

Complex Analysis TSC

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<https://aryamanmaithani.github.io/tuts/ma-205>

IIT Bombay

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Greetings

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You can find a link to this document on bit.ly/ca-205. Both with and without pauses. You may keep it open alongside for quick reference.

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Lecture 1

Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ -neighbourhood of z_0 , denoted by $B_\delta(z_0)$ is the set

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Definition 2 (Open sets)

A set $U \subset \mathbb{C}$ is said to be open if:
for every $z_0 \in \mathbb{C}$, there exists *some* $\delta > 0$ such that

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Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P . (A continuous function from $[0, 1]$ to P .)

Definition 4 (Differentiable)

Let $\Omega \subset \mathbb{C}$ be open. Let

$$f : \Omega \rightarrow \mathbb{C}$$

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exists. In this case, it is denoted by $f'(z_0)$.

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A function f is said to be holomorphic on an open set Ω if it is differentiable at every $z_0 \in \Omega$.

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For sets, however, there is no difference.

End of Lecture 1

Any questions?

From this point on, Ω will always denote an open subset of \mathbb{C} .
Whenever I write some complex number z as $z = x + iy$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + iv(z)$.

Lecture 2: CR Equations

Let $f : \Omega \rightarrow \mathbb{C}$ be a function. We can decompose f as

$$f(z) = u(z) + \iota v(z),$$

where $u, v : \Omega \rightarrow \mathbb{R}$ are real valued functions.

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The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v .

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The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v . Now, if we know that f is holomorphic, then we have the following result.

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Theorem 1 (CR equations)

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Also note that all the equalities are only at the point z_0 .

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Note the subscript is x for both in the above.

Also note that all the equalities are only at **the point** z_0 . In particular, we are only assuming differentiability at z_0 .

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An example for you to check is

$$f(z) := \begin{cases} \frac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

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Check that u and v satisfy the CR equations at $(0,0)$ but f is not differentiable at $0 + 0i$. (Page 23 of slides.)

Lecture 2: CR Equations

We recall MA 105 now.

Definition 6 (Total derivative)

If $f : \Omega \rightarrow \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega \rightarrow \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if

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The matrix A was called the *total derivative of f at (x_0, y_0)* .

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Theorem 2

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Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

End of Lecture 2

Any questions?

Lecture 3: Power Series

Definition 8 (Convergence of series)

A series of the form

$$\sum_{n=0}^{\infty} a_n$$

of complex numbers is said to converge if the sequence of partial sums

$$s_n = \sum_{k=0}^n a_k$$

converges (to a finite complex number).

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Check that $\sum (-1)^n$ and $\sum n$ both diverge.

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If $\lim_{n \rightarrow \infty} x_n$ itself exists, then it equals the limsup as well.

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We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (*)$$

where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

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What is the radius of convergence, though?

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What is the radius of convergence, though? (The definition, that is.)

Theorem 4 (Radius of convergence)

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- 1 $(*)$ converges for any z with $|z - z_0| < R$

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Lecture 3: Power Series

We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

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where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

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Note the **brackets**.

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If $\alpha = 0$, then $R = \infty$ and vice-versa.

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Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1} .

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Differentiability of power series is what one should expect.

Theorem 7 (Differentiability)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the **open disc** of radius R , let $f(z)$ denote this sum. Then, on this disc, we have

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Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

End of Lecture 3

Any questions?

Lecture 4: Exponential function

I shall just recall the facts from the lecture.

Definition 10 (Exponential function)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges on all of \mathbb{C} . This sum is denoted by $\exp(z)$.

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Theorem 10 (Final fact)

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

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Definition 11 (Domain)

A subset $\Omega \subset \mathbb{C}$ is said to be a *domain* if it is open and path-connected.

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End of Lecture 4

Any questions?

Lecture 5: Integration

Definition 12

Let $f : [a, b] \rightarrow \mathbb{C}$ be a piecewise continuous function. Writing $f = u + \iota v$ as usual, we define

$$\int_a^b f(t)dt := \int_a^b u(t)dt + \iota \int_a^b v(t)dt.$$

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Then, we have

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Lecture 5: Integration

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Existence of a primitive is a strong condition, by the way. A holomorphic function need not have a primitive on all of Ω .

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If Ω is simply-connected, then the interior condition is automatically met. This gives us the next result.

Theorem 15 (“General” Cauchy Theorem)

Let Ω be a simply-connected domain. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a simple, closed contour and $f : \Omega \rightarrow \mathbb{C}$ holomorphic. Then,

$$\int_{\gamma} f(z) dz = 0.$$

End of Lecture 5

Any questions?

