ODEs TSC

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ODEs

We know what an ODE is. The order of an ODE is the order of the highest derivative in the equation.

$$\sin\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^3$$
 has order _____.

The ODE is said to be linear if it of the form

$$a_n(x)y^{(n)}(x)+\cdots+a_0(x)y=b(x)$$

for some $n \ge 0$ and functions a_0, \ldots, a_n, b of x.

Solutions

Consider the ODE to be given as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

For example, y' = -x/y.

An explicit solution of the above ODE on an interval I is a function ϕ defined on I such that

$$\phi^{(n)}(x) = f(x, \phi(x), \dots, \phi^{(n-1)}(x))$$

for all $x \in I$. Example: $\phi(x) = \sqrt{25 - x^2}$ on the interval (-5, 5).

An implicit solution is a <u>relation</u> g(x,y)=0 if this relation defines at least one function ϕ which is an explicit solution on some nonempty interval. Example: $x^2+y^2=25$.

Orthogonal trajectories

Suppose we are given a family of curves, indexed by a parameter λ : $F(x, y, \lambda) = 0$. We wish to find the family of orthogonal trajectories.

First, differentiate the above and eliminate the parameter λ . This will now give you an equation involving x, y, y'. Replace y' with -1/y'. Solving this ODE now gives you the family of orthogonal trajectories.

Example: $x^2 + y^2 = \lambda^2$. Differentiating gives x + yy' = 0. Replacing y with -1/y' gives

$$xy'=y$$
.

Solving it gives y = cx $(c \in \mathbb{R})$ as the family of orthogonal trajectories.

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Separable ODEs

An ODE of the form

$$M(x) + N(y)y' = 0$$

is called a separable ODE. It may also be suggestively written as

$$M(x)dx + N(y)dy = 0.$$

The above is solved by "simply integrating". More precisely, if H_1 and H_2 are functions such that $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$, then the general solution is

$$H_1(x) + H_2(y) = c$$

for $c \in \mathbb{R}$.

Homogeneous functions

Recall that a function f of n-variables is called homogeneous of degree d if

$$f(tx_1,\ldots,tx_n)=t^df(x_1,\ldots,x_n)$$

for all $t \neq 0$. Examples: $f(x, y) = (x - y)^2 + xy$, $f(x, y) = y^2 + x^2 \exp(x/y)$.

Definition 1

The first order ODE

$$M(x,y) + N(x,y)y' = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

To solve: put y = xv and things "magically" fall in place by becoming a separable ODE in v.

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Definition

Definition 2

A first order ODE

$$M(x,y) + N(x,y)y' = 0$$

is called exact if there exists a function u(x, y) such that

$$u_x = M$$
 and $u_y = N$.

The general solution to the above ODE is then u(x,y) = c for $c \in \mathbb{R}$.

A <u>necessary</u> condition for the ODE to be exact is $M_y = N_x$.

The above is *also* <u>sufficient</u> if the domain is "nice": for example, if the domain is convex. (More generally, it suffices for the domain to be simply-connected, if you still remember what that means.)

Solving

The question is: how to find u? This is simple, just go by instincts.

You know that $u_x(x,y) = M(x,y)$. So, integrate M with respect to x. Remember that the arbitrary constant you add will be a function of y now. This will leave you with something like

$$u(x,y)=\int M(x,y)dx+k(y).$$

Now, differentiate the above with respect to y and equate it to N(x, y). Things will "magically" get cancelled and you will be left with

$$k'(y) =$$
some function of y .

Just integrate the above to get k(y) and in turn, get u(x, y).

Integrating Factors

Sometimes, the ODE M(x,y)dx + N(x,y)dy = 0 may not be exact. To combat this, we try to find an integrating factor, $\mu(x,y)$, such that the equation

$$\mu M dx + \mu N dy = 0$$

is exact. The above gives us the equation

$$\mu_{\mathbf{y}}\mathbf{M} + \mu\mathbf{M}_{\mathbf{y}} = \mu_{\mathbf{x}}\mathbf{N} + \mu\mathbf{N}_{\mathbf{x}}.$$

Now, we typically assume either $\mu_y=0$ (or $\mu_x=0$) and hope that the remaining terms cancel out nicely in a way that we are actually left with μ_x/μ being only a function of x (or the other way around). More precisely, if $\frac{M_y-N_x}{N}$ is a function of x, then we have an integrating factor μ given by

$$\mu = \exp\left(\int \frac{M_{\rm y}-N_{\rm x}}{\rm N} d{\rm x}\right). \label{eq:mu_pot}$$

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Definition and existence

Definition 3

An initial value problem (IVP) is an ODE of the form

$$y' = f(x, y), y(x_0) = y_0.$$
 (1)

We now see a condition telling us when the above has a solution.

