MA 109: Calculus I

Tutorial Solutions

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Contents

§0. Notations

1. $\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
2. $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
3. $\mathbb{Q}$ denotes the set of rational numbers.
4. $\mathbb{R}$ denotes the set of real numbers.
5. $\subset$ is used for subset, not necessarily proper.

   $$[0,1] \subset [0,1]$$

   is correct.
6. $\subsetneq$ is used for "proper subset."

§1. Tutorial 1

25th November, 2020

Sheet 1

2. (iv) $\lim\limits_{n \to \infty} (n)^{1/n}$.

   Define $h_n := n^{1/n} - 1$.
   Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. (Why?)

   Now, for $n > 2$, we have

$$\begin{aligned} n &= (1 + h_n)^n \\ &= 1 + nh_n + \binom{n}{2} h_n^2 + \cdots + \binom{n}{n} h_n^n \\ &\geq 1 + nh_n + \binom{n}{2} h_n^2 \\ &> \binom{n}{2} h_n^2 \\ &= \frac{n(n-1)}{2} h_n^2. \end{aligned}$$

   Thus, $h_n < \sqrt{\dfrac{2}{n-1}}$ for all $n > 2$.

   Using Sandwich Theorem, we get that $\lim\limits_{n \to \infty} h_n = 0$ which gives us that

$$\lim_{n \to \infty} n^{1/n} = 1.$$

   (Where did we use that $h_n \geq 0$?)

3. (ii) We show that $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is *not* convergent.

Solution. Note that from the difference formula, we know that if $\{a_n\}$ converges, then

$$\lim_{n\to\infty}|a_{n+1}-a_n|=0.$$

(The limit *exists* and equals 0.)

We show that this is not true for the given sequence. We define

$$b_n := a_{n+1}-a_n,$$

where $\{a_n\}$ is the sequence given in the question.

Then, b_n is given as

$$\begin{aligned}
b_n &= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)-(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\\
&= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)+(-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n}\right)\\
&= (-1)^{n+1}+(-1)^n\left(\frac{1}{n+1}+\frac{1}{n}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|b_n| &= \left|1-\left(\frac{1}{n+1}+\frac{1}{n}\right)\right|\\
&= \left|1-\frac{2n+1}{n(n+1)}\right|
\end{aligned}$$

From the above, we conclude that

$$\lim_{n\to\infty}|b_n|=1.$$

This shows that a_n does not converge. □

5. (iii) $a_1 = \sqrt{2},\ a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \geq 1.$

Solution. I first describe the general idea.

> The idea in these questions is to first prove a bound on a_n by induction. Then, using that bound we prove that the sequence is convergent.
>
> Once we do that, we then know that $\lim_{n\to\infty} a_n$ exists. Since that also equals $\lim_{n\to\infty} a_{n+1}$, we can take limit on both sides of the equation and solve for the limit L.

First, we prove that the sequence is bounded above.

> Claim 1. $a_n < 6$ for all $n \in \mathbb{N}$.
>
> *Proof.* We shall prove this via induction. The base case $n = 1$ is immediate as $2 < 6$.
> Assume that it holds for $n = k$. Then,
>
> $$a_{k+1} = 3 + \frac{a_n}{2} < 3 + \frac{6}{2} = 6.$$
>
> By principle of mathematical induction, we have proven the claim. □

> Claim 2. $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
>
> *Proof.* $a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n.$ □

Thus, we now know that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Then taking the limit on both sides of

$$a_{n+1} = 3 + \frac{a_n}{2}$$

gives us

$$L = 3 + \frac{L}{2},$$

which we can solve to get $L = 6$. □

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \qquad \text{for all } n \geq n_0.$$

Solution. Choose $\epsilon = \frac{|L|}{2}$. Note that this is indeed greater than 0.

By the $\epsilon - N$ definition, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon = \frac{|L|}{2}$$

for all $n > N$. Using triangle inequality, we get

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}.$$

Thus, we get

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}.$$

Adding $|L|$ on both sides gives us

$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2}$$

for all $n > N$, as desired. $\square$

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   1. $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
   2. $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

   Solution. Both the statements are false. We give one counterexample for both.

$$
\begin{aligned}
a_n &:= 1 && \text{for all } n \in \mathbb{N}, \\
b_n &:= (-1)^n && \text{for all } n \in \mathbb{N}.
\end{aligned}
$$

   Clearly, $\{a_n\}_{n\geq 1}$ converges and $\{b_n\}_{n\geq 1}$ is bounded. However, the product is again the latter sequence which does not converge. $\square$

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x \to c} f(x) = 0$ for some $c \in [a, b]$. Prove or disprove the following statements.

    1. $\lim\limits_{x \to c}[f(x)g(x)] = 0.$
    2. $\lim\limits_{x \to c}[f(x)g(x)] = 0$, if g is bounded.
    3. $\lim\limits_{x \to c}[f(x)g(x)] = 0$, if $\lim\limits_{x \to c} g(x)$ exists.

