Tutorial 1

07 August 2021 14:20

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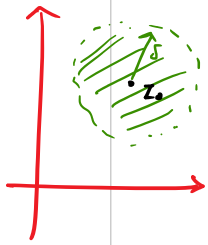
Lecture 1

Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ-neighbourhood of z_0, denoted by $B_\delta(z_0)$ is the set

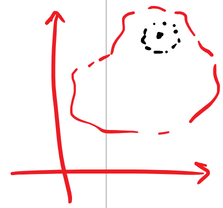
$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$



Definition 2 (Open sets)

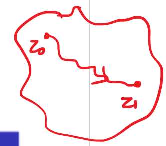
A set $U \subset \mathbb{C}$ is said to be open if:
for *every* $z_0 \in \mathbb{C}$, there exists *some* $\delta > 0$ such that

$$B_\delta(z_0) \subset U.$$



Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P. (A continuous function from $[0, 1]$ to P.)



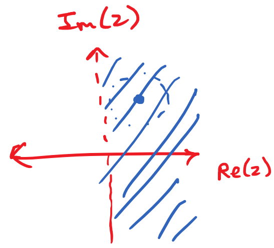
Examples.

① $B_\delta(z_0)$ are open for any $z_0 \in \mathbb{C}$ and $\delta > 0$.

② $\mathbb{C}$ is open. $\emptyset$ is open.

③ Strict right half plane $\mathbb{H}$ is open

$\mathbb{H} := \{z \in \mathbb{C} : Re(z) > 0\}$.



④ $\{z \in \mathbb{C} : Re(z) \geq 0\}$ is NOT open.

Lecture 1

Definition 4 (Differentiable)

Let $\Omega \subset \mathbb{C}$ be open. Let

$$f : \Omega \to \mathbb{C}$$

be a function. Let $z_0 \in \Omega$. f is said to be *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is denoted by $f'(z_0)$.

$f : (a, b) \to \mathbb{R}$

$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

$\Omega = \mathbb{C}$, $\quad z, z^2, \quad z^n, \ldots$

exp, sin, cos, ... ?

Non-diff: $|z|$, $\bar{z}$, ...

Lecture 1

Definition 5 (Holomorphic)

1. A function f is said to be holomorphic on an open set Ω if it is differentiable at every $z_0 \in \Omega$.
2. A function f is said to be holomorphic at z_0 if it is holomorphic on some neighbourhood of z_0.

Remark 1

A function may be differentiable at z_0 but not holomorphic at z_0. For example, $f(z) = |z|^2$ is differentiable only at 0. Thus, it is differentiable at 0 but holomorphic nowhere.

For sets, however, there is no difference.

Points: Holo. $\Rightarrow$ Diff but $\not\Leftarrow$

Notation

From this point on, Ω be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number z as $z = \underline{x} + \iota \underline{y}$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = \underline{u(z)} + \iota \underline{v(z)}$.

Lecture 2: CR Equations

$$\begin{bmatrix} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 \\ x+\iota y & \longleftrightarrow & (x,y) \end{bmatrix}$$

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose f as

$$f(z) = \underline{u(z)} + \iota v\underline{(z)},$$

where $u, v : \underline{\Omega} \to \mathbb{R}$ are real valued functions.

$\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$

The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x,y) = u(x+\iota y)$ and similarly for v. Now, if we know that f is holomorphic, then we have the following result.

$$\left[u, v : \Omega^{\subset \mathbb{R}^2} \longrightarrow \mathbb{R} \right]$$ MA 109, 111

u_x, u_y, v_x, v_y make sense.

Lecture 2: CR Equations

Cauchy – Riemann

Theorem 1 (CR equations)

Let $f : \Omega \to \mathbb{C}$ be differentiable at a point $z_0 \in \Omega$. Let $z_0 = x_0 + \iota y_0$.
Then, we have

$$① \quad u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, we have

$$② \quad f'(z_0) = u_x(x_0, y_0) + \iota v_x(x_0, y_0).$$

$f' = u_x + i v_x$

Existence of u_x, u_y, v_x, v_y is part of the theorem.

Note the subscript is x for both in the above.
Also note that all the equalities are only at the point z_0. In particular, we are only assuming differentiability at z_0.

