Calculus I Recap

Aryaman Maithani
https://aryamanmaithani.github.io/tuts/ma-109

IIT Bombay

Autumn Semester 2020-21

Start recording!

Week 1

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Definition 2 (Convergence)

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If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

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Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

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If no such l exists, then we say that f does not have any limit at x_0.

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Take doubts.

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Note carefully that the domain is an interval.

Now, we state another property, called the extreme value theorem.

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Once again, note that this only talks about "interior points."

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Stop recording. Start a new one.
Take doubts.

Start recording!

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If $f''(x_0) = 0$, then nothing can be concluded.

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$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

More graphically, given any two points on the graph of the function, the line segment joining the two points lies above the graph.

The definition of a *concave* function is obtained by replacing $\leq$ with $\geq$ and "above" with "below."

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Read it some day.

Proposition 2

Suppose $f : I \to \mathbb{R}$ is differentiable. Then

1. f' is increasing on $I \iff f$ is convex on I.
2. f' is decreasing on $I \iff f$ is concave on I.
3. f' is strictly increasing on $I \iff f$ is strictly convex on I.
4. f' is strictly decreasing on $I \iff f$ is strictly concave on I.

Corollary 1

Suppose $f : I \to \mathbb{R}$ is twice differentiable. Then

1. $f'' \geq 0$ on $I \iff f$ is convex on I.
2. $f'' \leq 0$ on $I \iff f$ is concave on I.
3. $f'' > 0$ on $I \implies f$ is strictly convex on I.
4. $f'' < 0$ on $I \implies f$ is strictly concave on I.

Let's now talk about inflection points.

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Theorem 10 (Derivative tests)

1. (First derivative test) Suppose f is differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(x_0 - \delta, x_0)$ and f' is decreasing on $(x_0, x_0 + \delta)$, or vice-versa.
2. (Second derivative test) Suppose f is twice differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that $f'' \geq 0$ on $(x_0 - \delta, x_0)$ and $f'' \leq 0$ on $(x_0, x_0 + \delta)$, or vice-versa.

Thus, if f is twice differentiable, then x_0 is inflection point iff f'' changes sign. (Note that $f''(x_0)$ is not required to exist. Recall the crazy example.)

The previous slide gives us a **necessary** condition for inflection point. We have the same notation as earlier.

Theorem 11 (Another second derivative test)

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&\vdots \\
P_n(x) =& f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
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$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where P_n is as in the previous slide.

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Week 3

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What was the example seen in class that illustrated this?

Week 3

Stop recording. Start a new one.
Take doubts.

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$$M_i := \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i := \inf_{x \in [x_{i-1}, x_i]} f(x),$$

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Thus, m_i and M_i denote the infimum and supremum of f over the i-th interval, respectively.

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Using the earlier sums, we now define the upper and lower Darboux *integrals.* The notations are continuing from earlier.

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We now turn to the definition of Riemann integrals.

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On the next slide, we state two equivalent definitions of Riemann integrability.

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Now, we see how derivatives and integrals relate. These are the two parts of the Fundamental Theorem of Calculus.

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In particular, if f is continuous, then Riemann integrability of f is guaranteed and the above equation is true for *all* $c \in (a, b)$.

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Some pathological remarks:

1. If a function is Riemann integrable, it doesn't mean that it is the derivative of a function. (That is, it needn't have an anti-derivative.)
2. If a function has an anti-derivative, it doesn't mean that it is Riemann integrable. (That is, derivatives needn't be Riemann integrable.)

For the first, take $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$. It cannot be the derivative of any function because it doesn't have IVP. (Recall Theorem 8.)

For the second, consider the derivative of $F : [-1, 1] \to \mathbb{R}$ defined by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$. F' here isn't bounded.

Start recording!

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Note that if $m = 1$, then $\|f(x) - L\|$ is just $|f(x) - L|$. In fact, for $n = m = 1$, the definition above coincides with the earlier one. (Definition 7.)

Week 5

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To see this, consider $n = 1$ and $U = [0, 1) \cup \{2\}$. Then, 1 is a limit point of U while 2 is not.

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As before, the case $n = m = 1$ recovers the original one.

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Note that the limit above is an ordinary one-variable limit of a real function, as we had seen earlier.

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The partial derivative with respect to x_2 is defined similarly. Note that the limit above is an ordinary one-variable limit of a real function, as we had seen earlier. Also note that b is fixed in the numerator.

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Note that the above says "there exists" and not "for every." Compare this with the definition of "limit point."

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provided it exists.

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As before, this is an ordinary limit. Taking $v = (1, 0)$ and $(0, 1)$ recovers the usual the partial derivatives with respect to x_1 and x_2, respectively.

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In this case, we write $Df(x_0, y_0) = A$ and call A the *total derivative* of f at (x_0, y_0).

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Suppose that f is differentiable is (x_0, y_0). Then, both the partial derivatives of f at (x_0, y_0) exist and

$$Df(x_0, y_0) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(x_0, y_0) & \dfrac{\partial f}{\partial x_2}(x_0, y_0) \end{bmatrix}.$$

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The above matrix is also called the *gradient* and denoted by $\nabla f(x_0, y_0)$.

Stop recording. Start a new one.
Take doubts.