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MA 526

Commutative Algebra

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Lecture 1 (07-01-2021)

07 January 2021 08:07

Noetherian Rings and Modules

Def^n (Poset) A set S with a relation $\leq$ which is
(i) Reflexive
(ii) Anti-symmetric
(iii) Transitive

A total order is a poset in which any two elements are comparable.
A subset of a poset is called a chain if it is totally ordered.

Prop^n. Let S be a poset.
TFAE
(1) $x_1 \leq x_2 \leq x_3 \leq \cdots \Rightarrow \exists N \in \mathbb{N}$ s.t. $x_n = x_{n+1}$ $\forall n \geq N$
(2) $T \subset S$, $T \neq \phi \Rightarrow T$ has a maximal element.

Proof. (1) $\Rightarrow$ (2)
Let $\phi \subsetneq T \subsetneq S$. Suppose, for the sake of contradiction, that T has no maximal element.

Pick any $x_1 \in T$. x_1 not maximal. $\therefore \exists x_2 \in T$ s.t. $x_2 > x_1$.
x_2 not maximal. $\exists x_3 \in T$ with $x_3 > x_2$. ...
We get a chain $x_1 < x_2 < \cdots$ which does not stabilise.

(2) $\Rightarrow$ (1) Let $x_1 \leq x_2 \leq x_3 \leq \cdots$ be a chain.
Consider $T = \{x_i : i \in \mathbb{N}\}$. This has a maximal element.
Let $N \in \mathbb{N}$ be s.t. x_N is maximal.
By assumption, $x_N \leq x_{N+1}$ but also maximal.
$\therefore x_N = x_{N+1}$.
In fact, for any $M > N$, the above argument holds. ∎

(1) is called the ascending chain condition. (a.c.c.)
(2) — " — maximal condition.

Def^n. Let R be a commutative ring with 1.
Let M be an R-module.
Let P be the poset of submodules of M (w.r.t. inclusion).
M is said to be Noetherian if P satisfies a.c.c.

(Equivalently, P satisfies maximal condition.)

If R is a Noetherian R-module, R is called a Noetherian ring.

There are the dual properties: descending chain condition (d.c.c.)
minimal condition.

Def^n. If submodules of an R-module M satisfy d.c.c., M is called an Artinian module.
Similarly, if R is Artinian as an R-module, it is called an Artinian ring.

Note that R-submodules of R are precisely ideals.
Thus, the Art./Noe. conditions are a.c.c./d.c.c. on ideals.

We shall soon see that Noe. rings are Art. but converse not true.

Examples.

(1) R PID. $R = \mathbb{Z}$ or $\mathbb{K}[x]$, for example.
Let us consider $\mathbb{Z}$.

$$0 \subsetneq (n_1) \subsetneq (n_2) \subsetneq \cdots$$

$n_2 \mid n_1$ with $n_2 \neq \pm n_1$, ...
At each stage, at least one prime is exhausted

Similar argument works in $\mathbb{k}[x]$ or any PID.

$\mathbb{Z}$ is NOT Noetherian. $(2) \supsetneq (2^2) \supsetneq (2^3) \supsetneq \cdots$
Can do the same in any PID which is not a field.

(2) $\mathbb{k}$ a field. $\mathbb{k}$ is both. } have only finitely many ideals. Satisfy acc & dcc trivially.

(3) $\mathbb{Z}/n\mathbb{Z}$ $\leftarrow$ both
$n > 1$

(4) Any finite abelian group G is a $\mathbb{Z}$-module.
Only finitely many subgroups ($\mathbb{Z}$-submodules) and hence, both.

(5) $\mathbb{Q}/\mathbb{Z}$. $\quad 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$

$$\mathbb{Q}/\mathbb{Z} = \left\{ \frac{r}{s} + \mathbb{Z} \;\middle|\; r, s \in \mathbb{Z} \text{ with } s \neq 0 \right\}$$

is an infinite abelian group.

Fix a prime $p > 0$. Define $G_n \subset \mathbb{Q}/\mathbb{Z}$ as

$$G_n := \left\{ \frac{a}{p^n} + \mathbb{Z} \;\middle|\; a \in \mathbb{Z} \right\}.$$

$$G_0 = 0 \lneq G_1 \lneq G_2 \lneq \cdots$$

$$\left(\frac{1}{p^n} + \mathbb{Z} \in G_n \setminus G_{n-1} \right)$$

Thus, $\mathbb{Q}/\mathbb{Z}$ is not Noetherian. (as a $\mathbb{Z}$-module)

Moreover, $G = \bigcup_{n=1}^{\infty} G_n \leq \mathbb{Q}/\mathbb{Z}$. This subgroup is also not a Noetherian $\mathbb{Z}$-module.

However, G does satisfy d.c.c.

(Ex. Every subgroup of G is of the form G_n.)

Thus, G is Artinian but not Noetherian!

(6) Hilbert Basis Theorem. $\mathbb{k}[x_1, \ldots, x_n]$ is Noe. ($n=1$ done above)

However, $\mathbb{k}[x_1, \ldots]$ is not Noetherian.

$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$

Not Artinian either. $R \supsetneq (x_1, x_2, \ldots) \supsetneq (x_2, \ldots) \supsetneq (x_3, \ldots) \supsetneq \cdots$

$(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \cdots$

(7) $0 \to \mathbb{Z} \to H^{\subseteq \mathbb{Q}} \to G \to 0$

$H = \left\{ \frac{m}{p^n} : m \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\} \right\}$ (p fixed prime)

Then H is not Art because $\mathbb{Z}$ is not.

