MA 108: ODEs

The Methods

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§0 INTRODUCTION

This file has been made to serve as a compilation of methods taught in the course MA 108.

The main purpose for making this file is the following - given the scenario regarding the abrupt end of this (Spring 2020) semester, this course had to be abandoned. However, this is a useful course for many and thus, one must at least be aware of the basics.

While this cannot serve as a complete substitute for the course in terms of theory, it's the next best thing to have. Ignoring the finer details of when certain things exist (converge), this document covers the things that one should remember from this course.

Due to this, I also skip proofs and derivations for the most part. For these, I refer the following two sources:

1. Professor Preeti Raman's page for this course - http://www.math.iitb.ac.in/~preeti/ma108-2019/.
2. Professor Gopala K. Srinivasan's notes for this course - http://www.math.iitb.ac.in/~gopal/MA108/ma108_handwritten_notes_2008.pdf.

These are also the references I have used for making this document.

Few things are written in a red box. These can be ignored. The last section can also be ignored.

Also, I have skipped the extremely basic ~~JEE~~ things about classification of ODEs and solving homogeneous ODEs. These can be found at the beginning of the above notes.

A note on abuse of notation

In this document, the following abuse of notation is often done - I write something like “the function $f(t)$” or “the function $u(x, y)$.”
The correct thing to write is “the function f” or “the function u.”
However, sometimes it is more illustrating to abuse notation.

These notes have not been thoroughly proofread and thus, it is possible that there are misteaks[1] (mostly typos). If you find any, do let me know.
You can anonymously send a feedback here - `https://forms.gle/nif2qPuB7GfSbqxt5`.

[1] intended

§1 Exact ODEs

§§1.1 Introduction

Definition 1.1: Exact ODEs

A first order ODE $M(x,y) + N(x,y)y' = 0$ is called exact if there is a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N.$$

Identifying such an ODE

If the functions (M and N) along with the domain are “good enough”, then the ODE is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The Solution

Given any such scalar function $u(x, y)$ as mentioned above, the solution is then given by

$$u(x, y) = c.$$

Thus, the question now reduces to finding such a $u(x, y)$.
This function can be found either via inspection or via the following method:

I: Integrate $\dfrac{\partial u}{\partial x} = M(x, y)$ with respect to x to get

$$u(x, y) = \int M(x, y)dx + k(y),$$

where $k(y)$ is a constant of integration.

II: To determine $k(y)$, differentiate the above equation in Step I with respect to y, to get:

$$\frac{\partial u}{\partial y} = k'(y) + \frac{\partial}{\partial y}\left(\int M(x,y)dx\right).$$

As the given ODE is exact, the LHS is simply $N(x,y)$. We rearrange this to get

$$k'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)dx\right).$$

This can now be used to determine $k(y)$ and hence, u.

Remark. Even though the RHS *looks* like a function of x and y both, it will simplify to just a function of y. This happens precisely because u is exact to begin with.

Exercise(s)

Solve the following ODEs.

1. $(2x + y^2) + 2xyy' = 0.$

2. $(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0.$

§§1.2 Integrating factors

Definition 1.2: Integrating factor

Consider the scenario where we have the ODE

$$M(x,y) + N(x,y)y' = 0$$

and $M_y \neq N_x$, id est, the ODE is not exact.

Sometimes, we may find a scalar function $\mu(x,y)$ such that

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

is exact, id est,

$$(\mu M)_y = (\mu N)_x.$$

Such a function $\mu(x,y)$ is called an integrating factor of the original ODE.

Actually solving it

In practice, we usually try to find an integrating factor which is only a function of x (or y). In this case, we have $\mu_y = 0$ (or $\mu_x = 0$, resp.).
In the respective cases, the equations simplify to:

1. $$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.$$

   Thus, our assumption that μ is just a function of x is valid precisely when the term in the bracket is independent of y.
   Similarly, the other case gives us:

2. $$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

Exercise(s)

Solve the following ODEs.

1. $(8xy - 9y^2) + (2x^2 - 6xy)y' = 0.$

2. $-y + xy' = 0.$

§2 LINEAR ODES

§§2.1 Introduction

Definition 2.1: Linear differential equations

A first order differential equation of the type

$$\frac{dy}{dx} + p(x)y = g(x)$$

is called a linear differential equation in standard form.

We shall assume that $p(x)$ and $g(x)$ are continuous on an open interval $I \subset \mathbb{R}$.

Solving such an ODE

Verify that $\exp\left(\int p(x)dx\right)$ is an integrating factor of the above ODE.

Thus, we can now turn back to technique of the previous section.

§§2.2 Bernoulli's DE

Definition 2.2: Bernoulli's DE

A first order differential equation of the type

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

is called a Bernoulli's differential equation.

Note that if $n = 0$ or 1, then it is a linear DE as well and we already know how to solve that. Thus, we assume that $n \notin \{0, 1\}$.

