$\mathbb{C}$omplex Analysis TSC - 1

Aryaman Maithani
https://aryamanmaithani.github.io/tuts/ma-205

IIT Bombay

18th September, 2020

Greetings

Hi, welcome to this

complex discussion.

Here are some "guidelines" for this TSC -

1. Unmute your mic at any time and ask your doubt.
2. I will not be checking chat often (or maybe at all), so posting it there might not be helpful.

You can find a link to this document on `bit.ly/ca-205`. Both with and without pauses. You may keep it open alongside for quick reference.

Warning

This is primarily going to be a quick recap of the facts important. It is, of course, not possible to go through everything in just 2 hours.
In particular, this will *not* be a substitute for all the lectures done so far.
I will also not be going through the proofs. We can discuss these finer things at the end, if time permits.
Though I'm not a fan of this - this session is pretty much going to cover things important from the point of view of an exam. I may also skip things from the lectures if I think that they are not important. They *might* turn out to be important, though.

Of course, I will not (intentionally) say anything which is mathematically incorrect.

Lecture 1

Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ-neighbourhood of z_0, denoted by $B_\delta(z_0)$ is the set

$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$

Definition 2 (Open sets)

A set $U \subset \mathbb{C}$ is said to be open if:
for *every* $z_0 \in \mathbb{C}$, there exists *some* $\delta > 0$ such that

$$B_\delta(z_0) \subset U.$$

Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P. (A continuous function from $[0, 1]$ to P.)

Definition 4 (Differentiable)

Let $\Omega \subset \mathbb{C}$ be open. Let

$$f : \Omega \to \mathbb{C}$$

be a function. Let $z_0 \in \Omega$. f is said to be *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is denoted by $f'(z_0)$.

Definition 5 (Holomorphic)

A function f is said to be holomorphic on an open set Ω if it is differentiable at every $z_0 \in \Omega$.
A function f is said to be holomorphic at z_0 if it is holomorphic on some neighbourhood of z_0.

Remark 1

A function may be differentiable at z_0 but not holomorphic at z_0. For example, $f(z) = |z|^2$ is differentiable only at 0. Thus, it is differentiable at 0 but holomorphic nowhere.

For sets, however, there is no difference.

End of Lecture 1

Any questions?

Notation

From this point on, Ω be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number z as $z = x + \iota y$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + \iota v(z)$.

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose f as

$$f(z) = u(z) + \iota v(z),$$

where $u, v : \Omega \to \mathbb{R}$ are real valued functions.

The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v. Now, if we know that f is holomorphic, then we have the following result.

Theorem 1 (CR equations)

Let $f : \Omega \to \mathbb{C}$ be differentiable at a point $z_0 \in \Omega$. Let $z_0 = x_0 + \iota y_0$.
Then, we have

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, we have

$$f'(z_0) = u_x(x_0, y_0) + \iota v_x(x_0, y_0).$$

Existence of u_x, u_y, v_x, v_y is part of the theorem.

Note the subscript is x for both in the above.
Also note that all the equalities are only at the point z_0. In particular, we are only assuming differentiability at z_0.

Converse? What is the converse? Is it true?

No. The converse is not true.

An example for you to check is

$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that u and v satisfy the CR equations at $(0, 0)$ but f is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

We recall MA 105 now.

Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega \to \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if there exits a 2×2 real matrix A such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(x_0+h, y_0+k) - f(x_0, y_0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h,k)\|} = 0.$$

The matrix A was called the *total derivative of f at (x_0, y_0).*

Theorem 2

If f is (complex) differentiable at a point $z_0 = x_0 + \iota y_0$, then f is real differentiable at (x_0, y_0).

Once again, this is only talking about differentiability at a point. The converse is again not true.
Take the example $f(z) = \bar{z}$. Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

Theorem 3

Let $f : \Omega \to \mathbb{C}$ be a function and let $z_0 = x_0 + \iota y_0 \in \Omega$. If the CR equations hold at the point (x_0, y_0) and if f is real differentiable at the point (x_0, y_0), then f is complex differentiable at the point z_0.

Definition 7 (Harmonic functions)

Let $u : \Omega \to \mathbb{R}^2$ be a twice continuously differentiable function. u is said to be *harmonic* if $u_{xx} + u_{yy} = 0$.

