

# Extra Questions for MA 109

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Notation:

$\mathbb{N} = \{1, 2, \dots\}$  denotes the set of natural numbers.

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$  denotes the set of integers.

$\mathbb{Q}$  denotes the set of rational numbers.

$\mathbb{R}$  denotes the set of real numbers.

## §1. Sequences

1. Let  $(a_n)$  be a sequence of real numbers. We say that  $(a_n)$  is *slack-convergent* if there is an  $a \in \mathbb{R}$  such that the following condition holds.  
For every  $\epsilon > 0$ , there is  $n_0 \in \mathbb{N}$  such that  $|a_n - a| \leq \epsilon$  for all  $n \geq n_0$ .  
Prove or disprove that a sequence is convergent (in the normal sense)  $\iff$  it is slack-convergent.

**(Additional)** What happens if we change  $n \geq n_0$  to  $n > n_0$ ?

2. Let  $(a_n)$  be a sequence of real numbers. We say that  $(a_n)$  is *reciprocal-convergent* if there is an  $a \in \mathbb{R}$  such that the following condition holds.  
For every  $\epsilon > 0$ , there is  $n_0 \in \mathbb{N}$  such that  $|a_n - a| < 1/\epsilon$  for all  $n \geq n_0$ .  
Prove or disprove that a sequence is convergent (in the normal sense)  $\iff$  it is reciprocal-convergent.
3. Let  $(a_n)$  be a sequence of real numbers. We say that  $(a_n)$  is *natural-convergent* if the following condition holds.  
For every  $k \in \mathbb{N}$ ,  $\lim_{n \rightarrow \infty} |a_{n+k} - a_n| = 0$ .  
Prove or disprove that a sequence is convergent (in the normal sense)  $\iff$  it is natural-convergent.
4. Let  $(a_n)$  be a sequence of real numbers. We say that  $(a_n)$  is *weirdly-convergent* if there is an  $a \in \mathbb{R}$  such that the following condition holds.  
For every  $\epsilon > 0$ , there is  $n_0 \in \mathbb{N}$  such that  $|a_n - a| < \epsilon$  for infinitely many  $n \geq n_0$ .  
Prove or disprove that a sequence is convergent (in the normal sense)  $\iff$  it is weirdly-convergent.
5. Let  $(a_n)$  be a sequence of real numbers. We say that  $(a_n)$  is *reverse-convergent* if there is an  $a \in \mathbb{R}$  such that the following condition holds.  
For every  $n_0 \in \mathbb{N}$ , there is  $\epsilon > 0$  such that  $|a_n - a| < \epsilon$  for all  $n \geq n_0$ .  
Prove or disprove that a sequence is convergent (in the normal sense)  $\iff$  it is reverse-convergent.
6. Let  $(a_n)$  be a bounded sequence such that

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n)$$

exists and equals 0.

Prove/disprove:  $(a_n)$  is convergent.

7. Let  $f$  be any bijection from  $\mathbb{N}$  to  $\mathbb{Q} \cap [0, 1]$ .  
Define the sequence  $(a_n)$  of real numbers as:  $a_n := f(n) \quad \forall n \in \mathbb{N}$ .  
Prove that  $(a_n)$  diverges or find an example of  $f$  such that  $(a_n)$  converges.

8. Let  $S$  be a nonempty subset of  $\mathbb{R}$  which is bounded above. Let  $(a_n)$  be an increasing sequence in  $S$  such that  $\lim_{n \rightarrow \infty} a_n = L \notin S$ . (The sequence converges to a point outside  $S$ .)  
Prove or disprove that  $L = \sup S$ .

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?

## §2. Continuity

1. Show that  $f : \mathbb{N} \rightarrow \mathbb{R}$  is continuous for any  $f$ .
2. Let  $f : \mathbb{Q} \rightarrow \mathbb{R}$  be a continuous function such that the image (range) of  $f$  is a subset of  $\mathbb{Q}$ . Let  $a, b, r \in \mathbb{Q}$  be such that  $a < b$  and  $f(a) < r < f(b)$ . Show (with the help of an example) that it is not necessary that there exists some  $c \in \mathbb{Q} \cap [a, b]$  such that  $f(c) = r$ .
3. (Dirichlet's function)  
Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  as

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

Show that  $f$  is discontinuous everywhere.

4. (Thomae's function)  
Define  $f : [0, 1] \rightarrow \mathbb{R}$  as follows:

$$f(x) := \begin{cases} 0 & x \text{ is irrational,} \\ \frac{1}{n} & x = \frac{m}{n} \text{ in simplest form.} \end{cases}$$

By "simplest form," we mean that  $m \in \mathbb{N} \cup \{0\}$ ,  $n \in \mathbb{N}$  with  $\gcd(m, n) = 1$ . (For 0, it will be  $0/1$ .)