Test $f(z) = z$
$= x + iy$
to see what the signs should be.

Lecture 2: CR Equations

Converse? What is the converse? Is it true?

$\boxed{\text{Converse}}$

NO!

CR equations hold at the point (x_0, y_0)

$\Downarrow$?

f is differentiable at z_0?

No. The converse is not true.

An example for you to check is

$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that u and v satisfy the CR equations at $(0, 0)$ but f is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

Lecture 2: CR Equations

We recall MA ~~105~~ 109 + 111 now.

Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \underbrace{\Omega}_{\subseteq \mathbb{R}^2} \to \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if there exits a 2×2 real matrix A such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(x_0 + h, y_0 + k) - f(x_0, y_0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

The matrix A was called the *total derivative of f at* (x_0, y_0).

Theorem 2

If f is (complex) differentiable at a point $z_0 = x_0 + \iota y_0$, then f is real differentiable at (x_0, y_0).

Once again, this is only talking about differentiability at a point.
The converse is again not true.
Take the example $f(z) = \bar{z}$. Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

$u(x,y) = x$ $\quad u_x \equiv 1$

$v(x,y) = -y$ $\quad v_y \equiv -1$

Lecture 2: CR Equations

CD $\Rightarrow$ CR + RD

CR $\not\Rightarrow$ CD, RD $\not\Rightarrow$ CD

Theorem 3

Let $f : \Omega \to \mathbb{C}$ be a function and let $z_0 = x_0 + \iota y_0 \in \Omega$. If the CR equations hold at the point (x_0, y_0) and if f is real differentiable at the point (x_0, y_0), then f is complex differentiable at the point z_0.

(CR + RD) $\Rightarrow$ CD

Recall from MA 109, 111:

If $f : \Omega \to \mathbb{R}^2$ is a function s.t. $f = (u, v)$

u_x, u_y, v_x, v_y are continuous on Ω,

then f is real diff. on Ω.

Note: if $f : \Omega \to \mathbb{R}^2$, then f_x, f_y, etc. are meaningless.

Definition 7 (Harmonic functions)

Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function. u is said to be *harmonic* if $u_{xx} + u_{yy} = 0$.

Proposition 1

The real and imaginary parts of a holomorphic function are harmonic.

$u_x = v_y$ $\quad$ $u_y = -v_x$

$u_{xx} = v_{yx}$ $\quad$ $u_{yy} = -v_{xy}$

but $v_{xy} = v_{yx}$ by assumption of u, v being C^2

Suppose u and v are harmonic on Ω. v is said to be a harmonic conjugate of u if $f = u + \iota v$ is holomorphic on Ω.

If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.

Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

Harmonic Conjugate need not exist.

Example. Consider $\Omega = \mathbb{R}^2 - \{(0,0)\}$ and $u : \Omega \to \mathbb{R}$ defined as

$$u(x, y) = \frac{1}{2}\log(x^2 + y^2).$$

If u had a harmonic conjugate v, then

$$v_y(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v_x(x, y) = -\frac{y}{x^2 + y^2}.$$

But $\nexists\, v : \Omega \to \mathbb{R}$ s.t.

$$\nabla v = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

(Proof?)

Claim 1. Arbitrary union of open sets is open.

Proof. Let $\{U_i : i \in I\}$ be a collection of open sets.

Define $U := \bigcup_{i \in I} U_i$

$= \{x : x \in U_i \text{ for some } i \in I\}$.

TS: U is open.

Proof. Let $x \in U$ be arbitrary.

Then, $\exists i_0 \in U$ s.t. $x \in U_{i_0}$.

Since U_{i_0} is open, $\exists \delta > 0$ s.t.

$$B_\delta(x) \subseteq U_{i_0}.$$

But $U_{i_0} \subseteq U$. Thus, $B_\delta(x) \subseteq U$.

Thus, U is open. ∎

Claim 2. Finite intersection of open sets is open.

Proof. It suffices to prove that intersection of two open sets is open.

