MA 205: Complex Analysis
Tutorial Solutions

Aryaman Maithani
https://aryamanmaithani.github.io/tuts/ma-205

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§0. Notations

1. Given $z \in \mathbb{C}$, $\Re z$ and $\Im z$ will denote the real and imaginary parts of z, respectively.
2. Given $z \in \mathbb{C}$, $\bar{z}$ will denote the complex conjugate of z.
3. Given $z \in \mathbb{C}$, $|z|$ will denote the modulus of z, defined as $\sqrt{z\bar{z}}$ or $\sqrt{(\Re z)^2 + (\Im z)^2}$.

§1. Tutorial 1

25th August, 2020

Notation: The set $\mathbb{C}[x]$ is the set of all polynomials (with indeterminate x) with complex coefficients. Similarly, $\mathbb{R}[x]$ is defined.

1. Show that complex polynomial of degree n has exactly n roots. (Assuming fundamental theorem of algebra.)
   Remark (my own): The above is counting the roots *with* multiplicity. That is, if $f(z) = (z - \iota)^2(z - 2)$, then ι is counted twice and 2 once.

   Solution. Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n. We prove this via induction on n.
   $n = 1$. Then, $f(x) = a_0 + a_1x$ for some $a_0, a_1 \in \mathbb{C}$ and $a_1 \neq 0$.
   Note that

   $$\begin{aligned} & f(x) = 0 \\ \iff & a_0 + a_1x = 0 \\ \iff & a_1x = -a_0 \\ \iff & x = -\frac{a_0}{a_1}. \end{aligned}$$

   Thus, $f(x)$ has exactly 1 root.

   Let us assume that whenever $g(x) \in \mathbb{C}[x]$ is a polynomial of degree n, then $g(x)$ has exactly n roots. (Counted with multiplicity.)
   Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n + 1$. By FTA, there exists a root $x_0 \in \mathbb{C}$. Thus, we can write

   $$f(x) = (x - x_0)g(x)$$

   for some polynomial $g(x) \in \mathbb{C}[x]$ of degree n. Moreover, note that

   $$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$

   By induction, the latter is possible for exactly n values of x. Thus, in total, $f(x)$ has $n + 1$ roots. (Both counts are with multiplicity.) □

2. Show that a real polynomial that is irreducible has degree at most two. i.e., if

   $$f(x) = a_0 + a_1x + \cdots + a_nx^n, \quad a_i \in \mathbb{R}$$

   then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.
   Remark (my own): $a_n \neq 0$, of course.

Solution. Let $f(x) \in \mathbb{R}[x]$ with degree ≥ 3 as above.
If $f(x)$ has a real root, then we are done by factoring as in the earlier question.

Thus, let us assume that $f(x) = 0$ has no real solution.
We may view $f(x) \in \mathbb{C}[x]$. Now, using FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we must have $x_0 \notin \mathbb{R}$ or that $x_0 \neq \overline{x_0}$.

> **Claim.** $f(\overline{x_0}) = 0$.
>
> *Proof.* Note that
>
> $$\begin{aligned} f(\overline{x_0}) &= a_0 + a_1\overline{x_0} + \cdots + a_n(\overline{x_0})^n \\ &= a_0 + a_1\overline{x_0} + \cdots + a_n\overline{x_0^n} \\ &= \overline{a_0} + \overline{a_1}\,\overline{x_0} + \cdots + \overline{a_n}\overline{x_0^n} \\ &= \overline{f(x_0)} \\ &= \bar{0} \\ &= 0 \end{aligned}$$
>
> $\because \overline{z^n} = \bar{z}^n$
>
> $\because a_i \in \mathbb{R}$ *and so,* $a_i = \overline{a_i}$
>
> $\overline{z_1 z_2 + z_3} = \overline{z_1}\,\overline{z_2} + \overline{z_3}$
>
> $\square$

Define $g(x) = (x - x_0)(x - \overline{x_0})$. A priori, this is a polynomial in $\mathbb{C}[x]$. However, upon multiplication, we see that the polynomial is actually an element of $\mathbb{R}[x]$. Indeed, we have

$$(x - x_0)(x - \overline{x_0}) = (x^2 - (2\Re x_0)x + |x_0|^2) \in \mathbb{R}[x].$$

