Calculus I Recap

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Start recording!

Definition 1 (Sequences)

A sequence in X is a function $a : \mathbb{N} \to X$. We usually write a_n instead of $a(n)$.

Definition 2 (Convergence)

Let X be a space. Let (a_n) be a sequence in X. Let $L \in X$. We write

$$\lim_{n\to\infty} a_n = L$$

if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon$$

for every $n > N$. L is said to be the *limit* of the sequence.

In this case, we say that (a_n) converges in X.

Note the highlights. They are important. Consider $X = \mathbb{R}$ and the sequence $a_n := 1/n$.
As we saw in class, (a_n) converges to $0 \in \mathbb{R}$. Thus, (a_n) converges in $\mathbb{R}$.

However, consider $X = (0, 1]$ and (a_n) be as earlier. This sequence does not converge (in X) anymore.

Similarly, consider $X = \mathbb{Q}$ and define $a_n = \dfrac{\lfloor 10^n \pi \rfloor}{10^n}$.

$$3.1, 3.14, 3.141, \ldots$$

The above is a sequence in $\mathbb{Q}$. However, it does not converge in $\mathbb{Q}$.

Definition 3 (Cauchy Sequences)

Let X be a space. Let (a_n) be a sequence in X. (a_n) is said to be *Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon$$

for all $n, m > N$.

Proposition 1 (Convergence $\implies$ Cauchy)

If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

Definition 4 (Completeness)

A space X is said to be *complete* if every Cauchy sequence in X converges in X.

Theorem 1 ($\mathbb{R}$ is complete)

$\mathbb{R}$ is complete.

This theorem is trivial and not trivial at the same time. You don't know what $\mathbb{R}$ *truly* is. So you can't really prove this.

Non-examples: We saw some examples earlier. Go back and see that $\mathbb{Q}$ and $(0, 1]$ are **not** complete.

Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

What is $\mathbb{R}$? Well, all one really needs is to know the following two slides about $\mathbb{R}$.

$\mathbb{R}$ is a field. This means that the familiar properties of addition/multiplication are true. (Commutativity, associativity, existence of identity, inverses, and distributivity.)

$\mathbb{R}$ is ordered. There is a binary operation $\leq$ on $\mathbb{R}$ which is reflexive, anti-symmetric, transitive, *and any two elements can be compared.*

$\mathbb{R}$ is an ordered field. All this means is that there is an order which is actually compatible with $+$ and $\cdot$. What does this mean?

$x < y \implies x + z < y + z$ for all $x, y, z \in \mathbb{R}$,
$x < y \implies x \cdot z < y \cdot z$ for all $x, y \in \mathbb{R}$ and $z \in \mathbb{R}_{>0}$.

Week 1

Note that all the properties earlier are also satisfied by $\mathbb{Q}$. Here's what sets $\mathbb{R}$ apart:

> $\mathbb{R}$ is complete.

There's another way of defining completeness of $\mathbb{R}$, which coincides with the usual. It is the following:

> Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

The least upper bound is called *supremum.*

Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

What one must really ask at this point is: how do we know that $\mathbb{R}$ exists?

That is, how do we know that there is some set $\mathbb{R}$ with some operations $+, \cdot$ and binary relation $<$ which satisfies all the listed properties?

That is what I refer to as a non-trivial part. It can be done but is not useful to us at the moment.

Back to sequences now.

Definition 5 (Monotonically increasing sequences)

A sequence (a_n) is said to be *monotonically increasing* if

$$a_{n+1} \geq a_n$$

for all $n \in \mathbb{N}$.

Similarly, one defines a monotonically decreasing sequence. A sequence is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Definition 6 (Eventually monotonically increasing sequences)

A sequence (a_n) is said to be *eventually monotonically increasing* if there exists $N \in \mathbb{N}$ such that

$$a_{n+1} \geq a_n$$

for all $n \geq N$.

As earlier, we can define eventually monotonically decreasing sequences and simply, eventually monotonic sequences.

Theorem 2

An eventually monotonic sequence in $\mathbb{R}$ which is bounded converges in $\mathbb{R}$.