Proposition 1

The real and imaginary parts of a holomorphic function are harmonic.

Suppose u and v are harmonic on Ω. v is said to be a harmonic conjugate of u if $f = u + \iota v$ is holomorphic on Ω.
If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.
Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

End of Lecture 2

Any questions?

Definition 8 (Convergence of series)

A series of the form

$$\sum_{n=0}^{\infty} a_n$$

of complex numbers is said to converge if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges (to a finite complex number).

The sequence of partial sums is just the following sequence:

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots.$$

"Divergent" is simply used to mean "not convergent."
Check that $\sum(-1)^n$ and $\sum n$ both diverge.

Definition 9 (limsup)

Given a sequence (x_n) of real numbers, we may define a new sequence (y_n) as

$$y_n = \sup\{x_m : m \geq n\}.$$

The limit of this sequence always exists and we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Remark 2

Each y_n might be ∞. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.
If $\lim_{n\to\infty} x_n$ itself exists, then it equals the $\limsup$ as well.

We will be interested in discussing radius of convergence of *power series.* We all know what that is. It is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad (*)$$

where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

What is the radius of convergence, though? (The definition, that is.)

Theorem 4 (Radius of convergence)

Given any power series as $(*)$, there exists $R \in [0, \infty]$ such that

1. $(*)$ converges for any z with $|z - z_0| < R$, and
2. $(*)$ diverges for any z with $|z - z_0| > R$.

This R is called the radius of convergence.

Note the brackets.

We would now like to be able to calculate the radius of convergence.

Theorem 5 (Root test)

Let $(*)$ be as earlier. Define

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then, $R = \alpha^{-1}$ is the radius of convergence.

This test *always works.* We had no assumptions of any kind on $(*)$.
Note that $^{-1}$.
If $\alpha = 0$, then $R = \infty$ and vice-versa.

We have another test. This is simpler (to calculate) but mightn't always work.

Theorem 6 (Ratio test)

Let $(*)$ be as earlier.
Assume that the limit

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists. (Possibly as ∞.)
Then, R is the radius of convergence.

Note that here we assume that the limit does exist. This may not always be true.
Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Differentiability of power series is what one should expect.

Theorem 7 (Differentiability)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the open disc of radius R, let $f(z)$ denote this sum. Then, on this disc, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

End of Lecture 3

Any questions?

Lecture 4: Exponential function

I shall just recall the facts from the lecture.

Definition 10 (Exponential function)

The power series
$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$
converges on all of $\mathbb{C}$. This sum is denoted by $\exp(z)$.

Theorem 8 (Facts)

1. $\exp'(z) = \exp(z)$,
2. $\exp'(bz) = b\exp(bz)$, for $b \in \mathbb{C}$,
3. $\exp(z) \cdot \exp(-z) = 1$ for all $z \in \mathbb{C}$,
4. $\exp(z)$ is always nonzero.

Now, we some "converse" facts.

Theorem 9 (Characterisations)

1. If $f'(z) = bf(z)$, then $f(z) = a\exp(bz)$ for some $a, b \in \mathbb{C}$,
2. If $f' = f$ and $f(0) = 1$, then $f(z) = \exp(z)$.

Theorem 10 (Final fact)

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

Definition 11 (Domain)

A subset $\Omega \subset \mathbb{C}$ is said to be a *domain* if it is open and path-connected.

More discussion - informal-tut.
We had one very nice result on the zeroes of a analytic functions.

Theorem 11 (Zeroes are isolated)

Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be a non-constant analytic function. Let $z_0 \in \Omega$ be such that $f(z_0) = 0$. Then, there exists $\delta > 0$ such that f has no other zero in $B_\delta(z_0)$.

The above is saying that around every zero of f, we can draw a (sufficiently small) circle such that f has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete.*

End of Lecture 4

Any questions?

Definition 12

Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Writing $f = u + \iota v$ as usual, we define

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + \iota \int_a^b v(t)\mathrm{d}t.$$

This is naturally what one would have wanted to define. Now, we define integration over a *contour*. (What is a contour?)

Definition 13

Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \to \Omega$ be a contour. We define

$$\int_\gamma f(z)\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

We have a useful inequality called the ML inequality.

