Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

Bruns & Herzog → Cohen-Macaulay rings
 - 1st Part

Affine algebra

X

Derived Category

- Gelfand & Manin
- E. Riehl's : Cat theory in context
- Dwyer & Spalinski

$R \to$ ring (possibly non comm.)

R-complexes :

$$\cdots \to M_{i+1} \xrightarrow{\partial} M_i \xrightarrow{\partial} M_{i-1} \to \cdots \qquad \partial^2 = 0$$

$$\mathrm{im}(\partial_{i+1}) \subseteq \ker(\partial_i)$$

$$H_i(M) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

$$H(M) = (H_i(M))_{i \in \mathbb{Z}}$$

$\mathcal{C}(R)$ = category of R-complexes (morphisms as usual)

If $f : M \to N$, we get an induced map $H(f) : H(M) \to H(N)$.

Def^n f is a quasi isomorphism (or weak equivalence) if $H(f)$ is bijective. (Automatically iso.)

$W :=$ collection of weak equivalences in $\mathcal{C}(R)$

$D(R) := \mathcal{C}(R)[W^{-1}]$. (or $W^{-1}\mathcal{C}(R)$)

- Key property : W has the 2-out-of-6 property:

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot,$

- Key property: W has the 2-out-of-6 property:

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,
if gf and $hg \in W$, then f, g, h, hgf are in W.

Ex: $\Rightarrow$ 2-out-of-3 property
If f, g, fg are defined and 2 are in W, then so is the third.

Concretely: $\mathcal{C}(R) \rightsquigarrow K(R) \rightsquigarrow D(R)$.

$\uparrow$ homotopy category

$M, N \to$ R-complexes

$\mathrm{Hom}_R(M, N) :=$ Hom-complex of abelian groups (when R is comm. this is an R-complex)

$\mathrm{Hom}_R(M, N)_n :=$ Maps of degree n from M to N (no compatibility!)

$$\begin{tikzcd}
\cdots \arrow[r] & M_i \arrow[r] \arrow[dl] & M_{i-1} \arrow[r] \arrow[dl] & \cdots \\
\cdots \arrow[r] & N_{i+n} \arrow[r] & N_{i+n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$

$$\partial : \mathrm{Hom}_R(M, N)_{n+1} \longrightarrow \mathrm{Hom}_R(M, N)$$

$$\partial(f) = \partial^N f - (-1)^{n+1} f \partial^M .$$

Check: $\partial^2 = 0$.

Observe: $Z_0(\mathrm{Hom}_R(M, N)) = \mathrm{Hom}_{\mathcal{C}}(M, N)$.

Def. $f, g \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ are homotopic if
$f - g \in B_0(\mathrm{Hom}_R(M, N))$, i.e.,
$f - g = \partial h$ for some $h \in \mathrm{Hom}_R(M, N)$,

$K(R) := \mathcal{C}(R) / \text{homotopy relation.}$

Object = R-complexes

$\mathrm{Hom}_K(M,N) = H_0(\mathrm{Hom}_R(M,N))$.

- $f \sim g$ in $\mathcal{C} \Rightarrow f = g$ in $K(R)$.

  $\Rightarrow H(f) = H(g)$

Defn. M an R-complex.

ΣM (or $M[1]$) is the R-complex

$$(\Sigma M)_i = M_{i-1}$$

with $\partial^{\Sigma M} = -\partial^M$.

$\mathrm{Proj}\, R :=$ Projective R-modules

$$\begin{tikzcd} K(R) \arrow[dr, "q"] & \\ \cup & \\ K(\mathrm{Proj}\, R) \arrow[r, shift right, "q"'] & D(R) \arrow[l, dashed, shift right, "p"'] \end{tikzcd}$$

q = localisation

$\exists p : D(R) \to K(\mathrm{Proj}\, R)$, a full and faithful embedding

"projective resolutions"

left adjoint to q.

$$\mathrm{Hom}_K(pM, N) = \mathrm{Hom}_D(M, qN)$$

$f : M \to N$ morphism

$\mathrm{cone}(f) := N \oplus \Sigma M$ with differential

$$\begin{matrix} N_{i+1} \\ \oplus \\ M_i \end{matrix} \longrightarrow \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \qquad \partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$$0 \to N \hookrightarrow \mathrm{cone}(f) \to \Sigma M \to 0.$$

f is w.e. $\iff H(\mathrm{cone}(f)) = 0$.

Image of p?

K-projectives.

P an R-complex is K-projective if given any solid diagram

$$\begin{tikzcd} & M \arrow[d, "\pi", "?"'] \\ P \arrow[r, "\alpha"'] \arrow[ur, dashed, "\tilde{\alpha}"] & N \end{tikzcd} \quad \text{w.e.}$$

$\exists$ lift $\tilde{\alpha}$.

FACT: $p : D(R) \xrightarrow{\sim} K\text{-}Proj(R) \subseteq K(Proj\ R)$.

morphisms here are up to homotopy

- $Hom_{\mathbb{Z}}(R, M) = Z_0(M)$

$$\begin{tikzcd} 0 \arrow[d] & \\ R \arrow[r] \arrow[d] & M_0 \arrow[d, "\partial"] \\ 0 \arrow[r] & M_{-1} \end{tikzcd}$$

Using this, check: R is K-projective.

Use: surjective + w.e. $\Downarrow$ $Z(M) \twoheadrightarrow Z(N)$ onto

- $(P_\lambda)_\lambda$ family of K-projectives

  Then, $\bigoplus P_\lambda$ is also K-projective.

  Conversely closed under direct summands.

- K-projectives are closed under suspensions.

  $\cdots \xrightarrow{\partial} P_{i+1} \xrightarrow{\partial} P_i \xrightarrow{\partial} P_{i-1} \xrightarrow{\partial} \cdots$

  is K-proj, if P_i projective $\forall i$.

Ex: $0 \to P' \xrightarrow{f} P \xrightarrow{g} P'' \to 0$ exact seq. of complexes.

Then, if P' and P'' are K-projective, so is P.

If P', P, P'' are complexes of projectives, then any two being K-projective $\Rightarrow$ third is K-proj.

of ...

any two being K-projective $\Rightarrow$ third is K-proj.

<u>Corollary</u>. Any bounded complex of projectives is K-projective.

<u>Proof</u>. $P_\bullet : \; 0 \to P_b \dashrightarrow P_a \to 0$, each P_i proj.

Induce on $b-a$.

$b-a = 0$ done earlier.

$$0 \longrightarrow P_{\leq b-1} \longrightarrow P \longrightarrow \Sigma^b P_b \to 0. \quad \blacksquare$$

$$P_{\leq b-1} = \quad 0 \to P_{b-1} \to \cdots \to P_a \to 0$$

Next: Any complex of projectives with $P_i = 0 \;\; \forall\, i \ll 0$ is K-projective.

$$P_\bullet : \quad \cdots \to P_{a+1} \to P_a \to 0$$

$P = \varinjlim_{n \geq a} P_{\leq n}$, each $P_{\leq n}$ is projective since bounded.

$$0 \to \bigoplus_n P_{\leq n} \xrightarrow{1-\delta} \bigoplus_n P_{\leq n} \to P \to 0.$$

K-proj

projectives

Use 2-out-of-3.

Do directly...

- $\begin{array}{c} M \\ \pi \downarrow \text{ w.e.} \\ N \end{array}$ $\qquad$ P is K-projective $\Leftrightarrow$ $\mathrm{Hom}_K(P, M) \cong \mathrm{Hom}_K(P, N)$.

Lecture 2 (11-01-2023)

11 January 2023 13:24

R ring.

$D(R) \simeq k\mathrm{Proj}(R)$.

Recall: $P \in \mathcal{C}(\mathrm{Proj}\ R)$ is K-projective if

$$\begin{tikzcd} & X \arrow[d, "\simeq"] \\ P \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Example $P \in \mathcal{C}(\mathrm{Proj}\ R)$ with $P_i = 0$ for all $i << 0$.

$$\cdots \to P_{a-1} \to P_a \to 0 \to 0 \to \cdots$$

Sketch. Construct lifting one step at a time.

Suppose $\tilde{\alpha} : P_{\leq n} \to X$ is a lifting.

Want $\tilde{\alpha} : P_{\leq n+1} \to X$ compatibly.

Let $s \in P_{n+1}$

We must have

- $\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$
- $\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$

$$\begin{tikzcd} & & X \arrow[d, "\varepsilon", "\simeq"'] \\ \cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[ur, "\tilde{\alpha}"] & (P_n \to P_{n-1} \to \cdots \to P_a \to 0) \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

Check that the above can be solved.

This uses three things:
- ε surjective $\Rightarrow$ ε surjective on boundaries
- $H(\varepsilon)$ iso $\Rightarrow$ ε surjective on cycles
- $\ker(\varepsilon)$ is acyclic.

$\Rightarrow$ Every module has a K-projective resolution.

$$\begin{tikzcd}[column sep=small] F: \cdots & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[dr, "\varepsilon"] & \\ & & \ker \partial_1 \arrow[ur] & & \ker \varepsilon \arrow[ur] & & M \end{tikzcd}$$

(Can avoid choices by taking generating set to be M, $\ker\varepsilon$, $\ker\partial_1, \ldots$)

$$F \xrightarrow{\varepsilon} M.$$

Def^n A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism $\varepsilon : P_\cdot \to M$ s.t.

① ε is a quasi iso, (Not insisting surjective.)

② $P_\cdot$ is K-projective.

② $P_\cdot$ is K-projective.

This makes it functorial!

Thm $\forall\, M_\cdot \in \mathcal{C}(R)$, $\exists$ surjective K-projective resolution:

$$\underset{\text{K-proj.}}{P_\cdot} \xrightarrow{\sim} M_\cdot$$

Defn. An R-complex F is semi-free if F admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \cdots \subseteq \bigcup_{n\geq 0} F(n) = F$$

s.t. ① $F(n) \subseteq F$ is a subcomplex,

② $F(n+1)/F(n)$ graded free module with $\partial = 0$,

i.e. $\partial(F(n+1)) \subseteq F(n)$

Fact/Ex. Semifree $\Rightarrow$ K-proj.

Example. $\cdots \to F_{a+1} \to F_a \to 0$

$$F(n) = F_{\leq n} = 0 \to F_n \to \cdots \to F_a \to 0$$

$$\frac{F(n+1)}{F(n)} = \cdots \to 0 \to F(n+1) \to 0 \to \cdots$$

$$= \Sigma^{n+1} F_{n+1}.$$

Thm. Each $M_\cdot \in \mathcal{C}(R)$ has a surjective semi-free resolution

$$\underset{\text{K-projective}}{F_\cdot} \xrightarrow{\sim} M_\cdot.$$

Corollary. Every K-projective is a retract of a semifree.

Proof. P is K-projective:

$$\begin{tikzcd} & F \arrow[d, "\wr"] \\ P \arrow[ur, dashed] \arrow[r, "id"', equal] & P \end{tikzcd}$$

$\blacksquare$

→ **Sketch** (Baby ver of Quillen's "small object argument")

Construct $0 = F(0) \hookrightarrow F(1) \xrightarrow{i(1)} F(2) \xrightarrow{i(2)} \cdots \hookrightarrow F(n) \xrightarrow{i(n)} \cdots \hookrightarrow \bigcup F(n) = F$

Construct

$$0 = F(0) \hookrightarrow F(1) \xrightarrow{i(1)} F(2) \overset{i(2)}{\hookrightarrow} \cdots \hookrightarrow F(n) \overset{i(n)}{\hookrightarrow} \cdots \hookrightarrow \bigcup_{n \geq 0} F(n) = F$$

with maps $\varepsilon(0), \varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n)$ to M

s.t. ① $F(n+1)/F(n)$ is graded free with $\partial = 0$,

② $\varepsilon(1)$ is surjective on homology.

(In turn, each $\varepsilon(n)$ is surjective on homology.)

③ $\ker(H(\varepsilon(n))) \subseteq H(F(n))$ maps to 0 under $H(i(n))$.

This does the job. ["Something 0 in colim is 0 at finite stage."]

Why is ε surjective?

Remark. $\varepsilon : X \to Y$ s.t. $Z(\varepsilon)$ surjective + $H(\varepsilon)$ bijective. Then, ε is surjective.

Indeed, we have s.e.s.es:

$$\begin{array}{ccccccccc} 0 & \to & B(X) & \to & Z(X) & \to & H(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B(Y) & \to & Z(Y) & \to & H(Y) & \to & 0 \end{array}$$

Snake lemma $\Downarrow$ $B(X) \to B(Y)$ is epi

$$\begin{array}{ccccccccc} 0 & \to & Z(X) & \to & X & \to & \Sigma B(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z(Y) & \to & Y & \to & \Sigma B(Y) & \to & 0 \end{array}$$

$\Downarrow$ $X \to Y$ is surj

Construction of $F(n), \varepsilon(n)$:

$\varepsilon(1) : F(1) \to M$ free cover of cycles.
0 diffs
($\varepsilon(1)$ morphism since mapping on cycles)

Say we have constructed $\varepsilon(n) : F(n) \longrightarrow M$.

Choose cycles $(z_\lambda)_\lambda \subseteq F(n)$ that map to a generating set of $\ker(H(\varepsilon(n)))$

Pick w_λ s.t. $\partial(w_\lambda) = \varepsilon(n)(z_\lambda)$.

Set $F(n+1) = F(n) \oplus Re_\lambda$ $\quad \deg(e_\lambda) = \deg(z_\lambda) + 1$.
with $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_\lambda) = z_\lambda.$$

Define $\varepsilon(n+1) : F(n+1) \to M$

$$\varepsilon(n+1)|_{F(n)} = \varepsilon(n),$$

$$\varepsilon(n+1)(e_\lambda) = \omega_\lambda.$$

$\boxtimes$

Remarks: ① As before the above construction can be made functorial by avoiding choices (consider all choices!).

② Depending on what we wish to do with the resolution, there are other constructions.

Given a module, we have the graded homology module $H(M) = \langle H_i(M) \rangle_{i \in \mathbb{Z}}$.

(Recall: for us, a graded module is a collection of modules.)

If $P_\cdot \xrightarrow{\sim} H(M)$ is a projective resol^n, one can "perturb" the differentials of $P_\cdot$ to construct a K-projective resolution of M.

(Adam's resolution, Cartan/Eilenberg resolution)

Exercise. $P_\cdot$ K-proj $\Rightarrow$ P_i projective $\forall i$.

Converse of above NOT true.

Example (Dold's): Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the complex

$$P_\cdot : \quad \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} \cdots$$

One way of seeing that the above is not K-projective is to do the following exercise and note that $P_\cdot$ is acyclic but not contractible.

is to do the following exercise ...

$P_\cdot$ is acyclic but not contractible.

Exercise. If P is K-proj and $H(P) = 0$, then P is contractible, i.e., $\mathrm{id}_P \sim 0$.

(or: P is the mapping cone of some idc.)

We saw we have an inclusion

$$\mathrm{KProj}(R) \hookrightarrow K(\mathrm{Proj}\ R).$$

FACT. Let R be comm. Noetherian. The above inclusion is an equality iff R is regular.

Examples of reg. rings: $\mathbb{Z}$, $k[x_1, \ldots, x_n]$.

Derived functors

Let $M_\cdot$ be an R-complex.

FACT. If $P_\cdot \xrightarrow{\sim} M_\cdot$ and $Q_\cdot \xrightarrow{\sim} M_\cdot$ are K-projective resolutions, then $P_\cdot \simeq Q_\cdot$ in $K(\mathrm{Proj}\ R)$.

Given any $N_\cdot \in \mathcal{C}(R)$, set

$$\mathrm{RHom}_R(M_\cdot, N_\cdot) := \mathrm{Hom}_R(P_\cdot, N_\cdot),$$

where $P_\cdot$ is a K-proj. res^n of $M_\cdot$

The object on the right is defined in the homotopy category of abelian groups, i.e.,

$\mathrm{RHom}_R(-, N_\cdot)$ is a functor

$\mathcal{C}(R) \to K(\mathbb{Z})$.

$R\mathrm{Hom}_R(-, N.)$ is a functor

$$\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$$

(If R is comm, then $\longrightarrow K(R)$.)

Define $\quad \mathrm{Ext}^i_R(M., N.) := H^i(R\mathrm{Hom}_R(M., N.))$

$$= H_{-i}(\mathrm{Hom}_R(P., N.))$$

$$\mathrm{Ext}^0_R(M, N) = H_0(\mathrm{Hom}_R(P., N.))$$

$$= \text{morphisms } P. \longrightarrow N., \text{ up to homotopy}$$

$$\mathrm{Ext}^i_R(M, N) = \text{—}''\text{——— } P. \to \Sigma^i N., \text{ —}''\text{—}$$

If $Q \xrightarrow{\sim} N$ is a K-proj resol^n, then

$$\mathrm{Hom}_R(P., Q.) \cong \mathrm{Hom}_R(P., N.).$$

quasi-iso

Tensors $\quad$ Let $M.$ be a chain complex of <u>right</u> R-modules.

Let $N. \in \mathcal{C}(R)$.

$M. \otimes N.$ is a complex of $\mathbb{Z}$-modules defined by

$$(M. \otimes N.)_i = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-1)^{|m|} m \otimes \partial n$$

<u>FACT</u>. If $X. \xrightarrow{\sim} Y.$ is a quasi iso, then $P. \otimes_R X. \xrightarrow{\sim} P. \otimes Y.$ for any K-proj $P.$

Defn. $M_\cdot \otimes_R^L N_\cdot := P_\cdot \otimes_R N_\cdot$ where $P_\cdot \xrightarrow{\sim} M_\cdot$ is a K-proj. resolⁿ.

$$\mathrm{Tor}_i^R(M_\cdot, N_\cdot) = H_i(P_\cdot \otimes_R N_\cdot).$$

$X \xrightarrow{\sim} Y$ quasi iso $\Rightarrow \mathrm{Tor}_i^R(M_\cdot, X_\cdot) = \mathrm{Tor}_i^R(M_\cdot, Y_\cdot)$.

Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \to$ comm. Noetherian ring

$M \to$ R-module

$r \in R$ is a zerodivisor on M if $r \cdot m = 0$ for some $m \neq 0$.

nzd = "not a zero divisor"

$$Z_R(M) = \{ r \in R : r \text{ is a z.d. on } M \}$$

$$= \bigcup_{p \in \mathrm{Ass}(M)} p.$$

(M need not be finite. Union need not be)

Fix R, M. Let $\underline{x} = x_1, \ldots, x_n$ be a sequence in R.

$\underline{x}$ is weakly M-regular or a weakly regular sequence on M

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \ldots, x_i)M} \quad \text{for } 0 \leq i \leq n-1.$$

$\underline{x}$ is M-regular (or ...) if further $M/(\underline{x})M \neq 0$.

Ex. $R = k[x_1, \ldots, x_n]$.
$\underline{x} := x_1, \ldots, x_n$ is a regular sequence on R.

Koszul Complexes. Given $r \in R$,

$$K(r; R) = 0 \to \underset{\deg 1}{R} \xrightarrow{r} \underset{\deg 0}{R} \to 0.$$

$$H_1(K(r;R)) = 0 \iff r \text{ is nzd on } R.$$

Given $\underline{x} = x_1, \ldots, x_n$, we define

$$K(\underline{x}; R) = \bigotimes_{i=1}^{n} K(x_i; R).$$

$$K(\underline{x}; R) = 0 \to R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{\cdot (x_1 \cdots x_n)} R \to 0$$

$$K(\underline{x}; R) = 0 \to R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{\cdot (x_1 \cdots x_n)} R \to 0$$

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mapsto \sum r_i x_i$$

Now, given $M \in \mathcal{C}(R)$,

$$K(\underline{x}; M) := K(\underline{x}, R) \otimes M.$$

$\hookrightarrow$ Koszul complex on $\underline{x}$ with coefficients in M.

$$H_i(\underline{x}; M) := H_i(K(\underline{x}; M)).$$ $\to$ Koszul homology

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If M is simply an R-module (viewed in degree 0), then

$K(\underline{x}; M)$:

$$0 \to M \to M^n \to \cdots \to M^n \to M \to 0$$

"same" differentials

$$H_0(\underline{x}; M) = M/\underline{x}M,$$

$$H_n(\underline{x}; M) = \{m \in M : x_i m = 0 \ \forall i\}$$

$$= (0 :_M (\underline{x})).$$

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① $K(\underline{x}; M) = K(x_1; R) \otimes_R K(x_2; R) \otimes_R \cdots \otimes K(x_n; R) \otimes M$

$$= K(x_1; R) \otimes K(x_{\geq 2}, M)$$

$$= K(x_1; K(x_{\geq 2}, M))$$

② $X, Y \in \mathcal{C}(R) \rightsquigarrow X \otimes_R Y \xrightarrow{\sim} Y \otimes_R X$ as R-complexes.

$$x \otimes y \longmapsto (-1)^{|x||y|} y \otimes x$$

$\therefore$ $K(\underset{\sim}{x}; R) \cong K(\underset{\sim}{x}^{\sigma}; R)$ for any $\sigma \in S_n$.

$\Rightarrow$ $K(\underset{\sim}{x}; M) \cong K(\underset{\sim}{x}^{\sigma}; M)$ — " —.

(Can apply this to Obs ①.)

<u>2nd Perspective</u>: "Koszul complexes are iterative mapping cones."

$f: X \to Y$ morphism of complexes

$$\mathrm{Cone}(f) = \left(Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ & -\partial^X \end{pmatrix}\right).$$

s.e.s. : $0 \to Y \to \mathrm{Cone}(f) \to \Sigma X \to 0$.

$y \mapsto \binom{y}{0}$

$\binom{y}{x} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow[H_i(f)]{} H_i(Y) \to H_i(\mathrm{Cone}(f)) \to \underset{\substack{\cong \\ H_{i-1}(X)}}{H_i(\Sigma X)} \to \cdots$$

connecting map

<u>Consider</u>: $x \in R$

$f: R \xrightarrow{x} R$

$1 \mapsto x$

$$\mathrm{Cone}(R \xrightarrow{x} R) = \left(\underset{\substack{\uparrow \\ \deg 0}}{R} \oplus \underset{\substack{\uparrow \\ \deg 1}}{R}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = K(x; R).$$

<u>Ditto</u>: If $x \in R$, and $M \in \mathcal{C}(R)$ → complex, not necessarily in deg 0

$$\mathrm{Cone}(M \xrightarrow{x} M) = K(x; M)$$

$\underline{x} = x_1, x_2, \ldots, x_n$

$$K(\underline{x}; M) = K(x_1 ; K(x_{\geq 2}; M))$$
$$= \operatorname{Cone}\left(K(x_{\geq 2}; M) \xrightarrow{x_1} K(x_{\geq 2}; M)\right)$$

on homology

iterate
⋮

$$H_i(x_{\geq 2}; M) \xrightarrow{\pm x_1} H_i(x_{\geq 2}; M) \to H_i(\underline{x}; M)$$
$$\downarrow$$
$$H_{i-1}(x_{\geq 2}; M)$$
$$\downarrow \pm x_1$$
$$\vdots$$

s.e.s.

$$0 \to \frac{H_i(x_{\geq 2}; M)}{x_1 H_i(x_{\geq 2}; M)} \to H_i(\underline{x}; M) \to (0 :_{H_{i-1}(x_{\geq 2}; M)} x_1) \to 0$$

$M \to R$-module

$$K(\underline{x}; M) \rightsquigarrow H_0(x; M) = \frac{M}{\underline{x} M}$$

So, $K(\underline{x}, M) \to \dfrac{M}{\underline{x} M}$ is a w.e. quasi iso

$\Leftrightarrow H_i(\underline{x}; M) = 0 \quad \forall i > 0.$

Def^n. $\underline{x}$ is Koszul-regular on M (or...) if

$$H_i(\underline{x}; M) = 0 \ \forall\, i \geq 1.$$

Lemma. When $\underline{x}$ is weakly M-reg, ($M \to R$-mod)

(weakly-reg $\Rightarrow$ Koszul-reg)

$$K(\underline{x}, M) \twoheadrightarrow M/(\underline{x})M \text{ is a w.eq.}$$

Proof: $n=1$: $0 \to M \xrightarrow{x} M \to 0.$

$H_1(x; M) = 0 \Leftrightarrow x$ nzd on M

$n \geq 2$: $K(\underline{x}; M) = K(x_n ; K(x_{<n}; M)).$

By induction,

$$K(x_{<n};M) \xrightarrow{\sim} \frac{M}{(x_{<n})M}.$$

Now, $0 \to R \xrightarrow{x_n} R \to 0$ is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x};M) = K(x_n;R) \otimes_R K(x_{<n};M)$$

$$\simeq K(x_n;R) \otimes_R \frac{M}{(x_{<n})M}$$

now note that x_n is a nzd on $\cancel{\frac{M}{(x_{<n})M}}$ and we are done. ∎

Instead of semifree, can use l.e.s. of homology and induction.

(Semifree business $\Rightarrow$ ($M \simeq N$ quasi $\Downarrow$ $K(\underline{x};M) \simeq K(\underline{x};N)$ quasi))

Note: ① $\underline{x}$ Koszul-reg $\Rightarrow$ $\underline{x}^\sigma$ is Koszul-reg $\forall \sigma \in S_n$.

② Not true for weakly regular.

Theorem. Say $\underline{x} \subseteq J(R)$ and $M \neq 0$ f.g. R-module.

TFAE:

1) $\underline{x}$ is M-regular. ($\equiv$ weakly M-reg, by NAK.)
2) $\underline{x}$ is Koszul M-regular, i.e., $H_i(\underline{x};M) = 0$ $\forall i \geq 1$.
3) $H_1(\underline{x};M) = 0$.

In particular, take R local and x_i nonunits.

Proof. ① $\Rightarrow$ ② $\Rightarrow$ ③ is clear.

Only need to prove ③ $\Rightarrow$ ①.

Already saw for $n=1$.

Induction:

$$K(\underline{x};M) = K(x_n; K(x_{<n};M)).$$

l.e.s.

$$0 \to \frac{H_i(\underline{x}_{<n}; M)}{x_n H_i(\underline{x}_{<n}; M)} \to H_i(\underline{x}; M) \to (0 :_{H_{i-1}(x_{<n}; M)} x_n) \to 0. \quad (*)$$

Put $i=1$ to get $\dfrac{H_1(x_{<n}; M)}{x_n H_1(x_{<n}; M)} = 0$

$$\xRightarrow{\text{NAK}} H_1(x_{<n}; M) = 0.$$

(Note: Koszul homology modules are f.g. when M is f.g.)

$\xRightarrow{\text{induction}}$ $x_1, \ldots, x_{n-1}$ is M-reg. — (1)

Moreover, $(*)$ now tells us

$$(0 :_{H_0(x_{<n}; M)} x_n) = 0.$$
$$\| $$
$$\ker\left(\frac{M}{x_{<n} M} \xrightarrow{x_n} \frac{M}{x_{<n} M}\right).$$

$\therefore$ x_n is nzd on $M/(x_{<n})M$. — (2)

(1) & (2) finish. ∎

Corollary. $I \subseteq J(R)$, M f.g., the property of $\underline{x}$ being regular is not dependent on the order of x_i.

(Permutation of regular is regular.)

$R = k[x, y, z]$

$x, \; y(1-x), \; z(1-x)$ reg

$y(1-x), \; z(1-x), \; x$ NOT

Lemma. If $\underline{x} = x_1, \ldots, x_n \subseteq R$, M an R-module.

TFAE:

① $\underline{x}$ is M-Koszul-regular.

② $\underline{x}^a$ is M-Koszul-regular for some $\underline{a} \geq \underline{1}$.

Proof. Suffices to prove:

$x_1, \boxed{x_2, \ldots, x_n}$ is M-KR

$\iff x_1^a, \boxed{x_2, \ldots, x_n}$ is M-KR for some $a \geq 1$.
(/all)

$x_1, x_2, \ldots, x_n$ KR

$\Rightarrow K(x_1 ; K(x_{\geq 2} ; M)) \xrightarrow{\sim} K\left(x_1 ; \dfrac{M}{(x_{\geq 2})M}\right)$.

