

# MA 109: Calculus I

## Tutorial Solutions

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## §0. Notations

1.  $\mathbb{N} = \{1, 2, \dots\}$  denotes the set of natural numbers.
2.  $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$  denotes the set of integers.
3.  $\mathbb{Q}$  denotes the set of rational numbers.
4.  $\mathbb{R}$  denotes the set of real numbers.
5.  $\subset$  is used for subset, not necessarily proper.

$$[0, 1] \subset [0, 1]$$

is correct.

6.  $\subsetneq$  is used for “proper subset.”

## §1. Tutorial 1

25th November, 2020

### Sheet 1

2. (iv)  $\lim_{n \rightarrow \infty} (n)^{1/n}$ .

Define  $h_n := n^{1/n} - 1$ .

Then,  $h_n \geq 0$  for all  $n \in \mathbb{N}$ .

(Why?)

Now, for  $n > 2$ , we have

$$\begin{aligned} n &= (1 + h_n)^n \\ &= 1 + nh_n + \binom{n}{2} h_n^2 + \cdots + \binom{n}{n} h_n^n \\ &\geq 1 + nh_n + \binom{n}{2} h_n^2 \\ &> \binom{n}{2} h_n^2 \\ &= \frac{n(n-1)}{2} h_n^2. \end{aligned}$$

Thus,  $h_n < \sqrt{\frac{2}{n-1}}$  for all  $n > 2$ .

Using Sandwich Theorem, we get that  $\lim_{n \rightarrow \infty} h_n = 0$  which gives us that

$$\lim_{n \rightarrow \infty} n^{1/n} = 1.$$

(Where did we use that  $h_n \geq 0$ ?)

3. (ii) We show that  $\left\{(-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n \geq 1}$  is *not* convergent.

*Solution.* Note that from the difference formula, we know that if  $\{a_n\}$  converges, then

$$\lim_{n \rightarrow \infty} |a_{n+1} - a_n| = 0.$$

(The limit *exists* and equals 0.)

We show that this is not true for the given sequence. We define

$$b_n := a_{n+1} - a_n,$$

where  $\{a_n\}$  is the sequence given in the question.

Then,  $b_n$  is given as

$$\begin{aligned} b_n &= (-1)^{n+1} \left(\frac{1}{2} - \frac{1}{n+1}\right) - (-1)^n \left(\frac{1}{2} - \frac{1}{n}\right) \\ &= (-1)^{n+1} \left(\frac{1}{2} - \frac{1}{n+1}\right) + (-1)^{n+1} \left(\frac{1}{2} - \frac{1}{n}\right) \\ &= (-1)^{n+1} + (-1)^n \left(\frac{1}{n+1} + \frac{1}{n}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} |b_n| &= \left| 1 - \left(\frac{1}{n+1} + \frac{1}{n}\right) \right| \\ &= \left| 1 - \frac{2n+1}{n(n+1)} \right| \end{aligned}$$

From the above, we conclude that

$$\lim_{n \rightarrow \infty} |b_n| = 1.$$

This shows that  $a_n$  does not converge. □

5. (iii)  $a_1 = \sqrt{2}$ ,  $a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \geq 1$ .

*Solution.* I first describe the general idea.

The idea in these questions is to first prove a bound on  $a_n$  by induction. Then, using that bound we prove that the sequence is convergent.

Once we do that, we then know that  $\lim_{n \rightarrow \infty} a_n$  exists. Since that also equals  $\lim_{n \rightarrow \infty} a_{n+1}$ , we can take limit on both sides of the equation and solve for the limit  $L$ .

First, we prove that the sequence is bounded above.

Claim 1.  $a_n < 6$  for all  $n \in \mathbb{N}$ .

*Proof.* We shall prove this via induction. The base case  $n = 1$  is immediate as  $2 < 6$ .

Assume that it holds for  $n = k$ . Then,

$$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6.$$

By principle of mathematical induction, we have proven the claim.  $\square$

Claim 2.  $a_n < a_{n+1}$  for all  $n \in \mathbb{N}$ .

*Proof.*  $a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n$ .  $\square$

Thus, we now know that the sequence converges. Let  $L = \lim_{n \rightarrow \infty} a_n$ . Then taking the limit on both sides of

$$a_{n+1} = 3 + \frac{a_n}{2}$$

gives us

$$L = 3 + \frac{L}{2},$$

which we can solve to get  $L = 6$ .  $\square$

7. If  $\lim_{n \rightarrow \infty} a_n = L \neq 0$ , show that there exists  $n_0 \in \mathbb{N}$  such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

*Solution.* Choose  $\epsilon = \frac{|L|}{2}$ . Note that this is indeed greater than 0.