Again, the above is not true if we take $\mathbb{Q}$ instead of $\mathbb{R}$. The π sequence shows this. In fact, the above is really a consequence of completeness.

We also saw series in the lectures. There's nothing much to be said about it. (As far as this course is concerned.) In reality, there is a lot more to be said about series and various tests for seeing if a series converges. Some of you will see this in future courses like MA 205. Those taking a minor in Mathematics will also come across it in MA 403. Of course, the ones in the Mathematics department will also see it in various courses.

For us, all we need to know is that convergence of a series is just the convergence of the sequence of its *partial sums.* Thus, we are back in the case where we study sequences!

We then moved on to the definition of limits of functions defined on intervals.

For the remainder, we fix $a, b \in \mathbb{R}$ such that $a < b$. (Just to recall, ∞ is not an element of $\mathbb{R}$.)

Definition 7 (Limit)

Let $f : (a, b) \to \mathbb{R}$ be a function. Let $x_0 \in [a, b]$ and $L \in \mathbb{R}$. Then, we write

$$\lim_{x \to x_0} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.

Note in the above that we can still talk about limits at points at which is the function is *not* defined.

If the thing in the previous slide does happen, then we say that $f(x)$ tends to l as x tends to x_0. Or that f has a limit l at x_0.

If no such l exists, then we say that f does not have any limit at x_0.

We then also defined limit at $\pm\infty$.

Definition 8 (Limit at ∞)

Let $A \subset \mathbb{R}$ be a set which is not bounded above. Let $f : A \to \mathbb{R}$ be a function and let $L \in \mathbb{R}$. We say

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists $X \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

for all $x \in A$ such that $x > X$.

Similarly, we have the limit at $-\infty$.

Stop recording. Start a new one.
Take doubts.

Start recording!

Last week, we had *limited* ourselves to *limits*. Today, we *continue* with *continuity*. Ba-dum-tss.

This is quite simple, using whatever we've already seen.

Definition 9 (Continuity)

If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be *continuous at the point* c if (and only if)

$$\lim_{x \to c} f(x) = f(c).$$

We simply say "f is continuous" if it is continuous at every point in the domain. If f is not continuous at a point c *in the domain*, then we say that f is discontinuous at c.

We have the usual rules which tell us that sum/product/composition of continuous functions is continuous. If f is continuous at c and $f(c) \neq 0$, then $1/f$ is continuous at c. We had also seen that the square root function is continuous. We now state an important property of continuous functions.

Definition 10 (Intermediate Value Property)

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Let $u \in \mathbb{R}$ be between $f(a)$ and $f(b)$. Then, there exists $c \in [a, b]$ such that $f(c) = u$.

Note carefully that the domain is an interval.

Now, we state another property, called the extreme value theorem.

Theorem 3 (Extreme value theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, there exist $x_1, x_2 \in [a, b]$ such that
$$f(x_1) \leq f(x) \leq f(x_2)$$
for all $x \in [a, b]$.

Note very carefully that the above not only shows that the image of f is bounded but also that the bounds are attained! Note that the domain was a <u>closed and bounded</u> interval.

Week 2

Recall that a (non-empty) set which is bounded above can have many upper bounds. However, completeness of $\mathbb{R}$ tells us that there is a *least* upper bound. We had called this the *supremum*.

Similarly, we had defined *infimum*.

By abuse of notation, given a function $f : X \to \mathbb{R}$, if the image $f(X) \subset \mathbb{R}$ is bounded above, then the supremum of the image is called the supremum of f on X.
Analogous comments hold for infimum.

Thus, what the previous theorem told us was that not only is the image bounded but the supremum and infimum are actually attained. (If the function is continuous and defined on a closed and bounded interval, that is.)

Non-examples of the previous theorem:

Consider $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = x.$$

The image is bounded but the infimum/supremum are not attained.

Consider $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}.$$

The image is not bounded above. It is bounded below but the infimum is not attained.

We saw one interesting result that helps simplify our life in some scenarios.