Replacing M by $M/(x_{\geq 2})M$ we are reduced to $n = 1$.

But

x is reg on M

$\iff$ x is nzd on M

$\iff$ x^a is nzd on M for some $a \geq 1$

$\iff$ x^a is reg on M ———— " ————.

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$ and M f.g. R-module.

Let $i \geq 0$ be s.t. $H_i(\underline{x} ; M) = 0$.

Then, $H_j(\underline{x} ; M) = 0 \quad \forall j \geq i$.

HW.

Lecture 4 (23-01-2023)

Monday, January 23, 2023 1:19 PM

$R \to$ commutative Noetherian

Given complexes, $M, N \in \mathcal{C}(R)$.

$$R\mathrm{Hom}_R(M,N) := \mathrm{Hom}_R(pM, N)$$

$pM \xrightarrow{\sim} M$ is a K-proj res^n

$$\mathrm{Ext}^*_R(M,N) := H^i(R\mathrm{Hom}_R(pM,N)).$$

Given $M, N, P \in \mathcal{C}(R)$, we have a morphism

$$\theta : R\mathrm{Hom}_R(M,N) \otimes^L_R P \longrightarrow R\mathrm{Hom}_R(M, N \otimes^L_R P).$$

Lemma. θ is a w.e. when P is perfect, i.e,

$$P \simeq (0 \to P_b \to \cdots \to P_a \to 0),$$

each P_i f.g. R-module.

$$\mathrm{Hom}_R(pM, N) \otimes_R pP \longrightarrow \mathrm{Hom}_R(pM, N \otimes_R pP)$$

$$f \otimes x \longmapsto m \mapsto (-1)^{|m||x|} f(m) \otimes x.$$

in $\mathcal{C}(R)$

Check this is a morphism.

Proof. Replace P with $0 \to P_b \to \cdots \to P_a \to 0$.

"Things true for $R \rightsquigarrow$ true for perfect."

Then it reduces to checking for one f.g. projective P.

But that can be reduced to f.g. free.

That reduces to R. Obvious. ∎

Rees' Lemma

Setup. $\underline{x} = x_1, \dots, x_c \subset R$ finite subset.

$N \rightsquigarrow R$-module s.t. $\underline{x} N = 0$.

$M \rightsquigarrow R$-module s.t. $\underline{x}$ is Koszul-regular on M.

i.e. $K(\underline{x}; M) \xrightarrow{\sim} M/\underline{x}M$

Lemma. Then, $\mathrm{RHom}_R(N, M/\underline{x}M) \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

In particular, $\mathrm{Ext}^*_R\left(N, \frac{M}{\underline{x}M}\right) \cong \mathrm{Ext}^*_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

$\Lambda^*(\Sigma R^c)$: $0 \to \underset{\text{deg } c}{R} \xrightarrow{0} R^c \xrightarrow{0} \cdots \xrightarrow{0} R^{\binom{c}{2}} \xrightarrow{0} R^c \to \underset{\text{deg } 0}{R} \to 0$

Proof. $\mathrm{RHom}_R(N, M/\underline{x}M) \simeq \mathrm{RHom}_R(N, K(\underline{x}; M))$

$\simeq \mathrm{RHom}_R(N, M \otimes^L_R K(\underline{x}; R))$

$\simeq \mathrm{RHom}_R(N, M) \otimes^L_R K(\underline{x}; R)$ perfect

Since $\underline{x} N = 0$, $\underline{x}\, \mathrm{Ext}^*_R(N, M) = 0$.

Alter: $\mathrm{RHom}_R(N, M) \simeq \mathrm{Hom}_R(N, I)$ where $M \xrightarrow{\sim} I$ is an injective resol^n.

Now, $\underline{x} \cdot \mathrm{Hom}_R(N, I) = 0$.

$\therefore$

$\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{x}; R)$

$\simeq K(\underline{x}; \mathrm{Hom}_R(N, I))$

$\simeq K(\underset{\text{length } c}{\underline{0}}; \mathrm{Hom}_R(N, I))$

$\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{0}; R)$

$$\cong \operatorname{Hom}_R(N, I) \otimes_R K(Q; R)$$
$$= \operatorname{Hom}_R(N, I) \otimes_R \Lambda^*(\Sigma R^c)$$

Last statement means:

$$\operatorname{Ext}^n_R\left(N, \frac{M}{xM}\right) \cong \left(\operatorname{Ext}_R(N, M) \otimes_R \Lambda(\Sigma R^c)\right)^n$$
$$= \bigoplus_i \operatorname{Ext}^i_R(N, M) \otimes_R \left(\Lambda(\Sigma R^c)\right)^{n-i} \quad \} \text{think}$$
$$= \bigoplus_i \operatorname{Ext}^i_R(N, M) \otimes_R R^{\binom{c}{i-n}}.$$

$V \to \mathbb{Z}$-graded object

View V as having both an upper grading and lower grading via $V^i = V_{-i}$.

Notation:

$$\sup V^* = \sup\{i : V^i \neq 0\}$$
$$\inf V^* = \inf\{i : V^i \neq 0\}.$$
$$\sup V_* = \sup\{i : V_i \neq 0\} \quad \ldots$$
$$= -\inf V^*$$

Corollary. $\rho := \inf \operatorname{Ext}^*_R(N, M) = \inf \operatorname{Ext}^*_R(N, M/xM) + c.$

$$\operatorname{Ext}^{\rho}_R(N, M) = \operatorname{Ext}^{\rho-c}_R(N, M/xM)$$

Defn. Fix $I \subseteq R$ ideal. $M \in \mathcal{C}(R)$.

$$\operatorname{depth}_R(I, M) := \inf \operatorname{Ext}^*(R/I, M).$$

I-depth of M

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences of complexes on . .

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences (of complexes or modules).

Then, $\operatorname{depth}_R(I, M) \geq \min\{\operatorname{depth}_R(I,L), \operatorname{depth}_R(I,N)\}$.

Similarly relations with other two using l.e.s.

$$\to \operatorname{Ext}^i_R(R/I, L) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, N)$$
$$\to \operatorname{Ext}^{i+1}_R(R/I, L) \to \cdots$$

Thm (depth).

② Let $\underline{x} = x_1, \ldots, x_c$ be a gen. set for the ideal I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M) \qquad \forall M \in \mathcal{C}(R).$$

Will prove when M is an R-module.

(recall Koszul homology)

Koszul complexes revisited

Giving $x_1, \ldots, x_c$ in R

$\leftrightarrow$ Giving a map $f : F \to C$, where F is a free mod of rank c and chosen basis

$K(f) := (\Lambda^*(\Sigma F), \partial)$, where ∂ is

$$e_{i_1} \wedge \cdots \wedge e_{i_n} \longmapsto \sum_{j=1}^{n} (-1)^{j-1} f(e_{i_j})\, x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_n}.$$

Lemma. Fix $\underline{x} = x_1, \ldots, x_c \subseteq R$. For any $y \in (\underline{x})$, we have

$$K(\underbrace{\underline{x}, y}_{c+1 \text{ seq.}}; M) \cong K(\underline{x}, 0; M)$$
$$\cong K(\underline{x}; M) \otimes (0 \to R \to R \to 0).$$

$\uparrow$ deg 0

Proof.

$$\begin{array}{ccc} R^{c+1} & \xrightarrow{[x_1 \ \cdots \ x_c \ \ y]} & R \\ \uparrow \cong & \text{?} & \| \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & & & -r_1 \\ & \ddots & & \vdots \\ & & 1 & -r_c \\ \hline & & & 1 \end{array}\right]$$

$y = \sum r_i x_i$

$$\left[\begin{array}{c|c} \ddots & \begin{matrix} -x_1 \\ \vdots \\ -x_c \end{matrix} \\ \hline & 1 \end{array}\right] \quad \uparrow \cong \quad R^{c+1} \xrightarrow[{[x_1 \cdots x_c \; 0]}]{\quad 2 \quad} R \qquad y = \sum r_i x_i$$

In particular,

$$\sup H_*(\underline{x}, y; M) = 1 + \sup H_*(\underline{x}; M)$$

Thus,

$$c+1 - \sup H_*(\underline{x}, y; M) = c - \sup H_*(\underline{x}; M).$$

Corollary. $c - \sup_* H(\underline{x}, y; M)$ is independent of a generating set of I. (If $\underline{x}$, $\underline{y}$ generate, look at $\underline{x}, \underline{y}$ & $\underline{y}, \underline{x}$.)

Proof of Thm (Depth) When M is a module

$$\begin{aligned} \operatorname{depth}(I, M) = 0 &\iff \operatorname{Hom}(R/I, M) \neq 0 \\ &\iff I \subseteq \operatorname{Zdv}_R(M) \\ &\iff H_c(\underline{x}; M) \neq 0 \\ &\iff \sup H_*(\underline{x}; M) = c. \end{aligned}$$

Actually, $H_c(\underline{x}; M) \cong \operatorname{Hom}(R/I, M)$

Can assume $\operatorname{depth}_R(I, M) \geq 1$, i.e., $\exists y \in I$ nzd on M

Then, in particular, y is Koszul-reg on M.

By Rees, $\because y(R/I) = 0$

$$\inf \operatorname{Ext}^*_R(R/I, M/yM) = \inf \operatorname{Ext}^*_R(R/I, M) - 1.$$

$$\therefore \operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R\left(I, \frac{M}{yM}\right)$$

induction now applies

$$= 1 + \left(c - \sup H_*\left(\underline{x}; \frac{M}{yM}\right)\right)$$

$$\cong H_*(\underline{x}; M/yM)$$

$$\left(\qquad\qquad yM)\right)$$

$$\begin{aligned} &H_*(\underline{x};\ M/yM) \\ &\cong H_*(\underline{x};\ K(y;M)) \\ &\cong H_*(K(\underline{x}, y; M)) \\ &\cong H_*(K(\underline{x}, 0; M)) \end{aligned}$$

$$= c + 1 - \sup H_*(\underline{x}, 0; M)$$

$$= c - \sup H_*(\underline{x}; M). \qquad \blacksquare$$

Ex. Show $\operatorname{depth}_R(I, M)$ is the length of longest ___ M-regular in I.

Lecture 5 (25-01-2023)

Wednesday, January 25, 2023 1:19 PM

$R \to$ comm. noetherian ring

$I \subsetneq R$ ideal

$M \to$ R-complex

$$\operatorname{depth}(I, M) = \inf \operatorname{Ext}_R^*(R/I, M)$$

$I = (x_1, \dots, x_c)$

$$= c - \sup H_*(\underline{x}; M)$$

- When M is a module, $\operatorname{depth}(I, M) =$ length of any maximal M-Koszul-regular sequence in I

(Maybe $IM \neq M$ needed?)

If M f.g. and $I \subseteq \operatorname{Jac}(R)$

$=$ length of any maximal M-reg. seq in I

<u>Observation</u>. $\underline{x} = x_1, \dots, x_c$
$\underline{y} = y_1, \dots, y_d$

$$\sup H_*(\underline{x}, \underline{y}; M) \leq \sup H_*(\underline{x}; M) + d.$$

(Can use l.e.s. to see this.)

$K(\underline{x}, \underline{y}; M) \cong K(\underline{x}; K(\underline{y}; M))$

This implies

$$I \subseteq J \Rightarrow \operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(J, M).$$

Also, $\sqrt{I} = \sqrt{J} \Rightarrow \operatorname{depth}_R(I, M) = \operatorname{depth}_R(J, M)$.

<u>Today</u>. $(R, \mathfrak{m}, k)$ is a local ring:

- R is commutative Noetherian,
- $\mathfrak{m}$ is the unique maximal ideal of R,

- $k = R/\mathfrak{m}$.

In this case, $\operatorname{depth}(M) = \mathfrak{m}$-depth of M

$$= \inf \operatorname{Ext}_R^*(k, M).$$

It suffices to compute the above using $\underline{x} = x_1, \ldots, x_c$

s.t. $\sqrt{(\underline{x})} = \mathfrak{m}$.

Thus we can take c minimal as $c = \dim(R)$.

(Then, $\underline{x}$ is a system of parameters.)

$\therefore$ Can compute using $\dim(R)$ elements.

×

Ausland-Buchsbaum Equality

- $F \to$ an R-complex

F has finite flat dimension if

$$F \simeq (0 \to F_b \to \cdots \to F_a \to 0)$$

with F_i flat.

We write $\operatorname{flat\,dim}_R F < \infty$.

Examples
- Flat modules.
- Perfect complexes.
- Koszul complexes.

If $\operatorname{flat\,dim}_R F < \infty$, then $\operatorname{Tor}_i(-, F) = 0 \quad \forall\, |i| \gg 0$

on Mod R.

$$\mathrm{Tor}_i^R(M, F) = H_i\big(M \otimes_R (0 \to F_b \to \cdots \to F_a \to 0)\big)$$
$$= 0 \quad \text{for } i \notin [a, b].$$

In fact, the above characterises flat $\dim_R F < \infty$.

Theorem (AB equality) $(R, \mathfrak{m}, k)$ local.

$F \to$ finite flat dimension. Then,

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R M - \underbrace{\sup H_*(k \otimes_R^L F)}_{\mathrm{Tor}_*^R(k, F)}$$

for **ANY** R-complex M.

Specialise: ① $M = R$.

$$\operatorname{depth}_R(F) = \operatorname{depth}_R(R) - \sup H_*(k \otimes_R^L F).$$

Now, let N be a f.g. R-module.

Such an N has a minimal free resolution.

$$\begin{tikzcd}
\cdots \arrow[r] & R^{b_2} \arrow[rr, "\partial_2"] \arrow[dr, two heads] & & R^{b_1} \arrow[rr, "\partial_1"] \arrow[dr, two heads, "\min"] & & R^{b_0} \arrow[r, two heads, "\varepsilon"] & N \\
& & \ker \partial_1 \arrow[ur] & & \ker \varepsilon \arrow[ur] & &
\end{tikzcd}$$

minimally

$\ker \partial_1 \subseteq \mathfrak{m} R^{b_1}$, $\ker \varepsilon \subseteq \mathfrak{m} R^{b_0}$, by minimality

This gives a complex

$$G : (\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to 0)$$
$$\downarrow \simeq$$
$$N$$

$\partial G \subseteq \mathfrak{m} G$. G turns out to be unique up to isomorphism of complexes.

up to isomorphism of complexes.

"The" minimal free resolution of N.

$$\operatorname{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes G)_i.$$

H_i = everything since $\partial \otimes k = 0$

$$\therefore \quad \operatorname{Tor}_i^R(k, N) = 0 \iff G_i = 0.$$

$\therefore$ $\text{flat dim}_R N < \infty \iff$ N has a finite free resolution, i.e., N is perfect.

$$\sup \operatorname{Tor}_*^R(k, N) = \text{length of } G$$

$$=: \text{proj dim}_R N.$$

② If N is a f.g. R-module with $\text{proj dim}_R N < \infty$, then

$$\text{proj dim}_R(N) + \text{depth}_R(N) = \text{depth}_R(R).$$

(Classical AB Equality.)

Corollary. $\text{proj dim}_R N < \infty \Rightarrow \text{depth}(N) \leq \text{depth}(R)$.

Furthermore, equality iff N is projective.

False without assumption of proj dim $< \infty$

Note we had:

$$\text{depth}_R(F) = \text{depth}_R(R) - \sup H_*(k \otimes_R^L F)$$

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

Subtract:

$$\operatorname{depth}_R(F) - \operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}(R) - \operatorname{depth}(M).$$

Note: Some terms above may be ∞.

when $H_i(\underline{x}; M) = 0 \ \forall i$, then

$\sup H_*(\underline{x}; M) = -\infty$.

Proof of AB Equality

$$\mathrm{RHom}_R(k, M \otimes_R^L F) \xleftarrow{\simeq} \mathrm{RHom}_R(k, M) \otimes_R^L F$$

quasi iso because k is a f.g. module

$\therefore \exists\, G \xrightarrow{\simeq} k$ where each G_i is finite free

AND flat dim $F < \infty$.

[More generally:

$\mathrm{Hom}_R(N, M) \otimes R \xrightarrow{\simeq} \mathrm{Hom}_R(N, M \otimes_R F)$

N f.g. R-mod, F flat]

Key observation:

$\mathrm{RHom}_R(k, M) \simeq$ complexes of k-vector spaces

(take injective resolution of M)

$$\mathrm{R\,Hom}_R(k, M) \otimes_R^L F \simeq \mathrm{R\,Hom}_R(k, M) \otimes_k (k \otimes_R^L F)$$

$$\therefore \operatorname{Ext}_R^*(k, M \otimes_R^L F) = \operatorname{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^L F).$$

Observation Let M be an R-complex.

Suppose $s := \sup H_*(M)$ is finite.

Then, $\operatorname{depth}_R M \geq -s$

Then, $\operatorname{depth}_R M \geq -s$

with equality iff $\operatorname{depth}_R H_s(M)$.

Note. For an R-module M, $\operatorname{depth} M = 0 \Leftrightarrow \inf \operatorname{Ext}^{i}(k, M) = 0$

$\Leftrightarrow \operatorname{Hom}(k, M) \neq 0$

$\Leftrightarrow k \hookrightarrow M$

$\Leftrightarrow \mathfrak{m} \in \operatorname{Ass}_R M.$

One proof. M as above.

$$\operatorname{Ext}_R^{-s}(N, M) = \operatorname{Hom}_R(N, H_s(M))$$

N any R-module.

- key. $M \simeq M'$ with $M_i' = 0 \ \forall \ i > s.$

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "0"] & M_{s+1} \arrow[r, "\partial"] \arrow[d, "0"'] & M_s \arrow[r] \arrow[d] & M_{s-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[d, equal] & = M \arrow[d, "\wr"] \\
\cdots \arrow[r] & 0 \arrow[r] & \dfrac{M_s}{\partial(M_{s+1})} \arrow[r] & M_{s-1} \arrow[r] & \cdots & = M'
\end{tikzcd}$$

$\therefore$ Can assume $M_i = 0$ for $i \geq s+1.$

In particular,

$$0 \to \Sigma^s H_s(M) \hookrightarrow M \to M'' \to 0.$$

iso on homology in degrees $\leq s-1$

$H_i(M'') = 0 \quad i \geq s.$

Let $\underline{x} = x_1, \ldots, x_n$ gen set for $\mathfrak{m}$.

Then,

Then,

$$H_{i+1}(\underline{x}; M'') \to H_i(\underline{x}, \Sigma^* H_s(M)) \to H_i(\underline{x}; M) \to H_i(\underline{x}; M'') \quad \text{— } (*)$$

$H_j(M'') = 0 \ \forall j \geq s.$

So, $M'' \simeq M'''$ with $M'''_j = 0$ for $j \geq s.$

$K(\underline{x}; M''')_j = 0$ for $j \geq s+n+1.$

Thus, $H_j(\underline{x}; M''') = 0 \quad \forall j \geq s+n+1.$

$$\Rightarrow H_i(\underline{x}; \Sigma^s H_s(M)) = 0 \quad \forall j \geq s+n+1.$$

Put $i \geq n+s$ in $(*)$:

$$H_j(\underline{x}; M) = 0 \quad \forall j \geq n+s+1$$

$$\Rightarrow \sup H_*(\underline{x}; M) \leq n+s$$

$$\Rightarrow -s \leq n - \sup H_*(\underline{x}; M) = \operatorname{depth} M.$$

Moreover,

$$H_{n+s}(\underline{x}; M) \cong H_{n+s}(\underline{x}; \Sigma^s H_s(M))$$
$$\cong H_n(\underline{x}, H_s(M))$$

The above is nonzero iff $\operatorname{depth} H_s(M) = 0.$ ☒

---

① flat $\dim_R F < \infty$. Then $\forall M$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F).$$

② $s = \sup H_*(M)$ is finite.

Then, $\operatorname{depth}_R(M) \geq -s$.

Equality $\Leftrightarrow$ $\operatorname{depth}(H_s(M)) = 0$.

Application. Say

$$F = \quad 0 \to F_d \to \cdots \to F_0 \to 0$$

minimal, $\partial F \subseteq \mathfrak{m} F$

is a finite free complex s.t.

$$0 < \operatorname{length}_R H_*(F) < \infty.$$

(All $H_i(F)$ are finite length and at least one is nonzero.)

Then, for any M,

$$\operatorname{depth}_R M = d - \sup H_*(F \otimes_R M).$$

Thus, such an F is depth sensitive.

When $\mathfrak{m}M \neq M$, one can check some $H_*(F \otimes_R M) \neq 0$.

Then, one gets

$$d \geq \operatorname{depth}_R(M).$$

Note: Over any local ring, $\exists M$ s.t. $\mathfrak{m}M \neq M$ and $\operatorname{depth}_R M = \dim R$.

M need not be f.g.

In particular, $d \geq \dim(R)$. $\longrightarrow$ New Intersection Theorem

Hochster ('70)
André (2016)
Bhatt (2021)

<u>Proof.</u> Assume $s = \sup H_*(F \otimes_R M)$.

Take any prime $\mathfrak{p} \neq \mathfrak{m}$.

$$H_i(F \otimes_R M)_{\mathfrak{p}} \cong H_i(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$$

$$H_i(F_{\mathfrak{p}}) = 0 \quad (\because \text{length } H_i(F) < \infty)$$

i.e. $F_{\mathfrak{p}} \simeq 0$ in $D(R_{\mathfrak{p}})$.

$$\therefore H_*(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$$

Thus, $H_i(F \otimes_R M)$ is $\mathfrak{m}$-power torsion.

(I.e., each $a \in H_i(F \otimes_R M)$ is killed by some $\mathfrak{m}^n$.)

$\therefore$ depth $H_s(F \otimes_R M) = 0$.

Thus, by previous result,

$$\operatorname{depth}_R(F \otimes_R M) = -s$$

$$\| \; \text{AB}$$

$$\operatorname{depth}(M) - \sup H(k \otimes^L_R F)$$

$$\Rightarrow \operatorname{depth}(M) = \underbrace{\sup H_*(k \otimes^L_R F)}_{d} - s. \qquad \square$$

Lecture 6 (30-01-2023)

Monday, January 30, 2023 1:26 PM

Recap. $I \subseteq R$ Comm. Noe
M an R-complex.

$$\operatorname{depth}_R(I, M) = \inf \operatorname{Ext}_R^*(R/I, M).$$

- Choose any fin gen set $\underline{x} = x_1, \dots, x_c$ of I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M).$$

(Focus on the case $H_*(M)$ bounded.)

$(R, \mathfrak{m}_R, k) \to$ local

$$\operatorname{depth}_R M := \operatorname{depth}_R(\mathfrak{m}_R, M).$$

$$\operatorname{depth}_R(M) \geq -\sup H_*(M). \quad \text{—— (1)}$$

Equality $\iff \mathfrak{m} \in \operatorname{Ass}(H_s(M))$

$s = \sup H_*(M).$

Exercise 1.

Suppose $M = 0 \to M_b \to \cdots \to M_a \to 0$

$$\operatorname{depth}_R M \geq \inf\{\operatorname{depth}(M_i) - i : a \leq i \leq b\}.$$

$$\operatorname{depth}_R M \geq \inf\{\operatorname{depth} H_i(M) - i : \inf H_*(M) \leq i \leq \sup H_*M\}. \quad \text{—— (2)}$$

(Note (2) $\Rightarrow$ (1).)

Setup. $H_*(M)$ bounded. $\underline{x} = x_1, \dots, x_c$. ($R$ not necessarily local.)

$$\sup H_*(M) \underset{(i)}{\leq} \sup H_*(\underline{x}; M) \underset{(ii)}{\leq} \sup H_*(M) + c$$

Lemma 2. (a) Inequality (ii) always holds.
Equality iff $\operatorname{depth}_R(\underline{x}; H_s(M)) = 0$ $\quad s = \sup H_*(M)$

(b) (i) holds if $\underline{x} \subseteq J(R)$, each $H_i(M)$ is f.g.

(b) (i) holds if $\underline{x} \subseteq J(R)$, each $H_i(M)$ is f.g.

Equality holds iff $\underline{x}$ is $H_s(M)$-regular.

Proof. (a) $H(\underline{x}; M) = H(x_1 ; K(x_2, \dots, x_c ; M))$.

Reduce to $c = 1$. $x := x_1$.

In this case, we have

$$H_{i+1}(M) \xrightarrow{x} H_{i+1}(M) \to H_{i+1}(x;M) \to H_i(M) \xrightarrow{x} H_i(M)$$

For $i \geq s$, we get $H_{i+1}(x;M) = 0$.

$\therefore$ (ii) follows.

Moreover, $H_{s+1}(x;M) \neq 0 \iff x$ is a zd on $H_s(M)$.

(b) $H_i(M) \neq 0 \overset{\text{NAK}}{\Longrightarrow} H_i(x;M) \neq 0$

$\overset{?}{\supseteq} \dfrac{H_i(M)}{x H_i(M)}$

$\therefore \sup H_*(x;M) \geq \sup H_*(M)$.

Moreover,

$$0 \to H_{s+1}(x;M) \to H_s(M) \xrightarrow{x} H_s(M).$$ ∎

Corollary 3. $\operatorname{depth}_R(\underline{x}; M) \geq -\sup H_*(M)$.

Equality $\iff \operatorname{depth}(\underline{x}, H_s(M)) = 0$

$\overset{?}{\iff} (\underline{x}) \subseteq \mathfrak{p} \in \operatorname{Ass} H_s(M)$.

Prop^n 4. $(R, \mathfrak{m}, k)$ local.

Let M be any bounded complex.

Then, for any $I \subset R$,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

In particular, if M is a f.g. module

$$\operatorname{depth}_R(M) \leq \inf\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M\}$$

$\leq \dim_R(M)$. $\rightsquigarrow \sup\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M\}$

Proof. Say $I = (y_1, \dots, y_c)$ and $d := \dim(R/I)$.

... s.t. they form a SoP in R/I

Proof. Say $I = (y_1, \ldots, y_c)$ and $d := \dim(R/I)$.

Let $x_1, \ldots, x_d \in R$ s.t. they form a SoP in R/I.

Thus, $\sqrt{(\underline{y}, \underline{x})} = \sqrt{m_R}$.

Apply Lemma (2) (b) to $K(\underline{y}\,; M)$ to get

$$\sup H_*(\underline{y}\,; M) \le \sup H_*(\underline{x}\,; K(\underline{y}\,; M)) = \sup H_*(\underline{x}, \underline{y}\,; M).$$

$$\therefore\ d + \underbrace{c - \sup H_*(\underline{y}\,; M)} \geq \underbrace{d + c - \sup H_*(\underline{x}, \underline{y}\,; M)}$$

$$\Rightarrow d + \operatorname{depth}(I, M) \geq \operatorname{depth} M. \qquad \blacksquare$$

Local case:

AB Equality. F an R-complex with $\operatorname{flatdim}_R F < \infty$.

Then, for any R-complex M,

$$\operatorname{depth}(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

$$\operatorname{depth}_R M \geq -\sup H_*(M) =s$$

with equality iff $\operatorname{depth}(H_s(M)) = 0$.

Prop^n. $R \to$ comm. Noe.

$I \subseteq R$ ideal.

$$\operatorname{depth}(I, M) = \inf \{\operatorname{depth}_{R_p} M_p \ : \ p \in V(I)\}.$$

Proof. $I \subset p \Rightarrow \operatorname{depth}_R(I, M) \le \operatorname{depth}_R(p, M) \le \operatorname{depth}_{R_p}(pR_p, M_p)$

check using Koszul

This gives $\le$.

We now construct p achieving $\operatorname{depth}_R(I, M)$.

Let $I = (x_1, \ldots, x_c)$. Let $s := \sup H_*(\underline{x}\,; M)$.

We now construct $\mathfrak{p}$ achieving $\operatorname{depth}_R(I, M)$.

Let $I = (x_1, \ldots, x_c)$. Let $s := \sup H_*(\underline{x}; M)$.