Solving such an ODE

Substitute

$$u(x) = \frac{1}{y^{n-1}}.$$

Calculate the derivative and rearrange the original ODE to obtain:

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u(x) = q(x).$$

Now, we are back to the case of a linear DE which we can solve.

Exercise(s)

Solve the following DE:

$$6y^2y' - yx = 2x^4.$$

§3 Picard's Iterates

§§3.1 Introduction

The Setup

Consider the initial value problem (IVP)

$$y' = f(t, y); \quad y(a) = b.$$

Corresponding to this, we set up the following *integral equation*

$$\phi(t) = b + \int_a^t f(s, \phi(s))ds.$$

It can be verified a solution ϕ_0 to the integral equation is a solution of the original DE as well. (And vice-versa.)

Now, we describe a method to solve the integral equation.

Picard's Iteration Method

We recursively define a family of functions $\phi_n(t)$ for every $n \geq 0$ as follows:

$$\phi_0 \equiv b,$$

$$\phi_{n+1}(t) = b + \int_a^t f(s, \phi_n(s))ds \quad \text{for } n \geq 0.$$

Under suitable conditions, the sequence of functions (ϕ_n) converges to a function

$$\phi(t) = \lim_{n\to\infty} \phi_n(t),$$

which is a solution to the IVP.

Exercise(s)

Solve the IVP:

$$y'(t) = 2t(1+y); \quad y(0) = 0.$$

§4 Higher order linear ODEs - linear algebra

This section has more theory as compared to the other sections. However, these are some key concepts that must be known.

§§4.1 Solutions

In this section, we shall be considering equations of the following form.

Definition 4.1: Linear ODE

An ordinary differential equation of the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y^{(1)} + p_0(x)y = r(x) \tag{4.1}$$

is a linear ODE of n^{th} order.

We shall be assuming that all the p_js are continuous on an open interval $I \subset \mathbb{R}$.
Note very carefully that the coefficient of $y^{(n)}$ is assumed to be 1. (This could have been replaced with any nonzero constant.)
The ODE (4.1) is said to be homogeneous iff $r \equiv 0$, that is, $r(x) = 0$ for all $x \in I$.

Given such a linear ODE, we have the associated homogeneous linear ODE defined as

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y^{(1)} + p_0(x)y = 0. \tag{4.2}$$

Now, we make the following observation:

> If $y_g(x)$ is the *general* solution of (4.2) and $y_p(x)$ is a *particular* solution of (4.1), then
>
> $$y_p(x) + y_g(x)$$
>
> is the general solution of (4.1).

(Compare the above with what you saw in MA 106 regarding the null-space of A and a particular solution of $A\mathbf{x} = \mathbf{b}$.)

§§4.2 Dimensions

Recall the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Let V denote the set of functions which are solutions of (4.2). It can be verified that V is a vector space with the usual addition and scalar multiplication. (We use the fact that the ODE is homogeneous.)

Theorem 4.2

$$\dim V = n.$$

To elaborate, the dimension of the solution space of

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$$

is equal to n.

§§4.3 Linear independence and the Wronskian

We recall the following from Linear Algebra.

Definition 4.3: Linear Independence

Let $f_1, \ldots, f_n : I \to \mathbb{R}$ be functions defined on some open interval I. The functions are said to be linearly dependent if there exist real numbers $a_1, \ldots, a_n$

not all zero such that

$$a_1 f_1(x) + \cdots + a_n f_n(x) = 0 \quad \forall x \in I.$$

The functions are said to be linearly independent if they are not linearly dependent.

A rephrasing

In other words, the functions are linearly independent if

$$a_1 f_1(x) + \cdots + a_n f_n(x) = 0 \quad \forall x \in I$$

implies $a_1 = \cdots = a_n = 0$.
In yet another words, their linear combination being identically zero is possible if and only if every scalar is 0.

Definition 4.4: Wronskian

Let $f_1, \ldots, f_n : I \to \mathbb{R}$ be sufficiently differentiable functions defined on some open interval I. Their Wronskian is another function defined on I as:

$$W(f_1, \ldots, f_n)(x) := \det \begin{bmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{bmatrix}.$$

Exercise(s)

1. Show that if $f_1, \ldots, f_n$ are linearly independent, then their Wronskian is identically zero. This can be written as $W(f_1, \ldots, f_n) \equiv 0$.

2. Show that the converse is not true.

We now state a case when the converse ***is*** true.

Theorem 4.5

If $y_1(x), \ldots, y_n(x)$ are solutions of a linear homogeneous ODE as (4.2), then they are linearly dependent if and only if $W(y_1, \ldots, y_n) \equiv 0$.

A particular application

Suppose y_1 and y_2 are solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

over an open interval $I \subset \mathbb{R}$. Let $a \in I$ be such that

$$y_1(a) = 0, y_1'(a) = 1, \quad y_2(a) = 1, y_2'(a) = 0.$$

Then, any other solution of the ODE is given by

$$c_1 y_1(x) + c_2 y_2(x).$$

The hypothesis tells us that y_1 and y_2 are two linearly independent solutions of the ODE. As the dimension of the solution space is $2,$ the functions must form a basis.

