Q.3.

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Q.3. Find the differential equation of the form $y'' + ay' + by = 0$, where a and b are constants for which the following functions are solutions:

(i) $e^{-2x}, 1$

Solution space here is spanned by e^{-2x} and e^{0x}.
Thus, the charac. polynomial is $(m+2)(m-0)$.
Expanding, we get $m^2 + 2m$.
Thus, the corresponding ODE is

$$y^{(2)} + 2y^{(1)} = 0.$$

Exercise: Find a, b s.t. the solutions to

$$x^2 y'' + a x y' + by = 0$$

are spanned by $\{x^2, 1\}$.

Q.6.

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Q.6. Solve the following:

(i) $y'' - 4y' + 3y = 0$, $\underline{y(0) = 1}$, $\underline{y'(0) = -5}$;

$\downarrow$

Constant coefficients ODE.

$y = e^{mx}$ gives us the characteristic polynomial

$$m^2 - 4m + 3 = (m-3)(m-1).$$

Thus, the general solution is $y = ae^{3x} + be^{x}$.

Now, we need the initial data.

$y(0) = 1 \implies 1 = a + b$ coeff ma

$y'(0) = -5 \implies -5 = 3a + b$

$$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

Solving gives us: $\boxed{\begin{aligned} a &= -3, \\ b &= 4. \end{aligned}}$

Thus, the solution is

$$\boxed{y(x) = -3e^{3x} + 4e^{x}.}$$

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

(Vandermonde)

$$= \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j).$$

Q.7.

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Q.7. Solve the following initial value problems.

(ii) $(D+1)^2y = 0, \quad y(0) = 1, y'(0) = 2$

$$(D+1)^2 f = (D+1)\left[(D+1) f\right]$$
$$= f'' + 2f' + f$$
$$= (D^2 + 2D + 1) f.$$

Q: $(D+1)(D+1) = D^2 + \underbrace{D \cdot 1}_{} + 1 \cdot D + 1 \cdot 1$ ← identity operator

$= D^2 + 2D + 1.$

Aside:
$$(xD) f = x f'$$
$$(Dx) f = D(xf) = xf' + f$$
$$= (xD + 1) f.$$

$$xD \neq Dx. \qquad (xD)^2 \neq x^2 D^2.$$

$$xD(xD - 1) = x^2 D^2$$
$$xD(xD - 1)(xD - 2) = x^3 D^3, \ldots$$

Back to the question: $(D+1)^2 y = 0, \quad y(0) = 1,$
$y'(0) = 2.$

Directly, the char poly is $(m+1)^2 \to -1$ is repeated with multiplicity 2.

Thus, the general solution is

$$y = a e^{-x} + b x e^{-x}.$$

$y(0) = 1 \Rightarrow 1 = a$

$y'(0) = 2 \Rightarrow 2 = -a + b$ $\cancel{\Rightarrow 3b}$

$$\therefore \quad b = 3$$

Thus, the solution is
$$\boxed{\begin{aligned} y(x) &= e^{-x} + 3x e^{-x} \\ &= (1+3x)e^{-x}. \end{aligned}}$$

Q.8.

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Q.8. Solve the following initial value problems.

(ii) $(4x^2D^2 + 4xD - 1)y = 0, y(4) = 2, y'(4) = -1/4$

$y = x^m$: $4m(m-1) + 4m - 1 = 0$

$\Rightarrow 4m^2 - 1 = 0 \Rightarrow m^2 - 1/4 = 0$

$\Rightarrow (m - 1/2)(m + 1/2) = 0.$

Thus, the general solution is $y = a x^{1/2} + b x^{-1/2}.$

$y(4) = 2 \Rightarrow 2 = 2a + b/2$

$y'(4) = -1/4 \Rightarrow -4 = 4a - b$

Thus, $a = 0,$

$b = 4.$

Thus, $\boxed{y(x) = 4x^{-1/2} = 4/\sqrt{x}.}$

Q.12.

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Q.12. Solve the Cauchy-Euler equations: (ii) $x^2y'' + 2xy' - 6y = 0$.

Again, we get the auxillary equation as

$$m(m-1) + 2m - 6 = 0.$$

$$\Rightarrow \quad m^2 + m - 6 = 0$$

$$\Rightarrow \quad (m+3)(m-2) = 0$$

Thus, the general solution is

$$\boxed{y(x) = a x^{-3} + b x^2.}$$

Q.13.

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Q.13. Find the solution of $x^2y'' - xy' - 3y = 0$ satisfying $y(1) = 1$ and $y(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Again, Cauchy-Euler. We get

$$m(m-1) - m - 3 = 0$$

$$\Rightarrow m^2 - 2m - 3 = 0$$

$$\Rightarrow (m+1)(m-3) = 0.$$

Thus, the general solution is

$$y = a/x + bx^3.$$

$$\lim_{y \to \infty} y(x) = 0 \Rightarrow b = 0.$$

Now, $y(1) = 1 \Rightarrow a = 1.$

Thus, the desired solution is

$$\boxed{y(x) = 1/x}.$$