Pick $\mathfrak{p} \in \operatorname{Ass} H_s(\underline{x}; M)$. (Maybe even minimal.)

Then, $\operatorname{depth}_{R_\mathfrak{p}} H_s(\underline{x}; M)_\mathfrak{p} = 0$.

Consider $K(\underline{x}; M)_\mathfrak{p} = K(\underline{x}; M_\mathfrak{p})$.

$\sup H_*(K(\underline{x}; M_\mathfrak{p})) = \sup H_*(\underline{x}; M_\mathfrak{p}) = s$

$$\operatorname{depth}_{R_\mathfrak{p}} K(\underline{x}; M_\mathfrak{p}) = -s$$

$$\begin{aligned}
&\quad\ \| \\
&\operatorname{depth}_{R_\mathfrak{p}} \left(K(\underline{x}; R_\mathfrak{p}) \otimes M_\mathfrak{p} \right) \\
&\quad\ \| \ \text{AB} \\
&\operatorname{depth}(M_\mathfrak{p}) - \sup H_*\left(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} K(\underline{x}; R_\mathfrak{p})\right) \\
&\geqslant \operatorname{depth}(M_\mathfrak{p}) - c
\end{aligned}$$

$$\Rightarrow \quad -s \geqslant \operatorname{depth}(M_\mathfrak{p}) - c$$

$$\Leftrightarrow \quad c - s \geqslant \operatorname{depth}(M_\mathfrak{p})$$

$$\underset{\operatorname{depth}_R(I, M)}{\|}$$

$\blacksquare$

Remark. The proof says
$$\operatorname{depth}_R(I, M) = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})$$
$$\forall\, \mathfrak{p} \in \operatorname{Ann} H_s(\underline{x}; M)$$

Thm. $(R, \mathfrak{m}_R) \longrightarrow (S, \mathfrak{m}_S)$ local map.

(i.e., $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$.)

Let M be an R-complex, N an S-module s.t. N is flat as an R-module.

Then,
$$\operatorname{depth}_S(N \otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N).$$

Corollary When $(R, m_R) \to (S, m_S)$ is flat, then ($M = R$, $N = S$ gives)

$$\operatorname{depth}_S(S) = \operatorname{depth}_R(R) + \operatorname{depth}_{S/m_RS}(S/m_RS).$$

"fiber"

$R \to S \to S/m_RS$

$$\begin{array}{ccc} \operatorname{Spec}(R/m_RS) & \hookrightarrow & \operatorname{Spec} S \\ \downarrow & & \downarrow \\ \{m_R\} & \hookrightarrow & \operatorname{Spec}(R) \end{array}$$

Under the same hypothesis, we have

$$\dim(S) = \dim(R) + \dim(S/m_RS).$$

Proof. $\underline{x} = x_1, \dots, x_c \in R$ s.t. $m_R = (\underline{x})$.

Pick $\underline{y} = y_1, \dots, y_d \in S$ s.t. $\underline{y}\left(\frac{S}{m_RS}\right)$ is the maximal ideal of S/m_RS (i.e., m_S/m_RS).

Then, $(\underline{x}S, \underline{y}) = m_S$.

$$K(\underline{x}S, \underline{y}; N \otimes_R M) \cong K(\underline{y}; N) \otimes_R K(\underline{x}; M)$$

assoc. of $\otimes$

N flat over $R \Rightarrow K(\underline{y}, N)$ has fin flat $\dim_R$

$$\operatorname{depth}_R K(\underline{x}, \underline{y}; N \otimes_R M) = \operatorname{depth}_R K(\underline{x}; M) - \sup H_*(k \otimes K(\underline{y}, N))$$

$(k = R/m_R)$

Note. $(\underline{x}, \underline{y}) \cdot H_*(\underline{x}, \underline{y}; N \otimes_R M) = 0$

$m_S \, H_*(\underline{x}, \underline{y}; N \otimes_R M) = 0$

$\Rightarrow m_R \, H_*(\underline{x}, \underline{y}; N \otimes_R M) = 0$

Similarly $m_R H_*(\underline{x}; M) = 0$.

$$-\sup H_*(\underline{x}, \underline{y}; N \otimes_R M) = -\sup H_*(\underline{x}, M) - \sup H_*\left(\underline{y}; \frac{N}{\underline{m}_R N}\right)$$

Add $c+d$

$\hookrightarrow$ depths $(N \otimes_R M)$

Exercise R local. M f.g. R-module

$$\operatorname{depth} R - \dim_R M \le \operatorname{grade}_R M \le \operatorname{codim}_R M$$

$\le \dim R - \dim M \le \operatorname{pdim}_R M.$

use intersection thm

$\operatorname{depth}_R(\operatorname{ann}_R M, R)$

$\operatorname{height}(\operatorname{ann} M)$

$\|$

$\inf\{\dim R_{\mathfrak p} : \mathfrak p \supseteq \operatorname{ann} M\}$

Recall. New Intersection Theorem (P. Roberts '86)

$$0 \to F_d \to \cdots \to F_0 \to 0 \quad \text{finite free}$$

$$0 < \operatorname{length} H_*(F) < \infty.$$

Then, $d \ge \dim \dim(R)$. ⊛

"Simple consequence" of AB + Existence of big CM modules

Corollary. (Intersection theorem)

R local.

$M \ne 0$ f.g. R-module s.t. $\operatorname{pdim}_R M < \infty$.

Then, for any f.g. R-module N,

$$\dim_R(N) - \dim_R(M \otimes_R N) \le \operatorname{pdim}_R(M).$$

Inspired by (Serre): R regular local (e.g. $k[[x_1, \dots, x_n]]$)

M, N any finite R-module.

(Serre) M, N any finite R-module.

$$\dim_R N - \dim_R (M \otimes_R N) \leq \dim R - \dim M.$$

Lecture 7 (01-02-2023)

Wednesday, February 1, 2023 1:24 PM

$R \to$ Commutative noetherian.

$I \subseteq R$ ideal

__Invariance of domain__

Let $R \longrightarrow S$ finite map. (S is a finite R-module.)

Let M be an S-complex.

Then,
$$\operatorname{depth}_R(I, M) = \operatorname{depth}_S(IS, M).$$

[Only need S noetherian for this.]

Obvious using Koszul.

$I = (\underline{x})$

$$K(\underline{x}; M) = K(\underline{x}; R) \otimes_R R$$
$$= (K(\underline{x}; R) \otimes_R S) \otimes_S M$$
$$= K(\underline{x}; S) \otimes_S M. \quad \blacksquare$$

$\dim_R M = \dim_S M.$ [Finiteness needed here.]

Say $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ is finite and local.

$$\begin{aligned} \operatorname{depth}_R(M) &= \operatorname{depth}(\mathfrak{m}_R, M) \\ &= \operatorname{depth}(\mathfrak{m}_R S, M) \\ &= \operatorname{depth}(\mathfrak{m}_S, M) \\ &= \operatorname{depth}_S(M). \end{aligned}$$

finiteness $\Rightarrow \sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$

__Special case.__ R, M an R-module.

Take $S = R/\operatorname{ann}_R(M)$.

__Recall.__ $(R, \mathfrak{m}_R)$ local, $M \neq 0$ a f.g R-module.

$\operatorname{depth}(R) - \dim(M) \overset{①}{\leq} \operatorname{grade}(M) \overset{②}{\leq} \operatorname{codim}(M) \overset{③}{\leq} \dim R - \dim M$

$$\operatorname{depth}(R) - \dim(M) \overset{①}{\leq} \operatorname{grade}(M) \overset{②}{\leq} \operatorname{codim}(M) \overset{③}{\leq} \dim R - \dim M \underset{④}{\leq} \operatorname{pdim}_R M.$$

$$\operatorname{depth}(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

Using invariance of domain, can tighten the above to

$$\operatorname{depth}_R M \leq \operatorname{depth}_R(I, M) + \underbrace{\dim(M/IM)}_{= \dim\left(R/I + \operatorname{ann} M\right)}.$$

Examples. ① $R = \dfrac{k[\![x, y]\!]}{(x^2, xy)}$

$\mathfrak{m}_R = (x, y)$.

$\dim(R) = 1$: $\sqrt{(x^2, xy)} = (x)$

$\therefore \dim R = \dim \dfrac{k[\![x, y]\!]}{(x)} = \dim k[\![y]\!] = 1.$

$\operatorname{depth}(R) = 0$, i.e., $\operatorname{Hom}_R(k, R) \neq 0$

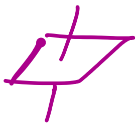
i.e, $(0 : \mathfrak{m}_R) \neq 0$. ← socle

Taking $M = R$ shows
$\operatorname{depth} R - \dim M < \operatorname{grade}(M)$ is strict.

② Take same R.

$M = R/\mathfrak{m}$. Then, $\operatorname{grade} M < \operatorname{codim} M$.

③ $k[\![x, y, z]\!] / (x) \cap (y, z) = R$.



Spec $R = V(x) \cup V(y, z)$

Pick $M = R/\mathfrak{p}$ where $\mathfrak{p}$ is a min'l prime but $\dim(R/\mathfrak{p}) < \dim(R)$.

$\mathfrak{p} = (y, z)$.

⑤ Say $\operatorname{pdim}_R M < \infty$.

$\dim R - \dim M \leq \operatorname{pdim}_R M$

⇑ ?, AB

$$\dim R - \dim M \le \operatorname{pdim}_R M$$
$$\Updownarrow \qquad\qquad \Big\} AB$$
$$\dim R - \dim M \le \operatorname{depth} R - \operatorname{depth} M$$
$$\Updownarrow$$
$$\dim R - \operatorname{depth} R \le \dim M - \operatorname{depth} M \quad \text{---} \quad (5)$$

Now, take $R = k[\![x,y]\!]$ and $M = R/(x^2, xy)$.

Then, LHS of (5) is 0 but RHS is not.

Def^n: $(R, \mathfrak{m})$ local, $M \overset{\neq 0}{}$ f.g. over R.

$$\operatorname{Cmd}_R(M) := \dim_R M - \operatorname{depth}_R M.$$

Cohen-Macaulay defect

If $\operatorname{Cmd}_R(M) = 0$, i.e., $\dim_R(M) = \operatorname{depth}_R(M)$, then

M is Cohen-Macualay (CM).

M is maximal Cohen-Macaulay (MCM) if $\operatorname{depth}(M) = \dim(R)$.

0 is also considered CM and MCM.

$\big\{$ $\underset{\infty}{\operatorname{depth} M} \geq \underset{-\infty}{\dim M}$ for $M = 0$.

One can define dim etc for complexes as well.

$(R, \mathfrak{m}_R)$ local. $M \overset{\neq 0}{}$ f.g. module.

$$\operatorname{depth}_R(M) \le \inf\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R M\} \le \dim_R M.$$

Corollary. M MCM $\Rightarrow$ $\operatorname{Ann}_R M = \operatorname{Min}_R M$

and $\dim(R/\mathfrak{p}) = \dim(M)$ $\forall\, \mathfrak{p} \in \operatorname{Min}(M)$.

Geometrically: $\operatorname{Supp}(M)$ has no "embedded components" and all components have the same dimension.

Prop^n. $(R, \mathfrak{m})$ local, M f.g. and CM.

① $\operatorname{depth}_R(I,M) = \dim M - \dim (M/IM) \quad \forall I \subseteq R$

② Given $\underline{x} = x_1, \dots, x_c \in \mathfrak{m}$, then

$\underline{x}$ is M-regular $\iff \dim(M) - \dim(M/\underline{x}M) = c$.

Note ($\Rightarrow$) is always true.

($\Leftarrow$) Apply ① to $I = (\underline{x})$.

The hypothesis gives

$\operatorname{depth}_R(\underline{x}, M) = c$

$\iff H_i(\underline{x}; M) = 0 \quad \forall i \geq 1$

$\iff \underline{x}$ is regular. ∎

③: $\underline{x}$ is part of an SoP for M ⇝ SoP for $R/\operatorname{ann}(M)$

$\iff \underline{x}$ is M-regular.

Thm. $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ local map.

M f.g. R-module, $\neq 0$, N f.g. S-module, $\neq 0$, flat over R.

Then,

$$\operatorname{Cmd}_S(N \otimes_R M) = \operatorname{Cmd}_R(M) + \operatorname{Cmd}_{(S/\mathfrak{m}_R S)}(N/\mathfrak{m}_R N).$$

Proof: We saw the above for depth instead of Cmd.

Same holds for dim. ∎

Corollary. Under same hypothesis,

$N \otimes_R M$ is CM$/S$ $\iff$ M CM$/R$ + $N/\mathfrak{m}_R N$ CM$/S/\mathfrak{m}_R S$

Special case: $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_R)$ flat local.

S CM $\iff$ R and $S/\mathfrak{m}_R S$ are CM.

Def^n A local ring R is CM if it is so as a module over itself.

Key Consequence: R CM: for any f.g. R-module M,

$$\text{grade}_R(M) = \text{codim}_R(M) = \dim(R) - \dim(M).$$

Exercise $(R, \mathfrak{m})$ M f.g.

M CM $\Rightarrow$ $M_{\mathfrak{p}}$ CM over $R_{\mathfrak{p}}$ $\forall\, \mathfrak{p} \in \text{Supp}\, M$.

Def^n $R \to$ comm. noe.
$M \to$ f.g. R-module

Then, M is CM if $M_{\mathfrak{p}}$ CM$/R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Supp}_R M$.

Equivalently, for all $\mathfrak{m} \in \text{Max}(\text{Supp}_R\, M)$.

R is CM if . . .

Examples: · $k[x_1, \dots, x_c]$ is CM. (How?)

· $k[[x_1, \dots, x_c]]$ is CM. $\because$ dim $= c$
depth $= c$ since $x_1, \dots, x_c$.

· R CM $\Leftrightarrow$ $\Lambda^I R$ is CM for some ($\equiv$ all) I.

$\hookrightarrow$ completion wrt I

If R local,

$(R, \mathfrak{m}) \longrightarrow \Lambda^I R$ is flat and local.
$\phantom{(R, \mathfrak{m}) \longrightarrow \Lambda^I R}$ "S"

Check $S/\mathfrak{m}_R S = R/\mathfrak{m}_R \to$ field (CM)

$\therefore$ $\Lambda^I R$ is CM $\Leftrightarrow$ R is CM.

Thm: $(R, \mathfrak{m})$ CM local.
M f.g. R module s.t. $\text{pdim}_R M < \infty$

Then,

M is CM $\Leftrightarrow$ $\text{pdim}_R M = \text{grade}_R M$.

Proof. $\operatorname{pdim}_R M = \operatorname{grade}_R M$

$\Updownarrow$

$\operatorname{depth} R - \operatorname{depth} M = \dim R - \dim M$

$\Updownarrow$

$\operatorname{depth} M = \dim M$. ☒

Lecture 8 (06-02-2023)

Monday, February 6, 2023 1:25 PM

Regular Rings

For now, (R, m, k) is a comm. noetherian local ring.

Recall.

$$\operatorname{depth}(R) \overset{①}{\leq} \dim(R) \overset{②}{\leq} \underset{\substack{\| \\ \dim_k(m/m^2)}}{\operatorname{edim}(R)}.$$

When equality holds in ①, then R is Cohen-Macaulay. (By defn.)

② holds by Krull's height theorem.

Notation: (embedding) codepth of R is

$$\operatorname{codepth}(R) = \operatorname{edim}(R) - \operatorname{depth}(R).$$

Defn. R is regular if $\operatorname{codepth}(R) = 0$.

Exercise. R is regular $\iff$ m is generated by a regular sequence

$\iff \operatorname{edim}(R) = \dim(R)$

Example.

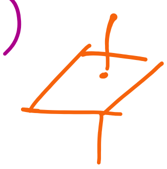
① $k[[x_1, \dots, x_n]]$ with k a field is regular

since m is gen by a reg. seq.

② $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ —— " ——.

③ $\mathbb{Z}_{(p)}$ similar.

(4) Regular $\Rightarrow$ CM. (Non CM $\Rightarrow$ Non reg.)



$$\frac{k[[x,y,z]]}{(xz,yz)} \quad \text{not reg.}$$

(5) $R = \dfrac{k[[x,y]]}{(xy)}$. codepth $= 1 > 0$.

NOT regular.

Flat Maps

Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local flat extension.

Recall

S is CM $\iff$ R is CM and $S/\mathfrak{m}S$ are CM.

Analog fails for regularity. ($\Leftarrow$) does hold.

Consider $\underset{R}{k[[x^2]]} \hookrightarrow k[[x]] \overset{S}{=}$. (flat ext^n since S is free)

$k[[x^2]]$: regular

$S/\mathfrak{m}S = k[[x]]/(x^2)$ $\hookrightarrow$ not regular, codepth $=1$

Also, if R is CM, then $R_\mathfrak{p}$ is CM $\forall \mathfrak{p}$.

Question. ['30s, Krull, Zariski, ...]

Does R regular imply $R_\mathfrak{p}$ regular $\forall \mathfrak{p} \in \operatorname{Spec} R$?

Homological characterisation of regularity. (Used to prove ↑)

Recall: for a f.g. R-module M, a minimal free resolution over $(R, \mathfrak{m}, k)$ is a free resolution

$$F \xrightarrow{\sim} M$$

with $\partial(F) \subseteq \mathfrak{m} F$.

Recall/Prove. Minimal resolutions exist and are unique up to isomorphism of complexes.

The ith Betti number of M is

$$\beta_i^R(M) = \mathrm{rank}_R(F_i)$$
$$= \mathrm{rank}_k(\mathrm{Tor}_i^R(M, k))$$
$$= \mathrm{rank}_k(\mathrm{Ext}_R^i(M, k)).$$

Example. (1) $R = k[\![x_1, \dots, x_n]\!]$. $f \in \mathfrak{m} \setminus \{0\}$.

$M := R/(f)$.

Then the min'l free res^n of M is

$$0 \to R \xrightarrow{\cdot f} R \to 0.$$

Koszul

$$\beta_i(M) = \begin{cases} 1 & ; \quad i = 0, 1 \\ 0 & ; \quad \text{else} \end{cases}$$

The min'l res^n of k is $K(\underline{x})$.

$$\beta_i(k) = \binom{n}{i} \quad \text{for } i \geq 0.$$

(2) $R = \dfrac{k[\![x, y]\!]}{(xy)}$.

$M = R/(x)$ has min'l free res^n:

$$\cdots \xrightarrow{\cdot y} \overset{3}{R} \xrightarrow{\cdot x} \overset{2}{R} \xrightarrow{\cdot y} \overset{1}{R} \xrightarrow{\cdot x} \overset{0}{R} \to 0$$

$\therefore \beta_i(M) = 1 \quad \forall i.$

Res^n for k: $k = R/(x,y)$

$$\text{repeat} \cdots \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x \;\; y)} R \longrightarrow 0$$

$$\beta_i(M) = \begin{cases} 1 & ; i = 0 \\ 2 & ; i \geq 1 \end{cases}$$

FACT. $0 \neq M$ f.g.:
(1 Dep^n)

$$\operatorname{projdim}_R(M) = \sup\{i \geq 0 : \beta_i^R(M) \neq 0\}.$$

$\parallel$

length of min'l free res^n

$\parallel$

inf (length (free res^n))

Theorem. [Auslander-Buchsbaum-Serre, '50s] (ABS)

$(R, \mathfrak{m}, k)$ local. TFAE:

① R is regular.

② $\operatorname{projdim}_R(M) < \infty$ for all f.g. M.

③ $\operatorname{projdim}_R(k) < \infty$.

Proof. ① ⇒ ②

Let $\underline{x} = x_1, \ldots, x_d$ be a min'l gen set for $\mathfrak{m}$. (Hence reg. seq.)

Then, $K(\underline{x})$ is a min'l free res^n for k.

$$\beta_i^R(M) = \operatorname{rank}_k(\operatorname{Tor}_i^R(M, k))$$

$$= \operatorname{rank}_k H_i(M \otimes_R K(\underline{x}))$$

$$= 0 \quad \text{for} \quad i > d.$$

$$\therefore \operatorname{projdim}_R(M) < \infty.$$

② ⇒ ③. ✓

② $\Rightarrow$ ③. ✓

③ $\Rightarrow$ ①. Serre's proof of ABS Theorem relies on:

Lemma. $(R, \mathfrak{m}, k)$ local.

Then, $\beta_i^R(k) \geq \binom{\text{edim}(R)}{i}$ for $i \geq 0$.

Proof of Lemma. Let $F \xrightarrow{\simeq} k$ be a min'l resol^n.

($\because \beta_i^R(k) = \text{rk}_R F_i$.)

Let $\underline{x} = x_1, \dots, x_e$ be a min gen set for $\mathfrak{m}$.

$$\begin{tikzcd} K := K(\underline{x}) \arrow[r] \arrow[dr, dashed] & k \\ & F \arrow[u, "\simeq"'] \end{tikzcd}$$

$\exists \varphi$ a map of complexes ($\because$ perfect)

Claim. φ_i is a split injection $\forall i$. (Then it is clear.)

$\varphi_i : K_i \to F_i$ is split injective

$\overset{\text{NAK}}{\Leftrightarrow}$ $k \otimes_R K_i \to k \otimes_R F_i$ is an injection.

By induction, we show φ_i is split inj.

$i = 0$ ✓ $R \xrightarrow{\simeq} R$

$i > 0$: Let $a \in K_i$ with $\varphi_i(a) \in \mathfrak{m} F_i$. (we are using the NAK result)

WTS: $a \in \mathfrak{m} K_i$.

$\therefore \partial\varphi(a) \in \mathfrak{m}^2 F_{i-1}$

$\quad \parallel$

$\varphi(\partial a)$

$$\begin{tikzcd} a \in K_i \arrow[r] \arrow[d] & K_{i-1} \arrow[d] \\ F_i \arrow[r] & F_{i-1} \end{tikzcd}$$

$F_i \supseteq \varphi(a) \in \mathfrak{m} F_i \qquad \partial(\varphi(a)) \subseteq \mathfrak{m}^2 F_{i-1}$

$\partial(F) \subseteq \mathfrak{m} F$

By induction,

$\partial^K a \in \mathfrak{m}^2 K_{i-1}$. $(*)$

Notice that by def^n of ∂^K and since x is a min gen set for $\mathfrak{m}$, Notice that $\mathfrak{m}$ [illegible] of [illegible] and

since $\underline{x}$ is a min gen set for $\mathfrak{m}$,

$(*) \Rightarrow a \in \mathfrak{m} K_i$. This does it. ∎

Back to : $\operatorname{proj\,dim}_R (k) < \infty \Rightarrow R$ is regular.

From Serre's inequality:

$$\operatorname{proj\,dim}_R k \geq \operatorname{edim} R.$$

$\|$ $\rightsquigarrow$ By AB Equality, since $\operatorname{proj\,dim} k < \infty$.

$\operatorname{depth}(R)$

But $\operatorname{depth}(R) \leq \operatorname{edim}(R)$ always true. ∎

The above now solves the localisation problem.

Corollary. R regular $\Rightarrow R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. R is reg.

$\Rightarrow \operatorname{proj\,dim}_R R/\mathfrak{p} < \infty$

$\Rightarrow \operatorname{proj\,dim}_{R_{\mathfrak{p}}} (k(\mathfrak{p})) < \infty \Rightarrow R_{\mathfrak{p}}$ is reg. ∎

Prop^n. $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{m}_S)$ local flat.

① R and $S/\mathfrak{m}S$ reg $\Rightarrow S$ is reg.

② S is regular $\Rightarrow R$ is regular.

Proof. ① $\bar{S} = S/\mathfrak{m}S$

$$\operatorname{depth} S \overset{\text{flat}}{=} \operatorname{depth}(R) + \operatorname{depth}(\bar{S})$$

regular hypothesis

$$= \operatorname{edim}(R) + \operatorname{edim}(\bar{S})$$

There is always a right exact sequence

[illegible]

$$\frac{m_R}{m_R^2} \otimes_k \ell \to \frac{m_S}{m_S^2} \longrightarrow \frac{m_{\bar S}}{m_{\bar S}^2} \longrightarrow 0$$

$\ell = S/m_S$

$$\therefore \ \operatorname{edim} S \leq \operatorname{edim} R + \operatorname{edim}(\bar S) = \operatorname{depth}(\bar S).$$

② Let $F \xrightarrow[R]{\sim} k$ min'l, since φ flat AND local,

$$F \otimes_R S \longrightarrow \bar S \quad \text{min'l.}$$

$$\beta_i^R(k) = \beta_i^S(\bar S) = 0 \quad \text{for } i \gg 0.$$

$\therefore$ projdim$_R\, k < \infty$.

$\Rightarrow R$ is reg. ∎

Lecture 9 (08-02-2023)

Wednesday, February 8, 2023 1:26 PM

Recall: R local is regular if $\operatorname{codepth}(R) = 0$,

i.e., $\mathfrak{m}$ generated by a regular sequence.

We proved Auslander-Buchsbaum Theorem:
$(R, \mathfrak{m}, k)$ TFAE:

① R is regular,

② $\operatorname{projdim}_R M < \infty$ for all f.g. M

③ $\operatorname{projdim}_R(k) < \infty$.

---

① $\Rightarrow$ ② $\Rightarrow$ ③ was ok.

③ $\Rightarrow$ ①: showed $\operatorname{projdim}_R(k) \geq \operatorname{edim}(R)$ and then AB Equality.

Sketch of second proof.

Theorem [Nagata]. Let $(R, \mathfrak{m}, k)$ be local (not necessarily regular) and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be nzd.

Set $\bar{R} = R/x$.

Then, for any f.g. R/x-module M, we have

$$\operatorname{Tor}^R_*(M, k) \cong \operatorname{Tor}^{\bar{R}}_*(M, k) \otimes_k \Lambda(\Sigma k).$$

$$= \quad 0 \to k \to k \to 0$$

In particular, $\beta^R_i(M) = \beta^{\bar{R}}_i(M) + \beta^{\bar{R}}_{i-1}(M)$.

In fact, one can show the min'l R-free res'n of M

has the form

$$\cdots \to \begin{matrix} G_3 \\ \oplus \\ G_2 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_3 & x \\ \beta_3 & -\alpha_2 \end{pmatrix}} \begin{matrix} G_2 \\ \oplus \\ G_1 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_2 & x \\ \beta_2 & -\alpha_1 \end{pmatrix}} \begin{matrix} G_1 \\ \oplus \\ G_0 \end{matrix} \xrightarrow{(\alpha_1 \ \ x)} G_0$$

multiplication by x

$G_i \to$ free

and $\cdots \to \overline{G_3} \xrightarrow{\alpha_3} \overline{G_2} \xrightarrow{\alpha_2} \cdots$ is the min'l $\overline{R}$-resol^n of M.

(Stronger result but not proving this.)

Example. $R = k[[x,y]]/(xy)$ (non regular)

Minimal R-free resol^n of k:

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x \ \ y)} R \to 0$$

Consider $y - x^2 \in m \setminus m^2$ (nzd on R).

By row/col operations, the min'l resol^n is iso to,

$$\cdots \xrightarrow{\begin{pmatrix} x & y-x^2 \\ & -y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & y-x^2 \\ & -x \end{pmatrix}} R^2 \xrightarrow{(x \quad y-x^2)} R \to 0$$

Now, going mod $y-x^2$, [using the result] we get

$$\cdots \to \overline{R} \xrightarrow{x} \overline{R} \xrightarrow{y} \overline{R} \xrightarrow{x} \overline{R} \to 0$$

$\overline{R} = k[[x,y]]$

$$\cdots \to \bar R \to \bar R$$

$$= \quad \cdots \to \bar R \xrightarrow{x} \bar R \xrightarrow{x^2} \bar R \xrightarrow{x} \bar R \to 0$$

$$\bar R = \frac{k[[x,y]]}{(xy,\ y-x^2)} \cong \frac{k[[x]]}{(x^3)}$$

Proof of Nagata's Thm.

If $a \in R$ is nzd, and set $\bar R = R/a$.

$M \rightsquigarrow \bar R$-module.