A result

Suppose y_1 and y_2 are solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

over an open interval $I \subset \mathbb{R}$. Let $a \in I$ be such that

$$y_1(a) = y_2(a), \quad y_1'(a) = y_2'(a).$$

Then, $y_1 = y_2$.

This is to say that if two functions satisfy an ODE and have the same initial conditions, then they are identically equal.

Exercise(s)

Show that the following sets of functions are linearly independent on $\mathbb{R}$ (unless otherwise mentioned):

1. $\{1, x, \ldots, x^n\}$.
2. $\{e^{m_1 x}, \ldots, e^{m_n x}\}$ where $m_1, \ldots, m_n$ are distinct real numbers.
3. $\{e^{mx}, xe^{mx}, \ldots, x^n e^{mx}\}$.
4. $\{\sin x, \sin 2x, \ldots, \sin nx\}$. (Wronskian could get quite messy. Try integration. Recall inner products.)
5. $\{x^m, x^m(\ln x), \ldots, x^m(\ln x)^n\}$. Show that this is linearly independent on $(0, \infty)$.

§§4.4 Abel-Liouville Formula

The formula

Suppose $y_1, \ldots, y_n$ are solutions of

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

over an open interval $I \subset \mathbb{R}$.
Let W denote their Wronskian. Then,

$$\frac{dW}{dx} = -P_1(x)W.$$

Hence, we have

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)dt\right)$$

(Note that here, P_1 is the coefficient of $y^{(n-1)}$ and not y'.)

Corollary 4.6

The above formula shows that either $W \equiv 0$ or that the Wronskian **never** vanishes. (In fact, it never changes sign.)

An application

Suppose y_1 is a solution of

$$y'' + P(x)y' + Q(x)y = 0.$$

Then, a second *linearly independent* solution is given by

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(-\int P(x)dx\right) dx.$$

§5 The Method of Undetermined Coefficients

§§5.1 The first challenge

The Setup

Consider a linear ODE of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0, \tag{5.1}$$

where $a_0, \ldots, a_{n-1}$ are constants. (Real or complex.)
This is a constant coefficients ODE.

Solving it

We plug the trial solution $y = e^{mx}$. This gives us the polynomial equation:

$$m^n + a_{n-1}m^{n-1} + \cdots + a_0 = 0 \tag{5.2}$$

If $m_1, \ldots, m_n$ are *distinct* solutions to the above equation, then $e^{m_1 x}, \ldots, e^{m_n x}$

are n linearly independent solutions of the ODE and thus, we are done.

Repeated roots

Suppose m_0 is a repeated root of (5.2). Suppose that it is repeated k times. Then, one can show that the following k functions are solutions of the original ODE:

$$e^{m_0x}, xe^{m_0x}, \ldots, x^{k-1}e^{m_0x}.$$

These are linearly independent as well.

Thus, we are now done for any case as the total number of roots is always going to be n when counted with multiplicity. (This is assuming that we work in $\mathbb{C}$, which we shall do.)

Getting real

Suppose we are in the case where each a_i is real.

In this case, if $m = a + ib$ is a solution, then so is $m' = a - ib$. Moreover, the

"amount of repetition" will also be same.
Thus, we can replace solutions having $\{e^{(a+ib)x}, e^{(a-ib)x}\}$ with $\{e^{ax}\sin bx, e^{ax}\cos bx\}$.

Similarly considerations apply to functions like $x^k e^{(a+ib)x}$.

Exercise(s)

Solve the following ODEs.

1. $y'' - 3y' + 2y = 0.$
2. $y^{(4)} + y = 0.$
3. $y^{(4)} + y^{(2)} + y = 0.$

§§5.2 Annihilators

Before going into the main problem, let us study annihilators.
In the following we shall use the following notation:

Notation

From here on, we shall write D to mean $\dfrac{d}{dx}$.

Similarly, we have $D^n = \dfrac{d^n}{dx^n}$.

Note that D is an "operator" which "acts" on smooth[a] functions to give another smooth function.

[a] Recall that a smooth function is a function that is infinitely differentiable.

Some arithmetic

The operator follows the usual rules like $DD = D^2$ which is the same as saying

$$\frac{d}{dx}\left(\frac{d}{dx}(f)\right) = \frac{d^2 f}{dx^2}.$$

We also have things like

$$(D+1)^2 = D^2 + 2D + 1.$$

Note that the 1 above is the operator 1, that is, $1f = f$ for any function f. (Don't make the mistake of thinking something like $D(D+1) = D^2$ because $D1 = 0$; the 1 here is not the constant function.)