H is not Noe. because G is not.

Lecture 2 (12-01-2021)

12 January 2021 14:02

Thm. Suppose R is a ring and M an R-module. Then M is Noetherian iff every submodule of M is f.g.

Proof ($\Rightarrow$) Suppose M is Noetherian and $N \leq M$ a submodule.
To show: N is not f.g.
Suppose not.
Then, $N \neq \{0\}$. $\quad (\because \langle \phi \rangle = \{0\})$
$\Rightarrow \exists x_1 \in N$ s.t. $x_1 \neq 0$.
$N_1 = Rx_1 \subsetneq N$. Thus, $\exists x_2 \in N \setminus N_1$.
$N_1 \subsetneq N_2 = Rx_1 + Rx_2 \subsetneq N$.

Similarly, we can construct $x_3, \ldots$

Thus, $0 \subsetneq N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \cdots \subseteq N \subseteq M$.
$\rightarrow\leftarrow$

Thus, N is f.g.. As N was arbitrary every submodule of M is f.g.

($\Leftarrow$) Suppose every submodule of M is f.g.
We show that a.c.c. holds

Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M$ be a seq. of submodules.

Put $N := \bigcup_{i=1}^{\infty} M_i$. $\leftarrow$ This is a submodule of M since $\{M_i\}_{i=1}^{\infty}$ is a chain.

Thus, N is f.g. Then, $R = \langle x_1, \ldots, x_g \rangle$ for some $x_1, \ldots, x_g \in N$.

$\therefore N = \bigcup_{i=1}^{\infty} M_i$, for some x_j, $\exists M_j$ s.t. $x_j \in M_j$.

$\therefore \quad N = \bigcup_{i=1}^{\infty} M_i$, for some x_j, $\exists M_j$ s.t. $x_j \in M_j$.

However, note that $\{M_i\}$ is a chain and $\exists t \in \mathbb{N}$ s.t.

$$x_1, \ldots, x_g \in M_t.$$

Thus, $x_1, \ldots, x_g \in M_T \quad \forall\, T \geq t$.

$$\Rightarrow \quad M_t = M = M_T \quad \forall\, T \geq t.$$

Thus, M is Noetherian.

__Cor.__ A ring is Noetherian iff every ideal of R is f.g.

__Prop^n.__ Suppose $0 \to N \xrightarrow{f} M \xrightarrow{g} P \to 0$ is an exact sequence. (That is, $\ker f = 0$, $\operatorname{im} f = \ker g$, $\operatorname{im} f = P$.)

(i) M is Noetherian $\Leftrightarrow$ N and P are Noetherian

(ii) M is Artinian $\Leftrightarrow$ N and P are Artinian

__Proof.__ We prove (i). (ii) is similar.

($\Rightarrow$) $N \simeq f(N)$ as f is injective.
Enough to prove $f(N)$ is Noetherian. But $f(N) \leq M$.
Thus, any chain in $f(N)$ is also in M. Thus, $f(N)$ is Noetherian because M is so.

$P \simeq M/\ker g$. Note any submodule of $M/\ker g$ is of the form $L/\ker g$ for some $L \leq M$ with $\ker g \leq L$.
Conclude.

↑ sufficient to show this is Noetherian

($\Leftarrow$) Let N and P be Noetherian modules.

Let $M_0 \le M_1 \le \cdots \le M$ be an icreasing sequence.

$\Rightarrow \quad f^{-1}(M_0) \le f^{-1}(M_1) \le \cdots \le N.$

N is Noe., thus $\exists\, n \in \mathbb{N}$ s.t. $f^{-1}(M_{n+i}) = f^{-1}(M_n) \ \forall i \geq 0.$

Similarly,

$$g(M_0) \le g(M_1) \le \cdots \le P$$

$\Rightarrow \quad \exists\, m \in \mathbb{N}$ s.t. $g(M_m) = g(M_{m+i}) \quad \forall i \geq 0$
with $m \geq n$

Then, $\left.\begin{array}{l} f^{-1}(M_m) = f^{-1}(M_{m+i}) \\ g(M_m) = g(M_{m+i}) \end{array}\right\} \forall i \geq 0$

<u>Claim.</u> $M_m = M_{m+i} \quad \forall i \geq 0.$

($\subseteq$) is given

($\supseteq$) Let $x \in M_{m+i}$. $\quad g(x) \in g(M_{m+i}) = g(M_m)$

$\Rightarrow \quad g(x) = g(y)$ for some $y \in M_m$

$\Rightarrow \quad x - y \in \ker g = \operatorname{im} f \cap M_{n+i}$

$\Rightarrow \quad x - y = f(z)$ for some $z \in N$

$\Rightarrow \quad z \in f^{-1}(M_{n+i}) = f^{-1}(M_n)$

$\Rightarrow \quad \underset{x-y}{\underset{\parallel}{f(z)}} \in M_n$

$\Rightarrow \quad x - y \in M_n$ but $y \in M_n$

$\therefore \quad x \in M_n$, as desired.

