

# MA 205: Complex Analysis

## Tutorial Solutions

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### §0. Notations

1. Given  $z \in \mathbb{C}$ ,  $\Re z$  and  $\Im z$  will denote the real and imaginary parts of  $z$ , respectively.
2. Given  $z \in \mathbb{C}$ ,  $\bar{z}$  will denote the complex conjugate of  $z$ .
3. Given  $z \in \mathbb{C}$ ,  $|z|$  will denote the modulus of  $z$ , defined as  $\sqrt{z\bar{z}}$  or  $\sqrt{(\Re z)^2 + (\Im z)^2}$ .

## §1. Tutorial 1

25th August, 2020

**Notation:** The set  $\mathbb{C}[x]$  is the set of all polynomials (with indeterminate  $x$ ) with complex coefficients. Similarly,  $\mathbb{R}[x]$  is defined.

1. Show that complex polynomial of degree  $n$  has exactly  $n$  roots. (Assuming fundamental theorem of algebra.)

Remark (my own): The above is counting the roots *with* multiplicity. That is, if  $f(z) = (z - \iota)^2(z - 2)$ , then  $\iota$  is counted twice and 2 once.

*Solution.* Let  $f(x) \in \mathbb{C}[x]$  be a polynomial of degree  $n$ . We prove this via induction on  $n$ .

$n = 1$ . Then,  $f(x) = a_0 + a_1x$  for some  $a_0, a_1 \in \mathbb{C}$  and  $a_1 \neq 0$ .

Note that

$$\begin{aligned} f(x) &= 0 \\ \iff a_0 + a_1x &= 0 \\ \iff a_1x &= -a_0 \\ \iff x &= -\frac{a_0}{a_1}. \end{aligned}$$

Thus,  $f(x)$  has exactly 1 root.

Let us assume that whenever  $g(x) \in \mathbb{C}[x]$  is a polynomial of degree  $n$ , then  $g(x)$  has exactly  $n$  roots. (Counted with multiplicity.)

Let  $f(x) \in \mathbb{C}[x]$  be a polynomial of degree  $n + 1$ . By FTA, there exists a root  $x_0 \in \mathbb{C}$ . Thus, we can write

$$f(x) = (x - x_0)g(x)$$

for some polynomial  $g(x) \in \mathbb{C}[x]$  of degree  $n$ . Moreover, note that

$$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$

By induction, the latter is possible for exactly  $n$  values of  $x$ . Thus, in total,  $f(x)$  has  $n + 1$  roots. (Both counts are with multiplicity.)  $\square$

2. Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n, \quad a_i \in \mathbb{R}$$

then there are non-constant real polynomials  $g$  and  $h$  such that  $f(x) = g(x)h(x)$  if  $n \geq 3$ .

Remark (my own):  $a_n \neq 0$ , of course.

*Solution.* Let  $f(x) \in \mathbb{R}[x]$  with degree  $\geq 3$  as above.

If  $f(x)$  has a real root, then we are done by factoring as in the earlier question.

Thus, let us assume that  $f(x) = 0$  has no real solution.

We may view  $f(x) \in \mathbb{C}[x]$ . Now, using FTA, we know that  $f(x)$  has a complex root  $x_0 \in \mathbb{C}$ . By assumption, we must have  $x_0 \notin \mathbb{R}$  or that  $x_0 \neq \overline{x_0}$ .

**Claim.**  $f(\overline{x_0}) = 0$ .

*Proof.* Note that

$$\begin{aligned}
 f(\overline{x_0}) &= a_0 + a_1 \overline{x_0} + \cdots + a_n (\overline{x_0})^n \\
 &= a_0 + a_1 \overline{x_0} + \cdots + a_n \overline{x_0^n} \\
 &= \overline{a_0} + \overline{a_1} \overline{x_0} + \cdots + \overline{a_n} \overline{x_0^n} \\
 &= \overline{f(x_0)} \\
 &= \overline{0} \\
 &= 0
 \end{aligned}
 \quad \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \because \overline{z^n} = \overline{z}^n \\ \because a_i \in \mathbb{R} \text{ and so, } a_i = \overline{a_i} \\ \overline{z_1 z_2 + z_3} = \overline{z_1} \overline{z_2} + \overline{z_3} \end{array}$$

□

Define  $g(x) = (x - x_0)(x - \overline{x_0})$ . A priori, this is a polynomial in  $\mathbb{C}[x]$ . However, upon multiplication, we see that the polynomial is actually an element of  $\mathbb{R}[x]$ . Indeed, we have

$$(x - x_0)(x - \overline{x_0}) = (x^2 - (2\Re x_0)x + |x_0|^2) \in \mathbb{R}[x].$$

By our claim, we see that  $g(x)$  divides  $f(x)$  in  $\mathbb{C}[x]$ . (Since  $x_0$  and  $\overline{x_0}$  are distinct, the polynomials  $x - x_0$  and  $x - \overline{x_0}$  are “coprime” and thus, if they individually divide  $f(x)$ , then their product must too.)