Theorem 12 (ML Inequality)

Let γ be a contour of length L and f be a continuous function defined on the image of γ.
Suppose that

$$|f(\gamma(t))| \leq M, \qquad \text{for all } t \in [a, b].$$

Then, we have

$$\left| \int_\gamma f(z) \mathrm{d}z \right| \leq ML.$$

Theorem 13 (Primitives and integrals)

Suppose $f : \Omega \to \mathbb{C}$ has a *primitive* on Ω. That is, there exists a function $F : \Omega \to \mathbb{C}$ such that $F' = f$. (The complex derivative.) Then, we have

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

If γ is *closed*, that is, if $\gamma(b) = \gamma(a)$, then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Existence of a primitive is a strong condition, by the way. A holomorphic function need not have a primitive on all of Ω.

Now, we come to Cauchy's theorem.

Theorem 14 (Cauchy's Theorem)

Let γ be a simple, closed contour and let f be a holomorphic function defined on an open set Ω containing γ as well as its interior. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

If Ω is simply-connected, then the interior condition is automatically met. This gives us the next result.

Theorem 15 ("General" Cauchy Theorem)

Let Ω be a simply-connected domain. Let $\gamma : [a, b] \to \mathbb{C}$ be a simple, closed contour and $f : \Omega \to \mathbb{C}$ holomorphic. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

End of Lecture 5

Any questions?

Theorem 16 (Cauchy Integral Formula)

Let f be holomorphic everywhere on an open set Ω. Let γ be a simple closed curve in Ω, oriented positively. If z_0 is interior to γ and Ω contains the interior of γ, then

$$f(z_0) = \frac{1}{2\pi\iota}\int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z$$

We then saw a consequence of CIF which I state as a theorem below.

Theorem 17 (Holomorphic $\implies$ Analytic)

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$. Let $R > 0$ be the largest such that $B_R(z_0) \subset \Omega$. (The case $R = \infty$ is allowed. That just means $\Omega = \mathbb{C}$.)
Then, on the disc $B_R(z_0)$, we may write $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where each a_n is given by

$$a_n = \frac{1}{2\pi\iota} \int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w.$$

The above also gives us (what I call) the "generalised" Cauchy Integral Formula.

Theorem 18 ("Generalised" CIF)

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0),$$

where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

Remark 3

Note that, as usual, we require f to be holomorphic within the circle as well.

Theorem 19 (Cauchy's estimate)

Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

An easy application of this give us:

Theorem 20 (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

End of Lectures 6 and 7

Any questions?

We discuss logarithm a bit.

Definition 14 (Branch of the logarithm)

Let $\Omega \subset \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\exp(f(z)) = z, \qquad \text{for all } z \in \Omega.$$

Then, f is called a *branch of the logarithm.*

Theorem 21 (Uniqueness of branches)

Assume that $f, g : \Omega \to \mathbb{C}$ are two branches of the logarithm. Then, $f - g$ is a constant function. Moreover, this constant is an integer multiple of $2\pi\iota$.

The last theorem also assumed that Ω is a domain.

Logarithm

The previous theorem talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible.

Theorem 22 (Existence of a branch)

Let Ω be a simply-connected domain in $\mathbb{C}$. Assume that $1 \in \Omega$ and $0 \notin \Omega$.
There exists a unique function $F : \Omega \to \mathbb{C}$ such that

1. $F(1) = 0$,
2. $F'(z) = 1/z$,
3. $\exp(F(z)) = z$ for all $z \in \Omega$,
4. $F(r) = \log(r)$ for all $r \in \Omega \cap \mathbb{R}^+$.

The log in the last point is the usual log for real numbers as seen in 105. The above F is then denoted by log.

Definition 15 (Singularities)

Let $f : \Omega \to \mathbb{C}$ be a function. A point $z_0 \in \mathbb{C}$ is said to be a singularity of f if

1. $z_0 \notin \Omega$, i.e., f is not defined at z_0, or
2. $z_0 \in \Omega$ and f is not holomorphic at z_0.

Definition 16 (Isolated singularity)

A singularity $z_0 \in \mathbb{C}$ is said to be *isolated* if there exists *some* $\delta > 0$ such that f is holomorphic on $B_\delta(z_0) \setminus \{z_0\}$.

The above is saying that "f is holomorphic on some *punctured disc* around z_0."
Compare this "isolation" with what we saw earlier when we said that "zeroes are isolated."