Cor. Let $M_1, \dots, M_n$ be R-modules.
Then

$$\bigoplus_{i=1}^{n} M_i \text{ is Noe} \iff M_i \text{ is Noe } \forall i.$$

Similar statetement holds for Artinian.

Proof. ($\Rightarrow$) $\pi_i : \bigoplus_{j=1}^{n} M_j \twoheadrightarrow M_i$ is onto.

$$0 \to \ker \pi_i \xrightarrow{\text{incl}} \bigoplus_{j=1}^{n} M_j \xrightarrow{\pi} M_i \to 0$$

shows M_i is Noe. (or Art).

($\Leftarrow$) Induction on n. $n=1$ true. Assume for n. Then,

$$0 \to M_{n+1} \xrightarrow{\text{ind}} \bigoplus_{i=1}^{n+1} M_i \twoheadrightarrow \bigoplus_{i=1}^{n} M_i \to 0$$

Noetherian (assumption) Noetherian (induction)

$\therefore \bigoplus_{i=1}^{n+1} M_i$ is Noe. ∎

Cor. Let R be a Noetherian (resp Artinian) ring and M a f.g. R-module. Then, M is Noetherian (resp. Artinian)

Proof. Since M is f.g., we can write M as a quotient of $R^{\oplus n}$. (*)
But $R^{\oplus n}$ is Noe. (resp Art.) since R is.
Thus, so is M.

(*) Let $M = Rm_1 + \cdots + Rm_n$ for $m_1, \dots, m_n \in M$

$$0 \to \ker f \to \bigoplus_{i=1}^{n} Re_i \xrightarrow{f} M \to 0$$ is an exact sequence.

$$e_i \mapsto m_i$$

[Note that for Noe., it is necessary that M be f.g. Thus, it is necessary & suff. if R is Noetherian.
However, for Art., M need not be f.g.]

Remark Subrings of Noetherian rings need not be Noetherian.

$R = \mathbb{k}[x, y]$ $\qquad$ $\mathbb{k}$ field ; $\quad$ x, y indeterminate

R is Noetherian. (Hilbert's basis theorem)

$S = \mathbb{k}[x, xy, xy^2, \dots]$ is a subring of R.

Note that

$$\langle x\rangle \subsetneq \langle x, xy\rangle \subsetneq \langle x, xy, xy^2\rangle \subsetneq \cdots$$

are strictly increasing ideals in S.

Note that in R, $\langle x\rangle = \langle x, xy\rangle$ since $y \in R$.

Thus, S is not Noetherian even though R is.

EXAMPLE. Let $X = [0, 1]$. $\quad \mathcal{C}(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous}\}$ is a comm. ring with 1. (Pointwise operations.)

$\mathcal{C}(X)$ is not Noetherian.

Define $F_n := \left[0, \frac{1}{n}\right]$ for $n \in \mathbb{N}$.

$$F_1 \supset F_2 \supset F_3 \supset \cdots$$

Define

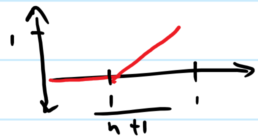
$$I_n = \{ f \in \mathcal{C}(X) : f(F_n) = 0\}.$$

Note I_n is an ideal. Moreover

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

($\subset$) is clear because $f_{n+1} \subset f_n$

($\neq$) because



Thus, R is not Noetherian.

———— x ————

R : Noetherian ring, I is an ideal
$\Rightarrow R/I$ is Noetherian (as a ring)

(What NOT to do: $0 \to I \to R \to R/I \to 0$
This only shows R/I is a Noe. R-module, not as ring.)
(However, this can be improved.)
See note.

Proof $\Rightarrow$ let $K \trianglelefteq R/I$ be an ideal. Then, $K = J/I$ for some $I \subseteq J \trianglelefteq R$.

R is Noe $\Rightarrow J$ is f.g. $\Rightarrow I$ is f.g. $\square$

NOTE. Let M be an R-module.
$\operatorname{ann} M := \{ r \in R : rm = 0 \ \forall m \in M \}$.
(E.g. R/I is an R-module and $\operatorname{ann}(R/I) = I$.)
M is also an $R/\operatorname{ann} M$ - module with operation

$$(r + \operatorname{ann} M)\, m = rm. \qquad \text{(well-defined)}$$

Then, the module structure is the "same". This shows that the previous argument actually works.

———— x ————

Thm. (Hilbert Basis Theorem) (Hilbert's Basis Theorem)
Let R be a Noetherian ring and X an indeterminate.
Then $R[X]$ is Noetherian.

<u>Remark.</u> Note the converse is trivial since $R \simeq \frac{R[X]}{\langle X \rangle}$.

($x = X$.)

<u>Proof.</u> Suppose $R[x]$ is not Noetherian.
Then, $\exists\, I \trianglelefteq R[x]$ s.t. I is not f.g.
In particular, $I \neq 0$. $\exists f_1 \in I \setminus \{0\}$

Pick f_1 of least degree. (May be many such f_1. Does not matter.)

$f_1 = a_1 x^{d_1} + (\text{smaller terms})$

($d_1 = \deg f_1$)

$I \neq (f_1)$. Choose $f_2 \in I \setminus (f_1)$ of least degree.
(d_2)

$f_2 = a_2 x^{d_2} + (\text{smaller terms})$

$I \neq (f_1, f_2)$. Continue picking $f_3, f_4, \ldots$ similarly

Note $a_1 \neq 0$, $a_2 \neq 0$, ...

