

MA 205: Complex Analysis

Tutorial Solutions

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§0. Notations

1. Given $z \in \mathbb{C}$, $\Re z$ and $\Im z$ will denote the real and imaginary parts of z , respectively.
2. Given $z \in \mathbb{C}$, \bar{z} will denote the complex conjugate of z .
3. Given $z \in \mathbb{C}$, $|z|$ will denote the modulus of z , defined as $\sqrt{z\bar{z}}$ or $\sqrt{(\Re z)^2 + (\Im z)^2}$.

§1. Tutorial 1

25th August, 2020

Notation: The set $\mathbb{C}[x]$ is the set of all polynomials (with indeterminate x) with complex coefficients. Similarly, $\mathbb{R}[x]$ is defined.

1. Show that complex polynomial of degree n has exactly n roots. (Assuming fundamental theorem of algebra.)

Remark (my own): The above is counting the roots *with* multiplicity. That is, if $f(z) = (z - \iota)^2(z - 2)$, then ι is counted twice and 2 once.

Solution. Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n . We prove this via induction on n .

$n = 1$. Then, $f(x) = a_0 + a_1x$ for some $a_0, a_1 \in \mathbb{C}$ and $a_1 \neq 0$.

Note that

$$\begin{aligned} f(x) &= 0 \\ \iff a_0 + a_1x &= 0 \\ \iff a_1x &= -a_0 \\ \iff x &= -\frac{a_0}{a_1}. \end{aligned}$$

Thus, $f(x)$ has exactly 1 root.

Let us assume that whenever $g(x) \in \mathbb{C}[x]$ is a polynomial of degree n , then $g(x)$ has exactly n roots. (Counted with multiplicity.)

Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n + 1$. By FTA, there exists a root $x_0 \in \mathbb{C}$. Thus, we can write

$$f(x) = (x - x_0)g(x)$$

for some polynomial $g(x) \in \mathbb{C}[x]$ of degree n . Moreover, note that

$$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$

By induction, the latter is possible for exactly n values of x . Thus, in total, $f(x)$ has $n + 1$ roots. (Both counts are with multiplicity.) \square

2. Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n, \quad a_i \in \mathbb{R}$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

Remark (my own): $a_n \neq 0$, of course.

Solution. Let $f(x) \in \mathbb{R}[x]$ with degree ≥ 3 as above.

If $f(x)$ has a real root, then we are done by factoring as in the earlier question.

Thus, let us assume that $f(x) = 0$ has no real solution.

We may view $f(x) \in \mathbb{C}[x]$. Now, using FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we must have $x_0 \notin \mathbb{R}$ or that $x_0 \neq \overline{x_0}$.

Claim. $f(\overline{x_0}) = 0$.

Proof. Note that

$$\begin{aligned}
 f(\overline{x_0}) &= a_0 + a_1 \overline{x_0} + \cdots + a_n (\overline{x_0})^n \\
 &= a_0 + a_1 \overline{x_0} + \cdots + a_n \overline{x_0^n} \\
 &= \overline{a_0} + \overline{a_1} \overline{x_0} + \cdots + \overline{a_n} \overline{x_0^n} \\
 &= \overline{f(x_0)} \\
 &= \overline{0} \\
 &= 0
 \end{aligned}
 \quad \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \because \overline{z^n} = \overline{z}^n \\ \because a_i \in \mathbb{R} \text{ and so, } a_i = \overline{a_i} \\ \overline{z_1 z_2 + z_3} = \overline{z_1} \overline{z_2} + \overline{z_3} \end{array}$$

□

Define $g(x) = (x - x_0)(x - \overline{x_0})$. A priori, this is a polynomial in $\mathbb{C}[x]$. However, upon multiplication, we see that the polynomial is actually an element of $\mathbb{R}[x]$. Indeed, we have

$$(x - x_0)(x - \overline{x_0}) = (x^2 - (2\Re x_0)x + |x_0|^2) \in \mathbb{R}[x].$$

By our claim, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Since x_0 and $\overline{x_0}$ are distinct, the polynomials $x - x_0$ and $x - \overline{x_0}$ are “coprime” and thus, if they individually divide $f(x)$, then their product must too.)

Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. However, since $f(x)$ and $g(x)$ are both real polynomials, so is $h(x)$. (Why?)

Thus, we get that

$$f(x) = g(x)h(x)$$

for real polynomials $g(x)$ and $h(x)$. Moreover, note that $\deg g(x) = 2$ and $\deg h(x) = n - 2 \geq 1$. Thus, both are non-constant. □

3. Show that if U is a path connected open set in \mathbb{C} , so is U minus any finite set.

Solution. We will first prove the following claim:

Claim: Let $U \subset \mathbb{C}$ be open and $w \in U$. Then, $U \setminus \{w\}$ is open.

Proof. Let $z_0 \in U \setminus \{w\}$ be arbitrary. Since U was open, there exists $\delta_1 > 0$ such that

$$B_{\delta_1}(z_0) \subset U.$$

Since $z_0 \neq w$, we have that $\delta_2 := |z_0 - w| > 0$.

Choose $\delta := \min\{\delta_1, \delta_2\}$. Clearly, $\delta > 0$. Moreover, we have

$$w \notin B_{\delta_2}(z_0) \supset B_{\delta}(z_0)$$

and thus, $w \notin B_{\delta}(z_0)$. Also,

$$B_{\delta}(z_0) \subset B_{\delta_1}(z_0) \subset U.$$

Thus, we get that

$$B_{\delta}(z_0) \subset U \setminus \{w\},$$

proving that $U \setminus \{w\}$ is open. □

By the above proof, we see that removing one point from an open set keeps it open. Thus, if we show that removing one point from an open path-connected set leaves it path-connected, then we are done since we can induct to get any other **finite**¹ set.

Thus, we now show that if U is open and path-connected, so is $U \setminus \{w\}$. (Where $w \in U$ is any arbitrary element.)

Let $z_0, z_1 \in U \setminus \{w\}$. We wish to show that there is a path in $U \setminus \{w\}$ connecting z_0 to z_1 .

Since U was path-connected to begin with, there exists a path $\sigma : [0, 1] \rightarrow U$ such that

$$\sigma(0) = z_0, \quad \sigma(1) = z_1.$$

If $\sigma(x) \neq w$ for any $x \in [0, 1]$, then we are done since σ is a path in $U \setminus \{w\}$ as well.

Suppose that this is not the case.

Then, we choose a $\delta > 0$ such that the *closed* ball

$$B := \{z \in \mathbb{C} : |z - w| \leq \delta\}$$

has the following properties:

¹Finiteness is important. Induction cannot prove this result for a countably infinite set.

- (a) $z_0 \notin B$,
- (b) $z_1 \notin B$,
- (c) $B \subset U$.

(Why must such a δ exist? There exists a δ_1 for which we get the first two properties since z_0 and z_1 are distinct from w . For the last property, let δ_2 be any such that $B_{\delta_2}(w) \subset U$, which exists since U is open. Then, consider $\delta_2/2$. The *closed* ball of this radius must again be completely within U . Take the minimum of δ_1 and $\delta_2/2$.)

Note that

$$\sigma^{-1}(B) = \{x \in [0, 1] : \sigma(x) \in B\}$$

is nonempty since $w \in B$ and $\sigma(c) = w$ for some $c \in [0, 1]$, by our assumption. Moreover, $\sigma^{-1}(B)$ must be closed. (Why?) Since it is a subset of $[0, 1]$, it is clearly bounded. Define

$$s := \inf \sigma^{-1}(B), \quad t := \sup \sigma^{-1}(B).$$

Since the set is closed, both s and t are elements of $\sigma^{-1}(B)$. Note that $\sigma(0) \notin B$ and $\sigma(1) \notin B$ and thus,

$$0 < s < t < 1.$$

(Why is the inequality $s < t$ strict?)

