Tutorial 1

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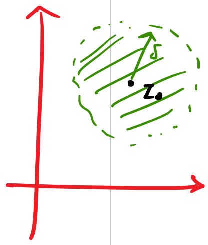
slides can be found at bit.ly/ca-205 under 2020 ≫ TSC Slides

Lecture 1

Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ-neighbourhood of z_0, denoted by $B_\delta(z_0)$ is the set

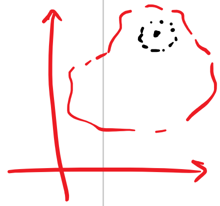
$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$



Definition 2 (Open sets)

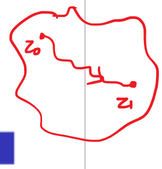
A set $U \subset \mathbb{C}$ is said to be open if:
for *every* $z_0 \in \mathbb{C}$, there exists *some* $\delta > 0$ such that

$$B_\delta(z_0) \subset U.$$



Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P. (A continuous function from $[0, 1]$ to P.)



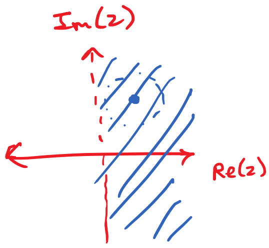
Examples.

① $B_\delta(z_0)$ are open for any $z_0 \in \mathbb{C}$ and $\delta > 0$.

② $\mathbb{C}$ is open. $\emptyset$ is open.

③ Strict right half plane $\mathbb{H}$ is open

$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.



④ $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is <u>NOT</u> open.

Lecture 1

Definition 4 (Differentiable)

Let $\Omega \subset \mathbb{C}$ be open. Let

$$f : \Omega \to \mathbb{C}$$

be a function. Let $z_0 \in \Omega$. f is said to be *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is denoted by $f'(z_0)$.

$f : (a, b) \to \mathbb{R}$

$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

$\Omega = \mathbb{C}$, $z, z^2, z^n, \ldots$

exp, sin, cos, ... ?

Non-diff: $|z|, \bar{z}, \ldots$

Lecture 1

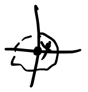
Definition 5 (Holomorphic)

1. A function f is said to be holomorphic on an open set Ω if it is differentiable at every $z_0 \in \Omega$.
2. A function f is said to be holomorphic at z_0 if it is holomorphic on some neighbourhood of z_0.

Remark 1

A function may be differentiable at z_0 but not holomorphic at z_0. For example, $f(z) = |z|^2$ is differentiable only at 0. Thus, it is differentiable at 0 but holomorphic nowhere.

For sets, however, there is no difference.



Points: Holo. $\Rightarrow$ Diff but $\not\Leftarrow$

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Notation

From this point on, Ω be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number z as $z = x + \iota y$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + \iota v(z)$.

Lecture 2: CR Equations

$$\left[\begin{array}{ccc} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 \\ x+\iota y & \longleftrightarrow & (x,y) \end{array}\right]$$

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose f as

$$f(z) = \underline{u(z)} + \iota v\underline{(z)},$$

where $u, v : \underline{\Omega} \to \mathbb{R}$ are real valued functions.

$\subseteq \mathbb{C} \cong \mathbb{R}^2$

The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v. Now, if we know that f is holomorphic, then we have the following result.

$$\left[u, v : \Omega^{\subset \mathbb{R}^2} \longrightarrow \mathbb{R} \right] \quad \underline{\text{MA 109, 111}}$$

u_x, u_y, v_x, v_y make sense.

Lecture 2: CR Equations

Cauchy – Riemann

Theorem 1 (CR equations)

Let $f : \Omega \to \mathbb{C}$ be differentiable at a point $z_0 \in \Omega$. Let $z_0 = x_0 + \iota y_0$.
Then, we have

$$① \quad u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, we have

$$② \quad f'(z_0) = u_x(x_0, y_0) + \iota v_x(x_0, y_0).$$

$f' = u_x + i v_x$

Existence of u_x, u_y, v_x, v_y is part of the theorem.

Note the subscript is x for both in the above.
Also note that all the equalities are only at the point z_0. In particular, we are only assuming differentiability at z_0.

Test $f(z) = z$
$= x + iy$
to see what the sign should be.

Lecture 2: CR Equations

Converse? What is the converse? Is it true?

Converse

No!

CR equations hold at the point (x_0, y_0)

$\Downarrow$?

f is differentiable at z_0?

No. The converse is not true.

An example for you to check is

$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that u and v satisfy the CR equations at $(0,0)$ but f is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

Lecture 2: CR Equations

We recall MA ~~105~~ 109 + 111 now.

Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega^{\subseteq \mathbb{R}^2} \to \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if there exits a 2×2 real matrix A such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(x_0 + h, y_0 + k) - f(x_0, y_0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

The matrix A was called the *total derivative of f at* (x_0, y_0).

Theorem 2

If f is (complex) differentiable at a point $z_0 = x_0 + \iota y_0$, then f is real differentiable at (x_0, y_0).