By the  $\epsilon - N$  definition, there exists  $N \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon = \frac{|L|}{2}$$

for all  $n > N$ . Using triangle inequality, we get

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}.$$

Thus, we get

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}.$$

Adding  $|L|$  on both sides gives us

$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2}$$

for all  $n > N$ , as desired. □

9. For given sequences  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$ , prove or disprove the following:

1.  $\{a_n b_n\}_{n \geq 1}$  is convergent, if  $\{a_n\}_{n \geq 1}$  is convergent.
2.  $\{a_n b_n\}_{n \geq 1}$  is convergent, if  $\{a_n\}_{n \geq 1}$  is convergent and  $\{b_n\}_{n \geq 1}$  is bounded.

*Solution.* Both the statements are false. We give one counterexample for both.

$$\begin{array}{ll} a_n := 1 & \text{for all } n \in \mathbb{N}, \\ b_n := (-1)^n & \text{for all } n \in \mathbb{N}. \end{array}$$

Clearly,  $\{a_n\}_{n \geq 1}$  converges and  $\{b_n\}_{n \geq 1}$  is bounded. However, the product is again the latter sequence which does not converge.  $\square$

11. Let  $f, g : (a, b) \rightarrow \mathbb{R}$  be functions and suppose that  $\lim_{x \rightarrow c} f(x) = 0$  for some  $c \in [a, b]$ . Prove or disprove the following statements.

1.  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$ .
2.  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$ , if  $g$  is bounded.
3.  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$ , if  $\lim_{x \rightarrow c} g(x)$  exists.

*Solution.* 1. No. Consider  $a = c = 0$  and  $b = 1$ . Let  $f, g$  be defined as

$$f(x) = x, \quad g(x) = \frac{1}{x}.$$

Verify that this works as a counterexample.

2. We prove this statement. Since  $g$  is bounded, there exists  $M > 0$  such that

$$|g(x)| < M$$

for all  $x \in (a, b)$ . Thus, we have

$$|f(x)g(x)| \leq M|f(x)|$$

for all  $x \in (a, b)$ . Since the LHS is clearly non-negative, using Sandwich theorem proves that

$$\lim_{x \rightarrow c} |f(x)g(x)| = 0.$$

This also gives us that

$$\lim_{x \rightarrow c} f(x)g(x) = 0.$$

(Why?)

3. This is also true. We can simply use that limit of products is the product of limits if the individual limits exist.

□



## §2. Tutorial 2

2nd December, 2020

### Sheet 1

13. (ii) Discuss the continuity of the following function:

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

*Solution.* For  $x \neq 0$ , the continuity of  $f$  at  $x$  follows from the fact that  $f$  is the product and composition of continuous functions.

For  $x = 0$ , we prove continuity using  $\epsilon - \delta$ . We show that

$$\lim_{x \rightarrow 0} f(x) = 0.$$

Since  $f(0) = 0$ , the continuity of  $f$  at 0 will follow.

To this end, let  $\epsilon > 0$  be given. We show that  $\delta := \epsilon$  works. Indeed, if  $0 < |x - 0| < \delta$ , then

$$\begin{aligned} |f(x) - 0| &= \left| x \sin\left(\frac{1}{x}\right) \right| \quad \left. \begin{array}{l} \leq |x| \\ = |x - 0| \\ < \delta = \epsilon. \end{array} \right\} |\sin| \leq 1 \end{aligned}$$

Thus, we have shown that

$$0 < |x - 0| < \delta \implies |f(x) - 0| < \epsilon,$$

proving that

$$\lim_{x \rightarrow 0} f(x) = 0 = f(0),$$

as desired. □

15. Let  $f(x) = x^2 \sin(1/x)$  for  $x \neq 0$  and  $f(0) = 0$ . Show that  $f$  is differentiable on  $\mathbb{R}$ . Is  $f'$  a continuous function?

*Solution.* As earlier, differentiability of  $f$  at  $x \neq 0$  follows due to product/composition rules.

Now, for  $h \neq 0$ , note that

$$\frac{f(0+h) - f(0)}{h} = h \sin\left(\frac{1}{h}\right).$$

As saw earlier, the limit of the above as  $h \rightarrow 0$  exists and is 0. Thus, we get that  $f$  is differentiable at 0 as well with  $f'(0) = 0$ .

