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MA 408

Measure Theory

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Spring 2020-21

Lecture 1

06 January 2021 22:06

Idea behind measure

Simplified case: Subsets of $\mathbb{R}$

Given $E \subseteq \mathbb{R}$, want to assign "length" or "content" to E.

Ideally, want a map

$$\mu : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$$

s.t.

(1) $\mu(\emptyset) = 0$

(2) For any $E \subset \mathbb{R}$ and $x \in E$,

$$\mu(E) = \mu(x + E).$$

$(x + E := \{x + y : y \in E\})$ ← translation by x

(3) Given a countable collection $\{E_i\}_{i=1}^{\infty}$ of subsets of $\mathbb{R}$, we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

(So far, $\mu \equiv 0$ will satisfy above properties!)

(4) $\mu([0, 1]) = 1$. ("Normalisation")

Any such μ would be a 'candidate' for our content.

However, no such μ exists!

Consider the following sets:

(1) Define $\sim$ on $\mathbb{R}$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
Clearly, $\sim$ is an equiv. relation.

Let $E \subseteq [0,1]$ be a set containing exactly one element from each equivalence class in $\mathbb{R}/\sim$.
(Existence is given by Axiom of Choice. Note that distinct equiv. classes are disjoint.)

and a small argument that lets you conclude $E \subset [0,1]$.

Q. What could $\mu(E)$ be?

Note that $\{E + r\}_{r \in \mathbb{Q} \cap [0,1]}$ is a collection of pairwise disjoint sets.

(Sketch. If $x \in (E + r_1) \cap (E + r_2)$, $(r_2 \geq r_1)$ then $x = r_1 + e_1 = r_2 + e_2$ for some $e_1, e_2 \in E$

$$\Rightarrow e_1 - e_2 = r_2 - r_1 \in \mathbb{Q}$$
$$\Rightarrow e_1 \sim e_2 \Rightarrow e_1 = e_2$$)

Moreover, $[0,1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [0,1]} (E + r) \subseteq [0,2] = [0,1] \cup [1,2]$

(An easy consequence of (1) – (3) is that $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$.

Proof. $\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$. $\square$)

$$\Rightarrow \quad \mu([0,1]) \leq \mu\left(\bigcup_{i=1}^{\infty} (E + r_i)\right) \leq \mu([0,1]) + \mu([1,2])$$

enumerate $\mathbb{Q} \cap [0,1]$ as $\{r_1, \dots\}$

$$\Rightarrow \quad 1 \leq \sum_{i=1}^{\infty} \mu(E + r_i) \leq 2 \qquad [1,2] = [0,1]+1$$

$$\Rightarrow \quad 1 \leq \sum_{i=1}^{\infty} \mu(E) \leq 2$$

If $\mu(E) = 0$

$1 \leq 0$ →←

If $\mu(E) = r > 0$

$\sum_{i=1}^{\infty} \mu(E) = \infty \leq 2$ →←

Possible way to salvage: Replace (3) to have "finite union" instead of "countable".

Turns out that that's still not enough.

(2) BANACH - TARSKI THEOREM (1924): (Using AC)

~~PARADOX~~

For any open sets $U, V \subseteq \mathbb{R}^n$ where $n \geq 3$, there exists $k \in \mathbb{N}$ and set $U_1, \dots, U_k$, $V_1, \dots, V_k$ s.t.

(1) $U_i \cap U_j = \emptyset$, $V_i \cap V_j = \emptyset$, $1 \leq i \neq j \leq k$.

(2) $U = \bigsqcup_{i=1}^{k} U_i$, $V = \bigsqcup_{i=1}^{k} V_i$.

(3) $U_i \simeq V_i$, i.e.,

U_i is obtained from V_i by a sequence of rotations, reflections, and translations.

In other words, by isometries.

Thus, the analogue of (2) implies $\mu(U_i) = \mu(V_i)$ $\forall i$.

$\Rightarrow \quad \mu(U) = \mu(V).$

$\rightsquigarrow$ Absurd conclusions.

As it turns out, the problem is NOT in the infinite union but rather the demand that μ is defined on all of $\mathcal{P}(\mathbb{R})$!

Thus, we restrict our attention to a smaller collection of subsets of $\mathbb{R}$. (Not too small!)

σ - ALGEBRAS

Let X be an arbitary set.

Def^n. (1) An algebra ("field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

① $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$

② $A_1, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$ for any $n \in \mathbb{N}$.

