$\mathbb{C}$omplex Analysis TSC

Aryaman Maithani

https://aryamanmaithani.github.io/tuts/ma-205

IIT Bombay

Autumn Semester 2020-21

Greetings

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You can find a link to this document on `bit.ly/ca-205`. Both with and without pauses. You may keep it open alongside for quick reference.

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Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ-neighbourhood of z_0, denoted by $B_\delta(z_0)$ is the set

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A set $U \subset \mathbb{C}$ is said to be open if:
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Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P. (A continuous function from $[0, 1]$ to P.)

Definition 4 (Differentiable)

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exists. In this case, it is denoted by $f'(z_0)$.

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For sets, however, there is no difference.

End of Lecture 1

Any questions?

Notation

From this point on, Ω be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number z as $z = x + \iota y$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + \iota v(z)$.

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose f as

$$f(z) = u(z) + \iota v(z),$$

where $u, v : \Omega \to \mathbb{R}$ are real valued functions.

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The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v. Now, if we know that f is holomorphic, then we have the following result.

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Also note that all the equalities are only at the point z_0. In particular, we are only assuming differentiability at z_0.

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An example for you to check is

$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

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$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that u and v satisfy the CR equations at $(0, 0)$ but f is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

We recall MA 105 now.

Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega \to \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if

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The matrix A was called the *total derivative of f at (x_0, y_0).*

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Theorem 3

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Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

End of Lecture 2

Any questions?

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$$\sum_{n=0}^{\infty} a_n$$

of complex numbers is said to converge if the sequence of partial sums

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Check that $\sum (-1)^n$ and $\sum n$ both diverge.

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Each y_n might be ∞. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.
If $\lim\limits_{n\to\infty} x_n$ itself exists, then it equals the $\limsup$ as well.

We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (*)$$

where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

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2. $(*)$ diverges for any z with $|z - z_0| > R$.

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What is the radius of convergence, though? (The definition, that is.)

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Given any power series as $(*)$, there exists $R \in [0, \infty]$ such that

1. $(*)$ converges for any z with $|z - z_0| < R$, and
2. $(*)$ diverges for any z with $|z - z_0| > R$.

This R is called the radius of convergence.

Note the brackets.

We would now like to be able to calculate the radius of convergence.

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This test *always works.* We had no assumptions of any kind on $(*)$.
Note that $^{-1}$.
If $\alpha = 0$, then $R = \infty$ and vice-versa.

We have another test. This is simpler (to calculate) but mightn't always work.

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Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Differentiability of power series is what one should expect.

Theorem 7 (Differentiability)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the open disc of radius R, let $f(z)$ denote this sum. Then, on this disc, we have

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Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

End of Lecture 3

Any questions?

I shall just recall the facts from the lecture.

Definition 10 (Exponential function)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges on all of $\mathbb{C}$. This sum is denoted by $\exp(z)$.

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4. $\exp(z)$ is always nonzero.

Now, we some "converse" facts.

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Theorem 10 (Final fact)

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

Definition 11 (Domain)

A subset $\Omega \subset \mathbb{C}$ is said to be a *domain* if it is open and path-connected.

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The above is saying that around every zero of f, we can draw a (sufficiently small) circle such that f has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete.*

End of Lecture 4

Any questions?

Definition 12

Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Writing $f = u + \iota v$ as usual, we define

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + \iota \int_a^b v(t)\mathrm{d}t.$$

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Then, we have

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Existence of a primitive is a strong condition, by the way. A holomorphic function need not have a primitive on all of Ω.

Now, we come to Cauchy's theorem.

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If Ω is simply-connected, then the interior condition is automatically met. This gives us the next result.

Theorem 15 ("General" Cauchy Theorem)

Let Ω be a simply-connected domain. Let $\gamma : [a, b] \to \mathbb{C}$ be a simple, closed contour and $f : \Omega \to \mathbb{C}$ holomorphic. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

End of Lecture 5

Any questions?