There is a long exact sequence

$$\cdots \xrightarrow{\chi} \mathrm{Tor}^{\bar R}_{n-1}(M,k) \to \mathrm{Tor}^{R}_{n}(M,k) \to \mathrm{Tor}^{\bar R}_{n}(M,k)$$

$$\xrightarrow{\chi} \mathrm{Tor}^{\bar R}_{n-2}(M,k) \to \mathrm{Tor}^{R}_{n-1}(M,k) \to \mathrm{Tor}^{\bar R}_{n-1}(M,k)$$

Specialising to our setting, we wish to show that each χ is 0.

One can compute χ in the following way:

$\bar F \xrightarrow{\sim} M$ min'l $\bar R$-res^n.

Lift this to a sequence of free R-modules:
(not saying complex!)

$$\cdots \to F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \to \cdots$$

∂^2 may not be 0 but

∂^i may not be 0 on

$\partial^2 = x\,\Theta$ where

$\Theta = \{\theta_i : F_i \longrightarrow F_{i-2}\}$ is a chain map

and the following diagram commutes: (up to sign)

$$\begin{tikzcd}
F_i \otimes_R k \arrow[r, "\theta_i \otimes_R k"] \arrow[d, phantom, "\cong"] & F_{i-2} \otimes_R k \arrow[dd, "\cong"] \\
\overline{F}_i \otimes_{\overline{R}} k \arrow[d, phantom, "="] & \\
\mathrm{Tor}_i^{\overline{R}}(M,k) \arrow[r, "x"'] & \mathrm{Tor}_{i-2}^{\overline{R}}(M,k)
\end{tikzcd} \qquad (*)$$

Now, since x is linear and $\partial^2 = x\,\Theta$,
we have that $\Theta(F) \subseteq \mathfrak{m} F$.

$\therefore$ the top map in $(*)$ is zero. $\therefore$ $x = 0$. $\blacksquare$

Second proof of ③ $\Rightarrow$ ① in ABT.

Given $\mathrm{projdim}_R(k) < \infty$. WTS: R is regular.

Induct on $d := \mathrm{depth}(R)$

$\underline{d=0}$: AB Equality gives $\mathrm{projdim}_R(k) = 0$.
$\therefore$ $R = k$ is a field. R is reg.

$\underline{d>0}$: By prime avoidance, $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$ nzd.

$\therefore$ $\mathrm{codepth}(R) = \mathrm{codepth}(R/x)$.

By Nagata's thm,
$\mathrm{projdim}_{\overline{R}}\, k = \mathrm{projdim}_R\, k - 1 < \infty.$

By Nagata's Thm,

$$\operatorname{projdim}_{R/x} k = \operatorname{projdim}_R k - 1 < \infty.$$

But R/x has smaller depth. ☒

$(R, \mathfrak{m}, k)$ local

Prop^n: $x \in \mathfrak{m}$ nzd.

R is CM $\iff$ R/x is CM.

($\mathrm{CMD}(R) = \mathrm{CMD}(R/x)$.)

($\mathrm{CMD} = \dim - \operatorname{depth}$.)

Proposition. $x \in \mathfrak{m}$ nzd.

① Suppose R is regular.

R/x is regular $\iff$ $x \notin \mathfrak{m}^2$.

② If R/x is regular, then R is regular.
(Hence, $x \notin \mathfrak{m}^2$.)

Proof.

$$\operatorname{codepth}(R/x) = \begin{cases} \operatorname{codepth}(R) & x \notin \mathfrak{m}^2, \\ \operatorname{codepth}(R)+1 & x \in \mathfrak{m}^2. \end{cases}$$

☒

Global Setting

$R \to$ comm. noetherian, not necessarily local or of finite Krull dim.

Def^n. R is regular if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$

($\iff \forall\, \mathfrak{m} \in \operatorname{mSpec}(R)$).

($\Leftrightarrow \forall\, \mathfrak{m} \in \mathrm{mSpec}(R)$).

(Since regularity localises for local rings, the above makes sense.)

Exercise. R is regular $\Leftrightarrow R[x]$ is regular
$\Leftrightarrow R[[x]]$ is regular

Example ① $k[x_1, \ldots, x_n]$ is regular. (k a field.)

② $\mathbb{Z}[x_1, \ldots, x_n]$ —— " ——. ($\mathbb{Z}$ is regular since every localisation is $\mathbb{Q}$ or a DVR.)

③ Nagata's example. (Infinite Krull dimension but regular)

Thm. [Bass–Murthy '60s].

$R \to$ comm noetherian
$M \to$ f.g.

$$\mathrm{projdim}_R M < \infty \iff \mathrm{projdim}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty \text{ for all } \mathfrak{p} \in \mathrm{Spec}\, R.$$

Corollary. $R \to$ comm noe.

$$R \text{ is regular} \iff \mathrm{projdim}_R M < \infty \text{ for all } M \text{ f.g.}$$

Proof. $(\Rightarrow)$ ✓

$(\Leftarrow)$ Let $F \xrightarrow{\sim} M$ be a free res^n of M, F_i f.g. for all $i \geq 0$.

For $n \geq 0$, define

For $n \geq 0$, define

$$D_n := \{ \mathfrak{p} \in \operatorname{Spec} R : \operatorname{proj\,dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leq n \}$$

$$= \{ \mathfrak{p} \in \operatorname{Spec} R : \operatorname{im}(\partial_n^F)_\mathfrak{p} \text{ is free over } R_\mathfrak{p} \}.$$

Since $\operatorname{im}(\partial_n^F)$ is f.g., the above set is open in $\operatorname{Spec}(R)$. "free locus is open"

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$$

and $\operatorname{Spec} R = \bigcup_{n \geq 0} D_n$.

Since R is noetherian, $\operatorname{Spec} R$ is a noe. top space.

$\therefore \operatorname{Spec} R = D_n$ for some n.

$\Rightarrow \operatorname{im}(\partial_n^F)$ is locally free.

$\Rightarrow \operatorname{im}(\partial_n^F)$ is projective.

Now, $0 \to \operatorname{im}(\partial_n^F) \hookrightarrow F_{n-1} \to \cdots \to F_0 \to 0$ is a proj. resolⁿ. $\boxtimes$

Lecture 10 (13-02-2023)

13 February 2023 13:26

$R \to$ comm noetherian

$\varphi : F_1 \longrightarrow F_0$ free modules of finite rank

$\begin{array}{ccc} F_1 & \longrightarrow & F_0 \\ \cong & & \cong \\ R^s & \xrightarrow{(a_{ij})} & R^r \end{array}$

$I_c(\varphi) =$ ideal generated by $c \times c$ minors of (a_{ij}).

(well defined)

$I_0(R) = R \supseteq I_1(R) = (a_{11}, \dots, a_{rs}) \supseteq \dots \supseteq I_i(R) \supseteq I_{i+1}(R) = 0$

$i = \min(r, s)$

M f.g. R-module

$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0$ (rank $= r$)

finite free presentations

$G_1 \xrightarrow{\psi} G_0 \longrightarrow M \longrightarrow 0$ (rank $= s$)

Then, $I_{r-c}(\varphi) = I_{s-c}(\psi)$ for all c.

(Exercise.)

Defn. $\mathrm{Fitt}_c^R(M) := I_{r-c}(\varphi)$, c^{th} Fitting ideal of M.

We get $\mathrm{Fitt}_0(M) \subseteq \mathrm{Fitt}_1(M) \subseteq \cdots$ eventually R.

$\mathrm{Fitt}_0(M) \neq 0$ only if $r \leq s$

$\mathrm{Fitt}_c(M) = R$ for $c \geq r$.

Examples.

① $R = k$ a field, V some vector space (f.g.) of rank n.

$0 \to k^n$ is a presentation

$\mathrm{Fitt}_c(V) = k \iff c \geq n.$

② R a PID.

$$0 \to R^s \xrightarrow{\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \\ \hline & 0 & \end{pmatrix}} R^r \to M \to 0$$

(can assume $s \leq r$)

can put in normal form

$\mathrm{Fitt}_0(M) \neq 0 \iff r = s \iff M$ is torsion.

In this case, $\mathrm{Fitt}_0(M) = (d_1 \cdots d_r)$.

Properties. $R^s \xrightarrow{\varphi} R^r \to M \to 0.$

① If $R \to S$ is any map of rings.

$$\mathrm{Fitt}_c^S(S \otimes_R M) = S \cdot \mathrm{Fitt}_c^R(M).$$

② $(\mathrm{ann}_R(M))^r \subseteq \mathrm{Fitt}_0(M) \subseteq \mathrm{ann}_R(M).$

Proof. Pick an $(r\times r)$-minor 'a' in φ.

WTP: $a \cdot R^r \subseteq \mathrm{im}(\varphi)$

Can assume

$$\varphi = \underbrace{\left(\underbrace{\diagup\!\diagup\!\diagup}_{r\times r} \;\Big|\; \right)}_{r \times s} = A$$

$\det = a$

Take the $s\times r$ matrix $\left(\begin{array}{c} \text{signed co-factor matrix}_{r\times r} \\ \hline 0 \end{array} \right)_{s\times r} = B$

Then, $AB = \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}.$

$\therefore\ aR^n \subseteq \operatorname{im}\varphi.$ Proves $\mathrm{Fitt}_0 \subseteq \operatorname{ann}.$

<u>Next</u>: $\operatorname{ann}(M)^r \subseteq \mathrm{Fitt}_0(M).$

Fix $a_1, \dots, a_r \in \operatorname{ann}(M).$

$$\begin{tikzcd}
 & R^r \arrow[dl, dashed] \arrow[d, "\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix}"] \\
R^s \arrow[r, "\varphi"'] & R^r
\end{tikzcd}$$

Apply $\Lambda^r(\cdot)$ to the above to get

$$\begin{tikzcd}
 & R \arrow[dl] \arrow[d, "a_1 \cdots a_r"] \\
R^{\binom{s}{r}} \arrow[r, "\Lambda^r(\varphi)"'] & R
\end{tikzcd}$$

③ Fix $c \geq 0$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.
TFAE:

ⓘ $\mathrm{Fitt}_c(M) \not\subseteq \mathfrak{p}$.

ⓘⓘ $\operatorname{im}(\varphi)_{\mathfrak{p}}$ contains a free summand of $R_{\mathfrak{p}}^r$ of rank $\geq r-c$.

ⓘⓘⓘ $\nu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq c.$

min'l # gens

<u>Sketch.</u> Can assume $(R, \mathfrak{p}, k)$ local. $\mathfrak{m} := \mathfrak{p}$.

$\mathrm{Fitt}_c(M) \not\subseteq \mathfrak{m} \iff \mathrm{Fitt}_c(R) = 0 \iff \mathrm{Fitt}_c(k \otimes_R M) \neq 0$

$\Updownarrow$ → first example

$\nu_R(M) \leq c \overset{\text{NAK}}{\iff} \nu_k(k \otimes_R M) \leq c$

④ Fix $c \geq 0$ and $\mathfrak{p} \in \operatorname{Spec} R$. TFAE

ⓘ $\mathrm{Fitt}_{c-1}(M)_{\mathfrak{p}} = 0$ and $\mathrm{Fitt}_c(M)_{\mathfrak{p}} = R_{\mathfrak{p}}.$

ⓘⓘ $\operatorname{im}(\varphi)_{\mathfrak{p}}$ is a free summand of $R_{\mathfrak{p}}^r$ of rank $r-c$.

(iii) $M_p \cong (R_p)^c$.

Deduce from (3).

(5) $c \geq 0$. M is projective of rank c.

$\iff \operatorname{Fitt}_{c-1}(M) = 0$ and $\operatorname{Fitt}_c(M) = R$.

Hilbert-Birch Theorem

$R \to$ comm noetherian

Given $I \subseteq R$ with free resolution

$$0 \to R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \to 0.$$

Then, $\exists$ nzd $a \in R$ s.t. $I = a \cdot I_n(\varphi)$.

Moreover, if I is projective, then I is principal.

If $\operatorname{projdim}(I) = 1$, then $\operatorname{depth}(I_n(\varphi), R) \geq 2$.

Conversely, if $\varphi: R^n \to R^{n+1}$ is a matrix s.t. $\operatorname{depth}(I_n(\varphi), R) \geq 2$,

then $0 \to R^n \to R^{n+1} \longrightarrow I_n(\varphi) \to 0$

is a free resolution.

Regular Local Rings are UFDs.

Thm. $M \to$ projective module

If M has a finite free res^n, then it must have a free resol^n of length 1.

Say $0 \to F_s \to F_{s-1} \to \dots F_1 \to 0$.

$\downarrow$ $\cong M$

M projective $\Rightarrow \partial(F_i)$ proj $\forall i \geq 1 \dots$ ∎

So, Projective + FFR $\Rightarrow$ stably free.
↳ finite free resolution

Corollary. $I \subseteq R$ proj + FFR $\Rightarrow$ I principal.

Proof. $0 \to R^n \to R^{n+1} \to I \to 0$.

I proj $\Rightarrow I = (a)$. ∎

Thm. R comm. noe. domain s.t. each f.g. R-modules has FFR. Then, R is a UFD.

Corollary. Regular local rings are UFDs.
↳ not true otherwise

Proof. Suppose R is local. ($\because$ Regular.)

- Induction on $\dim R$.
  - $\dim R \leq 1$ is clear.

Pick $w \in m_R \setminus m_R^2$. Then, R/w is also RLR and hence a domain.

$\therefore w$ is prime.
Suffices to verify R_w is a UFD.
Note $\dim R_w < \dim R$.

Pick $p \in \operatorname{Spec}(R_w)$ with ht $= 1$.

Suffices to show p is principal.

Note. p has a FFR. (For p comes from R.)

It now suffices to prove it is projective.

(Earlier corollary.)

This can be tested locally. But localisations are RLRs of dim < dim R.

$\therefore$ UFD.

$\therefore \mathfrak{p}$ is locally free and hence projective.

For general R, again fix $\mathfrak{p}$ of ht = 1.

WANT: $\mathfrak{p}$ is principal.

Since $\mathfrak{p}$ has FFR, suffices to prove projective.

This is local. $\blacksquare$

$R \to$ PID, M torsion module.

$$0 \to R^r \xrightarrow{\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}} R^r \to M \to 0 \qquad d_i \neq 0$$

$$\mathrm{Fitt}_0(M) = \left(\prod d_i\right) \subseteq \mathrm{ann}_R(M).$$

$$\mathrm{length}_R(M) = \mathrm{length}\left(R/\mathrm{Fitt}_0(M)\right)$$

Lecture 11 (15-02-2023)

Wednesday, February 15, 2023 1:18 PM

Complete Intersections

Cohen Structure Theory

$(R, \mathfrak{m}, k)$ local

- R is equicharacteristic if R contains a field (as a subring).
  - $\mathbb{Z} \longrightarrow R$ structure map.
    If char $R = p > 0$ prime, then char $k = p$.
  - When char $k = 0$, $\mathbb{Q} \hookrightarrow R$.
- Mixed characteristic : $\mathbb{Z} \hookrightarrow R$ and char $k = p > 0$.
  $\mathbb{Q} \not\hookrightarrow R$

When R is $\mathfrak{m}$-adically complete:

- If R equicharacteristic, then R contains a copy of k as a subring.
  $k \hookrightarrow R \twoheadrightarrow k$, $=$
- R mixed characteris, then R contains a complete DVR $\mathcal{O}$
  s.t. $\mathcal{O} \hookrightarrow R \twoheadrightarrow k$
  field of fractions

$$R = k[\![x_1, \dots, x_n]\!]/I \quad \text{(equi. case)}$$

$$R = \mathcal{O}[\![x_1, \dots, x_n]\!]/I .$$

Thus, R is of the form S/I, where S is a regular local ring.

We call this a Cohen presentation of R. (S/I, where S is an RLR.)

Say $R = S/I$ is a Cohen presentation.

$I \subseteq \mathfrak{m}_S$. Suppose $I \nsubseteq \mathfrak{m}_S^2$. Pick $x \in I \setminus \mathfrak{m}_S^2$.

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[dr, two heads] & R \\ & S/xS \arrow[u, two heads] \end{tikzcd}$$

S/xS → regular local

Thus, we can assume that $I \subseteq \mathfrak{m}_S^2$.

Equivalently, $\operatorname{edim} R = \operatorname{edim} S$.

$$\operatorname{edim} R = \operatorname{rank}_k\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}\right) = \operatorname{rank}_k\left(\frac{\mathfrak{m}_S}{I+\mathfrak{m}_S^2}\right), \qquad \operatorname{edim} S = \operatorname{rank}_k\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2}\right)$$

In this case, we say that S/I is a minimal Cohen presentation.

Defn. $(R, \mathfrak{m}, k)$ local is said to be a complete intersection (c.i.) if in some Cohen presentation

$$\hat{R} = S/I$$

$\mathfrak{m}$-adic

$\hat{R} = S/I$

$\mathfrak{m}$-adic Completion of R

and I can be generated by a regular sequence in S.

[Not every c.i. is a quotient of a regular ring.]

Intrinsic characterisation?

Let $K^R :=$ Koszul complex on a minimal gen for $\mathfrak{m}_R$.

(Well defined up to iso of complexes (with multiplicative structure).)

- $\hat{R} \otimes_R K^R = K^{\hat{R}}$ $\quad (\because \mathfrak{m}\hat{R} = \mathfrak{m}_{\hat{R}}.)$

- $R \to \hat{R}$ induces

$$K^R \longrightarrow \hat{R} \otimes_R K^R = K^{\hat{R}}.$$

Then,

$$H(K^R) = H(\hat{R} \otimes_R K^R) \overset{\text{flatness of } R \to \hat{R}}{=} \hat{R} \otimes_R H(K^R)$$

Since $\mathfrak{m} H(K^R) = 0$, the map

$$H(K^R) \longrightarrow \hat{R} \otimes_R H(K^R)$$

is an iso.

- Thus $K^R \xrightarrow[\text{quasi}]{\sim} K^{\hat{R}}$.

Say, $R = S/I$ is a minimal Cohen presentation.
$(I \subseteq \mathfrak{m}_S^2)$

<u>Lemma.</u> $\operatorname{rank}_k \left(H_i(K^R)\right) = \beta_i^S(R)$.

In particular, the RHS is independent of the min'l Cohen presentation.

<u>Proof.</u> Let $K^S \to$ Koszul complex on S.
S regular, so $K^S \xrightarrow{\sim} k$.
Since $I \subseteq \mathfrak{m}_S^2$,

$$R \otimes_S K^S = K^R.$$

(min'l gen set for $\mathfrak{m}_S$ gives a min'l gen set for $\mathfrak{m}_R$.)

$$\therefore H_*(K^R) = H_*(R \otimes_S K^S) \quad \because K^S \xrightarrow{\sim} k$$
$$= \operatorname{Tor}_*^S(R, k).$$
⊠

$R = S/I \longrightarrow$ min'l Cohen presentation

$$\operatorname{ht} I \leq \nu_S(I)$$

$\operatorname{ht} I = \operatorname{depth}_S(I, S)$, since S reg.

$\nu_S(I) \to$ #gens of I as an S-module (Krull)

Recall: I is generated by a regular seq iff equality holds.

Thus, I is gen by regular sequence

$$\begin{aligned} \iff \nu_S(I) &= \dim(S) - \dim(S/I) \quad \text{regular} \\ &= \operatorname{edim}(S) - \dim(R) \\ &= \operatorname{edim}(R) - \dim(R) \quad \because \text{min'l} \end{aligned}$$

$\nu_S(I) = \beta_1^S(R) = \operatorname{rank}_k H_1(K^R)$

Therefore, if $R = S/I$ min'l Cohen presentation, then

I is gen. by a regular sequence

$$\iff \operatorname{rank}_k H_1(K^R) = \operatorname{edim}(R) - \dim(R).$$

Corollary. R is a local ring.
R is c.i. iff

$$\operatorname{rank}_k H_1(K^R) = \operatorname{edim}(R) - \dim(R).$$

Proof. All the numbers above are same for R and $\hat{R}$. $\blacksquare$

Remark. Say $\hat{R} = S/I$ is some Cohen presentation where $I = \langle \text{reg seq} \rangle$.

Then, we can reduce it to a min'l Cohen presentation where ideal $= \langle \text{reg seq} \rangle$.

Proof.

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \hat{R} \\ & S/xS = S' \arrow[u] \end{tikzcd}$$

$x \in I \setminus m_S^2$.

Then, $x \notin m_S I$ and so can be extended to a min gen set for I.

(x, y)

Then, y is a regular seq...

Theorem. R c.i. $\Rightarrow R_p$ c.i. $\forall\, p \in \operatorname{Spec} R$.

Nontrivial!!!

This is clear if R itself has a Cohen presentation.

$R = S/I$, I gen. by reg. seq.

$p \in \operatorname{Spec} R \subseteq \operatorname{Spec} S$. $R_p \cong S_p / I S_p \ldots$

$(R, \mathfrak{m}, k)$:
$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R}^{\mathfrak{m}} \arrow[d, "?"] \\ R_{\mathfrak{p}} \arrow[r] & \hat{R_{\mathfrak{p}}}^{\mathfrak{p}R_{\mathfrak{p}}} \end{tikzcd}$$

← not a localisation of $\hat{R}^{\mathfrak{m}}$.

Pick $\mathfrak{q} \in \operatorname{Spec} \hat{R}$ s.t. $\mathfrak{q} \cap R = \mathfrak{p}$.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R} \arrow[d] \\ R_{\mathfrak{p}} \arrow[r] & (\hat{R})_{\mathfrak{q}} \end{tikzcd}$$

↑ flat + local

Theorem $\varphi : R \to S$ flat local map.
Then, S c.i. $\iff$ R and $S/\mathfrak{m}_R S$ are c.i.

Fact. R c.i. $\iff$ $R[[x_1, \dots, x_n]]$ c.i.

$(R, \mathfrak{m}, k)$ local

$R = S/I$ min'l Cohen presentation (say it exists)

Write $\mathfrak{m} = (x_1, \dots, x_n)$ with $n = \operatorname{edim}(S)$.

$I = (f_1, \dots, f_c)$, min'l gen set.

$\mathfrak{m}_R = \underline{x} \cdot R$.

K^R : $\longrightarrow \bigoplus^n R e_i \longrightarrow R \to 0$

K^R : $\quad \cdots \longrightarrow \bigoplus_{i=1}^{n} Re_i \longrightarrow R \to 0$

$\partial e_i = x_i$

Since $I \subseteq m_S^2$, can write

$$f_j = \sum_{i=1}^{n} S_{ij} x_i, \quad \text{where} \quad S_{ij} \in m.$$

Let $z_j = \sum_{i=1}^{n} \overline{S_{ij}} e_i \quad \in \bigoplus_{i=1}^{n} Re_i.$

Note $\partial(z_j) = \sum_{i=1}^{n} \overline{S_{ij}} \overline{x_i} = \overline{f_j} = 0$

$\therefore z_1, \dots, z_c$ are cycles.

Claim. The classes $[z_1], \dots, [z_c] \in H_1$ form a k-basis.

Tate's result.

(R, m, k)

$R^n \xrightarrow{f} R \qquad \mathrm{im}(f) = m$, $n = \mathrm{edim}(R)$

$K^R = (\Lambda^* (\Sigma R^n), \partial).$

$\rightsquigarrow K^R$ is a dg (= differential graded) R-algebra.

(Even graded -commutative.)

$\rightsquigarrow H(K^R)$ is a graded-commutative k-algebra.

Universal property gives:

$$\chi^R : \Lambda_*(\Sigma H_1(K^R)) \longrightarrow H(K^R)$$

$H_1(K^R) = \Sigma k^c$

map of k-algebras

Theorem (Tate) $\quad R$ is c.i. $\iff \chi^R$ is an isomorphism.

(Assmus) $\quad \iff \Lambda^2 H_1(K^R) \twoheadrightarrow H_2(K^R).$

Lecture 12 (22-02-2023)

22 February 2023 13:26

$(R, m_R, k) \to$ local ring

R is c.i. $\equiv$ $\hat{R} \cong S/I$, where S regular
I gen by reg seq.
Can assume $I \subseteq m_S^2$.

Intrinsic characterisation:

$$\operatorname{rank}_k H_1(K^R) = \operatorname{edim}(R) - \dim(R)$$

(In general, $\geq$ holds.)

Key: $H_1(K^R) \cong I/m_S I$.

K^R is an exterior algebra with ∂ satisfying Leibniz rule.
K^R is a (commutative) differential graded R-algebra.

$\therefore$ $H(K^R)$ is also graded-commutative k-alg. $\left(ab = (-1)^{|a||b|} ba \text{ for homog. } a, b.\right)$

By the universal prop. of exterior algebra,
$H_1(K^R) \hookrightarrow H(K^R)$
induces

$$\Lambda_k(\Sigma H_1(K^R)) \longrightarrow H(K^R)$$ of graded k-algebra.

- Image is the k-subalgebra of $H(K^R)$ generated by $H_1(K^R)$.

Theorem (Tate-Asmus) (R, m_R, k) local. TFAE:

① R is complete intersection.

② $\Lambda_k(\Sigma H_1(K^R)) \xrightarrow{\cong} H(K^R)$, i.e., $H(K^R)$ is the exterior algebra on $H_1(K^R)$.

③ The map above is surjective in deg two, i.e.,
$$H_1(K^R)^{\wedge 2} = H_2(K^R).$$

Can assume R is complete. Let $R = S/I$ min'l Cohen presentation.

Then $H_*(K^R) \cong \mathrm{Tor}_*^S(k, R)$ as k-algebras.

- S regular, so m_S is generated by a regular sequence.

Now, suppose S is any local ring and $I \subseteq J \subseteq m_S$ are ideals generated by regular sequences.

Want to compute $\mathrm{Tor}_*^S(S/J, S/I)$. (In the case we care about, $J = m_S$.)

Let $I = (\underline{a})$, where $\underline{a} = a_1, \dots, a_m \in m_S$ reg,

$J = (\underline{b})$, —"— $\underline{b} = b_1, \dots, b_n$ —"— .

$$\left.\begin{array}{l} K(\underline{a}\,; S) \xrightarrow{\sim} S/I \quad \text{and} \\ K(\underline{b}\,; S) \xrightarrow{\sim} S/J. \end{array}\right\} \text{dg algebra maps}$$

$$\mathrm{Tor}_*^S(S/J, S/I) = H_*\left[K(\underline{b}\,; S) \otimes_S S/I\right] \quad \text{as algebras.}$$
$$= H_*\left(K(\underline{b}\,; S/I)\right)$$

$$\begin{array}{ccc} K(\underline{b}\,;S)\otimes_S S/I & \xleftarrow{\sim} & K(\underline{b}\,;S) \otimes_S K(\underline{a}\,;S) \\ & & \downarrow \sim \\ & & S/J \otimes_S K(\underline{a}\,;S) = K(\underline{a}\,; S/J) \\ & & \| \\ & & K(\underline{0}\,; S/J) \end{array}$$

map of dga

quasi iso $\because K(\underline{a};S) \xrightarrow{\sim} S/I$ and $K(\underline{b};S)$ fin. free

$\because I \subseteq J$

exterior algebra

$$H(K(\underline{0}; S/J)) = K(\underline{0}\,; S/J) = \wedge \underbrace{\Sigma (S/J)^m}_{H_1(K(\underline{0}\,;\, S/J))}$$

Therefore: $H(\underline{b}\,; S/I) \cong \wedge \Sigma H_1(\underline{b}\,; S/I)$

In summary: $\mathrm{Tor}^S(S/J, S/I) \cong \wedge \Sigma \,\mathrm{Tor}_1^S(S/J, S/I)$

$$\mathrm{Tor}_1^S(S/J, S/I) = \frac{I \cap J}{IJ} = I/IJ.$$

$\because I \subseteq J$

Thus:

$\because I \subseteq J$

Thus:

Lemma. If $I \subseteq J$ are gen. by a regular sequence, then

$$\mathrm{Tor}_*^S(S/J, S/I) \cong \Lambda(\Sigma\, I/IJ).$$

Moreover, I/IJ free S/J-module.