This also shows how the original ODE (5.1) relates to the polynomial (5.2). To make it clearer, note that the ODE can simply be written as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0.$$

Annihilators of special functions

From the discussion in the previous section, we can already see annihilators of some special functions as follows:

Function	Annihilator
e^{mx}	$D - m$
$x^k e^{mx}$	$(D - m)^{k+1}$
$\sin bx, \cos bx$	$D^2 + b^2$
$x^k \sin bx, x^k \cos bx$	$(D^2 + b^2)^{k+1}$
$e^{ax} \sin bx, e^{ax} \cos bx$	$(D - a)^2 + b^2$
$x^k e^{ax} \sin bx, x^k e^{ax} \cos bx$	$((D - a)^2 + b^2)^{k+1}$

Note that all the annihilators in the above table are polynomials in D, we will usually write an arbitrary such operator as $P(D)$.
With the above things in mind, we proceed to the next subsection.

§§5.3 The main problem

The Setup

We consider a linear ODE of the form

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = Q(x), \tag{5.3}$$

where each a_i is a constant and $Q(x)$ is one of the special functions listed in the table earlier.

The Solution

We do this quite systematically.

I. First, we consider the associated homogeneous equation

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0.$$

This can be solved completely by the methods we saw in §§5.1.
Let the *general* solution of this be $y_g(x)$.

II. As $Q(x)$ was a special function, we take its annihilator $P(D)$ from the table and apply it to both sides of (5.3). This gives us an equation of the form

$$P(D)(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0. \tag{5.4}$$

Note that (5.4) is again a constant coefficients ODE (why? Recall the arithmetic.) and hence, we can solve it completely.
Let this solution by $y_{g'}(x)$.

III. This $y_{g'}$ will be a sum of special functions. It will have all n of the original special functions in y_g and k more. ($k = \deg P(D)$.)
Thus,

$$y_{g'}(x) = y_g(x) + c_1 y_1(x) + \cdots + c_k y_k(x),$$

where the y_is are the k new functions. We now solve for **undetermined coefficients** c_is by substituting the above solution back in (5.3). (Note that y_g

will get completely annihilated and can be ignored for better calculations.)
Thus, once we solve for the c_is, we are done and the final *general* solution is

$$y(x) = y_g(x) + c_1 y_1(x) + \cdots + c_k y_k(x).$$

Slightly more general

Note that the original restriction that $Q(x)$ be a special function was unnecessary. Indeed, we can do better and allow $Q(x)$ to be a *linear combination* of the special functions.
We can solve the smaller equations individually and finally add them (with appropriate scaling) to get the final solution.
To be more explicit in terms of an example:
Consider the ODE $(D^2 + 1)y = e^x + 2\sin x$.
We shall first solve to get the general solution y_g of $(D^2 + 1)y = 0$.
Then, we get a particular solution y_{p_1} of $(D^2 + 1)y = e^x$ using II and III from above.

Similarly, we get a particular solution y_{p_2} of $(D^2+1)y = \sin x$.
Then, the final solution is $y_g + y_{p_1} + 2y_{p_2}$.

A note about calculations

Sometimes, it may be computationally easier to *not* break $Q(x)$ into all of its components.
For example, if $Q(x) = e^x + xe^x + e^{2x}$, it would be better to break the problem into that for $(e^x + xe^x)$ and that for e^{2x}.
The reason for this is that $(D-1)^2$ is an annihilator for the former and would minimise repeated calculations.
(Try some problems yourself, it's easier to just do it and realise what's best!)

§6 Cauchy Euler equations

§§6.1 The first challenge

The Setup

An ODE of the form

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

is called a Cauchy Euler equation. (Each a_i is a constant.)

We shall be interested in solving the above equation only on $(0, \infty)$.

The Solution

We plug the trial solution $y = x^m$. This gives us the polynomial equation:

$$\begin{aligned} m(m-1)\cdots(m-(n-1)) + a_{n-1}m(m-1)\cdots(m-(n-2))+ \\ \cdots + a_1 m + a_0 = 0. \end{aligned} \tag{6.1}$$

Note that if m is a root of the above, then x^m is a solution of the ODE.
Thus, if the above equation has n distinct roots $m_1, \ldots, m_n$, then the original ODE has n linearly independent solutions $x^{m_1}, \ldots, x^{m_n}$ and we are done.

Repeated roots

Suppose m_0 is a repeated root of (6.1). Suppose that it is repeated k times. Then, one can show that the following k functions are solutions of the original ODE:

$$x^{m_0}, x^{m_0} \ln x, \ldots, x^{m_0} (\ln x)^{k-1}.$$

These are linearly independent as well.
Thus, we are now done for any case as the total number of roots is always going to be n when counted with multiplicity. (This is assuming that we work in $\mathbb{C}$, which we shall do.)