Thus,

$$f(x) = g(x)h(x)$$

for some  $h(x) \in \mathbb{C}[x]$ . However, since  $f(x)$  and  $g(x)$  are both real polynomials, so is  $h(x)$ . (Why?)

Thus, we get that

$$f(x) = g(x)h(x)$$

for real polynomials  $g(x)$  and  $h(x)$ . Moreover, note that  $\deg g(x) = 2$  and  $\deg h(x) = n - 2 \geq 1$ . Thus, both are non-constant. □

3. Show that if  $U$  is a path connected open set in  $\mathbb{C}$ , so is  $U$  minus any finite set.

*Solution.* We will first prove the following claim:

**Claim:** Let  $U \subset \mathbb{C}$  be open and  $w \in U$ . Then,  $U \setminus \{w\}$  is open.

*Proof.* Let  $z_0 \in U \setminus \{w\}$  be arbitrary. Since  $U$  was open, there exists  $\delta_1 > 0$  such that

$$B_{\delta_1}(z_0) \subset U.$$

Since  $z_0 \neq w$ , we have that  $\delta_2 := |z_0 - w| > 0$ .

Choose  $\delta := \min\{\delta_1, \delta_2\}$ . Clearly,  $\delta > 0$ . Moreover, we have

$$w \notin B_{\delta_2}(z_0) \supset B_{\delta}(z_0)$$

and thus,  $w \notin B_{\delta}(z_0)$ . Also,

$$B_{\delta}(z_0) \subset B_{\delta_1}(z_0) \subset U.$$

Thus, we get that

$$B_{\delta}(z_0) \subset U \setminus \{w\},$$

proving that  $U \setminus \{w\}$  is open. □

By the above proof, we see that removing one point from an open set keeps it open. Thus, if we show that removing one point from an open path-connected set leaves it path-connected, then we are done since we can induct to get any other **finite**<sup>1</sup> set.

Thus, we now show that if  $U$  is open and path-connected, so is  $U \setminus \{w\}$ . (Where  $w \in U$  is any arbitrary element.)

Let  $z_0, z_1 \in U \setminus \{w\}$ . We wish to show that there is a path in  $U \setminus \{w\}$  connecting  $z_0$  to  $z_1$ .

Since  $U$  was path-connected to begin with, there exists a path  $\sigma : [0, 1] \rightarrow U$  such that

$$\sigma(0) = z_0, \quad \sigma(1) = z_1.$$

If  $\sigma(x) \neq w$  for any  $x \in [0, 1]$ , then we are done since  $\sigma$  is a path in  $U \setminus \{w\}$  as well.

Suppose that this is not the case.

Then, we choose a  $\delta > 0$  such that the *closed* ball

$$B := \{z \in \mathbb{C} : |z - w| \leq \delta\}$$

has the following properties:

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<sup>1</sup>Finiteness is important. Induction cannot prove this result for a countably infinite set.

- (a)  $z_0 \notin B$ ,
- (b)  $z_1 \notin B$ ,
- (c)  $B \subset U$ .

(Why must such a  $\delta$  exist? There exists a  $\delta_1$  for which we get the first two properties since  $z_0$  and  $z_1$  are distinct from  $w$ . For the last property, let  $\delta_2$  be any such that  $B_{\delta_2}(w) \subset U$ , which exists since  $U$  is open. Then, consider  $\delta_2/2$ . The *closed* ball of this radius must again be completely within  $U$ . Take the minimum of  $\delta_1$  and  $\delta_2/2$ .)