    Solution. 1. No. Consider $a = c = 0$ and $b = 1$. Let f, g be defined as

    $$f(x) = x, \quad g(x) = \frac{1}{x}.$$

    Verify that this works as a counterexample.

    2. We prove this statement. Since g is bounded, there exists $M > 0$ such that

    $$|g(x)| < M$$

    for all $x \in (a, b)$. Thus, we have

    $$|f(x)g(x)| \leq M|f(x)|$$

    for all $x \in (a, b)$. Since the LHS is clearly non-negative, using Sandwich theorem proves that

    $$\lim_{x \to c} |f(x)g(x)| = 0.$$

    This also gives us that

    $$\lim_{x \to c} f(x)g(x) = 0.$$

    (Why?)

    3. This is also true. We can simply use that limit of products is the product of limits if the individual limits exist.

    □

§2. Tutorial 2

2nd December, 2020

Sheet 1

13. (ii) Discuss the continuity of the following function:

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Solution. For $x \neq 0$, the continuity of f at x follows from the fact that f is the product and composition of continuous functions.

For $x = 0$, we prove continuity using $\epsilon - \delta$. We show that

$$\lim_{x \to 0} f(x) = 0.$$

Since $f(0) = 0$, the continuity of f at 0 will follow.

To this end, let $\epsilon > 0$ be given. We show that $\delta := \epsilon$ works. Indeed, if $0 < |x - 0| < \delta$, then

$$\begin{aligned} |f(x) - 0| &= \left| x \sin\left(\frac{1}{x}\right) \right| \quad |\sin| \leq 1 \\ &\leq |x| \\ &= |x - 0| \\ &< \delta = \epsilon. \end{aligned}$$

Thus, we have shown that

$$0 < |x - 0| < \delta \implies |f(x) - 0| < \epsilon,$$

proving that

$$\lim_{x \to 0} f(x) = 0 = f(0),$$

as desired. □

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

Solution. As earlier, differentiability of f at $x \neq 0$ follows due to product/composition rules.

Now, for $h \neq 0$, note that

$$\frac{f(0+h) - f(0)}{h} = h \sin\left(\frac{1}{h}\right).$$

As saw earlier, the limit of the above as $h \to 0$ exists and is 0. Thus, we get that f is differentiable at 0 as well with $f'(0) = 0$.

Thus, f is differentiable on $\mathbb{R}$.

Now, for $x \neq 0$, we can compute the derivative using product/chain rule. Putting this together, we get

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

We now show that f' is not continuous at 0. We use the sequential criterion for this. Consider the sequence

$$x_n := \frac{1}{2n\pi}, \qquad n \in \mathbb{N}.$$

Clearly, we have that $x_n \to 0$ and $x_n \neq 0$. Thus, we get

$$f'(x_n) = -\cos(2n\pi) = -1.$$

Thus, we see that $f'(x_n) \to -1 \neq f'(0)$.
This shows that f' is not continuous. $\square$

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

Solution. Putting $x = y = 0$, we note that $f(0) = (f(0))^2$. If $f(0) = 0$, show that $f(x) = 0$ for all x and conclude that the given thing is indeed true.

Now, assume that $f(0) \neq 0$. Then, $f(0) = 1$.

Let $c \in \mathbb{R}$ be arbitrary. For $h \neq 0$, we note that

$$\begin{aligned} \frac{f(c+h) - f(c)}{h} &= \frac{f(c)f(h) - f(c)}{h} \\ &= f(c)\frac{f(h) - 1}{h} \\ &= f(c)\frac{f(h) - f(0)}{h}. \end{aligned}$$

Since f is given to be differentiable at 0, the above limit as $h \to 0$ exists and equals $f(c)f'(0)$. Thus, we see that $f'(c)$ exists and equals $f(c)f'(0)$. □

Sheet 1 Optional

7. Let $f : (a, b) \to \mathbb{R}$ be differentiable and $c \in (a, b)$. Show that the following are equivalent:

   (i) f is differentiable at c.

   (ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$, and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and

   $$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

   (iii) There exists $\alpha \in \mathbb{R}$ such that

   $$\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0.$$

Solution. We prove this by a usual technique in math by showing that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii)
First, we pick $\delta := \min\{c - a, b - c\}$. Note that $\delta > 0$ and $(c-\delta, c+\delta) \subset (a, b)$.

Now, since f is differentiable at c, $f'(c)$ exists. We define $\alpha := f'(c) \in \mathbb{R}$. Now, we define $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ as

$$\epsilon_1(h) := \begin{cases} \dfrac{f(c+h) - f(c)}{h} - \alpha & h \neq 0, \\ 0 & h = 0. \end{cases}$$

(Note that $f(c + h)$ above makes sense because $(c - \delta, c + \delta) \subset (a, b)$.)

