Question 1

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1. Find Laurent expansions for the function $f(z) = \dfrac{2(z-1)}{z^2-2z+3}$ valid on the annuli

   (a) $0 \le |z| < 1$,

   (b) $1 < |z| < 3$,

   (c) $|z| > 3$.

idea

$$\frac{f(z)}{g(z)} = \frac{a_0 + a_1 x + \cdots}{b_n x^n + b_{n+1}x^{n+1} + \cdots} \quad (b_n \neq 0)$$

$$= \frac{(a_0 + a_1 x + \cdots)}{b_n x^n}\left(1 + \frac{b_{n+1} x}{b_n} + \cdots\right)^{-1}$$

$\downarrow$ geo.

$$f(z) = \frac{2(z-1)}{(z+1)(z-3)} = \frac{(z+1)+(z-3)}{(z+1)(z-3)}$$

$$= \frac{1}{z+1} + \frac{1}{z-3}$$

(a) $0 \le |z| < 1$.

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots, \quad \text{since } |z|<1$$

$$= \sum_{n=0}^{\infty} (-z)^n$$

$$\frac{1}{z-3} = -\frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n, \quad \text{since } \left|\frac{z}{3}\right| < \frac{1}{3} < 1$$

Thus, $$f(z) = \sum_{n=0}^{\infty}\left((-1)^n - \frac{1}{3^{n+1}}\right) z^n.$$

(b) $1 < |z| < 3$

$$\frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}, \quad \text{since } \left|\frac{1}{z}\right| < 1$$

$$\frac{1}{z-3} = -\frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n, \quad \text{since } \left|\frac{z}{3}\right| < \frac{3}{3} = 1$$

Thus,
$$f(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n} + \left(-\frac{1}{3}\right)\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$

$$= \sum_{n=-\infty}^{-1}(-1)^{-n-1}z^n - \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}.$$

(c) $3 < |z|$

$$\frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}, \quad \text{since } \left|\frac{1}{z}\right| < \frac{1}{3} < 1$$

$$\frac{1}{z-3} = \frac{1}{z}\cdot\frac{1}{\left(1-\frac{3}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n, \quad \text{since } \left|\frac{3}{z}\right| < \frac{3}{3} = 1$$

Thus,
$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n + 3^n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1} + 3^{n-1}}{z^n} = \sum_{n=-\infty}^{-1}\left[(-1)^{-n-1} + 3^{-n-1}\right]z^n.$$

Question 2

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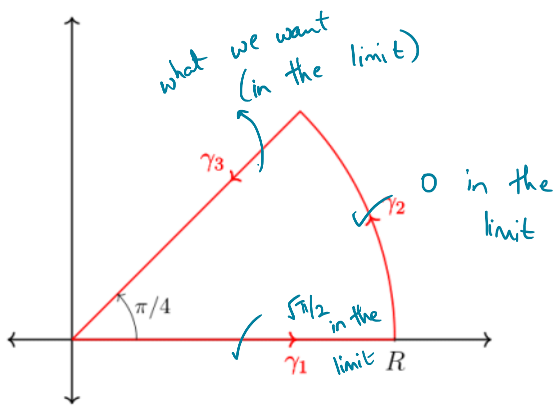
2. By integrating e^{-z^2} around a sector of radius R, one arm of which is along the real axis and the other making an angle $\pi/4$ with the real axis, show that:

$$\int_0^\infty \sin(x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos(x^2)\mathrm{d}x.$$

(Here, use the well-known integral $\left[\int_{-\infty}^{\infty} \exp(-x^2)\mathrm{d}x = \sqrt{\pi}\right]$) → Ex. Compute this using $\mathbb{C}$ Analysis



Let $f(z) = e^{-z^2}$ ← entire

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f = 0$$

(Cauchy's Theorem) ↓ f is entire

γ_1: $\displaystyle\int_0^R e^{-z^2}\,dz =: I_1(R)$

$$\lim_{R\to\infty} I_1(R) = \frac{\sqrt{\pi}}{2}.$$

γ_2: $\displaystyle\int_0^{\pi/4} f(Re^{i\theta})\,(Rie^{i\theta})\,d\theta = iR\int_0^{\pi/4} e^{-R^2e^{2i\theta}} \cdot e^{i\theta}\,d\theta =: I_2(R)$