Specialising to our c.i. case, we get ① ⇒ ② of Tate-Assmus ...

② ⇒ ③ ok.

Proof of ③ ⇒ ①: Hypothesis:

$$\Lambda^2(\Sigma H_1(K^R)) \twoheadrightarrow H_2(K^R).$$

Equivalently,

$$\Lambda^2\, \Sigma\, \mathrm{Tor}_1^S(k,R) \twoheadrightarrow \mathrm{Tor}_2^S(k,R).$$

($R = S/I$ Cohen pres.)

Let $I = (\underline{a})$ with $\underline{a}$ min'l gen set.

Let $F \xrightarrow{\simeq} R$ be a min'l free resolution of R over S.

$$F: \quad \cdots \to S^n \longrightarrow S^m \xrightarrow{(a_1 \cdots a_m)} S \longrightarrow 0$$

same as Koszul

$$\begin{array}{ccc} K(\underline{a}; S) & \dashrightarrow & F \\ \downarrow & & \downarrow \simeq \\ R = S/\underline{a}S & = & R \end{array}$$

lift of identity on R.

A morphism of S-complexes.

WANT TO DEDUCE: $H_1(K(\underline{a}; S)) = 0$

$\because \underline{a}$ reg.

Can ensure that the lifting is id in degrees 0, 1.

$$\begin{array}{lcccccccc} K(\underline{a};S): & \cdots \to & S^{\binom{m}{2}} & \longrightarrow & S^m & \longrightarrow & S & \longrightarrow & 0 \\ & & \downarrow * & & \| & & \| & & \\ F: & \cdots \to & S^n & \longrightarrow & S^m & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

ETP. $(*)$ is onto. Then a diagram chase shows $H_1 = 0$.

$\Rightarrow \underline{a}$ is reg.

$\Rightarrow R$ is c.i.

By NAK, suffices to prove

$$K_2(\underline{a}; S) \otimes_S k \longrightarrow F_2 \otimes_S k \quad \text{is onto}$$

Equivalently, surjectivity on H_2. ($\because$ differential becomes zero)

$$K(\underline{a}; S) \otimes_S k \longrightarrow F \otimes_S k.$$

In homology:

$$H_*(K(\underline{a}; k)) \longrightarrow H_*(F \otimes_S k) = \mathrm{Tor}_*^S(R, k)$$

$$\| $$

$$\wedge \Sigma H_1(\underline{a}; k) \longrightarrow \mathrm{Tor}_*^S(R, k) \qquad (+)$$

Note. $H_1(\underline{a}; k) \xrightarrow{\simeq} \mathrm{Tor}_1^S(R, k)$.

Hypothesis: Tor_2^S generated by Tor_1^S.

$\therefore (+)$ is onto. #

(R, m_R, k) local ring.

$$\sup\{i \geq 0 : H_i(K^R) \neq 0\} = \operatorname{edim}(R) - \operatorname{depth}(R).$$

$\therefore H_1(K^R)^s = 0$ for $s > \operatorname{edim}(R) - \operatorname{depth}(R)$.

Theorem of Wiebe: R is c.i. $\Leftrightarrow H_1(K^R)^s \neq 0$ for $s = \operatorname{edim}(R) - \operatorname{depth}(R)$.

Proof $(\Rightarrow)$ By Tate.

Indeed, R c.i. $\Rightarrow H(K^R) = \wedge \Sigma H_1(K^R)$

$H_1(K^R) \cong k^c, \quad c = \operatorname{edim} R - \dim R$
$= \operatorname{edim} R - \operatorname{depth} R.$

($\Leftarrow$) Can reduce to the case $\operatorname{depth}(R) = 0$.

Ek,
we can choose a min gen set $x_1, \dots, x_n$ ($n = \operatorname{edim} R$)
for m_R s.t. $x_1, \dots, x_d$ is a reg seq. on R, $d = \operatorname{depth} R$.

$$K(\underline{x}; R) = K(x_1, \dots, x_d; R) \otimes K(x_{d+1}, \dots, x_n; R)$$
$$\downarrow \simeq$$
$$\underbrace{R/(x_1, \dots, x_d)R}_{R'} \otimes_R K(x_{d+1}, \dots, x_n; R)$$

$$\therefore \underset{K^R}{\overset{\|}{K(\underline{x}; R)}} \xrightarrow{\simeq} \underset{K^{R'}}{\overset{\|}{K(x_{d+1}, \dots, x_n; R')}}$$

$\therefore H(K^R) \cong H(K^{R'})$ as k-algebras.

$\operatorname{edim} R - \operatorname{depth} = \operatorname{edim} R'$ (& $\operatorname{depth} R' = 0$).

R is c.i. $\iff$ R' is c.i.

Thus, can assume $\operatorname{depth} R = 0$. The hypothesis is

$$H_1(K^R)^{\operatorname{edim}(R)} \neq 0.$$

The desired conclusion is that R is c.i.

This hypothesis is equivalent to $\operatorname{Fitt}_0(M) \neq 0$.

Fitting ideals and Koszul homology

Say $J \subseteq R$ is an ideal. (R noe., possibly not local)

$$R^n \xrightarrow[(a_{ij})]{} R^m \xrightarrow[(r_1 \cdots r_m)]{} J \to 0 \qquad \text{presentation}$$

$$\mathrm{Fitt}_0(J) = I_{m\times m}((a_{ij})).$$

Koszul cx on $\underline{r} = r_1, \ldots, r_m$.

$$K(\underline{r};R): \quad \cdots \to R^{\binom{m}{2}} \longrightarrow \overset{= \bigoplus Re_i}{R^m} \xrightarrow[(r_1 \cdots r_m)]{} R \longrightarrow 0.$$

Cycles in $K_1(\underline{x};R)$ are precisely the syzygies of J.

$\left[\partial(\sum b_i e_i) = 0 \iff \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathrm{im}(a_{ij})\right.$

In particular

$$z_j := \sum_{i=1}^{m} a_{ij} e_i \quad \text{are cycles in } K_1(\underline{r};R)$$

for $j = 1, \ldots, n$.

$$\langle z_{i_1} \wedge \cdots \wedge z_{i_m} : i_1 < i_2 < \cdots < i_m \rangle = I_{m\times m}((a_{ij}))\cdot K_m(\underline{r};R)$$

In summary: $H_1(\underline{r};R)^m = \mathrm{Fitt}_0(J)\cdot K_m(\underline{r};R)$.

Recall: $H_m(\underline{r};R) = \mathrm{ann}_R(J)$.

So, $H_1(\underline{r};R)^m \subseteq H_m(\underline{r};R)$ reflects

$$\mathrm{Fitt}_0(J) \subseteq \mathrm{ann}(J).$$

One can restate (depth 0 case) of Wiebe as:

R is c.i. with $\dim R = 0$ $\iff$ $\mathrm{Fitt}_0(\mathfrak{m}) \neq 0$.

Lecture 13 (27-02-2023)

Monday, February 27, 2023 1:26 PM

Wiebe's Theorem

$S \to$ comm ring

$I \subseteq J$ where

$$I = (a_1, \dots, a_n),$$

$$J = (x_1, \dots, x_n), \quad \text{where } \underline{a}, \underline{x} \text{ are reg.}$$

We can write $\underline{a} = U \underline{x}$.

Lemma. $$(I : J) = I + (\det U),$$ and $\det U \notin I$.

Example. $R = k[x_1, \dots, x_n]$.

$$(x_1^l, \dots, x_n^l) \subseteq (x_1, \dots, x_n).$$

$$U = \begin{bmatrix} x_1^{l-1} & & 0 \\ & \ddots & \\ 0 & & x_n^{l-1} \end{bmatrix}.$$

$$\left((x_1^l, \dots, x_n^l) : (x_1, \dots, x_n)\right) = (x_1^l, \dots, x_n^l) + x_1^{l-1} \cdots x_n^{l-1}.$$

Proof of Lemma: Equivalent:

$$\left(0 :_{S/I} J/I\right) = (\det U) \cdot S/I.$$

Note $\left(0 :_{S/I} J/I\right) = H_n(\underline{x}, S/I)$.

Also,

$$\begin{tikzcd} S^n \arrow[r, "(a_1,\dots,a_n)"] \arrow[d, "u"'] & S \arrow[d, equal] \\ S^n \arrow[r, "(x_1,\dots,x_n)"'] & S \end{tikzcd}$$

This induces $\Lambda^* u : K(\underline{a}; S) \longrightarrow K(\underline{x}; S)$.

The top degree map is $K_n(\underline{a}; S) \xrightarrow{\det u} K_n(\underline{x}; S)$.

This induces $\Lambda^* u : K(\underline{a}; S/I) \longrightarrow K(\underline{x}; S/I)$.

all differentials 0 here
$\therefore H_n(*) = *$ for this

Need to prove $H_n(\Lambda^* u)$ is onto.
(The map is still $\xrightarrow{\det u}$.)

Moreover :

$$\begin{tikzcd} H_*(\underline{a}; S/I) \arrow[r] & H_*(\underline{x}; S/I) \\ \Lambda^* H_1(a; S/I) \arrow[r] \arrow[u, "\cong"] & \Lambda^* H_1(\underline{x}; S/I) \arrow[u, "\cong"', "\text{Tate}"] \end{tikzcd}$$

Suffices to now prove surjectivity at degree one.

$$H_1(\underline{a}; S/I) \longrightarrow H_1(\underline{x}; S/I)$$

$$\begin{tikzcd} & K(\underline{a}; S) \otimes K(\underline{x}; S) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ K(\underline{a}; S/J) & & K(\underline{x}; S/I) \end{tikzcd}$$

$$\begin{tikzcd} K(\underline{a}; S/J) & & K(\underline{x}; S/I) \\ & K(\underline{a}; S/I) \arrow[ul] \arrow[ur, "\Lambda^* u"'] & \end{tikzcd}$$

both have zero diff.

$\therefore$ surjective on H^1

(or could've directly applied $H_n(-)$ here.)

Special Case

$R = S/I$, S is RLR, $I \subseteq \mathfrak{m}_S^2$. R is c.i.

(Assume R complete.)

- $\dim R = 0$
- $I = \underbrace{(a_1, \dots, a_n)}_{\text{regular}} \subseteq \underbrace{(x_1, \dots, x_n)}_{\text{regular}} = \mathfrak{m}_S$.

$n = \operatorname{edim} R$.

By lemma: $(0 :_R \mathfrak{m}_R) = (\det u)$.

$\neq 0$

Def^n. $(R, \mathfrak{m}_R, k)$ local.

$$\underset{\text{socle}}{\operatorname{Soc}(R)} = (0 :_R \mathfrak{m}_R) \cong \operatorname{Hom}_R(k, R).$$

Note. $\mathfrak{m}$ annihilates $\operatorname{soc}(R)$ by def^n.

$\therefore \operatorname{soc}(R)$ is a k-vector space. $\operatorname{rank}_k(\operatorname{soc}(R)) =: \operatorname{type}(R)$.

Corollary. R c.i. with $\dim R = 0$, then

$(0 :_R \mathfrak{m}_R) \subseteq$ any nonzero ideal.

(if $\dim \neq 0$, then $\operatorname{soc} = 0$ anyway.)

"c.i. rings are gorenstein"

Corollary. $\varphi : (R, m_R) \longrightarrow T$.

Suppose R is c.i.

If $\varphi(\operatorname{soc} R) \neq 0$, then φ is an iso.

Weibe's Thm. (R, m_R) local

If $\operatorname{image}\left(\Lambda^c H_1(K) \rightarrow H_c(K)\right) \neq 0$,

where $c = \operatorname{edim} R - \operatorname{depth} R$

then R is c.i.

We showed that we can reduce to depth $R = 0$.

That is,

$$\Lambda^n H_1(K^R) \longrightarrow H_n(K^R) = Z_n(K^R) \quad \text{nonzero}$$

$$\Rightarrow R \text{ is c.i.} \quad \left(\begin{array}{l}\text{necessarily} \\ \dim R = 0\end{array}\right)$$

$$\left(\equiv \operatorname{Fitt}_0 (m_R) \neq 0 \right)$$

Can assume R is complete. Write

$R = S/I$ min'l Cohen presentation.

$$\frac{I}{m_S I} \longleftrightarrow H_1(K^R).$$

Hypothesis, $\exists z_1, \ldots, z_n \in Z_1(K^R)$ s.t.

$$z_1 \wedge \cdots \wedge z_n \neq 0 \quad \text{in} \quad K_n^R.$$

Let $a_1, \dots, a_n \in I \setminus m_S I$ be some rep^n of $z_1, \dots, z_n$.

<u>Case 1.</u> Suppose $\dim(R) = 0$, i.e., $\text{height}(I) = n$.

Then, $\text{depth}_S(I, S) = n$ $\quad (\because S \text{ regular})$.

<u>Exercise.</u> $\exists\, a_1', \dots, a_n' \in m_S I$ s.t.

$a_1 + a_1', \dots, a_n + a_n'$ is a reg. seq. in S.

$\therefore$ Can assume $a_1, \dots, a_n$ is a reg. seq.

$S/(\underline{a}) = R' \longrightarrow R$ induce

$$\begin{tikzcd}
\Lambda^n H_1(K^{R'}) \arrow[r] \arrow[d] \arrow[dr, "\neq 0"] & H_n(K^{R'}) \arrow[d, "\neq 0"] & \subseteq R' = \mathrm{Soc}(R') \arrow[d] \\
\Lambda^n H_1(K^R) \arrow[r, "\neq 0"'] & H_n(K^R) & \subseteq R
\end{tikzcd}$$

image of this map contains $z_1 \wedge \dots \wedge z_n$

$\neq 0 \Rightarrow$

$\therefore\ \mathrm{soc}(R')R \neq 0$.

$\therefore\ R' = R$ by earlier part.

$\therefore\ R = S/(\underline{a})$ with $\underline{a}$ regular!

<u>Case 2</u> $\dim R > 0$ (will prove that this cannot be).

$z_1 \wedge \dots \wedge z_n \neq 0$ in $K_n^R \cong R$.

$\therefore\ \exists\, s \gg 0$ s.t. $z_1 \wedge \dots \wedge z_n \notin m_R^s$. by Krull int. thm.

$$\Lambda^n H_1(R) \longrightarrow H_n(R) \qquad z_1 \wedge \dots \wedge z_n \qquad R$$

$$
\begin{tikzcd}
\Lambda^n H_1(R) \arrow[r] \arrow[d] \arrow[ddr, bend left, "\neq 0"'] & H_n(R) \arrow[d] & z_1\wedge\cdots\wedge z_n \arrow[dd, mapsto] & R \arrow[d] \\
\Lambda^n H_1(K^{R/m_R^{s+1}}) \arrow[r, "①"'] \arrow[d] & H_n(R/m_R^{s+1}) \arrow[d, "②"] & & R/m_R^{s+1} \arrow[d] \\
\Lambda^n H_1(K^{R/m^s}) \arrow[r] & H_n(R/m^s) & \neq 0 & R/m_R^s
\end{tikzcd}
$$

$\therefore$ ① $\neq 0$.

$\Rightarrow$ R/m_R^{s+1} is c.i by ①.

But ② $\neq 0$ $\Rightarrow$ $R/m^{s+1} \to R/m^s$ is an iso (by socle corollary).

$\therefore m^s = m^{s+1}$.

But then NAK forces $m^s = 0$.

But then dim $= 0$. ∎

<u>Wiebe</u>.

$$\Lambda^i H(K^R) \longrightarrow H_i(K^R)$$

<u>Tate</u> : R c.i. $\Rightarrow$ iso for all i.

<u>Assmus</u> : Onto for $i = 2$ $\Rightarrow$ R c.i.

<u>Wiebe</u> : Non zero for $i =$ edim R $-$ depth R $\Rightarrow$ R c.i.

<u>Bruns</u> : Map is zero for $i >$ edim R $-$ dim R.

Lecture 14 (01-03-2023)

Wednesday, March 1, 2023 1:28 PM

$A \longrightarrow$ any ring (possibly noncomm.)

$E \longrightarrow$ A-module (left)

Def^n. E is injective if $\mathrm{Hom}_A(-, E)$ is exact.

I.e., given any solid diagram

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[dl, dashed, "\tilde{f}"] \\ E & \end{tikzcd}$$

a dotted extension exists.

Thm. (Baer's Criterion)

E is injective $\iff$ $I \hookrightarrow A$ has the extension property for all left ideals I.

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & A \arrow[dl] \\ E & \end{tikzcd} \qquad \begin{array}{c} 1 \\ \downarrow \\ x \end{array}$$

Thus: any map $I \to E$ is multiplication by some $x \in E$.

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$)

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[dl, dashed] \\ E & \end{tikzcd}$$

Consider all submodules U s.t.

$$N \leq U \leq M$$

$$\begin{tikzcd} \cdot \arrow[d, "f"'] & \cdot \arrow[dl, dashed, "f_U"] \\ E & \end{tikzcd}$$

... Take max'l extension (U, f_U).

If $U = M$, done.

Else pick $x \in M \setminus U$.

Extend to $U + Ax$... ∎

Thus, enough to check extension along I.

Now, suppose R is a commutative domain.

① E injective $\Rightarrow$ E divisible.

Given $x \in E$, $r \in R \setminus \{0\}$, $\exists\, y \in E$ s.t. $ry = x$.

② Converse holds if R is a PID.

Equivalently,
$$\begin{tikzcd} R \arrow[r, hook, "r"] \arrow[d] & R \arrow[dl, dashed] \\ E & \end{tikzcd}$$
has extension property for all $r \neq 0$.

<u>Example.</u> Over $\mathbb{Z}$, $\mathbb{Q}$ is divisible and $\therefore$ injective.

<u>Lemma.</u> Any $\mathbb{Z}$-module can be embedded into an injective $\mathbb{Z}$-module.

<u>Proof.</u> $\mathbb{Q}$ divisible $\Rightarrow$ $\oplus \mathbb{Q}$ divisible

$\Rightarrow$ Any quotient of $\oplus \mathbb{Q}$ is also divisible

$\Rightarrow$ Any quotient of $\oplus\mathbb{Q}$ is also divisible

$\Rightarrow$ Any quotient of $\oplus\mathbb{Q}$ is injective.

Now, any free F can be embedded $F \hookrightarrow \oplus\mathbb{Q}$.

Now if $U \subseteq F$ submodule, then

$$F/U \hookrightarrow \oplus\mathbb{Q}/U.$$

$\boxtimes$

Lemma. Let $A \to B$ map of rings.

E : injective A-module.

Then,

$$\mathrm{Hom}_A(B, E)$$

viewed as a left B-module, is injective.

[If $f: B \to E$ is A-linear, and $b \in B$ then
$$(b \cdot f)(x) = f(x b).$$
Check $b \cdot f$ is A-linear and this is an action.]

Proof. Need to check

$$\mathrm{Hom}_B(\_, \mathrm{Hom}_A(B, E)) \text{ is exact on mod-}B$$

But

$$\cong$$
$$\mathrm{Hom}_A(\_, E).$$

$\boxtimes$

Prop^n. $A \to$ any ring

Any A-module embeds into an injective A-module.

Proof. $\mathbb{Z} \to A$ structure map.

$M \to A$-module

$M\downarrow_{\mathbb{Z}}$ $\rightsquigarrow$ think of M as a $\mathbb{Z}$-module

$M\downarrow_{\mathbb{Z}} \hookrightarrow E$ ← injective $\mathbb{Z}$-module

$$\mathrm{Hom}_{\mathbb{Z}}(A, M\downarrow_{\mathbb{Z}}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, E).$$

↳ A-modules

The map is also A-linear.

Suffices to embed
$$M \overset{\lambda}{\hookrightarrow} \mathrm{Hom}_{\mathbb{Z}}(A, M\downarrow_{\mathbb{Z}}).$$

$$\lambda(x)(a) := a \cdot x.$$

Essential Extensions

Def^n. An extension $N \hookrightarrow M$ is essential if
$\forall\, U \overset{\neq 0}{\subseteq} M$ submodule
if $U \cap N \neq 0$.

It is proper if $N \neq M$.

Lemma. An A-module E is injective
$\iff$ E admits no proper essential extensions.

Proof. $(\Rightarrow)$ E injective. If $E \overset{i}{\hookrightarrow} M$, essential then i split.

$F \oplus E = M$. But $F \cap E = 0$.

$\therefore$ $F = 0$ by essentiality.

$\therefore$ $M = E$.

($\Leftarrow$) Say E admits no proper essential inclusions.

Given
$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \\ E \end{tikzcd}$$
, consider pushforward

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[d] \\ E \arrow[r, hook] & W \end{tikzcd} = \frac{M \oplus E}{\langle (n, f(n)) : n \in N \rangle}$$

In W, let $U \subseteq W$ be a max'l submodule s.t. $U \cap E = 0$.

But now
$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & W/U \\ & W \arrow[ur] & \end{tikzcd}$$
is injective and essential.

$\therefore$ Isomorphism.

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[d, "g"] \\ E \arrow[r, hook] \arrow[d, "\varphi"', "\cong"] & W \arrow[dl, "p"] \\ W/U & \end{tikzcd}$$

Define $M \longrightarrow E$ as $\varphi^{-1} \circ p \circ g$. Desired extension. $\blacksquare$

Defn. M any A-module.

An injective hull of M is an essential extension
$$M \hookrightarrow E$$
with E injective.

(This E is unique up to iso.)

Prop^n. Let M be an A-module.

1. Take $M \hookrightarrow I$ with I injective.
   Then, any maximal essential extension of M in I is injective. (And hence an essential hull of M.)

2. $M \hookrightarrow I$ with I injective, and $M \hookrightarrow E$ is an injective hull, then $\exists$ mono $E \hookrightarrow I$ s.t.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, hook] & E \arrow[d, hook] \\ & I \end{tikzcd}$$ commutes.

$\therefore E$ is a "minimal injective module containing M".

3. $M \hookrightarrow E$ and $M \hookrightarrow E'$ injective hulls, then $\exists$ comm diagram

$$\begin{tikzcd} & M \arrow[dl, hook] \arrow[dr, hook] & \\ E \arrow[rr, "\cong"'] & & E' \end{tikzcd}.$$

Proof. 2, 3 OK.

1. Let $M \hookrightarrow E \hookrightarrow I$.
   E : max'l ess. ext^n of M in I.

<u>ETP</u>: E admits no proper essential extension.

Say $E \hookrightarrow U$ essential.

Then,
$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, hook] & U \arrow[d, "i"] \\ & I \end{tikzcd}.$$

Moreover, $\ker(i) \cap E = 0$.
$\therefore \ker i = 0$.
$\Rightarrow i$ is injective.

n.t the $E \subset U \subseteq I$ in I.

But the $E \subseteq U \subseteq I$ in I.

maximality $\Rightarrow E = U$. $\blacksquare$

($M \hookrightarrow E$ ess + $E \hookrightarrow U$ ess $\Rightarrow M \hookrightarrow U$ ess.)

Key Consequence.

$A \to$ any ring

$M \to$ any module.

Then, M admits a unique min'l injective resolution, which is unique up to iso (of complexes). $\blacksquare$

$$\begin{tikzcd}[column sep=small]
 & \boxed{E_A(M)} \arrow[rr] \arrow[dr, two heads] & & E_A(M^1) \arrow[rr] \arrow[dr, two heads] & & E_A(M^2) \arrow[dr, two heads] & \\
M \arrow[ur, hook, "i^0"'] & & M^1 \arrow[ur, hook, "i^1"'] & & M^2 \arrow[ur, hook, "i^2"'] & & \cdots
\end{tikzcd}$$

$M^1 = \operatorname{coker}(i^0)$

$$M \simeq 0 \to E^0 \to E^1 \to \cdots.$$

$$\operatorname{Ext}^n_A(-, M) = H^n(\operatorname{Hom}_A(-, M)).$$

Remark. $\{E_i\}$ any family of injectives, $\prod E_i$ is injective.

Baer's Criterion $\Rightarrow$ If A is (left) Noetherian, then $\bigoplus E_i$ is injective.

Lecture 15 (13-02-2023)

Monday, March 13, 2023 1:24 PM

Structure of Injectives

R : M, N

$$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(P,N)).$$

where $P \xrightarrow{\sim} M$

E is injective $\iff \mathrm{Ext}^1_R(-,E)=0$ on Mod R

$\iff \mathrm{Ext}^1_R(R/I, E) = 0$ for all ideals $I \subseteq R$.

$\iff \mathrm{Ext}^i_R(-,E)=0$ on Mod R $\forall i \geq 1$

Note. $0 \to X \to Y \to Z \to 0$ is exact in Mod R

gives $0 \to \mathrm{Hom}(Z,E) \to \mathrm{Hom}(Y,E) \to \mathrm{Hom}(X,E) \to \mathrm{Ext}^1(Z,E) \to \cdots$

R commutative noetherian.

1. $\{E_\lambda\}_\lambda$ injective $\Rightarrow \bigoplus_\lambda E_\lambda$ injective.

2. Any injective can be written as a direct sum of indecomposable injectives.

indecomposable injectives.

③ The indecomposable injectives are precisely those of the form

$$E_R(R/\mathfrak{p})$$

for $\mathfrak{p} \in \operatorname{Spec} R$. (These are pairwise non-iso.)

④ $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion.

M is $\mathfrak{p}$-local if $M \xrightarrow{\sim} M_\mathfrak{p}$ is an iso.

$\Leftrightarrow$ $M \xrightarrow{r} M$ is an iso for $r \in R \setminus \mathfrak{p}$

$\Leftrightarrow$ R action on M factors through $R \to R_\mathfrak{p}$

every $x \in M$ is annihilated by $\mathfrak{p}^n$ $(n = n(x))$

i.e.,

$$\bigcup_{n \geq 0} (0 :_M \mathfrak{p}^n) = M$$

$$\| $$

$$\Gamma_{V(\mathfrak{p})}(M)$$

Lemma. $S := U^{-1}R$, where $U \subseteq R$ mult. closed. $R \to \hat{R}$ is an example.

$R \to S$.

Not true for arbit flat maps.

① E injective R-module $\Rightarrow$ $S \otimes_R E$ injective S-module.

② Any injective S-module is injective over R.

③ $M \hookrightarrow N$ is essential over R

$\Rightarrow$ $S \otimes_R M \hookrightarrow S \otimes_R N$ is essential.

In particular, $S \otimes_R E_R(M) = E_S(S \otimes_R M)$.

Proof. ② Holds for any flat map $R \to S$.

Let I be inj S-module.

$\operatorname{Ext}^1_R(M, I) \cong \operatorname{Ext}^1_S(S \otimes_R M, I) = 0.$ ✓

$$\mathrm{Ext}^1_R(M, I) \cong \mathrm{Ext}^1_S(S \otimes_R M, I) = 0. \quad \checkmark$$

$\because$ I S-module and $R \to S$ flat

① Any f.g. S-module is of the form $S \otimes_R M$ with M f.g. over R.

("Clear denominators of generators")

$$\mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R E) \cong \mathrm{Ext}^1_R(M, S \otimes_R E)$$
$$\cong S \otimes_R \mathrm{Ext}^1_R(M, E) = 0.$$

We used the following key property of localisation:

$R \to S$ flat
$S \otimes_R S \xrightarrow{\cong} S$ iso.

Defines "absolutely flat map"

③ WTS $U^{-1}M \hookrightarrow U^{-1}N$ is essential.

Equiv: $S\left(\frac{x}{u}\right) \cap U^{-1}M \neq 0$ for $\frac{x}{u} \neq 0$.

($x \in N$, $u \in U$.)

Can even just check when $u = 1$.

Of course, $x \neq 0$ in N.

Consider $\{\mathrm{ann}_R(ux) : u \in U\} =: \Sigma$

Note $ux \neq 0$ since $\frac{x}{1} \neq 0$.

Noetherian $\Rightarrow$ Σ has maximal element(s).

(Proper ideals.)

Replace x by a suitable ux to ensure that $\mathrm{ann}_R(x)$ is a maximal el't of Σ.