Consider the following ideals of R:

$$(a_1) \subseteq (a_1, a_2) \subseteq (a_1, a_2, a_3) \subseteq \cdots$$

R is Noetherian. Thus, the above chain stabilises

$\Rightarrow$ $(a_1, \ldots, a_k) = (a_1, \ldots, a_k, \ldots, a_{k+i})$ $\forall i \geqslant 0$

$a_{k+1} = b_1 a_1 + \cdots + b_k a_k$ for some $b_1, \ldots, b_k \in R$.

$$f_1 = a_1 x^{d_1} + (\cdots)$$
$$\vdots$$
$$f_k = a_k x^{d_k} + (\cdots)$$
$$f_{k+1} = a_k x^{d_{k+1}} + (\cdots)$$

Note $d_1 \leq d_2 \leq \cdots$

Thus, $d_{k+1} \geq d_k \geq \cdots$

Now, look at
$$g = b_1 f_1 x^{d_{k+1} - d_1} + \cdots + b_k a_k f x^{d_{k+1} - d_k} - f_{k+1}$$

Note : $\deg g < \underset{\deg f_{k+1}}{\overset{\shortparallel}{d_{k+1}}}$ but $g \notin (f_1, \ldots, f_k)$.

else $f_{k+1} \in (f_1, \ldots, f_k)$ $\rightarrow\leftarrow$

Thus, $R[x]$ is Noetherian.

Cor. R Noetherian $\Rightarrow$ $R[x_1, \ldots, x_n]$ is Noetherian.
Moreover, quotients are also Noetherian.

Cor. R Noetherian $\Rightarrow$ any f.g. R-alg is Noetherian.

$$S = R[s_1, \ldots, s_n] \simeq \frac{R[x_1, \ldots, x_n]}{I}.$$

Remark. Analogous result __NOT__ true for Artinian. $\mathbb{k}$ & $\mathbb{k}[x]$.

Lecture 3 (15-01-2021)

15 January 2021 14:03

Lemma. Let $I \trianglelefteq R$ be an ideal and $b \in R$ be s.t.

$$I : b = \{ r \in R \mid rb \in I \}$$ and

$\langle I, b \rangle$ are finitely generated. Then, I is also f.g.

Proof.

$$\begin{tikzcd}[column sep=small] I : b \arrow[dr, dash] & & \langle I, b \rangle \arrow[dl, dash] \\ & I & \end{tikzcd}$$

$$\langle I, b \rangle = \{ x + yb \mid x \in I, y \in R \}$$

Generators of $\langle I, b \rangle$ can be of the form $a_1, \ldots, a_r \in I$, b.

$$\langle I, b \rangle = \langle a_1, \ldots, a_r, b \rangle.$$

$$(I : b) = (c_1, \ldots, c_s) \Rightarrow c_i b \in I \quad \forall i$$

Put $J = \langle a_1, \ldots, a_r, c_1 b, \ldots, c_s b \rangle \subseteq I$.

We show $I \subseteq J$ and conclude. ($\because J$ is f.g.)

Let $a \in I \subseteq \langle I, b \rangle = \langle J, b \rangle$. Then, $a = c + rb$, $c \in J, r \in R$

$$\Rightarrow rb = a - c \in I$$

$$\Rightarrow r \in I : b$$

Thus, $r = d_1 c_1 + \cdots + d_s c_s$ $\quad (I : b = \langle c_1, \ldots, c_s \rangle)$

$$\Rightarrow a = c + rb = \underbrace{c}_{\in J} + d_1 \underbrace{bc_1}_{\in J} + \cdots + d_s \underbrace{bc_s}_{\in J}$$

$\therefore a \in J$. ∎

Thm. (Cohen's Theorem)

If prime ideals of a commutative ring are f.g., then the ring is Noetherian.

Proof. We show that all ideals are f.g.
Suppose not. Define

$$\Sigma = \{ I \mid I \trianglelefteq R \text{ s.t. } I \text{ is not f.g.} \}$$

$\Sigma \neq \phi$ by hypothesis. Σ is a poset, under $\subseteq$.

Suppose $\{I_\alpha\}_{\alpha \in \Lambda}$ is a chain of ideals in Σ.
We show that

$$I = \bigcup_{\alpha \in \Lambda} I_\alpha \quad \text{is not f.g.}$$

(That it is an ideal is clear.)

This is simple for if $I = \langle x_1, \ldots, x_r \rangle$, then one can find a suitable $\alpha \in \Lambda$ s.t. $I_\alpha \ni x_1, \ldots, x_r$. ($\because \{I_\alpha\}$ is a chain)
In that case

$$I = \langle x_1, \ldots, x_r \rangle \subseteq I_\alpha \subseteq I.$$

Thus, $I_\alpha = \langle x_1, \ldots, x_r \rangle$ is f.g. $\rightarrow\leftarrow$

Thus, Σ has a maximal element, by Zorn's Lemma.
Let J be a maximal element of Σ.
Since $J \in \Sigma$, J is not f.g and hence, not prime.

$\therefore \exists\, a, b \in R$ s.t. $a \notin J$, $b \notin J$ but $ab \in J$.

$$ab \in J \Rightarrow a \in J : b \supsetneq J \quad \text{since } a \notin J$$

Also, $\langle J, b \rangle \supsetneq J$ since $b \notin J$.