$$|I_2(R)| \le R\int_0^{\pi/4} \left|e^{-R^2e^{2i\theta}}\right| d\theta$$

$|\exp(z)| = \exp(\operatorname{Re} z)$

$$= R\int_0^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta$$

$$= R\int^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta$$

$$= R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\, d\theta$$

$$= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\theta}\, d\theta$$

Theorem 3

If $x \in \left[0, \frac{\pi}{2}\right]$, then

$$\sin x \geq \frac{2}{\pi}x.$$

$$-\sin x \leq -\frac{2}{\pi}x$$

$$\leq \frac{R}{2}\int_0^{\pi/2} e^{-R^2\frac{2\theta}{\pi}}\, d\theta$$

$$= \frac{R}{2}\left(\frac{\pi}{2R^2}\right)\left(1 - e^{-R^2}\right)$$

$$= \frac{1}{R}\cdot\frac{\pi}{4}\cdot\left(1 - e^{-R^2}\right)$$

$$\therefore \lim_{R\to\infty} I_2(R) = 0.$$

γ_3 :

$$\int_R^0 f\left(te^{i\pi/4}\right)\left(e^{i\pi/4}\right) dt =: I_3(R)$$

$$I_3(R) = e^{i\pi/4}\int_R^0 e^{-\left(te^{i\pi/4}\right)^2} dt = e^{i\pi/4}\int_R^0 e^{-it^2} dt$$

$$= e^{i\pi/4}\int_R^0 \left[\cos(t^2) - i\sin(t^2)\right] dt$$

$$I_1(R) + I_2(R) + I_3(R) = 0 \qquad \forall R > 0$$

$\downarrow$ take limit

$$I_2(R) \to 0, \quad I_1(R) \to \frac{\sqrt{\pi}}{2}$$

$$\lim_{R\to\infty}\left(e^{i\pi/4}\int_0^R \cos t^2 - i\sin t^2\, dt\right) = \frac{\sqrt{\pi}}{2}$$

$$e^{i\pi/4}\left(C - iS\right) = \sqrt{\pi}/_2 \quad \longrightarrow \quad C - iS = \frac{\sqrt{\pi}}{2}e^{-i\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{2}}(1-i)$$

$$C = \int_0^\infty \cos(t^2)\,dt, \qquad S = \int_0^\infty \sin(t^2)\,dt$$

$\Downarrow$

$C = S = \frac{\sqrt{\pi}}{2\sqrt{2}}$

$$\frac{1}{\sqrt{2}}(1+i)(C - iS) = \frac{\sqrt{\pi}}{2}$$

$$\frac{1}{\sqrt{2}}\left((C+S) + i(C-S)\right) = \frac{\sqrt{\pi}}{2}$$

two lin. eq^{n}s

solve!

$$\boxed{C = S = \frac{\sqrt{\pi}}{2\sqrt{2}}}$$

"Fresnel integral"

Question 3

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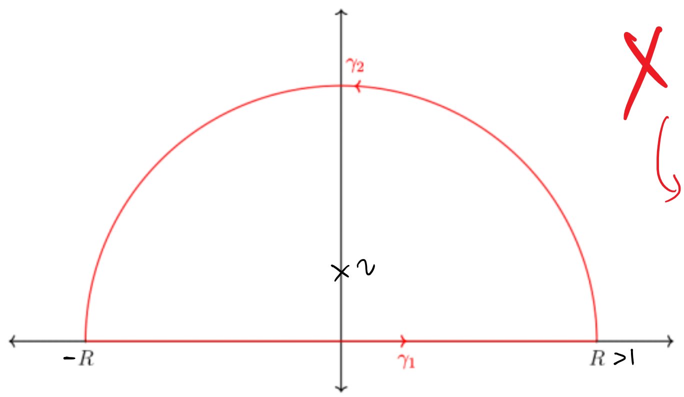
3. Compute using residue theory $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)^2}\mathrm{d}x.$



$$\bigg\{ f(z) = \frac{\cos(z)}{(1+z^2)^2}$$

poles $= \{\pm i\}$

trouble along γ_2.

