Week 1

11 May 2022 13:30

→ Hi. I'm Aryaman. Almost an alumnus.

→ bit.ly/ma-108 (Recordings of recap and tut - MS Team)

PDFs of whatever I write

MA-108

ODE: A relation involving $x, y, y', \ldots, y^{(n)}$.

$$F(x, y, \ldots, y^{(n)}) = 0.$$

interested in

eg. ① $y' + y = 0$

② $(y')^2 + xe^y + \sin(y'') = 0$

NOT

NOT example: $y(y(x)) = y'(x)$.

$$y^{(n)} = G(x, y, \ldots, y^{(n-1)}) \quad \text{— } (*)$$

EXPLICIT: (explicit)

A solution of $(*)$ is a function ϕ which is defined on some (open) interval I s.t.

$$\phi^{(n)}(x) = G(x, \phi(x), \ldots, \phi^{(n-1)}(x))$$

for all $x \in I$.

IVP:
$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Here, we will try to find an interval I containing x_0 s.t. $\exists\, \phi : I \to \mathbb{R}$ satisfying the above.

Linear ODE: Is an ODE of the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = b(x).$$

The above is said to have order n if $a_n(x) \neq 0$.

EXAMPLE: $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (n \in \mathbb{Z})$

NON-EXAMPLES: ① $yy' + x = 43$

EXAMPLE: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ → Legendre's equation (MA 207)

NON-EXAMPLES: ① $yy' + x = 4^3$

② $e^y = y'$

EXPLICIT SOLUTION:

IVP: $y' = e^{x^2}$; $y(0) = 0$.

$\phi(x) := \int_0^x e^{t^2}\,dt$ is an EXPLICIT SOLUTION.

IMPLICIT: "Don't have $y = \phi(x)$.
Rather, a relation $F(x, y) = 0$
s.t. on some interval I, we can
get a ϕ s.t. $F(x, \phi(x)) = 0$
$\forall x \in I$."

EXAMPLE: ① $yy' + x = 0$.

$y^2 + x^2 = c$

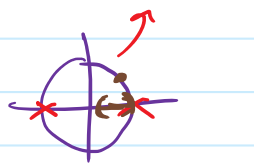
Particular

$y^2 + x^2 = 4$ → Implicit

But this

$y = \sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$

as two solutions.



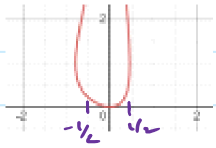
② $\left[F(x, y) = y\sin x + x^2e^y - y\right]$

Let's say $y\sin x + x^2e^y - y = 0$

From the graph, it is "clear" that we can get a function of x on $(-1/2, 1/2)$.

Formally: IMPLICIT FUNCTION THEOREM

Formally: IMPLICIT FUNCTION THEOREM



Note: ① $F(0,0) = 0.$

② $\dfrac{\partial F}{\partial y}(0,0) = \left(\sin(x) + x^2 e^y - 1\right)\Big|_{(x,y)=(0,0)}$

$= -1 \neq 0.$

The above conditions tell you that you can get y as a function x <u>locally</u>.

$F(x, y) = c$

Given (x_0, y_0) s.t.

$F(x_0, y_0) = c.$

<u>Q:</u> Can you can get ϕ on some interval I around x_0

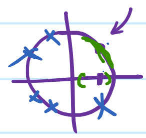
s.t. $F(x, \phi(x)) = c$

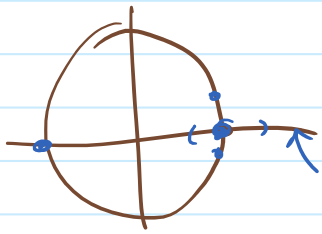
$\forall x \in I.$

$x^2 + y^2 = 4$

$\partial/\partial y$

$2y$





<u>A:</u> (IFT) Yes, if $\dfrac{\partial F}{\partial y}(x_0, y_0) \neq 0.$

Week 2

18 May 2022 12:41

1. Integrating factor
2. Homogeneous ODEs $\rightsquigarrow$ substitute $v = y/x$
3. Orthogonal trajectory $\rightsquigarrow$ example
4. Lipschitz continuity $\rightsquigarrow$

1. Integrating Factors.

$$M\,dx + N\,dy = 0. \qquad (1)$$

functions of x and y

(1) is said to be exact if $M_y = N_x$.

(I will assume domain is nice. For example, convex (or simply-connected, more generally).)

We can then solve the above, i.e., we get a function ϕ s.t. $\phi_x = M$ and $\phi_y = N$.
Then, $\phi = C$ is the gen. solution.

If (1) is NOT exact, we try to find an integrating factor μ s.t.

$$\mu M\,dx + \mu N\,dy = 0$$

is exact.

Thus, we want

$$(\mu M)_y = (\mu N)_x$$

or

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

For simplicity, we assume μ is a function of (say) x alone. Then, $\mu_y \equiv 0$ and we get

$$\mu_x N = \mu(M_y - N_x) \quad \text{or}$$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}.$$

μ $\sim$

4. Let $D \subseteq \mathbb{R}^2$ (say a rectangle).
$f : D \to \mathbb{R}$.

f is said to be Lipschitz (in y) if
$\exists M \geq 0$ s.t.

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

Let $I \subseteq \mathbb{R}$. $g : I \to \mathbb{R}$ is said to be Lipschitz if $\exists L \geq 0$ s.t.

$$|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$$

for all $x_1, x_2 \in I$.

Example:
- $I = \mathbb{R}$, $g(x) = x$. ($L = 1$ works.)
- $I = [1, 2]$, $g(x) = x^2$.

$(x_1 \neq x_2)$ $$\frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|} = |x_1 + x_2| \leq 4.$$

$L = 4$ works.

- $I = [0, 1]$, $g(x) = \sqrt{x}$.

Not Lipschitz.

Suppose not. Then, $\exists L \geq 0$ s.t.

$$|g(x) - g(0)| \leq L|x - 0| \quad \forall x \in I.$$

$\Leftrightarrow$ $\sqrt{x} \leq Lx$ $\forall x \in I$

$\Rightarrow$ $\frac{1}{\sqrt{x}} \leq L$ for all $x \in I \setminus \{0\}$.

But this is a contradiction, choose
$x = \frac{1}{L^2+1} \in I \setminus \{0\}$.

Digression:

$\nearrow$ complete metric space.

$f: X \longrightarrow X$ is conti

s.t.

$$|f(x) - f(y)| \leq \alpha |x-y|$$

for some $0 < \alpha < 1$.

Then, f has a unique fixed pt,

$x \in X$ s.t. $f(x) = x$.

DE : $y' = f(x,y)$

$\Updownarrow$

$$\phi(x) = \int_{x_0}^{x} f(s, \phi(s))\, ds$$

$\|f-g\| = \sup_{x\in[0,1]} |f(x)-g(x)|.$

X = appropriate metric space

$f: X \to X$

$$f(\phi) := \int_{x_0}^{x} f(s, \phi(s))\, ds.$$

Show f is a contraction.

Done.