Theorem 4 (Sequential criterion)

Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Then, f is continuous at a iff given any sequence (a_n) in A such that $a_n \to a$, we have $f(a_n) \to f(a)$.

This makes life simpler because it is sometimes easier to deal with sequences. We had seen an example of this when we proved that a certain oscillatory function does not have a limit. Do you remember which?

Next, we defined derivative. This was also not difficult.

Definition 11 (Derivative)

Let $f : (a, b) \to \mathbb{R}$ be a function and let $c \in (a, b)$. f is said to be *differentiable at the point c* if the following limit exists:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

In such a case, we call the value of the above limit the derivative of f at c and denote it by $f'(c)$.

We then have the usual rules about product/sum/composition of differentiable functions again being differentiable. Of course, we **don't** have the naïve product rule but rather $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$. We then looked at minima/maxima.

Definition 12 (Local maximum)

Let $f : X \to \mathbb{R}$ be a function and let $x_0 \in X$. Suppose that there is an interval $(c, d) \subset X$ containing x_0. If we have $f(x_0) \geq f(x)$ for all $x \in (c, d)$, then we say that f has a *local maximum* at x_0.

Of course, we have an analogous definition for minimum. Note that here, we have that x_0 is an "interior point." That is, there is an interval *around* x_0 contained within the domain.

Theorem 5 (Fermat's Theorem)

If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maxmimum at a point $x_0 \in X$, then $f'(x_0) = 0$.

Once again, note that this only talks about "interior points."

We then saw Rolle's Theorem. Note the hypothesis carefully.

Theorem 6 (Rolle's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is a *continuous* function. Further, assume that it is differentiable on (a, b). In this case, if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Using the above, we have a more general result.

Theorem 7 (Mean Value Theorem)

Let f be continuous and differentiable as above. There exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We then saw a theorem which said "derivatives have IVP." To be more precise:

Theorem 8 (Darboux's Theorem)

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Let $c < d$ be points in (a, b). Let u be between $f'(c)$ and $f'(d)$. Then, there exists $x_0 \in (c, d)$ such that

$$f'(x_0) = u.$$

Note that the derivative of a (differentiable) function need not be continuous. We shall see an example in the tutorial today, in fact. However, the above theorem tells us how the derivative can't have "jump" discontinuity.

Stop recording. Start a new one.
Take doubts.

Start recording!

Week 3

What did we see last week? Continuity, IVP, EVT, sequential criterion, derivative, (local) maximum and minimum, Fermat's not-Last Theorem, Rolle's and Mean Value Theorems, Darboux's Theorem. (Whew!)

We also saw an example of a function with non-continuous derivative. What was it?

$f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Keep this in mind. please.

We turn back to maximum and minimum and recall a theorem you must have seen in your previous life.

Theorem 9 (Second derivative test)

Assume that $f : [a, b] \to \mathbb{R}$ is continuous.
Suppose that $x_0 \in (a, b)$ is such that $f'(x_0) = 0$ and $f''(x_0)$ exists.
Then,

1. $f''(x_0) > 0 \implies f$ has a local minimum at x_0,
2. $f''(x_0) < 0 \implies f$ has a local maximum at x_0.

If $f''(x_0) = 0$, then nothing can be concluded.

We now look at concavity and convexity. For what follows, I will always denote an interval. (It could be open/close/neither/unbounded.)

Definition 13 (Convex)

A function $f : I \to \mathbb{R}$ is said to be *convex* if for every $x_1, x_2 \in I$ and every $t \in [0, 1]$, we have

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

More graphically, given any two points on the graph of the function, the line segment joining the two points lies above the graph.

The definition of a *concave* function is obtained by replacing $\leq$ with $\geq$ and "above" with "below."

Note that the definition does not even assume continuity. In particular, the function need not be differentiable, much less twice differentiable.

However, if we do assume that it's differentiable, then we can say some things. If we assume twice differentiability, we can say some more things. I have put a summary of these on the next slide.

Read it some day.