Note that $\sigma(s)$ and $\sigma(t)$ must lie on the circumference of B . (Why?) (This also shows why $s < t$.)

Now consider the path $\sigma' : [0, 1] \rightarrow U$ defined as follows:

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in [0, s] \cup [t, 1] \\ \gamma(x) & \text{if } x \in [s, t], \end{cases}$$

where $\gamma : [s, t] \rightarrow B$ is the path which is the arc joining $\sigma(s)$ to $\sigma(t)$. (Note that $\sigma(s) = \sigma(t)$ is possible in which case, it's the constant path.)

Clearly, σ' avoids w and is continuous. (Why?)

Moreover, $\sigma'(0) = \sigma(0) = z_0$ and $\sigma'(1) = \sigma(1) = z_1$ and thus, σ' is a path from z_0 to z_1 in $U \setminus \{w\}$, showing that $U \setminus \{w\}$ is path-connected. \square

4. Check for real differentiability and holomorphicity:

- (a) $f(z) = c$,
- (b) $f(z) = z$,

- (c) $f(z) = z^n, n \in \mathbb{Z}$,
- (d) $f(z) = \Re z$,
- (e) $f(z) = |z|$,
- (f) $f(z) = |z|^2$,
- (g) $f(z) = \bar{z}$,
- (h) $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

Solution. Not going to do all.

- (a) Real differentiable and holomorphic, both.
- (b) Real differentiable and holomorphic, both.
- (c) For $n \geq 0$:

Real differentiable and holomorphic, both. Let us see why.

As we know, holomorphicity implies real differentiability, so we only check that f is holomorphic on \mathbb{C} .

Let $z_0 \in \mathbb{C}$ be arbitrary. We show that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

This is clear because for $z_0 \neq z$, we have

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

The limit $z \rightarrow z_0$ of the RHS clearly exists.

$n < 0$: The function is now defined on $\mathbb{C} \setminus \{0\}$. It is still holomorphic and real differentiable everywhere (in its domain!).

To see this, we just use the quotient rule and appeal to the previous case of $n \geq 0$.

- (d) Real differentiable but not holomorphic. Note that f can be written as

$$f(x + iy) = x + 0i.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$.

This is clearly real differentiable everywhere since all the partial derivatives

exist everywhere and are continuous.

However, we show that f is not complex differentiable at any point. Thus, it is not holomorphic.

This is easy because one sees that $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$ and thus, the CR equations don't hold.

- (e) $|z|$ is real differentiable everywhere except 0 and complex differentiable nowhere. Breaking the function as earlier, we have

$$u(x, y) = \sqrt{x^2 + y^2}, \quad v(x, y) = 0.$$

On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous. At $(0, 0)$, u_x and u_y fail to exist.

This clearly shows that f is not complex differentiable at $0 \in \mathbb{C}$ since it is not even real differentiable there.

However, we see that $v_y = 0 = v_x$ everywhere else but at least one of u_x or u_y is nonzero on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and thus, the CR equations prevent f from being complex differentiable anywhere else.

- (f) Real differentiable everywhere.
Complex differentiable precisely at 0.
Holomorphic nowhere.

Same steps as above.

- (g) Real differentiable everywhere. Complex differentiable nowhere. Use CR equations again.
- (h) f is real differentiable precisely on $\mathbb{R}^2 \setminus \{(0, 0)\}$.
However, it is not complex differentiable anywhere.

Breaking as earlier, we get

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad v(x, y) = \frac{2xy}{x^2 + y^2},$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and

$$u(0, 0) = 0 = v(0, 0).$$

Note that u and v aren't even continuous at $(0, 0)$. Thus, neither is f . Hence, f is neither real nor complex differentiable at $(0, 0)$.