$A_1, \ldots, A_n \to$ open

$\Downarrow$

$(A_1 \cap A_2), A_3, \ldots, A_n \to$ open

$\Downarrow$

$A_1 \cap A_2 \cap A_3, \ldots, A_n \to$ open

$\Downarrow$

$\vdots$

$\Downarrow$

$A_1 \cap \ldots \cap A_n \to$ open

Let U_1 and U_2 be open and $x \in U_1 \cap U_2$.

Let U_1 and U_2 be open and $x \in U_1 \cap U_2$.

$\because U_1 \,\&\, U_2$ are open

$\exists \delta_1 > 0$ s.t. $B_{\delta_1}(x) \subseteq U_1$ and

$\exists \delta_2 > 0$ s.t. $B_{\delta_2}(x) \subseteq U_2$.

Pick $\delta := \min(\delta_1, \delta_2) > 0$.

Then, $B_\delta(x) \subseteq B_{\delta_1}(x) \subseteq U_1$ and

$B_\delta(x) \subseteq B_{\delta_2}(x) \subseteq U_2$.

$\therefore B_\delta(x) \subseteq (U_1 \cap U_2)$. ∎

"Dual" statements for closed sets.

$U_1, U_2 \to$ open You can say: $U_1 \cup U_2$ and $U_1 \cap U_2$ are open

$U_1, U_2, \ldots, U_n \to$ open $\Rightarrow$ $U_1 \cup \cdots \cup U_n$ & $U_1 \cap \cdots \cap U_n$ are open.

$U_1, U_2, U_3, \ldots \to$ open $\Rightarrow$ $\bigcup_{i=1}^{\infty} U_i$ is open but $\bigcap_{i=1}^{\infty} U_i$ may not be.

$$\mathbb{C} - \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{C} - U_i)$$

closed $\Leftrightarrow$ complement is open.

$\boxed{U_i := B_{1/i}(0).}$

$\bigcap_{i \in \mathbb{N}} U_i = \{0\}$ ← not open.

Tutorial 2

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Lecture 3: Power Series

Definition 8 (Convergence of series)

A series of the form

$$\rightarrow \sum_{n=0}^{\infty} a_n$$

$(a_n)_{n\geq 0} \rightarrow$ sequence in $\mathbb{C}$

of complex numbers is said to converge if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges (to a finite complex number).

The sequence of partial sums is just the following sequence:

$$\underline{a_0}, \underline{a_0 + a_1}, \underline{a_0 + a_1 + a_2}, \ldots.$$

"Divergent" is simply used to mean "not convergent."
Check that $\sum(-1)^n$ and $\sum n$ both diverge.

$a_n = (-1)^n$ for $n \geq 0$

$1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$

partial sums $\rightarrow 1, 0, 1, 0, 1, 0, 1, 0 \ldots$

$\hookrightarrow$ Diverges

Similarly

$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ diverges

$1 + 2 + 3 + 4 + \cdots$ diverges

Lecture 3: Power Series

Definition 9 (limsup)

Given a sequence (x_n) of real numbers, we may define a new sequence (y_n) as

$$y_n = \sup\{x_m : m \geq n\}.$$

The limit of this sequence always exists and we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

$x_0, x_1, x_2, x_3, x_4, \ldots$

sup $= y_4$

$\hookrightarrow$ sup $= y_3$

$\hookrightarrow$ sup $= y_2$

$\hookrightarrow$ sup $= y_1$

Remark 2

Each y_n might be ∞. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.

Each y_n might be ∞. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.
If $\lim_{n\to\infty} x_n$ itself exists, then it equals the $\limsup$ as well.

I know: $\lim_{n\to\infty} n^{1/n} = 1.$

Thus, $\limsup_{n\to\infty} n^{1/n} = 1.$

$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

$\Downarrow$

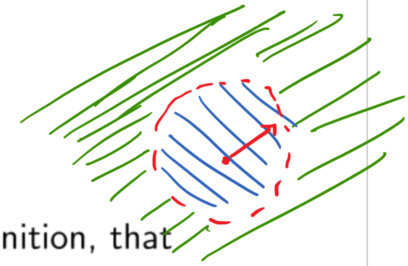
$\limsup_{n\to\infty} \frac{\sin(n)}{n} = 0$

lim sup of a real sequence always exists!