Now, from the definition above, it is clear that

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

We only need to show that $\lim_{h\to 0} \epsilon_1(h) = 0$. However, note that, for $h \neq 0$, we have

$$\epsilon_1(h) = \frac{f(c+h) - f(c)}{h} - \alpha.$$

Since $f'(c) = \alpha$, we know that

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

which gives us that $\epsilon_1(h) \to 0$ as $h \to 0$, as desired.

(ii) $\implies$ (iii)
Let α be as in (ii). Then, for $h \neq 0$, we note that

$$\begin{aligned} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \frac{|h\epsilon_1(h)|}{|h|} \\ &= |\epsilon_1(h)|. \end{aligned}$$

Since $\lim_{h\to 0} \epsilon_1(h) = 0$, we get that $\lim_{h\to 0} |\epsilon_1(h)| = 0$, which proves the desired limit.

(iii) $\implies$ (i)
We show that the α in (iii) is the derivative of f at c. Note that we are given

$$\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

or

$$\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0.$$

The above gives us that

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

or

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} \right) = \alpha.$$

Thus, $f'(c)$ exists and equals α. □

In the above, we used the following implicitly:

$$\lim_{x\to c} f(x) = 0 \iff \lim_{x\to c} |f(x)| = 0.$$

10. Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Solution. We need to show that there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$. Consider $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) := f(x) - x.$$

Then, showing that f has a fixed point is equivalent to showing that g has a zero.

Note that

$$g(0) = f(0) \geq 0$$

and

$$g(1) = f(1) - 1 \leq 0.$$

If either of the equalities hold, then we are done. Otherwise, we have

$$g(0) > 0 \quad \text{and} \quad g(1) < 0.$$

By intermediate value property, $g(x_0) = 0$ for some $x_0 \in [0,1]$, as desired. $\square$

Sheet 2

2 Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a,b)$, show that there is a unique $x_0 \in (a,b)$ such that $f(x_0) = 0$.

Solution. The existence of x_0 is given by the intermediate value theorem since 0 lies between $f(a)$ and $f(b)$.

We now show uniqueness. Suppose that there exists $x_1 \in (a,b)$ such that $f(x_1) = 0$ and $x_1 \neq x_0$. We show that this leads to a contraction.
By LMVT, there exists c between x_0 and x_1 such that

$$\begin{aligned} f'(c) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= 0. \end{aligned}$$

A contraction since $c \in (a,b)$ and we were given that $f'(x) \neq 0$ for any $x \in (a,b)$. $\square$

5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

   Solution. If $a = b$, then the inequality is clear. Suppose that $a \neq b$.

   Then, there exists c between a and b such that
$$\sin'(c) = \frac{\sin a - \sin b}{a - b}.$$
   Note that $\sin' = \cos$ and thus,
$$\left|\frac{\sin a - \sin b}{a - b}\right| = |\cos(c)| \leq 1.$$
   Cross-multiplying gives us the desired result. □

§3. Tutorial 3

9th December, 2020

Sheet 2

8. In each case, find a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies all the given conditions, or else show that no such function exists.

   (ii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$.

   Solution. $f(x) := x + \dfrac{x^2}{2}$ is one such. Justify. □

   (iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$.

   Solution. Not possible.
   Assume not. As f'' is nonnegative, f' must be increasing everywhere. We are given that $f'(0) = 1$.

   Thus, given any $c > 0$, we know that

   $$f'(c) \geq 1. \tag{$*$}$$

   Let $x \in (0, \infty)$. By MVT, we know that there exists $c \in (0, x)$ such that

   $$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$

   Thus, by ($*$), we have it that $f(x) \geq x + f(0)$ for all positive x.

   This contradicts that $f(x) \leq 100$ for all positive x. (How?) □

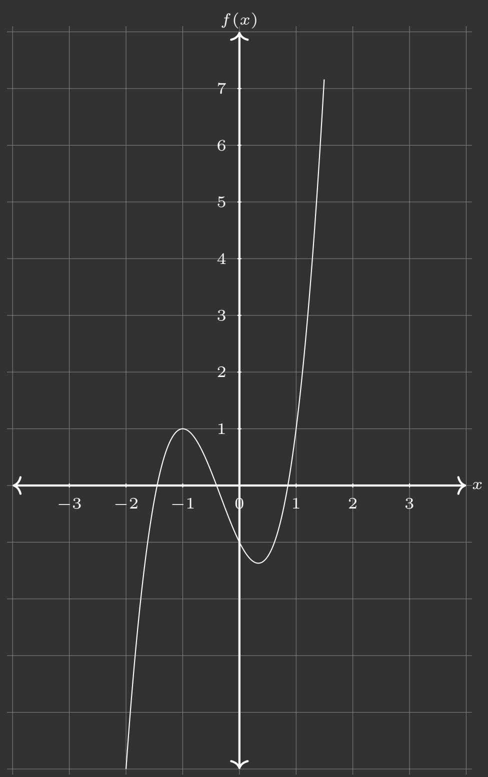
10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

    (i) $f(x) = 2x^3 + 2x^2 - 2x - 1$

    Solution. Note that this is a cubic and can have at most 3 roots. It is easy to locate that they're in $(-2, -1)$, $(-1, 0)$ and $(0, 1)$ since f changes signs consecutively at $-2, -1, 0, 1$.