Since J is maximal, $(J : b), \langle J, b \rangle \notin \Sigma$.
Thus, both are f.g. By the earlier lemma,

so is J.

Thus, we have a contradiction.

Thus, all ideals are f.g. and hence, R is Noetherian. ∎

Cor. R is Noetherian $\Rightarrow$ $R[\![x_1, \dots, x_n]\!]$ is Noetherian.

Proof. Enough to prove for $n = 1$.
Using Cohen's, it is sufficient to show that prime ideals in $R[\![x]\!]$ are f.g.

Consider the evaluation map $\phi : R[\![x]\!] \longrightarrow R$
$$f(x) \mapsto f(0)$$

Let $\mathfrak{p} \in \operatorname{Spec}(R[\![x]\!])$. Then, $\phi(\mathfrak{p})$ is an ideal of R (ϕ is onto)
and hence, $\phi(\mathfrak{p})$ is f.g.
(since R is Noetherian)

$$\phi(\mathfrak{p}) = \langle a_1, \dots, a_r \rangle$$ ← ideal of all constant terms in $\mathfrak{p}$.

Case 1. $x \in \mathfrak{p}$.
Let $f(x) \in \mathfrak{p}$ be arbitrary

Write $f(x) = b_0 + b_1 x + \cdots = b_0 + x(b_1 + b_2 x + \cdots)$
Then, $b_0 \in \phi(\mathfrak{p})$. $\langle b_0, x \rangle$

$$b_0 = c_1 a_1 + \cdots + c_r a_r$$

$$f(x) \in \langle a_0, \dots, a_r, x \rangle \subset \mathfrak{p}$$

$$\therefore \mathfrak{p} = \langle a_0, \dots, a_r, x \rangle \text{ is f.g.}$$

Case 2. $x \notin \mathfrak{p}$

$\phi(\mathfrak{p}) = \langle a_1, \dots, a_r \rangle$

For each $i = 1, \dots, r$, we have $f_i(x) \in \mathfrak{p}$

s.t.

$$f_i(x) = a_i + x\, g_i(x) \; ; \qquad g_i(x) \in R[\![x]\!].$$

<u>Claim.</u> $\mathfrak{p} = \langle f_1, \dots, f_r \rangle$. (2) is obvious.

<u>Proof.</u> Let $g(x) \in \mathfrak{p}$.

Write $g(x) = b + x h(x)$, $\quad h(x) \in R[\![x]\!]$.

$$b = \sum_{i=1}^{r} b_i a_i$$

$$g - \sum b_i f_i = [b + x h(x)] - \sum b_i (a_i + x g_i(x))$$

$$\underbrace{\overset{\mathfrak{p} \ni}{g} - \sum b_i \overset{\in \mathfrak{p}}{f_i}}_{\in \mathfrak{p}} = \underset{\notin \mathfrak{p}}{x} \underbrace{\left[h(x) - \sum_{i=1}^{r} b_i\, g_i(x) \right]}_{\therefore \in \mathfrak{p}}$$

call this $h_1(x)$

$$g(x) = \sum b_i f_i + x\, h_1(x)$$

Can repeat the process on $h_1(x) \in \mathfrak{p}$ to give

$$h_1(x) = \sum c_i f_i + x\, h_2(x) \qquad \text{for } h_2(x) \in R[\![x]\!].$$

$$g(x) = \sum b_i f_i + x \sum c_i f_i + x^2 h_2(x)$$

Can continue so on to get $g(x) \in \langle f_1, \dots, f_r \rangle$.

$$g(x) = f_1 (b_1 + x c_1 + x^2 d_1 + \cdots)$$
$$+ \ldots$$

$\vdots$

$$+ f_r \left(b_r + x \left(c_r + x^2 d_r + \cdots \right) \right.$$

Lecture 4 (19-01-2021)

19 January 2021 13:52

Chapter 2: Associated primes of ideals and modules

$R \to$ commutative ring with 1. $I, J \trianglelefteq R$ are ideals.

Recall the colon of two ideals I, J is the ideal

(colon)

$$I :_R J := \{ r \in R \mid rJ \subset I \}.$$

(Analog of division.)

Suppose M, N are R-submodules of some R-module M'.

We define

$$M :_R N := \{ r \in R \mid rN \subseteq M \}.$$

$M :_R N$ is an ideal of R.

$$\operatorname{ann} M = 0 :_R M = \{ r \in R \mid rM = 0 \}.$$

(ann M or annihilator of M)

Example: $R = \mathbb{Z}$, $M = \mathbb{Z}/n\mathbb{Z}$ is a R-module

Suppose $n = p^a q^b$ p, q primes

$$(n : p^a q^{b-1}) = (q) \qquad (n : p^{a-1} q^b) = (p)$$

Note $I : J \supseteq I$ in general. Thus, can go modulo n.

$$\frac{(n : p^a q^{b-1})}{(n)} = \frac{(q)}{(n)} \qquad \frac{(n : p^{a-1} q^b)}{(n)} = \frac{(p)}{(n)}$$

prime ideal in $\mathbb{Z}/n\mathbb{Z}$

$$(q) = 0 :_{\mathbb{Z}} x \qquad (p) = 0 :_{\mathbb{Z}} y$$

$$x = p^a b^{q-1} + (n) \qquad y = p^{a-1} q^b + (n)$$

Def^n Let M be an R-module and $0 \neq x \in M$.
If $0 :_R x = \mathfrak{p}$ is a prime ideal in R, then we say that P is an associated prime of M.