However, at all other points, all partial derivatives exist and are continuous. Thus, f is real differentiable at all those points. However, computing u_x, u_y, v_x, v_y explicitly shows that the CR equations are not satisfied anywhere. Thus, f is not complex differentiable anywhere. \square

5. Show that the CR equations take the form

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

in polar coordinates.

Solution. We shall follow the same idea as in the slides. We first write

$$f(r, \theta) = f(re^{i\theta}) = u(r, \theta) + i v(r, \theta).$$

Suppose that f is differentiable at $z_0 = r_0 e^{i\theta_0} \neq 0$. (Note that it wouldn't make sense to talk at 0 since there's a r^{-1} factor in the question anyway.)

Thus, we know that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We shall calculate it in two ways:

(a) Fix $\theta = \theta_0$ and let $r \rightarrow r_0$. Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} + i \frac{v(r, \theta_0) - v(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} \right\} \\ &= e^{-i\theta_0} \lim_{r \rightarrow r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + i \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0} \right\} \\ &= e^{-i\theta_0} (u_r(r_0, \theta_0) + i v_r(r_0, \theta_0)). \end{aligned} \quad (*)$$

(b) Fix $r = r_0$ and let $\theta \rightarrow \theta_0$. Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{\theta \rightarrow \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} \right\} \\ &= \frac{1}{r_0} \lim_{\theta \rightarrow \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \right\} \end{aligned} \quad (**)$$

We concentrate on the first term of the limit. Note that

$$\begin{aligned} &\lim_{\theta \rightarrow \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \\ &= \lim_{\theta \rightarrow \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}. \end{aligned}$$

In the product, the first term is clearly $u_\theta(r_0, \theta_0)$, after taking the limit. The second term can be calculated to be

$$\frac{1}{\iota e^{\iota\theta_0}}.$$

(How? Write $e^{\iota\theta}$ in terms of \cos and \sin and differentiate those and put it back.)

Of course, a similar argument goes through for the v term as well.

Thus, we get that $(**)$ transforms to

$$f'(z_0) = \frac{e^{-\iota\theta_0}}{r_0} (\iota u_\theta(r_0, \theta_0) + v_\theta(r_0, \theta_0)).$$

Equating the above with $(*)$, cancelling $e^{-\iota\theta_0}$, and comparing the real and imaginary parts, we get

$$u_r(r_0, \theta_0) = \frac{1}{r_0} v_\theta(r_0, \theta_0), \quad v_r(r_0, \theta_0) = -\frac{1}{r_0} u_\theta(r_0, \theta_0),$$

as desired. □

§2. Tutorial 2

1st September, 2020

1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions.

Remark (my own): This is true iff u and v are defined on domains, that is, open and path-connected sets.

Solution. Since v is a harmonic conjugate of u , we get that

$$u_X = v_Y, \quad u_Y = -v_X.$$

On the other hand, since u is a harmonic conjugate of v , we get that

$$v_X = u_Y, \quad v_Y = -u_X.$$

(Note that the equalities mean that they're true for every (X_0, Y_0) in the domain.)
Thus, we get that

$$u_X = u_Y = v_X = v_Y \equiv 0,$$

identically.

Since the domain is connected, this implies that u and v are constant. \square

2. Show that $u = XY - 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

Solution.

Smart way: If we can show that the above function is the real (or imaginary) part of a holomorphic function f , then we have shown that u is harmonic.

Writing $Z = X + iY$, it is not too tough to see that the above is the **imaginary** part of $\frac{1}{2}Z^2 + Z^3$. Since

$$f(Z) = \frac{1}{2}Z^2 + Z^3$$

is holomorphic on \mathbb{C} , this gives us that u is harmonic.