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$$f(z_0) = \frac{1}{2\pi\iota} \int_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z$$

We then saw a consequence of CIF which I state as a theorem below.

Theorem 17 (Holomorphic $\implies$ Analytic)

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$.

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The above also gives us (what I call) the "generalised" Cauchy Integral Formula.

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$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0),$$

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Remark 3

Note that, as usual, we require f to be holomorphic within the circle as well.

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Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then

Theorem 19 (Cauchy's estimate)

Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

An easy application of this give us:

Theorem 20 (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

End of Lectures 6 and 7

Any questions?

Logarithm

We discuss logarithm a bit.

Definition 14 (Branch of the logarithm)

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The last theorem also assumed that Ω is a domain.

The previous theorem talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible.

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Definition 15 (Singularities)

Let $f : \Omega \to \mathbb{C}$ be a function. A point $z_0 \in \mathbb{C}$ is said to be a singularity of f if

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Compare this "isolation" with what we saw earlier when we said that "zeroes are isolated."

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Remark 4

The above classification is only for isolated singularities.

Definition 18 (Removable singularity)

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

Theorem 23 (Riemann's Removable Singularity Theorem)

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$$f(z) = \frac{\sin z}{z}$$

defined on $\mathbb{C} \setminus \{0\}$ has 0 as a removable singularity.

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If z_0 is a pole of f, then there exists an integer $m > 0$ such that

$$f(z) = (z - z_0)^{-m} f_1(z)$$

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on a punctured neighbourhood of z_0, for some function f_1 which is holomorphic on the complete neighbourhood.

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If the order is 1, then z_0 is said to be *simple* pole.

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An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

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Theorem 26 (Casorati-Weierstrass Theorem)

If z_0 is an isolated singularity, then it is essential iff the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

End of Lecture 8

Any questions?

Theorem 27 (Modified CIF)

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$$f(z) = \frac{1}{2\pi\imath} \int_{|w-z_0|=R'} \frac{f(w)}{z-w} \mathrm{d}w - \frac{1}{2\pi\imath} \int_{|w-z_0|=r'} \frac{f(w)}{z-w} \mathrm{d}w,$$

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where $r < r' < |z| < R' < R$.

Just like how the usual CIF gave us the power series, this CIF gives us the Laurent series.

Theorem 28 (Laurent Series)

With the same setup as earlier, for $z \in A$, we can write $f(z)$ as

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where $r < r_0 < R$.

Note that the above is valid for $n < 0$ as well.

Definition 21 (Laurent series expansion at z_0)

If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $\{z : 0 < |z - z_0| < r\}$ for some $r > 0$. The Laurent series expansion on this annulus is called the Laurent series expansion **at** z_0.

Definition 22 (Principal part)

Let $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be the Laurent series expansion *at* z_0. Its principal part is

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

The most interesting coefficient of the principal part is the -1^{st} one.

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This is what is usually called the *residue* and written as

$$a_{-1} = \operatorname{Res}(f; z_0).$$

With residues, calculation of integrals becomes easier.

Theorem 29 (Cauchy's Residue Theorem)

Suppose f is given and has finitely many singularities $z_1, \ldots, z_n$ within a simple closed contour γ.

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Suppose f is given and has finitely many singularities $z_1, \ldots, z_n$ within a simple closed contour γ.Then, we have

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\iota \sum_{i=1}^{n} \mathrm{Res}(f; z_i).$$

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Theorem 29 (Cauchy's Residue Theorem)

Suppose f is given and has finitely many singularities $z_1, \ldots, z_n$ within a simple closed contour γ. Then, we have

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\iota \sum_{i=1}^{n} \mathrm{Res}(f; z_i).$$

Note that the above is implicitly implying that f is holomorphic at all other points within γ.

Recall that given an isolated singularity, we can expand the function as a Laurent series *around* that point on a punctured neighbourhood.