(Associated primes)

$$\mathrm{Ass}(M) := \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = 0 : x \text{ for some } x \in M \setminus \{0\} \}$$

$$\mathrm{Ass}(\mathbb{Z}/n\mathbb{Z}) = \{ (p), (q) \}.$$

$$\mu_x : R \xrightarrow{x} M \qquad \mu_x = \text{the homothety by } x$$
$$r \longmapsto rx$$

μ_x is an R-linear map.

$$\ker \mu_x = \{ r \in R : rx = 0 \}$$
$$= \mathrm{ann}_R(x) = (0 : x)$$

If $(0 : x) = \mathfrak{p}$ is prime, then

$$\frac{R}{\ker \mu_x} = \frac{R}{\mathfrak{p}} \hookrightarrow \underset{\simeq}{M} \; Rx$$

Conversely, if $R/\mathfrak{p} \overset{\varphi}{\hookrightarrow} M$, φ R-linear, then $\mathfrak{p} = 0 : x$ for some $0 \neq x \in M$.

If $\varphi(1 + \mathfrak{p}) = x$, then
$\varphi(r + \mathfrak{p}) = rx$. $\therefore \tilde{\varphi} : R \longrightarrow M$ is μ_x
and $\mathfrak{p} = \ker \mu_x$.

Thus, alternate def^n:

$$\mathrm{Ass}\, M = \{ \mathfrak{p} \in \mathrm{Spec}\, R \mid R/\mathfrak{p} \hookrightarrow M \}.$$

(Associated primes are those $\mathfrak{p}$ s.t. $R/\mathfrak{p}$ injects into M as a submodule.)

Def^n: $a \in R$ is a zero divisor on M if

$ax = 0$ for some $0 \neq x \in M$.

(Zero divisors)

$Z(M)$ = set of zero divisors. $\left(Z(M) \subseteq R\right)$ not necessarily an ideal

Note that a is a zero divisor $\Leftrightarrow$ μ_a is not injective.

If μ_a is injective, then μ_a is called a non zero divisor on M, or M-regular.

(Non zero divisors or M-regular)

Note $\mathfrak{p} \in \text{Ass } M \Rightarrow \mathfrak{p} = 0 : x$ for some $x \in M \setminus \{0\}$
$\Rightarrow \mathfrak{p} \subseteq Z(M)$

Prop^n. (Existence of associated primes)

Let R be a Noetherian ring and $M \neq 0$ a f.g. R-module.
Then,

(a) Maximal elements among $\{(0:x) \mid x \in M\}$ are prime ideals.
Hence, $\text{Ass } M \neq \phi$.

(b) $Z(M) = \bigcup_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}$.

Proof. (a) $\mathcal{F} := \{(0:x) \mid x \in M \setminus \{0\}\}$.
Note that $\mathcal{F}$ is non empty $(M \neq 0)$ and contains only proper ideals. $(1 \notin (0:x)$ if $x \neq 0)$

As R is Noetherian, $\mathcal{F}$ has a maximal element (w.r.t. $\subseteq$).

Claim. Any maximal member of $\mathcal{F}$ is prime. Hence, $\text{Ass } M \neq \phi$.

Proof. Let $0:x$ be a maximal member.
Let $a, b \in R$ be s.t. $ab \in 0:x$, $a \notin 0:x$.
To show: $b \in 0:x$. That is, $bx = 0$.

$abx = 0 \quad (\because ab \in 0:x)$

$\Rightarrow b \in 0:ax \supseteq 0:x$

$\hookrightarrow$ proper since $ax \neq 0$ since $a \notin 0:x$

By maximality, $0:x = 0:ax \ni b$.

$\therefore$ $b \in 0:x$. Thus, $(0:x)$ is prime. ✓

(Didn't use M f.g. here. We usually keep the blanket assumption anyway.)

(b) We saw $Z(M) \supseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$ already.

($\subseteq$) Let $b \in Z(M)$. Thus, $bx = 0$ for some $x \in M \setminus \{0\}$.

$\in R$

That is, $b \in 0:x$.

Since $\mathcal{F}$ has maximal members, $0:x \subseteq 0:y$ for some maximal member $0:y$. But that is prime.

Thus, $b \in 0:y \subset \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$. ∎

Ex. $\mathbb{Z}/n\mathbb{Z}$, $n = p^a q^b$, then $Z(\mathbb{Z}/n\mathbb{Z}) = (p) \cup (q)$.

Ex. Let $\Bbbk$ be a field. $R = \Bbbk[x, y]$. $\rightarrow$ UFD

Consider $I = (x^2, xy) = (x) \cap (x^2, y)$

$I : x \ni x, y$. But $\frac{R}{(x, y)} \simeq k \to$ field.

Thus, (x, y) is maximal.

However, $x \notin I$. Thus, $I : x \neq R$.

Thus, $I : x = (x, y) = \mathfrak{m}$.

$M = R/I$, $0 : \bar{x} = \mathfrak{m}$. $\Rightarrow \mathfrak{m} \in \operatorname{Ass}(R/I)$.

$I : y \underset{\text{prove!}}{=} (x) = \mathfrak{p}$ prime

$\therefore$ $\mathfrak{p} \in \operatorname{Ass}(R/I)$. $\mathfrak{p} \subset \mathfrak{m}$.

