

# Complex Analysis TSC

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Hi, welcome to this

*complex* discussion.

Here are some “guidelines” for this TSC -

- 1 Unmute your mic at any time and ask your doubt.
- 2 I will not be checking chat often (or maybe at all), so posting it there might not be helpful.

You can find a link to this document on [bit.ly/ca-205](https://bit.ly/ca-205). Both with and without pauses. You may keep it open alongside for quick reference.

# Warning

This is primarily going to be a quick recap of the facts important. It is, of course, not possible to go through everything in just 2 hours.

In particular, this will *not* be a substitute for all the lectures done so far.

I will also not be going through the proofs. We can discuss these finer things at the end, if time permits.

Though I'm not a fan of this - this session is pretty much going to cover things important from the point of view of an exam. I may also skip things from the lectures if I think that they are not important. They *might* turn out to be important, though.

Of course, I will not (intentionally) say anything which is mathematically incorrect.

# Lecture 1

## Definition 1 (Some notation)

Given  $z_0 \in \mathbb{C}$  and  $\delta > 0$ , the  $\delta$ -neighbourhood of  $z_0$ , denoted by  $B_\delta(z_0)$  is the set

$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$

## Definition 2 (Open sets)

A set  $U \subset \mathbb{C}$  is said to be open if:  
for every  $z_0 \in \mathbb{C}$ , there exists *some*  $\delta > 0$  such that

$$B_\delta(z_0) \subset U.$$

## Definition 3 (Path-connected sets)

A set  $P \subset \mathbb{C}$  is said to be path-connected if any two points in  $P$  can be joined by a path in  $P$ . (A continuous function from  $[0, 1]$  to  $P$ .)

## Definition 4 (Differentiable)

Let  $\Omega \subset \mathbb{C}$  be **open**. Let

$$f : \Omega \rightarrow \mathbb{C}$$

be a function. Let  $z_0 \in \Omega$ .  $f$  is said to be *differentiable* at  $z_0$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is denoted by  $f'(z_0)$ .

# Lecture 1

## Definition 5 (Holomorphic)

A function  $f$  is said to be holomorphic on an open set  $\Omega$  if it is differentiable at every  $z_0 \in \Omega$ .

A function  $f$  is said to be holomorphic at  $z_0$  if it is holomorphic on some neighbourhood of  $z_0$ .

## Remark 1

A function may be differentiable at  $z_0$  but not holomorphic at  $z_0$ . For example,  $f(z) = |z|^2$  is differentiable only at 0. Thus, it is differentiable at 0 but holomorphic nowhere.

For sets, however, there is no difference.

# End of Lecture 1

Any questions?

From this point on,  $\Omega$  will always denote an open subset of  $\mathbb{C}$ .

Whenever I write some complex number  $z$  as  $z = x + iy$ , it will be assumed that  $x, y \in \mathbb{R}$ .

Similarly for  $f(z) = u(z) + iv(z)$ .



## Lecture 2: CR Equations

Let  $f : \Omega \rightarrow \mathbb{C}$  be a function. We can decompose  $f$  as

$$f(z) = u(z) + \iota v(z),$$

where  $u, v : \Omega \rightarrow \mathbb{R}$  are real valued functions.

The idea now is to consider  $u$  and  $v$  as functions of two variables. We can do so by simply considering  $u(x, y) = u(x + \iota y)$  and similarly for  $v$ . Now, if we know that  $f$  is holomorphic, then we have the following result.

## Lecture 2: CR Equations

### Theorem 1 (CR equations)

Let  $f : \Omega \rightarrow \mathbb{C}$  be differentiable at a point  $z_0 \in \Omega$ . Let  $z_0 = x_0 + iy_0$ .

Then, we have

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, we have

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Existence of  $u_x, u_y, v_x, v_y$  is part of the theorem.

Note the subscript is  $x$  for both in the above.

Also note that all the equalities are only at the point  $z_0$ . In particular, we are only assuming differentiability at  $z_0$ .

