# Tutorial 1

07 August 2021 14:20

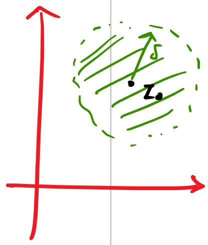
slides can be found at bit.ly/ca-205 under 2020 >> TSC Slides

## Lecture 1

**Definition 1 (Some notation)**

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the $\delta$-neighbourhood of $z_0$, denoted by $B_\delta(z_0)$ is the set

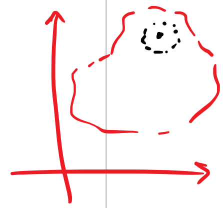
$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$




**Definition 2 (Open sets)**

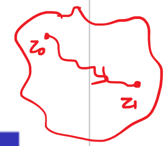
A set $U \subset \mathbb{C}$ is said to be open if:
for *every* $z_0 \in \mathbb{C}$, there exists *some* $\delta > 0$ such that

$$B_\delta(z_0) \subset U.$$



**Definition 3 (Path-connected sets)**

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in $P$ can be joined by a path in $P$. (A continuous function from $[0, 1]$ to $P$.)




Aryaman Maithani | Complex Analysis TSC

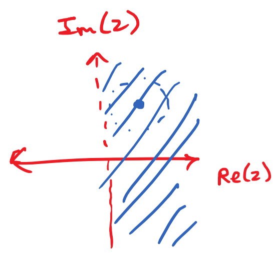
**Examples.**

① $B_\delta(z_0)$ are open for any $z_0 \in \mathbb{C}$ and $\delta > 0$.

② $\mathbb{C}$ is open. $\emptyset$ is open.

③ Strict right half plane $\mathbb{H}$ is open

$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.




④ $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is NOT open.

## Lecture 1

### Definition 4 (Differentiable)

Let $\Omega \subset \mathbb{C}$ be open. Let

$$f : \Omega \to \mathbb{C}$$

be a function. Let $z_0 \in \Omega$. $f$ is said to be *differentiable* at $z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is denoted by $f'(z_0)$.

$f : (a,b) \to \mathbb{R}$

$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.



$\Omega = \mathbb{C}$, $z, z^2, z^n, \ldots$

exp, sin, cos, ... ?

Non-diff: $|z|$, $\bar{z}$, ...

### Definition 5 (Holomorphic)

1 A function $f$ is said to be holomorphic on an open set $\Omega$ if it is differentiable at every $z_0 \in \Omega$.

2 A function $f$ is said to be holomorphic at $z_0$ if it is holomorphic on some neighbourhood of $z_0$.

### Remark 1

A function may be differentiable at $z_0$ but not holomorphic at $z_0$. For example, $f(z) = |z|^2$ is differentiable only at 0. Thus, it is differentiable at 0 but holomorphic nowhere.

For sets, however, there is no difference.

Points: Holo. $\Rightarrow$ Diff but $\not\Leftarrow$

# Notation

From this point on, $\Omega$ be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number $z$ as $z = x + \iota y$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + \iota v(z)$.

## Lecture 2: CR Equations

$$\left[\begin{array}{ccc} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 \\ x+\iota y & \longleftrightarrow & (x, y) \end{array}\right]$$

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose $f$ as

$$f(z) = \underline{u(z)} + \iota v\underline{(z)},$$

where $u, v : \underline{\Omega} \to \mathbb{R}$ are real valued functions. $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$

The idea now is to consider $u$ and $v$ as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for $v$. Now, if we know that $f$ is holomorphic, then we have the following result.

$$\left[ u, v : \Omega^{\subset \mathbb{R}^2} \longrightarrow \mathbb{R} \right]$$

MA 109, 111

$u_x, u_y, v_x, v_y$ make sense.

# Lecture 2: CR Equations

Cauchy–Riemann

## Theorem 1 (CR equations)

Let $f : \Omega \to \mathbb{C}$ be differentiable at a point $z_0 \in \Omega$. Let $z_0 = x_0 + \iota y_0$.
Then, we have

$$① \quad u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, we have

$$② \quad f'(z_0) = u_x(x_0, y_0) + \iota v_x(x_0, y_0).$$

$f' = u_x + i v_x$

Existence of $u_x, u_y, v_x, v_y$ is part of the theorem.

Note the subscript is $x$ for both in the above.
Also note that all the equalities are only at the point $z_0$. In particular, we are only assuming differentiability at $z_0$.

Test $f(z) = z$
$= x + iy$
to see what the signs should be.

## Lecture 2: CR Equations

Converse? What is the converse? Is it true?

Converse

NO!

CR equations hold at the point $(x_0, y_0)$

$\Downarrow$?

$f$ is differentiable at $z_0$?

No. The converse is not true.