Getting real

Suppose $m = a + ib$ is a root. Then, we have

$$\begin{aligned} x^m &:= \exp(m \ln x) && \text{(By definition)} \\ &= \exp(a \ln x + ib \ln x) \\ &= \exp(a \ln x) \cdot \exp(ib \ln x) \\ &= \exp(a \ln x) \cdot \big(\cos(b \ln x) + i \sin(b \ln x)\big) \\ &= x^a (\cos(b \ln x) + i \sin(b \ln x)) \end{aligned}$$

As before, if each a_i is real, then the roots appear in the same multiplicity as their conjugates which will give us the real pair of solutions as:

$$x^a \cos(b \ln x), x^a \sin(b \ln x).$$

In case of repetition twice, we get the four solutions:

$$x^a \cos(b \ln x), x^a \sin(b \ln x), \quad x^a (\ln x) \cos(b \ln x), x^a (\ln x) \sin(b \ln x).$$

The general case is (hopefully) clear.

§§6.2 Welcome back, annihilators

This time, we consider annihilators of polynomial (polylogmial?) functions.
As before, we shall see that it suffices to consider the case of just the monomials first.
The annihilator table is particularly simple this time:

Annihilators of special functions

Function	Annihilator
x^n	$xD - n$
$x^n (\ln x)^k$	$(xD - n)^{k+1}$

A word on xD

Note that xD is the operator which acts on a function as:

$$(xD)(f) = xf'.$$

On the other hand, Dx is an operator which acts as:

$$(Dx)(f) = D(xf) = xf' + f.$$

In particular, $xD \neq Dx$. This means that $(xD)^2 \neq x^2D^2$ and so on.

Note: xD is an operator but there is no xP operator. (A joke.)

Some arithmetic

For ease of calculations, one may note the following useful identities‘:

$$x^2D^2 = xD(xD - 1)$$

$$x^3 D^3 = xD(xD-1)(xD-2)$$

$$\vdots$$

§§6.3 The main problem

The Setup

We consider a linear ODE of the form

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = Q(x), \tag{6.2}$$

where each a_i is a constant and $Q(x)$ is a linear combination of functions of the form $x^k(\ln x)^m$.

The Solution

The steps are now identical as the case of The Method of Undetermined Coefficients .

We shall assume that $Q(x) = x^k(\ln x)^m$ and not a linear combination. The general case follows as before by considering the linear combination of solutions.

I. We first consider the equation

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = 0.$$

This can be solved by the method we saw in §§6.1.

II. We then apply the annihilator of $Q(x)$ from the table earlier.
This again gives us a Cauchy Euler equation.
We solve this again using the method of §§6.1.

III. We follow the step III as in the case of The Method of Undetermined Coefficients to determine the coefficients of the new functions we obtain in Step II.

Note that D^{11} is also an annihilator of x^{10}, however we do not use that.

The first reason is that the calculations would be a nightmare.

Secondly, applying this annihilator to both sides of the equation wouldn't technically give us a Cauchy Euler equation again.

Also, note that the method we have used doesn't require k to be an integer. This would work even if we wish to solve something like $xy'' + y = x^{1/2}$. Another reason to prefer the annihilator $xD - 1/2$.

§7 The Method of Variation of Parameters

§§7.1 Introduction

This is a powerful method to find a particular solution to a linear ODE when the general solution of the associated homogeneous equation is known.

The Setup

Consider the second order ODE

$$y'' + P(x)y' + Q(x)y = R(x). \tag{7.1}$$

Assume that the general solution $y_g(x)$ of the associated homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{7.2}$$

is known.

The Solution - derivation. (Can be ignored.)

Let y_1 and y_2 be two linearly independent solutions of (7.2).
We seek to find a particular solution $y_p(x)$ of (7.1) of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{7.3}$$

(It is our *assumption - I* that we would have a solution of this form.)
Now, we place the following restriction

$$v_1'y_1 + v_2'y_2 \equiv 0. \tag{7.4}$$

(This is *assumption - II*.)
Under this assumption, we get

$$\begin{aligned} y_p' &= v_1y_1' + v_2y_2', \\ y_p'' &= v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2'. \end{aligned}$$

Substituting this back into (7.1) and using the fact that y_1 and y_2 are solutions of (7.2) gives us that

$$v_1' y_1' + v_2' y_2' = R(x). \tag{7.5}$$

Now, note that the system of equations (7.4) and (7.5) can be uniquely solved for v_1' and v_2'. (Since $\{y_1, y_2\}$ is linearly independent and hence, the Wronskian is always nonzero. Recall Corollary 4.6.)
Integrate to obtain v_1 and v_2 and finally, a particular solution y_p is given by (7.3).
The general solution is, as usual, $y_g + y_p$.

The Solution

Let y_1 and y_2 be two linearly independent solutions of (7.2).
Solve the following system of equations for functions $v_1'(x)$ and $v_2'(x)$:

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0, \\ v_1' y_1' + v_2' y_2' &= R(x). \end{aligned}$$

Integrate v_1' and v_2' to get v_1 and v_2. Then, a particular solution of (7.1) is:

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

The general solution of (7.1) is then $y_g + y_p$.

