Calculus I Recap

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IIT Bombay

Autumn Semester 2020-21

Week 1

Start recording!

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Definition 1 (Sequences)

Definition 2 (Convergence)

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If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

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Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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$x < y \implies x \cdot z < y \cdot z$ for all $x, y \in \mathbb{R}$ and $z \in \mathbb{R}_{>0}$.

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Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

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For us, all we need to know is that convergence of a series is just the convergence of the <u>sequence</u> of its *partial sums.* Thus, we are back in the case where we study sequences!

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If the thing in the previous slide does happen, then we say that $f(x)$ tends to l as x tends to x_0. Or that f has a limit l at x_0.

If no such l exists, then we say that f does not have any limit at x_0.

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Similarly, we have the limit at $-\infty$.

Week 1

Stop recording. Start a new one.
Take doubts.