$\mathbb{C}$omplex Analysis TSC - 1

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IIT Bombay

18th September, 2020

Greetings

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You can find a link to this document on `bit.ly/ma-205`. Both with and without pauses. You may keep it open alongside for quick reference.

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Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ-neighbourhood of z_0, denoted by $B_\delta(z_0)$ is the set

$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$

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A set $U \subset \mathbb{C}$ is said to be open if:
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Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P. (A continuous function from $[0, 1]$ to P.)

Definition 4 (Differentiable)

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exists. In this case, it is denoted by $f'(z_0)$.

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For sets, however, there is no difference.

End of Lecture 1

Any questions?

Notation

From this point on, Ω be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number z as $z = x + \iota y$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + \iota v(z)$.

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose f as

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where $u, v : \Omega \to \mathbb{R}$ are real valued functions.

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The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v. Now, if we know that f is holomorphic, then we have the following result.

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Lecture 2: CR Equations

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$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that u and v satisfy the CR equations at $(0,0)$ but f is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

We recall MA 105 now.

Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega \to \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if

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The matrix A was called the *total derivative of f at (x_0, y_0)*.

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Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient.

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Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

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If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.
Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

End of Lecture 2

Any questions?

Definition 8 (Convergence of series)

A series of the form
$$\sum_{n=0}^{\infty} a_n$$
of complex numbers is said to converge if the sequence of partial sums
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Check that $\sum(-1)^n$ and $\sum n$ both diverge.

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$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad (*)$$

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Lecture 3: Power Series

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What is the radius of convergence, though? (The definition, that is.)

Theorem 4 (Radius of convergence)

Given any power series as $(*)$, there exists $R \in [0, \infty]$ such that

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Note the brackets.

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If $\alpha = 0$, then $R = \infty$ and vice-versa.

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Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Differentiability of power series is what one should expect.

Theorem 7 (Differentiability)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the open disc of radius R, let $f(z)$ denote this sum. Then, on this disc, we have

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Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

End of Lecture 3

Any questions?

I shall just recall the facts from the lecture.

Definition 10 (Exponential function)

The power series
$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$
converges on all of $\mathbb{C}$. This sum is denoted by $\exp(z)$.

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Now, we some "converse" facts.

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Theorem 10 (Final fact)

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

Definition 11 (Domain)

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The above is saying that around every zero of f, we can draw a (sufficiently small) circle such that f has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete.*

End of Lecture 4

Any questions?

Definition 12

Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Writing $f = u + \iota v$ as usual, we define

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + \iota \int_a^b v(t)\mathrm{d}t.$$

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Theorem 12 (Primitives and integrals)

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Now, we come to Cauchy's theorem.

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$$\int_\gamma f(z)\mathrm{d}z = 0.$$

If Ω is simply-connected, then the interior condition is automatically met. This gives us the next result.

Theorem 14 ("General" Cauchy Theorem)

Let Ω be a simply-connected domain. Let $\gamma : [a, b] \to \mathbb{C}$ be a simple, closed contour and $f : \Omega \to \mathbb{C}$ holomorphic. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

End of Lecture 5

Any questions?

Theorem 15 (Cauchy Integral Formula)

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Let f be holomorphic everywhere on an open set Ω. Let γ be a simple closed curve in Ω, oriented positively. If z_0 is interior to γ and Ω contains the interior of γ, then

$$f(z_0) = \frac{1}{2\pi\iota}\int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z$$

We then saw a consequence of CIF which I state as a theorem below.

Theorem 16 (Holomorphic $\implies$ Analytic)

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$.

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$$a_n = \frac{1}{2\pi\iota} \int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w.$$

The above also gives us (what I call) the "generalised" Cauchy Integral Formula.

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$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0),$$

where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

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Remark 3

Note that, as usual, we require f to be holomorphic within the circle as well.

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An easy application of this give us:

Theorem 19 (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

End of Lectures 6 and 7

Any questions?

We discuss logarithm a bit.

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Theorem 20 (Uniqueness of branches)

Assume that $f, g : \Omega \to \mathbb{C}$ are two branches of the logarithm. Then, $f - g$ is a constant function. Moreover, this constant is an integer multiple of $2\pi\iota$.

We discuss logarithm a bit.

Definition 14 (Branch of the logarithm)

Let $\Omega \subset \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\exp(f(z)) = z, \qquad \text{for all } z \in \Omega.$$

Then, f is called a *branch of the logarithm.*

Theorem 20 (Uniqueness of branches)

Assume that $f, g : \Omega \to \mathbb{C}$ are two branches of the logarithm. Then, $f - g$ is a constant function. Moreover, this constant is an integer multiple of $2\pi\iota$.

The last theorem also assumed that Ω is a domain.

The previous theorem talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible.

Theorem 21 (Existence of a branch)

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4. $F(r) = \log(r)$ for all $r \in \Omega \cap \mathbb{R}$.

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Definition 15 (Singularities)

Let $f : \Omega \to \mathbb{C}$ be a function. A point $z_0 \in \mathbb{C}$ is said to be a singularity of f if

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Compare this "isolation" with what we saw earlier when we said that "zeroes are isolated."

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Remark 4

The above classification is only for isolated singularities.

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If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

Theorem 22 (Riemann's Removable Singularity Theorem)

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$$f(z) = \frac{\sin z}{z}$$

defined on $\mathbb{C} \setminus \{0\}$ has 0 as a removable singularity.

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An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

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Theorem 25 (Casorati-Weierstrass Theorem)

If z_0 is an isolated singularity, then it is essential iff the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

End of Lecture 8

Any questions?

Theorem 26 (Modified CIF)

Suppose that z_0 is an isolated singularity of f.

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Just like how the usual CIF gave us the power series, this CIF gives us the Laurent series.

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With the same setup as earlier, for $z \in A$, we can write $f(z)$ as

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where $r < r_0 < R$.

Note that the above is valid for $n < 0$ as well.

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$$a_{-1} = \mathrm{Res}(f; z_0).$$

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Suppose f is given and has finitely many singularities $z_1, \ldots, z_n$ within a simple closed contour γ. Then, we have

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Note that the above is implicitly implying that f is holomorphic at all other points within γ.

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Given a Laurent series as earlier, its principal part is

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In particular, the residue at a removable singularity is 0.

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Thus,

$$g(z) = (z - z_0)^m f(z)$$

is holomorphic at z_0 and

$$a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(z_0).$$

The End

Doubts?