Note that

$$\sigma^{-1}(B) = \{x \in [0, 1] : \sigma(x) \in B\}$$

is nonempty since  $w \in B$  and  $\sigma(c) = w$  for some  $c \in [0, 1]$ , by our assumption. Moreover,  $\sigma^{-1}(B)$  must be closed. (Why?) Since it is a subset of  $[0, 1]$ , it is clearly bounded. Define

$$s := \inf \sigma^{-1}(B), \quad t := \sup \sigma^{-1}(B).$$

Since the set is closed, both  $s$  and  $t$  are elements of  $\sigma^{-1}(B)$ . Note that  $\sigma(0) \notin B$  and  $\sigma(1) \notin B$  and thus,

$$0 < s < t < 1.$$

(Why is the inequality  $s < t$  strict?)

Note that  $\sigma(s)$  and  $\sigma(t)$  must lie on the circumference of  $B$ . (Why?) (This also shows why  $s < t$ .)

Now consider the path  $\sigma' : [0, 1] \rightarrow U$  defined as follows:

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in [0, s] \cup [t, 1] \\ \gamma(x) & \text{if } x \in [s, t], \end{cases}$$

where  $\gamma : [s, t] \rightarrow B$  is the path which is the arc joining  $\sigma(s)$  to  $\sigma(t)$ . (Note that  $\sigma(s) = \sigma(t)$  is possible in which case, it's the constant path.)

Clearly,  $\sigma'$  avoids  $w$  and is continuous. (Why?)

Moreover,  $\sigma'(0) = \sigma(0) = z_0$  and  $\sigma'(1) = \sigma(1) = z_1$  and thus,  $\sigma'$  is a path from  $z_0$  to  $z_1$  in  $U \setminus \{w\}$ , showing that  $U \setminus \{w\}$  is path-connected.  $\square$

4. Check for real differentiability and holomorphicity:

- (a)  $f(z) = c$ ,
- (b)  $f(z) = z$ ,

- (c)  $f(z) = z^n, n \in \mathbb{Z}$ ,
- (d)  $f(z) = \Re z$ ,
- (e)  $f(z) = |z|$ ,
- (f)  $f(z) = |z|^2$ ,
- (g)  $f(z) = \bar{z}$ ,
- (h)  $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

*Solution.* Not going to do all.

- (a) Real differentiable and holomorphic, both.
- (b) Real differentiable and holomorphic, both.
- (c) For  $n \geq 0$  :

Real differentiable and holomorphic, both. Let us see why.

As we know, holomorphicity implies real differentiability, so we only check that  $f$  is holomorphic on  $\mathbb{C}$ .

Let  $z_0 \in \mathbb{C}$  be arbitrary. We show that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

This is clear because for  $z_0 \neq z$ , we have

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

The limit  $z \rightarrow z_0$  of the RHS clearly exists.

$n < 0$  : The function is now defined on  $\mathbb{C} \setminus \{0\}$ . It is still holomorphic and real differentiable everywhere (in its domain!).

To see this, we just use the quotient rule and appeal to the previous case of  $n \geq 0$ .

- (d) Real differentiable but not holomorphic. Note that  $f$  can be written as

$$f(x + iy) = x + 0i.$$

Thus,  $u(x, y) = x$  and  $v(x, y) = 0$ .

This is clearly real differentiable everywhere since all the partial derivatives

exist everywhere and are continuous.

However, we show that  $f$  is not complex differentiable at any point. Thus, it is not holomorphic.

This is easy because one sees that  $u_x(x_0, y_0) = 1$  and  $v_y(x_0, y_0) = 0$  for all  $(x_0, y_0) \in \mathbb{R}^2$  and thus, the CR equations don't hold.

- (e)  $|z|$  is real differentiable everywhere except 0 and complex differentiable nowhere. Breaking the function as earlier, we have

$$u(x, y) = \sqrt{x^2 + y^2}, \quad v(x, y) = 0.$$

On  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , all partial derivatives exist and are continuous. At  $(0, 0)$ ,  $u_x$  and  $u_y$  fail to exist.

This clearly shows that  $f$  is not complex differentiable at  $0 \in \mathbb{C}$  since it is not even real differentiable there.

However, we see that  $v_y = 0 = v_x$  everywhere else but at least one of  $u_x$  or  $u_y$  is nonzero on  $\mathbb{R}^2 \setminus \{(0, 0)\}$  and thus, the CR equations prevent  $f$  from being complex differentiable anywhere else.

- (f) Real differentiable everywhere.  
Complex differentiable precisely at 0.  
Holomorphic nowhere.

Same steps as above.