## Lecture 2: CR Equations

Converse? What is the converse? Is it true?

**No.** The converse is **not** true.

An example for you to check is

$$f(z) := \begin{cases} \frac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that  $u$  and  $v$  satisfy the CR equations at  $(0,0)$  but  $f$  is not differentiable at  $0 + 0i$ . (Page 23 of slides.)

## Lecture 2: CR Equations

We recall MA 105 now.

### Definition 6 (Total derivative)

If  $f : \Omega \rightarrow \mathbb{C}$  is a function, we may view it as a function

$$f : \Omega \rightarrow \mathbb{R}^2.$$

Recall that  $f$  is said to be real differentiable at  $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$  if there exists a  $2 \times 2$  real matrix  $A$  such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{\left\| f(x_0 + h, y_0 + k) - f(x_0, y_0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

The matrix  $A$  was called the *total derivative of  $f$  at  $(x_0, y_0)$* .

## Lecture 2: CR Equations

### Theorem 2

If  $f$  is (complex) differentiable at a point  $z_0 = x_0 + iy_0$ , then  $f$  is real differentiable at  $(x_0, y_0)$ .

Once again, this is only talking about differentiability at a point. The converse is again **not** true.

Take the example  $f(z) = \bar{z}$ . Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

## Lecture 2: CR Equations

### Theorem 3

Let  $f : \Omega \rightarrow \mathbb{C}$  be a function and let  $z_0 = x_0 + iy_0 \in \Omega$ . If the CR equations hold **at the point**  $(x_0, y_0)$  and if  $f$  is real differentiable **at the point**  $(x_0, y_0)$ , then  $f$  is complex differentiable **at the point**  $z_0$ .

# Lecture 2: CR Equations

## Definition 7 (Harmonic functions)

Let  $u : \Omega \rightarrow \mathbb{R}$  be a twice continuously differentiable function.  $u$  is said to be *harmonic* if  $u_{xx} + u_{yy} = 0$ .

## Proposition 1

The real and imaginary parts of a holomorphic function are harmonic.

Suppose  $u$  and  $v$  are harmonic on  $\Omega$ .  $v$  is said to be a harmonic conjugate of  $u$  if  $f = u + iv$  is holomorphic on  $\Omega$ .

If  $v$  is a harmonic conjugate of  $u$ , then  $-u$  is a harmonic conjugate of  $v$ .

Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

## End of Lecture 2

Any questions?



## Lecture 3: Power Series

### Definition 8 (Convergence of series)

A series of the form

$$\sum_{n=0}^{\infty} a_n$$

of complex numbers is said to converge if the sequence of partial sums

$$s_n = \sum_{k=0}^n a_k$$

converges (to a finite complex number).

The sequence of partial sums is just the following sequence:

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$$

“Divergent” is simply used to mean “not convergent.”  
Check that  $\sum (-1)^n$  and  $\sum n$  both diverge.

# Lecture 3: Power Series

## Definition 9 (limsup)

Given a sequence  $(x_n)$  of **real numbers**, we may define a new sequence  $(y_n)$  as

$$y_n = \sup\{x_m : m \geq n\}.$$

The limit of this sequence always exists and we define

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n.$$

## Remark 2

Each  $y_n$  might be  $\infty$ . That is allowed.

The limsup might be  $\pm\infty$ . This is also allowed.

If  $\lim_{n \rightarrow \infty} x_n$  itself exists, then it equals the limsup as well.

## Lecture 3: Power Series

We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \quad (*)$$

where  $z_0 \in \mathbb{C}$  and each  $a_n \in \mathbb{C}$ .

What is the radius of convergence, though? (The definition, that is.)

### Theorem 4 (Radius of convergence)

Given any power series as  $(*)$ , there exists  $R \in [0, \infty]$  such that

- ①  $(*)$  converges for any  $z$  with  $|z - z_0| < R$ , and
- ②  $(*)$  diverges for any  $z$  with  $|z - z_0| > R$ .

This  $R$  is called the radius of convergence.

Note the **brackets**.