Proposition 2

Suppose $f : I \to \mathbb{R}$ is differentiable. Then

1. f' is increasing on $I \iff f$ is convex on I.
2. f' is decreasing on $I \iff f$ is concave on I.
3. f' is strictly increasing on $I \iff f$ is strictly convex on I.
4. f' is strictly decreasing on $I \iff f$ is strictly concave on I.

Corollary 1

Suppose $f : I \to \mathbb{R}$ is twice differentiable. Then

1. $f'' \geq 0$ on $I \iff f$ is convex on I.
2. $f'' \leq 0$ on $I \iff f$ is concave on I.
3. $f'' > 0$ on $I \implies f$ is strictly convex on I.
4. $f'' < 0$ on $I \implies f$ is strictly concave on I.

Let's now talk about inflection points.

Definition 14 (Inflection point)

Let x_0 be an *interior point* of I. Then, x_0 is called an inflection for f if there exists $\delta > 0$ such that either

1. f is convex on $(x_0 - \delta, x_0)$ and concave on $(x_0, x_0 + \delta)$, or
2. f is concave on $(x_0 - \delta, x_0)$ and convex on $(x_0, x_0 + \delta)$.

As a crazy example, note that 0 is an inflection point of:
$f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Note that f is not even continuous at 0. Let alone twice differentiable. Also note that every point is a point of inflection for an affine function $x \mapsto ax + b$. (Even if $a = 0$.)

Week 3

Here's some more information being thrown at you. Look at it some day. Let $x_0 \in I$ be an *interior point*, and $f : I \to \mathbb{R}$.

Theorem 10 (Derivative tests)

1. (First derivative test) Suppose f is differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(x_0 - \delta, x_0)$ and f' is decreasing on $(x_0, x_0 + \delta)$, or vice-versa.
2. (Second derivative test) Suppose f is twice differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that $f'' \geq 0$ on $(x_0 - \delta, x_0)$ and $f'' \leq 0$ on $(x_0, x_0 + \delta)$, or vice-versa.

Thus, if f is twice differentiable, then x_0 is inflection point iff f'' changes sign. (Note that $f''(x_0)$ is not required to exist. Recall the crazy example.)

The previous slide gives us a **necessary** condition for inflection point. We have the same notation as earlier.

Theorem 11 (Another second derivative test)

Suppose f is twice differentiable at x_0. If x_0 is a point of inflection for f, then $f''(x_0) = 0$.

In the above, we are not assuming the existence of f'' at other points. The following is now a **sufficient** condition.

Theorem 12 (A third derivative test)

Suppose f is thrice differentiable at x_0 such that $f''(x_0) = 0$ and $f'''(x_0) \neq 0$. Then, x_0 is an inflection point for f.

Okay, that's enough about convex/concave/inflection points. Hopefully, any possible doubt about these is covered in the previous slides. Read them some day and ask doubts, if any.

Once again, keep in mind the crazy example. The definitions of these concepts do not require any continuity or anything at the point. However, we do have the theorem that a convex function on an open interval is continuous. Proof:
https://unapologetic.wordpress.com/2008/04/15/convex-functions-are-continuous/

We also have the theorem that a convex function is differentiable at all but at most countably many points. Proof:
https://math.stackexchange.com/questions/946311

Let's now look at Taylor polynomials. From now, I will be an open interval, a an interior point of I, and $f : I \to \mathbb{R}$ a function.

Definition 15 (Taylor polynomials)

Let f be n times differentiable at x_0. We define the $n+1$ Taylor polynomials as

$$\begin{aligned}
P_0(x) =& f(x_0) \\
P_1(x) =& f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) \\
P_2(x) =& f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 \\
&\vdots \\
P_n(x) =& f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
\end{aligned}$$

Note that all the Taylor **polynomials** have only **finite**ly many terms, as a polynomial should have. Also note that so far, we have just defined some polynomials. We are yet to see how it actually connects with the function itself. This is given by the following theorem.

Theorem 13 (Taylor's theorem)

Suppose that f is $n+1$ times differentiable on I. Suppose that $b \in I$. Then, there exists $c \in (a, b) \cup (b, a)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where P_n is as in the previous slide.

