MA 205: $\mathbb{C}$omplex Analysis

Extra questions

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Contents

§0. Notations

1. $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers.
2. $\mathbb{Z}$ is the set of integers.
3. $\mathbb{Q}$ is the set of rational numbers.
4. $\mathbb{R}$ is the set of real numbers.
5. $A \subset B$ is read as "A is a subset of B." In particular, note that $A \subset A$ is true for any set A.
6. $A \subsetneq B$ is read "A is a *proper* subset of B."
7. $\supset$ and $\supsetneq$ are defined similarly.
8. Given a function $f : X \to Y$, $A \subset X$, $B \subset Y$, we define

$$\begin{aligned} f(A) &= \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y, \\ f^{-1}(B) &= \{x \in X \mid f(x) \in B\} \subset X. \end{aligned}$$

   (Note that this f^{-1} is different from the inverse of a function. In particular, this is always defined, even if f is not bijective. However, the f and f^{-1} above need not be "inverses.")
9. A *domain*, as a subset of $\mathbb{C}$ will always refer to a set which is open and path connected.
   (Note that this is different from domain of a function.)

§1. Topology

1. Is the interval $(0, 1)$ open as a subset of $\mathbb{C}$?
   HIDDEN: No

2. Is the interval $(0, 1)$ closed as a subset of $\mathbb{C}$?
   HIDDEN: No

3. Consider the following four properties that a subset of $\mathbb{C}$ can have:

   (a) Open

   (b) Closed

   (c) Bounded

   (d) Path connected

   Thus, we can classify all the subsets of $\mathbb{C}$ into 2^4 classes on the basis of what properties they have (and what they don't).
   Give an example of each or a proof that some certain class cannot have anything.
   You may assume that $\varnothing$ and $\mathbb{C}$ are the only subsets of $\mathbb{C}$ which are both open and closed.

§2. Cauchy Riemann Equations

1. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined as
$$f(z) = \bar{z}.$$
Show that f is continuous at each point.
Show that f is differentiable at no point.
(This has given us a very easy example of a function which is continuous everywhere but differentiable nowhere. On the contrary, one has to put a lot more effort to construct an example in the case of real analysis.)

2. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as
$$f(x, y) = (x, -y)$$
is differentiable in the sense that you saw in MA 105. (That is, its total derivative exists at every point.)
Compare this with the previous question.

3. Let Ω be open (and not necessarily path-connected).
Let $f : \Omega \to \mathbb{C}$ be holomorphic such that $f'(z) = 0$ for all $z \in \Omega$.
Show that it is *not* necessary that f is constant.

   Show that if Ω is also assumed to be path-connected (that is, Ω is a domain), then it *is* necessary that f is constant.

4. Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose
$$f(z) \in \mathbb{R} \quad \text{for all } z \in \Omega.$$
Show that f is constant. (That is, if a complex differentiable function takes only real values, then it must be constant on path-connected sets.)

5. Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose that $|f|$ is constant. Show that f is constant.