## Lecture 3: Power Series

We would now like to be able to calculate the radius of convergence.

### Theorem 5 (Root test)

Let  $(*)$  be as earlier. Define

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

Then,  $R = \alpha^{-1}$  is the radius of convergence.

This test *always works*. We had no assumptions of any kind on  $(*)$ . Note that  $\alpha^{-1}$ .

If  $\alpha = 0$ , then  $R = \infty$  and vice-versa.

## Lecture 3: Power Series

We have another test. This is simpler (to calculate) but mightn't always work.

### Theorem 6 (Ratio test)

Let  $(*)$  be as earlier.

Assume that the limit

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists. (Possibly as  $\infty$ .)

Then,  $R$  is the radius of convergence.

Note that here we assume that the limit does exist. This may not always be true.

Note that I'm not taking any inverse here but also note the way the ratio is taken. We have  $a_n/a_{n+1}$ .

# Lecture 3: Power Series

Differentiability of power series is what one should expect.

## Theorem 7 (Differentiability)

Let  $\sum_{n=0}^{\infty} a_n z^n$  be a power series with radius of convergence  $R > 0$ . On the **open disc** of radius  $R$ , let  $f(z)$  denote this sum. Then, on this disc, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

## End of Lecture 3

Any questions?

# Lecture 4: Exponential function

I shall just recall the facts from the lecture.

## Definition 10 (Exponential function)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges on all of  $\mathbb{C}$ . This sum is denoted by  $\exp(z)$ .

## Theorem 8 (Facts)

- 1  $\exp'(z) = \exp(z)$ ,
- 2  $\exp'(bz) = b \exp(bz)$ , for  $b \in \mathbb{C}$ ,
- 3  $\exp(z) \cdot \exp(-z) = 1$  for all  $z \in \mathbb{C}$ ,
- 4  $\exp(z)$  is always nonzero.



# Lecture 4: Exponential function

Now, we some “converse” facts.

## Theorem 9 (Characterisations)

- 1 If  $f'(z) = bf(z)$ , then  $f(z) = a \exp(bz)$  for some  $a, b \in \mathbb{C}$ ,
- 2 If  $f' = f$  and  $f(0) = 1$ , then  $f(z) = \exp(z)$ .

## Theorem 10 (Final fact)

Let  $z, w \in \mathbb{C}$ , then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

# Lecture 4: Exponential function

## Definition 11 (Domain)

A subset  $\Omega \subset \mathbb{C}$  is said to be a *domain* if it is open and path-connected.

More discussion - informal-tut.

We had one very nice result on the zeroes of a analytic functions.

## Theorem 11 (Zeroes are isolated)

Let  $\Omega$  be a **domain** and  $f : \Omega \rightarrow \mathbb{C}$  be a **non-constant** analytic function. Let  $z_0 \in \Omega$  be such that  $f(z_0) = 0$ . Then, there exists  $\delta > 0$  such that  $f$  has no other zero in  $B_\delta(z_0)$ .

The above is saying that around every zero of  $f$ , we can draw a (sufficiently small) circle such that  $f$  has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete*.

# End of Lecture 4

Any questions?

# Lecture 5: Integration

## Definition 12

Let  $f : [a, b] \rightarrow \mathbb{C}$  be a piecewise continuous function. Writing  $f = u + \iota v$  as usual, we define

$$\int_a^b f(t) dt := \int_a^b u(t) dt + \iota \int_a^b v(t) dt.$$

This is naturally what one would have wanted to define. Now, we define integration over a *contour*. (What is a contour?)

## Definition 13

Let  $f : \Omega \rightarrow \mathbb{C}$  be a continuous function. Let  $\gamma : [a, b] \rightarrow \Omega$  be a contour. We define

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

# Lecture 5: Integration

We have a useful inequality called the  $ML$  inequality.

## Theorem 12 ( $ML$ Inequality)

Let  $\gamma$  be a contour of length  $L$  and  $f$  be a continuous function defined on the image of  $\gamma$ .