- (g) Real differentiable everywhere. Complex differentiable nowhere. Use CR equations again.
- (h)  $f$  is real differentiable precisely on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ .  
However, it is not complex differentiable anywhere.

Breaking as earlier, we get

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad v(x, y) = \frac{2xy}{x^2 + y^2},$$

for  $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  and

$$u(0, 0) = 0 = v(0, 0).$$

Note that  $u$  and  $v$  aren't even continuous at  $(0, 0)$ . Thus, neither is  $f$ . Hence,  $f$  is neither real nor complex differentiable at  $(0, 0)$ .

However, at all other points, all partial derivatives exist and are continuous. Thus,  $f$  is real differentiable at all those points. However, computing  $u_x, u_y, v_x, v_y$  explicitly shows that the CR equations are not satisfied anywhere. Thus,  $f$  is not complex differentiable anywhere.  $\square$

5. Show that the CR equations take the form

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

in polar coordinates.

*Solution.* We shall follow the same idea as in the slides. We first write

$$f(r, \theta) = f(re^{i\theta}) = u(r, \theta) + i v(r, \theta).$$

Suppose that  $f$  is differentiable at  $z_0 = r_0 e^{i\theta_0} \neq 0$ . (Note that it wouldn't make sense to talk at 0 since there's a  $r^{-1}$  factor in the question anyway.)

Thus, we know that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We shall calculate it in two ways:

(a) Fix  $\theta = \theta_0$  and let  $r \rightarrow r_0$ . Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} + i \frac{v(r, \theta_0) - v(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} \right\} \\ &= e^{-i\theta_0} \lim_{r \rightarrow r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + i \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0} \right\} \\ &= e^{-i\theta_0} (u_r(r_0, \theta_0) + i v_r(r_0, \theta_0)). \end{aligned} \quad (*)$$

(b) Fix  $r = r_0$  and let  $\theta \rightarrow \theta_0$ . Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{\theta \rightarrow \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} \right\} \\ &= \frac{1}{r_0} \lim_{\theta \rightarrow \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \right\} \end{aligned} \quad (**)$$

We concentrate on the first term of the limit. Note that

$$\begin{aligned} &\lim_{\theta \rightarrow \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \\ &= \lim_{\theta \rightarrow \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}. \end{aligned}$$



In the product, the first term is clearly  $u_\theta(r_0, \theta_0)$ , after taking the limit. The second term can be calculated to be

$$\frac{1}{\iota e^{\iota\theta_0}}.$$

(How? Write  $e^{\iota\theta}$  in terms of  $\cos$  and  $\sin$  and differentiate those and put it back.)

Of course, a similar argument goes through for the  $v$  term as well.

Thus, we get that  $(**)$  transforms to

$$f'(z_0) = \frac{e^{-\iota\theta_0}}{r_0} (\iota u_\theta(r_0, \theta_0) + v_\theta(r_0, \theta_0)).$$

Equating the above with  $(*)$ , cancelling  $e^{-\iota\theta_0}$ , and comparing the real and imaginary parts, we get

$$u_r(r_0, \theta_0) = \frac{1}{r_0} v_\theta(r_0, \theta_0), \quad v_r(r_0, \theta_0) = -\frac{1}{r_0} u_\theta(r_0, \theta_0),$$

as desired. □

## §2. Tutorial 2

1st September, 2020

1. If  $u(X, Y)$  and  $v(X, Y)$  are harmonic conjugates of each other, show that they are constant functions.

Remark (my own): This is true iff  $u$  and  $v$  are defined on domains, that is, open and path-connected sets.

*Solution.* Since  $v$  is a harmonic conjugate of  $u$ , we get that

$$u_X = v_Y, \quad u_Y = -v_X.$$

On the other hand, since  $u$  is a harmonic conjugate of  $v$ , we get that

$$v_X = u_Y, \quad v_Y = -u_X.$$

(Note that the equalities mean that they're true for every  $(X_0, Y_0)$  in the domain.)  
Thus, we get that

$$u_X = u_Y = v_X = v_Y \equiv 0,$$

identically.

Since the domain is connected, this implies that  $u$  and  $v$  are constant.  $\square$

2. Show that  $u = XY - 3X^2Y - Y^3$  is harmonic and find its harmonic conjugate.

*Solution.*

**Smart way:** If we can show that the above function is the real (or imaginary) part of a holomorphic function  $f$ , then we have shown that  $u$  is harmonic.

