

MA 205: \mathbb{C} Complex Analysis

Extra questions

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§0. Notations

1. $\mathbb{N} = \{1, 2, 3, \dots\}$, the set of positive integers.
2. \mathbb{Z} is the set of integers.
3. \mathbb{Q} is the set of rational numbers.
4. \mathbb{R} is the set of real numbers.
5. $A \subset B$ is read as “ A is a subset of B .” In particular, note that $A \subset A$ is true for any set A .
6. $A \subsetneq B$ is read “ A is a *proper* subset of B .”
7. \supset and \supsetneq are defined similarly.
8. Given a function $f : X \rightarrow Y$, $A \subset X$, $B \subset Y$, we define

$$\begin{aligned} f(A) &= \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y, \\ f^{-1}(B) &= \{x \in X \mid f(x) \in B\} \subset X. \end{aligned}$$

(Note that this f^{-1} is different from the inverse of a function. In particular, this is always defined, even if f is not bijective. However, the f and f^{-1} above need not be “inverses.”)

§1. Topology

1. Is the interval $(0, 1)$ open as a subset of \mathbb{C} ?

HIDDEN: No

2. Is the interval $(0, 1)$ closed as a subset of \mathbb{C} ?

HIDDEN: No

3. Consider the following four properties that a subset of \mathbb{C} can have:

- (a) Open
- (b) Closed
- (c) Bounded
- (d) Path connected

Thus, we can classify all the subsets of \mathbb{C} into 2^4 classes on the basis of what properties they have (and what they don't).

Give an example of each or a proof that some certain class cannot have anything. You may assume that \emptyset and \mathbb{C} are the only subsets of \mathbb{C} which are both open and closed.

§2. Cauchy Riemann Equations

1. Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined as

$$f(z) = \bar{z}.$$

Show that f is continuous at each point.

Show that f is differentiable at no point.

(This has given us a very easy example of a function which is continuous everywhere but differentiable nowhere. On the contrary, one has to put a lot more effort to construct an example in the case of real analysis.)

2. Show that the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as

$$f(x, y) = (x, -y)$$

is differentiable in the sense that you saw in MA 105. (That is, its total derivative exists at every point.)

Compare this with the previous question.

3. Let Ω be a region and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic.

Suppose

$$f(z) \in \mathbb{R} \quad \text{for all } z \in \Omega.$$

Show that f is constant. (That is, if a complex differentiable function takes only real values, then it must be constant on path-connected sets.)

4. Let Ω be a region and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic.

Suppose that $|f|$ is constant. Show that f is constant.