Theorem 4 (Existence)

Let R be a rectangle of the form $(x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b)$. Suppose that f is continuous and bounded on R, say $|f(x, y)| \leq K$ for all $(x, y) \in R$.

Then, (1) has an explicit solution defined on $(x_0 - \delta, x_0 + \delta)$, where $\delta := \min\{a, b/K\}$.

Note that a solution *may* exist on a larger interval. Furthermore, there may be multiple solutions on that given interval itself. We now see when the solution is unique.

Lipschitz

Let f be a function of one variable defined on some interval $I \subseteq \mathbb{R}$. f is said to be Lipschitz continuous if there exists some $L \geqslant 0$ such that

$$|f(x_1)-f(x_2)| \leqslant L|x_1-x_2|$$

for all $x_1, x_2 \in I$.

Now, if f is a function of two variables defined on some $D \subseteq \mathbb{R}^2$, then we say that f is Lipschitz continuous with respect to y if there exists some $L \geqslant 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

Remarks and examples

Any Lipschitz continuous function (of one variable) is continuous. Consequently, if f is Lipschitz continuous with respect to y, then for every fixed x, the function f(x,y) is a continuous in y. However, f may not be continuous in x. For example,

$$f(x, y) = \lfloor x \rfloor + y$$

is Lipschitz continuous in y but f(x,1) is not continuous function.

If f is a differentiable function of one variable with f' bounded, then f is Lipschitz. Consequently, if f is a function of two variables with $\frac{\partial f}{\partial y}$ bounded, then f is Lipschitz with respect to y.

An non-example of Lipschitz function (in y) is: $f(x,y) = \sqrt{|y|}$ defined on $[-1,1] \times [-1,1]$. Similarly, $f(x,y) = y^2$ is not Lipschitz w.r.t. y on \mathbb{R}^2 but is so on bounded domains.

Back to uniqueness

Theorem 5 (Uniqueness)

Suppose that we have the IVP

$$y' = f(x, y), y(x_0) = y_0.$$

As before, suppose f is continuous on

 $R = (x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b)$ and bounded by K. We already saw that the above IVP has a solution defined on $(x_0 - \delta, x_0 + \delta)$.

Furthermore, if f also satisfies the Lipschitz condition with respect to y on R, then the solution is *unique* on that interval.

As before, there may a solution on a larger interval. Moreover, there may still be a larger interval where the solution is unique.

Picard's iteration method

As before, suppose we have the IVP: y' = f(x, y), $y(x_0) = y_0$.

The above differential equation is equivalent to solving the integral equation

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt.$$

We define the Picard's iterates recursively as

$$y_0(x) := y_0,$$

 $y_{n+1}(x) := y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) dt.$

Under the assumptions of the existence-uniqueness theorem, the above converges to the solution y of the IVP defined by $y(x) := \lim_{n \to \infty} y_n(x)$.

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Definition and convention

We had seen what a linear ODE was. A linear ODE of degree n in standard form is one of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = b(x).$$

For example, xy'-10y=0 is *not* in standard form. However, if we are interested in solving the ODE on $(0,\infty)$, then we can put it in standard form as $y'-\frac{10}{x}y=0$.

Disclaimer

Our results will always assume that the ODE is in standard form. This is crucial.

Homogeneous

The standard ODE is said to be homogeneous if b(x) = 0, i.e., it is of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

From now on, "homogeneous" will refer to the above, not the one we had defined earlier.

First order

A first order linear ODE is particular simple, it is of the form

$$y' + P(x)y = Q(x).$$

The above can be solved by multiplying with the integrating factor

$$\mu(x) := \exp\left(\int_{x_0}^x P(t) dt\right).$$

The final solution is also explicitly given by

$$y(x) = \frac{1}{\mu(x)} \left(\int Q(x) \mu(x) dx + c \right).$$

(Bernoulli) If the ODE was instead $y' + P(x)y = Q(x)y^n$ for some $n \neq 0, 1$, then substitute $v = y^{1-n}$ and it will "magically" get reduced to the above.

Second order

Consider the following second order homogeneous linear ODE:

$$y'' + p(x)y' + q(x)y = 0,$$
 (2)

where the functions p and q are continuous on some open interval I.

Theorem 6 (Existence-uniqueness result)

Let $x_0 \in I$, and fix $a, b \in \mathbb{R}$. There is a unique solution y, defined on I, satisfying (2) along with $y(x_0) = a$ and $y'(x_0) = b$.