Suppose that

$$|f(\gamma(t))| \leq M, \quad \text{for all } t \in [a, b].$$

Then, we have

$$\left| \int_{\gamma} f(z) dz \right| \leq ML.$$

# Lecture 5: Integration

## Theorem 13 (Primitives and integrals)

Suppose  $f : \Omega \rightarrow \mathbb{C}$  has a *primitive* on  $\Omega$ . That is, there exists a function  $F : \Omega \rightarrow \mathbb{C}$  such that  $F' = f$ . (The complex derivative.) Then, we have

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

If  $\gamma$  is *closed*, that is, if  $\gamma(b) = \gamma(a)$ , then

$$\int_{\gamma} f(z) dz = 0.$$

Existence of a primitive is a strong condition, by the way. A holomorphic function need not have a primitive on all of  $\Omega$ .

## Lecture 5: Integration

Now, we come to Cauchy's theorem.

### Theorem 14 (Cauchy's Theorem)

Let  $\gamma$  be a simple, closed contour and let  $f$  be a holomorphic function defined on an open set  $\Omega$  containing  $\gamma$  **as well as its interior**. Then,

$$\int_{\gamma} f(z) dz = 0.$$

If  $\Omega$  is simply-connected, then the interior condition is automatically met. This gives us the next result.

### Theorem 15 (“General” Cauchy Theorem)

Let  $\Omega$  be a simply-connected domain. Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a simple, closed contour and  $f : \Omega \rightarrow \mathbb{C}$  holomorphic. Then,

$$\int_{\gamma} f(z) dz = 0.$$



# End of Lecture 5

Any questions?

### Theorem 16 (Cauchy Integral Formula)

Let  $f$  be holomorphic everywhere on an open set  $\Omega$ . Let  $\gamma$  be a simple closed curve in  $\Omega$ , oriented positively. If  $z_0$  is interior to  $\gamma$  and  $\Omega$  contains the interior of  $\gamma$ , then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$

## Lecture 6: CIF and Consequences

We then saw a consequence of CIF which I state as a theorem below.

### Theorem 17 (Holomorphic $\implies$ Analytic)

Let  $\Omega \subset \mathbb{C}$  be open and  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic. Pick any  $z_0 \in \Omega$ . Let  $R > 0$  be the largest such that  $B_R(z_0) \subset \Omega$ . (The case  $R = \infty$  is allowed. That just means  $\Omega = \mathbb{C}$ .) Then, on the disc  $B_R(z_0)$ , we may write  $f(z)$  as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where each  $a_n$  is given by

$$a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw.$$

## Lecture 6: CIF and Consequences

The above also gives us (what I call) the “generalised” Cauchy Integral Formula.

### Theorem 18 (“Generalised” CIF)

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw = \frac{2\pi i}{n!} f^{(n)}(z_0),$$

where  $f$  is a function which is holomorphic on an open disc  $B_R(z_0)$  and  $r < R$ .

### Remark 3

Note that, as usual, we require  $f$  to be holomorphic within the circle as well.

# Lecture 7: CIF and Consequences

## Theorem 19 (Cauchy's estimate)

Suppose that  $f$  is holomorphic on  $|z - z_0| < R$  and bounded by  $M > 0$  on this disc. Then,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{R^n}.$$

An easy application of this give us:

## Theorem 20 (Liouville's Theorem)

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic. If  $f$  is bounded, then  $f$  is constant!

# End of Lectures 6 and 7

Any questions?

# Logarithm

We discuss logarithm a bit.

## Definition 14 (Branch of the logarithm)

Let  $\Omega \subset \mathbb{C}$  be a **domain**. Let  $f : \Omega \rightarrow \mathbb{C}$  be a continuous function such that

$$\exp(f(z)) = z, \quad \text{for all } z \in \Omega.$$

Then,  $f$  is called a *branch of the logarithm*.

## Theorem 21 (Uniqueness of branches)

Assume that  $f, g : \Omega \rightarrow \mathbb{C}$  are two branches of the logarithm. Then,  $f - g$  is a constant function. Moreover, this constant is an integer multiple of  $2\pi i$ .

