Calculus I Recap

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Autumn Semester 2020-21

Week 1

Start recording!

Definition 1 (Sequences)

A sequence in X is a function $a : \mathbb{N} \to X$. We usually write a_n instead of $a(n)$.

Definition 2 (Convergence)

Let X be a space. Let (a_n) be a sequence in X. Let $L \in X$. We write

$$\lim_{n\to\infty} a_n = L$$

if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon$$

for every $n > N$. L is said to be the *limit* of the sequence.

In this case, we say that (a_n) converges in X.

Note the highlights. They are important. Consider $X = \mathbb{R}$ and the sequence $a_n := 1/n$.
As we saw in class, (a_n) converges to $0 \in \mathbb{R}$. Thus, (a_n) converges in $\mathbb{R}$.

However, consider $X = (0, 1]$ and (a_n) be as earlier. This sequence does not converge (in X) anymore.

Similarly, consider $X = \mathbb{Q}$ and define $a_n = \dfrac{\lfloor 10^n \pi \rfloor}{10^n}$.

$$3.1, 3.14, 3.141, \ldots$$

The above is a sequence in $\mathbb{Q}$. However, it does not converge in $\mathbb{Q}$.

Definition 3 (Cauchy Sequences)

Let X be a space. Let (a_n) be a sequence in X. (a_n) is said to be *Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon$$

for all $n, m > N$.

Proposition 1 (Convergence $\implies$ Cauchy)

If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

Definition 4 (Completeness)

A space X is said to be *complete* if every Cauchy sequence in X converges in X.

Theorem 1 ($\mathbb{R}$ is complete)

$\mathbb{R}$ is complete.

This theorem is trivial and not trivial at the same time. You don't know what $\mathbb{R}$ *truly* is. So you can't really prove this.

Non-examples: We saw some examples earlier. Go back and see that $\mathbb{Q}$ and $(0, 1]$ are **not** complete.

Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

What is $\mathbb{R}$? Well, all one really needs is to know the following two slides about $\mathbb{R}$.

$\mathbb{R}$ is a field. This means that the familiar properties of addition/multiplication are true. (Commutativity, associativity, existence of identity, inverses, and distributivity.)

$\mathbb{R}$ is ordered. There is a binary operation $\leq$ on $\mathbb{R}$ which is reflexive, anti-symmetric, transitive, *and any two elements can be compared.*

$\mathbb{R}$ is an ordered field. All this means is that there is an order which is actually compatible with $+$ and $\cdot$. What does this mean?

$x < y \implies x + z < y + z$ for all $x, y, z \in \mathbb{R}$,
$x < y \implies x \cdot z < y \cdot z$ for all $x, y \in \mathbb{R}$ and $z \in \mathbb{R}_{>0}$.

Note that all the properties earlier are also satisfied by $\mathbb{Q}$. Here's what sets $\mathbb{R}$ apart:

> $\mathbb{R}$ is complete.

There's another way of defining completeness of $\mathbb{R}$, which coincides with the usual. It is the following:

> Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

The least upper bound is called *supremum*.

Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

What one must really ask at this point is: how do we know that $\mathbb{R}$ exists?

That is, how do we know that there is some set $\mathbb{R}$ with some operations $+, \cdot$ and binary relation $<$ which satisfies all the listed properties?

That is what I refer to as a non-trivial part. It can be done but is not useful to us at the moment.

Back to sequences now.

Definition 5 (Monotonically increasing sequences)

A sequence (a_n) is said to be *monotonically increasing* if

$$a_{n+1} \geq a_n$$

for all $n \in \mathbb{N}$.

Similarly, one defines a monotonically decreasing sequence. A sequence is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Definition 6 (Eventually monotonically increasing sequences)

A sequence (a_n) is said to be *eventually monotonically increasing* if there exists $N \in \mathbb{N}$ such that

$$a_{n+1} \geq a_n$$

for all $n \geq N$.

As earlier, we can define eventually monotonically decreasing sequences and simply, eventually monotonic sequences.

Theorem 2

An eventually monotonic sequence in $\mathbb{R}$ which is bounded converges in $\mathbb{R}$.

Again, the above is not true if we take $\mathbb{Q}$ instead of $\mathbb{R}$. The π sequence shows this. In fact, the above is really a consequence of completeness.

We also saw series in the lectures. There's nothing much to be said about it. (As far as this course is concerned.) In reality, there is a lot more to be said about series and various tests for seeing if a series converges. Some of you will see this in future courses like MA 205. Those taking a minor in Mathematics will also come across it in MA 403. Of course, the ones in the Mathematics department will also see it in various courses.

For us, all we need to know is that convergence of a series is just the convergence of the sequence of its *partial sums.* Thus, we are back in the case where we study sequences!

We then moved on to the definition of limits of functions defined on intervals.

For the remainder, we fix $a, b \in \mathbb{R}$ such that $a < b$. (Just to recall, ∞ is not an element of $\mathbb{R}$.)

Definition 7 (Limit)

Let $f : (a, b) \to \mathbb{R}$ be a function. Let $x_0 \in [a, b]$ and $l \in \mathbb{R}$. Then, we write

$$\lim_{x \to x_0} f(x) = l$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.