Exercise(s)

Solve $y'' + y = \tan x$.

§8 SOME INTEGRAL FUNCTIONS

Here we study some *integral*[1] functions that will help us later.
The reader may skip this section for now and return to this when referenced later.
We skip the formal discussion about convergence of improper Riemann integrals. (These are different from the integrals done in MA 105.) A discussion can be found in Professor GKS' notes mentioned.

§§8.1 The Gamma Function

Definition 8.1: The Gamma Function

The Gamma function is defined for $a > 0$ as follows:

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt.$$

[1] pun intended

Some results

1. $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{Z}^+ = \{1, 2, \ldots\}$.

2. $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

4. The last two relations now let us calculate Γ for all half-integers.

5. And that's as far as we can get. Other values of Gamma are not known in such elementary terms.

6. Note that we can still calculate ratios such as

$$\Gamma\left(\frac{5}{4}\right) \Big/ \Gamma\left(\frac{1}{4}\right).$$

§§8.2 The Beta Function

Definition 8.2: The Beta Function

This is a function of two variables defined for $a > 0, b > 0$ as

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

Some identities

1. $B(a,b) = B(b,a)$.
2. $B(a+1,b) = \frac{a}{b}B(a,b+1)$.
3. $B(a+1,b) + B(a,b+1) = B(a,b)$.

4. $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$

Theorem 8.3: The Beta-Gamma Relation

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b) \quad \forall\, a, b > 0.$$

Theorem 8.4: Euler's Reflection Formula

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)} \quad \forall\, 0 < a < 1$$

An integral computation

Theorem 8.5

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin \pi a} \quad \text{for } 0 < a < 1$$

Proof. Put $t = \dfrac{1}{1+x}$. The integral transforms as

$$\begin{aligned}
&\int_0^1 (1-t)^{a-1} t^{-a} dt \\
&= B(a, 1-a) \\
&= \Gamma(a)\Gamma(1-a) \\
&= \frac{\pi}{\sin \pi a}.
\end{aligned}$$

$\square$

§9 LAPLACE TRANSFORMS

§§9.1 The Laplace Transform

Definition 9.1: The Laplace Transform

Let $f : (0, \infty) \to \mathbb{R}$ be a good enough function.
The Laplace transform of f is another function denoted as $\mathcal{L}(f)$ or $\mathcal{L}f$ or simply, F. It is defined as follows:

$$(\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t)dt. \tag{9.1}$$

Remarks

1. It is customary to use uppercase letters for the Laplace transform and use the variable s as the argument of the transform. Thus, we write $F(s), X(s), Y(s)$, et cetera for the Laplace transforms of $f(t), x(t), y(t)$, et cetera.

2. We have not stated what “good enough” means. Also we have not stated the

domain of $\mathcal{L}f$. It is precisely whenever the integral in (9.1) exists. More details about sufficient conditions can be found in the notes linked.

3. Notations will be abused a lot and we'll write things like $\mathcal{L}(f) = F(s)$ or $\mathcal{L}(f(t)) = F(s)$. (If you don't realise why this is abuse of notation, then ignorance is bliss and continue.)

A remark

Note that in general, the function $F(s)$ need not exist on the whole real line. Often, there exists some $a \in \mathbb{R}$ such that $F(s)$ exists for $s > a$.

§§9.2 Properties

The advantage of studying Laplace transforms will be seen when we see its many different properties.
We have the following property that's easy to verify:

Linearity of Laplace Transforms

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g).$$

Of course, a and b are real numbers.

Recall from calculus that changing the value of a function at finitely many points has no effect on the integral. Thus, it is possible for two different functions to have the same Laplace transform. However, we do have the following theorem if we demand continuity.

Theorem 9.2: Equality of Laplace transforms

Let $f, g : [0, \infty) \to \mathbb{R}$ be continuous functions such that

$$\mathcal{L}f = \mathcal{L}g.$$

In this case, we have

$$f = g.$$

A trick

One useful trick is differentiate with respect to the parameter. For example, consider:

$$\mathcal{L}(e^{at})(s) = \frac{1}{s-a}$$

Differentiating with respect to a :

$$\mathcal{L}(te^{at})(s) = \frac{1}{(s-a)^2}.$$

Note that the interchanging of $\dfrac{\partial}{\partial a}$ and $\mathcal{L}$ needs justification. (Which we do not provide.)

Theorem 9.3: Derivatives of Laplace

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s).$$

In general,

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}F(s).$$

Theorem 9.4: Laplace of derivatives

$$\mathcal{L}(f'(t)) = sF(s) - f(0),$$
$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0),$$
$$\mathcal{L}(f^{(n)}(t)) = s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0).$$

Theorem 9.5: First Shift Theorem

If

$$\mathcal{L}(f(t)) = F(s),$$

then

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Definition 9.6: Heaviside step function

The Heaviside unit step function $u : \mathbb{R} \to \{0, 1\}$ is defined as

$$u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

For $c \in \mathbb{R}$, the function $u_c(t)$ is defined as $u(t - c)$.