An example for you to check is

$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that $u$ and $v$ satisfy the CR equations at $(0, 0)$ but $f$ is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

# Lecture 2: CR Equations

We recall MA ~~105~~ 109 + 111 now.

## Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega\,{}^{\subseteq \mathbb{R}^2} \to \mathbb{R}^2.$$

Recall that $f$ is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if there exits a $2 \times 2$ real matrix $A$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(x_0 + h, y_0 + k) - f(x_0, y_0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

The matrix $A$ was called the *total derivative of $f$ at* $(x_0, y_0)$.

# Lecture 2: CR Equations

**Theorem 2**

If $f$ is (complex) differentiable at a point $z_0 = x_0 + \iota y_0$, then $f$ is real differentiable at $(x_0, y_0)$.

Once again, this is only talking about differentiability at a point.
The converse is again not true.
Take the example $f(z) = \bar{z}$. Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

$u(x,y) = x$ $\quad u_x \equiv 1$

$v(x,y) = -y$ $\quad v_y \equiv -1$

## Lecture 2: CR Equations

CD $\Rightarrow$ CR + RD

CR $\not\Rightarrow$ CD, RD $\not\Rightarrow$ CD

**Theorem 3**

Let $f : \Omega \to \mathbb{C}$ be a function and let $z_0 = x_0 + \iota y_0 \in \Omega$. If the CR equations hold at the point $(x_0, y_0)$ and if $f$ is real differentiable at the point $(x_0, y_0)$, then $f$ is complex differentiable at the point $z_0$.

(CR + RD) $\Rightarrow$ CD

Recall from MA 109, 111:

If $f : \Omega \to \mathbb{R}^2$ is a function s.t.

$f = (u, v)$

$u_x, u_y, v_x, v_y$ are continuous on $\Omega$,

then $f$ is real diff. on $\Omega$.

## Lecture 2: CR Equations

Note : if $f: \Omega \to \mathbb{R}^2$, then $f_x$, $f_y$, etc. are meaningless.

**Definition 7 (Harmonic functions)**

Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function. $u$ is said to be *harmonic* if $u_{xx} + u_{yy} = 0$.

**Proposition 1**

The real and imaginary parts of a holomorphic function are harmonic.

$u_x = v_y \Rightarrow u_{xx} = v_{yx}$ $\quad$ $u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$ $\quad$ but $v_{xy} = v_{yx}$ by assumption of $u, v$ being $C^2$

Suppose $u$ and $v$ are harmonic on $\Omega$. $v$ is said to be a harmonic conjugate of $u$ if $f = u + \iota v$ is holomorphic on $\Omega$.
If $v$ is a harmonic conjugate of $u$, then $-u$ is a harmonic conjugate of $v$.
Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.



Harmonic Conjugate need not exist.

Example. Consider $\Omega = \mathbb{R}^2 - \{(0,0)\}$ and $u : \Omega \to \mathbb{R}$ defined as

$$u(x, y) = \frac{1}{2}\log(x^2 + y^2).$$

If $u$ had a harmonic conjugate $v$, then

$$v_y(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad v_x(x,y) = \frac{-y}{x^2+y^2}.$$

But $\nexists\, v : \Omega \to \mathbb{R}$ s.t.

$$\nabla v = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

(Proof?)

$\oint$ $\quad$ $\oplus$

**Claim 1.** Arbitrary union of open sets is open.

**Proof.** Let $\{U_i : i \in I\}$ be a collection of open sets.

Define $U := \bigcup_{i \in I} U_i$

$= \{x : x \in U_i \text{ for some } i \in I\}$.

IS: $U$ is open.

**Proof.** Let $x \in U$ be arbitrary.

Then, $\exists i_0 \in U$ s.t. $x \in U_{i_0}$.

Since $U_{i_0}$ is open, $\exists \delta > 0$ s.t.

$$B_\delta(x) \subseteq U_{i_0}.$$

But $U_{i_0} \subseteq U$. Thus, $B_\delta(x) \subseteq U$.

Thus, $U$ is open. ∎

**Claim 2.** Finite intersection of open sets is open.

**Proof.** It suffices to prove that intersection of two open sets is open.

$A_1, \ldots, A_n \to$ open

$\Downarrow$

$(A_1 \cap A_2), A_3, \ldots, A_n \to$ open

$\Downarrow$

$A_1 \cap A_2 \cap A_3, \ldots, A_n \to$ open

$\Downarrow$

$\vdots$

$\Downarrow$

$A_1 \cap \ldots \cap A_n \to$ open

Let $U_1$ and $U_2$ be open and $x \in U_1 \cap U_2$.

Let $U_1$ and $U_2$ be open and $x \in U_1 \cap U_2$.