By our claim, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Since x_0 and $\overline{x_0}$ are distinct, the polynomials $x - x_0$ and $x - \overline{x_0}$ are "coprime" and thus, if they individually divide $f(x)$, then their product must too.)
Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. However, since $f(x)$ and $g(x)$ are both real polynomials, so is $h(x)$. (Why?)
Thus, we get that

$$f(x) = g(x)h(x)$$

for real polynomials $g(x)$ and $h(x)$. Moreover, note that $\deg g(x) = 2$ and $\deg h(x) = n - 2 \geq 1$. Thus, both are non-constant. $\square$

3. Show that if U is a path connected open set in $\mathbb{C}$, so is U minus any finite set.

Solution. We will first prove the following claim:

> **Claim:** Let $U \subset \mathbb{C}$ be open and $w \in U$. Then, $U \setminus \{w\}$ is open.
>
> *Proof.* Let $z_0 \in U \setminus \{w\}$ be arbitrary. Since U was open, there exists $\delta_1 > 0$ such that
> $$B_{\delta_1}(z_0) \subset U.$$
> Since $z_0 \neq w$, we have that $\delta_2 := |z_0 - w| > 0$.
> Choose $\delta := \min\{\delta_1, \delta_2\}$. Clearly, $\delta > 0$. Moreover, we have
> $$w \notin B_{\delta_2}(z_0) \supset B_\delta(z_0)$$
> and thus, $w \notin B_\delta(z_0)$. Also,
> $$B_\delta(z_0) \subset B_{\delta_1}(z_0) \subset U.$$
> Thus, we get that
> $$B_\delta(z_0) \subset U \setminus \{w\},$$
> proving that $U \setminus \{w\}$ is open. □

By the above proof, we see that removing one point from an open set keeps it open. Thus, if we show that removing one point from an open path-connected set leaves it path-connected, then we are done since we can induct to get any other **finite**[1] set.

Thus, we now show that if U is open and path-connected, so is $U \setminus \{w\}$. (Where $w \in U$ is any arbitrary element.)

Let $z_0, z_1 \in U \setminus \{w\}$. We wish to show that there is a path in $U \setminus \{w\}$ connecting z_0 to z_1.
Since U was path-connected to begin with, there exists a path $\sigma : [0,1] \to U$ such that
$$\sigma(0) = z_0, \quad \sigma(1) = z_1.$$
If $\sigma(x) \neq w$ for any $x \in [0,1]$, then we are done since σ is a path in $U \setminus \{w\}$ as well.
Suppose that this is not the case.
Then, we choose a $\delta > 0$ such that the *closed* ball
$$B := \{z \in \mathbb{C} : |z - w| \leq \delta\}$$
has the following properties:

[1] Finiteness is important. Induction cannot prove this result for a countably infinite set.

(a) $z_0 \notin B$,

(b) $z_1 \notin B$,

(c) $B \subset U$.

(Why must such a δ exist? There exists a δ_1 for which we get the first two properties since z_0 and z_1 are distinct from w. For the last property, let δ_2 be any such that $B_{\delta_2}(w) \subset U$, which exists since U is open. Then, consider $\delta_2/2$. The *closed* ball of this radius must again be completely within U. Take the minimum of δ_1 and $\delta_2/2$.)

Note that
$$\sigma^{-1}(B) = \{x \in [0,1] : \sigma(x) \in B\}$$
is nonempty since $w \in B$ and $\sigma(c) = w$ for some $c \in [0,1]$, by our assumption.
Moreover, $\sigma^{-1}(B)$ must be closed. (Why?)
Since it is a subset of $[0,1]$, it is clearly bounded. Define
$$s := \inf \sigma^{-1}(B), \quad t := \sup \sigma^{-1}(B).$$
Since the set is closed, both s and t are elements of $\sigma^{-1}(B)$. Note that $\sigma(0) \notin B$ and $\sigma(1) \notin B$ and thus,
$$0 < s < t < 1.$$
(Why is the inequality $s < t$ strict?)
Note that $\sigma(s)$ and $\sigma(t)$ must lie on the circumference of B. (Why?) (This also shows why $s < t$.)
Now consider the path $\sigma' : [0,1] \to U$ defined as follows:
$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in [0,s] \cup [t,1] \\ \gamma(x) & \text{if } x \in [s,t], \end{cases}$$
where $\gamma : [s,t] \to B$ is the path which is the arc joining $\sigma(s)$ to $\sigma(t)$. (Note that $\sigma(s) = \sigma(t)$ is possible in which case, it's the constant path.)
Clearly, σ' avoids w and is continuous. (Why?)