Lecture 3: Power Series

We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n (z - z_0)^n} \qquad (*)$$



where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

What is the radius of convergence, though? (The definition, that is.)

Theorem 4 (Radius of convergence)

Given any power series as $(*)$, there exists $R \in [0, \infty]$ such that
1. $(*)$ converges for any z with $|z - z_0| < R$, and → absolutely
2. $(*)$ diverges for any z with $|z - z_0| > R$.

This R is called the radius of convergence.

→ on the circle, behaviour is weird.
↳ may converge for some
↳ may diverge for others

Note the brackets.

$\sum_{n=1}^{\infty} \frac{z^n}{n} \rightsquigarrow$ RoC $= 1$

at -1 : converges

at 1 : diverges

$\sum z^n \rightsquigarrow$ RoC $= 1$

↳ diverges for all z with $|z| = 1$

Lecture 3: Power Series

We would now like to be able to calculate the radius of convergence.

Theorem 5 (Root test)

Let $(*)$ be as earlier. Define

$$\boxed{\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.}$$

ALWAYS WORKS.

Then, $\boxed{R = \alpha^{-1}}$ is the radius of convergence.

This test *always works.* We had no assumptions of any kind on $(*)$.
Note that $^{-1}$.
If $\alpha = 0$, then $R = \infty$ and vice-versa.

$\sum_{n=1}^{\infty} \frac{z^n}{n}$. $\alpha = \limsup \left(\frac{1}{n}\right)^{1/n} \overset{?}{=}$

Note: limit rules of $+, \cdot, (\,)^{-1}$ do not apply to limsup.

We know $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \frac{1}{1} = 1.$ $\therefore \limsup \left(\frac{1}{n}\right)^{1/n} = 1$

$\limsup (a_n + b_n) \le \limsup (a_n) + \limsup (b_n)$

Lecture 3: Power Series

We have another test. This is simpler (to calculate) but mightn't always work.

Theorem 6 (Ratio test)

Let $(*)$ be as earlier.
Assume that the limit

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$\rightarrow$ can apply to $\sum \frac{z^n}{n}$ or $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

exists. (Possibly as ∞.)
Then, R is the radius of convergence.

$\dfrac{\left(\frac{1}{n!}\right)}{\frac{1}{(n+1)!}} = n+1 \longrightarrow \infty.$

Note that here we assume that the limit does exist. This may not always be true.
Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Take $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Then, $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$.

$\hookrightarrow$ R.O.C = 1

Converges at 1

does not converge at 1.

Lecture 3: Power Series

Differentiability of power series is what one should expect.

Theorem 7 (Differentiability)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the open disc of radius R, let $f(z)$ denote this sum. Then, on this disc, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Boundary: NO COMMENTS.

Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

$\limsup |a_n|^{1/n} = \limsup n^{1/n} |a_n|^{1/n}$

Digression. Let $(x_n)_{n\geq 0}$ be a real sequence.

Consider

$E := \{$ limits of all possible convergent subsequences $\} \subseteq \mathbb{R} \cup \{\pm\infty\}$.

Then, $\limsup_{n\to\infty} x_n = \sup E$.

Lecture 4: Exponential function

I shall just recall the facts from the lecture.

Definition 10 (Exponential function)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

by ratio test

converges on all of $\mathbb{C}$. This sum is denoted by $\exp(z)$.

Theorem 8 (Facts)

1. $\exp'(z) = \exp(z)$,
2. $\exp'(bz) = b\exp(bz)$, for $b \in \mathbb{C}$,
3. $\exp(z) \cdot \exp(-z) = 1$ for all $z \in \mathbb{C}$,
4. $\exp(z)$ is always nonzero.

Lecture 4: Exponential function

Now, we some "converse" facts.

Theorem 9 (Characterisations)

1. If $f'(z) = bf(z)$, then $f(z) = a\exp(bz)$ for some $a, b \in \mathbb{C}$,
2. If $f' = f$ and $f(0) = 1$, then $f(z) = \exp(z)$.

Theorem 10 (Final fact)

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

$\exp : \underset{+}{\mathbb{C}} \longrightarrow \underset{\cdot}{\mathbb{C}^\times}$ is a group homomorphism.