We will see later: $\operatorname{Ass}(R/I) = \{\mathfrak{p}, \mathfrak{m}\}$.

Behaviour of associated primes under localisation

Let R be a Noetherian ring and $M \neq 0$ an R-module.
Let $S \subseteq R$ be an m.c.s. of R.

$$R \longrightarrow S^{-1}R \qquad\qquad M \longrightarrow S^{-1}M \cong S^{-1}R \otimes_R M$$

$$r \longmapsto \frac{r}{1} \qquad\qquad x \longmapsto \frac{x}{1}$$

$$S^{-1}M = \left\{ \frac{m}{s} \;\middle|\; \begin{array}{l} m \in M \\ s \in S \end{array} \right\}$$

$$\frac{m}{s} = \frac{m'}{s'} \Leftrightarrow \exists t \in S \text{ s.t. } t(ms' - sm') = 0.$$

(We will assume $1 \in S$.)

What is the connection $\mathrm{Ass}_R M \longleftrightarrow \mathrm{Ass}_{S^{-1}R} S^{-1}M$?

Recall : $\mathrm{Spec}(S^{-1}R) = \{ S^{-1}\mathfrak{p} : \mathfrak{p} \cap S = \phi \}$

What we will show is :

$$\mathrm{Ass}\ M \longleftrightarrow \mathrm{Ass}_{S^{-1}R}\ S^{-1}M$$

$$\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} \longleftrightarrow \{ S^{-1}\mathfrak{p}_i \mid \mathfrak{p}_i \cap S = \phi \}$$

Prop^n. ① $\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = \{ S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}\ M,\ \mathfrak{p} \cap S = \phi \}$

② $\mathfrak{p} \in \mathrm{Ass}\ M \Leftrightarrow \mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}} M_\mathfrak{p}$ → Reduces the problem to solving over local rings.

Proof. ①(⊇) Let $\mathfrak{p} \in \mathrm{Ass}\ M$, $\mathfrak{p} \cap S = \phi$.
Thus, $R/\mathfrak{p} \hookrightarrow M$.

(Recall that localisation preserves exactness)

$$0 \to R/\mathfrak{p} \to M \to \mathrm{coker} \to 0$$

localise

(S^{-1} commutes with this quotienting)

$$0 \to S^{-1}R/S^{-1}\mathfrak{p} \to S^{-1}M \to S^{-1}\,\mathrm{coker} \to 0$$

$\Rightarrow \quad S^{-1}\mathfrak{p} \in \mathrm{Ass}_{S^{-1}R}\, S^{-1}M$

$R \longrightarrow S^{-1}R$ $\qquad$ $S^{-1}M$ is an $S^{-1}R$ module,
also an R module (restriction of scalars)

Ex. $\mathrm{Ass}_R\, S^{-1}M = \{ \mathfrak{p} \in \mathrm{Ass}\, M \mid \mathfrak{p} \cap S = \phi \}$

($\subseteq$) Let $S^{-1}\mathfrak{p} \in \mathrm{Ann}_{S^{-1}R}\, S^{-1}M$ where $\mathfrak{p} \in \mathrm{Spec}\, R$ and $\mathfrak{p} \cap S = \phi$.
we know primes of $S^{-1}R$ are of this form

$S^{-1}P = 0 :_{S^{-1}R} \frac{x}{s}$ $\quad = \left\{ \frac{a}{t} \;\middle|\; \frac{a}{t} \cdot \frac{x}{s} = \frac{0}{1} \right\}$

for some $\frac{x}{s}$

$= \left\{ \frac{a}{t} \;\middle|\; \frac{ax}{1} = \frac{0}{1} \right\}$

$= \left\{ \frac{a}{t} \mid \exists u \in S \text{ s.t. } uax = 0 \right\}$

Write $\mathfrak{p} = (a_1, \dots, a_s)$.
Then, $S^{-1}\mathfrak{p}$ kills $\frac{x}{s}$.

That is, $\frac{a_i}{1}\frac{x}{s} = 0 \quad \forall i$

$\Rightarrow \exists s_i \in S$ s.t. $s_i a_i x = 0 \quad \forall i$

Put $s = s_1 \cdots s_n$. Then, $s a_i x = 0 \quad \forall i$.

$\Rightarrow a_i \in 0 : sx \quad \forall i$

$\Rightarrow \mathfrak{p} \subseteq (0 : sx)$.

We now show $\supseteq$.

Let $b \in (0 : sx)$. Then, $bsx = 0$.

$\Rightarrow \frac{b}{1} \cdot \frac{x}{1} = 0$ in $S^{-1}M$

$\Rightarrow \frac{b}{1} \cdot \frac{x}{s} = 0$

$\Rightarrow \frac{b}{1} \in 0 : \frac{x}{s} = S^{-1}\mathfrak{p}$

$\Rightarrow \frac{b}{1} = \frac{a}{1\ldots}$; $a \in \mathfrak{p}$

$t \in S$

$\Rightarrow \exists u \in S, \quad u(bt - a) = 0$

$\Downarrow$

$\underset{\notin \mathfrak{p}}{u}\ \underset{\notin \mathfrak{p}}{bt} = \underbrace{ua}_{\in \mathfrak{p}}$

$\Rightarrow b \in \mathfrak{p}$ ☒

(b) Take $S = R \setminus \mathfrak{p}$. $\quad \mathfrak{p} \in \operatorname{Ass} M \iff \underset{\mathfrak{p}R_{\mathfrak{p}}}{\underset{\shortparallel}{S^{-1}\mathfrak{p}}} \in \operatorname{Ass}_{R_{\mathfrak{p}}} \underbrace{S^{-1}M}_{M_{\mathfrak{p}}}$

Connection between Ass M and supp M

Recall: $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

If M is f.g., the $\operatorname{Supp} M = V(\operatorname{ann} M)$.
In particular, $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} R$.
in Zariski topology

Note that the complement is open and dense.