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Let f be holomorphic everywhere on an open set Ω . Let γ be a simple closed curve in Ω , oriented positively. If z_0 is interior to γ and Ω contains the interior of γ , then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$

Lecture 6: CIF and Consequences

We then saw a consequence of CIF which I state as a theorem below.

Theorem 17 (Holomorphic \implies Analytic)

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$.

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Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$. Let $R > 0$ be the largest such that $B_R(z_0) \subset \Omega$. (The case $R = \infty$ is allowed. That just means $\Omega = \mathbb{C}$.) Then, on the disc $B_R(z_0)$, we may write $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where each a_n is given by

$$a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw.$$

Lecture 6: CIF and Consequences

The above also gives us (what I call) the “generalised” Cauchy Integral Formula.

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$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw = \frac{2\pi i}{n!} f^{(n)}(z_0),$$

where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

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where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

Remark 3

Note that, as usual, we require f to be holomorphic within the circle as well.

Lecture 7: CIF and Consequences

Theorem 19 (Cauchy's estimate)

Theorem 20 (Liouville's Theorem)

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Suppose that f is holomorphic on $|z - z_0| < R$

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Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

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Lecture 7: CIF and Consequences

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Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

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Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic.

Lecture 7: CIF and Consequences

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An easy application of this give us:

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Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. If f is bounded, then

Lecture 7: CIF and Consequences

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An easy application of this give us:

Theorem 20 (Liouville's Theorem)

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

End of Lectures 6 and 7

Any questions?

We discuss logarithm a bit.

Definition 14 (Branch of the logarithm)

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Assume that $f, g : \Omega \rightarrow \mathbb{C}$ are two branches of the logarithm.

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The last theorem also assumed that Ω is a **domain**.

The previous theorem talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible.

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Lecture 8: Singularities

Definition 15 (Singularities)

Let $f : \Omega \rightarrow \mathbb{C}$ be a function. A point $z_0 \in \mathbb{C}$ is said to be a singularity of f if

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Lecture 8: Singularities

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Compare this “isolation” with what we saw earlier when we said that “zeroes are isolated.”

Lecture 8: Singularities

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Remark 4

The above classification is only for **isolated** singularities.

Lecture 8: Singularities

Definition 18 (Removable singularity)

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

Theorem 23 (Riemann's Removable Singularity Theorem)

Lecture 8: Singularities

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$$f(z) = \frac{\sin z}{z}$$

defined on $\mathbb{C} \setminus \{0\}$ has 0 as a removable singularity.

Lecture 8: Singularities

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An isolated singularity z_0 is a pole of f iff $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$.

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Theorem 25 (Order of a pole)

If z_0 is a pole of f , then there exists an integer $m > 0$ such that

$$f(z) = (z - z_0)^{-m} f_1(z)$$

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If z_0 is a pole of f , then there exists an integer $m > 0$ such that

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on a punctured neighbourhood of z_0 , for some function f_1 which is holomorphic on the complete neighbourhood.

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If the order is 1, then z_0 is said to be *simple* pole.

Lecture 8: Singularities

Definition 20 (Essential singularity)

An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

Theorem 26 (Casorati-Weierstrass Theorem)

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Theorem 26 (Casorati-Weierstrass Theorem)

If z_0 is an isolated singularity, then it is essential iff the values of f come arbitrarily close to every complex number in a neighborhood of z_0 .

End of Lecture 8

Any questions?

Lecture 9: Laurent Series

Theorem 27 (Modified CIF)

Suppose that z_0 is an isolated singularity of f .

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$$A = \{z : r < |z - z_0| < R\},$$

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$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{-n} + \sum_{n=1}^{\infty} b_n (z - z_0)^n,$$

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Suppose that z_0 is an isolated singularity of f . Consider an annulus of the form

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where $0 \leq r < R \leq \infty$. Assume that f is holomorphic on this open annulus A . Then, CIF takes the form

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=R'} \frac{f(w)}{z-w} dw - \frac{1}{2\pi i} \int_{|w-z_0|=r'} \frac{f(w)}{z-w} dw,$$

where $r < r' < R' < R$.

Just like how the usual CIF gave us the power series, this CIF gives us the Laurent series.

Lecture 9: Laurent Series

Theorem 28 (Laurent Series)

With the same setup as earlier, for $z \in A$, we can write $f(z)$ as

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where $r < r_0 < R$.

Note that the above is valid for $n < 0$ as well.

Lecture 9: Laurent Series

Definition 21 (Laurent series expansion at z_0)

If z_0 is an isolated singularity of f , then f is holomorphic in an annulus $\{z : 0 < |z - z_0| < r\}$ for some $r > 0$. The Laurent series expansion on this annulus is called the Laurent series expansion **at** z_0 .