Note in the above that we can still talk about limits at points at which is the function is *not* defined.

If the thing in the previous slide does happen, then we say that $f(x)$ tends to l as x tends to x_0. Or that f has a limit l at x_0.

If no such l exists, then we say that f does not have any limit at x_0.

We then also defined limit at $\pm\infty$.

Definition 8 (Limit at ∞)

Let $A \subset \mathbb{R}$ be a set which is not bounded above. Let $f : A \to \mathbb{R}$ be a function and let $l \in \mathbb{R}$. We say

$$\lim_{x\to\infty} f(x) = l$$

if for every $\epsilon > 0$, there exists $X \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

for all $x \in A$ such that $x > X$.

Similarly, we have the limit at $-\infty$.

Stop recording. Start a new one.
Take doubts.

Week 2

Start recording!

Last week, we had ended on limits. Today, we *continue* with *continuity*. Ba-dum-tss.

This is quite simple, using whatever we've already seen.

Definition 9 (Continuity)

If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be *continuous at the point c* if (and only if)

$$\lim_{x \to c} f(x) = f(c).$$

We simply say "f is continuous" if it is continuous at every point in the domain. If f is not continuous at a point c *in the domain*, then we say that f is discontinuous at c.

We have the usual rules which tell us that sum/product/composition of continuous functions is continuous. If f is continuous at c and $f(c) \neq 0$, then $1/f$ is continuous at c. We had also seen that the square root function is continuous. We now state an important property of continuous functions.

Definition 10 (Intermediate Value Property)

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Let $u \in \mathbb{R}$ be between $f(a)$ and $f(b)$. Then, there exists $c \in [a, b]$ such that $f(c) = u$.

Note carefully that the domain is an interval.

Now, we state another property, called the extreme value theorem.

Theorem 3 (Extreme value theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, there exist $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$.

Note very carefully that the above not only shows that the image of f is bounded but also that the bounds are attained! Note that the domain was a <u>closed and bounded</u> interval.

Recall that a (non-empty) set which is bounded above can have many upper bounds. However, completeness of $\mathbb{R}$ tells us that there is a *least* upper bound. We had called this the *supremum*.

Similarly, we had defined *infimum*.

By abuse of notation, given a function $f : X \to \mathbb{R}$, if the image $f(X) \subset \mathbb{R}$ is bounded above, then the supremum of the image is called the supremum of f $\underline{\text{on } X}$.
Analogous comments hold for infimum.

Thus, what the previous theorem told us was that not only is the image bounded but the supremum and infimum are actually attained. (If the function is continuous and defined on a close and bounded interval, that is.)

Non-examples of the previous theorem:

Consider $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = x.$$

The image is bounded but the infimum/supremum are not attained.

Consider $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}.$$

The image is not bounded above. It is bounded below but the infimum is not attained.

We saw one interesting result that helps simplify our life in some scenarios.

Theorem 4 (Sequential criterion)

Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Then, f is continuous at a iff given any sequence (a_n) in A such that $a_n \to a$, we have $f(a_n) \to f(a)$.

This makes life simpler because it is sometimes easier to deal with sequences. We had seen an example of this when we proved that a certain oscillatory function does not have a limit. Do you remember which?

Next, we defined derivative. This was also not difficult.

Definition 11 (Derivative)

Let $f : (a, b) \to \mathbb{R}$ be a function and let $c \in (a, b)$. f is said to be *differentiable at the point c* if the following limit exists:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

In such a case, we call the value of the above limit the derivative of f at c and denote it by $f'(c)$.

We then have the usual rules about product/sum/composition of differentiable functions again being differentiable. Of course, we **don't** have the naïve product rule but rather $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$. We then looked at minima/maxima.

Definition 12 (Local maximum)

Let $f : X \to \mathbb{R}$ be a function and let $x_0 \in X$. Suppose that there is an interval $(c, d) \subset X$ containing x_0. If we have $f(x_0) \geq f(x)$ for all $x \in (c, d)$, then we say that f has a *local maximum* at x_0.

Of course, we have an analogous definition for minimum. Note that here, we have that x_0 is an "interior point." That is, there is an interval *around* x_0 contained within the domain.

Theorem 5 (Fermat's Theorem)

If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maxmimum at a point $x_0 \in X$, then $f'(x_0) = 0$.

Once again, note that this only talks about "interior points."

We then saw Rolle's Theorem. Note the hypothesis carefully.

Theorem 6 (Rolle's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is a *continuous* function. Further, assume that it is differentiable on (a, b). In this case, if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Using the above, we have a more general result.

Theorem 7 (Mean Value Theorem)

Let f be continuous and differentiable as above. There exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We then saw a theorem which said "derivatives have IVP." TO be more precise:

Theorem 8 (Darboux's Theorem)

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Let $c < d$ be points in (a, b). Let u be between $f'(c)$ and $f'(d)$. Then, there exists $x_0 \in (c, d)$ such that

$$f'(x_0) = u.$$

Note that the derivative of a (differentiable) function need not be continuous. We shall see an example in the tutorial today, in fact. However, the above theorem tells us how the derivative can't have "jump" discontinuity.

Week 2

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Take doubts.