Given a function f (which is $n+1$ times differentiable) and a Taylor polynomial P_n, we can define the n^{th} remainder as

$$R_n(x) := f(x) - P_n(x), \qquad \text{for } x \in I.$$

By the previous theorem, we know that

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1},$$

for some c_x between x and a.

Sometimes, assuming $f \in \mathcal{C}^\infty(I)$, we can bound $f^{(n+1)}(c_x)$ in a nice enough way to get that

$$R_n(x) \to 0$$

for *some* $x \in I$.

If the previous thing happens, then we get that

$$f(x) = \lim_{n\to\infty} P_n(x)$$

for all such x.

Thus, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all those x. For some nice functions, we get an $R > 0$ such that the above happens for all $x \in (a - R, a + R)$. If such an R exists for all $a \in I$, then f is said to be *analytic*. (The R may depend on a.)

Note that the Taylor series about some point a may still converge but *not* to f. Such a function is not called analytic.

Some final remarks:

The last thing written $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is not a Taylor **polynomial**.
It is the Taylor **series**.

It may happen to not converge for *any* $x \neq a$. It may also happen to converge for all $x \in \mathbb{R}$.

Suppose that the series converges on some interval J such that $a \in J \subset I$. It is not necessary that the Taylor series converges to f on J.

What was the example seen in class that illustrated this?

Stop recording. Start a new one.
Take doubts.

In the following, it will be tacitly assumed that $a, b \in \mathbb{R}$ with $a < b$.

Definition 16 (Partitions)

Given a closed interval $[a, b]$, a *partition* P of $[a, b]$ is a finite collection of points

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

Note that a partition P is really just a subset of $[a, b]$ with the requirement that it must be finite and contain a and b. It is customary to then list it in increasing order.

Definition 17 (Refinements)

Given two partitions P and P' of $[a, b]$, we say that P' is a *refinement* of P if $P \subset P'$.

The "$\subset$" makes sense because of our earlier remark about partitions just being subsets of $[a, b]$. In other words, it means that every point of P is also a point in P'. Thus, we have "refined" the partition by further "chopping" it up.

Given two partitions P_1 and P_2 of $[a, b]$, we see that $P = P_1 \cup P_2$ is also a partition of $[a, b]$. Moreover, P is a refinement of both P_1 and P_2. In other words, any two partitions have a common refinement.

Definition 18

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

a partition of $[a, b]$.
We define the following quantities:

$$M_i := \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i := \inf_{x \in [x_{i-1}, x_i]} f(x),$$

for $i = 1, \ldots, n$.

Thus, m_i and M_i denote the infimum and supremum of f over the i-th interval, respectively.

Week 4

Given everything as in the previous slide, we define lower/upper sums as following.

Definition 19 (Lower/Upper sum)

The *lower sum* of f with respect to the partition P is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

The *upper sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

In the above, note that we have both f and P in the notation. This is crucial because the sums depend on the partition.

Using the earlier sums, we now define the upper and lower Darboux *integrals.* The notations are continuing from earlier.

Definition 20 (Lower/Upper Darboux integrals)

The *lower Darboux integral of f* is defined as

$$L(f) := \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\},$$

and the *upper Darboux integral of f* is defined as

$$U(f) := \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Note that the sup / inf is over all the partitions P of $[a, b]$.

Note that the notation now does not have any P. This is because $L(f)$ and $U(f)$ don't depend on any specific partition.

Definition 21 (Darboux integrable)

A <u>bounded</u> function $f : [a, b] \to \mathbb{R}$ is said to be *Darboux integrable* if $L(f) = U(f)$.
In this case, we define

$$\int_a^b f(t)\mathrm{d}t := U(f) = L(f).$$

This value is called the Darboux integral.

Theorem 14 (Criteria for Darboux integrable)

A bounded function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

A corollary of the previous is the following.

Corollary 2

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that (P_n) is a sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Then, f is Darboux integrable.

We now turn to the definition of Riemann integrals.

Some jargon.

