Calculus I Recap

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https://aryamanmaithani.github.io/tuts/ma-109

IIT Bombay

Autumn Semester 2020-21

Start recording!

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Exercise: Show that \mathbb{N}, \mathbb{Z} are complete. (What property do you really need? Can you generalise this?)



Now, we digress a bit to see what $\mathbb R$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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Note that **neither** of the above grey boxes is true if we replace \mathbb{R} by \mathbb{Q} .



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For us, all we need to know is that convergence of a series is just the convergence of the <u>sequence</u> of its *partial sums*. Thus, we are back in the case where we study sequences!

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for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.



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Similarly, we have the limit at $-\infty$.



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Note carefully that the domain is an interval.



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The image is not bounded above.

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Consider $f:(0,1)\to\mathbb{R}$ defined by

$$f(x) = x$$
.

The image is bounded but the infimum/supremum are not attained.

Consider $f:(0,1)\to\mathbb{R}$ defined by

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The image is not bounded above. It is bounded below but the infimum is not attained.



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Of course, we have an analogous definition for minimum. Note that here, we have that x_0 is an "interior point." That is, there is an interval *around* x_0 contained within the domain.

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Once again, note that this only talks about "interior points."

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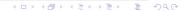
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Stop recording. Start a new one. Take doubts.

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If $f''(x_0) = 0$, then nothing can be concluded.

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The definition of a *concave* function is obtained by replacing \leq with \geq and "above" with "below."



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Read it some day.

Proposition 2

Suppose $f: I \to \mathbb{R}$ is differentiable. Then

- **1** f' is increasing on $I \iff f$ is convex on I.
- 2 f' is decreasing on $I \iff f$ is concave on I.
- **3** f' is strictly increasing on $I \iff f$ is strictly convex on I.
- f' is strictly decreasing on $I \iff f$ is strictly concave on I.

Corollary 1

Suppose $f: I \to \mathbb{R}$ is twice differentiable. Then

- **1** $f'' \ge 0$ on $I \iff f$ is convex on I.
- 2 $f'' \le 0$ on $I \iff f$ is concave on I.
- **3** f'' > 0 on $I \implies f$ is strictly convex on I.
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Let's now talk about inflection points.

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Theorem 10 (Derivative tests)

- **1** (First derivative test) Suppose f is differentiable on $(x_0 r, x_0) \cup (x_0, x_0 + r)$ for some r > 0. Then, x_0 is a point of inflection \iff there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(x_0 \delta, x_0)$ and f' is decreasing on $(x_0, x_0 + \delta)$, or vice-versa.
- ② (Second derivative test) Suppose f is twice differentiable on $(x_0 r, x_0) \cup (x_0, x_0 + r)$ for some r > 0. Then, x_0 is a point of inflection \iff there is $\delta > 0$ with $\delta < r$ such that $f'' \ge 0$ on $(x_0 \delta, x_0)$ and $f'' \le 0$ on $(x_0, x_0 + \delta)$, or vice-versa.

Thus, if f is twice differentiable, then x_0 is inflection point iff f'' changes sign. (Note that $f''(x_0)$ is not required to exist. Recall the crazy example.)

The previous slide gives us a **necessary** condition for inflection point. We have the same notation as earlier.

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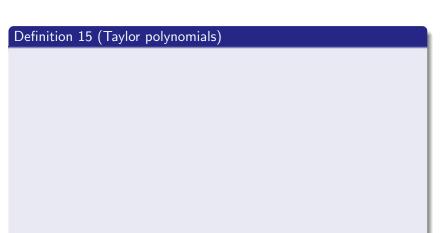
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What was the example seen in class that illustrated this?

Stop recording. Start a new one. Take doubts.