$\therefore U_1 \& U_2$ are open

$\exists \delta_1 > 0$ s.t. $B_{\delta_1}(x) \subseteq U_1$ and

$\exists \delta_2 > 0$ s.t. $B_{\delta_2}(x) \subseteq U_2$.

Pick $\delta := \min(\delta_1, \delta_2) > 0$.

Then, $B_\delta(x) \subseteq B_{\delta_1}(x) \subseteq U_1$ and

$B_\delta(x) \subseteq B_{\delta_2}(x) \subseteq U_2$.

$\therefore B_\delta(x) \subseteq (U_1 \cap U_2)$. ∎

---

"Dual" statements for closed sets.

$U_1, U_2 \to$ open You can say: $U_1 \cup U_2$ and $U_1 \cap U_2$ are open

$U_1, U_2, \dots, U_n \to$ open $\Rightarrow$ $U_1 \cup \cdots \cup U_n$ & $U_1 \cap \cdots \cap U_n$ are open.

$U_1, U_2, U_3, \dots \to$ open $\Rightarrow$ $\bigcup_{i=1}^{\infty} U_i$ is open but $\bigcap_{i=1}^{\infty} U_i$ may not be.

---

$$\mathbb{C} - \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{C} - U_i)$$

closed $\Leftrightarrow$ complement is open.

$\boxed{U_i := B_{1/i}(0).}$

$\bigcap_{i \in \mathbb{N}} U_i = \{0\}$

$\uparrow$ not open.

Tutorial 2

14 August 2021 11:55

# Lecture 3: Power Series

## Definition 8 (Convergence of series)

A series of the form

$(a_n)_{n\geq 0} \to$ sequence in $\mathbb{C}$

$$\to \sum_{n=0}^{\infty} a_n$$

of complex numbers is said to converge if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges (to a finite complex number).

The sequence of partial sums is just the following sequence:

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots.$$

"Divergent" is simply used to mean "not convergent."
Check that $\sum(-1)^n$ and $\sum n$ both diverge.

Aryaman Maithani Complex Analysis TSC

$a_n = (-1)^n$ for $n \geq 0$

$1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$

partial sums $1, 0, 1, 0, 1, 0, 1, 0 \ldots$

↳ Diverges

Similarly

$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ diverges

$1 + 2 + 3 + 4 + \cdots$ diverges

# Lecture 3: Power Series

## Definition 9 (limsup)

Given a sequence $(x_n)$ of real numbers, we may define a new sequence $(y_n)$ as

$$y_n = \sup\{x_m : m \geq n\}.$$

The limit of this sequence always exists and we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

$x_0, x_1, x_2, x_3, x_4, \ldots$

$\sup = y_4$

↳ $\sup = y_3$

↳ $\sup = y_2$

↳ $\sup = y_1$

## Remark 2

Each $y_n$ might be $\infty$. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.

Each $y_n$ might be $\infty$. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.
If $\lim_{n\to\infty} x_n$ itself exists, then it equals the lim sup as well.



I know: $\lim_{n\to\infty} n^{1/n} = 1$.

Thus, $\limsup_{n\to\infty} n^{1/n} = 1$.

$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

$\Downarrow$

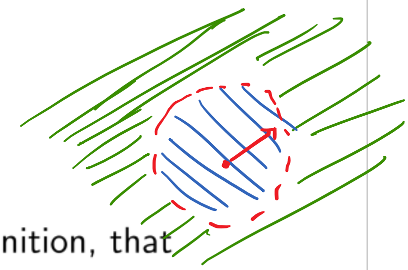
$\limsup_{n\to\infty} \frac{\sin(n)}{n} = 0$

lim sup of a real sequence always exists!

# Lecture 3: Power Series

We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n (z - z_0)^n} \quad (*)$$



where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

What is the radius of convergence, though? (The definition, that is.)

**Theorem 4 (Radius of convergence)**

Given any power series as $(*)$, there exists $R \in [0, \infty]$ such that

1. $(*)$ converges for any $z$ with $|z - z_0| < R$, and → absolutely
2. $(*)$ diverges for any $z$ with $|z - z_0| > R$.

This $R$ is called the radius of convergence.

→ on the circle, behaviour is weird.
↳ may converge for some
→ may diverge for others

Note the brackets.



$\sum_{n=1}^{\infty} \frac{z^n}{n} \rightsquigarrow$ RoC $= 1$

at $-1$ : converges

at $1$ : diverges

$\sum z^n \rightsquigarrow$ RoC $= 1$

↳ diverges for all $z$ with $|z| = 1$

## Lecture 3: Power Series

We would now like to be able to calculate the radius of convergence.

**Theorem 5 (Root test)**

Let $(*)$ be as earlier. Define

$$\boxed{\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.}$$

ALWAYS WORKS.