The last theorem also assumed that  $\Omega$  is a **domain**.

The previous theorem talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible.

## Theorem 22 (Existence of a branch)

Let  $\Omega$  be a **simply-connected** domain in  $\mathbb{C}$ . Assume that  $1 \in \Omega$  and  $0 \notin \Omega$ .

There exists a unique function  $F : \Omega \rightarrow \mathbb{C}$  such that

- ①  $F(1) = 0$ ,
- ②  $F'(z) = 1/z$ ,
- ③  $\exp(F(z)) = z$  for all  $z \in \Omega$ ,
- ④  $F(r) = \log(r)$  for all  $r \in \Omega \cap \mathbb{R}^+$ .

The log in the last point is the usual log for real numbers as seen in 105. The above  $F$  is then denoted by  $\log$ .



# Lecture 8: Singularities

## Definition 15 (Singularities)

Let  $f : \Omega \rightarrow \mathbb{C}$  be a function. A point  $z_0 \in \mathbb{C}$  is said to be a singularity of  $f$  if

- ①  $z_0 \notin \Omega$ , i.e.,  $f$  is not defined at  $z_0$ , or
- ②  $z_0 \in \Omega$  and  $f$  is not holomorphic at  $z_0$ .

## Definition 16 (Isolated singularity)

A singularity  $z_0 \in \mathbb{C}$  is said to be *isolated* if there exists *some*  $\delta > 0$  such that  $f$  is holomorphic on  $B_\delta(z_0) \setminus \{z_0\}$ .

The above is saying that “ $f$  is holomorphic on some *punctured disc* around  $z_0$ .”

Compare this “isolation” with what we saw earlier when we said that “zeroes are isolated.”

# Lecture 8: Singularities

## Definition 17 (Non-isolated singularity)

A singularity which is not an isolated singularity is called a non-isolated singularity.

*The floor is made of floor.*

Note that if  $f$  has only finitely many singularities, then all the singularities are isolated.

We classify **isolated** singularities into three types:

- 1 Removable singularities,
- 2 Poles,
- 3 Essential singularities.

## Remark 4

The above classification is only for **isolated** singularities.

# Lecture 8: Singularities

## Definition 18 (Removable singularity)

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

These are characterised by the following theorem.

## Theorem 23 (Riemann's Removable Singularity Theorem)

$z_0$  is a removable singularity of  $f$  iff  $\lim_{z \rightarrow z_0} f(z)$  exists.

In the above, we mean that it exists as a (finite) complex number.

$$f(z) = \frac{\sin z}{z}$$

defined on  $\mathbb{C} \setminus \{0\}$  has 0 as a removable singularity.

## Lecture 8: Singularities

### Definition 19 (Pole)

An isolated singularity  $z_0$  is said to be a pole if  $|f(z)| \rightarrow \infty$  as  $z \rightarrow z_0$ .

### Theorem 24

An isolated singularity  $z_0$  is a pole of  $f$  iff  $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$ .

### Theorem 25 (Order of a pole)

If  $z_0$  is a pole of  $f$ , then there exists an integer  $m > 0$  such that

$$f(z) = (z - z_0)^{-m} f_1(z)$$

on a punctured neighbourhood of  $z_0$ , for some function  $f_1$  which is holomorphic on the complete neighbourhood. The **smallest** such integer  $m$  is called the *order* of the pole.

If the order is 1, then  $z_0$  is said to be *simple* pole.

# Lecture 8: Singularities

## Definition 20 (Essential singularity)

An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

## Theorem 26 (Casorati-Weierstrass Theorem)

If  $z_0$  is an isolated singularity, then it is essential iff the values of  $f$  come arbitrarily close to every complex number in a neighborhood of  $z_0$ .

# End of Lecture 8

Any questions?

