Calculus I Recap

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https://aryamanmaithani.github.io/tuts/ma-109

IIT Bombay

Autumn Semester 2020-21

Start recording!

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Exercise: Show that \mathbb{N}, \mathbb{Z} are complete. (What property do you really need? Can you generalise this?)



Now, we digress a bit to see what $\mathbb R$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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Note that **neither** of the above grey boxes is true if we replace \mathbb{R} by \mathbb{Q} .



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For us, all we need to know is that convergence of a series is just the convergence of the <u>sequence</u> of its *partial sums*. Thus, we are back in the case where we study sequences!

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for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.



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Similarly, we have the limit at $-\infty$.



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Note carefully that the domain is an interval.



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Thus, what the previous theorem told us was that not only is the image bounded but the supremum and infimum are actually attained. (If the function is continuous and defined on a close and bounded interval, that is.)

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Consider $f:(0,1)\to\mathbb{R}$ defined by

$$f(x)=\frac{1}{x}.$$

The image is not bounded above. It is bounded below but the infimum is not attained.



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Once again, note that this only talks about "interior points."

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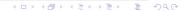
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Stop recording. Start a new one. Take doubts.