Then, $\boxed{R = \alpha^{-1}}$ is the radius of convergence.

This test *always works.* We had no assumptions of any kind on $(*)$.
Note that $^{-1}$.
If $\alpha = 0$, then $R = \infty$ and vice-versa.

$\sum_{n=1}^{\infty} \frac{z^n}{n}$. $\alpha = \limsup \left(\frac{1}{n}\right)^{1/n} \overset{?}{=}$



Note: limit rules of $+, \cdot, (\;)^{-1}$ do not apply to limsup.

$\limsup (a_n + b_n) \leq \limsup (a_n) + \limsup (b_n)$

We know $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \frac{1}{1} = 1$. $\therefore \limsup \left(\frac{1}{n}\right)^{1/n} = 1$

## Lecture 3: Power Series

We have another test. This is simpler (to calculate) but mightn't always work.

**Theorem 6 (Ratio test)**

Let $(*)$ be as earlier.
Assume that the limit

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$\rightarrow$ can apply to $\sum \frac{z^n}{n}$ or $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

exists. (Possibly as $\infty$.)
Then, $R$ is the radius of convergence.

$\frac{\left(\frac{1}{n!}\right)}{\frac{1}{(n+1)!}} = n+1 \longrightarrow \infty$.

Note that here we assume that the limit does exist. This may not always be true.
Note that I'm not taking any inverse here but also note the way the ratio is taken. We have $a_n/a_{n+1}$.

Note that I'm not taking any inverse here but also note the way the ratio is taken. We have $a_n/a_{n+1}$.



Take $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Then, $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$.

ROC = 1

converges at 1

does not converge at 1.

## Lecture 3: Power Series

Differentiability of power series is what one should expect.

**Theorem 7 (Differentiability)**

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the open disc of radius $R$, let $f(z)$ denote this sum. Then, on this disc, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Boundary: NO COMMENTS.

Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

$\limsup |a_n|^{1/n} = \limsup n^{1/n} |a_n|^{1/n}$



Digression. Let $(x_n)_{n\geq 0}$ be a real sequence.

Consider

$E := \{$ limits of all possible convergent subsequences $\} \subseteq \mathbb{R} \cup \{\pm\infty\}$.

Then, $\limsup_{n\to\infty} x_n = \sup E$.

## Lecture 4: Exponential function

I shall just recall the facts from the lecture.

**Definition 10 (Exponential function)**

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

by ratio test

converges on all of $\mathbb{C}$. This sum is denoted by $\exp(z)$.

**Theorem 8 (Facts)**

1. $\exp'(z) = \exp(z)$,
2. $\exp'(bz) = b\exp(bz)$, for $b \in \mathbb{C}$,
3. $\exp(z) \cdot \exp(-z) = 1$ for all $z \in \mathbb{C}$,
4. $\exp(z)$ is always nonzero.

# Lecture 4: Exponential function

Now, we some "converse" facts.

**Theorem 9 (Characterisations)**

1. If $f'(z) = bf(z)$, then $f(z) = a\exp(bz)$ for some $a, b \in \mathbb{C}$,
2. If $f' = f$ and $f(0) = 1$, then $f(z) = \exp(z)$.

**Theorem 10 (Final fact)**

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

$\exp : \underset{+}{\mathbb{C}} \longrightarrow \underset{\cdot}{\mathbb{C}^\times} = \mathbb{C} \setminus \{0\}$ is a group homomorphism.

## Lecture 4: Exponential function

**Definition 11 (Domain)**

A subset $\Omega \subset \mathbb{C}$ is said to be a *domain* if it is open and path-connected.

We had one very nice result on the zeroes of a analytic functions.

**Theorem 11 (Zeroes are isolated)**

Let $\Omega$ be a domain and $f : \Omega \to \mathbb{C}$ be a non-constant analytic function. Let $z_0 \in \Omega$ be such that $f(z_0) = 0$. Then, there exists $\delta > 0$ such that $f$ has no other zero in $B_\delta(z_0)$.


The above is saying that around every zero of $f$, we can draw a (sufficiently small) circle such that $f$ has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete*.

## Logarithm

We discuss logarithm a bit.

**Definition 14 (Branch of the logarithm)**

Let $\Omega \subset \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\exp(f(z)) = z, \qquad \text{for all } z \in \Omega.$$

Then, $f$ is called a *branch of the logarithm.*

**Theorem 21 (Uniqueness of branches)**

Assume that $f, g : \Omega \to \mathbb{C}$ are two branches of the logarithm. Then, $f - g$ is a constant function. Moreover, this constant is an integer multiple of $2\pi\iota$.

The last theorem also assumed that $\Omega$ is a domain.

Branch of log may not exit on a given domain.



no branch on $\mathbb{C}$. Also, there is no branch on $\mathbb{C}^\times$.