$0 \neq M \cap Rx = (r_1 x, \dots, r_n x)$, $\quad r_i \in R$.

Say $r \in R$ is s.t.

$rx = 0$ in $U^{-1} N$.

$\Rightarrow urx = 0$ for some $u \in \mathcal{U}$, i.e.,

$\Rightarrow r \in \mathrm{ann}_R(ux)$ ----

[always contains $\mathrm{ann}_R(x)$.

$\therefore r \in \mathrm{ann}_R(x)$.

Now, pick $r_i \in \{r_1, \dots, r_n\}$ s.t. $r_i x \neq 0$.

Then, $r_i x \in U^{-1} M \cap S(\frac{x}{1}) \setminus \{0\}$. ∎

Lemma. Fix $\mathfrak{p} \in \operatorname{Spec} R$.

① $E_R(R/\mathfrak{p})$ is indecomposable.

② $\operatorname{Ass}_R(E_R(R/\mathfrak{p})) = \{\mathfrak{p}\}$.

③ $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion.

④ $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E(R/\mathfrak{p})) \cong \kappa(\mathfrak{p})$.

Proof.

① A stronger property holds: any two nonzero modules in $E_R(R/\mathfrak{p})$ has nontrivial intersection.

($\equiv$ any submodule is indecomposable)

Let $M, N \neq 0$ be nonzero submodules.

$R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p})$ essential.

So, $M \cap R/\mathfrak{p} \neq 0$,

$$N \cap R/\mathfrak{p} \neq 0.$$

$$(M \cap R/\mathfrak{p}) \cap (N \cap R/\mathfrak{p}) \neq 0$$

nonzero ideals in domain

$$\therefore \quad M \cap N \neq 0.$$

This proves ①

$$\cdot \quad \underbrace{R/\mathfrak{p} \hookrightarrow k(\mathfrak{p})}_{\text{ess}} \underbrace{\hookrightarrow}_{\text{ess}} E_{R_\mathfrak{p}}(k(\mathfrak{p}))$$

$\hookrightarrow$ inj $/R_\mathfrak{p}$

$\Rightarrow$ inj $/R$

$$\therefore E_R(R/\mathfrak{p}) = E_{R_\mathfrak{p}}(k(\mathfrak{p}))$$

previous

$$\cong R_\mathfrak{p} \otimes_R E_R(k(\mathfrak{p})).$$

② follows from: $\forall M \subset \mathrm{Mod}\, R$

$$\mathrm{Ass}_R(E_R(M)) = \mathrm{Ass}_R(M)$$

($\supseteq$) true since $M \hookrightarrow E_R(M)$.

($\subseteq$) Let $\mathfrak{p} \in \mathrm{Ass}_R(E_R(M))$.

$R/\mathfrak{p} \hookrightarrow M$. Let $U \cong R/\mathfrak{p}$ be the image.

$$U \cap M \neq 0.$$

$\therefore \mathrm{Ass}(U \cap M)$ has an ass prime.

But $\mathrm{Ass}(U \cap M) \subseteq \mathrm{Ass}(U) = \{\mathfrak{p}\}$.

$$\therefore \quad \mathfrak{p} \in \mathrm{Ass}(U \cap M) \subset \mathrm{Ass}(M). \quad \blacksquare$$

This also implies $\mathfrak{p}$-power torsion.

④ May assume R local and $\mathfrak{p}$ max'l.

(4) May assume R local and $\mathfrak{p}$ max'l.

NTP $\quad \operatorname{Hom}_R(k, E_R(k)) \cong k.$

The Hom is $\neq 0$ since $i : k \hookrightarrow E_R(k)$
is in the Hom.

Moreover, $\operatorname{Hom}_R(k, E_R(k))$ is a k-subspace
of $E_R(k)$. But it's indecomposable
by our earlier statement.

$\therefore$ it is k. ∎

Thm. Any injective can be written as a direct sum of indecomposable injectives.

Proof $E \to$ inj. R-module.

Pick $\mathfrak{p} \in \operatorname{Ass}_R(E)$. $\therefore R/\mathfrak{p} \hookrightarrow E$. This factors:

$$R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p}) \hookrightarrow E.$$

Now Zorn... ∎

This finishes proof all of the main properties!

Uniqueness?

Suppose

$$E \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E(R/\mathfrak{p})^{\nu(\mathfrak{p})}.$$

$\neq 0$ iff
$\mathfrak{p} \in \operatorname{Ass} E$

Then, $\nu(\mathfrak{p}) = \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E_\mathfrak{p})$ $\neq 0$ iff $\mathfrak{p} \in \operatorname{Ass}_R E$.

and hence independent of decomposition.

Key: $\operatorname{Hom}_{R_\mathfrak{p}}\big(k(\mathfrak{p}), E(R/\mathfrak{q})_\mathfrak{p}\big) = \begin{cases} k(\mathfrak{p}) & \mathfrak{p} = \mathfrak{q} \\ 0 & \text{else} \end{cases}$ by earlier

~~Say $\mathfrak{q} \neq \mathfrak{p}$.~~

~~If $\mathfrak{q} \notin \mathfrak{p}$, $E(R/\mathfrak{q})$ is $\mathfrak{q}$-pow torsion~~

~~but $\exists\, a \in \mathfrak{q} \setminus \mathfrak{p}$. so~~

~~a is inverted but kills $E(R/\mathfrak{q})$.~~

Use the following:

$\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M) \neq 0 \iff \mathfrak{p} \in \operatorname{Ass}(M)$.

$$M \hookrightarrow E_R(M) = \bigoplus_{\mathfrak{p} \in \operatorname{Ass}_R(M)} E(R/\mathfrak{p})^{\mu(\mathfrak{p})}$$

$$\mu(\mathfrak{p}) = \operatorname{rank}_{R_\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}\big(k(\mathfrak{p}), E_R(M)_\mathfrak{p}\big)$$

$$= \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

In particular, if M is f.g., then $\mu(\mathfrak{p})$ is finite.

Defn. $M \in \operatorname{Mod} R$. The ith Bass number of M is wrt $\mathfrak{p}$

$$\mu_R^i(\mathfrak{p}; M) := \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Ext}_{R_\mathfrak{p}}^i(k(\mathfrak{p}), M_\mathfrak{p}).$$

$$\mu_R^i(\mathfrak{p}; M) := \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

Let $M \xrightarrow{\simeq} I$ be a min'l inj. resolution.

Then $$I^i \cong \bigoplus E(R/\mathfrak{p})^{\mu^i(\mathfrak{p}; M)}.$$

Lecture 16 (15-03-2023)

Wednesday, March 15, 2023 1:26 PM

Matlis Duality

$R \to$ comm noetherian

① $M \otimes_R \mathrm{Hom}_R(N, E) \xrightarrow{\text{natural map}} \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), E)$. (+)

$x \otimes \alpha \longmapsto [f \mapsto \alpha f(x)]$

adjoint

$\mathrm{Hom}(M, N) \otimes_R M \otimes_R \mathrm{Hom}_R(N, E) \longrightarrow E$

$N \otimes \mathrm{Hom}_R(N, E)$

(+) is bijective when E is injective and M is f.g

Sketch. True when $M = R$.

$\therefore$ true when $M = R^{\oplus n}$.

Next, present M as

$G \to F \to M \to 0$ $\quad$ F, G finite free

Then,

$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F, N) \to \mathrm{Hom}_R(G, N)$ is exact.

$\therefore E$ injective

$\mathrm{Hom}_R(\mathrm{Hom}_R(G,N), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(F,N), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,N), E) \to 0.$

tensoring $\quad$ $\cong$ isom.s since $\cong$ $\quad \Longrightarrow \quad \cong$

tensoring right exact

$\cong$ isoms since F, G fin free $\qquad \cong \qquad \Longrightarrow \qquad \cong$

$$G \otimes_R \mathrm{Hom}_R(N, E) \longrightarrow F \otimes_R \mathrm{Hom}_R(N,E) \longrightarrow M \otimes_R \mathrm{Hom}_R(N,E) \longrightarrow 0$$

Corollary. If E and E' are injective $\Rightarrow$ $\mathrm{Hom}_R(E', E)$ flat.

Proof. $- \otimes_R \mathrm{Hom}_R(E', E) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(-, E'), E)$ on f.g. mod R.

But RHS is an exact functor.

$\therefore$ $- \otimes_R \mathrm{Hom}_R(E', E)$ is exact on fg mod R.

$\Longrightarrow$ —"— on Mod-R. ∎

Thus, $\mathrm{Hom}_R(-, E) : \mathrm{Inj}\ R \longrightarrow \mathrm{Flat}\ R$.

Check. $\mathrm{Hom}_R(-, E) : \mathrm{Flat}\ R \longrightarrow \mathrm{Inj}\ R$.

---

② $(R, m, k) \twoheadrightarrow S \simeq R/I$.

Claim $\mathrm{Hom}_R(S, E_R(k)) \cong E_S(k)$. ← Injective over S.

$$\mathrm{Hom}_R(S, E_R(k)) = (0 :_{E_R(k)} I)$$

← largest S-module of $E_R(k)$

$$\left(k \hookrightarrow E_R(k), \quad k \hookrightarrow (0 :_{E_R(k)} I) \hookrightarrow E_R(k) \right)$$

Fix a local (comm noe) ring $(R, \mathfrak{m}, k)$.

Fix $E := E_R(k)$.

$(\_)^\vee := \mathrm{Hom}_R(\_, E)$.

all R-modules

$$\begin{array}{ccc} \mathrm{Mod}\ R & \xrightarrow{(\ )^\vee} & \mathrm{Mod}\ R. \\ \cup\!\mid & & \\ \mathrm{mod}\ R & \longrightarrow & ? \end{array}$$

f.g. R-modules

Consider $M \xrightarrow{\text{eval}} M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$

$x \longmapsto (f \mapsto f(x))$

Obs.

- $R^\vee = E$ flat, by earlier.
- $R^{\vee\vee} = E^\vee = \mathrm{Hom}_R(E, E) = \mathrm{End}(E)$

$R \xrightarrow{\text{eval}} \mathrm{End}_R(E)$ → ring homomorphism even

$r \longmapsto (E \xrightarrow{\cdot r} E)$

will see $\mathrm{End}_R(E) \cong \hat{R}$ as rings (and R-mod)

If M is f.g., we have a comm diagram

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E) \\ \cong \uparrow & & \uparrow \cong \\ M \otimes_R R & \longrightarrow & M \otimes_R \mathrm{End}_R(E) \end{array}$$

Lemma. ① When N has finite length, then

(a) $\ell_R(N) = \ell_R(N^\vee)$.

(R = field recovers usual duality)

Lemma. ①

(recovers usual duality)

(a) $\ell_R(N) = \ell_R(N^\vee)$,

(b) $\beta_0^R(N) = \mu_R^0(N^\vee)$,

$\mu_R^0(N) = \beta_0^R(N^\vee)$.

$\beta_0^R(M) :=$ min'l # gens of N, or $\dim_k(M/m_M M)$

$\mu^0(M) := \operatorname{rank}_k \operatorname{Hom}_R(k, M)$

$= \operatorname{rank}_k \operatorname{Soc}_R(M)$

(c) $N \xrightarrow{\cong} N^{\vee\vee}$.

② When R is artinian,

$$(\ )^\vee : (\underset{\text{f.g.}}{\text{mod}}\ R)^{op} \longrightarrow \text{mod}\ R$$

is an equiv of categories.

(If we drop "artinian", it's an equiv on fin length R-mod.)

Proof. ① Recall $\operatorname{Hom}_R(k, E) \cong k$.

$\therefore\ k^{\vee\vee} \cong k$.

$\Rightarrow k \xrightarrow{\cong} k^{\vee\vee}$ ($\because$ nonzero and dim = 1.)

$\therefore$ (a), (c) hold for k.

Now induce on $\ell_R(N)$.

$$0 \to k \hookrightarrow N \to N' \to 0.$$

Apply $(-)^\vee$ $\qquad \ell_R(N') = \ell_R(N) - 1.$

$$0 \to (N')^\vee \to N^\vee \to k \to 0.$$

$$\Rightarrow \quad \ell_R(N^\vee) = \ell_R((N')^\vee) + 1 \quad \text{induction}$$
$$= \ell_R(N') + 1$$
$$= \ell_R(N).$$

Similarly use

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] \arrow[d, "\cong"'] & N \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & k^{\vee\vee} \arrow[r] & N^{\vee\vee} \arrow[r] & N'^{\vee\vee} \arrow[r] & 0 \end{tikzcd}$$

for (c).

(b) $\beta_0^R(N) = \operatorname{rank}_k (k \otimes_R N)$
$$= \operatorname{rank}_k \operatorname{Hom}_R(k \otimes N, E)$$
$$= \operatorname{rank}_k \operatorname{Hom}_R(k, N^\vee)$$
$$= \mu_R^0(N^\vee).$$

Suppose R is $\mathfrak{m}$-adically complete.

Lemma $(R, \mathfrak{m}, k)$ complete local. Then
$$R \xrightarrow{\cong} \operatorname{End}_R(E).$$

Proof. For each $n \geq 1$, consider

$$E_n := \operatorname{Hom}(R/\mathfrak{m}^n R, E) \subseteq E$$
$$E_n = E_{R/\mathfrak{m}^n R}(k)$$

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{End}_R(E) \arrow[d, "\text{restriction (onto because...)}"] \\ \dfrac{R}{\mathfrak{m}^n R} \arrow[r, "\cong"'] & \operatorname{End}_{R/\mathfrak{m}^n R}(E_n) \end{tikzcd}$$

$\because$ Artinian

We get $R \xrightarrow{\cong} \varprojlim_n \frac{R}{m^n R} \xrightarrow{\cong} \lim_n \mathrm{End}(E_n) \xleftarrow{\cong} \mathrm{End}_R(E)$.

iso $\because$ R-complete

have to check

Follows since $E = \bigcup_{n \geq 0} E_n$.

Theorem. R complete local ring.
One has an equivalence of categories

$$\operatorname{mod} R \underset{(\ )^\vee}{\overset{(\ )^\vee}{\rightleftarrows}} \operatorname{art} R.$$

f.g.

artinian

(In particular, E is Artinian.)

Proof. ① We show E is Artinian.

Say $E \supseteq M_0 \supseteq M_1 \supseteq \cdots$.

This gives $E^\vee \to M_0 \twoheadrightarrow M_1 \twoheadrightarrow \cdots$.
$E^\vee = R$

Surjections correspond to ascending chain of ideals

$\therefore$ Stabilise eventually.

$0 \to M_n \hookrightarrow M_{n+1} \to \frac{M_{n+1}}{M_n} \to 0$

$\Rightarrow M_n = M_{n+1} \quad \forall n \gg 0.$

dualising
$\Rightarrow \left(\frac{M_{n+1}}{M_n}\right)^\vee = 0.$

But M_{n+1}/M_n is m-torsion being a subquotient of E.

$\therefore$ if nonzero, then $k \hookrightarrow \frac{M_{n+1}}{M_n}$.

But then, $\left(M_{n+1}/M_n\right)^\vee \twoheadrightarrow k^\vee$. $\to\leftarrow$

But then, $\left(M_{n+1}/M_n \right)^\vee \twoheadrightarrow k^\vee$. $\rightarrow\leftarrow$

$\therefore$ E is Artinian.

This now implies $M^\vee$ artinian for all f.g. M.

Now, if M artinian, note

$$\begin{array}{ccc} \mathrm{soc}(M) & \hookrightarrow & M \\ \cong & & \\ k^\mu & & \end{array}$$

$\therefore$ $\mathrm{soc}(M) \hookrightarrow E^\mu$.

$\because$ M is artinian, $\mathrm{soc}(M) \hookrightarrow M$ is essential. $(*)$

$\therefore$ $M \hookrightarrow E^\mu$.

Now, dualise to get $R^\mu \twoheadrightarrow M^\vee$.

$\Rightarrow M$ is f.g.

For f.g. M, we have

$$\begin{array}{ccc} M & \longrightarrow & M^{\vee\vee} \\ \cong \uparrow & & \uparrow \cong \\ M \otimes_R R & \xrightarrow[\cong]{} & M \otimes_R \mathrm{End}_R(E) \end{array}$$

M f.g.

By lemma

$\therefore$ top is $\cong$.

Need to check for artinian.

Okay for E. For general artinian, do copresentation.

$R \twoheadrightarrow k$ $\qquad$ mod k

$\operatorname{Hom}_k(-, k)$

mod $k \subseteq$ mod R

"Can we lift the vector space duality to mod R?"

Is there is an R-module I s.t.

$\operatorname{Hom}_R(-, I) \cong \operatorname{Hom}_R(-, k)$ on mod k.

exact on mod R

$I = E$ does the job, by Matlis duality.

Lecture 17 (20-03-2023)

Monday, March 20, 2023 1:24 PM

Matlis Duality

$(R, \mathfrak{m}, k)$. $E = E_R(k)$.

Ex. $M \neq 0 \Rightarrow \mathrm{Hom}_R(M, E) \neq 0$.

i.e., E is a cogenerator for Mod R.

Ex. R comm noetherian (not necessarily local).

Then, $\bigoplus_{\mathfrak{m} \in \mathrm{Max}(R)} E_R(R/\mathfrak{m})$ is a cogenerator.

$\rightsquigarrow$ turns out to be $\mathbb{Q}/\mathbb{Z}$

Key computation:

$$\hat{R} \xrightarrow{\cong} \mathrm{End}_R(E).$$

Matlis duality followed from ↑ and E being injective.

$A \to$ any ring (possibly non comm, non noe ...)

I an A-complex.

- I is *K-injective* if $\mathrm{Hom}_A(-, I)$ preserves quasi isomorphisms.

  That is, if $f : X \to Y$ is a map of A-cx's s.t. $H(f)$ is bijective, then

  $$\mathrm{Hom}_A(f, I) : \mathrm{Hom}_A(Y, I) \to \mathrm{Hom}_A(X, I)$$

  is bijective in homology.

- I is *semi-injective* if $\mathrm{Hom}_A(-, I)$ takes

- I is semi-injective if $\mathrm{Hom}_A(-, I)$ takes $\{$mono + quasi iso$\}$ to $\{$epi + quasi iso$\}$.

FACT: I semi-injective
$\iff$ I K-injective + I^n injective $\forall n$

FACT: Any complex has a semi-injective resolution:

$\forall M$: $\quad M \xrightarrow{\sim} I.$ $\leftarrow$ semi-injective

Can also assume $\rightarrow$ is one-one.

Any two such resolutions are homotopy equivalent.

M, N A-complexes.

$P_M \xrightarrow{\sim} M$ and $P_N \xrightarrow{\sim} N$ semi proj resol^{n}s.

Then,

$$\begin{array}{ccc} R\mathrm{Hom}_A(M, N) & = & \mathrm{Hom}_A(P_M, P_N) \\ & & \downarrow \simeq \\ & & \mathrm{Hom}_A(P_M, N). \end{array}$$

Similarly, if $M \xrightarrow{\sim} I_M$ and $N \xrightarrow{\sim} I_N$ are semi-inj resol^{n}s, then

$$\begin{array}{ccc} \mathrm{Hom}_A(M, I_N) & \xrightarrow{\sim} & \mathrm{Hom}_A(P_M, I_N) \\ \uparrow \simeq & & \uparrow \simeq \\ \mathrm{Hom}_A(I_M, I_N) & & \mathrm{Hom}_A(P_M, N) \end{array}$$

$$\mathrm{Ext}^i(M,N) = H^i(R\mathrm{Hom}_A(M,N)).$$

R commutative noetherian

$M \in D^b(\mathrm{mod}\ R)$

$M \xrightarrow{\sim} I$ minimal injective resol^n.

Minimality (defn): induced differential on $\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), I_\mathfrak{p})$ is zero for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

In the case R local and $\mathfrak{p} = \mathfrak{m}$:

$\mathrm{Hom}_R(k, I)$:

$$0 \to I^a \to \cdots \to I^n \to I^{n+1} \to \cdots$$

$$\Big\downarrow \mathrm{Hom}_R(k, -)$$

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_R(k, I^a) & \to \cdots \to & \mathrm{Hom}_R(k, I^n) & \to \cdots \\ & \| & & \| & \\ 0 \to & \mathrm{soc}(I^a) & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & \mathrm{soc}(I^n) & \to \cdots \end{array}$$

Minimality means $\partial(\mathrm{soc}(I^n)) = 0$.

Minimality gives:

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = H^i(\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), I_\mathfrak{p})) \quad \text{differentials} = 0$$

$$= \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), I^i_\mathfrak{p})$$

$\Rightarrow \mathrm{rank}_{k(\mathfrak{p})} \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = \#$ copies of $E(R/\mathfrak{p})$ in I^i

$\|$

$$\mu_R^i(\mathfrak{p}, M) \qquad \text{(Recall Bass number)}$$

Ex. $M, N \in D^b(\text{mod } R)$, then the R-module $\text{Ext}_R^i(M, N)$ is f.g. $\forall i$.

In particular, $\mu_R^i(\mathfrak{p}, M) < \infty$.

- $\text{injdim}_R(M) = \underset{\text{sup or inf}}{?} \{ n : M \xrightarrow{\sim} I \text{ semi inje resol}^n \text{ with } I^i = 0 \ \forall i > n \}$

WANT injdim $(0) = -\infty$.

(Possibly incorrect for $M = 0 \ldots$)

$$= ? \{ n \mid \mu_R^i(\mathfrak{p}, M) = 0 \quad \forall i > n, \ \forall \mathfrak{p} \in \text{Spec } R \}.$$

Theorem 1 Fix $M \overset{\neq 0}{\in} D^b(\text{mod } R)$.

Then,

$$\text{injdim}_R M = \inf \{ n \mid \mu_R^i(\mathfrak{m}, M) = 0 \quad \forall i > n, \ \forall \mathfrak{m} \in \text{Max}(R) \}.$$

In particular, if $(R, \mathfrak{m})$ is local, then

$$\text{injdim}_R M = \sup \{ n : \mu_R^i(k, M) \neq 0 \}.$$

Recall: $\text{depth}_R(M) = \inf \{ n : \text{Ext}_R^n(k, M) \neq 0 \}$

So, $\text{depth}_R M \leq \text{injdim}_R M$.

Theorem 2 $M \in \text{Mod } R$.

Theorem 2 $M \in \mathrm{Mod}\ R$.

$$\mu_R^i(\mathfrak{m}, M) \neq 0 \quad \forall\ \mathrm{depth}_R M \leq i \leq \mathrm{injdim}_R(M).$$

Will prove this later.

Theorem 3. R commutative noetherian.
$M \in D^b(\mathrm{mod}\ R)$.

$\mathfrak{p} \subsetneq \mathfrak{q}$ in $\mathrm{Spec}\ R$ s.t. there are no primes between $\mathfrak{p}$ and $\mathfrak{q}$.

Then, if

$$\mu_R^i(\mathfrak{p}, M) \neq 0 \Rightarrow \mu_R^{i+1}(\mathfrak{q}, M) \neq 0.$$

(This gives Theorem 1.)

Proof. Can localise at $\mathfrak{q}$ to assume $(R, \mathfrak{m}, k)$ local with $\mathfrak{m} = \mathfrak{q}$ and $\dim(R/\mathfrak{p}) = 1$.

We prove:

$$\mu^{i+1}(\mathfrak{m}, M) = 0 \Rightarrow \mu^i(\mathfrak{p}, M) = 0.$$

Ie, $\mathrm{Ext}_R^{i+1}(k, M) = 0 \Rightarrow \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M) = 0$

$\Downarrow$

$\mathrm{Ext}_R^{i+1}(-, M) = 0$ on finite length R-modules.

(Say N has finite length ≥ 1. Can embed $0 \to k \hookrightarrow N \to N' \to 0$ with length N' = length $N - 1$. ...)

Pick $r \in \mathfrak{m} \setminus \mathfrak{p}$. We have an exact seq.

$$0 \to \frac{R}{} \xrightarrow{r} \frac{R}{} \to \frac{R}{} \to 0.$$

$$0 \to \frac{R}{\mathfrak{p}} \xrightarrow{r} \frac{R}{\mathfrak{p}} \to \frac{R}{(\mathfrak{p}, r)} \to 0.$$

$\frac{R}{\mathfrak{p}}$: 1-dim'l

$\frac{R}{(\mathfrak{p}, r)}$: $\therefore$ 0 dim'l $\Rightarrow$ Artinian $\Rightarrow$ finite length

$$\therefore \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \xrightarrow{r} \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \to 0.$$

$\therefore \cdot r$ is surjective. By NAK, it is zero.

$\blacksquare$

Now localise ...

Theorem. $(R, \mathfrak{m}, k)$ local. $M, N \in D^b(\operatorname{mod} R)$.

(Ischebeck) If $\operatorname{injdim}_R N < \infty$, then

$$\sup \operatorname{Ext}^*_R(M, N) = \operatorname{injdim}_R N - \operatorname{depth}_R M.$$

[Auslander-Buchsbaum for injectives.]

Recall: $M \otimes_R \operatorname{Hom}_R(N, I) \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,N), I)$.

iso when M f.g. and I injective. (modules)

This extends to

$M, N, I \to R$-complexes.

$$M \overset{L}{\otimes}_R \operatorname{RHom}(N, I) \to \operatorname{RHom}_R(\operatorname{RHom}_R(M, N), I)$$

is an iso

when $M, N \in D^b(\operatorname{mod} R)$,

$I \simeq$ bounded complex of injectives.

Back to Ischebeck:

Let $E = E_R(k)$. Then,

$\operatorname{RHom}_R(N, E) \simeq$ bounded cx of flat modules

[[illegible] takes inj' to flat]

$R\mathrm{Hom}_R(N, E) \simeq$ bounded cx of flat modules

[$\mathrm{Hom}_R(-, E)$ takes inj to flat]

Now, we apply Auslander-Bu...

$$\mathrm{depth}\left(M \overset{L}{\underset{R}{\otimes}} R\mathrm{Hom}(N,E)\right) = \mathrm{depth}_R M - \sup H_*\left(k \overset{L}{\underset{R}{\otimes}} R\mathrm{Hom}_R(N,E)\right)$$

$$\mathrm{depth}\left(\mathrm{Hom}_R(R\mathrm{Hom}_R(M,N), E)\right) \qquad \mathrm{depth}\, M - \sup H_*\left(\mathrm{Hom}(R\mathrm{Hom}_R(k,N), E)\right) \quad (1)$$

$X := R\mathrm{Hom}_R(M, N)$.

$H^i(X) = \mathrm{Ext}_R^i(M, N)$ ← f.g.

$X^\vee := \mathrm{Hom}_R(X, E)$. ∵ E inj

$$H_i(X^\vee) = \mathrm{Hom}_R(H^i(X), E)$$
$$= \mathrm{Hom}_R(\mathrm{Ext}_R^i(M,N), E)$$

∵ E cogenerator

$$s := \sup\{i : H_i(X^\vee) \neq 0\} = \sup\{i : \mathrm{Ext}_R^i(M,N) \neq 0\}$$

$H_s(X^\vee)$ artinian

∴ $\mathrm{depth}\, H_s(X^\vee) = 0$. So, $\mathrm{depth}_R(X^\vee) = -s$.

∴ LHS of (1) is $-\sup \mathrm{Ext}_R^*(M,N)$.

Do similar for RHS. ☒

Now, specialising to $M = R$:

$$0 = \mathrm{injdim}_R N - \mathrm{depth}\, R.$$

$$\Rightarrow \boxed{\operatorname{injdim}_R N = \operatorname{depth} R.}$$ (If $\operatorname{injdim}_R N < \infty$.)

<u>In fact:</u> If $(R, \mathfrak{m}, k)$ has a f.g. module $N \neq 0$ of finite injective dim, then R is C.M.

So, $\operatorname{inj\,dim} N = \dim R.$

Lecture 18 (22-03-2023)

Wednesday, March 22, 2023 1:23 PM

Dualising Complexes

$R \to$ comm. noetherian

Def^n A complex $\omega_R^\cdot$ is a dualising complex if

① $\omega_R \in D^b(\text{mod } R)$, homology is nonzero in finitely many places and is f.g.