$$f(z) = \frac{e^{iz}}{(1+z^2)^2}$$

poles $= \{\pm i\}$

$R>1$

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 2\pi i \operatorname{Res}(f;i)$$

$$I_2(R) := \int_{\gamma_2} f.$$

$$I_2(R) = \int_0^{\pi} f(Re^{i\theta})\,(Rie^{i\theta})\,d\theta$$

$$= Ri \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{(1+R^2e^{i\theta})^2} e^{i\theta}\,d\theta$$

$$|I_2(R)| \leq R \int_0^{\pi} \left| \frac{e^{iR(\cos\theta + i\sin\theta)}}{(1+R^2e^{i\theta})^2} \right| d\theta$$

$$\leq \frac{R}{R^2-1} \int_0^{\pi} e^{-R\sin\theta}\,d\theta$$

$\sin\theta \geq 0$

$$\leq \frac{R}{R^2-1}\left[\int_0^{\pi} 1\,d\theta\right] \xrightarrow[R\to\infty]{} 0$$

$$\lim_{R\to\infty}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz\right) = 2\pi i \operatorname{Res}(f;i)$$

$$\lim_{R\to\infty}\left(\int_{\gamma_1} f(z)\,dz + \underbrace{\int_{\gamma_2} f(z)\,dz}_{=0,\ \text{in the lim}}\right) = 2\pi i\,\mathrm{Res}(f;i)$$

$$\lim_{R\to\infty}\int_{-R}^{R}\left[\frac{\cos(x)}{(1+x^2)^2} + \underbrace{\frac{i\sin x}{(1+x^2)^2}}_{\text{odd } f^n.\ \therefore \int = 0}\right]dx = 2\pi i\,\mathrm{Res}(f;i)$$

$$\Rightarrow \lim_{R\to\infty}\int_{-R}^{R}\frac{\cos(x)}{(1+x^2)^2}\,dx = 2\pi i\,\mathrm{Res}(f;i) = 2\pi i\left(-\frac{i}{2}e^{-1}\right)$$

$$= \boxed{\frac{\pi}{e}}$$

Residue: Note that i is a pole of order 2.

$$f(z) = \frac{e^{iz}}{(z+i)^2(z-i)^2}$$

$\Rightarrow \lim_{z\to i}(z-i)^2 f(z)$ exists and is non-zero.

Thus, $\mathrm{Res}(f;i) = \dfrac{g^{(1)}(i)}{1!}$, where $g(z) = (z-i)^2 f(z)$

$$g(z) = \frac{e^{iz}}{(z+i)^2}\ ;\quad g'(z) = \frac{(z+i)^2(ie^{iz}) - e^{iz}(2)(z+i)}{(z+i)^4}$$

$$g'(i) = \frac{-4ie^{-1} - e^{-1}(4i)}{16} = \frac{-8ie^{-1}}{16} = -\frac{ie^{-1}}{2}$$

When you select the incorrect ~~branch~~ function to contour integrate over

Question 4

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4. Show by transforming into an integral over the unit circle, that

$$\int_0^{2\pi} \frac{1}{a^2 - 2a\cos\theta + 1}\, d\theta = \frac{2\pi}{a^2-1},$$

where $a > 1$. Also compute the value when $a < 1$.

$0 < a \neq 1$

$$(a-\cos\theta)^2 + (\sin\theta)^2 = |a - e^{i\theta}|^2$$

$$\int_0^{2\pi} \frac{1}{|a-e^{i\theta}|^2}\, d\theta = \int_0^{2\pi} \frac{1}{(a-e^{i\theta})(a-e^{-i\theta})}\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{2e^{i\theta}}{(a-e^{i\theta})(ae^{i\theta}-1)}\, d\theta$$

$$= \frac{1}{2} \int_{|z|=1} \frac{1}{(a-z)(az-1)}\, dz$$

$$= -\frac{1}{ai} \int_{|z|=1} \frac{1}{(z-a)(z-1/a)}\, dz$$

$a<1$ $a>1$ CIF

$$= \frac{i}{a} \int_{|z|=1} \frac{1/(z-a)}{\left(z-\frac{1}{a}\right)}\, dz$$

$$= \left(\frac{i}{a}\right)(2\pi i)\left(\frac{1}{\frac{1}{a}-a}\right)$$

similarly

$$\frac{2\pi}{1-a^2}$$

$$= \frac{-2\pi}{1-a^2} = \frac{2\pi}{a^2-1}$$ ✓

Question 5

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5. Show that if $a_1, \ldots, a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n, for each $1 \leq k \leq n$ we have the formula:

$$\prod_{j \neq k} (a_k - a_j) = P'(a_k).$$

$$P(z) = \underbrace{C}_{=1,\ \text{monic}} (z - a_1) \cdots (z - a_n) = (z - a_1) \cdots (z - a_n)$$

Fix $k \in \{1, \ldots, n\}$.

$$P(z) = (z - a_k) \underbrace{\prod_{j \neq k} (z - a_j)}_{=: P_k(z)}$$

$$P(z) = (z - a_k) P_k(z)$$

$$\Rightarrow \quad P'(z) = (z - a_k) P_k'(z) + P_k(z)$$

$$\big\downarrow z = a_k$$

$$P'(a_k) = P_k(a_k)$$

$$P'(a_k) = \prod_{j \neq k} (a_k - a_j)$$

Question 6

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6. ① Show that an entire function $f(z)$ has a pole at ∞ if and only if $|f(z)| \to \infty$ as $|z| \to \infty$. ② Also show that such entire functions are necessarily non-constant polynomials.