$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

Lecture 4: Exponential function

Definition 11 (Domain)

A subset $\Omega \subset \mathbb{C}$ is said to be a *domain* if it is open and path-connected.

We had one very nice result on the zeroes of a analytic functions.

Theorem 11 (Zeroes are isolated)

Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be a non-constant analytic function. Let $z_0 \in \Omega$ be such that $f(z_0) = 0$. Then, there exists $\delta > 0$ such that f has no other zero in $B_\delta(z_0)$.

The above is saying that around every zero of f, we can draw a (sufficiently small) circle such that f has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete*.

Logarithm

We discuss logarithm a bit.

Definition 14 (Branch of the logarithm)

Let $\Omega \subset \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\exp(f(z)) = z, \qquad \text{for all } z \in \Omega.$$

Then, f is called a *branch of the logarithm.*

Theorem 21 (Uniqueness of branches)

Assume that $f, g : \Omega \to \mathbb{C}$ are two branches of the logarithm. Then, $f - g$ is a constant function. Moreover, this constant is an integer multiple of $2\pi\iota$.

The last theorem also assumed that Ω is a domain.

Branch of log may not exit on a given domain.

no branch on $\mathbb{C}$. Also, there is no branch on $\mathbb{C}^\times$.

Tutorial 3

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Lecture 5: Integration

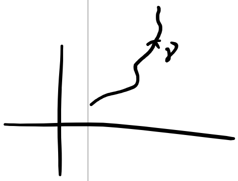
Definition 12

Let $f : \underbrace{[a, b]}_{} \overset{\in \mathbb{R}}{\to} \mathbb{C}$ be a piecewise continuous function. Writing $f = u + \iota v$ as usual, we define

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + \iota \int_a^b v(t)\mathrm{d}t.$$

This is naturally what one would have wanted to define. Now, we define integration over a *contour*. (What is a contour?)

↳ continuous and piece wise diff.



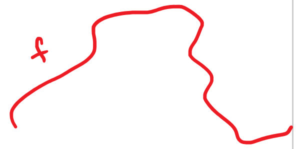
Definition 13

Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \to \Omega$ be a contour. We define

$$\int_\gamma f(z)\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

Aryaman Maithani Complex Analysis TSC

Lecture 5: Integration



We have a useful inequality called the ML inequality.

Theorem 12 (ML Inequality) length

Let γ be a contour of length L and f be a continuous function defined on the image of γ.
Suppose that

$$\underline{|f(\gamma(t))| \leq M}, \qquad \text{for all } t \in [a, b].$$

Then, we have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq ML.$$

"Real analogue": If $f : [a,b] \to \mathbb{R}$ is bounded by M, then $\int_a^b f \leq M(b-a)$.

Lecture 5: Integration

"FTC"

Theorem 13 (Primitives and integrals)

Suppose $f : \Omega \to \mathbb{C}$ has a *primitive* on Ω. That is, there exists a function $F : \Omega \to \mathbb{C}$ such that $F' = f$. (The complex derivative.) hol.
Then, we have

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

If γ is *closed*, that is, if $\gamma(b) = \gamma(a)$, then

$\gamma(a) = \gamma(b)$

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Primitive on Ω $\Downarrow$ Integral over closed curves in Ω is 0.

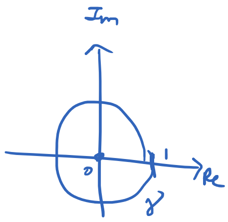
Note: Ω need not contain interior of γ. Thus, we can conclude $\int_\gamma \frac{1}{z^2}dz = 0$ over the unit circle.

Existence of a primitive is a strong condition, by the way. A holomorphic function need not have a primitive on all of Ω.

Consider $\Omega = \mathbb{C} - \{0\}$.

Let $f : \Omega \to \mathbb{C}$ be $f(z) := \frac{1}{z}$.

Then, f has no primitive on Ω.



Proof. Let $\gamma(t) := e^{2\pi i t}$, $t \in [0, 1]$.