This also shows a harmonic conjugate of u is

$$v(X, Y) = -\Re f(Z) = \frac{1}{2}(Y^2 - X^2) + 3XY^2 - X^3.$$

(Note the negative sign! If we had gotten u as the *real* part of a holomorphic function, then for finding harmonic conjugate, we would've simply taken the imaginary part *without* the negative sign.)

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Laborious way: This is the way to do it if observing is difficult.
First, we show that u is harmonic by manual calculation. Note that

$$u_{XX}(X_0, Y_0) = 6Y_0 \text{ and } u_{YY}(X_0, Y_0) = -6Y_0.$$

Thus, $u_{XX} + u_{YY} \equiv 0$ and u is indeed harmonic.

To find its harmonic conjugate, we perform the procedure as given in slides.
Note that $u_X = v_Y$. Here, we get $u_X = Y + 6XY = v_Y$.
Integrating with respect to Y gives us

$$v = \frac{1}{2}Y^2 + 3XY^2 + g(X)$$

for some function g . Then, we need $v_X = -u_Y$. Computing each individually, we get

$$3Y^2 + g'(X) = -X - 3X^2 + 3Y^2.$$

Thus, up to a constant, we get

$$g(X) = -\frac{1}{2}X^2 - X^3.$$

Finally, this gives

$$v = \frac{1}{2}Y^2 + 3XY^2 - \frac{1}{2}X^2 - X^3.$$

□

3. Find the radius of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} nz^n,$

(b) $\sum_{p \text{ prime}} z^p,$

(c) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$

Solution. We shall be using the root test in the first two cases and ratio test in the third.

One thing to recall is that if the limit $\lim_{n \rightarrow \infty} a_n$ exists, then $\limsup_{n \rightarrow \infty} a_n$ is equal

to that limit. This will be helpful in the first and third parts since the limits will themselves exist.

Moreover, we recall that if

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|},$$

then the radius of convergence R is given by

$$R = \alpha^{-1}.$$

(The case $\alpha = 0$ corresponds to $R = \infty$ and vice-versa.)

Similar analysis holds for

$$\alpha = \lim_{n \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right|.$$

(Here, however, note that I need the existence of α . In the case of \limsup , that was always guaranteed.)

(a) Note that we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

(MA 105 Tutorial Sheet 1, Question 2 (iv))

Thus, we also have

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

and thus,

$$R = \alpha^{-1} = \boxed{1}.$$

(b) Note that first we can rewrite the series in the form

$$\sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n := \begin{cases} 0 & n \text{ is not a prime,} \\ 1 & n \text{ is a prime.} \end{cases}$$

For this, we clearly have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} 1 = 1.$$

(To see this, note that there are infinitely many primes and thus, given any n , there exists $m \geq n$ such that $a_m = 1$.)

Thus, as before, the radius of convergence is 1.

(c) Here, we have

$$a_n = \frac{n!}{n^n}.$$

Thus, we get

$$\begin{aligned}\alpha &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} (n+1) \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-n} \\ &= e^{-1}.\end{aligned}$$

Thus, the limit actually exists and we get

$$R = \alpha^{-1} = \boxed{e}. \quad \square$$

4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

Solution. Consider the sequence

$$\frac{1}{1^3}, \frac{1}{1^2}, \frac{1}{2^3}, \frac{1}{2^2}, \dots, \frac{1}{n^3}, \frac{1}{n^2}, \dots$$

That is, let (a_n) be the sequence defined by

$$a_{2n} = \frac{1}{n^2}, \quad a_{2n-1} = \frac{1}{n^3}.$$

Note that $\sum a_n$ converges, since $\sum n^{-2}$ and $\sum n^{-3}$ converge. (This can be checked via the integral test.)

On the other hand, note that that

$$L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \geq \limsup_{n \rightarrow \infty} \left| \frac{a_{2n}}{a_{2n-1}} \right| = \limsup_{n \rightarrow \infty} n = \infty.$$

Thus, $L = \infty$, clearly > 1 .

