MA 205: $\mathbb{C}$omplex Analysis

Extra questions

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Contents

§0. Notations

1. $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers.
2. $\mathbb{Z}$ is the set of integers.
3. $\mathbb{Q}$ is the set of rational numbers.
4. $\mathbb{R}$ is the set of real numbers.
5. $A \subset B$ is read as "A is a subset of B." In particular, note that $A \subset A$ is true for any set A.
6. $A \subsetneq B$ is read "A is a *proper* subset of B."
7. $\supset$ and $\supsetneq$ are defined similarly.
8. Given a function $f : X \to Y$, $A \subset X$, $B \subset Y$, we define

   $$f(A) = \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y,$$
   $$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X.$$

   (Note that this f^{-1} is different from the inverse of a function. In particular, this is always defined, even if f is not bijective. However, the f and f^{-1} above need not be "inverses.")
9. A *domain*, as a subset of $\mathbb{C}$ will always refer to a set which is open and path connected.
   (Note that this is different from domain of a function.)

§1. Topology

1. Is the interval $(0,1)$ open as a subset of $\mathbb{C}$?
   HIDDEN: No

2. Is the interval $(0,1)$ closed as a subset of $\mathbb{C}$?
   HIDDEN: No

3. Consider the following four properties that a subset of $\mathbb{C}$ can have:

   (a) Open

   (b) Closed

   (c) Bounded

   (d) Path connected

   Thus, we can classify all the subsets of $\mathbb{C}$ into 2^4 classes on the basis of what properties they have (and what they don't).
   Give an example of each or a proof that some certain class cannot have anything. You may assume that $\varnothing$ and $\mathbb{C}$ are the only subsets of $\mathbb{C}$ which are both open and closed.

4. Let $U \subset \mathbb{C}$ be open and nonempty. Show that U is not countable.

5. Let $U \subset \mathbb{C}$ be open and K be countably open. Give examples to show that $U \setminus K$ may or not be open.

§2. Cauchy Riemann Equations

1. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined as
$$f(z) = \bar{z}.$$
Show that f is continuous at each point.
Show that f is differentiable at no point.
(This has given us a very easy example of a function which is continuous everywhere but differentiable nowhere. On the contrary, one has to put a lot more effort to construct an example in the case of real analysis.)

2. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as
$$f(x, y) = (x, -y)$$
is differentiable in the sense that you saw in MA 105. (That is, its total derivative exists at every point.)
Compare this with the previous question.

3. Let Ω be open (and not necessarily path-connected).
Let $f : \Omega \to \mathbb{C}$ be holomorphic such that $f'(z) = 0$ for all $z \in \Omega$.
Show that it is *not* necessary that f is constant.

   Show that if Ω is also assumed to be path-connected (that is, Ω is a domain), then it *is* necessary that f is constant.

4. Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose
$$f(z) \in \mathbb{R} \quad \text{for all } z \in \Omega.$$
Show that f is constant. (That is, if a complex differentiable function takes only real values, then it must be constant on path-connected sets.)

5. Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose that $|f|$ is constant. Show that f is constant.

§3. Series

1. (Cauchy criterion for series.) "Recall" Cauchy criterion for convergence from MA 105. (Prove or assume that the analogous thing holds for complex sequences as well.)

   Let (a_n) be a sequence of complex numbers. Show that $\sum_{n=1}^{\infty} a_n$ converges iff for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
   $$\left| \sum_{k=n}^{m} a_n \right| < \epsilon, \qquad \text{for all } m \geq n \geq N.$$

2. Let (a_n) be a sequence of complex numbers such that $\sum |a_n|$ converges. Use the above Cauchy criteria to show that $\sum a_n$ converges.

3. Let (a_n) and (b_n) be complex sequences such that $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$. Show that if $\sum |b_n|$ converges, then so does $\sum |a_n|$ and hence, so does $\sum a_n$. Show that you can weaken the "for all $n \in \mathbb{N}$" condition to "for all n sufficiently large." (Formulating what we mean by "sufficiently large" is part of the exercise.)

4. Use the above to show that
   $$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$
   converges for all $z \in \mathbb{C}$ satisfying $|z| = 1$.

5. Show that
   $$\sum_{n=1}^{\infty} \frac{1}{n}$$
   diverges.
   HIDDEN: Compare it with the sequence $1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, \ldots$.

6. Let (a_n) be a sequence of real numbers and (b_n) a sequence of complex numbers satisfying

   (a) (a_n) is monotonic,

   (b) $\lim_{n\to\infty} a_n = 0$,

   (c) there exists $M \geq 0$ such that
   $$\left| \sum_{n=1}^{N} b_n \right| \leq M$$
   for every $N \in \mathbb{N}$.

Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Here's an outline of what you can do:

(a) Define the partial sums $S_n = \sum_{k=1}^{n} a_k b_k$ and $B_n = \sum_{k=1}^{n} b_k$.

Show that

$$S_n = a_n B_n + \sum_{k=1}^{n-1} B_k (a_k - a_{k+1}).$$

(This is called summation by parts.)

(b) Note that B_n is bounded by M and $a_n \to 0$. Conclude that the first term $\to 0$ as $n \to \infty$.

(c) Note that give any k, we have $|B_k(a_k - a_{k+1})| \leq M|a_k - a_{k+1}|$.

(d) Using (a_n) is monotonic, conclude that

$$\sum_{k=1}^{n-1} |a_k - a_{k+1}| = \sum_{k=1}^{n-1} |a_1 - a_n|.$$

(e) Conclude that $\lim_{n\to\infty} S_n$ exists.

The above is called **Dirichlet's test**.

7. Let $z \in \mathbb{C}$ be such that $|z| = 1$ and $z \neq 1$. Define the sequences (a_n) and (b_n) as

$$a_n := \frac{1}{n}, \quad b_n := z^n.$$

Show that (a_n) and (b_n) satisfy the hypothesis of Dirichlet's test. Conclude that

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges.

8. Compute the radius of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

These should come out to be 1. By the previous questions, conclude that the first converges everywhere on the boundary of the disc except at 1. However,

the second one converges everywhere on the boundary.
Do the same for the power series

$$\sum_{n=1}^{\infty} z^n.$$

HIDDEN: You should get that it converges nowhere on the boundary.
(Note that these series are (more or less) derivatives and anti-derivatives of each other on the *open* disc. However, they show very different behaviour on the boundary of the disc.)

§4. Properties of holomorphic functions

1. Let $\mathbb{H} = \{z \in \mathbb{C} : \Re z > 0\}$ be the open right plane.
   Construct a non-constant holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

   $$f\left(\frac{1}{n}\right) = 0, \quad \text{for all } n \in \mathbb{N}.$$

   (Does this contradict what we saw in slides? Why not?)

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that

   $$f\left(\frac{1}{n}\right) = 0, \quad \text{for all } n \in \mathbb{N}.$$

   Show that f is constant (and that the constant is 0).