# Lecture 9: Laurent Series

## Theorem 27 (Modified CIF)

Suppose that  $z_0$  is an isolated singularity of  $f$ . Consider an annulus of the form

$$A = \{z : r < |z - z_0| < R\},$$

where  $0 \leq r < R \leq \infty$ . Assume that  $f$  is holomorphic on this open annulus  $A$ . Then, CIF takes the form

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=R'} \frac{f(w)}{z-w} dw - \frac{1}{2\pi i} \int_{|w-z_0|=r'} \frac{f(w)}{z-w} dw,$$

where  $r < r' < R' < R$ .

Just like how the usual CIF gave us the power series, this CIF gives us the Laurent series.

# Lecture 9: Laurent Series

## Theorem 28 (Laurent Series)

With the same setup as earlier, for  $z \in A$ , we can write  $f(z)$  as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,$$

where each  $a_n$  is given, as before, by

$$a_n = \frac{1}{2\pi i} \int_{|z-w|=r_0} \frac{f(w)}{(z-w)^{n+1}} dw,$$

where  $r < r_0 < R$ .

Note that the above is valid for  $n < 0$  as well.



# Lecture 9: Laurent Series

## Definition 21 (Laurent series expansion at $z_0$ )

If  $z_0$  is an isolated singularity of  $f$ , then  $f$  is holomorphic in an annulus  $\{z : 0 < |z - z_0| < r\}$  for some  $r > 0$ . The Laurent series expansion on this annulus is called the Laurent series expansion **at**  $z_0$ .

## Definition 22 (Principal part)

Let  $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$  be the Laurent series expansion at  $z_0$ . Its principal part is

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

## Lecture 9: Laurent Series

The most interesting coefficient of the principal part is the  $-1^{\text{st}}$  one. When we integrate a Laurent series along a circle centered at  $z_0$  (which contains no other singularity), only  $a_{-1}$  remains (with a factor of  $2\pi i$ ). This is given by

$$a_{-1} = \frac{1}{2\pi i} \int_{|z-z_0|=r_0} f(w)dw.$$

This is what is usually called the *residue* and written as

$$a_{-1} = \text{Res}(f; z_0).$$

## Lecture 9: Laurent Series

With residues, calculation of integrals becomes easier.

### Theorem 29 (Cauchy's Residue Theorem)

Suppose  $f$  is given and has finitely many singularities  $z_1, \dots, z_n$  within a **simple** closed contour  $\gamma$ . Then, we have

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{i=1}^n \text{Res}(f; z_i).$$

Note that the above is implicitly implying that  $f$  is holomorphic at all other points within  $\gamma$ .

## Lecture 9: Laurent Series

Recall that given an isolated singularity, we can expand the function as a Laurent series *around* that point on a punctured neighbourhood. We had defined the principal part of this series to be the part containing the negative powers of  $z - z_0$ . We now see how they are related to the nature of the isolated singularity.

### Theorem 30 (Isolated singularities and their principal parts)

The isolated singularity  $z_0$  is

- ① removable iff the principal part has no terms,
- ② a pole iff the principal part has finitely many (and at least one) terms, and
- ③ essential iff the principal part has infinitely many terms.

In particular, the residue at a removable singularity is 0.

## Lecture 9: Laurent Series

Now, we see how one can calculate residue at a pole.

By the previous theorem, we know that  $f$  can be written as

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots ,$$

for some integer  $m > 0$ .

Thus,

$$g(z) = (z - z_0)^m f(z)$$

is holomorphic at  $z_0$  (after redefining; note that  $z_0$  is a removable singularity for  $g$ ) and

$$a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(z_0).$$

## Definition 23 (Neighbourhood of $\infty$ )

A neighbourhood of  $\infty$  is a set of the form

$$A(0, R, \infty) := \{z \in \mathbb{C} : |z| > R\}$$

for some  $R > 0$ .

## Definition 24 (Isolated singularity at $\infty$ )

$f$  is said to have an isolated singularity at  $\infty$  if  $f$  is (defined and) holomorphic on some neighbourhood of  $\infty$ . Equivalently,

$z \mapsto f\left(\frac{1}{z}\right)$  has an isolated singularity at 0.