Show that  $f$  is discontinuous at all rationals in  $[0, 1]$  and continuous at all other points in  $[0, 1]$ .

5. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$ . We say that  $f$  is *reverse continuous* at  $c$  if for all  $\delta > 0$ , there exists  $\epsilon > 0$  such that  $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$ .  
Is this notion of continuity the same as the normal notion?  
If not, then give an example of a function which is reverse continuous at a point but not continuous or vice-versa.
6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$ . We say that  $f$  is *upper continuous* at  $c$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|x - c| < \delta \implies f(c) \leq f(x) < f(c) + \epsilon$ .  
  - (a) Prove that a function is continuous at a point if it is upper continuous at that point.
  - (b) Show that the converse may not be true.
  - (c) Give an example of a function that is upper continuous at only one point.
  - (d) Given any  $n \in \mathbb{N}$ , show that there exists a function that is upper continuous at exactly  $n$  points.
  - (e) Show that there exists a function that is upper continuous at infinitely many points.
  - (f) Give an example of a function  $f$  that is upper continuous everywhere.
  - (g) Can you give an example of another function  $g$  such that  $g$  is upper continuous everywhere but  $f - g$  is not constant?
7. Let  $A, B \subset \mathbb{R}$  and  $f : A \rightarrow B$  be a bijection. Show with the help of an example that  $f$  is continuous  $\not\Rightarrow f^{-1}$  is continuous.
8. Show that there exists a bijection from  $(0, 1)$  to  $[0, 1]$ .
9. Show that there exists no continuous bijection from  $(0, 1)$  to  $[0, 1]$  or from  $[0, 1]$  to  $(0, 1)$ .
10. Let  $f : A \rightarrow B$  be a continuous surjective function. Show that it is possible for  $A$  to be a bounded open interval and  $B$  to be a bounded closed interval.  
Is it possible for  $A$  to be a bounded closed interval and  $B$  to be a bounded open interval?
11. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function with the intermediate value property. Is it necessary that  $f$  is continuous *somewhere*?
12. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that given any  $c \in \mathbb{R}$ , the limit  $\lim_{x \rightarrow c} f(x)$  exists. Is it necessary that  $f$  is continuous *somewhere*?

The last two questions are just for one to think about. I do not expect solutions for those.

### §3. Differentiation

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function. Let  $c \in \mathbb{R}$ . Is it necessary that there exist  $a, b \in \mathbb{R}$  such that  $a < c < b$  and  $f'(c) = \frac{f(b) - f(a)}{b - a}$ ?
2. Let  $k \in \mathbb{N}$ . Construct a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that is  $k$  times differentiable everywhere but not  $(k + 1)$  times differentiable somewhere.
3. Construct a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is differentiable at only one point.
4. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable. Suppose there is  $\alpha \in \mathbb{R}$  such that for all  $x \in \mathbb{R}$ ,  $|f'(x)| \leq \alpha < 1$ . Let  $a_1 \in \mathbb{R}$  and set  $a_{n+1} := f(a_n)$  for all  $n \in \mathbb{N}$ . Show that the sequence  $(a_n)$  converges.
5. Let  $D \subset \mathbb{R}$ . A function  $f : D \rightarrow \mathbb{R}$  is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in D, \forall \lambda \in [0, 1].$$

Prove that if  $I$  is an open interval and  $f : I \rightarrow \mathbb{R}$  is convex, then  $f$  is continuous. Where did you use that  $I$  is an open interval?

Give an example to show that if  $J$  is not an open interval, then a convex function  $f : J \rightarrow \mathbb{R}$  need not be continuous.