Prop^n. Let $0 \to N \to M \to L \to 0$ be an exact sequence of R-modules. Then,

$$\operatorname{supp} M = \operatorname{supp} N \cup \operatorname{supp} P.$$

Proof. ($\subseteq$) Let $\mathfrak{p} \in \operatorname{Supp} M$ and suppose $\mathfrak{p} \notin \operatorname{supp} N$.
That is, $M_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} = 0$. We show $L_{\mathfrak{p}} \neq 0$.

Note that $0 \to N \to M \to L \to 0$ is exact.

$\Rightarrow \quad 0 \to N_{\mathfrak{p}} \to M_{\mathfrak{p}} \to L_{\mathfrak{p}} \to 0$ is exact

$\Rightarrow \quad 0 \to M_{\mathfrak{p}} \to L_{\mathfrak{p}} \to 0$ is exact

$\Rightarrow \quad \underset{\neq 0}{M_{\mathfrak{p}}} \simeq L_{\mathfrak{p}} \qquad \therefore L_{\mathfrak{p}} \neq 0$ or $\mathfrak{p} \in \operatorname{Supp} L$.

($\supseteq$) Suppose $\mathfrak{p} \in \operatorname{Supp} N$.

$0 \to N \to M$ exact $\Rightarrow$ $0 \to \underset{\neq 0}{N_p} \xrightarrow{1\text{-}1} M_p$ exact

$\therefore M_p \neq 0$
$\Rightarrow p \in \operatorname{Supp} M$

Similarly, let $p \in \operatorname{Supp} L$. $M \to L \to 0$ exact

$\Rightarrow M_p \to L_p$ is surjective

and $L_p \neq 0$.

Thus, $M_p \neq 0$ or $p \in \operatorname{Supp} M$. ∎

Prop^n. Let L, K be f.g R-modules

Then,

$$\operatorname{Supp}(L \otimes_R K) = \operatorname{Supp} L \cap \operatorname{Supp} K.$$

In particular, $\operatorname{Supp} M/IM = \operatorname{Supp} M \cap V(I)$

Proof. ($\subseteq$) Let $p \in \operatorname{Supp}(L \otimes K)$.

Note $(L \otimes_R K)_p \simeq L_p \otimes_{R_p} K_p$.

Thus, $L_p, K_p \neq 0$. Thus, $p \in \operatorname{Supp} L \cap \operatorname{Supp} K$.

($\supseteq$) Let $p \in \operatorname{Supp} L \cap \operatorname{Supp} K$.

To show: $(L \otimes_R K)_p \neq 0$.

Note

$$(L \otimes_R K)_p \simeq L_p \otimes_{R_p} K_p$$

R_p is a local ring with maximal ideal pR_p.

Moreover, L_p, K_p are f.g. R_p-modules

Suffices to prove the following:

Prop^n. Let (R, m) be local and L, K f.g. R-modules.

Then, $L \otimes_R K \neq 0$.

Proof. Look at $\dfrac{L}{mL} \otimes_{R/m} \dfrac{K}{mK}$. L/mL & K/mK are fin dim R/m v. spaces.

Note $\dim_{\mathbb{k}}(V_1 \otimes_{\mathbb{k}} V_2) = \dim_{\mathbb{k}} V_1 \dim_{\mathbb{k}} V_2$

Thus, $\frac{L}{\mathfrak{m}L} \otimes_{R/\mathfrak{m}} \frac{K}{\mathfrak{m}K} \neq 0$. In turn, $L \otimes_R K \neq 0$. ∎

Connection of Ass M and Supp M.

$\mathfrak{p} \in \text{Ass } M \Rightarrow R/\mathfrak{p} \hookrightarrow M$ — localization preserves exactness

$\Downarrow$

field $\rightarrow \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \hookrightarrow M_{\mathfrak{p}}$ $\qquad \therefore M_{\mathfrak{p}} \neq 0$

$\neq 0$

Thus, $\text{Ass } M \subset \text{Supp } M$.

Converse not true. Take $\mathbb{k}$ as an inf. field.

$(x) \subseteq k[x, y] = R$

$\text{Supp}(R/(x)) = V((x)) \ni (x, y - \alpha)_{\alpha \in k}$

$\uparrow$ maximal ideals

$\text{Ass}_R(R/(x)) = \{ (0 : \bar{f}) \text{ prime} \quad x \nmid f \}$

$gf \in (x)$

$x \mid gf$ but $x \nmid f$

$\Rightarrow x \mid g$

$\Rightarrow (g) \subseteq (x)$

Ex. $\text{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$

Thus, $\text{Ass}_R(R/(x))$ is a singleton, whereas Supp is infinite.