Calculus I Recap

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IIT Bombay

Autumn Semester 2020-21

Start recording!

Week 1

Definition 1 (Sequences)

Definition 2 (Convergence)

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A sequence in X

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If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

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Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

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Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

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For us, all we need to know is that convergence of a series is just the convergence of the sequence of its *partial sums.* Thus, we are back in the case where we study sequences!

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for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.

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If no such l exists, then we say that f does not have any limit at x_0.

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Similarly, we have the limit at $-\infty$.

Stop recording. Start a new one.
Take doubts.

Week 2

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Note carefully that the domain is an interval.

Now, we state another property, called the extreme value theorem.

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Thus, what the previous theorem told us was that not only is the image bounded but the supremum and infimum are actually attained. (If the function is continuous and defined on a closed and bounded interval, that is.)

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Once again, note that this only talks about "interior points."

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Stop recording. Start a new one.
Take doubts.

Start recording!

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If $f''(x_0) = 0$, then nothing can be concluded.

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$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

More graphically, given any two points on the graph of the function, the line segment joining the two points lies above the graph.

The definition of a *concave* function is obtained by replacing $\leq$ with $\geq$ and "above" with "below."

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Read it some day.

Proposition 2

Suppose $f : I \to \mathbb{R}$ is differentiable. Then

1. f' is increasing on $I \iff f$ is convex on I.
2. f' is decreasing on $I \iff f$ is concave on I.
3. f' is strictly increasing on $I \iff f$ is strictly convex on I.
4. f' is strictly decreasing on $I \iff f$ is strictly concave on I.

Corollary 1

Suppose $f : I \to \mathbb{R}$ is twice differentiable. Then

1. $f'' \geq 0$ on $I \iff f$ is convex on I.
2. $f'' \leq 0$ on $I \iff f$ is concave on I.
3. $f'' > 0$ on $I \implies f$ is strictly convex on I.
4. $f'' < 0$ on $I \implies f$ is strictly concave on I.

Let's now talk about inflection points.

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Theorem 10 (Derivative tests)

1. (First derivative test) Suppose f is differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(x_0 - \delta, x_0)$ and f' is decreasing on $(x_0, x_0 + \delta)$, or vice-versa.
2. (Second derivative test) Suppose f is twice differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that $f'' \geq 0$ on $(x_0 - \delta, x_0)$ and $f'' \leq 0$ on $(x_0, x_0 + \delta)$, or vice-versa.

Thus, if f is twice differentiable, then x_0 is inflection point iff f'' changes sign. (Note that $f''(x_0)$ is not required to exist. Recall the crazy example.)

The previous slide gives us a **necessary** condition for inflection point. We have the same notation as earlier.

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$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where P_n is as in the previous slide.

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What was the example seen in class that illustrated this?

Stop recording. Start a new one.
Take doubts.