Definition 22 (Principal part)

Let $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be the Laurent series expansion *at* z_0 . Its principal part is

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

Lecture 9: Laurent Series

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$$a_{-1} = \text{Res}(f; z_0).$$

Lecture 9: Laurent Series

With residues, calculation of integrals becomes easier.

Theorem 29 (Cauchy's Residue Theorem)

Suppose f is given and has finitely many singularities z_1, \dots, z_n within a **simple** closed contour γ .

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Suppose f is given and has finitely many singularities z_1, \dots, z_n within a **simple** closed contour γ . Then, we have

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Note that the above is implicitly implying that f is holomorphic at all other points within γ .

Lecture 9: Laurent Series

Recall that given an isolated singularity, we can expand the function as a Laurent series *around* that point on a punctured neighbourhood.

Theorem 30 (Isolated singularities and their principal parts)

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Recall that given an isolated singularity, we can expand the function as a Laurent series *around* that point on a punctured neighbourhood. We had defined the principal part of this series to be the part containing the negative powers of $z - z_0$. We now see how they are related to the nature of the isolated singularity.

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- ① removable iff the principal part has no terms,
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In particular, the residue at a removable singularity is 0.

Lecture 9: Laurent Series

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$$a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(z_0).$$

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A neighbourhood of ∞ is a set of the form

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We didn't define the residue at ∞ . Check Wikipedia for what the definition is, if interested. It is not the same as the residue of $f(1/z)$ at 0.

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An “application:” Suppose that f is defined on the closed unit disc such that it is continuous on the closed disc and holomorphic on the open disc. Since the closed disc is closed and bounded and f is continuous, $|f|$ must attain a maximum on the closed disc. By MMT, this maximum must be on the boundary.

End of Lectures 10 and 11

Any questions?

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for $z \in \mathbb{D}$.

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Moreover, if $|f(z)| = |z|$ for some $z \in \mathbb{D}$ or if $|f'(0)| = 1$, then $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

Definition 26 (Open maps)

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As before, note that the zeroes are counted with multiplicity. For example, z^{43} has 43 zeroes within the curve $|z| = 1$.

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As a corollary, we had gotten that harmonic functions are infinitely differentiable since open discs are simply connected.

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As a corollary, we obtain MMT for harmonic functions which says that u cannot obtain a maximum at any interior point unless it is constant.

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Let u be a harmonic function on a domain $\Omega \subset \mathbb{C}$. If $u = 0$ on a non-empty open subset $U \subset \Omega$,

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Let u be a harmonic function on a domain $\Omega \subset \mathbb{C}$. If $u = 0$ on a non-empty open subset $U \subset \Omega$, then $u = 0$ throughout Ω .

End of Lectures 12 and 13

Any questions?

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In other words, if an entire function misses two points, then it must be constant.

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Let f, g be continuous complex valued functions defined on the upper semicircular contour $C_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$ for some $R > 0$.

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$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} \max_{\theta \in [0, \pi]} |g(Re^{i\theta})|.$$

This is useful in the cases that the quantity on the right goes to 0 in the limit $R \rightarrow \infty$.

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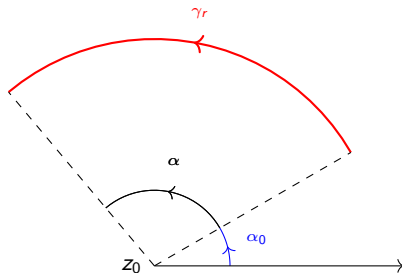
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Integration theorems

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For $r > 0$, define $\gamma_r(\theta) := z_0 + re^{i(\theta+\alpha_0)}$ for $\theta \in [0, \alpha]$.



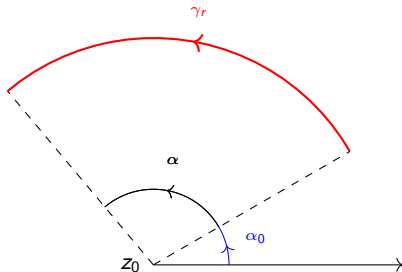
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For $r > 0$, define $\gamma_r(\theta) := z_0 + re^{i(\theta+\alpha_0)}$ for $\theta \in [0, \alpha]$. Then,

$$\lim_{r \rightarrow 0^+} \int_{\gamma_r} f(z) dz = \alpha i \operatorname{Res}(f; z_0).$$



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Usually, we will be interested in the upper half semi-circle. ML inequality will tell us that the integral over the semicircle goes to 0 in the limit $R \rightarrow \infty$.

The End

Doubts?