behind the proof was the following:

Take any closed loop $C \subset D$. Then, the points contained "within" C are also points in D. Thus, the vector field (M, N) is defined completely "within" C. Then, we used Green's Theorem to compute $\displaystyle\oint (M, N) \cdot d\mathrm{r}$, which turns out to be zero as $N_x - M_y = 0$.

Thus, we got that the line integrals of (M, N) are path-independent in D and hence, (M, N) is the gradient of a scalar field.

An example

Note that the additional hypothesis of D being simply connected was indeed required. To see this recall the following tutorial question from MA 105:
Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $M, N : D \to \mathbb{R}$ be given by

$$(M(x,y), N(x,y)) := \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Then, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. However, (M, N) is not the gradient of any scalar field on D.

(How had we shown this? Hint: integrate (M, N) along the unit circle centered at origin.)

Summary

If M and N are good enough functions, then we have

$$\text{exact} \implies \text{closed.}$$

If the domain is simply connected, then we have

$$\text{exact} \iff \text{closed.}$$

Exercise

Note that the domain $D = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is simply connected. (The upper half plane.)
Let $M, N : D \to \mathbb{R}$ be defined as in the earlier example. Find a scalar field u such that $\nabla u = (M, N)$. (The existence of such a field is guaranteed as we have a closed form on a simply connected domain.)

Do the above for the case that $D = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. (The right half plane.)

Mixed Partial Theorem

Theorem

Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Then there is $r > 0$ such that

$$S := \{(x, y) \in \mathbb{R}^2 : |x - x_0| < r \text{ and } |y - y_0| < r\} \subset D.$$

Consider $f : S \to \mathbb{R}$ and suppose f_x and f_y exist on S. If one of the mixed partials f_{xy} or f_{yx} exists on S, and it is continuous at (x_0, y_0), then the other mixed partial exists at (x_0, y_0), and $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.