Theorem 7 (Dimension result)

The solution space of (2) is a two-dimensional real vector space.

Wronskian

Definition 8

Let y_1 and y_2 be differentiable on I. The Wroskian of y_1 and y_2 is defined by

$$W(y_1, y_2)(x) := \det \begin{bmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{bmatrix}.$$

Note that the Wronskian is defined for any two functions, without any mention of any DE.

Wronskian and linear dependence

Recall that two functions y_1 and y_2 are said to be linearly dependent (LD) on I if there exists constants $c_1, c_2 \in \mathbb{R}$ not both zero such that

$$c_1y_1(x) + c_2y_2(x) = 0$$

for all $x \in I$.

Theorem 9

If y_1 and y_2 are LD on I, then $W(y_1, y_2)(x) = 0$ for all $x \in I$.

However, even if $W(y_1, y_2)(x) = 0$ for all $x \in I$, it is **not** necessary that y_1 and y_2 are linearly dependent on I.

Consider I = (-1, 1) and the functions $y_1(x) = x^3$ and $y_2(x) = |x|^3$.

Again, note that no reference to any DE has been made.

Wronskian, linear dependence, and an ODE

Now we make reference to an ODE and also see a (strong!) converse to the previous theorem.

Theorem 10

Let y_1 and y_2 be solutions to y'' + p(x)y' + q(x)y = 0 on an open interval I (as before, p and q are continuous on I). The following are equivalent:

- **1** y_1 and y_2 are linearly dependent on I.
- 2 Their Wronskian vanishes everywhere on *I*.
- 3 Their Wronskian vanishes at one point in I.

What the above theorem tells us about x^3 and $|x|^3$ is that they cannot be the solutions to a standard linear ODE on (-1,1). Note that they are solutions to $x^2y'' - 5xy' + 6y = 0$.

Similarly, x^2 and x^3 are not LD on (-1,1) but their Wronskian vanishes at 0. (Again, both of them are solutions to that non-standard ODE written above.)

Abel's formula

On the previous slide, we saw that if the Wronskian is nonzero at a point, then it must nonzero everywhere. We actually have a more precise relation given by Abel's formula. The notations I, p, q continue to be as before.

Theorem 11 (Abel-Liouville)

Let y_1 and y_2 be any two solutions of y'' + p(x)y' + q(x)y = 0. Then, the Wronskian $W := W(y_1, y_2)$ satisfies the differential equation

$$W'(x) = -p(x)W(x).$$

Consequently, if $x_0 \in I$, then

$$W(x) = W(x_0) \exp\left(-\int_{x_0}^x p(t) dt\right).$$

Getting a second solution

A consequence of the earlier is the following: If y_1 is one solution of

$$y'' + p(x)y' + q(x)y = 0,$$

then a linearly independent solution y_2 to the above (homogeneous) equation is given by

$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int p(x) dx\right)}{y_1(x)^2} dx.$$

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Constant coefficients

ODE in question:

$$y'' + py' + qy = 0.$$

Here p and q are real numbers.

Solution: Find the roots of the quadratic $m^2 + pm + q = 0$. Call them m_1 and m_2 .

Case 1: Roots are real and distinct. A basis for solution is $\{e^{m_1x}, e^{m_2x}\}$.

Case 2: Real repeated root. A basis for solution is $\{e^{m_1x}, xe^{m_1x}\}$.

Case 3: Roots are distinct and not real. In this case, the roots are of the form $a \pm \iota b$. A basis for solution is $\{e^{ax}\cos(bx), e^{ax}\sin(bx)\}$.

Note that basis being $\{y_1, y_2\}$ means that the general solution is given by $c_1y_1 + c_2y_2$ for $c_1, c_2 \in \mathbb{R}$.

Cauchy-Euler

ODE in question:

$$x^2y'' + pxy' + qy = 0.$$

Here p and q are real numbers. The above is **not** in standard form. However, we wish to solve the above on $(0, \infty)$, where it can be put in standard form by dividing by x^2 .

Solution: Find the roots of the quadratic m(m-1) + pm + q = 0. Call them m_1 and m_2 .

- Case 1: Roots are real and distinct. A basis for solution is $\{x^{m_1}, x^{m_2}\}$.
- Case 2: Real repeated root. A basis for solution is $\{x^{m_1}, x^{m_1} \log(x)\}$.
- Case 3: Roots are distinct and not real. In this case, the roots are of the form $a \pm \iota b$. A basis for solution is $\{x^a \cos(b \log(x)), x^a \sin(b \log(x))\}$.