6. Let  $D \subset \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$  be a differentiable function. Show by example that  $f'(x) = 0 \quad \forall x \in D$  does not imply that  $f$  is constant.
7. Let  $D \subset \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$  be a differentiable function.  
We say that  $f$  is increasing if  $\forall x, y \in D : x \leq y \implies f(x) \leq f(y)$ .  
Show by example that  $f'(x) \geq 0 \quad \forall x \in D$  does not imply that  $f$  is increasing.
8. Show that the implication in the last two questions would be true if  $D$  were an interval.
9. Let  $A$  and  $B$  be open intervals in  $\mathbb{R}$  and  $f : A \rightarrow B$  be a bijection such that  $f$  is differentiable. Show that it is not necessary that  $f^{-1}$  is differentiable.
10. \* Construct a function  $f_1 : \mathbb{R} \rightarrow \mathbb{R}$  with the following properties or show that no such function exists:
  1.  $f_1$  is differentiable everywhere except one point  $x_1$ .
  2. Define  $f_2 : \mathbb{R} \setminus \{x_1\} \rightarrow \mathbb{R}$  as  $f_2(x) :=$  derivative of  $f_1$  at  $x$ . This  $f_2$  must be differentiable everywhere in its domain except one point  $x_2$ .
  3. Define  $f_3 : \mathbb{R} \setminus \{x_1, x_2\} \rightarrow \mathbb{R}$  as  $f_3(x) :=$  derivative of  $f_2$  at  $x$ . This  $f_3$  must be differentiable everywhere in its domain except one point  $x_3$ .
  - $\vdots$
  - $n$ . Define  $f_n : \mathbb{R} \setminus \{x_1, \dots, x_{n-1}\} \rightarrow \mathbb{R}$  as  $f_n(x) :=$  derivative of  $f_{n-1}$  at  $x$ . This  $f_n$  must be differentiable everywhere in its domain except one point  $x_n$ .
  - $\vdots$

(Note that we do not stop at any  $n$ .)

### §4. Riemann integration

1. Define  $f : [0, 2] \rightarrow \mathbb{R}$  as

$$f(x) := \begin{cases} 0 & x \neq 1, \\ 1 & x = 1. \end{cases}$$

Show that  $f$  is Riemann integrable on  $[0, 2]$  and that the integral is 0.

Note that we had  $f \geq 0$  with  $\int_a^b f = 0$  and we still didn't get that  $f$  is identically zero.

2. Suppose  $a < b$  and  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Show that if  $f \geq 0$  and  $\int_a^b f = 0$ , then  $f \equiv 0$ . That is,  $f(x) = 0$  for all  $x \in [a, b]$ .  
Compare this with the previous question.

3. Recall Dirichlet's function  $f : \mathbb{R} \rightarrow \mathbb{R}$  from earlier:

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

Show that  $f$  is not integrable over  $[0, 1]$ .

## §5. General

Most of these questions are above the level of the course.

1. Let  $D \subset \mathbb{R}$ . We say a function  $f : D \rightarrow \mathbb{R}$  is *uniformly continuous* if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $x, y \in D$  and  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .
  - (a) Understand how this definition is different from the definition of (usual) continuity.
  - (b) Give an example of a function which is continuous but not uniformly continuous.
  - (c) Show that any uniformly continuous function is also continuous.
2. Let  $(f_n)$  be a sequence of real valued functions defined on  $[a, b]$  such that each  $f_n$  is continuous. Moreover, you are given that for each  $x \in [a, b]$ , the limit  $\lim_{n \rightarrow \infty} f_n(x)$  exists. Define the function  $f : [a, b] \rightarrow \mathbb{R}$  as follows:

$$f(x) := \lim_{n \rightarrow \infty} f_n(x).$$

Show with the help of an example that it is not necessary that  $f$  is continuous.

3. Let  $f_n : D \rightarrow \mathbb{R}$  be a sequence of functions from the set  $D \subset \mathbb{R}$  to  $\mathbb{R}$ . We say that the sequence  $(f_n)$  *converges uniformly* to the function  $f : D \rightarrow \mathbb{R}$  if given  $\epsilon > 0$ , there exists an integer  $N$  such that

$$|f_n(x) - f(x)| < \epsilon$$

for all  $n > N$  and all  $x \in D$ .

Prove that if  $(f_n)$  is a sequence of continuous functions that converges uniformly to  $f$ , then  $f$  is continuous.

If you have solved the previous question, show that  $(f_n)$  didn't uniformly converge to  $f$  for that example.

4. Let  $f : [a, b] \rightarrow \mathbb{R}$  be any function. Then, we know that if

- (a)  $f$  is monotonic, or
- (b)  $f$  is bounded and has at most a finite number of discontinuities in  $[a, b]$ ,

then  $f$  is (Riemann) integrable.

Is the converse true?