Thus,  $f$  is differentiable on  $\mathbb{R}$ .

Now, for  $x \neq 0$ , we can compute the derivative using product/chain rule. Putting this together, we get

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

We now show that  $f'$  is not continuous at 0. We use the sequential criterion for this. Consider the sequence

$$x_n := \frac{1}{2n\pi}, \quad n \in \mathbb{N}.$$

Clearly, we have that  $x_n \rightarrow 0$  and  $x_n \neq 0$ . Thus, we get

$$f'(x_n) = -\cos(2n\pi) = -1.$$

Thus, we see that  $f'(x_n) \rightarrow -1 \neq f'(0)$ .

This shows that  $f'$  is not continuous. □

18. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfy

$$f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

If  $f$  is differentiable at 0, then show that  $f$  is differentiable at  $c \in \mathbb{R}$  and  $f'(c) = f'(0)f(c)$ .

*Solution.* Putting  $x = y = 0$ , we note that  $f(0) = (f(0))^2$ . If  $f(0) = 0$ , show that  $f(x) = 0$  for all  $x$  and conclude that the given thing is indeed true.

Now, assume that  $f(0) \neq 0$ . Then,  $f(0) = 1$ .

Let  $c \in \mathbb{R}$  be arbitrary. For  $h \neq 0$ , we note that

$$\begin{aligned} \frac{f(c + h) - f(c)}{h} &= \frac{f(c)f(h) - f(c)}{h} \\ &= f(c) \frac{f(h) - 1}{h} \\ &= f(c) \frac{f(h) - f(0)}{h}. \end{aligned}$$

Since  $f$  is given to be differentiable at 0, the above limit as  $h \rightarrow 0$  exists and equals  $f(c)f'(0)$ . Thus, we see that  $f'(c)$  exists and equals  $f(c)f'(0)$ .  $\square$

## Sheet 1 Optional

7. Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable and  $c \in (a, b)$ . Show that the following are equivalent:

(i)  $f$  is differentiable at  $c$ .

(ii) There exists  $\delta > 0$ ,  $\alpha \in \mathbb{R}$ , and a function  $\epsilon_1 : (-\delta, \delta) \rightarrow \mathbb{R}$  such that  $\lim_{h \rightarrow 0} \epsilon_1(h) = 0$  and

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

(iii) There exists  $\alpha \in \mathbb{R}$  such that

$$\lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0.$$

*Solution.* We prove this by a usual technique in math by showing that (i)  $\implies$  (ii)  $\implies$  (iii)  $\implies$  (i).

(i)  $\implies$  (ii)

First, we pick  $\delta := \min \{c - a, b - c\}$ . Note that  $\delta > 0$  and  $(c - \delta, c + \delta) \subset (a, b)$ .

Now, since  $f$  is differentiable at  $c$ ,  $f'(c)$  exists. We define  $\alpha := f'(c) \in \mathbb{R}$ .

Now, we define  $\epsilon_1 : (-\delta, \delta) \rightarrow \mathbb{R}$  as

$$\epsilon_1(h) := \begin{cases} \frac{f(c+h) - f(c)}{h} - \alpha & h \neq 0, \\ 0 & h = 0. \end{cases}$$

(Note that  $f(c+h)$  above makes sense because  $(c - \delta, c + \delta) \subset (a, b)$ .)

Now, from the definition above, it is clear that

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

We only need to show that  $\lim_{h \rightarrow 0} \epsilon_1(h) = 0$ . However, note that, for  $h \neq 0$ , we have

$$\epsilon_1(h) = \frac{f(c+h) - f(c)}{h} - \alpha.$$

Since  $f'(c) = \alpha$ , we know that

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

which gives us that  $\epsilon_1(h) \rightarrow 0$  as  $h \rightarrow 0$ , as desired.

(ii)  $\implies$  (iii)

Let  $\alpha$  be as in (ii). Then, for  $h \neq 0$ , we note that

$$\begin{aligned} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \frac{|h\epsilon_1(h)|}{|h|} \\ &= |\epsilon_1(h)|. \end{aligned}$$

Since  $\lim_{h \rightarrow 0} \epsilon_1(h) = 0$ , we get that  $\lim_{h \rightarrow 0} |\epsilon_1(h)| = 0$ , which proves the desired limit.