(So, not only did we show that $L > 1$ doesn't imply **divergence** but also that even $L = \infty$ is not good enough to conclude divergence.) \square

5. Construct a infinitely differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

Solution. Recall the function $g : \mathbb{R} \rightarrow \mathbb{R}$ from the lectures given as

$$g(x) := \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0. \end{cases}$$

As we saw, this is an infinitely differentiable function. Now, consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) := g(x)g(1-x).$$

Clearly, f is infinitely differentiable, being the product of two such functions. Moreover, $f(x) = 0$ if $x \leq 0$ or $x \geq 1$. In other words, f is 0 outside the bounded set

$$(0, 1).$$

However, f is non-zero since

$$f\left(\frac{1}{2}\right) = \left(g\left(\frac{1}{2}\right)\right)^2 = e^{-4} \neq 0.$$

On the other hand, let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function which is zero outside some bounded set K . We show that g is zero everywhere.

Since K is bounded, there exists $M > 0$ such that

$$|z| \leq M \quad \text{for all } z \in K.$$

Thus, choosing the point $z_0 = M+43$, we see that f is zero in the neighbourhood of z_0 of radius 42. (Why?)

However, since \mathbb{C} is path-connected, this implies that f is zero *everywhere*, as desired. \square

6. Show that $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ is onto.

Solution. Let $z_0 \in \mathbb{C}^\times$. We show that $\exp(z) = z_0$ for some $z \in \mathbb{C}$.

Note that $r = |z_0| \neq 0$.

Then,

$$w = \frac{z_0}{r}$$

has modulus 1. In other words,

$$w = x_0 + iy_0$$

for some $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0^2 + y_0^2 = 1$.

Thus, $x_0 = \cos \theta$ and $y_0 = \sin \theta$ for some $\theta \in [0, 2\pi)$.

Define $z = \log(r) + \iota\theta$. Note that this \log is the real-valued \log . Thus, we get

$$\begin{aligned}\exp(z) &= \exp(\log(r) + \exp(\iota\theta)) = \exp(\log(r)) \cdot \exp(\iota\theta) \\ &= r \cdot (\cos \theta + \iota \sin \theta) \\ &= rw = z_0.\end{aligned}$$

Thus, \exp is surjective. □

7. Show that $\sin, \cos : \mathbb{C} \rightarrow \mathbb{C}$ are surjective. (In particular, note the difference with real sine and cosine which were bounded by 1).

Solution. We show this for \cos . The method works the same for \sin . Recall that

$$\cos(z) = \frac{1}{2} (e^{\iota z} + e^{-\iota z}).$$

Let $z_0 \in \mathbb{C}$. We show that $\cos(z) = z_0$ for some $z \in \mathbb{C}$.

Consider the quadratic equation

$$\frac{1}{2} \left(t + \frac{1}{t} \right) = z_0. \quad (*)$$

Rearranging this gives

$$t^2 - 2z_0t + 1 = 0.$$

Note that the above has complex solutions t_1 and t_2 . (Since every complex number has a square root in \mathbb{C} !)

Moreover, note that $t_1 \neq 0$. Thus, by the previous part, there exists $z \in \mathbb{C}$ such that $e^z = t_1$.

Plugging $t_1 = e^z$ in $(*)$ shows that

$$\cos(z) = z_0,$$

as desired. □

8. Show that for any complex number z , $\sin^2(z) + \cos^2(z) = 1$.

Solution. Recall the definitions

$$\iota \sin(z) = \frac{1}{2} (e^{\iota z} - e^{-\iota z}), \quad \cos(z) = \frac{1}{2} (e^{\iota z} + e^{-\iota z}).$$

Squaring and subtracting gives

$$(\cos(z))^2 - (\iota \sin(z))^2 = \frac{1}{4} (4e^{\iota z} e^{-\iota z}) = 1$$

or

$$\sin^2(z) + \cos^2(z) = 1. \quad \square$$