Not local

② $\text{injdim } \omega_R < \infty$, i.e., $\omega_R \simeq$ bdd complex of injectives,

③ $R \xrightarrow{\simeq} R\mathrm{Hom}_R(\omega_R, \omega_R) =: R\,\mathrm{End}_R(\omega_R)$.

local

quasi iso

[ω_R itself could be an unbounded complex.]

① ω_R dualising $\Rightarrow \Sigma^n \omega_R$ dualising $\forall n$.

(starting with any one ω_R, doing these processes get all dual.--)

$\Downarrow$

$\omega_R \otimes_R L$ dualising

(L a f.g. proj R-module of rank 1.)

(L_p is free R_p module of rank 1 for all associated p.)

② ω_R dualising $\Rightarrow U^{-1}\omega_R$ dualising over $U^{-1}R$ for any multiplicative closed $U \subseteq R$.

Theorem. (Bass, Murthy ~60s)
$M \in D^b(\text{mod } R)$. $\quad M_{\mathfrak p}$ perfect $/R_{\mathfrak p}$ $\forall \mathfrak p$

$\Longrightarrow M$

(perfect = in $D^b(\text{mod } R)$ and $\operatorname{pdim}_R < \infty$.)

Not so for inj $\dim_R$.

Def^n. Semidualising $\rightarrow$ if ① and ③ are satisfied.

Dualising complexes may not exist.

R is always a semidualising complex.

Local Duality

R, ω_R as above.

(Assume ω_R exists.)

$$(-)^{\dagger} := \operatorname{RHom}_R(-, \omega_R) : D(\text{Mod } R) \to D(\text{Mod } R)$$

restricts to equivalences (contrapositive)

$$\begin{tikzcd}
D^{fl}(\text{mod } R) \arrow[r, "\simeq"'] \arrow[d, phantom, "\cap"] & D^{fl}(\text{mod } R) \arrow[d, phantom, "\cap"] \\
D^b(\text{mod } R) \arrow[r, "(-)^{\dagger}", "\simeq"'] & D^b(\text{mod } R) \\
\operatorname{Perf}(R) \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & \operatorname{Finj}(R) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\operatorname{length}_R M < \infty$

P... ite

$$\mathrm{Perf}(R) \xrightarrow{\ \sim\ } \mathrm{Fep}(R)$$

finite proj dim $\to \mathrm{Perf}(R)$ $\qquad$ finite injective dim $\to \mathrm{Fep}(R)$

<u>Proof.</u> $M \in D^b(\mathrm{mod}\ R)$, then $M^\dagger \in D^b(\mathrm{mod}\ R)$.

$$M^\dagger = R\mathrm{Hom}_R(M, \omega_R)$$

Because: $M, \omega_R \in D^b(\mathrm{mod}\ R)$

$\Rightarrow$ each $\mathrm{Ext}^i_R(M, \omega_R)$ f.g. $\forall i$.

Moreover, inj dim $\omega_R < \infty \Rightarrow \mathrm{Ext}^i$ vanishes for large $|i|$.

Now, $\dagger\dagger$ factors as:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\simeq"'] & R\mathrm{Hom}_R(R\mathrm{Hom}_R(M, \omega_R), \omega_R) \\
M \otimes^{\mathbb{L}}_R R\mathrm{Hom}_R(\omega_R, \omega_R) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

$\simeq$ induced by $R \xrightarrow{\sim} R\,\mathrm{End}_R(\omega_R)$

$\because$ $M \in D^b(\mathrm{mod}\ R)$, $\mathrm{injdim}_R\, \omega_R < \infty$.

$\therefore$ top is $\simeq$

Similarly now check $\mathrm{Perf} \to \mathrm{Fep} \to \mathrm{Perf}$.

<u>Local rings</u> $(R, \mathfrak{m}, k)$.

$\omega_R \to$ dualising complex.

$$D^b(\mathrm{mod}\ R) \xrightarrow[\simeq]{(\ )^\dagger} D^b(\mathrm{mod}\ R)$$

$$k \xrightarrow{\simeq} k^{\dagger\dagger} \quad \text{(quasi iso)}$$

$$R\,\mathrm{Hom}_R(k,\omega_R) \simeq \text{graded } k\text{-vector space with } 0 \text{ diff}$$

$\omega_R \xrightarrow{\sim} I$ min'l

$$\mathrm{Hom}_R(k, I) = \mathrm{soc}(I)$$

$$\Rightarrow \quad R\,\mathrm{Hom}_R(k,\omega_R) \simeq \mathrm{Ext}_R^*(k,\omega_R)$$

$$k \xrightarrow{\sim} R\mathrm{Hom}_R\left(\mathrm{Ext}_R^*(k,\omega_R), \omega_R\right)$$

$$\cong \quad \mathrm{Hom}_k\left(\mathrm{Ext}_R^*(k,\omega_R), k\right) \otimes_R \mathrm{Ext}_R^*(k,\omega_R)$$

↳ has to be rank 1, i.e.,

$$\Rightarrow \mathrm{Ext}_R^*(k,\omega_R) \simeq \Sigma^a k.$$

<u>Normalising convention</u>: $\mathrm{Ext}_R^*(k,\omega_R) = k$ $(a=0)$, i.e.,

$$\mathrm{Ext}_R^i(k,\omega_R) = \begin{cases} 0 & i \neq 0, \\ 1 & i = 0. \end{cases}$$

$\omega_R \simeq I^{\cdot}$ min inj. resolution. Thus,

$$0 \to I^{-\dim R} \; --- \longrightarrow I^{-2} \longrightarrow I^{-1} \to I^0 = E_R(k) \longrightarrow 0$$

$$I^{-2} = \bigoplus_{\dim(R/p)=2} E(R/p), \qquad I^{-1} = \bigoplus_{\dim(R/p)=1} E(R/p)$$

← each p shows up!! Since localisation @ p gives dualizing for R_p (not normalized) and still minimal...

Every prime shows up exactly once, at location prescribed by its "co-height".

<u>Observe</u>. If

$$\begin{tikzcd}[column sep=small]
 & q \arrow[dl, dash] \arrow[dr, dash] & \\
q_1 \arrow[d, dotted, dash] & & q_1' \arrow[d, dotted, dash] \\
q_\ell \arrow[dr, dash] & & q_{\ell'}' \arrow[dl, dash] \\
 & p &
\end{tikzcd}$$

max'l chains, then $\ell = \ell'$.

Thus, if ω_R exists, then R is catenary!

Connection to Matlis Duality.

$(R, \mathfrak{m}, k)$ local.

ω_R normalised.

$\simeq I^{\cdot}$ min inj. resol^n

$M \in \operatorname{mod} R$, $\operatorname{length}_R M < \infty$.

$$M^{\dagger} = R\operatorname{Hom}_R(M, \omega_R) = \operatorname{Hom}_R(M, I^{\cdot}) = \operatorname{Hom}(M, E).$$

$\operatorname{Hom}_R(M, E(R/\mathfrak{p})) = 0$

$\forall\, \mathfrak{p} \neq \mathfrak{m}$

since the Hom is $\mathfrak{m}$-torsion and $\mathfrak{p}$-local

Theorem. $(R, \mathfrak{m}, k)$ ω_R normalised.

$M \in D^b(\operatorname{mod} R)$.

① $\inf H(M^{\dagger}) = \operatorname{depth}_R M$, ① $\inf H_*(M^\dagger) = \operatorname{depth}_R M$,

② $\sup H_*(M^\dagger) = \sup\{\dim(R/\mathfrak{p}) - \inf H_*(M)_\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} R\}$.

$$= \sup\{\dim_R H_i(M) - i : i \geq 0\}.$$

Definition of $\dim_R(M)$ for a complex M.

Thus,

$$\operatorname{amp} H_*(M^\dagger) := \sup H_*(M^\dagger) - \inf H_*(M^\dagger)$$
$$= \dim_R M - \operatorname{depth}_R M$$
$$=: \operatorname{cmd}_R(M).$$

Proof. ① $\operatorname{injdim} \omega_R = 0$ ($\because$ normalised)

By Ischebeck, $\sup \operatorname{Ext}_R^*(M, \omega_R) = \underbrace{\operatorname{injdim}}_{0} \omega_R - \operatorname{depth} M$

$$\Rightarrow \operatorname{depth} M = -\sup H^*(M^\dagger)$$
$$= \inf H_*(M^\dagger).$$

② Recall for any $X \in D^b(\operatorname{mod} R)$

$$\sup H_*(X) \underset{s}{=} \sup\{-\operatorname{depth} X_\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} R\}.$$

Note: $\operatorname{depth}(M^\dagger) = \inf H_*(M^{\dagger\dagger})$
$$= \inf H_*(M).$$

Also, $(\omega_R)_{\mathfrak{p}} = \underset{\substack{\uparrow \\ \text{normalising}}}{\sum^{\dim(R/\mathfrak{p})}} \omega_{R_{\mathfrak{p}}}$

$$\operatorname{depth}_{R_{\mathfrak{p}}} R\operatorname{Hom}_R(M, \omega_R)_{\mathfrak{p}} = \operatorname{depth}_{R_{\mathfrak{p}}} \left(R\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\omega_R)_{\mathfrak{p}})\right)$$
$$= \operatorname{depth}_{R_{\mathfrak{p}}} R\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, \Sigma^{\dim R/\mathfrak{p}} \omega_{R_{\mathfrak{p}}})$$
$$= -\dim(R/\mathfrak{p}) + \operatorname{depth}_{R_{\mathfrak{p}}} R\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}})$$
$$= -\dim(R/\mathfrak{p}) + \inf H_*(M_{\mathfrak{p}}). \quad \blacksquare$$

$$0 \to I_d \to \cdots \longrightarrow I_0 \to 0$$

$$I_n = \bigoplus_{\mathfrak{p}:\, \dim(R/\mathfrak{p}) = n} E(R/\mathfrak{p}).$$

· M an R-module.

$$0 \to \operatorname{Hom}_R(M, I_d) \to \cdots \to \operatorname{Hom}_R(M, \underset{\substack{\| \\ I_0}}{E}) \to 0$$

$$\operatorname{Hom}_R(M, E(R/\mathfrak{p})) = 0 \Leftarrow \dim R/\mathfrak{p} > \dim M.$$
$$\Downarrow$$
$$\operatorname{ann}(M) \not\subseteq \mathfrak{p}.$$

This gives one direction of $=$ in result above....

Lecture 19 (27-03-2023)

Monday, March 27, 2023 1:27 PM

$(R, \mathfrak{m}, k)$ local ring

$\omega_R \longrightarrow$ dualising complex (normalised)

$$\mathrm{Ext}_R^*(k, \omega_R) \cong k.$$

Equivalently, $\mu_R^i(\mathfrak{m}, \omega_R) = \begin{cases} 0 & ; \ i \neq 0, \\ 1 & ; \ i = 0. \end{cases}$

Then, $\forall M \in D^b(\mathrm{mod}\ R)$,

$$\mathrm{depth}_R(M) = \inf H_*(M^\dagger),$$

$$\dim_R(M) = \sup H_*(M^\dagger).$$

$(-)^\dagger = \mathrm{RHom}_R(-, \omega_R) : D^b(\mathrm{mod}\ R)^{op} \xrightarrow{\sim} D^b(\mathrm{mod}\ R).$

Ex. Let $X \in D(R)$.

If $\mathrm{RHom}_R(-, X)$ induces an autoequivalence $D^b(\mathrm{mod}\ R)^{op} \xrightarrow{\sim} D^b(\mathrm{mod}\ R)$,

then $X \simeq \Sigma^n \omega_R$ for some n.

Note: $\omega_R^\dagger = R \Rightarrow \omega_R$ is a ~~CM complex~~.

Elaboration of Local Duality:

$M \in \mathrm{mod}(R)$ that is CM, $\dim(M) =: d$.

Then, $\inf H_*(M^\dagger) = d = \sup H_*(M^\dagger)$.

$$\Rightarrow \quad M^+ = \Sigma^d H_d(M^+).$$

Claim. $H_d(M^+)$ is also CM of dimension d.

Proof. $(H_d(M^+))^+ \simeq (\Sigma^{-d} M^+)^+ \simeq \Sigma^d M^{++}$

$$\simeq \Sigma^d M.$$

$$\therefore \quad \Sigma^{-d}(\ )^+ : \left\{\begin{matrix}\text{CM modules of}\\ \text{dimension } d\end{matrix}\right\} \circlearrowleft \equiv$$

Poincare Series and Bass Series.

$M \in D^b(\text{mod } R)$.

$$P^R_M(t) := \sum_{n \in \mathbb{Z}} \operatorname{rank}_k \operatorname{Tor}^R_n(k, M)\, t^n.$$

Poincare Series

↪ Generating series of Betti numbers of M

$$I^M_R(t) := \sum_n \operatorname{rank}_k \operatorname{Ext}^n_R(k, M)\, t^n$$

$$= \sum_n \mu^n_R(\mathfrak{m}, M)\, t^n.$$

Bass series.

Obs.

$$\left.\begin{aligned} \mu^n_R(\mathfrak{m}, M^+) &= \operatorname{rank}_k \operatorname{Ext}^n_R(k, M^+) \\ &= \operatorname{rank}_k \operatorname{Ext}^n_R(M^+, k) \end{aligned}\right\} (*)$$

$$(*)\quad \mathrm{Ext}^n_R(k, M^+) \cong \mathrm{Hom}_{\mathcal{D}(R)}(k, \Sigma^n M^+)$$ local duality
$$\cong \mathrm{Hom}_{\mathcal{D}(R)}((\Sigma^n M^+)^+, k^+)$$
$$\cong \mathrm{Hom}_{\mathcal{D}(R)}(\Sigma^{-n} M^{++}, k)$$
$$\cong \mathrm{Hom}_{\mathcal{D}(R)}(\Sigma^{-n} M, k)$$

$$\mathrm{Tor}^R_n(k, M)^\vee \cong \mathrm{Ext}^n_R(M, k)$$

$$\therefore \quad \mu^n_R(\mathfrak{m}, M^+) = \beta^R_n(M) \quad \text{and}$$
$$\mu^n_R(\mathfrak{m}, M) = \beta^R_n(M^+).$$

Theorem. $P^R_M(t) = I^{M^+}_R(t)$.

No gaps theorem. $M \in \mathrm{mod}(R)$. Then,

$$\mu^i_R(\mathfrak{m}, M) \neq 0 \quad \forall \ \mathrm{depth}_R M \leq i \leq \mathrm{injdim}_R M.$$

Proof (Roberts). May assume M indecomposable.

Say $\mu^i_R(\mathfrak{m}, M) = 0$ for some $i > \mathrm{depth}_R(M)$. (Choose i least.)

Then, $\beta^R_i(M^+) = 0$.

Consider $F \simeq M^+$ min free resol^n.

$$\to F_{i+1} \to \underset{\substack{\parallel \\ 0}}{F_i} \to F_{i-1} \to \cdots \to F_{\mathrm{depth}(M)} \to 0$$

i.e., $F = F_{\leq i-1} \oplus F_{\geq i+1}$.

i.e., $F = F_{\leq i-1} \oplus F_{\geq i+1}$.

Thus, $M \simeq (M^\dagger)^\dagger \simeq F^\dagger$

$\simeq (F_{\leq i-1})^\dagger \oplus (F_{\geq i+1})^\dagger$.

Thus, $H_0(M) \simeq H_0(F_{\leq i-1}^\dagger) \oplus H_0(F_{\geq i+1}^\dagger)$ — (*)

and $0 = H_n(M) \simeq H_n(F_{\leq i-1}^\dagger) \oplus H_n(F_{\geq i+1}^\dagger)$.

$\therefore H_n(F_{\leq i-1}^\dagger) = H_n(F_{\geq i+1}^\dagger) = 0$ for all $n \neq 0$

(*) and M indec $\Rightarrow$ one of the two has H_0 zero as well.

Use degree argument: $H_0(F_{\geq i+1}^\dagger) = 0$.

$\therefore H_n(F_{\geq i+1}^\dagger) = 0$ for all n.

$\Rightarrow F_{\geq i+1}^\dagger = 0$.

$\Rightarrow$ projdim $M^\dagger < i$

$\Rightarrow$ injdim $M < i$. ∎

Existence of Dualising Complexes.

- R comm. noetherian.

ω_R dualising $\Rightarrow U^{-1}\omega_R$ dualising for $U^{-1}R$.

Lemma. If $R \to S$ finite map, then $\mathrm{Hom}_R(S, \omega_R)$ is a dualising complex for S.

Proof. Need to check three things:

① $\mathrm{Hom}_R(S, \omega_R) \in D^b(\mathrm{mod}\ S)$.

Pf. Viewed as R-module, $\mathrm{Hom}_R(S, \omega_R) \in D^b(\mathrm{mod}\ R)$.

$\Rightarrow$ —''— $\in D^b(\mathrm{mod}\ S)$. ∎

② $\mathrm{injdim}_S\ R\mathrm{Hom}_R(S, \omega_R) < \infty$.

Pf. $R\mathrm{Hom}_S(\_\_, R\mathrm{Hom}_R(S, \omega_R)) \simeq R\mathrm{Hom}_R(\_\_, \omega_R)$. ∎

↳ vanishes beyond $\mathrm{injdim}_R\ \omega_R$.

③
$$\begin{tikzcd} S \arrow[r, "\simeq"'] \arrow[dr, bend right, "(-)^{\dagger\dagger}"'] & R\mathrm{Hom}_S(R\mathrm{Hom}_R(S, \omega_R), R\mathrm{Hom}_R(S, \omega_R)) \arrow[d, "\wr\ \text{adjunction}"] \\ & R\mathrm{Hom}_R(R\mathrm{Hom}_R(S, \omega_R), \omega_R) \end{tikzcd}$$

The diagram above commutes. ∎ ∎

Uniqueness: ω_R and ω_R' dualising (for R).

Consider

$$\omega_R \otimes^{L}_{R} R\mathrm{Hom}_R(\omega_R, \omega_R') \longrightarrow \omega_R'.$$

evaluation

Claim. The above is a q. iso.

Proof. $\omega_R \otimes^{L}_{R} R\mathrm{Hom}_R(\omega_R, \omega_R') \to R\mathrm{Hom}_R(R, \omega_R') = \omega_R'$

$\because$ injdim $\omega_R' < \infty$ $\simeq$

$\simeq$ since ω_R dual

$\therefore$ inj dim $\omega_R' < \infty$ $\simeq$
$\omega_R \in D^b(\text{mod } R)$)

$\mathrm{RHom}_R(\mathrm{RHom}_R(\omega_R, \omega_R), \omega_R')$. $\simeq$ since ω_R dual

$\blacksquare$

Lemma. $\mathrm{RHom}_R(\omega_R, \omega_{R'}) \simeq \Sigma^n P$ with P a rank one projective module.

Proof. $\mathrm{RHom}_R(\omega_R, \omega_{R'}) \in D^b(\text{mod } R)$.

Enough to check after localisation that

$$\mathrm{RHom}_R(\omega_R, \omega_R') \simeq \Sigma^n R \quad \text{when } R \text{ is local.}$$

May assume ω_R and ω_R' are normalised.

$$\beta_i^R(\mathrm{RHom}_R(\omega_R, \omega_R')) = \mu_R^i(m, \omega_R)$$

$\because \omega_R'$ normalised dualising

$$= \begin{cases} 1 \; ; \; i = 0, \\ 0 \; ; \; \text{else}. \end{cases}$$

$\blacksquare$

Gorenstein

"R is Gorenstein if inj dim $R < \infty$." (Only for dim $R < \infty$.)

($\equiv$ R is dualising.)

Example R regular with $\dim R < \infty$

Lemma. $x \in R$ not a zero divisor

R Gorenstein $\Rightarrow R/xR$ Gorenstein.

If $x \in \mathrm{Jac}(R)$, then $\Leftarrow$ holds too.

Proof $R \twoheadrightarrow R/xR =: S$. finite map.

R dualising $\Rightarrow$ $\mathrm{RHom}_R(R, S)$ dualising for S

$$\begin{array}{c} \mathrm{RHom}_R(R,S) \\ \wr| \\ \Sigma^{-1} S \end{array}$$

$0 \to R \xrightarrow{x} R \to 0$ free resol^n

$\simeq S$

$\therefore \mathrm{RHom}_R(S,R) \simeq \mathrm{Hom}_R(0 \to R \xrightarrow{x} R \to 0, R)$

in $\mathcal{D}(R)$

$\simeq \underset{\deg 0}{0 \to R} \xrightarrow{x} \underset{-1}{R} \to 0$

$\simeq \Sigma^{-1} S.$

$\therefore \mathrm{RHom}_R(S,R) \simeq \Sigma^{-1} S$ in $\mathcal{D}(R)$.

Also as S-complexes.

Converse is exercise. $\blacksquare$

Thus, R regular $\Rightarrow$ c.i. $\Rightarrow$ Gorenstein.

Note: (R, m, k) local. Then,

R Gorenstein $\Leftrightarrow \hat{R}$ Gorenstein.

Theorem (R, m, k) local. Then,

Theorem. $(R, \mathfrak{m}, k)$ local. Then,

R Gorenstein $\iff$ R CM and $\operatorname{type}(R) = 1$.

Pf. $(\Rightarrow)$ R dualising.

$\therefore$ $\mathrm{RHom}_R(R, R) \simeq R$.

$\Rightarrow R$ CM.

Moreover, $\operatorname{type}(R) \overset{\text{depth} = \dim}{\underset{\text{defn}}{=}} \mu_R^d(\mathfrak{m}, R) = 1$ $\because$ R dualising.

(Thus, $\Sigma^d R$ is the normalised dualising cx for R.)

$(\Leftarrow)$ Suppose R is CM and type 1.

Passing to $\hat{R}$, we can assume R has a dualising complex (normalised).

R C.M. $\Rightarrow$ $\operatorname{amp} H_*(\omega_R) = 0$.

ETP: $\omega_R \simeq R$.

$$\beta_i^R(\omega_R) = \mu_R^i(\mathfrak{m}, R)$$

$$\beta_d^R(\omega_R) = \mu_R^d(\mathfrak{m}, R) = 1. \quad (d = \dim R)$$

$(d = \dim R)$

$F \xrightarrow{\sim} \omega_R$ min free resolution.

Since $H_i(\omega_R) = 0$ for $i < d$:

$$\cdots \to F_{d-1} \to \underset{= R}{F_d} \to \cdots$$

$$\Rightarrow H_d(\omega_R) \cong R/I.$$

Also, $\operatorname{ann}_R H_d(\omega_R) = 0$.

$$R \xrightarrow{\sim} \operatorname{RHom}_R(\omega_R, \omega_R).$$

$$\Rightarrow H_d(\omega_R) \cong R. \qquad \therefore\ R \simeq \Sigma^{-d} \omega_R.$$

■

Theorem (Roberts) $\operatorname{type}(R) = 1 \Rightarrow R$ Gorenstein. ■

Corollary. $(R, \mathfrak{m}, k)$ artinian.

R is Gorenstein $\iff \operatorname{type}(R) = 1$

$\iff \operatorname{Soc}(R) \simeq k$. ■

Lecture 20 (29-03-2023)

Wednesday, March 29, 2023 1:25 PM

Ex. Look up sharp equivalence and deduce from local duality.

Theorem If R is artinian, TFAE:

① R Gorenstein. (I.e., $\operatorname{injdim}_R R < \infty$.)

② $R \simeq E_R(k)$. ②' $E_R(k)$ free R-module.

③ $\operatorname{rank}_k \operatorname{soc}(R) = 1$. (I.e., $\operatorname{soc}(R) \cong k$.)

Example Say R a k-alg. of finite rank.

$$k \hookrightarrow R \twoheadrightarrow k.$$

$$E_R(k) \simeq \operatorname{Hom}_k(R, k)$$

- injective
- socle $(\operatorname{Hom}_k(R,k)) \simeq k$.

$\left(\begin{array}{c}\operatorname{Hom}_R(-, \operatorname{Hom}_k(R,k)) \\ \cong \operatorname{Hom}_k(-, k).\end{array}\right)$

Theorem tells us R Gorenstein $\Rightarrow \operatorname{Hom}_k(R, k) \cong R$ as R-modules.

What is a basis of ? (Treat $k \subset R$ splitting.)

Say $A = k \oplus A_1 \oplus \cdots \oplus A_g$.

- k field, $\operatorname{rank}_k(A) < \infty$
- A is k-alg; either commutative or graded commutative

$a \cdot b = (-1)^{|a||b|} ba$ for homog. a, b

$$0 \le i \le g : \quad A_i \times A_{g-i} \longrightarrow A_g$$ ⇝ "symmetric", bilinear pairing

This gives $A_i \to \operatorname{Hom}_k(A_{g-i}, A_g)$.

Prop^n A is Gorenstein $\Leftrightarrow$ is a bijection for $\forall\ 1 \le i \le g-1$.

$\Leftrightarrow \operatorname{Hom}_k(A, k)[\pm g] \cong A$ as A-modules.

$\Rightarrow$ $\mathrm{Hom}_k(A, k)[\pm g] \cong A$ as A-modules.

$\Leftrightarrow$ $A_g \cong k$ and the pairings are nondegenerate.

Local Cohomology

R commutative noetherian.

$I \subsetneq R$ ideal.

M an R-module (not necessarily finite).

$$\Gamma_I(M) := \{ m \in M : I^n \cdot x = 0 \text{ for some } n \geq 0 \}.$$

$\hookrightarrow$ I-power torsion submodule of M

$$\cong \bigcup_{n \geq 0} \mathrm{Hom}_R(R/I^n, M)$$

$$= \operatorname*{colim}_n \, (0 :_M I^n)$$

$$= \bigcup_{j \geq 0} \mathrm{Hom}_R(R/I_j, M), \quad \text{where}$$

$\cdots \subseteq I_{j+1} \subseteq I_j \subseteq \cdots$

is cofinal with $\langle I^n \rangle_n$.

Example. $I = (x_1, \ldots, x_c)$.

$$\{(x_1^s, \ldots, x_c^s)\}_{s \geq 0}.$$

char p: $\{(x_1^{p^e}, \ldots, x_c^{p^e})\}_{e \geq 0}$.

char p: $\{(x_1^{p^e}, \dots, x_c^{p^e})\}_{e \geq 0}$.

- $\Gamma_I(M) = \ker\left(M \to \prod_{\substack{\mathfrak{p} \notin V(I)}} M_{\mathfrak{p}}\right)$

i.e., $I \not\subseteq \mathfrak{p}$

- Γ_I is left-exact on Mod-R.

(Using left-exactness of Hom and exactness of direct colim.)

$0 \to M' \to M \to M'' \to 0$ induces

$$0 \to \Gamma_I(M') \to \Gamma_I(M) \to \Gamma_I(M'') \to \cdots.$$

$R\Gamma_I(M) = \Gamma_I(iM)$ where $M \to iM$ is a semi-injective res^n.

One gets

$$R\Gamma_I : D(R) \to D(R).$$

$$H_I^n(M) := H^n(R\Gamma_I(M)).$$

Computing $R\Gamma_I$:

① $\Gamma_I(-) = \bigcup_n \mathrm{Hom}(R/I^n, -)$

$$\Rightarrow \boxed{H_I^i(M) = \operatorname{colim}_n \mathrm{Ext}_R^i(R/I^n, M).}$$

Corollary. $H_I^i(M) = 0$ for $i < \operatorname{depth}_R(I, M)$.

Moreover, $H^i_I(M) \neq 0$ for $i = \operatorname{depth}_R(I, M)$.

Prove using Auslander type arguments...

Corollary. $H_I^{\,i}(E) = 0$ $\quad i \geq 1$, E injective.

In fact,

$$H^i_I\left(E(R/\mathfrak{p})\right) = \begin{cases} E(R/\mathfrak{p}) & ; \; i = 0, \; \mathfrak{p} \supseteq I \\ 0 & ; \; i = 0, \; \mathfrak{p} \not\supseteq I \\ 0 & ; \; i \geq 1 \end{cases}$$

Say $I = (x_1, \dots, x_c)$.

