

MA 205: Complex Analysis

Tutorial Solutions

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§0. Notations

1. Given $z \in \mathbb{C}$, $\Re z$ and $\Im z$ will denote the real and imaginary parts of z , respectively.
2. Given $z \in \mathbb{C}$, \bar{z} will denote the complex conjugate of z .
3. Given $z \in \mathbb{C}$, $|z|$ will denote the modulus of z , defined as $\sqrt{z\bar{z}}$ or $\sqrt{(\Re z)^2 + (\Im z)^2}$.

§1. Tutorial 1

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Notation: The set $\mathbb{C}[x]$ is the set of all polynomials (with indeterminate x) with complex coefficients. Similarly, $\mathbb{R}[x]$ is defined.

1. Show that complex polynomial of degree n has exactly n roots. (Assuming fundamental theorem of algebra.)

Remark (my own): The above is counting the roots *with* multiplicity. That is, if $f(z) = (z - \iota)^2(z - 2)$, then ι is counted twice and 2 once.

Solution. Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n . We prove this via induction on n .

$n = 1$. Then, $f(x) = a_0 + a_1x$ for some $a_0, a_1 \in \mathbb{C}$ and $a_1 \neq 0$.

Note that

$$\begin{aligned} f(x) &= 0 \\ \iff a_0 + a_1x &= 0 \\ \iff a_1x &= -a_0 \\ \iff x &= -\frac{a_0}{a_1}. \end{aligned}$$

Thus, $f(x)$ has exactly 1 root.

Let us assume that whenever $g(x) \in \mathbb{C}[x]$ is a polynomial of degree n , then $g(x)$ has exactly n roots. (Counted with multiplicity.)

Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n + 1$. By FTA, there exists a root $x_0 \in \mathbb{C}$. Thus, we can write

$$f(x) = (x - x_0)g(x)$$

for some polynomial $g(x) \in \mathbb{C}[x]$ of degree n . Moreover, note that

$$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$

By induction, the latter is possible for exactly n values of x . Thus, in total, $f(x)$ has $n + 1$ roots. (Both counts are with multiplicity.) \square

2. Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n, \quad a_i \in \mathbb{R}$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

Remark (my own): $a_n \neq 0$, of course.

Solution. Let $f(x) \in \mathbb{R}[x]$ with degree ≥ 3 as above.

If $f(x)$ has a real root, then we are done by factoring as in the earlier question.

Thus, let us assume that $f(x) = 0$ has no real solution.

We may view $f(x) \in \mathbb{C}[x]$. Now, using FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we must have $x_0 \notin \mathbb{R}$ or that $x_0 \neq \overline{x_0}$.

Claim. $f(\overline{x_0}) = 0$.

Proof. Note that

$$\begin{aligned}
 f(\overline{x_0}) &= a_0 + a_1 \overline{x_0} + \cdots + a_n (\overline{x_0})^n \\
 &= a_0 + a_1 \overline{x_0} + \cdots + a_n \overline{x_0^n} \\
 &= \overline{a_0} + \overline{a_1} \overline{x_0} + \cdots + \overline{a_n} \overline{x_0^n} \\
 &= \overline{f(x_0)} \\
 &= \overline{0} \\
 &= 0
 \end{aligned}
 \begin{array}{l}
 \left. \begin{array}{l} \cdot \cdot \cdot \overline{z^n} = \overline{z}^n \\ \cdot \cdot \cdot a_i \in \mathbb{R} \text{ and so, } a_i = \overline{a_i} \end{array} \right\} \\
 \left. \cdot \cdot \cdot \overline{z_1 z_2 + z_3} = \overline{z_1} \overline{z_2} + \overline{z_3} \right\}
 \end{array}$$

□

Define $g(x) = (x - x_0)(x - \overline{x_0})$. A priori, this is a polynomial in $\mathbb{C}[x]$. However, upon multiplication, we see that the polynomial is actually an element of $\mathbb{R}[x]$. Indeed, we have

$$(x - x_0)(x - \overline{x_0}) = (x^2 - (2\Re x_0)x + |x_0|^2) \in \mathbb{R}[x].$$

By our claim, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Since x_0 and $\overline{x_0}$ are distinct, the polynomials $x - x_0$ and $x - \overline{x_0}$ are “coprime” and thus, if they individually divide $f(x)$, then their product must too.)

Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. However, since $f(x)$ and $g(x)$ are both real polynomials, so is $h(x)$. (Why?)