    Moreover, f' has nice roots: -1 and $1/3$.

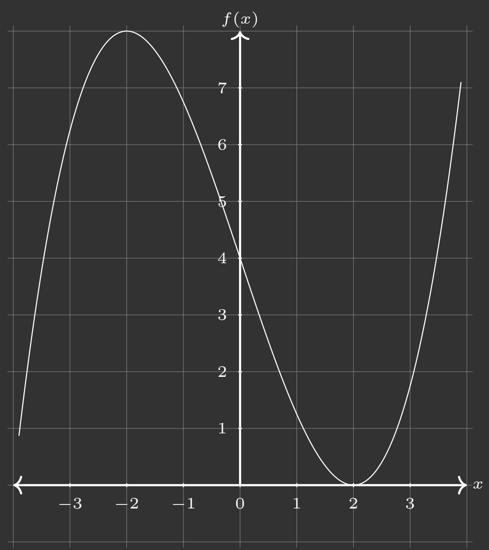
    Lastly, f'' has a root at $-1/3$. Using the above, we get pretty much all we want. Calculating $f(-1)$, $f(1/3)$ and $f(-1/3)$ also tells us the location of the roots with respect to minima/maxima and inflection point.



Above is the graph. $\square$

11. Sketch a continuous curve $y = f(x)$ having all the following properties:
$f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ f'(-2) = f(2) = 0;$
$f'(x) > 0$ for $|x| > 2,\ f'(x) < 0$ for $|x| < 2;$
$f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

Solution. Here is the graph:



I have actually graphed a polynomial that satisfies the given properties.
Can you come up with it?
Is there a unique such polynomial?
What's the minimum degree of such a polynomial?
Is there a unique polynomial with that degree?
Suppose you have two distinct polynomials f and g that satisfy the given conditions. Can you come up with a distinct third polynomial such that it satisfies the conditions as well? □

Sheet 3

1. Write down the Taylor series for $\arctan x$ about the point 0. Write down a precise remainder $R_n(x)$.

   Solution. For each of notation, let $f(x) := \arctan x$ and $g(x) := \dfrac{1}{1+x^2}$. Note that $f' = g$.

   Note that if $n \ge 1$, then $f^{(n)}(0) = g^{(n-1)}(0)$. For g, we have the easy Taylor expansion as
   $$g(x) = 1 - x^2 + x^4 - \cdots$$
   which is valid for $x \in (-1, 1)$.

   Thus, we easily see that
   $$g^{(n)}(0) = \begin{cases} 0 & n \text{ is odd,} \\ (-1)^{n/2} n! & n \text{ is even.} \end{cases}$$

   Thus,
   $$f^{(n)}(0) = \begin{cases} 0 & n \text{ is even,} \\ (-1)^{(n-1)/2}(n-1)! & n \text{ is odd.} \end{cases}$$
   (The above is for $n \ge 1$.) Using this, we get the $(2n+1)$-th Taylor polynomial as
   $$\begin{aligned} P_{2n+1}(x) &= \sum_{k=0}^{2n+1} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \sum_{k=1}^{2n+1} \frac{f^{(n)}(0)}{n!} x^n \\ &= 0 + x - \frac{2!}{3!} x^3 + \cdots + \frac{(-1)^n (2n)!}{(2n+1)!} x^{2n+1} \\ &= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n}{2n+1} x^{2n+1}. \end{aligned}$$
   Since $f^{(2n)} = 0$, we see that
   $$P_{2n}(x) = P_{2n-1}(x)$$
   for $n \ge 1$.

   This solves the problem for finding the Taylor polynomial. Now we solve for the remainder.

Once again, note that

$$g(t) = 1 - t^2 + t^4 - \cdots .$$

For $n \geq 1$, we note that

$$\begin{aligned} g(t) &= [1 - t^2 + \cdots + (-1)^n t^{2n}] + (-1)^{n+1} t^{2n+2} [1 - t^2 + \cdots] \\ &= [1 - t^2 + \cdots + (-1)^n t^{2n}] + (-1)^{n+1} \frac{t^{2n+2}}{1 + t^2} \end{aligned}$$

Integrating both sides from 0 to x gives

$$f(x) = P_{2n+1}(x) + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1 + t^2} \mathrm{d}t.$$

Thus, the term in red is the $(2n+1)$-th remainder $R_{2n+1}(x)$. Conclude as before, for $R_{2n}(x)$. $\square$

2. Write down the Taylor series of the polynomial $x^3 - 3x^2 + 3x - 1$ about the point 1.

   Solution. As one can easily calculate, we have

   $$f^{(n)}(1) = \begin{cases} 6 & n = 3 \\ 0 & n \neq 3, \end{cases}$$

   for $n \geq 0$. Thus, we get the Taylor "series" to actually be the following finite sum:

   $$\frac{f^{(3)}(1)}{3!}(x-1)^3.$$

   In other words, the Taylor series is simply $(x-1)^3$. $\square$

4. Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. Prove that it converges as follows. Choose $N > 2\,|x|$. We see that for all $n > N$,

$$\frac{x^{n+1}}{(n+1)!} \leq \frac{1}{2}\frac{|x|^n}{n!}.$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$), convergent.