(iii)  $\implies$  (i)

We show that the  $\alpha$  in (iii) is the derivative of  $f$  at  $c$ . Note that we are given

$$\lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

or

$$\lim_{h \rightarrow 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0.$$

The above gives us that

$$\lim_{h \rightarrow 0} \left( \frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

or

$$\lim_{h \rightarrow 0} \left( \frac{f(c+h) - f(c)}{h} \right) = \alpha.$$

Thus,  $f'(c)$  exists and equals  $\alpha$ . □

In the above, we used the following implicitly:

$$\lim_{x \rightarrow c} f(x) = 0 \iff \lim_{x \rightarrow c} |f(x)| = 0.$$

10. Show that any continuous function  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point.

*Solution.* We need to show that there exists  $x_0 \in [0, 1]$  such that  $f(x_0) = x_0$ . Consider  $g : [0, 1] \rightarrow \mathbb{R}$  defined by

$$g(x) := f(x) - x.$$

Then, showing that  $f$  has a fixed point is equivalent to showing that  $g$  has a zero.

Note that

$$g(0) = f(0) \geq 0$$

and

$$g(1) = f(1) - 1 \leq 0.$$

If either of the equalities hold, then we are done. Otherwise, we have

$$g(0) > 0 \quad \text{and} \quad g(1) < 0.$$

By intermediate value property,  $g(x_0) = 0$  for some  $x_0 \in [0, 1]$ , as desired.  $\square$

**Sheet 2**

- 2 Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a)$  and  $f(b)$  are of different signs and  $f'(x) \neq 0$  for all  $x \in (a, b)$ , show that there is a unique  $x_0 \in (a, b)$  such that  $f(x_0) = 0$ .

*Solution.* The existence of  $x_0$  is given by the intermediate value theorem since 0 lies between  $f(a)$  and  $f(b)$ .

We now show uniqueness. Suppose that there exists  $x_1 \in (a, b)$  such that  $f(x_1) = 0$  and  $x_1 \neq x_0$ . We show that this leads to a contradiction.

By LMVT, there exists  $c$  between  $x_0$  and  $x_1$  such that

$$\begin{aligned} f'(c) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= 0. \end{aligned}$$

A contradiction since  $c \in (a, b)$  and we were given that  $f'(x) \neq 0$  for any  $x \in (a, b)$ .  $\square$

5. Use the MVT to prove that  $|\sin a - \sin b| \leq |a - b|$ , for all  $a, b \in \mathbb{R}$ .

*Solution.* If  $a = b$ , then the inequality is clear. Suppose that  $a \neq b$ .

Then, there exists  $c$  between  $a$  and  $b$  such that

$$\sin'(c) = \frac{\sin a - \sin b}{a - b}.$$

Note that  $\sin' = \cos$  and thus,

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos(c)| \leq 1.$$

Cross-multiplying gives us the desired result. □



## §3. Tutorial 3

9th December, 2020

### Sheet 2

8. In each case, find a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which satisfies all the given conditions, or else show that no such function exists.

- (ii)  $f''(x) \geq 0$  for all  $x \in \mathbb{R}$ ,  $f'(0) = 1$ ,  $f'(1) = 2$ .

*Solution.*  $f(x) := x + \frac{x^2}{2}$  is one such. Justify. □

- (iii)  $f''(x) \geq 0$  for all  $x \in \mathbb{R}$ ,  $f'(0) = 1$ ,  $f(x) \leq 100$  for all  $x > 0$ .

*Solution.* Not possible.

Assume not. As  $f''$  is nonnegative,  $f'$  must be increasing everywhere. We are given that  $f'(0) = 1$ .

Thus, given any  $c > 0$ , we know that

$$f'(c) \geq 1. \quad (*)$$

Let  $x \in (0, \infty)$ . By MVT, we know that there exists  $c \in (0, x)$  such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$

Thus, by  $(*)$ , we have it that  $f(x) \geq x + f(0)$  for all positive  $x$ .