Thus, we get that

$$f(x) = g(x)h(x)$$

for real polynomials $g(x)$ and $h(x)$. Moreover, note that $\deg g(x) = 2$ and $\deg h(x) = n - 2 \geq 1$. Thus, both are non-constant. □

3. Show that if U is a path connected open set in \mathbb{C} , so is U minus any finite set.

Solution. We will first prove the following claim:

Claim: Let $U \subset \mathbb{C}$ be open and $w \in U$. Then, $U \setminus \{w\}$ is open.

Proof. Let $z_0 \in U \setminus \{w\}$ be arbitrary. Since U was open, there exists $\delta_1 > 0$ such that

$$B_{\delta_1}(z_0) \subset U.$$

Since $z_0 \neq w$, we have that $\delta_2 := |z_0 - w| > 0$.

Choose $\delta := \min\{\delta_1, \delta_2\}$. Clearly, $\delta > 0$. Moreover, we have

$$w \notin B_{\delta_2}(z_0) \supset B_{\delta}(z_0)$$

and thus, $w \notin B_{\delta}(z_0)$. Also,

$$B_{\delta}(z_0) \subset B_{\delta_1}(z_0) \subset U.$$

Thus, we get that

$$B_{\delta}(z_0) \subset U \setminus \{w\},$$

proving that $U \setminus \{w\}$ is open. □

By the above proof, we see that removing one point from an open set keeps it open. Thus, if we show that removing one point from an open path-connected set leaves it path-connected, then we are done since we can induct to get any other **finite**¹ set.

Thus, we now show that if U is open and path-connected, so is $U \setminus \{w\}$. (Where $w \in U$ is any arbitrary element.)

Let $z_0, z_1 \in U \setminus \{w\}$. We wish to show that there is a path in $U \setminus \{w\}$ connecting z_0 to z_1 .

Since U was path-connected to begin with, there exists a path $\sigma : [0, 1] \rightarrow U$ such that

$$\sigma(0) = z_0, \quad \sigma(1) = z_1.$$

If $\sigma(x) \neq w$ for any $x \in [0, 1]$, then we are done since σ is a path in $U \setminus \{w\}$ as well.

Suppose that this is not the case.

Then, we choose a $\delta > 0$ such that the *closed* ball

$$B := \{z \in \mathbb{C} : |z - w| \leq \delta\}$$

has the following properties:

¹Finiteness is important. Induction cannot prove this result for a countably infinite set.

- (a) $z_0 \notin B$,
- (b) $z_1 \notin B$,
- (c) $B \subset U$.

(Why must such a δ exist? There exists a δ_1 for which we get the first two properties since z_0 and z_1 are distinct from w . For the last property, let δ_2 be any such that $B_{\delta_2}(w) \subset U$, which exists since U is open. Then, consider $\delta_2/2$. The *closed* ball of this radius must again be completely within U . Take the minimum of δ_1 and $\delta_2/2$.)

Note that

$$\sigma^{-1}(B) = \{x \in [0, 1] : \sigma(x) \in B\}$$

is nonempty since $w \in \sigma^{-1}(B)$. Moreover, it must be closed. (Why?) Since it is a subset of $[0, 1]$, it is clearly bounded. Define

$$s := \inf \sigma^{-1}(B), \quad t := \sup \sigma^{-1}(B).$$

Since the set is closed, both s and t are elements of $\sigma^{-1}(B)$. Note that $\sigma(0) \notin B$ and $\sigma(1) \notin B$ and thus,

$$0 < s < t < 1.$$

(Why is the inequality $s < t$ strict?)

Note that $\sigma(s)$ and $\sigma(t)$ must lie on the circumference of B . (Why?) (This also shows why $s < t$.)

Now consider the path $\sigma' : [0, 1] \rightarrow U$ defined as follows:

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in [0, s] \cup [t, 1] \\ \gamma(x) & \text{if } x \in [s, t], \end{cases}$$

where $\gamma : [s, t] \rightarrow B$ is the path which is the arc joining $\sigma(s)$ to $\sigma(t)$. (Note that $\sigma(s) = \sigma(t)$ is possible in which case, it's the constant path.)

Clearly, σ' avoids w and is continuous. (Why?)

Moreover, $\sigma'(0) = \sigma(0) = z_0$ and $\sigma'(1) = \sigma(1) = z_1$ and thus, σ' is a path from z_0 to z_1 in $U \setminus \{w\}$, showing that $U \setminus \{w\}$ is path-connected. \square

4. Check for real differentiability and holomorphicity:

- (a) $f(z) = c$,
- (b) $f(z) = z$,
- (c) $f(z) = z^n$, $n \in \mathbb{Z}$,

- (d) $f(z) = \Re z$,
- (e) $f(z) = |z|$,
- (f) $f(z) = |z|^2$,
- (g) $f(z) = \bar{z}$,
- (h) $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

Solution. Not going to do all.