Definition 17 (Non-isolated singularity)

A singularity which is not an isolated singularity is called a non-isolated singularity.

The floor is made of floor.

Note that if f has only finitely many singularities, then all the singularities are isolated.

We classify isolated singularities into three types:

1. Removable singularities,
2. Poles,
3. Essential singularities.

Remark 4

The above classification is only for isolated singularities.

Definition 18 (Removable singularity)

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

These are characterised by the following theorem.

Theorem 23 (Riemann's Removable Singularity Theorem)

z_0 is a removable singularity of f iff $\lim_{z \to z_0} f(z)$ exists.

In the above, we mean that it exists as a (finite) complex number.

$$f(z) = \frac{\sin z}{z}$$

defined on $\mathbb{C} \setminus \{0\}$ has 0 as a removable singularity.

Definition 19 (Pole)

An isolated singularity z_0 is said to be a pole if $|f(z)| \to \infty$ as $z \to z_0$.

Theorem 24

An isolated singularity z_0 is a pole of f iff $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$.

Theorem 25 (Order of a pole)

If z_0 is a pole of f, then there exists an integer $m > 0$ such that

$$f(z) = (z - z_0)^{-m} f_1(z)$$

on a punctured neighbourhood of z_0, for some function f_1 which is holomorphic on the complete neighbourhood. The smallest such integer m is called the *order* of the pole.
If the order is 1, then z_0 is said to be *simple* pole.

Definition 20 (Essential singularity)

An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

Theorem 26 (Casorati-Weierstrass Theorem)

If z_0 is an isolated singularity, then it is essential iff the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

End of Lecture 8

Any questions?

Theorem 27 (Modified CIF)

Suppose that z_0 is an isolated singularity of f. Consider an annulus of the form

$$A = \{z : r < |z - z_0| < R\},$$

where $0 < r < R \leq \infty$. Assume that f is holomorphic on this open annulus A. Then, CIF takes the form

$$f(z) = \frac{1}{2\pi\iota} \int_{|w-z_0|=R'} \frac{f(w)}{z-w} \mathrm{d}w - \frac{1}{2\pi\iota} \int_{|w-z_0|=r'} \frac{f(w)}{z-w} \mathrm{d}w,$$

where $r < r' < R' < R$.

Just like how the usual CIF gave us the power series, this CIF gives us the Laurent series.

Theorem 28 (Laurent Series)

With the same setup as earlier, for $z \in A$, we can write $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where each a_n is given, as before, by

$$a_n = \frac{1}{2\pi\iota} \int_{|z-w|=r_0} \frac{f(w)}{(z-w)^{n+1}} \mathrm{d}w,$$

where $r < r_0 < R$.

Note that the above is valid for $n < 0$ as well.

The most interesting coefficient is the -1^{st} one. When we integrate a Laurent series along a circle centered at z_0, only a_{-1} remains (with a factor of $2\pi\iota$). This is given by

$$a_{-1} = \frac{1}{2\pi\iota} \int_{|z-z_0|=r_0} f(w)\mathrm{d}w.$$

This is what is usually called the *residue* and written as

$$a_{-1} = \mathrm{Res}(f; z_0).$$

With residues, calculation of integrals becomes easier.

Theorem 29 (Cauchy's Residue Theorem)

Suppose f is given and has finitely many singularities $z_1, \ldots, z_n$ within a simple closed contour γ.Then, we have

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\iota \sum_{i=1}^{n} \mathrm{Res}(f; z_i).$$

Note that the above is implicitly implying that f is holomorphic at all other points within γ.

Definition 21 (Principal part of a Laurent Series)

Given a Laurent series as earlier, its principal part is

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

Theorem 30 (Isolated singularities and their principal parts)

The isolated singularity z_0 is

1. removable iff the principal part has no terms,
2. a pole iff the principal part has finitely many (and at least one) terms, and
3. essential iff the principal part has infinitely many terms.

In particular, the residue at a removable singularity is 0.

Now, we see how one can calculate residue at a pole.
By the previous theorem, we know that f can be written as

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots ,$$

for some integer $m > 0$.
Thus,

$$g(z) = (z-z_0)^m f(z)$$

is holomorphic at z_0 and

$$a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(z_0).$$

The End

Doubts?