Solution. If $N > 2\,|x|$ and $n > N$, then

$$\begin{aligned}
\left|\frac{x^{n+1}}{(n+1)!}\right| &= \left|\frac{x^n}{n!}\right|\left|\frac{x}{n+1}\right| && n+1 > n > N \\
&\leq \left|\frac{x^n}{n!}\right|\left|\frac{x}{N}\right| && N > 2\,|x| \\
&\leq \frac{1}{2}\left|\frac{x^n}{n!}\right|.
\end{aligned}$$

Thus, we can repeatedly use the above to get:

$$\left|\frac{x^{n+1}}{(n+1)!}\right| \leq \frac{1}{2}\left|\frac{x^n}{n!}\right| \leq \cdots \leq \frac{1}{2^{n+1-N}}\left|\frac{x^N}{N!}\right|.$$

Let $s_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}$.

Now, given $m > n > N$, we have

$$\begin{aligned}
|s_m(x) - s_n(x)| &= \left|\sum_{k=n+1}^{m} \frac{x^k}{k!}\right| \\
&\leq \sum_{k=n+1}^{m} \left|\frac{x^k}{k!}\right| \\
&= \left|\frac{x^{n+1}}{(n+1)!}\right| + \cdots + \left|\frac{x^m}{m!}\right| \\
&\leq \frac{|x|^N}{N!}\left(\frac{1}{2} + \cdots + \frac{1}{2^{m-n}}\right) \\
&\leq \frac{|x|^N}{N!}.
\end{aligned}$$

Note that given any $\epsilon > 0$, we can pick $N \in \mathbb{N}$ such that $\dfrac{|x|^N}{N!} < \epsilon$. Conclude Cauchy-ness. ∎

5. Using Taylor series, write down a series for

$$\int \frac{e^x}{x} \mathrm{d}x.$$

Solution. Note that

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}.$$

Dividing by x gives

$$\frac{e^x}{x} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}.$$

Integrating both sides gives us

$$\int \frac{e^x}{x} \mathrm{d}x = C + \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$$

□

§4. Tutorial 4

16th December, 2020

Sheet 4

2. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)\mathrm{d}x = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

Solution. For the first part, let

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

be an arbitrary partition of $[a, b]$. Note that

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x) \geq 0$$

for all $0 \leq i \leq n - 1$. (This is because 0 is a lower bound of f.)

Thus, we get that $L(f, P) \geq 0$.

In turn, we see that $L(f) \geq 0$, since $L(f)$ is the supremum of $L(f, P)$ over <u>all</u> partitions P of $[a, b]$. Since f is given to be Riemann integrable, we know that the integral is $L(f)$ and we are done.

For the second part, we prove by contrapositive. That is, if $f(x) \neq 0$ for some $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \neq 0$.

Suppose $c \in [a, b]$ is such that $f(c) \neq 0$. As $f(x) \geq 0$ for all $x \in [a, b]$, we have that $f(c) > 0$. Let $\epsilon := f(c)$.

As f is continuous, there is a $\delta > 0$ such that if $x \in [a, b]$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon/2$ which implies that $\epsilon/2 < f(x)$.

Note that even if $c = a$ or $c = b$, the above shows that we can find $c \in (a, b)$ with $f(c) > 0$. Thus, WLOG we may assume that $c \in (a, b)$. Moreover, we may also assume that $\delta > 0$ is small enough so that $(c - \delta, c + \delta) \subset (a, b)$.

Now, consider the partition of $[a, b]$ given as

$$P = \{a, c - \delta/2, c + \delta/2, b\}.$$

Now, note that
$$\inf_{x\in[c-\delta/2,c+\delta/2]} f(x) \geq \frac{\epsilon}{2}.$$

Thus, $L(P,f) > 0$. As $L(f)$ is the supremum over all such $L(P,f)$, we see that $L(f) > 0$. Since f is given to be Riemann integrable, we know that the integral is $L(f)$ and we are done. ☐

Here is an alternate easier solution for both the parts:

Solution. Consider the trivial partition $P_0 = \{a, b\}$ of $[a, b]$. Clearly,
$$\inf_{x\in[a,b]} f(x) \geq 0.$$

Thus,
$$L(f, P_0) = \left[\inf_{x\in[a,b]} f(x)\right][b-a] \geq 0$$

and hence,
$$L(f) \geq L(f, P_0) \geq 0.$$

Since f is given to be Riemann integrable, we know that the integral is $L(f)$ and we are done.