Then, $\int_\gamma f = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 (e^{-2\pi i t})(2\pi i)(e^{2\pi i t})dt$

$= (2\pi i) \int_0^1 dt = 2\pi i \neq 0$

Lecture 5: Integration

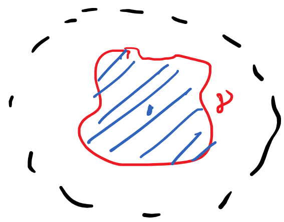
Now, we come to Cauchy's theorem.

Theorem 14 (Cauchy's Theorem)

Let γ be a simple, closed contour and let f be a holomorphic function defined on an open set Ω containing γ as well as its interior. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

If Ω is simply-connected, then the interior condition is automatically met. This gives us the next result.



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Lecture 5: Integration

Theorem 15 ("General" Cauchy Theorem)

Let Ω be a simply-connected domain. Let $\gamma : [a, b] \to \mathbb{C}$ be a simple, closed contour and $f : \Omega \to \mathbb{C}$ holomorphic. Then,

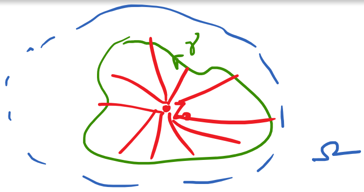
$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Lecture 6: CIF and Consequences

Theorem 16 (Cauchy Integral Formula)

Let f be holomorphic everywhere on an open set Ω. Let γ be a simple closed curve in Ω, oriented positively. If z_0 is interior to γ and Ω contains the interior of γ, then

$$f(z_0) = \frac{1}{2\pi\iota}\int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z.$$



"mean value"

Lecture 6: CIF and Consequences

We then saw a consequence of CIF which I state as a theorem below.

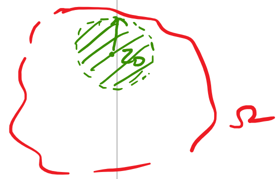
Theorem 17 (Holomorphic $\implies$ Analytic)

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$. Let $R > 0$ be the largest such that $B_R(z_0) \subset \Omega$. (The case $R = \infty$ is allowed. That just means $\Omega = \mathbb{C}$.) Then, on the disc $B_R(z_0)$, we may write $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

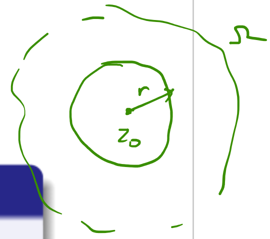
where each a_n is given by

$$a_n = \frac{1}{2\pi\iota}\int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}}\mathrm{d}w.$$



Lecture 6: CIF and Consequences

The above also gives us (what I call) the "generalised" Cauchy Integral Formula.



Theorem 18 ("Generalised" CIF)

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w = \frac{2\pi\iota}{n!}f^{(n)}(z_0),$$

where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

Remark 3

Note that, as usual, we require f to be holomorphic within the circle as well.

Just keep the "Generalised CIF" in mind, in case all these various theorems are too confusing! It will let you derive everything else quite simply!

In fact, the simplest thing is Cauchy's residue theorem which is the best generalisation of all these results, which we'll see later in the course and everything else becomes a very direct corollary of it.

Lecture 7: CIF and Consequences

Theorem 19 (Cauchy's estimate)

Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

An easy application of this give us:

Theorem 20 (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

Logarithm

from last tutorial

The previous theorem talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible. Short : Simply connected domain without 0 has a log.

Theorem 22 (Existence of a branch)

Let Ω be a simply-connected domain in $\mathbb{C}$. Assume that $1 \in \Omega$ and $0 \notin \Omega$.
There exists a unique function $F : \Omega \to \mathbb{C}$ such that

1. $F(1) = 0$,
2. $F'(z) = 1/z$,
3. $\exp(F(z)) = z$ for all $z \in \Omega$, F is a branch of log
4. $F(r) = \log(r)$ for all $r \in \Omega \cap \mathbb{R}^+$.

The log in the last point is the usual log for real numbers as seen in ~~105~~ 109/111. The above F is then denoted by log.

3. $F(r) = \log(r)$ for all $r \in \Omega \cap \mathbb{R}^+$.

The log in the last point is the usual log for real numbers as seen in ~~105~~. The above F is then denoted by log.

109/111

Special example: $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$= \mathbb{C} \setminus (-\infty, 0]$