Moreover, $\sigma'(0) = \sigma(0) = z_0$ and $\sigma'(1) = \sigma(1) = z_1$ and thus, σ' is a path from z_0 to z_1 in $U \setminus \{w\}$, showing that $U \setminus \{w\}$ is path-connected. $\square$

4. Check for real differentiability and holomorphicity:

   (a) $f(z) = c$,

   (b) $f(z) = z$,

(c) $f(z) = z^n$, $n \in \mathbb{Z}$,

(d) $f(z) = \Re z$,

(e) $f(z) = |z|$,

(f) $f(z) = |z|^2$,

(g) $f(z) = \bar{z}$,

(h) $f(z) = \begin{cases} \dfrac{z}{\bar{z}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

Solution. Not going to do all.

(a) Real differentiable and holomorphic, both.

(b) Real differentiable and holomorphic, both.

(c) For $n \geq 0$:
Real differentiable and holomorphic, both. Let us see why.
As we know, holomorphicity implies real differentiability, so we only check that f is holomorphic on $\mathbb{C}$.
Let $z_0 \in \mathbb{C}$ be arbitrary. We show that the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.
This is clear because for $z_0 \neq z$, we have

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

The limit $z \longrightarrow z_0$ of the RHS clearly exists.

$n < 0$: The function is now defined on $\mathbb{C} \setminus \{0\}$. It is still holomorphic and real differentiable everywhere (in its domain!).
To see this, we just use the quotient rule and appeal to the previous case of $n \geq 0$.

(d) Real differentiable but not holomorphic. Note that f can be written as

$$f(x + \iota y) = x + 0\iota.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$.
This is clearly real differentiable everywhere since all the partial derivatives

exist everywhere and are continuous.
However, we show that f is not complex differentiable at any point. Thus, it is not holomorphic.
This is easy because one sees that $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$ and thus, the CR equations don't hold.

(e) $|z|$ is real differentiable everywhere except 0 and complex differentiable nowhere. Breaking the function as earlier, we have

$$u(x, y) = \sqrt{x^2 + y^2}, \quad v(x, y) = 0.$$

On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous. At $(0, 0)$, u_x and u_y fail to exist.

This clearly shows that f is not complex differentiable at $0 \in \mathbb{C}$ since it is not even real differentiable there.
However, we see that $v_y = 0 = v_x$ everywhere else but at least one of u_x or u_y is nonzero on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and thus, the CR equations prevent f from being complex differentiable anywhere else.

(f) Real differentiable everywhere.
Complex differentiable precisely at 0.
Holomorphic nowhere.

Same steps as above.

(g) Real differentiable everywhere. Complex differentiable nowhere. Use CR equations again.

(h) f is real differentiable precisely on $\mathbb{R}^2 \setminus \{(0, 0)\}$.
However, it is not complex differentiable anywhere.

Breaking as earlier, we get

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad v(x, y) = \frac{2xy}{x^2 + y^2},$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and

$$u(0, 0) = 0 = v(0, 0).$$

Note that u and v aren't even continuous at $(0, 0)$. Thus, neither if f. Hence, f is neither real nor complex differentiable at $(0, 0)$.
However, at all other points, all partial derivatives exist and are continuous. Thus, f is real differentiable at all those points. However, computing u_x, u_y, v_x, v_y explicitly shows that the CR equations are not satisfied anywhere. Thus, f is not complex differentiable anywhere. $\square$

5. Show that the CR equations take the form
$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$
in polar coordinates.