- (a) Real differentiable and holomorphic, both.
- (b) Real differentiable and holomorphic, both.
- (c) Real differentiable and holomorphic, both. Let us see why.
As we know, holomorphicity implies real differentiability, so we only check that f is holomorphic on \mathbb{C} .
Let $z_0 \in \mathbb{C}$ be arbitrary. We show that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

This is clear because for $z_0 \neq z$, we have

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

The limit $z \rightarrow z_0$ of the RHS clearly exists.

- (d) Real differentiable but not holomorphic. Note that f can be written as

$$f(x + iy) = x + 0i.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$.

This is clearly real differentiable everywhere since all the partial derivatives exist everywhere and are continuous.

However, we show that f is not complex differentiable at any point. Thus, it is not holomorphic.

This is easy because one sees that $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$ and thus, the CR equations don't hold.

- (e) $|z|$ is real differentiable everywhere except 0 and complex differentiable nowhere. Breaking the function as earlier, we have

$$u(x, y) = \sqrt{x^2 + y^2}, \quad v(x, y) = 0.$$

On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous. At $(0, 0)$, u_x and u_y fail to exist.

This clearly shows that f is not complex differentiable at $0 \in \mathbb{C}$ since it is not even real differentiable there.

However, we see that $v_y = 0 = v_x$ everywhere else but at least one of u_x or u_y is nonzero on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and thus, the CR equations prevent f from being complex differentiable anywhere else.

- (f) Real differentiable everywhere.
Complex differentiable precisely at 0.
Holomorphic nowhere.

Same steps as above.

- (g) Real differentiable everywhere. Complex differentiable nowhere. Use CR equations again.
- (h) f is real differentiable precisely on $\mathbb{R}^2 \setminus \{(0, 0)\}$.
However, it is not complex differentiable anywhere.

Breaking as earlier, we get

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad v(x, y) = \frac{2xy}{x^2 + y^2},$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and

$$u(0, 0) = 0 = v(0, 0).$$

Note that u and v aren't even continuous at $(0, 0)$. Thus, neither is f . Hence, f is neither real nor complex differentiable at $(0, 0)$.

However, at all other points, all partial derivatives exist and are continuous. Thus, f is real differentiable at all those points. However, computing u_x, u_y, v_x, v_y explicitly shows that the CR equations are not satisfied anywhere. Thus, f is not complex differentiable anywhere. \square

5. Show that the CR equations take the form

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

in polar coordinates.

Solution. We shall follow the same idea as in the slides. We first write

$$f(r, \theta) = f(re^{i\theta}) = u(r, \theta) + i v(r, \theta).$$

Suppose that f is differentiable at $z_0 = r_0 e^{i\theta_0} \neq 0$. (Note that it wouldn't make sense to talk at 0 since there's a r^{-1} factor in the question anyway.)

Thus, we know that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We shall calculate it in two ways:

(a) Fix $\theta = \theta_0$ and let $r \rightarrow r_0$. Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} + i \frac{v(r, \theta_0) - v(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} \right\} \\ &= e^{-i\theta_0} \lim_{r \rightarrow r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + i \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0} \right\} \\ &= e^{-i\theta_0} (u_r(r_0, \theta_0) + i v_r(r_0, \theta_0)). \end{aligned} \quad (*)$$

(b) Fix $r = r_0$ and let $\theta \rightarrow \theta_0$. Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{\theta \rightarrow \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} \right\} \\ &= \frac{1}{r_0} \lim_{\theta \rightarrow \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \right\} \end{aligned} \quad (**)$$

We concentrate on the first term of the limit. Note that

$$\begin{aligned} &\lim_{\theta \rightarrow \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \\ &= \lim_{\theta \rightarrow \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}. \end{aligned}$$

In the product, the first term is clearly $u_\theta(r_0, \theta_0)$, after taking the limit. The second term can be calculated to be

$$\frac{1}{i e^{i\theta_0}}.$$

(How? Write $e^{i\theta}$ in terms of \cos and \sin and differentiate those and put it back.)

Of course, a similar argument goes through for the v term as well.

Thus, we get that $(**)$ transforms to

$$f'(z_0) = \frac{e^{-i\theta_0}}{r_0} (iu_\theta(r_0, \theta_0) + v_\theta(r_0, \theta_0)).$$

Equating the above with $(*)$, cancelling $e^{-i\theta_0}$, and comparing the real and imaginary parts, we get

$$u_r(r_0, \theta_0) = \frac{1}{r_0}v_\theta(r_0, \theta_0), \quad v_r(r_0, \theta_0) = -\frac{1}{r_0}u_\theta(r_0, \theta_0),$$

as desired. □