Theorem 30 (Isolated singularities and their principal parts)

Recall that given an isolated singularity, we can expand the function as a Laurent series *around* that point on a punctured neighbourhood. We had defined the principal part of this series to be the part containing the negative powers of $z - z_0$. We now see how they are related to the nature of the isolated singularity.

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In particular, the residue at a removable singularity is 0.

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Definition 23 (Neighbourhood of ∞)

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f is said to have an isolated singularity at ∞ if f is (defined and) holomorphic on some neighbourhood of ∞. Equivalently, $z \mapsto f\left(\frac{1}{z}\right)$ has an isolated singularity at 0.

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We didn't define the residue at ∞. Check Wikipedia for what the definition is, if interested. It is not the same as the residue of $f(1/z)$ at 0.

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An "application:" Suppose that f is defined on the closed unit disc such that it is continuous on the closed disc and holomorphic on the open disc. Since the closed disc is closed and bounded and f is continuous, $|f|$ must attain a maximum on the closed disc. By MMT, this maximum must be on the boundary.

End of Lectures 10 and 11

Any questions?

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Moreover, if $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ *or* if $|f'(0)| = 1$, then $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

Definition 26 (Open maps)

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As before, note that the zeroes are counted with multiplicity. For example, z^{43} has 43 zeroes within the curve $|z| = 1$.

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As a corollary, we had gotten that harmonic functions are infinitely differentiable since open discs are simply connected.

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As a corollary, we obtain MMT for harmonic functions which says that u cannot obtain a maximum at any interior point unless it is constant.

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Theorem 38 (Identity Principle for harmonic functions)

Let u be a harmonic function on a domain $\Omega \subset \mathbb{C}$. If $u = 0$ on a non-empty open subset $U \subset \Omega$, then $u = 0$ throughout Ω.

End of Lectures 12 and 13

Any questions?

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In other words, if an entire function misses two points, then it must be constant.

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for all $z \in C_R$.

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Let f, g be continuous complex valued functions defined on the upper semicircular contour $C_R = \{Re^{\iota\theta} : \theta \in [0, \pi]\}$ for some $R > 0$. Assume that there exists $a > 0$ such that

$$f(z) = e^{\iota az} g(z),$$

for all $z \in C_R$. Then,

$$\left| \int_{C_R} f(z) \mathrm{d}z \right| \leq \frac{\pi}{a} \max_{\theta \in [0,\pi]} \left| g(Re^{\iota\theta}) \right|.$$

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This is useful in the cases that the quantity on the right goes to 0 in the limit $R \to \infty$.

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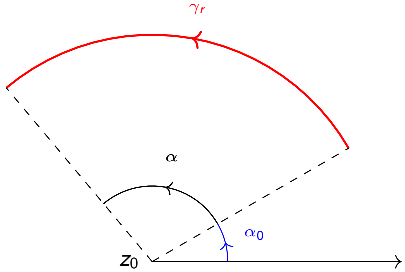

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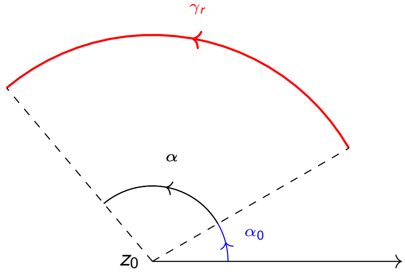
For $r > 0$, define $\gamma_r(\theta) := z_0 + re^{\iota(\theta+\alpha_0)}$ for $\theta \in [0, \alpha]$. Then,

$$\lim_{r \to 0^+} \int_{\gamma_r} f(z)\mathrm{d}z = \alpha\iota \operatorname{Res}(f; z_0).$$



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Usually, we will be interested in the upper half semi-circle. ML inequality will tell us that the integral over the semicircle goes to 0 in the limit $R \to \infty$.

The End

Doubts?