Theorem 9.7: Second Shift Theorem

Suppose $\mathcal{L}f = F(s)$ for $s > a \geq 0$.
If $c > 0$, then we have

$$\mathcal{L}(u_c(t)f(t - c)) = e^{-cs}F(s),$$

for $s > a$.

Theorem 9.8: Laplace transform of periodic functions

Let $p > 0$ be such that $f(t+p) = f(t)$.

$$\mathcal{L}f = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt.$$

§§9.3 Laplace table

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t	$1/s^2$	t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
$u_c(t)$	e^{-cs}/s	e^{at}	$\dfrac{1}{s-a}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$t\sin(\omega t)$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$t\cos(\omega t)$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$e^{at}\sin(\omega t)$	$\dfrac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$\sinh(\omega t)$	$\dfrac{\omega}{s^2-\omega^2}$	$\cosh(\omega t)$	$\dfrac{s}{s^2-\omega^2}$
$e^{at}\sinh(\omega t)$	$\dfrac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\cosh(\omega t)$	$\dfrac{s-a}{(s-a)^2-\omega^2}$

§§9.4 The convolution

Definition 9.9: Convolution

The convolution of functions f and g is another function $f * g$ defined as

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Properties

1. $f * g = g * f$.
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$.
3. $(f * g) * h = f * (g * h)$.
4. $f * 0 = 0 * f = 0$. (Here, 0 denotes the zero *function*.)

Caution

$f * 1 = f$ is **not** true in general.
For instance, $\sin t * 1 = 1 - \cos t$.

Theorem 9.10: Convolution of Laplace

Suppose $\mathcal{L}f$ and $\mathcal{L}g$ exist for all $s > a \geq 0$. Then,

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \qquad \text{for } s > a.$$

§10 SOME EXTRA RESULTS

This subsection can be skipped by the reader. However, I find this quite interesting and thus, I have kept it here anyway.

The sinc integral

In this box, we compute $\displaystyle\int_0^\infty \frac{\sin x}{x}dx.$

It is known that the improper integral cannot be expressed in terms of elementary functions. Thus, it is quite neat (to me) that we can compute the above (definite) integral.
We first define the following function for $t > 0$:

$$f(t) = \int_0^\infty \frac{\sin tx}{x}dx.$$

(It will have to be justified that the integral on the right does indeed exist for all $t > 0$. This is not too tough.)

Taking the Laplace transform on both sides (and interchanging $\int$ and $\mathcal{L}$), we get:

$$\begin{aligned}(\mathcal{L}f)(s) &= \int_0^\infty \frac{1}{x}\mathcal{L}(\sin tx)dx \\ &= \int_0^\infty \frac{1}{x}\cdot\frac{x}{x^2+s^2}dx \\ &= \int_0^\infty \frac{1}{x^2+s^2}dx \\ &= \frac{\pi}{2s} \\ &= \left(\mathcal{L}\left(\frac{\pi}{2}\right)\right)(s).\end{aligned}$$

Thus, $f(t) = \pi/2$, identically. (Both are continuous.)
In particular, we have:

$$f(1) = \int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Generalised Fresnel integrals

We calculate integrals of the form $\int_0^\infty \sin(x^a)dx$ for $a > 1$. The case $a = 2$ is of particular importance in optics or whatever.
Let $a > 1$ be given. For $t > 0$, we define

$$f(t) := \int_0^\infty \sin tx^a dx.$$

Taking Laplace transforms as earlier, we get

$$F(s) = \int_0^\infty \frac{x^a}{s^2 + x^{2a}} dx$$

put $x^{2a} = s^2 u$

$$F(s) = \int_0^\infty \frac{s\sqrt{u}}{s^2(1+u)} \cdot \frac{s^{1/a}}{2a} \cdot u^{\frac{1}{2a}-1} du$$

$$
\begin{aligned}
&= \frac{1}{s^{1-\frac{1}{a}}} \cdot \frac{1}{2a} \int_0^\infty \frac{u^{\frac{1}{2}+\frac{1}{2a}-1}}{1+u} du \\
&= \frac{1}{s^{1-\frac{1}{a}}} \frac{1}{2a} \pi \operatorname{cosec}\left(\frac{\pi}{2} + \frac{\pi}{2a}\right) \qquad \text{(Theorem 8.5.)} \\
&= \frac{\pi}{2a} \frac{1}{s^{1-\frac{1}{a}}} \sec\left(\frac{\pi}{2a}\right).
\end{aligned}
$$

Recall that

$$
\mathcal{L}(t^k) = \frac{\Gamma(k+1)}{s^{k+1}}.
$$

Thus, we see that

$$
f(t) = \frac{\pi}{2a} \sec\left(\frac{\pi}{2a}\right) \frac{t^{-1/a}}{\Gamma\left(1 - \frac{1}{a}\right)}.
$$