That is, if  $f$  is (Riemann) integrable, then is it necessary that one of (a) or (b) should be true? Prove or disprove via counterexample. (Credit: Amit Kumar)

5. Show that any function  $f : \mathbb{N} \rightarrow \mathbb{R}$  is uniformly continuous.
6. Let  $a \in \mathbb{R}$  and  $(a_n)$  be a sequence of real numbers with the following property: Given any subsequence  $(a_{n_k})$  of  $(a_n)$ , there exists a subsequence  $(a_{n_{k_l}})$  of  $(a_{n_k})$  with the property that  $\lim_{l \rightarrow \infty} a_{n_{k_l}} = a$ . Prove that  $\lim_{n \rightarrow \infty} a_n = a$ .
7. Let  $E$  be a bounded subset of  $\mathbb{R}$  with the following property: There exists  $x_0 \in \mathbb{R} \setminus E$  such that there exists a sequence  $(x_n)$  in  $E$  which converges to  $x_0$ . (For those familiar with the lingo,  $E$  is not a closed set.) Show that there exists:
  - (a) A function  $g : E \rightarrow \mathbb{R}$  which is continuous but not bounded.
  - (b) A function  $f : E \rightarrow \mathbb{R}$  such that  $f(E)$  is bounded but does not have a maximum.
  - (c) A function  $h : E \rightarrow \mathbb{R}$  such that  $h$  is continuous but not uniformly continuous.

8. Let  $f : (a, b) \rightarrow \mathbb{R}$  be a monotonically increasing function, that is,  $a < x < y < b \implies f(x) \leq f(y)$ . Show that for any  $x \in (a, b)$ , both  $\lim_{t \rightarrow x^-} f(t)$  and  $\lim_{t \rightarrow x^+} f(t)$  exist. Moreover, show that  $\lim_{t \rightarrow x^-} f(t) \leq f(x) \leq \lim_{t \rightarrow x^+} f(t)$ .
- Also show that if  $x < y$ , then  $\lim_{t \rightarrow x^+} f(t) \leq \lim_{t \rightarrow y^-} f(t)$ .
- (Hint: Try relating  $\lim_{t \rightarrow x^-} f(t)$  with  $\sup_{a < t < x} f(t)$ .)
9. Let  $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ . Show that given any  $x \in \mathbb{R}$ , there exists a sequence  $(s_n)$  in  $S$  that converges to  $x$ .
- Bonus 1: Generalise the argument by replacing  $\sqrt{2}$  by any irrational square root of a natural number.
- Bonus 2: Generalise the argument by replacing  $\sqrt{2}$  by any irrational number.
10. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic function with period  $p > 0$ . That is,  $f(x + p) = f(x)$  for all  $x \in \mathbb{R}$ . Moreover, assume that  $f$  is Riemann integrable on  $[x, x + p]$  for any  $x \in \mathbb{R}$ . Is it necessary that  $\int_x^{x+p} f(x) dx$  is independent of  $x$ ? (Note that  $f$  is not necessarily continuous.)
11. Let  $A \subset \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$  be a continuous and periodic function.
- Show that if  $A = \mathbb{R}$ , then  $f$  is bounded.
  - Show that there exists some  $A$  and some  $f$  for which the hypothesis holds but  $f$  is not bounded.
12. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that it is differentiable at 0. Is it necessary that there exist  $a < 0 < b$  such that  $f$  is continuous at every point in  $(a, b)$ ?
13. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a monotonically increasing function. Show that the set of discontinuities of  $f$  is countable. (A set  $E$  is said to be countable if there exists a one-to-one function from  $E$  to  $\mathbb{N}$ . Examples -  $\emptyset$ ,  $\{1, 5, 6\}$ ,  $\mathbb{Q}$ .)
14. Show with the help of an example that there exists a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is continuous and bounded but not uniformly continuous.
15. Suppose  $E \subset \mathbb{R}$ . Let  $f : E \rightarrow \mathbb{R}$  be a uniformly continuous function. Show that if  $(x_n)$  is a convergent sequence in  $E$ , then the sequence  $(f(x_n))$  converges in  $\mathbb{R}$ . (Hint: Cauchy)
- Show with the help of an example that the result need not hold if the function is just "continuous."
16. Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions such that  $f(q) = g(q)$  for all  $q \in \mathbb{Q}$ . Show that  $f(x) = g(x)$  for all  $x \in \mathbb{R}$ .
- Is the result true if we drop the continuity hypothesis?
- Can you think of a more general result? More simply, what sort of sets can we replace  $\mathbb{Q}$  with?
17. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function such that  $f'$  is bounded. Show that  $f$  is uniformly continuous.
18. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an infinitely differentiable function. Suppose  $f$  has the property that  $f^{(n)}(0) = 0$  for all  $n \in \mathbb{N}$ . Is it necessary that there exists  $\epsilon > 0$  such that  $f$  is constant in the interval  $(-\epsilon, \epsilon)$ ?
19. Does there exist  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x)$  is rational iff  $x$  is irrational?