$\mathbb{R}$eal Analysis

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§1. Sets and stuff

§2. Topology

1. Let X be a metric space and let $U \subset X$. Define the *boundary* of U as
$$\partial U = \bar{U} \cap \overline{(U^c)}.$$
Show that $\partial U = U \setminus U^\circ$.

2. Prove or disprove that
$$(\partial U)^\circ = \varnothing$$
for any subset U of any metric space X.
HINT: Disprove it. Even in the case that $X = \mathbb{R}^n$.

3. Let (X, d) be a metric space and $x \in X$. Let $\delta > 0$. Define the following sets:
$$B_\delta(x) := \{y \in X \mid d(x, y) < \delta\},$$
$$C_\delta(x) := \{y \in X \mid d(x, y) \leq \delta\}.$$
Show that $\overline{B_\delta(x)} \subset C_\delta(x)$.
Can this inclusion be proper?
HINT: Not if you stay in $\mathbb{R}^n$. Think about other spaces.

4. **Topological Nim**
You and your friend want to play Topological Nim. Here's how it works:
Let X be your favourite compact metric space and $r > 0$ your favourite (positive) real number.
Each player removes an open disk of radius r from the space on their turn (only the center of the disk must not have been removed in a prior move), until one player—the winner—removes what remains of the space on his turn.

Show that no matter what moves are played, the game stops after a finite number of moves. (In other words, there is no infinite sequence of legal moves.)

Bonus: Fix $n \in \mathbb{N}$ and $r > 0$. Assuming optimal play, who will win the game if
$$X = S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$
with the standard metric?
(The answer will depend on r.)

Credits: https://puzzling.stackexchange.com/questions/99859/

§3. Continuity and derivatives

§4. Integration

§5. Sequence and series of functions