Calculus I Recap

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https://aryamanmaithani.github.io/tuts/ma-109

IIT Bombay

Autumn Semester 2020-21

Week 1

Start recording!

Week 1

Definition 1 (Sequences)

Definition 2 (Convergence)

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If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

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Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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$x < y \implies x \cdot z < y \cdot z$ for all $x, y \in \mathbb{R}$ and $z \in \mathbb{R}_{>0}$.

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Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

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If no such l exists, then we say that f does not have any limit at x_0.

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Take doubts.

Start recording!

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Note carefully that the domain is an interval.

Now, we state another property, called the extreme value theorem.

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Once again, note that this only talks about "interior points."

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Stop recording. Start a new one.
Take doubts.

Week 3

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If $f''(x_0) = 0$, then nothing can be concluded.

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A function $f : I \to \mathbb{R}$ is said to be *convex* if for every $x_1, x_2 \in I$ and every $t \in [0, 1]$, we have

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

More graphically, given any two points on the graph of the function, the line segment joining the two points lies above the graph.

The definition of a *concave* function is obtained by replacing $\leq$ with $\geq$ and "above" with "below."

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Read it some day.

Proposition 2

Suppose $f : I \to \mathbb{R}$ is differentiable. Then

1. f' is increasing on $I \iff f$ is convex on I.
2. f' is decreasing on $I \iff f$ is concave on I.
3. f' is strictly increasing on $I \iff f$ is strictly convex on I.
4. f' is strictly decreasing on $I \iff f$ is strictly concave on I.

Corollary 1

Suppose $f : I \to \mathbb{R}$ is twice differentiable. Then

1. $f'' \geq 0$ on $I \iff f$ is convex on I.
2. $f'' \leq 0$ on $I \iff f$ is concave on I.
3. $f'' > 0$ on $I \implies f$ is strictly convex on I.
4. $f'' < 0$ on $I \implies f$ is strictly concave on I.

Let's now talk about inflection points.

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Theorem 10 (Derivative tests)

1. (First derivative test) Suppose f is differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(x_0 - \delta, x_0)$ and f' is decreasing on $(x_0, x_0 + \delta)$, or vice-versa.
2. (Second derivative test) Suppose f is twice differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that $f'' \geq 0$ on $(x_0 - \delta, x_0)$ and $f'' \leq 0$ on $(x_0, x_0 + \delta)$, or vice-versa.

Thus, if f is twice differentiable, then x_0 is inflection point iff f'' changes sign. (Note that $f''(x_0)$ is not required to exist. Recall the crazy example.)

The previous slide gives us a **necessary** condition for inflection point. We have the same notation as earlier.

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&\vdots \\
P_n(x) =& f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
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$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where P_n is as in the previous slide.

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What was the example seen in class that illustrated this?

Week 3

Stop recording. Start a new one.
Take doubts.

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$$M_i := \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i := \inf_{x \in [x_{i-1}, x_i]} f(x),$$

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Thus, m_i and M_i denote the infimum and supremum of f over the i-th interval, respectively.

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Using the earlier sums, we now define the upper and lower Darboux *integrals.* The notations are continuing from earlier.

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We now turn to the definition of Riemann integrals.

Week 4

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On the next slide, we state two equivalent definitions of Riemann integrability.

Week 4

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Now, we see how derivatives and integrals relate. These are the two parts of the Fundamental Theorem of Calculus.

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In particular, if f is continuous, then Riemann integrability of f is guaranteed and the above equation is true for *all* $c \in (a, b)$.

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Some pathological remarks:

1. If a function is Riemann integrable, it doesn't mean that it is the derivative of a function. (That is, it needn't have an anti-derivative.)
2. If a function has an anti-derivative, it doesn't mean that it is Riemann integrable. (That is, derivatives needn't be Riemann integrable.)

For the first, take $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$. It cannot be the derivative of any function because it doesn't have IVP. (Recall Theorem 8.)

For the second, consider the derivative of $F : [-1, 1] \to \mathbb{R}$ defined by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$. F' here isn't bounded.