MA 109: Calculus I

Tutorial Solutions

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§0. Notations

1. $\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
2. $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
3. $\mathbb{Q}$ denotes the set of rational numbers.
4. $\mathbb{R}$ denotes the set of real numbers.

§1. Tutorial 1

25th November, 2020

Sheet 1

2. (iv) $\lim_{n\to\infty} (n)^{1/n}$.

Define $h_n := n^{1/n} - 1$.
Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. (Why?)

Now, for $n > 2$, we have

$$\begin{aligned} n &= (1 + h_n)^n \\ &= 1 + nh_n + \binom{n}{2}h_n^2 + \cdots + \binom{n}{n}h_n^n \\ &\geq 1 + nh_n + \binom{n}{2}h_n^2 \\ &> \binom{n}{2}h_n^2 \\ &= \frac{n(n-1)}{2}h_n^2. \end{aligned}$$

Thus, $h_n < \sqrt{\dfrac{2}{n-1}}$ for all $n > 2$.

Using Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$ which gives us that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

(Where did we use that $h_n \geq 0$?)

3. (ii) We show that $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is *not* convergent.

Solution. Note that from the difference formula, we know that if $\{a_n\}$ converges, then

$$\lim_{n\to\infty}|a_{n+1}-a_n|=0.$$

(The limit *exists* and equals 0.)

We show that this is not true for the given sequence. We define

$$b_n := a_{n+1}-a_n,$$

where $\{a_n\}$ is the sequence given in the question.

Then, b_n is given as

$$\begin{aligned}
b_n &= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)-(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\\
&= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)+(-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n}\right)\\
&= (-1)^{n+1}+(-1)^n\left(\frac{1}{n+1}+\frac{1}{n}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|b_n| &= \left|1-\left(\frac{1}{n+1}+\frac{1}{n}\right)\right|\\
&= \left|1-\frac{2n+1}{n(n+1)}\right|
\end{aligned}$$

From the above, we conclude that

$$\lim_{n\to\infty}|b_n|=1.$$

This shows that a_n does not converge. □

5. (iii) $a_1 = \sqrt{2},\ a_{n+1} = 3 + \dfrac{a_n}{2} \quad \forall n \geq 1.$

Solution. I first describe the general idea.

> The idea in these questions is to first prove a bound on a_n by induction. Then, using that bound we prove that the sequence is convergent.
>
> Once we do that, we then know that $\lim_{n\to\infty} a_n$ exists. Since that also equals $\lim_{n\to\infty} a_{n+1}$, we can take limit on both sides of the equation and solve for the limit L.

First, we prove that the sequence is bounded above.

> Claim 1. $a_n < 6$ for all $n \in \mathbb{N}$.
>
> *Proof.* We shall prove this via induction. The base case $n = 1$ is immediate as $2 < 6$.
> Assume that it holds for $n = k$. Then,
>
> $$a_{k+1} = 3 + \frac{a_n}{2} < 3 + \frac{6}{2} = 6.$$
>
> By principle of mathematical induction, we have proven the claim. □

> Claim 2. $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
>
> *Proof.* $a_{n+1} - a_n = 3 - \dfrac{a_n}{2} = \dfrac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n.$ □

Thus, we now know that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Then taking the limit on both sides of

$$a_{n+1} = 3 + \frac{a_n}{2}$$

gives us

$$L = 3 + \frac{L}{2},$$

which we can solve to get $L = 6$. □

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

Solution. Choose $\epsilon = \dfrac{|L|}{2}$. Note that this is indeed greater than 0.

By the $\epsilon - N$ definition, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon = \frac{|L|}{2}$$

for all $n > N$. Using triangle inequality, we get

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}.$$

Thus, we get

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}.$$

Adding $|L|$ on both sides gives us

$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2}$$

for all $n > N$, as desired. □

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   1. $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
   2. $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

   Solution. Both the statements are false. We give one counterexample for both.

   $$\begin{aligned} a_n &:= 1 && \text{for all } n \in \mathbb{N}, \\ b_n &:= (-1)^n && \text{for all } n \in \mathbb{N}. \end{aligned}$$

   Clearly, $\{a_n\}_{n\geq 1}$ converges and $\{b_n\}_{n\geq 1}$ is bounded. However, the product is again the latter sequence which does not converge. □

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x \to c} f(x) = 0$ for some $c \in [a, b]$. Prove or disprove the following statements.