Writing  $Z = X + iY$ , it is not too tough to see that the above is the **imaginary** part of  $\frac{1}{2}Z^2 + Z^3$ . Since

$$f(Z) = \frac{1}{2}Z^2 + Z^3$$

is holomorphic on  $\mathbb{C}$ , this gives us that  $u$  is harmonic.

This also shows a harmonic conjugate of  $u$  is

$$v(X, Y) = -\Re f(Z) = \frac{1}{2}(Y^2 - X^2) + 3XY^2 - X^3.$$

(Note the negative sign! If we had gotten  $u$  as the *real* part of a holomorphic function, then for finding harmonic conjugate, we would've simply taken the imaginary part *without* the negative sign.)

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**Laborious way:** This is the way to do it if observing is difficult.  
First, we show that  $u$  is harmonic by manual calculation. Note that

$$u_{XX}(X_0, Y_0) = 6Y_0 \text{ and } u_{YY}(X_0, Y_0) = -6Y_0.$$

Thus,  $u_{XX} + u_{YY} \equiv 0$  and  $u$  is indeed harmonic.

To find its harmonic conjugate, we perform the procedure as given in slides.  
Note that  $u_X = v_Y$ . Here, we get  $u_X = Y + 6XY = v_Y$ .  
Integrating with respect to  $Y$  gives us

$$v = \frac{1}{2}Y^2 + 3XY^2 + g(X)$$

for some function  $g$ . Then, we need  $v_X = -u_Y$ . Computing each individually, we get

$$3Y^2 + g'(X) = -X - 3X^2 + 3Y^2.$$

Thus, up to a constant, we get

$$g(X) = -\frac{1}{2}X^2 - X^3.$$

Finally, this gives

$$v = \frac{1}{2}Y^2 + 3XY^2 - \frac{1}{2}X^2 - X^3.$$

□

3. Find the radius of convergence of the following power series:

(a)  $\sum_{n=0}^{\infty} nz^n,$

(b)  $\sum_{p \text{ prime}} z^p,$

(c)  $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$

*Solution.* We shall be using the root test in the first two cases and ratio test in the third.

One thing to recall is that if the limit  $\lim_{n \rightarrow \infty} a_n$  exists, then  $\limsup_{n \rightarrow \infty} a_n$  is equal

to that limit. This will be helpful in the first and third parts since the limits will themselves exist.

Moreover, we recall that if

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|},$$

then the radius of convergence  $R$  is given by

$$R = \alpha^{-1}.$$

(The case  $\alpha = 0$  corresponds to  $R = \infty$  and vice-versa.)

Similar analysis holds for

$$\alpha = \lim_{n \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right|.$$

(Here, however, note that I need the existence of  $\alpha$ . In the case of  $\limsup$ , that was always guaranteed.)

(a) Note that we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

(MA 105 Tutorial Sheet 1, Question 2 (iv))

Thus, we also have

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

and thus,

$$R = \alpha^{-1} = \boxed{1}.$$

(b) Note that first we can rewrite the series in the form

$$\sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n := \begin{cases} 0 & n \text{ is not a prime,} \\ 1 & n \text{ is a prime.} \end{cases}$$

For this, we clearly have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} 1 = 1.$$

(To see this, note that there are infinitely many primes and thus, given any  $n$ , there exists  $m \geq n$  such that  $a_m = 1$ .)

Thus, as before, the radius of convergence is 1.

(c) Here, we have

$$a_n = \frac{n!}{n^n}.$$

Thus, we get

$$\begin{aligned} \alpha &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} (n+1) \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^{-n} \\ &= e^{-1}. \end{aligned}$$

Thus, the limit actually exists and we get

$$R = \alpha^{-1} = \boxed{e}. \quad \square$$

4. Show that  $L > 1$  in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

*Solution.* Consider the sequence

$$\frac{1}{1^3}, \frac{1}{1^2}, \frac{1}{2^3}, \frac{1}{2^2}, \dots, \frac{1}{n^3}, \frac{1}{n^2}, \dots$$

That is, let  $(a_n)$  be the sequence defined by

$$a_{2n} = \frac{1}{n^2}, \quad a_{2n-1} = \frac{1}{n^3}.$$

Note that  $\sum a_n$  converges, since  $\sum n^{-2}$  and  $\sum n^{-3}$  converge. (This can be checked via the integral test.)

