MA 109: Calculus I

Tutorial Solutions

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Contents

§0. Notations

1. $\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
2. $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
3. $\mathbb{Q}$ denotes the set of rational numbers.
4. $\mathbb{R}$ denotes the set of real numbers.

§1. Tutorial 1

25th November, 2020

Sheet 1

2. (iv) $\lim_{n\to\infty} (n)^{1/n}$.

   Define $h_n := n^{1/n} - 1$.
   Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. (Why?)

   Now, for $n > 2$, we have

$$\begin{aligned} n &= (1 + h_n)^n \\ &= 1 + nh_n + \binom{n}{2}h_n^2 + \cdots + \binom{n}{n}h_n^n \\ &\geq 1 + nh_n + \binom{n}{2}h_n^2 \\ &> \binom{n}{2}h_n^2 \\ &= \frac{n(n-1)}{2}h_n^2. \end{aligned}$$

   Thus, $h_n < \sqrt{\dfrac{2}{n-1}}$ for all $n > 2$.

   Using Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$ which gives us that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

   (Where did we use that $h_n \geq 0$?)

3. (ii) We show that $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is *not* convergent.

Solution. Note that from the difference formula, we know that if $\{a_n\}$ converges, then

$$\lim_{n\to\infty}|a_{n+1}-a_n|=0.$$

(The limit *exists* and equals 0.)

We show that this is not true for the given sequence. We define

$$b_n := a_{n+1}-a_n,$$

where $\{a_n\}$ is the sequence given in the question.

Then, b_n is given as

$$\begin{aligned}
b_n &= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)-(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\\
&= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)+(-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n}\right)\\
&= (-1)^{n+1}+(-1)^n\left(\frac{1}{n+1}+\frac{1}{n}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|b_n| &= \left|1-\left(\frac{1}{n+1}+\frac{1}{n}\right)\right|\\
&= \left|1-\frac{2n+1}{n(n+1)}\right|
\end{aligned}$$

From the above, we conclude that

$$\lim_{n\to\infty}|b_n|=1.$$

This shows that a_n does not converge. □

5. (iii) $a_1 = \sqrt{2},\ a_{n+1} = 3 + \dfrac{a_n}{2} \quad \forall n \geq 1.$

Solution. I first describe the general idea.

> The idea in these questions is to first prove a bound on a_n by induction. Then, using that bound we prove that the sequence is convergent.
>
> Once we do that, we then know that $\lim_{n\to\infty} a_n$ exists. Since that also equals $\lim_{n\to\infty} a_{n+1}$, we can take limit on both sides of the equation and solve for the limit L.

First, we prove that the sequence is bounded above.

> Claim 1. $a_n < 6$ for all $n \in \mathbb{N}$.
>
> *Proof.* We shall prove this via induction. The base case $n = 1$ is immediate as $2 < 6$.
> Assume that it holds for $n = k$. Then,
>
> $$a_{k+1} = 3 + \frac{a_n}{2} < 3 + \frac{6}{2} = 6.$$
>
> By principle of mathematical induction, we have proven the claim. □

> Claim 2. $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
>
> *Proof.* $a_{n+1} - a_n = 3 - \dfrac{a_n}{2} = \dfrac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n.$ □

Thus, we now know that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Then taking the limit on both sides of

$$a_{n+1} = 3 + \frac{a_n}{2}$$

gives us

$$L = 3 + \frac{L}{2},$$

which we can solve to get $L = 6$. □

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

Solution. Choose $\epsilon = \dfrac{|L|}{2}$. Note that this is indeed greater than 0.

By the $\epsilon - N$ definition, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon = \frac{|L|}{2}$$

for all $n > N$. Using triangle inequality, we get

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}.$$

Thus, we get

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}.$$

Adding $|L|$ on both sides gives us

$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2}$$

for all $n > N$, as desired. $\square$

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:
    1. $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
    2. $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

    Solution. Both the statements are false. We give one counterexample for both.

    $$\begin{aligned} a_n &:= 1 && \text{for all } n \in \mathbb{N}, \\ b_n &:= (-1)^n && \text{for all } n \in \mathbb{N}. \end{aligned}$$

    Clearly, $\{a_n\}_{n\geq 1}$ converges and $\{b_n\}_{n\geq 1}$ is bounded. However, the product is again the latter sequence which does not converge. □

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for some $c \in [a, b]$. Prove or disprove the following statements.

    1. $\lim\limits_{x\to c}[f(x)g(x)] = 0$.
    2. $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if g is bounded.
    3. $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if $\lim\limits_{x\to c} g(x)$ exists.

Solution.

1. No. Consider $a = c = 0$ and $b = 1$. Let f, g be defined as

   $$f(x) = 0, \quad g(x) = \frac{1}{x}.$$

   Verify that this works as a counterexample.

2. We prove this statement. Since g is bounded, there exists $M > 0$ such that

   $$|g(x)| < M$$

   for all $x \in (a, b)$. Thus, we have

   $$|f(x)g(x)| \le M|f(x)|$$

   for all $x \in (a, b)$. Since the LHS is clearly non-negative, using Sandwich theorem proves that

   $$\lim_{x\to c} |f(x)g(x)| = 0.$$

   This also gives us that

   $$\lim_{x\to c} f(x)g(x) = 0.$$

   (Why?)

3. This is also true. We can simply use that limit of products is the product of limits if the individual limits exist.

□