(2) A σ-algebra ("σ-field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

① $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$

② $A_1, \ldots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Note that complements and unions give us intersections. Also, $\phi, X \in \mathcal{F}$.

EXAMPLES

① $\mathcal{F} = \mathcal{P}(X) \leftarrow$ both

② (Countable - cocountable σ-algebra)
$\mathcal{F} = \{ E \subseteq X : E \text{ or } E^c \text{ is countable} \}$

Proof. Clearly closed under complement.
Let $A_1, \dots \in \mathcal{F}$.
If all A_i are countable, then $\bigcup A_i$ is.
Suppose A_1 not countable. Then, A_1^c is.
But

$$A_1 \subset \bigcup A_i \Rightarrow \left(\bigcup A_i\right)^c \subset A_1^c$$

$$\Rightarrow \left(\bigcup A_i\right)^c \text{ is countable. } \square$$

③ Given any $\mathcal{F} \subseteq \mathcal{P}(X)$, we can talk about σ-algebra generated by $\mathcal{F}$ denoted $\mathcal{M}(\mathcal{F})$ defined by

$$\mathcal{M}(\mathcal{F}) = \bigcap_{\substack{\mathcal{F} \subseteq B \\ B \text{ is a } \sigma\text{-alg}}} B$$

(Note that the intersection is non-empty because of $\mathcal{P}(X)$. Easy to see that intersection of σ-algebrae is again a σ-alg.)

By construction, $\mathcal{M}(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$.

Borel σ-Algebra.

Def^n: Let $(X, \mathcal{T})$ be a topological space. The σ-algebra generated by $\mathcal{T}$ is called the Borel σ-algebra on X, denoted $\mathcal{B}(X)$.

(Abuse of notation that we don't mention $\mathcal{T}$.)

In other words, it is generated by the open sets of X.

Borel σ-algebra on $\mathbb{R}$: Smallest σ-alg on $\mathbb{R}$ containing all the open sets.

Consequences:

1. All open sets are in $\mathcal{B}(\mathbb{R})$.
2. All closed sets are in $\mathcal{B}(\mathbb{R})$.
3. All F_σ, G_δ sets are in $\mathcal{B}(\mathbb{R})$.

$$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i \ (F_i \text{ closed}) \ ; \quad G_\delta \equiv \bigcap_{i=1}^{\infty} G_i \ (G_i \text{ open})$$

Prop^n: Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$.
Then, $\mathcal{B}$ is also generated by any of the following:

(i) $\{(a, b) : a < b\}$ or $\{[a, b] : a < b\}$
(ii) $\{[a, b) : a < b\}$ or $\{(a, b] : a < b\}$
(iii) $\{(a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a) : a \in \mathbb{R}\}$
(iv) $\{[a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a] : a \in \mathbb{R}\}$

Proof Easy. $\square$

Borel σ-algebra on $\mathbb{R}^n$:

Suppose $\{X_i\}_{i=1}^{n}$ are metric spaces.

Let $X = \prod_{i=1}^{n} X_i$ with the product metric.

def^n

If ρ_i is the metric on X_i, then ρ on $\prod X_i$ is defined as

$$\rho(x, y) = \max \rho_i(x_i, y_i)$$ $\left| x = (x_1, \dots, x_n) \right|$

$$\rho(x, y) = \max_{1 \le i \le n} \rho_i(x_i, y_i) \qquad \left(\begin{array}{l} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{array} \right)$$

Def^n. Suppose $(X_i, \mathcal{M}_i)$ are σ-algebrae. One can define a σ-algebra on $X := \prod X_i$ as follows:

Consider the projection maps $\pi_i : X \to X_i$.
Let

$$\mathcal{F} = \{ \pi_i^{-1}(E) : E \in \mathcal{M}_i, \ i = 1, \dots, n \}$$

$$= \{ E \times X_2 \times \cdots \times X_n : E \in \mathcal{M}_1 \}$$
$$\cup \{ X_1 \times E \times \cdots \times X_n : E \in \mathcal{M}_2 \}$$
$$\cup \cdots \cup \{ X_1 \times \cdots \times X_{n-1} \times E : E \in \mathcal{M}_n \}.$$

$\mathcal{M} := \mathcal{M}(\mathcal{F}) \subseteq \mathcal{P}(X)$ is the product σ-algebra induced by $\{\mathcal{M}_i\}_{i=1}^n$.

We often write the above as $\mathcal{M} = \prod_{i=1}^n \mathcal{M}_i$.