Second part:

Define $F : [a, b] \to \mathbb{R}$ as
$$F(x) := \int_a^x f(t)\mathrm{d}t.$$

Note that since f is continuous, F is differentiable with $F' = f$. (FTC Part I)

Thus, we get that $F' = f \geq 0$ and hence, F is increasing. Thus, we get
$$F(a) \leq F(x) \leq F(b)$$

for all $x \in [a, b]$. However, note that $F(a) = 0 = F(b)$ and hence, F is constant. Thus,
$$f(x) = F'(x) = 0,$$

for all $x \in [a, b]$, as desired. ☐

(b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\displaystyle\int_a^b f(x)\mathrm{d}x = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

Solution. Let $a = 0, b = 2$ and $f : [a, b] \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 0 & x \neq 1, \\ 1 & x = 1. \end{cases}$$

Show that f is actually Riemann integrable on $[0, 2]$ with the integral equal to 0. $\square$

3. Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an appropriate Riemann sum for a suitable function over a suitable interval.

   (ii) $S_n = \sum_{i=1}^{n} \dfrac{n}{i^2 + n^2}$.

   (iv) $S_n = \dfrac{1}{n} \sum_{i=1}^{n} \cos\left(\dfrac{i\pi}{n}\right)$.

   Solution. For both the parts, we shall use the following theorem:

   Theorem 1

   Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose that (P_n, t_n) is a sequence of tagged partitions of $[a, b]$ such that $\|P_n\| \to 0$.
   Then,
   $$\lim_{n\to\infty} R(f, P_n, t_n) = \int_a^b f(x)\mathrm{d}x.$$

   Note very carefully in the above that we already need to know that f is Riemann integrable.

   (ii) Note that
   $$S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2} = \sum_{i=1}^{n} \frac{1}{\left(\frac{i}{n}\right)^2 + 1} \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

   Define $f : [0, 1] \to \mathbb{R}$ by $f(x) := \tan^{-1} x$.
   Then, we have that $f'(x) = \dfrac{1}{x^2 + 1}$.

   As f' is continuous and bounded, it is (Riemann) integrable.
   For $n \in \mathbb{N}$, let
   $$P_n := \{0, 1/n, \ldots, n/n\}.$$
   The tags are given as follows: For the interval $\left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$, we pick the point $\dfrac{i}{n}$; for each $i = 1, \ldots, n$.

   This collection corresponding to P_n is denoted by t_n. Thus, we get a sequence (P_n, t_n) of tagged partitions.

   Then, $S_n = R(f', P_n, t_n)$. Since $\|P_n\| = 1/n \to 0$, it follows that
   $$\lim_{n\to\infty} R(f', P_n, t_n) = \int_0^1 \frac{1}{x^2 + 1}\mathrm{d}x = \int_0^1 f'(x)\mathrm{d}x.$$

By FTC Part II, we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\mathrm{d}x = f(1) - f(0) = \frac{\pi}{4}.$$

(iv) Note that

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right) = \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \pi^{-1}\sin(\pi x)$.
Then, we have that $f'(x) = \cos(\pi x)$.

As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let

$$P_n := \{0, 1/n, \ldots, n/n\}.$$

The tags are given as follows: For the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, we pick the point $\frac{i}{n}$; for each $i = 1, \ldots, n$.

This collection corresponding to P_n is denoted by t_n. Thus, we get a sequence (P_n, t_n) of tagged partitions.

Then, $S_n = R(f', P_n, t_n)$. Since $\|P_n\| = 1/n \to 0$, it follows that

$$\lim_{n\to\infty} R(f', P_n, t_n) = \int_0^1 \cos(\pi x)\mathrm{d}x = \int_0^1 f'(x)\mathrm{d}x.$$

By FTC Part II, we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\mathrm{d}x = f(1) - f(0) = 0. \qquad \square$$

4. (b) Compute $F'(x)$, if for $x \in \mathbb{R}$

   (i) $F(x) = \displaystyle\int_1^{2x} \cos(t^2)\mathrm{d}t.$

   (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)\mathrm{d}t.$

Solution. For both the parts, we shall use the following theorem:

Theorem 2

Let $v : \mathbb{R} \to \mathbb{R}$ be differentiable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Fix $a \in \mathbb{R}$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) := \int_a^{v(x)} g(t)\mathrm{d}t.$$

Then,

$$F'(x) = g(v(x))v'(x).$$

Note that using the above, we can state the more general result for when the lower limit is also a differentiable function.

Proof. First, define $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) := \int_a^{x} g(t)\mathrm{d}t.$$

By FTC Part I, we know that G is differentiable and

$$G'(x) = g(x).$$

On the other hand, note that

$$F(x) = G(v(x)).$$

An application of chain rule yields

$$F'(x) = G'(v(x))v'(x) = g(v(x))v'(x).$$ □

Both the parts are now solved easily.