> **Definition 4: Limit is infinity**
>
> Let $a \in \mathbb{C}$ and f be a complex valued function defined on some deleted neighbourhood of a. We say
> $$\lim_{z\to a} f(z) = \infty$$
> if for every $M > 0$, there exists $\delta > 0$ such that
> $$0 < |z-a| < \delta \implies |f(z)| > M.$$

> **Definition 6: Limit at infinity is infinity**
>
> Let f be a complex valued function defined on some set of the form $\{z \in \mathbb{C} : |z| > R_0\}$ for some $R_0 > 0$. We say
> $$\lim_{z\to\infty} f(z) = \infty \quad \text{or} \quad \lim_{|z|\to\infty} f(z) = \infty$$
> if for every $M > 0$, there exists $R > R_0$ such that
> $$|z| > R \implies |f(z)| > M.$$

> **Theorem 7**
>
> Let f be a function defined on a neighbourhood of infinity, that is, on a set of the form $\{z \in \mathbb{C} : |z| > R_0\}$ for some $R_0 > 0$. Then,
> $$\lim_{|z|\to\infty} f(z) = \infty \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = \infty.$$

① Let $g(z) := f(\frac{1}{z})$ for $z \in \mathbb{C}^*$.

By def^n : $[\infty$ is pole of $f] \overset{\text{def}^n \text{ of pole at } \infty}{\iff} 0$ is a pole of g def^n of pole at (finite) complex no.

$$\iff \lim_{z\to 0} |g(z)| = \infty$$

$$\iff \left[\lim_{|z|\to\infty} |f(z)| = \infty\right] \quad \text{Thm. 7}$$

② f is entire, f has pole at ∞.

To show: f is a non-const poly.

Proof: Since f is entire, f has a pow. series rep. centered at 0 which is valid everywhere.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } z \in \mathbb{C}.$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } z \in \mathbb{C}.$$

Then, $$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad \text{for all } z \in \mathbb{C}^{\times}.$$

Since ∞ is a pole of f, 0 is a pole of g.

Thus, the above Laurent ser. exp. has only fin many non-zero terms

Thus, $\exists N \in \mathbb{N}$ s.t. $a_n = 0$ for all $n \geq N$.

Also, $\exists n \geq 1$ s.t. $a_n \neq 0$ since 0 was not a removable sing.

Thus, $$f(z) = \sum_{n=0}^{N} a_n z^n \longrightarrow \text{polynomial}$$

moreover, $a_n \neq 0$ for at least one $n \geq 1$ $\rightarrow$ non-const.

Pole $\Rightarrow$ fin. many neg. terms

Let $a \in \mathbb{C}$ be a pole of f.

Note that f can't be const on any deleted nbd. (otherwise RRST says it's rem.)

Since f is a holo on $B_\delta(a) \setminus \{a\}$, zeroes of f are isolated. Thus, $g = \frac{1}{f}$ makes sense on some (possibly) smaller del. nbd. $B_{\delta'}(a) \setminus \{a\}$.

Then, $\lim_{z \to a} g(z) = 0$. Thus, a is a remov. sing. of g. (RRST)

Thus, we can treat g to be holo. on $B_{\delta'}(a)$.

Here, we can write

$$g(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

$z \in B_{\delta'}(a)$

Note that g is not const.

Thus, choose the smallest $m \geq 0$ s.t. $a_m \neq 0$

Thus, $g(z) = (z-a)^m \left[\underset{\neq 0}{a_m} + a_{m+1}(z-a) + \cdots \right]$

$$\Rightarrow f(z) = \frac{1}{(z-a)^m} \frac{1}{(a_m + a_{m+1}(z-a) + \cdots)} = \frac{1}{(z-a)^m} \cdot \frac{1}{a_m} \cdot \left(1 + \underbrace{\frac{a_{m+1}}{a_m}(z-a) + \cdots}\right)^{-1}$$

$$= \sum_{n=-m}^{\infty} b_n (z-a)^n$$