## Definition 25 (Nature of isolated singularity at $\infty$ )

The nature of the singularity of  $f$  at  $\infty$  is defined to be the nature of the singularity of  $z \mapsto f\left(\frac{1}{z}\right)$  at 0.

### Examples.

- ❶  $f(z) = 0$  has a removable singularity at  $\infty$ .
- ❷  $f(z) = \frac{1}{z}$  has a removable singularity at  $\infty$ .
- ❸  $f(z) = z^n$  has a pole of order  $n$  at  $\infty$ . ( $n \in \mathbb{N}$ .)
- ❹  $\exp$  has an essential singularity at  $\infty$ .

We didn't define the residue at  $\infty$ . Check Wikipedia for what the definition is, if interested. It is not the same as the residue of  $f(1/z)$  at 0.

## Theorem 31 (Maximum Modulus Theorem)

Let  $\Omega$  be a **domain**. Let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic and non-constant. Then,  $|f|$  does not attain a maximum.

Said differently: If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and  $|f|$  attains a maximum, then  $f$  is constant.

An “application:” Suppose that  $f$  is defined on the closed unit disc such that it is continuous on the closed disc and holomorphic on the open disc. Since the closed disc is closed and bounded and  $f$  is continuous,  $|f|$  must attain a maximum on the closed disc. By MMT, this maximum must be on the boundary.



# End of Lectures 10 and 11

Any questions?

## Theorem 32 (Schwarz Lemma)

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disc.

Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be holomorphic such that

$$f(0) = 0 \quad \text{and} \quad |f(z)| \leq 1,$$

for  $z \in \mathbb{D}$ .

Then,  $|f(z)| \leq |z|$  for all  $z \in \mathbb{D}$  and  $|f'(0)| \leq 1$ .

Moreover, if  $|f(z)| = |z|$  for some  $z \in \mathbb{D} \setminus \{0\}$  or if  $|f'(0)| = 1$ , then  $f(z) = \lambda z$  for some  $\lambda \in \mathbb{C}$  such that  $|\lambda| = 1$ .

## Definition 26 (Open maps)

A function  $f : \Omega \rightarrow \mathbb{C}$  is said to be an open map if  $f(U)$  is open for any open subset  $U \subset \Omega$ .

## Theorem 33 (Open Mapping Theorem)

Let  $\Omega$  be open and  $f : \Omega \rightarrow \mathbb{C}$  be **non-constant** and holomorphic. Then,  $f$  is an open map.

In particular,  $f(\Omega)$  is open. As a corollary, if  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic such that  $f(\Omega)$  is not open, then  $f$  is constant.

## Theorem 34 (Argument principle)

Let  $f$  be a meromorphic on  $\Omega$ . That is, the only singularities of  $f$  in  $\Omega$  are poles.

Let  $\gamma$  be a simple closed curve in  $\Omega$ , oriented positively. Moreover, assume that  $f$  has no zero or pole along  $\gamma$ . Then,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = N_{\gamma}(f) - P_{\gamma}(f),$$

where  $N_{\gamma}(f)$  (resp.,  $P_{\gamma}(f)$ ) denotes the number of zeroes (resp., poles) of  $f$  within  $\gamma$  counted with multiplicity (resp., order).

## Theorem 35 (Rouché's Theorem)

Let  $f, g : \Omega \rightarrow \mathbb{C}$  be holomorphic. Let  $\gamma$  be closed curve in  $\Omega$ . Suppose that

$$|f(z) - g(z)| < |f(z)|,$$

for all  $z$  on the image of  $\gamma$ .

Then,

$$N_{\gamma}(f) = N_{\gamma}(g).$$

As before, note that the zeroes are counted with multiplicity. For example,  $z^{43}$  has 43 zeroes within the curve  $|z| = 1$ .

## Theorem 36 (Existence of harmonic conjugates)

Let  $\Omega \subset \mathbb{R}^2$  be a simply connected domain. Let  $u : \Omega \rightarrow \mathbb{R}$  be harmonic. Then,  $u$  admits a harmonic conjugate on  $\Omega$ . Moreover, this conjugate is unique, up to an additive constant.