    1. $\lim\limits_{x \to c} [f(x)g(x)] = 0$.
    2. $\lim\limits_{x \to c} [f(x)g(x)] = 0$, if g is bounded.
    3. $\lim\limits_{x \to c} [f(x)g(x)] = 0$, if $\lim\limits_{x \to c} g(x)$ exists.

    Solution. 1. No. Consider $a = c = 0$ and $b = 1$. Let f, g be defined as

    $$f(x) = x, \quad g(x) = \frac{1}{x}.$$

    Verify that this works as a counterexample.

    2. We prove this statement. Since g is bounded, there exists $M > 0$ such that

    $$|g(x)| < M$$

    for all $x \in (a, b)$. Thus, we have

    $$|f(x)g(x)| \leq M|f(x)|$$

    for all $x \in (a, b)$. Since the LHS is clearly non-negative, using Sandwich theorem proves that

    $$\lim_{x \to c} |f(x)g(x)| = 0.$$

    This also gives us that

    $$\lim_{x \to c} f(x)g(x) = 0.$$

    (Why?)

    3. This is also true. We can simply use that limit of products is the product of limits if the individual limits exist.

    $\square$

§2. Tutorial 2

2nd December, 2020

Sheet 1

13. (ii) Discuss the continuity of the following function:

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Solution. For $x \neq 0$, the continuity of f at x follows from the fact that f is the product and composition of continuous functions.

For $x = 0$, we prove continuity using $\epsilon - \delta$. We show that

$$\lim_{x \to 0} f(x) = 0.$$

Since $f(0) = 0$, the continuity of f at 0 will follow.

To this end, let $\epsilon > 0$ be given. We show that $\delta := \epsilon$ works. Indeed, if $0 < |x - 0| < \delta$, then

$$\begin{aligned} |f(x) - 0| &= \left| x \sin\left(\frac{1}{x}\right) \right| \qquad |\sin| \leq 1 \\ &\leq |x| \\ &= |x - 0| \\ &< \delta = \epsilon. \end{aligned}$$

Thus, we have shown that

$$0 < |x - 0| < \delta \implies |f(x) - 0| < \epsilon,$$

proving that

$$\lim_{x \to 0} f(x) = 0 = f(0),$$

as desired. $\square$

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

Solution. As earlier, differentiability of f at $x \neq 0$ follows due to product/composition rules.

Now, for $h \neq 0$, note that

$$\frac{f(0+h) - f(0)}{h} = h \sin\left(\frac{1}{h}\right).$$

As saw earlier, the limit of the above as $h \to 0$ exists and is 0. Thus, we get that f is differentiable at 0 as well with $f'(0) = 0$.

Thus, f is differentiable on $\mathbb{R}$.

Now, for $x \neq 0$, we can compute the derivative using product/chain rule. Putting this together, we get

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

We now show that f' is not continuous at 0. We use the sequential criterion for this. Consider the sequence

$$x_n := \frac{1}{2n\pi}, \qquad n \in \mathbb{N}.$$

Clearly, we have that $x_n \to 0$ and $x_n \neq 0$. Thus, we get

$$f'(x_n) = -\cos(2n\pi) = -1.$$

Thus, we see that $f'(x_n) \to -1 \neq f'(0)$.
This shows that f' is not continuous. $\square$

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

Solution. Putting $x = y = 0$, we note that $f(0) = (f(0))^2$. If $f(0) = 0$, show that $f(x) = 0$ for all x and conclude that the given thing is indeed true.

Now, assume that $f(0) \neq 0$. Then, $f(0) = 1$.

