MA 205: Complex Analysis Extra questions

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Contents

0	Notations	2
1	Topology	3
2	Cauchy Riemann Equations	4
3	Series	5
4	Properties of holomorphic functions	8

 \S **0** Notations

§0. Notations

- 1. $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers.
- 2. \mathbb{Z} is the set of integers.
- 3. \mathbb{Q} is the set of rational numbers.
- 4. \mathbb{R} is the set of real numbers.
- 5. $A \subset B$ is read as "A is a subset of B." In particular, note that $A \subset A$ is true for any set A.
- 6. $A \subsetneq B$ is read "A is a proper subset of B."
- 7. \supset and \supsetneq are defined similarly.
- 8. Given a function $f: X \to Y, A \subset X, B \subset Y$, we define

$$f(A) = \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y,$$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X.$$

(Note that this f^{-1} is different from the inverse of a function. In particular, this is always defined, even if f is not bijective. However, the f and f^{-1} above need not be "inverses.")

9. A *domain*, as a subset of $\mathbb C$ will always refer to a set which is open and path connected.

(Note that this is different from domain of a function.)

 $\S 1$ Topology 3

§1. Topology

1. Is the interval (0,1) open as a subset of \mathbb{C} ?

HIDDEN: No

2. Is the interval (0,1) closed as a subset of \mathbb{C} ?

HIDDEN: No

- 3. Consider the following four properties that a subset of $\mathbb C$ can have:
 - (a) Open
 - (b) Closed
 - (c) Bounded
 - (d) Path connected

Thus, we can classify all the subsets of \mathbb{C} into 2^4 classes on the basis of what properties they have (and what they don't).

Give an example of each or a proof that some certain class cannot have anything. You may assume that \varnothing and $\mathbb C$ are the only subsets of $\mathbb C$ which are both open and closed.

- 4. Let $U \subset \mathbb{C}$ be open and nonempty. Show that U is not countable.
- 5. Let $U\subset \mathbb{C}$ be open and K be countably open. Give examples to show that $U\setminus K$ may or not be open.

§2. Cauchy Riemann Equations

1. Consider the function $f:\mathbb{C}\to\mathbb{C}$ defined as

$$f(z) = \bar{z}$$
.

Show that f is continuous at each point.

Show that f is differentiable at no point.

(This has given us a very easy example of a function which is continuous everywhere but differentiable nowhere. On the contrary, one has to put a lot more effort to construct an example in the case of real analysis.)

2. Show that the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x,y) = (x, -y)$$

is differentiable in the sense that you saw in MA 105. (That is, its total derivative exists at every point.)

Compare this with the previous question.

3. Let Ω be open (and not necessarily path-connected).

Let $f: \Omega \to \mathbb{C}$ be holomorphic such that f'(z) = 0 for all $z \in \Omega$.

Show that it is *not* necessary that f is constant.

Show that if Ω is also assumed to be path-connected (that is, Ω is a domain), then it is necessary that f is constant.

4. Let Ω be a domain and $f:\Omega\to\mathbb{C}$ be holomorphic.

Suppose

$$f(z) \in \mathbb{R}$$
 for all $z \in \Omega$.

Show that f is constant. (That is, if a complex differentiable function takes only real values, then it must be constant on path-connected sets.)

5. Let Ω be a domain and $f:\Omega\to\mathbb{C}$ be holomorphic.

Suppose that |f| is constant. Show that f is constant.

§3. Series

1. (Cauchy criterion for series.) "Recall" Cauchy criterion for convergence from MA 105. (Prove or assume that the analogous thing holds for complex sequences as well.)

Let (a_n) be a sequence of complex numbers. Show that $\sum_{n=1}^{\infty} a_n$ converges iff for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=n}^{m} a_n \right| < \epsilon, \quad \text{ for all } m \ge n \ge N.$$

- 2. Let (a_n) be a sequence of complex numbers such that $\sum |a_n|$ converges. Use the above Cauchy criteria to show that $\sum a_n$ converges.
- 3. Let (a_n) and (b_n) be complex sequences such that $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$. Show that if $\sum |b_n|$ converges, then so does $\sum |a_n|$ and hence, so does $\sum a_n$. Show that you can weaken the "for all $n \in \mathbb{N}$ " condition to "for all n sufficiently large." (Formulating what we mean by "sufficiently large" is part of the exercise.)
- 4. Use the above to show that

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

converges for all $z \in \mathbb{C}$ satisfying |z| = 1.

5. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

HIDDEN: Compare it with the sequence 1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, ...

- 6. Let (a_n) be a sequence of real numbers and (b_n) a sequence of complex numbers satisfying
 - (a) (a_n) is monotonic,
 - (b) $\lim_{n\to\infty} a_n = 0$,
 - (c) there exists $M \ge 0$ such that

$$\left| \sum_{n=1}^{N} b_n \right| \le M$$

for every $N \in \mathbb{N}$.

§3 Series 6

Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Here's an outline of what you can do:

(a) Define the partial sums $S_n = \sum_{k=1}^n a_k b_k$ and $B_n = \sum_{k=1}^n b_k$.

Show that

$$S_n = a_n B_n + \sum_{k=1}^{n-1} B_k (a_k - a_{k+1}).$$

(This is called summation by parts.)

- (b) Note that B_n is bounded by M and $a_n \to 0$. Conclude that the first term $\to 0$ as $n \to \infty$.
- (c) Note that give any k, we have $|B_k(a_k a_{k+1})| \leq M|a_k a_{k+1}|$.
- (d) Using (a_n) is monotonic, conclude that

$$\sum_{k=1}^{n-1} |a_k - a_{k+1}| = \sum_{k=1}^{n-1} |a_1 - a_n|.$$

(e) Conclude that $\lim_{n\to\infty} S_n$ exists.

The above is called **Dirichlet's test**.

7. Let $z \in \mathbb{C}$ be such that |z| = 1 and $z \neq 1$. Define the sequences (a_n) and (b_n) as

$$a_n := \frac{1}{n}, \quad b_n := z^n.$$

Show that (a_n) and (b_n) satisfy the hypothesis of Dirichlet's test. Conclude that

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges.

8. Compute the radius of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

These should come out to be 1. By the previous questions, conclude that the first converges everywhere on the boundary of the disc except at 1. However,

§**3 Series** 7

the second one converges everywhere on the boundary. Do the same for the power series

$$\sum_{n=1}^{\infty} z^n.$$

HIDDEN: You should get that it converges nowhere on the boundary.

(Note that these series are (more or less) derivatives and anti-derivatives of each other on the *open* disc. However, they show very different behaviour on the boundary of the disc.)

§4. Properties of holomorphic functions

1. Let $\mathbb{H} = \{z \in \mathbb{C} : \Re z > 0\}$ be the open half right plane. Construct a non-constant holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f\left(\frac{1}{n}\right) = 0, \quad \text{ for all } n \in \mathbb{N}.$$

(Does this contradict what we saw in slides? Why not?)

2. Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that

$$f\left(\frac{1}{n}\right) = 0, \quad \text{ for all } n \in \mathbb{N}.$$

Show that f is constant (and that the constant is 0). Compare this with the previous question.

3. Suppose that the domains in the previous question was replaced by an arbitrary domain Ω such that $\{n^{-1}:n\in\mathbb{N}\}\subset\Omega$. Characterise Ω precisely such the above f(1/n)=0 condition would ensure that f is constant. (That is, come up with a rule such that if Ω follows that rule, then f has to be constant and that if Ω does not follow the rule, then f may be

HIDDEN: The rule should be (equivalent to): $0 \in \Omega$

non-constant.)

4. Let $f,g:\mathbb{C}\to\mathbb{C}$ be holomorphic functions which are nonzero everywhere. Suppose that f and g satisfy

$$\left(\frac{f'}{f}\right)\left(\frac{1}{n}\right) = \left(\frac{g'}{g}\right)\left(\frac{1}{n}\right), \quad \text{ for all } n \in \mathbb{N}.$$

(The LHS means that the function f'/f is evaluated at 1/n and similarly foe the RHS.)

Find a simpler relation between f and g. (Yes, "simpler" is subjective.)