Solution. We shall follow the same idea as in the slides. We first write
$$f(r,\theta) = f(re^{\iota\theta}) = u(r,\theta) + \iota v(r,\theta).$$
Suppose that f is differentiable at $z_0 = r_0 e^{\iota\theta_0} \neq 0$. (Note that it wouldn't make sense to talk at 0 since there's a r^{-1} factor in the question anyway.)
Thus, we know that the limit
$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists. We shall calculate it in two ways:

(a) Fix $\theta = \theta_0$ and let $r \to r_0$. Then, we get
$$f'(z_0) = \lim_{r\to r_0}\left\{\frac{u(r,\theta_0) - u(r_0,\theta_0)}{e^{\iota\theta_0}(r - r_0)} + \iota\frac{v(r,\theta_0) - v(r_0,\theta_0)}{e^{\iota\theta_0}(r - r_0)}\right\}$$
$$= e^{-\iota\theta_0}\lim_{r\to r_0}\left\{\frac{u(r,\theta_0) - u(r_0,\theta_0)}{r - r_0} + \iota\frac{v(r,\theta_0) - v(r_0,\theta_0)}{r - r_0}\right\}$$
$$= e^{-\iota\theta_0}\left(u_r(r_0,\theta_0) + \iota v_r(r_0,\theta_0)\right). \tag{$*$}$$

(b) Fix $r = r_0$ and let $\theta \to \theta_0$. Then, we get
$$f'(z_0) = \lim_{\theta\to\theta_0}\left\{\frac{u(r_0,\theta) - u(r_0,\theta_0)}{r_0(e^{\iota\theta} - e^{\iota\theta_0})} + \iota\frac{v(r_0,\theta) - v(r_0,\theta_0)}{r_0(e^{\iota\theta} - e^{\iota\theta_0})}\right\}$$
$$= \frac{1}{r_0}\lim_{\theta\to\theta_0}\left\{\frac{u(r_0,\theta) - u(r_0,\theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} + \iota\frac{v(r_0,\theta) - v(r_0,\theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}}\right\} \tag{$**$}$$

We concentrate on the first term of the limit. Note that
$$\lim_{\theta\to\theta_0}\frac{u(r_0,\theta) - u(r_0,\theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}}$$
$$= \lim_{\theta\to\theta_0}\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0}\frac{\theta - \theta_0}{e^{\iota\theta} - e^{\iota\theta_0}}.$$

In the product, the first term is clearly $u_\theta(r_0, \theta_0)$, after taking the limit. The second term can be calculated to be

$$\frac{1}{\iota e^{\iota\theta_0}}.$$

(How? Write $e^{\iota\theta}$ in terms of $\cos$ and $\sin$ and differentiate those and put it back.)
Of course, a similar argument goes through for the v term as well.
Thus, we get that $(**)$ transforms to

$$f'(z_0) = \frac{e^{-\iota\theta_0}}{r_0}\left(\iota u_\theta(r_0, \theta_0) + v_\theta(r_0, \theta_0)\right).$$

Equating the above with $(*)$, cancelling $e^{-\iota\theta_0}$, and comparing the real and imaginary parts, we get

$$u_r(r_0, \theta_0) = \frac{1}{r_0} v_\theta(r_0, \theta_0), \quad v_r(r_0, \theta_0) = -\frac{1}{r_0} u_\theta(r_0, \theta_0),$$

as desired. $\square$

§2. Tutorial 2

1st September, 2020

1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions.
   Remark (my own): This is true iff u and v are defined on domains, that is, open and path-connected sets.

   Solution. Since v is a harmonic conjugate of u, we get that
   $$u_X = v_Y, \quad u_Y = -v_X.$$
   On the other hand, since u is a harmonic conjugate of v, we get that
   $$v_X = u_Y, \quad v_Y = -u_X.$$
   (Note that the equalities mean that they're true for every (X_0, Y_0) in the domain.)
   Thus, we get that
   $$u_X = u_Y = v_X = v_Y \equiv 0,$$
   identically.
   Since the domain is connected, this implies that u and v are constant. $\square$

2. Show that $u = XY - 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

   Solution.
   Smart way: If we can show that the above function is the real (or imaginary) part of a holomorphic function f, then we have shown that u is harmonic.
   Writing $Z = X + iY$, it is not too tough to see that the above is the **imaginary** part of $\frac{1}{2}Z^2 + Z^3$. Since
   $$f(Z) = \frac{1}{2}Z^2 + Z^3$$
   is holomorphic on $\mathbb{C}$, this gives us that u is harmonic.
   This also shows *a* harmonic conjugate of u is
   $$v(X, Y) = -\Re f(Z) = \frac{1}{2}(Y^2 - X^2) + 3XY^2 - X^3.$$
   (Note the negative sign! If we had gotten u as the *real* part of a holomorphic function, then for finding harmonic conjugate, we would've simply taken the imaginary part *without* the negative sign.)