Caution. The above $\prod_{i=1}^n$ is NOT the set-theoretic cartesian product.

Now, we get two (possibly different) σ-algebrae on $\mathbb{R}^n$.

① Borel σ-alg. on $(\mathbb{R}^n, \mathcal{T})$

② Product of Borel σ-alg of $\mathcal{B}(\mathbb{R})$.

Prop^n: $\mathcal{B}(\mathbb{R}^n) = \prod_{i=1}^n \mathcal{B}(\mathbb{R})$. That is, both the σ-alg above are same.

Proof. We will prove this by a sequence of observations.

① Suppose $\{(X_i, \mathcal{M}_i)\}_{i=1}^{n}$ are σ-algebrae and $\mathcal{F}_i \subseteq \mathcal{M}_i$ are such that $\mathcal{M}_i = \mathcal{M}(\mathcal{F}_i)$. $(i=1,\dots,n)$

Then, if $X = \prod_{i=1}^{n} X_i$ and $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i$, then

$\mathcal{M}$ is generated by $\{\pi_i^{-1}(E) : E \in \mathcal{F}_i,\ i=1,\dots,n\}$.

② $\mathcal{M}$ is generated by $\{E_1 \times \cdots \times E_n : E_i \in \mathcal{F}_i\}$.

Assuming ① and ② for now, we now note the following.

Clearly, one has $\prod_{i=1}^{n} \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^n)$.

[Proof: using ②, $\prod_{i=1}^{n} \mathcal{B}(\mathbb{R})$ is gen. by sets of the form $U_1 \times \cdots \times U_n$, each $U_i \subseteq \mathbb{R}$ open

Each such set is open in the metric space $\mathbb{R}^n$.
Thus, it is in $\mathcal{B}(\mathbb{R}^n)$.]

We show $\mathcal{B}(\mathbb{R}^n) \subseteq \prod \mathcal{B}(\mathbb{R})$.

(*) It suffices to show that every set of the form

$$U_1 \times \cdots \times U_n \quad \text{where } U_i \subset \mathbb{R} \text{ are open}$$

are in the product $\prod \mathcal{B}(\mathbb{R})$.

[Why? Every open set in $\mathbb{R}^n$ is a countable union of sets of aforementioned form. In turn, the open sets generate $\mathcal{B}(\mathbb{R}^n)$.]

Proving (*) is easy because

$$U_1 \times \dots \times U_n = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \dots \cap \pi_n^{-1}(U_n).$$

these are in $\prod \mathcal{B}(\mathbb{R})$, by def^n

Proof of 1

Want to show that $\tilde{\mathcal{F}} = \{\pi_i^{-1}(E) : E \in \mathcal{F}_i, \ 1 \le i \le n\}$ gen. $\underbrace{\prod \mathcal{M}_i}_{\mathcal{M}}$.

Clearly $\mathcal{M}(\tilde{\mathcal{F}}) \subseteq \mathcal{M}$. ($\tilde{\mathcal{F}} \subset \mathcal{M}$ and $\mathcal{M}$ is a σ-alg)

It now suffices to show that every (standard) generator of $\mathcal{M}$ is in $\mathcal{M}(\tilde{\mathcal{F}})$.

Note $\mathcal{M} = \langle \pi_i^{-1}(E) : E \in \mathcal{M}_i, \ 1 \le i \le n \rangle$

$\tilde{\mathcal{M}} := \langle \pi_i^{-1}(E) : E \in \mathcal{F}_i, \ 1 \le i \le n \rangle = \mathcal{M}(\tilde{\mathcal{F}})$

Let $\tilde{\mathcal{M}}_i := \{E \in \mathcal{M}_i : \pi_i^{-1}(E) \in \tilde{\mathcal{M}}\} \subseteq \mathcal{P}(X_i)$.

We shall show that $\tilde{\mathcal{M}}_i = \mathcal{M}_i$.

We know, by def^n that $\mathcal{F}_i \subseteq \tilde{\mathcal{M}}_i$. $\left(E \in \mathcal{F}_i \overset{\subseteq \mathcal{M}_i}{\Rightarrow} \pi_i^{-1}(E) \in \tilde{\mathcal{M}} \cap \mathcal{M}_i \Downarrow E \in \tilde{\mathcal{M}}_i \right)$

Moreover, $\mathcal{M}(\mathcal{F}_i) = \mathcal{M}_i$. Thus, it suffices to show that $\tilde{\mathcal{M}}_i$ is a σ-alg.