Let $c \in \mathbb{R}$ be arbitrary. For $h \neq 0$, we note that

$$\begin{aligned}\frac{f(c+h) - f(c)}{h} &= \frac{f(c)f(h) - f(c)}{h}\\ &= f(c)\frac{f(h) - 1}{h}\\ &= f(c)\frac{f(h) - f(0)}{h}.\end{aligned}$$

Since f is given to be differentiable at 0, the above limit as $h \to 0$ exists and equals $f(c)f'(0)$. Thus, we see that $f'(c)$ exists and equals $f(c)f'(0)$. □

Sheet 1 Optional

7. Let $f : (a, b) \to \mathbb{R}$ be differentiable and $c \in (a, b)$. Show that the following are equivalent:

   (i) f is differentiable at c.

   (ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$, and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and

   $$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

   (iii) There exists $\alpha \in \mathbb{R}$ such that

   $$\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0.$$

Solution. We prove this by a usual technique in math by showing that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii)
First, we pick $\delta := \min\{c - a, b - c\}$. Note that $\delta > 0$ and $(c-\delta, c+\delta) \subset (a, b)$.

Now, since f is differentiable at c, $f'(c)$ exists. We define $\alpha := f'(c) \in \mathbb{R}$.
Now, we define $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ as

$$\epsilon_1(h) := \begin{cases} \dfrac{f(c+h) - f(c)}{h} - \alpha & h \neq 0, \\ 0 & h = 0. \end{cases}$$

(Note that $f(c + h)$ above makes sense because $(c - \delta, c + \delta) \subset (a, b)$.)

Now, from the definition above, it is clear that

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

We only need to show that $\lim_{h\to 0} \epsilon_1(h) = 0$. However, note that, for $h \neq 0$, we have

$$\epsilon_1(h) = \frac{f(c+h) - f(c)}{h} - \alpha.$$

Since $f'(c) = \alpha$, we know that

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

which gives us that $\epsilon_1(h) \to 0$ as $h \to 0,$ as desired.

(ii) $\implies$ (iii)
Let α be as in (ii). Then, for $h \neq 0,$ we note that

$$\begin{aligned}\frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \frac{|h\epsilon_1(h)|}{|h|} \\ &= |\epsilon_1(h)|.\end{aligned}$$

Since $\lim_{h\to 0} \epsilon_1(h) = 0,$ we get that $\lim_{h\to 0} |\epsilon_1(h)| = 0,$ which proves the desired limit.

(iii) $\implies$ (i)
We show that the α in (iii) is the derivative of f at c. Note that we are given

$$\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

or

$$\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0.$$

The above gives us that

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

or

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} \right) = \alpha.$$

Thus, $f'(c)$ exists and equals α. □

In the above, we used the following implicitly:

$$\lim_{x\to c} f(x) = 0 \iff \lim_{x\to c} |f(x)| = 0.$$

10. Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point.

    Solution. We need to show that there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$. Consider $g : [0,1] \to \mathbb{R}$ defined by

    $$g(x) := f(x) - x.$$

    Then, showing that f has a fixed point is equivalent to showing that g has a zero.

    Note that

    $$g(0) = f(0) \geq 0$$

    and

    $$g(1) = f(1) - 1 \leq 0.$$

    If either of the equalities hold, then we are done. Otherwise, we have

    $$g(0) > 0 \quad \text{and} \quad g(1) < 0.$$

    By intermediate value property, $g(x_0) = 0$ for some $x_0 \in [0,1]$, as desired. □

Sheet 2

2 Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Solution. The existence of x_0 is given by the intermediate value theorem since 0 lies between $f(a)$ and $f(b)$.

We now show uniqueness. Suppose that there exists $x_1 \in (a, b)$ such that $f(x_1) = 0$ and $x_1 \neq x_0$. We show that this leads to a contraction.
By LMVT, there exists c between x_0 and x_1 such that

$$\begin{aligned} f'(c) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= 0. \end{aligned}$$

A contraction since $c \in (a, b)$ and we were given that $f'(x) \neq 0$ for any $x \in (a, b)$. $\square$

5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

   Solution. If $a = b$, then the inequality is clear. Suppose that $a \neq b$.

   Then, there exists c between a and b such that
$$\sin'(c) = \frac{\sin a - \sin b}{a - b}.$$
   Note that $\sin' = \cos$ and thus,
$$\left|\frac{\sin a - \sin b}{a - b}\right| = |\cos(c)| \leq 1.$$
   Cross-multiplying gives us the desired result. $\square$