. . . .

Laborious way: This is the way to do it if observing is difficult.
First, we show that u is harmonic by manual calculation. Note that

$$u_{XX}(X_0, Y_0) = 6Y_0 \text{ and } u_{YY}(X_0, Y_0) = -6Y_0.$$

Thus, $u_{XX} + u_{YY} \equiv 0$ and u is indeed harmonic.

To find its harmonic conjugate, we perform the procedure as given in slides.
Note that $u_X = v_Y$. Here, we get $u_X = Y + 6XY = v_Y$.
Integrating with respect to Y gives us

$$v = \frac{1}{2}Y^2 + 3XY^2 + g(X)$$

for some function g. Then, we need $v_X = -u_Y$. Computing each individually, we get

$$3Y^2 + g'(X) = -X - 3X^2 + 3Y^2.$$

Thus, up to a constant, we get

$$g(X) = -\frac{1}{2}X^2 - X^3.$$

Finally, this gives

$$v = \frac{1}{2}Y^2 + 3XY^2 - \frac{1}{2}X^2 - X^3.$$

□

3. Find the radius of convergence of the following power series:

   (a) $\displaystyle\sum_{n=0}^{\infty} nz^n$,

   (b) $\displaystyle\sum_{p \text{ prime}} z^p$,

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

Solution. We shall be using the root test in the first two cases and ratio test in the third.
One thing to recall is that *if* the limit $\lim\limits_{n\to\infty} a_n$ exists, *then* $\limsup\limits_{n\to\infty} a_n$ is equal

to that limit. This will be helpful in the first and third parts since the limits will themselves exist.
Moreover, we recall that if

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

then the radius of convergence R is given by

$$R = \alpha^{-1}.$$

(The case $\alpha = 0$ corresponds to $R = \infty$ and vice-versa.)
Similar analysis holds for

$$\alpha = \lim_{n\to\infty} \left|\frac{a_{i+1}}{a_i}\right|.$$

(Here, however, note that I need the existence of α. In the case of $\limsup$, that was always guaranteed.)

(a) Note that we have

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

(MA 105 Tutorial Sheet 1, Question 2 (iv))

Thus, we also have

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{n} = 1$$

and thus,

$$R = \alpha^{-1} = \boxed{1.}$$

(b) Note that first we can rewrite the series in the form

$$\sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n := \begin{cases} 0 & n \text{ is not a prime,} \\ 1 & n \text{ is a prime.} \end{cases}$$

For this, we clearly have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} 1 = 1.$$

(To see this, note that there are infinitely many primes and thus, given any n, there exists $m \geq n$ such that $a_m = 1$.)
Thus, as before, the radius of convergence is 1.

(c) Here, we have

$$a_n = \frac{n!}{n^n}.$$

Thus, we get

$$\begin{aligned}\alpha = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}(n+1)\frac{n^n}{(n+1)^{n+1}}\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}\\ &= e^{-1}.\end{aligned}$$

Thus, the limit actually exists and we get

$$R = \alpha^{-1} = \boxed{e.}$$

□

4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

Solution. Consider the sequence

$$\frac{1}{1^3}, \frac{1}{1^2}, \frac{1}{2^3}, \frac{1}{2^2}, \ldots, \frac{1}{n^3}, \frac{1}{n^2}, \ldots.$$

That is, let (a_n) be the sequence defined by

$$a_{2n} = \frac{1}{n^2}, \quad a_{2n-1} = \frac{1}{n^3}.$$

Note that $\sum a_n$ converges, since $\sum n^{-2}$ and $\sum n^{-3}$ converge. (This can be checked via the integral test.)
On the other hand, note that that

$$L = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \geq \limsup_{n\to\infty}\left|\frac{a_{2n}}{a_{2n-1}}\right| = \limsup_{n\to\infty} n = \infty.$$