In particular (setting $t = 1$), we see that

$$\int_0^\infty \sin x^a dx = \frac{\pi}{2a} \sec\left(\frac{\pi}{2a}\right) \frac{1}{\Gamma\left(1 - \frac{1}{a}\right)}.$$

Thus, we also see that

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Exercise(s)

Show that $\displaystyle\int_0^\infty \cos x^2 dx = \frac{\pi}{2\sqrt{2}}.$

The Beta-Gamma Relation

Let $a, b > 0$. We first calculate the convolution $t^a * t^b$:

$$t^a * t^b = \int_0^t (t-x)^a x^b dx$$

$$\text{put } u = x/t$$

$$= \int_0^1 (t-ut)^a (ut)^b t du$$

$$= t^{a+b+1} \int_0^1 (1-u)^a u^b du$$

$$= t^{a+b+1} B(a+1, b+1).$$

Thus, $t^{a-1} * t^{b-1} = t^{a+b-1} B(a, b)$.
Taking the Laplace transform on both sides gives us:

$$\mathcal{L}(t^{a-1} * t^{b-1}) = B(a, b) \mathcal{L}(t^{a+b-1})$$

$$\implies \mathcal{L}(t^{a-1})\mathcal{L}(t^{b-1}) = B(a,b)\frac{\Gamma(a+b)}{s^{a+b}}$$

$$\implies \frac{\Gamma(a)}{s^a}\frac{\Gamma(b)}{s^b} = B(a,b)\frac{\Gamma(a+b)}{s^{a+b}}$$

$$\implies \Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b).$$

A little integral

Before the next nice result, let us prove the following result:

(10.1) $$\ln N = \int_0^\infty \frac{1}{t}(e^{-t} - e^{-Nt})dx.$$

(That the above exists for all $N > 0$ can be checked.)

$$\int_0^\infty \frac{1}{t}(e^{-t} - e^{-Nt})dx = \int_0^\infty \int_1^N e^{-tx}dxdt$$

$$= \int_1^N \int_0^\infty e^{-tx} dt dx$$
$$= \int_1^N \frac{1}{x} dx$$
$$= \ln N.$$

Another pesky integral

The nice result is almost here. However, first let's get this out of the way

$$\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t} dt = -\int_0^\infty e^{-t} \ln t dt \tag{10.2}$$

Note that

$$\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t} dt$$

$$= \int_0^\infty e^{-t} \frac{d}{dt}\left(\ln\left(\frac{e^t-1}{t}\right)\right) dt \quad \text{(Just compute)}$$

$$= e^{-t} \ln\left(\frac{e^t-1}{t}\right)\Bigg|_0^\infty + \int_0^\infty e^{-t} \ln\left(\frac{e^t-1}{t}\right) dt \quad \text{(by parts)}$$

$$= 0 + \int_0^\infty e^{-t} \ln\left(\frac{e^t-1}{t}\right) dt$$

$$= \int_0^\infty e^{-t} \ln(e^t-1) dt - \int_0^\infty e^{-t} \ln t dt.$$

Thus, it suffices to show that the first integral is zero. Call it I.
We have

$$I = \int_0^\infty e^{-t} \ln(e^t-1) dt$$

put $u = \ln(e^t - 1)$

$$= \int_{-\infty}^{\infty} \frac{ue^u}{(1+e^u)^2} du$$

$$= 0 \quad \text{(the integrand is odd)}$$

Thus, we have established the result.

Smol gamma and big Gamma

The Euler Mascheroni constant γ is defined as

$$\gamma := \lim_{N\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}-\ln N\right).$$

Note that $\displaystyle\int_0^\infty e^{-kt}dt = \frac{1}{k}$.

Thus,

$$1+\frac{1}{2}+\cdots+\frac{1}{N} = \int_0^\infty \sum_{k=1}^N e^{-kt}dt.$$

Now, using (10.1), we get the following:

$$1+\frac{1}{2}+\cdots+\frac{1}{N}-\ln N = \int_0^\infty \left(\sum_{k=1}^N e^{-kt} - \frac{1}{t}(e^{-t}-e^{-Nt})\right) dt$$

Letting $N \to \infty$, we get

$$\lim_{N\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}-\ln N\right)=\int_0^\infty\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)e^{-t}dt$$
$$=-\int_0^\infty e^{-t}\ln t dt.$$

We used (10.2) for the last equality.

Thus, we have another expression for γ, namely

$$\boxed{\gamma=-\int_0^\infty e^{-t}\ln t dt.}$$

Now, recalling big Γ, note that

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$$

$$
\begin{aligned}
\implies \Gamma'(x) &= \int_0^\infty e^{-t} t^{x-1} \ln t dt \\
\implies \Gamma'(1) &= \int_0^\infty e^{-t} \ln t dt \\
&= -\gamma.
\end{aligned}
$$

Thus, we also have

$$\boxed{\gamma = -\Gamma'(1).}$$