(i) We have $a = 1$, $g(t) = \cos(t^2)$ and $v(x) = 2x$. Thus, $v'(x) = 2$ and

$$\boxed{F'(x) = 2\cos(4x^2).}$$

(ii) We have $a = 0,$ $g(t) = \cos(t)$ and $v(x) = x^2$. Thus, $v'(x) = 2x$ and

$$\boxed{F'(x) = 2x\cos(x^2).}$$

□

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin\left[\lambda(x-t)\right] \mathrm{d}t.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Solution. Just brute calculation. Note that

$$\begin{aligned} g(x) &= \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \mathrm{d}t \\ &= \frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t \right) \mathrm{d}t \\ &= \frac{1}{\lambda} \sin \lambda x \int_0^x f(t) \cos \lambda t \mathrm{d}t - \frac{1}{\lambda} \cos \lambda x \int_0^x f(t) \sin \lambda t \mathrm{d}t. \end{aligned}$$

Now, we can differentiate g using product rule and FTC Part I.

$$g'(x) = \cos \lambda x \int_0^x f(t) \cos \lambda t \mathrm{d}t + \sin \lambda x \int_0^x f(t) \sin \lambda t \mathrm{d}t$$

Since the limits of integrals appearing in the expressions for g and g' are both from 0 to x, we see that $g(0) = 0 = g'(0)$.

We can differentiate g' in a similar way and get,

$$\begin{aligned} g''(x) = -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t dt + f(x) \cos^2 \lambda x + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t dt \\ + f(x) \sin^2 \lambda x \\ = f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t \right) dt \right) \\ = f(x) - \lambda^2 g(x). \end{aligned}$$

Rearranging the above gives

$$g''(x) + \lambda^2 g(x) = f(x) \qquad \square$$

§5. Tutorial 5

23rd December, 2020

Sheet 5

4. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions for $(x, y) \in \mathbb{R}^2$ are continuous:

   (i) $f(x) \pm g(x)$,

   (ii) $f(x)g(y)$,

   (iii) $\max\{f(x), g(y)\}$,

   (iv) $\min\{f(x), g(y)\}$.

Solution. The idea in all is to use sequential criterion. To recap:

> **Theorem 3: Sequential criterion**
>
> Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $(x_0, y_0) \in \mathbb{R}^2$. Then, h is continuous at (x_0, y_0) if and only if for every sequence $((x_n, y_n))$ converging to (x_0, y_0), we have that
>
> $$\lim_{n\to\infty} h(x_n, y_n) = h(x_0, y_0).$$

The proof of the above is identical to that for the case in one variable.

We now prove the first two parts.
Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary. Let $((x_n, y_n))$ be an arbitrary sequence converging to (x_0, y_0). Then we see that $x_n \to x_0$ and $y_n \to y_0$. Now, applying the (usual) sequential criterion of continuity to f and g, we see that

$$\lim_{n\to\infty} f(x_n) = f(x_0) \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = g(y_0).$$

Using the usual algebra of limits, we get that

$$\lim_{n\to\infty} [f(x_n) \pm g(y_n)] = \lim_{n\to\infty} f(x_n) \pm \lim_{n\to\infty} g(y_n) = f(x_0) \pm g(y_0),$$
$$\lim_{n\to\infty} [f(x_n)g(y_n)] = \lim_{n\to\infty} f(x_n) \lim_{n\to\infty} g(y_n) = f(x_0)g(y_0).$$

Since the sequence was arbitrary, we have shown continuity at(x_0, y_0). Since (x_0, y_0) was arbitrary, we have shown that the desired functions are continuous on $\mathbb{R}^2$.

For the third and fourth parts, use the fact that

$$\min\{a, b\} = \frac{a + b - |a - b|}{2} \quad \text{and} \quad \max\{a, b\} = \frac{a + b + |a - b|}{2}.$$

A similar argument gives the answer, since $|\cdot|$ is continuous. $\square$

For an elaboration of the last argument, see https://aryamanmaithani.github.io/ma-109-tut/handwritten/5.pdf.

6. Examine the following function for the existence of partial derivatives at $(0,0)$.

   (ii) $f(x,y) := \begin{cases} \dfrac{\sin^2(x+y)}{|x|+|y|} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

   Solution. We shall show that neither partial derivative exists at $(0,0)$. First, we show this for the partial derivative in the first direction.

   For $h \neq 0$, we note that

$$\begin{aligned} \frac{f(0+h,0) - f(0,0)}{\|(h,0)\|} &= \frac{\frac{\sin^2(h)}{h} - 0}{|h|} \\ &= \frac{\sin^2 h}{h\,|h|} \end{aligned}$$

   It is easy to see that

$$\lim_{h \to 0} \frac{\sin^2 h}{h\,|h|}$$

   does not exist. (Consider the RHL and LHL.)