This contradicts that  $f(x) \leq 100$  for all positive  $x$ . (How?) □

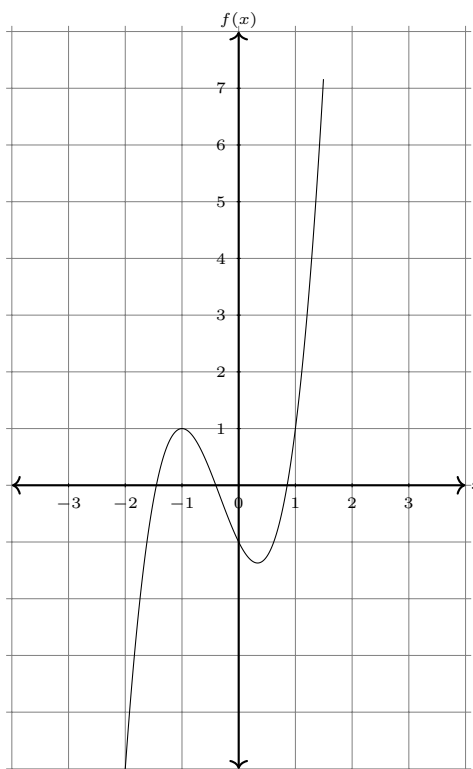
10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

(i)  $f(x) = 2x^3 + 2x^2 - 2x - 1$

*Solution.* Note that this is a cubic and can have at most 3 roots. It is easy to locate that they're in  $(-2, -1)$ ,  $(-1, 0)$  and  $(0, 1)$  since  $f$  changes signs consecutively at  $-2, -1, 0, 1$ .

Moreover,  $f'$  has nice roots:  $-1$  and  $1/3$ .

Lastly,  $f''$  has a root at  $-1/3$ . Using the above, we get pretty much all we want. Calculating  $f(-1)$ ,  $f(1/3)$  and  $f(-1/3)$  also tells us the location of the roots with respect to minima/maxima and inflection point.



Above is the graph.

□

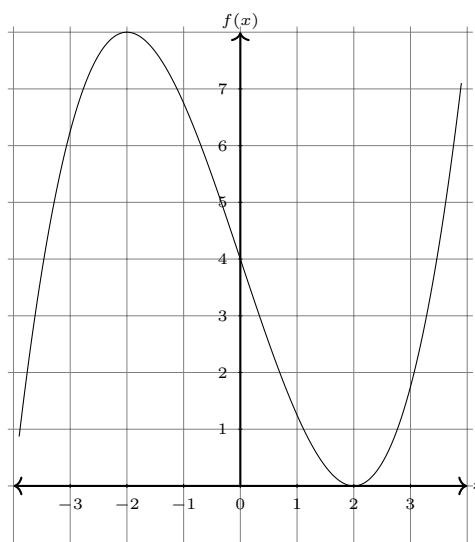
11. Sketch a continuous curve  $y = f(x)$  having all the following properties:

$$f(-2) = 8, f(0) = 4, f(2) = 0; f'(-2) = f(2) = 0;$$

$$f'(x) > 0 \text{ for } |x| > 2, f'(x) < 0 \text{ for } |x| < 2;$$

$$f''(x) < 0 \text{ for } x < 0 \text{ and } f''(x) > 0 \text{ for } x > 0.$$

*Solution.* Here is the graph:



I have actually graphed a polynomial that satisfies the given properties.

Can you come up with it?

Is there a unique such polynomial?

What's the minimum degree of such a polynomial?

Is there a unique polynomial with that degree?

Suppose you have two distinct polynomials  $f$  and  $g$  that satisfy the given conditions. Can you come up with a distinct third polynomial such that it satisfies the conditions as well? □

## Sheet 3

1. Write down the Taylor series for  $\arctan x$  about the point 0. Write down a precise remainder  $R_n(x)$ .

*Solution.* For each of notation, let  $f(x) := \arctan x$  and  $g(x) := \frac{1}{1+x^2}$ .

Note that  $f' = g$ .

Note that if  $n \geq 1$ , then  $f^{(n)}(0) = g^{(n-1)}(0)$ . For  $g$ , we have the easy Taylor expansion as

$$g(x) = 1 - x^2 + x^4 - \dots$$

which is valid for  $x \in (-1, 1)$ .

Thus, we easily see that

$$g^{(n)}(0) = \begin{cases} 0 & n \text{ is odd,} \\ (-1)^{n/2} n! & n \text{ is even.} \end{cases}$$

Thus,

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ is even,} \\ (-1)^{(n-1)/2} (n-1)! & n \text{ is odd.} \end{cases}$$

(The above is for  $n \geq 1$ .) Using this, we get the  $(2n+1)$ -th Taylor polynomial as

$$\begin{aligned} P_{2n+1}(x) &= \sum_{k=0}^{2n+1} \frac{f^{(k)}(0)}{k!} x^k \\ &= f(0) + \sum_{k=1}^{2n+1} \frac{f^{(k)}(0)}{k!} x^k \\ &= 0 + x - \frac{2!}{3!} x^3 + \dots + \frac{(-1)^n (2n)!}{(2n+1)!} x^{2n+1} \\ &= x - \frac{x^3}{3} + \dots + \frac{(-1)^n}{2n+1} x^{2n+1}. \end{aligned}$$

Since  $f^{(2n)} = 0$ , we see that

$$P_{2n}(x) = P_{2n-1}(x)$$

for  $n \geq 1$ .