Definition 22 (Norm of a partition)

Let $P = \{a = x_0 < \cdots < x_n = b\}$ be a partition of $[a, b]$. The *norm* of P is defined to be

$$\|P\| := \max_{1 \le i \le n} [x_i - x_{i-1}].$$

In other words, it is the length of the largest sub-interval.

Definition 23 (Tagged partition)

Given a partition P of $[a, b]$ as before, we get the intervals $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. For each i, we pick a point $t_i \in I_i$. This collection of points together is denoted by t. The pair (P, t) is called a *tagged partition* of $[a, b]$.

Definition 24 (Riemann sum)

Let $f : [a, b] \to \mathbb{R}$ be a function. Let (P, t) be a tagged partition of $[a, b]$. We define the *Riemann sum* associated to f and (P, t) by

$$R(f, P, t) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

Note that the notation here includes f, P, *and* t. Also note that here we didn't demand f be bounded.

On the next slide, we state two equivalent definitions of Riemann integrability.

Definition 25 (Riemann 1)

A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for there exists $R \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon$$

for all tagged partitions (P, t) such that $\|P\| < \delta$.

Definition 26 (Riemann 2)

A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for there exists $R \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ and a partition P such that

$$|R(f, P', t') - R| < \epsilon$$

for all tagged refinements (P', t') of P with $\|P'\| < \delta$.

Definition 27

In both the definitions on the earlier slide, the R is unique and it is called the *Riemann integral* of f over $[a, b]$.

Theorem 15 (Darboux and Riemann are friends)

Let $f : [a, b] \to \mathbb{R}$ be a function.
If f is bounded and Darboux integrable, then f is also Riemann integrable.
If f is Riemann integrable, then f is bounded and also Darboux integrable.
In both the cases above, the Darboux and Riemann integrals are the same.

Theorem 16 (Riemann sums approximating the integral)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose that (P_n, t_n) is a sequence of tagged partitions of $[a, b]$ such that $\|P_n\| \to 0$. Then,

$$\lim_{n\to\infty} R(f, P_n, t_n) = \int_a^b f(x)\mathrm{d}x.$$

Note that we assumed f to be Riemann integrable to begin with. Thus, we cannot use the above theorem if we don't already know that f is Riemann integrable. The next theorem helps us in determining when that happens.

Theorem 17

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is Riemann integrable.

The converse of the previous theorem is not true. In fact, the theorem is true even if we assume something less. Namely, if f is bounded and is discontinuous on a finite set, then it is Riemann integrable. The "finite" can even be replaced with "at most countable," if you know what that means.

The "at most countable" can actually be replaced with "measure zero." At this point, the converse also becomes true!

Now, we see how derivatives and integrals relate. These are the two parts of the Fundamental Theorem of Calculus.

Theorem 18 (FTC Part I)

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function, and let

$$F(x) := \int_a^x f(t)\mathrm{d}t$$

for $x \in [a, b]$.
Then, F is continuous. Moreover, if f is continuous at some $c \in (a, b)$, then F is differentiable at c and

$$F'(c) = f(c).$$

In particular, if f is continuous, then Riemann integrability of f is guaranteed and the above equation is true for *all* $c \in (a, b)$.

Theorem 19 (FTC Part II)

Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $F : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and satisfies $F' = f$ on (a, b). If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a).$$

Note that the if is crucial. It isn't necessary that the derivative of a function is Riemann integrable. It needn't even be bounded. (But even if it is bounded, it needn't be Riemann integrable. Although an example of this is harder.)

Some pathological remarks:

1. If a function is Riemann integrable, it doesn't mean that it is the derivative of a function. (That is, it needn't have an anti-derivative.)
2. If a function has an anti-derivative, it doesn't mean that it is Riemann integrable. (That is, derivatives needn't be Riemann integrable.)

For the first, take $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$. It cannot be the derivative of any function because it doesn't have IVP. (Recall Theorem 8.)

For the second, consider the derivative of $F : [-1, 1] \to \mathbb{R}$ defined by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$. F' here isn't bounded.