On the other hand, note that that

$$L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \geq \limsup_{n \rightarrow \infty} \left| \frac{a_{2n}}{a_{2n-1}} \right| = \limsup_{n \rightarrow \infty} n = \infty.$$

Thus,  $L = \infty$ , clearly  $> 1$ .

(So, not only did we show that  $L > 1$  doesn't imply convergence but even that  $L = \infty$  is not good enough.)  $\square$

5. Construct a infinitely differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

*Solution.* Recall the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  from the lectures given as

$$g(x) := \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0. \end{cases}$$

As we saw, this is an infinitely differentiable function. Now, consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$f(x) := g(x)g(1-x).$$

Clearly,  $f$  is infinitely differentiable, being the product of two such functions. Moreover,  $f(x) = 0$  if  $x \leq 0$  or  $x \geq 1$ . In other words,  $f$  is 0 outside the bounded set

$$(0, 1).$$

However,  $f$  is non-zero since

$$f\left(\frac{1}{2}\right) = \left(g\left(\frac{1}{2}\right)\right)^2 = e^{-4} \neq 0.$$

On the other hand, let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function which is zero outside some bounded set  $K$ . We show that  $g$  is zero everywhere.

Since  $K$  is bounded, there exists  $M > 0$  such that

$$|z| \leq M \quad \text{for all } z \in K.$$

Thus, choosing the point  $z_0 = M+43$ , we see that  $f$  is zero in the neighbourhood of  $z_0$  of radius 42. (Why?)

However, since  $\mathbb{C}$  is path-connected, this implies that  $f$  is zero *everywhere*, as desired.  $\square$

6. Show that  $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$  is onto.

*Solution.* Let  $z_0 \in \mathbb{C}^\times$ . We show that  $\exp(z) = z_0$  for some  $z \in \mathbb{C}$ .

Note that  $r = |z_0| \neq 0$ .

Then,

$$w = \frac{z_0}{r}$$

has modulus 1. In other words,

$$w = x_0 + iy_0$$

for some  $(x_0, y_0) \in \mathbb{R}^2$  such that  $x_0^2 + y_0^2 = 1$ .

Thus,  $x_0 = \cos \theta$  and  $y_0 = \sin \theta$  for some  $\theta \in [0, 2\pi)$ .

Define  $z = \log(r) + \iota\theta$ . Note that this  $\log$  is the real-valued  $\log$ . Thus, we get

$$\begin{aligned}\exp(z) &= \exp(\log(r) + \exp(\iota\theta)) = \exp(\log(r)) \cdot \exp(\iota\theta) \\ &= r \cdot (\cos \theta + \iota \sin \theta) \\ &= rw = z_0.\end{aligned}$$

Thus,  $\exp$  is surjective. □

7. Show that  $\sin, \cos : \mathbb{C} \rightarrow \mathbb{C}$  are surjective. (In particular, note the difference with real sine and cosine which were bounded by 1).

*Solution.* We show this for  $\cos$ . The method works the same for  $\sin$ . Recall that

$$\cos(z) = \frac{1}{2} (e^{\iota z} + e^{-\iota z}).$$

Let  $z_0 \in \mathbb{C}$ . We show that  $\cos(z) = z_0$  for some  $z \in \mathbb{C}$ .

Consider the quadratic equation

$$\frac{1}{2} \left( t + \frac{1}{t} \right) = z_0. \quad (*)$$

Rearranging this gives

$$t^2 - 2z_0t + 1 = 0.$$

Note that the above has complex solutions  $t_1$  and  $t_2$ . (Since every complex number has a square root in  $\mathbb{C}$ !)

Moreover, note that  $t_1 \neq 0$ . Thus, by the previous part, there exists  $z \in \mathbb{C}$  such that  $e^z = t_1$ .

Plugging  $t_1 = e^z$  in  $(*)$  shows that

$$\cos(z) = z_0,$$

as desired. □

8. Show that for any complex number  $z$ ,  $\sin^2(z) + \cos^2(z) = 1$ .

*Solution.* Recall the definitions

$$\iota \sin(z) = \frac{1}{2} (e^{\iota z} - e^{-\iota z}), \quad \cos(z) = \frac{1}{2} (e^{\iota z} + e^{-\iota z}).$$

Squaring and subtracting gives

$$(\cos(z))^2 - (\iota \sin(z))^2 = \frac{1}{4} (4e^{\iota z} e^{-\iota z}) = 1$$

or

$$\sin^2(z) + \cos^2(z) = 1. \quad \square$$