   Thus, we see that $\dfrac{\partial f}{\partial x_1}(0,0)$ does not exist. A similar computation shows the same for the second partial as well. □

8. Let $f(0,0)=0$ and

$$f(x,y)=\begin{cases} x\sin(1/x)+y\sin(1/y) & \text{if } x\neq 0,\ y\neq 0,\\ x\sin(1/x) & \text{if } x\neq 0,\ y=0,\\ y\sin(1/y) & \text{if } x=0,\ y\neq 0.\end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

Solution. To show continuity: First, for $(x,y)\neq(0,0)$, note that

$$|f(x,y)|\leq |x|+|y|\leq \sqrt{2}\sqrt{x^2+y^2}.$$

The first inequality follows by taking the three cases and the second by simply squaring and verifying.

The above can be written as

$$|f(x,y)-f(0,0)|\leq \sqrt{2}\|(x,y)-(0,0)\|.$$

Thus, given any $\epsilon>0$, $\delta=\epsilon/\sqrt{2}$ works in the definition of continuity.

Now, we show neither partial derivative exists. The calculations are similar and we show only the first. For $h\neq 0$, we note that

$$\frac{f(h,0)-f(0,0)}{h}=\frac{h\sin(1/h)}{h}=\sin\left(\frac{1}{h}\right).$$

The limit of the above expression as $h\to 0$ does not exist. Thus, we are done. $\square$

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}$$

otherwise. Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every unit vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Solution. For continuity: Let $(x,y) \neq (0,0)$. If $y = 0$, then

$$|f(x,y) - f(0,0)| = 0$$

and if $y \neq 0$, then

$$|f(x,y) - f(0,0)| = \sqrt{x^2+y^2} = \|(x,y) - (0,0)\|.$$

The cases put together give

$$|f(x,y) - f(0,0)| \leq \|(x,y) - (0,0)\|.$$

Thus, $\delta = \epsilon$ works as before.

Now to see the partial derivatives: We can write $\underline{u} = (u_1, u_2)$. Note that $u_1^2 + u_2^2 = 1$.

If $u_2 = 0$, then for $t \neq 0$, note that

$$\begin{aligned}
\frac{f(0+u_1t, 0+u_2t) - f(0,0)}{t} &= \frac{f(u_1t, 0) - 0}{t} \\
&= \frac{0-0}{t} = 0.
\end{aligned}$$

Clearly, the above limit exists as $t \to 0$ and is 0.

Now, for $u_2 \neq 0$ and $t \neq 0$, note that

$$\begin{aligned}
\frac{f(0+u_1t, 0+u_2t) - f(0,0)}{t} &= \frac{f(u_1t, u_2t) - 0}{t} \\
&= \frac{1}{t}\frac{u_2t}{|u_2t|}\sqrt{(u_1^2+u_2^2)t^2} \\
&= \frac{1}{t}\frac{u_2t}{|u_2t|}|t| \\
&= \frac{u_2}{|u_2|}.
\end{aligned}$$

Clearly, the above limit exists as $t \to 0$ and is $\frac{u_2}{|u_2|}$.

Thus, all directional derivatives exist and we have

$$D_{\underline{u}} f(0,0) = \begin{cases} 0 & u_2 = 0, \\ \frac{u_2}{|u_2|} & u_2 \neq 0, \end{cases}$$

Note that taking $\underline{u} = (1,0)$ and $(0,1)$ recovers the first and second partial derivatives, respectively. We now check for differentiability.

If f is differentiable at $(0,0)$, then the total derivative *must* be

$$A := \begin{bmatrix} \frac{\partial f}{\partial x_1}(0,0) & \frac{\partial f}{\partial x_2}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We now see whether that actually satisfies the limit condition. That is, we must check if

$$\lim_{(h,k)\to(0,0)} \frac{\left| f(0+h, 0+k) - f(0,0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right|}{\|(h,k)\|} = 0.$$

We show that that is not case.

For $(h,k) \neq (0,0)$ and $k \neq 0$, we note that

$$\begin{aligned} \frac{\left| f(0+h, 0+k) - f(0,0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right|}{\|(h,k)\|} &= \frac{|f(0+h, 0+k) - f(0,0) - 0h - 1k|}{\sqrt{h^2+k^2}} \\ &= \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right| \end{aligned}$$

Note that along the curve $h = k$ with $(h,k) \neq (0,0)$, we see that the above expression equals

$$\left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right| = \left| \frac{k}{|k|} - \frac{k}{\sqrt{2k^2}} \right| = \left(1 - \frac{1}{\sqrt{2}} \right)$$

and the limit of *that* is not 0 as $k \to 0$.

Thus, we see that the original limit (which was supposed to be 0) also does not equal 0. Thus, f is not differentiable at $(0,0)$. $\square$

Note that we haven't actually shown that the limit equals $1 - \frac{1}{\sqrt{2}}$. (In fact, it doesn't exist.) All we have shown is that the limit is not 0.