As a corollary, we had gotten that harmonic functions are infinitely differentiable since open discs are simply connected.

## Theorem 37 (Mean Value Property)

Let  $w \in \mathbb{R}^2$  and  $u$  be a function harmonic on  $B_R(w)$  for some  $R > 0$ . Let  $0 < r < R$ . Then, we have

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta.$$

Note that in CIF, we had a  $z$  in the denominator. No such thing here. Moreover, we have  $2\pi$  instead of  $2\pi i$ . The latter is of course expected since everything is  $\mathbb{R}$  real.

As a corollary, we obtain MMT for harmonic functions which says that  $u$  cannot obtain a maximum at any interior point unless it is constant.

Note that here, we are talking about  $u$  directly. *Not*  $|u|$ . Applying MMT to  $-u$  also gives us that  $u$  cannot attain a minimum at any interior point unless it is constant.

## Theorem 38 (Identity Principle for harmonic functions)

Let  $u$  be a harmonic function on a domain  $\Omega \subset \mathbb{C}$ . If  $u = 0$  on a non-empty open subset  $U \subset \Omega$ , then  $u = 0$  throughout  $\Omega$ .



# End of Lectures 12 and 13

Any questions?

# Little Picard Theorem

## Theorem 39 (Little Picard)

Let  $f$  be an entire function, i.e.,  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a holomorphic function. If  $f$  is nonconstant, then the image of  $f$  is either all of  $\mathbb{C}$  or  $\mathbb{C}$  minus a point.

In other words, if an entire function misses two points, then it must be constant.

## Theorem 40 (Jordan's Lemma)

Let  $f, g$  be continuous complex valued functions defined on the upper semicircular contour  $C_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$  for some  $R > 0$ . Assume that there exists  $a > 0$  such that

$$f(z) = e^{iaz}g(z),$$

for all  $z \in C_R$ . Then,

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} \max_{\theta \in [0, \pi]} |g(Re^{i\theta})|.$$

This is useful in the cases that the quantity on the right goes to 0 in the limit  $R \rightarrow \infty$ .

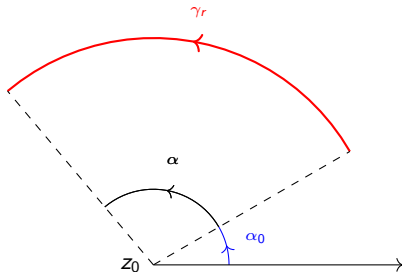
# Integration theorems

## Theorem 41 (Fractional residue theorem)

Let  $f$  have a simple pole at  $z_0$ . Fix  $\alpha \in (0, 2\pi]$  and  $\alpha_0 \in [0, 2\pi)$ .

For  $r > 0$ , define  $\gamma_r(\theta) := z_0 + re^{i(\theta+\alpha_0)}$  for  $\theta \in [0, \alpha]$ . Then,

$$\lim_{r \rightarrow 0^+} \int_{\gamma_r} f(z) dz = \alpha i \operatorname{Res}(f; z_0).$$



# Integration Theorems

Not exactly an integration theorem but something we saw in lectures that is helpful in computing integrals of rational functions.

## Theorem 42

Let  $P(z)/Q(z)$  be a rational function such that  $\deg Q(x) \geq \deg P(x) + 2$ . Then, there exist constants  $R_0$  and  $C$  such that

$$\left| \frac{P(z)}{Q(z)} \right| \leq \frac{C}{|z|^2},$$

whenever  $|z| > R_0$ .

Thus, if  $R > R_0$ , then  $\left| \frac{P(z)}{Q(z)} \right| \leq \frac{C}{R^2}$  on a circle of radius  $R$ .

Usually, we will be interested in the upper half semi-circle. ML inequality will tell us that the integral over the semicircle goes to 0 in the limit  $R \rightarrow \infty$ .

# The End

Doubts?