This solves the problem for finding the Taylor polynomial. Now we solve for the remainder.

Once again, note that

$$g(t) = 1 - t^2 + t^4 - \dots.$$

For  $n \geq 1$ , we note that

$$\begin{aligned} g(t) &= [1 - t^2 + \dots + (-1)^n t^{2n}] + (-1)^{n+1} t^{2n+2} [1 - t^2 + \dots] \\ &= [1 - t^2 + \dots + (-1)^n t^{2n}] + (-1)^{n+1} \frac{t^{2n+2}}{1 + t^2} \end{aligned}$$

Integrating both sides from 0 to  $x$  gives

$$f(x) = P_{2n+1}(x) + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1 + t^2} dt.$$

Thus, the term in red is the  $(2n+1)$ -th remainder  $R_{2n+1}(x)$ . Conclude as before, for  $R_{2n}(x)$ .  $\square$

2. Write down the Taylor series of the polynomial  $x^3 - 3x^2 + 3x - 1$  about the point 1.

*Solution.* As one can easily calculate, we have

$$f^{(n)}(1) = \begin{cases} 6 & n = 3 \\ 0 & n \neq 3, \end{cases}$$

for  $n \geq 0$ . Thus, we get the Taylor “series” to actually be the following finite sum:

$$\frac{f^{(3)}(1)}{3!}(x-1)^3.$$

In other words, the Taylor series is simply  $(x-1)^3$ . □

4. Consider the series  $\sum_{k=0}^{\infty} \frac{x^k}{k!}$  for a fixed  $x$ . Prove that it converges as follows. Choose  $N > 2|x|$ . We see that for all  $n > N$ ,

$$\frac{x^{n+1}}{(n+1)!} \leq \frac{1}{2} \frac{|x|^n}{n!}.$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of  $\mathbb{R}$ ), convergent.

*Solution.* If  $N > 2|x|$  and  $n > N$ , then

$$\begin{aligned} \left| \frac{x^{n+1}}{(n+1)!} \right| &= \left| \frac{x^n}{n!} \right| \left| \frac{x}{n+1} \right| \\ &\leq \left| \frac{x^n}{n!} \right| \left| \frac{x}{N} \right| \\ &\leq \frac{1}{2} \left| \frac{x^n}{n!} \right|. \end{aligned} \quad \begin{array}{l} \left. \begin{array}{l} n+1 > n > N \\ N > 2|x| \end{array} \right\} \end{array}$$

Thus, we can repeatedly use the above to get:

$$\left| \frac{x^{n+1}}{(n+1)!} \right| \leq \frac{1}{2} \left| \frac{x^n}{n!} \right| \leq \cdots \leq \frac{1}{2^{n+1-N}} \left| \frac{x^N}{N!} \right|.$$

$$\text{Let } s_n(x) = \sum_{k=0}^n \frac{x^k}{k!}.$$

Now, given  $m > n > N$ , we have

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{k=n+1}^m \frac{x^k}{k!} \right| \\ &\leq \sum_{k=n+1}^m \left| \frac{x^k}{k!} \right| \\ &= \left| \frac{x^{n+1}}{(n+1)!} \right| + \cdots + \left| \frac{x^m}{m!} \right| \\ &\leq \frac{|x|^N}{N!} \left( \frac{1}{2} + \cdots + \frac{1}{2^{m-n}} \right) \\ &\leq \frac{|x|^N}{N!}. \end{aligned}$$

Note that given any  $\epsilon > 0$ , we can pick  $N \in \mathbb{N}$  such that  $\frac{|x|^N}{N!} < \epsilon$ . Conclude Cauchy-ness.  $\square$

5. Using Taylor series, write down a series for

$$\int \frac{e^x}{x} dx.$$

*Solution.* Note that

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}.$$

Dividing by  $x$  gives

$$\frac{e^x}{x} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}.$$

Integrating both sides gives us

$$\int \frac{e^x}{x} dx = C + \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$$

□