Thus, $L = \infty$, clearly > 1.
(So, not only did we show that $L > 1$ doesn't imply divergence but also that even $L = \infty$ is not good enough to conclude divergence.) □

5. Construct a infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

Solution. Recall the function $g : \mathbb{R} \to \mathbb{R}$ from the lectures given as

$$g(x) := \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0. \end{cases}$$

As we saw, this is an infinitely differentiable function. Now, consider $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := g(x)g(1-x).$$

Clearly, f is infinitely differentiable, being the product of two such functions. Moreover, $f(x) = 0$ if $x \leq 0$ or $x \geq 1$. In other words, f is 0 outside the bounded set

$$(0, 1).$$

However, f is non-zero since

$$f\left(\frac{1}{2}\right) = \left(g\left(\frac{1}{2}\right)\right)^2 = e^{-4} \neq 0.$$

On the other hand, let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function which is zero outside some bounded set K. We show that g is zero everywhere.
Since K is bounded, there exists $M > 0$ such that

$$|z| \leq M \qquad \text{for all } z \in K.$$

Thus, choosing the point $z_0 = M+43$, we see that f is zero in the neighbourhood of z_0 of radius 42. (Why?)
However, since $\mathbb{C}$ is path-connected, this implies that f is zero *everywhere*, as desired. □

6. Show that $\exp : \mathbb{C} \to \mathbb{C}^\times$ is onto.

Solution. Let $z_0 \in \mathbb{C}^\times$. We show that $\exp(z) = z_0$ for some $z \in \mathbb{C}$.
Note that $r = |z_0| \neq 0$.
Then,

$$w = \frac{z_0}{r}$$

has modulus 1. In other words,

$$w = x_0 + \iota y_0$$

for some $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0^2 + y_0^2 = 1$.
Thus, $x_0 = \cos\theta$ and $y_0 = \sin\theta$ for some $\theta \in [0, 2\pi)$.

Define $z = \log(r) + \iota\theta$. Note that this $\log$ is the real-valued $\log$. Thus, we get

$$\begin{aligned}\exp(z) = \exp(\log(r) + \exp(\iota\theta)) &= \exp(\log(r)) \cdot \exp(\iota\theta) \\ &= r \cdot (\cos\theta + \iota \sin\theta) \\ &= rw = z_0.\end{aligned}$$

Thus, $\exp$ is surjective. □

7. Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and cosine which were bounded by 1).

Solution. We show this for $\cos$. The method works the same for $\sin$.
Recall that
$$\cos(z) = \frac{1}{2}\left(e^{\iota z} + e^{-\iota z}\right).$$
Let $z_0 \in \mathbb{C}$. We show that $\cos(z) = z_0$ for some $z \in \mathbb{C}$.
Consider the quadratic equation
$$\frac{1}{2}\left(t + \frac{1}{t}\right) = z_0. \quad (*)$$
Rearranging this gives
$$t^2 - 2z_0 t + 1 = 0.$$
Note that the above has complex solutions t_1 and t_2. (Since every complex number has a square root in $\mathbb{C}$!)
Moreover, note that $t_1 \neq 0$. Thus, by the previous part, there exists $z \in \mathbb{C}$ such that $e^z = t_1$.
Plugging $t_1 = e^z$ in $(*)$ shows that
$$\cos(z) = z_0,$$
as desired. □

8. Show that for any complex number z, $\sin^2(z) + \cos^2(z) = 1$.

Solution. Recall the definitions
$$\iota \sin(z) = \frac{1}{2}\left(e^{\iota z} - e^{-\iota z}\right), \quad \cos(z) = \frac{1}{2}\left(e^{\iota z} + e^{-\iota z}\right).$$
Squaring and subtracting gives
$$(\cos(z))^2 - (\iota \sin(z))^2 = \frac{1}{4}(4e^{\iota z}e^{-\iota z}) = 1$$
or
$$\sin^2(z) + \cos^2(z) = 1. \quad \square$$