Once again, this is only talking about differentiability at a point. The converse is again not true.
Take the example $f(z) = \bar{z}$. Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

$u(x,y) = x$ $\quad u_x \equiv 1$

$v(x,y) = -y$ $\quad v_y \equiv -1$

Lecture 2: CR Equations

CD $\Rightarrow$ CR + RD

CR $\not\Rightarrow$ CD, RD $\not\Rightarrow$ CD

Theorem 3

Let $f : \Omega \to \mathbb{C}$ be a function and let $z_0 = x_0 + \iota y_0 \in \Omega$. If the CR equations hold at the point (x_0, y_0) and if f is real differentiable at the point (x_0, y_0), then f is complex differentiable at the point z_0.

(CR + RD) $\Rightarrow$ CD

Recall from MA 109, 111:

If $f : \Omega \to \mathbb{R}^2$ is a function s.t.

$f = (u, v)$

u_x, u_y, v_x, v_y are continuous on Ω,

then f is real diff. on Ω.

Lecture 2: CR Equations

Note : if $f : \Omega \to \mathbb{R}^2$, then f_x, f_y, etc. are meaningless.

Definition 7 (Harmonic functions)

Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function. u is said to be *harmonic* if $u_{xx} + u_{yy} = 0$.

Proposition 1

The real and imaginary parts of a holomorphic function are harmonic.

$u_x = v_y \Rightarrow u_{xx} = v_{yx}$, $\quad u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$ } but $v_{xy} = v_{yx}$ by assumption of u, v being C^2

Suppose u and v are harmonic on Ω. v is said to be a harmonic conjugate of u if $f = u + \iota v$ is holomorphic on Ω.

If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.

Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

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Harmonic Conjugate need not exist.

Example. Consider $\Omega = \mathbb{R}^2 - \{(0,0)\}$ and $u : \Omega \to \mathbb{R}$ defined as

$$u(x, y) = \frac{1}{2} \log (x^2 + y^2).$$

If u had a harmonic conjugate v, then

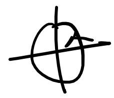
$$v_y(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad v_x(x,y) = -\frac{y}{x^2+y^2}.$$

But $\nexists\, v : \Omega \to \mathbb{R}$ s.t.

$$\nabla v = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

(Proof?)

$\oint$



Claim 1. Arbitrary union of open sets is open.

Proof. Let $\{U_i : i \in I\}$ be a collection of open sets.

Define $$U := \bigcup_{i \in I} U_i$$
$$= \{x : x \in U_i \text{ for some } i \in I\}.$$

TS: U is open.

Proof. Let $x \in U$ be arbitrary.

Then, $\exists i_0 \in U$ s.t. $x \in U_{i_0}$.

Since U_{i_0} is open, $\exists \delta > 0$ s.t.

$$B_\delta(x) \subseteq U_{i_0}.$$

But $U_{i_0} \subseteq U$. Thus, $B_\delta(x) \subseteq U$.

Thus, U is open. ∎

Claim 2. Finite intersection of open sets is open.

Proof. It suffices to prove that intersection of two open sets is open.

$A_1, \ldots, A_n \to$ open

$\Downarrow$

$(A_1 \cap A_2), A_3, \ldots, A_n \to$ open

$\Downarrow$

$A_1 \cap A_2 \cap A_3, \ldots, A_n \to$ open

$\Downarrow$

$\vdots$

$\Downarrow$

$A_1 \cap \ldots \cap A_n \to$ open

Let U_1 and U_2 be open and $x \in U_1 \cap U_2$.

Let U_1 and U_2 be open and $x \in U_1 \cap U_2$.

$\because U_1 \& U_2$ are open

$\exists \delta_1 > 0$ s.t. $B_{\delta_1}(x) \subseteq U_1$ and

$\exists \delta_2 > 0$ s.t. $B_{\delta_2}(x) \subseteq U_2$.

Pick $\delta := \min(\delta_1, \delta_2) > 0$.

Then, $B_\delta(x) \subseteq B_{\delta_1}(x) \subseteq U_1$ and

$B_\delta(x) \subseteq B_{\delta_2}(x) \subseteq U_2$.

$\therefore B_\delta(x) \subseteq (U_1 \cap U_2)$. ∎

"Dual" statements for closed sets.

$U_1, U_2 \to$ open You can say: $U_1 \cup U_2$ and $U_1 \cap U_2$ are open

$U_1, U_2, \ldots, U_n \to$ open $\Rightarrow$ $U_1 \cup \cdots \cup U_n$ & $U_1 \cap \cdots \cap U_n$ are open.

$U_1, U_2, U_3, \ldots \to$ open $\Rightarrow$ $\bigcup_{i=1}^{\infty} U_i$ is open but $\bigcap_{i=1}^{\infty} U_i$ may not be.

$$\mathbb{C} - \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{C} - U_i)$$

closed $\Leftrightarrow$ complement is open.

$\boxed{U_i := B_{1/i}(0).}$

$\bigcap_{i \in \mathbb{N}} U_i = \{0\}$

$\uparrow$ not open.