Also, $\tilde{\mathcal{M}}_i \subseteq \mathcal{M}_i$.

To that end, let $A \in \tilde{\mathcal{M}}_i$. Then, $\pi_i^{-1}(A) \in \tilde{\mathcal{M}}$.

Then, $\pi_i^{-1}(A)^c \in \tilde{\mathcal{M}}$. But $\pi_i^{-1}(A^c) = \pi_i^{-1}(A)^c \in \mathcal{M}$.

$\Rightarrow \pi_i^{-1}(A^c) \in \mathcal{M}_i$

Similarly, noting that $\pi_i^{-1}\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigcup_{j=1}^{\infty} \pi_i^{-1}(A_j)$ yields the result.

Proof of 2.

Now, put $\tilde{\mathcal{F}} := \{E_1 \times \cdots \times E_n : E_i \in \mathcal{F}_i\}$ and
$\tilde{\mathcal{M}} := \mathcal{M}(\tilde{\mathcal{F}})$.

Since $E_1 \times \cdots \times E_n = \bigcap_{i=1}^{n} \pi_i^{-1}(E_i)$, we see that $\tilde{\mathcal{F}} \subseteq \mathcal{M}$.

Thus, $\tilde{\mathcal{M}} \subseteq \mathcal{M}$.

REMARKS.

1. The argument above generalises for a separable metric spaces.

2. If $(X_i, \mathcal{M}_i)_{i \in A}$, and A is COUNTABLE then again, $X = \prod X_i$, $\mathcal{M} = \prod \mathcal{M}_i$ generated by $\{\pi_i^{-1}(E) : E \in \mathcal{M}_i, i \in A\}$ is also generated by sets of the form

$$\left(\prod_{i \in A} E_i\right), \quad E_i \in \mathcal{F}_i.$$

MEASURE

Def^n. Suppose $(X, \mathcal{M})$ is a measure space, i.e., $\mathcal{M}$ is a σ-algebra on X. A measure on X is a map $\mu : \mathcal{M} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) if $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

in $\mathcal{M}$

EXAMPLES.

(1) $X = \{x_1, x_2, \dots\}$ is countable. Suppose $p_i \geq 0$ are reals s.t. $\sum_{i=1}^{\infty} p_i = 1$. Let $\mathcal{M} = \mathcal{P}(X)$ and define $\mu : \mathcal{M} \to [0,1]$ as

$$\mu(E) = \sum_{i:\, x_i \in E} p_i.$$

(2) $(X, \mathcal{M})$ be s.t. $\mathcal{M}$ is the countable-cocountable σ-alg. s.t. X itself is uncountable

Define
$$\mu(E) := \begin{cases} 0 & ; E \text{ is countable} \\ 1 & ; E \text{ is uncountable} \end{cases}$$

Prop^n. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Then,

① $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$

② $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$ (μ is "sub-additive")

③ If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \cdots$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i).$$

Proof. ① & ② are trivial

③ Define $F_i = E_i \setminus E_{i-1}$ for $i \geq 2$.

$$F_1 = F_1$$

Then, $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. Also, $F_i \in \mathcal{M}$ for each i.

($n = \infty$ as well) Moreover, $F_i \cap F_j = \phi$ for $i \neq j$.

Thus, $$\mu\left(\bigcup E_i\right) = \mu\left(\bigcup F_i\right) = \sum_{i=1}^{\infty} \mu(F_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{\infty} E_i.$$

Def^n. ① A null set in a measure space $(X, \mathcal{M}, \mu)$ is a set E s.t. $E \subseteq F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.

($E \in \mathcal{M}$ NOT necessary.)

② Given a measure space $(X, \mathcal{M}, \mu)$, the completion of $\mathcal{M}$, denoted $\bar{\mathcal{M}}$ is the collection of all sets of the form $F \cup N$ where $F \in \mathcal{M}$ and N is a null set.

Prop^n. ① If $(X, \mathcal{M}, \mu)$ is a measure space, then $\bar{\mathcal{M}}$ is a σ-alg.

② Moreover, there exists a unique measure

$$\bar{\mu} : \bar{\mathcal{M}} \longrightarrow [0, \infty] \quad \text{s.t.}$$

$$\bar{\mu}\big|_{\mathcal{M}} = \mu.$$

(That is, there is a unique extension of μ to a measure $\bar{\mu}$ on $\bar{\mathcal{M}}$.)