Define $I_s := (x_1^s, \dots, x_c^s)$.

$\{I_s\}_{s \geq 0}$ cofinal to $\{I^n\}_{n \geq 0}$.

we have $K(\underline{x}^s) = K(x_1^s, \dots, x_c^s)$

$$\downarrow$$

$$R/I_s .$$

This gives

$$\begin{tikzcd} \operatorname{Hom}_R(R/I_s, M) \arrow[r] \arrow[d] & \operatorname{Hom}_R(K(\underline{x}^s), M) \arrow[d] \\ \operatorname{Hom}_R(R/I_{s+1}, M) \arrow[r] & \operatorname{Hom}_R(K(\underline{x}^{s+1}), M) \end{tikzcd}$$

$$\rightsquigarrow$$

$$\operatorname*{colim}_s \operatorname{Hom}_R(R/I_{s+1}, M) \to \operatorname*{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M).$$

$$\operatorname{colim}_s \operatorname{Hom}_R(R/I_{s+1}, M) \to \operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M).$$

$M \to J$ inj. resol^n.

$$\begin{array}{ccc} R\Gamma_I(M) & \longrightarrow & \operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), J) \\ & & \uparrow \simeq \\ & & \operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M). \end{array}$$

We get natural maps

$$H^i_I(M) \longrightarrow \operatorname{colim}_s H^i(\underline{x}^s; M).$$

$$\overset{\shortparallel}{H^i(\operatorname{Hom}_R(K(\underline{x}^s), M))}$$

Theorem. The above is an iso for all i and M.

Idea. For any module M, it is clear that the above is an iso for $i = 0$.

ETP: They both vanish on injectives. (δ-functor...)
We know it for H^i_I.
Just need to check for RHS.
Can prove for injective hulls! Check. ∎

$$\operatorname{Hom}_R(K(\underline{x}^s), M) \simeq \operatorname{Hom}_R(K(\underline{x}^s), R) \otimes_R M$$

$$\operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M) \simeq \left(\operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), R)\right) \otimes_R M.$$

$$K^\infty(\underline{x}) := \operatorname*{colim}_s \mathrm{Hom}_R(K(\underline{x}^s), R).$$

"stable Koszul complex"

$$K(x_1^s, \dots, x_c^s) = \bigotimes_{i=1}^{c} K(x_i^s).$$

$$\mathrm{Hom}_R(K(\underline{x}^s), R) = \mathrm{Hom}_R\Big(\bigotimes_{i=1}^{c} K(x_i^s), R\Big)$$

$$= \mathrm{Hom}_R\Big(\bigotimes_{i=1}^{c-1} K(x_i^s), \mathrm{Hom}_R(K(x_c^s), R)\Big)$$

$$= \mathrm{Hom}_R\Big(\bigotimes_{i=1}^{c-1} K(x_i^s), R\Big) \otimes_R \mathrm{Hom}_R(K(x_c^s), R).$$

$c=1$:

$$\begin{tikzcd} K(x^{s+1}) \arrow[d] \\ K(x^s) \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^{s+1}"] \arrow[d, "\cdot x"'] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^s"] & R \arrow[r] & 0 \end{tikzcd}$$

Dualise:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^s"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^{s+1}"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^{s+1}"] \arrow[d, equal] & R \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \end{tikzcd}$$

colim

$$0 \to R \longrightarrow R[\tfrac{1}{x}] \to 0$$

natural

localisation

natural map

localisation

$$\operatorname*{colim}_s \mathrm{Hom}_R(K(\underline{x}^s), R) \cong \bigotimes_{i=1}^{c} \left(0 \to \underset{0}{R} \to \underset{1}{R[\tfrac{1}{x}]} \to 0\right)$$

deg

This gives:

$$\overset{\infty}{K}(\underline{x}) : 0 \to R \to \underbrace{\bigoplus_{i=1}^{c} R\left[\frac{1}{x_i}\right] \to \cdots \to R\left[\frac{1}{x_1 \cdots x_c}\right]}_{} \to 0.$$

Čech complex computing sheaf cohomology

of $\operatorname{Spec} R \setminus V(\underline{x}) = \bigcup_{i=1}^{c} D(x_i)$.

call this cover $\mathcal{U}$

$= \check{C}(X, \mathcal{U})$

$$0 \to \check{C}(X, \mathcal{U}) \hookrightarrow K^\infty(\underline{x}) \to R \to 0.$$

Corollary. $H_I^i(M) = 0$ for $i \geq c+1$.

M module.

Note $H_I^i(M) = H_J^i(M)$ if $\sqrt{I} = \sqrt{J}$.

$\therefore H_I^i(M) = 0$ for $i > \mathrm{ara}(I)$

$$\parallel$$

$$\inf \left\{ c : \begin{array}{l} \exists x_1, \dots, x_c \\ \text{s.t. } \sqrt{(x_1, \dots, x_c)} = \sqrt{I} \end{array} \right\}.$$

Lecture 21 (03-04-2023)

Monday, April 3, 2023 1:20 PM

R comm. noe. ring

$\mathfrak{a} \subseteq R$ ideal.

$\Gamma_{\mathfrak{a}}(-) : \text{Mod-}R \to \text{Mod-}R$

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \geq 0} \mathrm{Hom}_R(R/\mathfrak{a}^n, M)$$

$$= \bigcup_{n \geq 0} \mathrm{Hom}_R(R/\mathfrak{a}_n, M).$$

where $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ is cofinal to $(\mathfrak{a}^n)_{n \geq 0}$.

Induces: $R\Gamma_{\mathfrak{a}} : D(R) \to D(R)$,

$R\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(iM)$

injective resol^n

$H^i_{\mathfrak{a}}(M) = H^i(R\Gamma_{\mathfrak{a}}(M))$

i^{th} local cohomology of M supported on $\mathfrak{a}$

(rather $V(\mathfrak{a})$)

$(\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{b}} \Leftarrow \sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}})$

$-\ R\Gamma_{\mathfrak{a}}(M) = \underset{n}{\mathrm{hocolim}}\ R\mathrm{Hom}_R(R/\mathfrak{a}^n, M)$

homotopy colimit

(usual colim DNE in derived...)

$\frac{R}{\mathfrak{a}^{n+1}} \twoheadrightarrow \frac{R}{\mathfrak{a}^n}$ induces

$R\mathrm{Hom}\left(\frac{R}{\mathfrak{a}^n}, M\right) \to R\mathrm{Hom}\left(\frac{R}{\mathfrak{a}^n}, M\right)$

- $\mathfrak{a} = (x_1, \dots, x_c)$

  $\mathfrak{a}_s = (x_1^s, \dots, x_c^s)$

  $K(\mathfrak{a}_s) := \text{Koszul c'x on } x_1^s, \dots, x_c^s.$

$$\begin{tikzcd} K(\mathfrak{a}_s) \arrow[r] & R/\mathfrak{a}_s \\ K(\mathfrak{a}_{s+1}) \arrow[r] \arrow[u] & R/\mathfrak{a}_{s+1} \arrow[u, two heads] \end{tikzcd}$$

$\exists$ canonical choice for this map

(e.g. $c=1$

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^s"] & R \arrow[r] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^{s+1}"'] \arrow[u, "x"] & R \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

Dualising:

$$\begin{tikzcd} R\mathrm{Hom}_R(R/\mathfrak{a}_s, -) \arrow[r] \arrow[d] & \mathrm{Hom}(K(\mathfrak{a}_s), -) \arrow[d] \\ R\mathrm{Hom}_R(R/\mathfrak{a}_{s+1}, -) \arrow[r] & \mathrm{Hom}_R(K(\mathfrak{a}_{s+1}), -) \end{tikzcd}$$

Induces:

$$\underset{s}{\mathrm{hocolim}}\ R\mathrm{Hom}_R(R/\mathfrak{a}_s, -) \to \underset{s}{\mathrm{hocolim}}\ \mathrm{Hom}(K(\mathfrak{a}_s), -).$$

$\quad \cong$

$R\Gamma_{\mathfrak{a}}(-)$

Thm. The above is an isomorphism of functors.

In cohomology:

Koszul cohomology

$$\mathrm{Ext}^*_R(R/\mathfrak{a}_s, -) \to H^*(\mathfrak{a}_s; -).$$

Induces $\quad \mathrm{colim}\ \mathrm{Ext}^*_R(R/\mathfrak{a}_s, -) \to \mathrm{colim}\ H^*(\mathfrak{a}_s; -)$

Induces $\operatorname{colim}_s \operatorname{Ext}^*_R(R/\mathfrak{a}_s, -) \longrightarrow \operatorname{colim}_s H^*(\mathfrak{a}_s; -)$.

$$\begin{tikzcd}
\operatorname{colim}_s \operatorname{Ext}^*_R(R/\mathfrak{a}_s, -) \arrow[r] \arrow[d, phantom, "\wr\wr"] & \operatorname{colim}_s H^*(\mathfrak{a}_s; -) \arrow[d, equal] \\
H^*_{\mathfrak{a}}(-) \arrow[r, "\cong"'] & \operatorname{colim}_s H^*(\mathfrak{a}_s; -).
\end{tikzcd}$$

Content of the above theorem.

$\operatorname{Hom}_R(K(\mathfrak{a}_s), M) \cong \operatorname{Hom}_R(K(\mathfrak{a}_s), R) \otimes_R M.$

($\because K$ perfect)

$\rightsquigarrow \operatorname{colim}_s \operatorname{Hom}_R(K(\mathfrak{a}_s), M) \cong \left[\operatorname{colim}_s \operatorname{Hom}(K(\mathfrak{a}_s), R)\right] \otimes M.$

Explicit description

$K^\infty(\underline{x}) := 0 \to R \to \bigoplus_i R\left[\frac{1}{x_i}\right] \to \cdots \to R\left[\frac{1}{x_1 \cdots x_c}\right] \to 0.$

Stable Koszul complex / Extended Čech complex.

Explicit desc above follows from examining $c = 1$.

$(R, \mathfrak{m}, k)$ local. Will look at $\Gamma_{\mathfrak{m}}(-)$.

<u>Key properties</u>: $M \in D^b(\operatorname{mod} R)$.

① $H^i_{\mathfrak{m}}(M)$ are artinian $\forall i$. (Not true for general $\mathfrak{a}$.)

② $H^i_{\mathfrak{m}}(-) = 0$ on $\operatorname{Mod} R$ $\forall i \geq \operatorname{ara}(\mathfrak{m}) + 1 = \dim(R) + 1$

③ $\inf H^*_{\mathfrak{m}}(M) = \operatorname{depth}(M)$.

④ $\sup H^*_{\mathfrak{m}}(M) = \dim_R(M)$.

Will check using local duality.

<u>Exercise</u>: $\hat{R} = \mathfrak{m}$-adic completion.

$H^*_{\mathfrak{m}}(M) = H^*_{\mathfrak{m}}(\hat{R} \otimes M)$ $\Big\}$ Use Čech complex description.

$\forall M \in D^b(\mathrm{mod}\, R)$

$\therefore$ May assume R complete and has a dualising complex.

So, assume $(R, \mathfrak{m}, k)$ local and ω_R a normalised dualising complex.

<u>Local duality</u>.

$$M \xrightarrow{\simeq} R\mathrm{Hom}_R(M^\dagger, \omega_R)$$

$M^\dagger = R\mathrm{Hom}_R(M, \omega_R)$.

$$R\Gamma_{\mathfrak{m}}(M) \xrightarrow{\simeq} R\Gamma_{\mathfrak{m}}(R\mathrm{Hom}_R(M^\dagger, \omega_R))$$

$$\uparrow\simeq \quad \because M^\dagger \in D^b(\mathrm{mod}\, R)$$

$$R\mathrm{Hom}_R(M^\dagger, \underbrace{R\Gamma_{\mathfrak{m}}(\omega_R)}))$$

(Accept this for now.)

can compute this explicitly:

$$\omega_R \simeq 0 \to \bigoplus_{\dim(R/\mathfrak{p})=0} E(R/\mathfrak{p}) \to \cdots \to E(R/\mathfrak{m}) \to 0.$$

$$R\Gamma_{\mathfrak{m}}(\omega_R) \simeq 0 \to \Gamma_{\mathfrak{m}}\left(\bigoplus E(R/\mathfrak{p})\right) \to \cdots \to \Gamma_{\mathfrak{m}}(E(R/\mathfrak{m})) \to 0$$

Recall: $\Gamma_{\mathfrak{m}}(E(R/\mathfrak{p})) \subseteq E(R/\mathfrak{p})$ ...

Recall: $\Gamma_{\mathfrak{a}}(E(R/\mathfrak{p})) = \begin{cases} E(R/\mathfrak{p}) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{else} \end{cases}$

Thus, $R\Gamma_{\mathfrak{m}}(\omega_R) \simeq E(R/\mathfrak{m})$.

Up shot:

$$\boxed{R\Gamma_{\mathfrak{m}}(M) \simeq R\mathrm{Hom}_R(M^{\dagger}, E(R/\mathfrak{m}))}$$

Theorem $H^i_{\mathfrak{m}}(M) \cong H_i(M^{\dagger})^{\vee}$

$$\cong \left[\mathrm{Ext}_R^{-i}(M, \omega_R)\right]^{\vee}.$$

$(-)^{\vee}$: matlis duality

Corollary. ① $M \in D^b(\mathrm{mod}\ R) \Rightarrow M^{\dagger} \in D^b(\mathrm{mod}\ R)$

$\Longrightarrow \mathrm{Ext}_R^{-i}(M, \omega_R)$ is a f.g. R-mod $\forall i$

$\Longrightarrow \mathrm{Ext}_R^{-i}(M, \omega_R)^{\vee}$ artinian $\forall i$.

i.e., $H^i_{\mathfrak{m}}(M)$ artinian.

② $\inf H^*_{\mathfrak{m}}(M) = \inf H_*(M^{\dagger}) = \mathrm{depth}_R(M)$.

③ $\sup H^*_{\mathfrak{m}}(M) = \sup H_*(M^{\dagger}) = \dim_R(M)$.

Grothendieck's Nonvanishing Result

Justification of "accept this"

$$R\Gamma_{\mathfrak{a}}\, R\mathrm{Hom}_R(M,N) \longleftarrow R\mathrm{Hom}_R(M, R\Gamma_{\mathfrak{a}} N)$$

$R\Gamma_{\mathfrak{a}}(N) \longrightarrow N$

$$\begin{array}{ccc} R\Gamma_{\mathfrak{a}}(N) & \longrightarrow & N \\ \| & & \downarrow \simeq \\ \Gamma(iN) & \hookrightarrow & iN \end{array} \quad \leftarrow \text{inj res}^{n}$$

$$\begin{array}{ccc} R\mathrm{Hom}_R(M, R\Gamma_{\mathfrak{a}}(N)) & \longrightarrow & R\mathrm{Hom}_R(M,N) \\ & \searrow & \uparrow \\ & & R\Gamma_{\mathfrak{a}} R\mathrm{Hom}_R(M,N) \end{array}$$

$M \in D^b(\mathrm{mod}\, R)$

is $\simeq$ when $N \in D_-(\mathrm{Mod}\, R)$

i.e., $H_i(N) = 0$ $\forall i \gg 0$

Another perspective :

$$R\Gamma_{\mathfrak{a}} R\mathrm{Hom}_R(M,N) \simeq K^{\infty} \otimes_R R\mathrm{Hom}_R(M,N)$$

$\because K^{\infty}$ is a bdd c'x of flat mod

$$\simeq R\mathrm{Hom}_R(M, N \overset{L}{\otimes}_R K^{\infty})$$

$- M \in D^b(\mathrm{mod}\, R)$ & $H_i(N) = 0$ $\forall i \gg 0$

$$F \otimes_R \mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(M, F \otimes_R N)$$

F flat R-module
M f.g.

Recall : $H^i_{\mathfrak{m}}(M) = H_i(M^{\dagger})^{\vee}$.

Suppose R is Gorenstein.

Then, $\omega_R \simeq \Sigma^d R$, $d = \dim(R)$.

$$H^i_{\mathfrak{m}}(M) \cong \left(H_i\left(R\mathrm{Hom}_R(M, \Sigma^d R)\right)\right)^{\vee}$$

$$\cong \left(H^{-i}\left(R\mathrm{Hom}_R(M, \Sigma^d R)\right)\right)^{\vee}$$

$$\cong \left(H^{-i}\left(\mathrm{RHom}_R(M, \Sigma^d R)\right)\right)^\vee$$

$$\cong \left(H^{d-i}\left(\mathrm{RHom}_R(M,R)\right)\right)^\vee$$

$$\cong \mathrm{Ext}_R^{d-i}(M,R)^\vee$$

Corollary. R Gorenstein, $M \in D^b(\mathrm{mod}\, R)$:

$$H^i_{\mathfrak{m}}(M) = \mathrm{Ext}_R^{d-i}(M,R)^\vee.$$ ∎

When R is C.M. with canonical module ω, then

$\Sigma^d \omega$ normalised dualising complex,

$$H^i_{\mathfrak{m}}(M) \cong \mathrm{Ext}_R^{d-i}(M,\omega)^\vee.$$ ∎

Remark. $R \longrightarrow S$, $N \in D(S)$, $\mathfrak{a} \subseteq R$.

$H^*_{\mathfrak{a}}(N) \cong H^*_{\mathfrak{a}S}(N)$. (Immediate from Koszul description)

Note: $\mathrm{Ext}_R(R/\mathfrak{a}^n, N)$ & $\mathrm{Ext}_S(S/\mathfrak{a}^n S, N)$ can be different.

Another perspective on $R\Gamma_{\mathfrak{a}}(-)$:

$\Gamma_{\mathfrak{a}}(M) \hookrightarrow M$ (M module)

M is $\mathfrak{a}$ power torsion if $\Gamma_{\mathfrak{a}}(M) = M$.

$M \in D(R)$ is $\mathfrak{a}$-power torsion if $H_i(M)$ is $\mathfrak{a}$-power torsion $\forall i$

$\Gamma_{\mathfrak{a}} D(R) := \{ M \in D(R) \ : \ M \ \mathfrak{a}\text{-power torsion} \}$

→ triangulated sub category of $D(R)$

→ closed under coproducts

i.e. $\Gamma_{\mathfrak{a}} D(R)$ is a localising sub cat. of $D(R)$.

$$\Gamma_{\mathfrak{a}} D(R) \underset{R\Gamma_{\mathfrak{a}}}{\rightleftarrows} D(R)$$

$R\Gamma_{\mathfrak{a}}$ is right adjoint to the inclusion,

i.e., $R\Gamma_{\mathfrak{a}} M \longrightarrow M \qquad \forall \ M \in D(R)$

and $\mathrm{Hom}_{\mathcal{D}}(-, R\Gamma_{\mathfrak{a}} M) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(-, M)$

is bijective on $\mathfrak{a}$-torsion objects.

<u>FACT</u>: $\Gamma_{\mathfrak{a}} D(R) = \mathcal{L}_{\mathfrak{a}}(\kappa(\mathfrak{a}))$.

localising subcategory?

Dwyer & Greenlees: Complete modules & torsion modules

Lecture 22 (05-04-2023)

Wednesday, April 5, 2023 1:13 PM

Kähler diff. & derivations

- Majadas & Rodicio: Smoothness, regularity, and complete intersection
- Matsumura's two books
- André - Quillen homology of comm. algebra

$K \to$ comm. ring (possibly non noetherian)

$R \to$ K-algebra

M an R-module

Defn A *derivation* $d: R \to M$ is a K-linear map satisfying $d(xy) = (dx)y + x\,dy$.

Remark. $d(1^2) = 2d(1) \Rightarrow d(1) = 0$.

Also, $d(K) = 0$.

$\mathrm{Der}_K(R, M) \subseteq \mathrm{Hom}_K(R, M)$.

$\uparrow$ R-submodule

($d: R \to M$ der, $f: M \to N$ linear $\Rightarrow f \circ d$ der.)

$\mathrm{Der}_K(R, \_) : \text{Mod-}R \to \text{Mod-}R$ functor.

Theorem This functor is representable.

That is, $\exists$ R-module Ω and a derivation $\delta: R \to \Omega$ s.t.

$$\mathrm{Hom}_R(\Omega, M) \to \mathrm{Der}(R, M)$$
$$f \longmapsto f \circ \delta$$

$$f \longmapsto f \circ \delta$$

is bijective for all M. That is,

$$\mathrm{Hom}_R(\Omega, -) \xrightarrow{\cong} \mathrm{Der}_K(R, -).$$

· Moreover, (Ω, δ) is unique (up to iso).

We write $\Omega =: \Omega_{R/K}$, ← module of Kähler differentials

$\delta =: \delta_{R/K}$. ← universal derivation

<u>Proof</u> $R^e := R \otimes_K R$. ← enveloping algebra

we have a surjection
$$\begin{aligned} R^e &\xrightarrow{\mu} R \\ x \otimes y &\longmapsto xy. \end{aligned}$$

Put $J := \ker \mu$.

Then, $\Omega = J/J^2$ and
$$\begin{aligned} \delta : R &\to J/J^2 \\ 1 &\longmapsto x \otimes 1 - 1 \otimes x \end{aligned}$$
does the job.

<u>Check:</u> $J = \langle x \otimes 1 - 1 \otimes x \mid x \in R \rangle$

In fact, suffices to take x that generate R as a K-algebra.

<u>Ex</u>. $R = k[x_\lambda]_{\lambda \in \Lambda}$, then $J = (x_\lambda \otimes 1 - 1 \otimes x_\lambda : \lambda \in \Lambda)$.

$R \overset{i_1}{\underset{i_2}{\rightrightarrows}} R \otimes_K R$, $\quad i_1(r) = r \otimes 1$,
$\quad i_2(r) = 1 \otimes r$. ↗ R-algebra maps

$i_2 \qquad\qquad i_2(r) = 1 \otimes r.$

$$J = \sum_{x \in R} R(x \otimes 1 - 1 \otimes x).$$

These give J two R-mod structures. Both same.

$$\begin{tikzcd} R \arrow[r, "i_1 - i_2"'] \arrow[d, dashed] & J \arrow[r, hook] \arrow[d] & R^e \\ \delta & J/J^2 & \end{tikzcd}$$

<u>Check</u>: δ is a derivation.

$$J = \sum_{x \in R} R\,\delta x.$$

Now, if $d : R \to M$ is a derivation,

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, "d"'] & \Omega \arrow[dl, dashed, "f"] \\ M & \end{tikzcd}$$

$f(\delta x) := dx$ for $x \in R$. ☒

<u>Example</u>. $R = K[\{x_\lambda\}_{\lambda \in \Lambda}]$.

M.

Derivation $R \to M$.

$\updownarrow$

Assignments $\{x_\lambda\} \xrightarrow{d} M$ (map of sets only)

Given an assignment, extend by

$$p(\underline{x}) = \sum \left(\frac{\partial p}{\partial x_\lambda}\right) d x_\lambda$$

R linear maps $\bigoplus R\,dx_\lambda \longrightarrow M$.

$$\Omega_{R/K} = \bigoplus R\,dx_\lambda$$

$$\delta_{R/K}(x_\lambda) = dx_\lambda.$$

Jacobi - Zariski sequences: (JZ)

$\varphi: R \longrightarrow S$ map of K-algebras.

We get a derivation $R \longrightarrow \Omega_{S/K}$.

$$R \xrightarrow{\delta} S \xrightarrow{\delta_{S/K}} \Omega_{S/K}$$

By the universal property we get

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\delta_{R/K}"'] & S \arrow[d, "\delta_{S/K}"] \\
\Omega_{R/K} \arrow[r, "R\text{-linear}"] \arrow[d] & \Omega_{S/K} \\
S \otimes_R \Omega_{R/K} \arrow[ur, "d\varphi"'] &
\end{tikzcd}$$

extend scalars

$d\varphi$ is S-linear

$$s \otimes \delta_{R/K}\, x \longmapsto s\, \delta_{S/K}(\varphi(x)).$$

First JZ sequence.

$$\boxed{S \otimes_R \Omega_{R/K} \xrightarrow{d\varphi} \Omega_{S/K} \longrightarrow \Omega_{S/R} \longrightarrow 0}$$

First JZ sequence.

$$\boxed{S\otimes_R \Omega_{R/K} \xrightarrow{d\varphi} \Omega_{S/K} \to \Omega_{S/R} \to 0}$$

Prop^n. The above sequence is exact.

Proof. $N \in$ Mod-S. $\mathrm{Hom}_S(-,N)$ applied to above yields

$$0 \to \mathrm{Hom}_S(\Omega_{S/R}, N) \to \mathrm{Hom}_S(\Omega_{S/K}, N) \to \mathrm{Hom}_S(S\otimes_R \Omega_{R/K}, N).$$

(Suffices to check $\uparrow$ is exact $\forall$ N.)

$\mathrm{Hom}_S(S\otimes_R \Omega_{R/K}, N) \cong \mathrm{Hom}_R(\Omega_{R/K}, N)$ (adjunction)

becomes

$$0 \to \mathrm{Der}_R(S,N) \to \mathrm{Der}_K(S,N) \to \mathrm{Der}_K(R,N).$$

Easy to see above is exact ■

II^nd JZ sequence.

$$R \twoheadrightarrow S = R/I.$$

$$\mathrm{Der}_R(S,-) = 0.$$

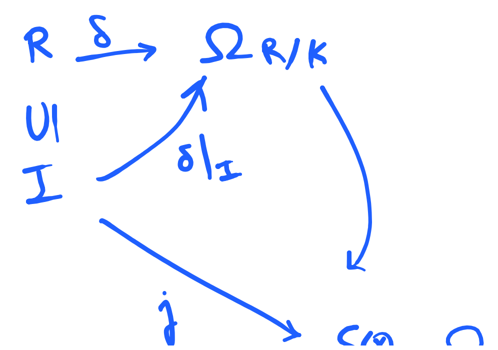
Thus, $\Omega_{S/R} = 0$.

$$S\otimes_R \Omega_{R/K} \to \Omega_{S/K} \to 0 \quad \text{exact.}$$

Moreover,

Prop^n $I/I^2 \xrightarrow{\bar{d}} S\otimes_R \Omega_{R/K} \to \Omega_{S/K} \to 0$ is exact.

(As a seq of S-modules.)



$$\xrightarrow{j} S \otimes_R \Omega_{R/K}$$

Check: j is R-linear.

This gives

$$\underset{I/I^2}{\overset{}{S \otimes_R I}} \xrightarrow[j]{} S \otimes_R \Omega_{R/K}$$

Proof of exactness. Apply $\mathrm{Hom}_S(-, N)$ again.

$$0 \to \mathrm{Der}_K(S, N) \to \mathrm{Der}_K(R, N) \to \mathrm{Hom}_S(I/I^2, N) = \mathrm{Hom}_S(S \otimes_R I, N) = \mathrm{Hom}_R(I, N).$$

restriction to I

Example. $S = \dfrac{K[x_1, \ldots, x_n]}{(f_1, \ldots, f_c)} \hookleftarrow I$.

$$0 \to \underset{(f)}{I} \hookrightarrow \overset{R}{K[x]} \twoheadrightarrow S.$$

$$I/I^2 = \sum R[f_i]$$

$$I/I^2 \xrightarrow{j} \bigoplus_i S\,dx_i \to \Omega_{S/K} \to 0.$$

$$j(f_h) = \sum_{i=1}^{c} \frac{\partial f_h}{\partial x_i}\, dx_i .$$

$$S^c \xrightarrow[\left(\frac{\partial f_h}{\partial x_i}\right)]{} S \to \Omega_{S/K} \to 0.$$

$\rightsquigarrow$ Jacobian matrix.

$\left(\frac{\partial}{\partial x_i}\right) \rightsquigarrow$ Jacobian matrix.

This gives $\Omega_{S/K}$ for any algebra S.
(Didn't need any finiteness.)

Ex. $R := \dfrac{K[x,y]}{(y^2 - x^3)}$.

$$R \xrightarrow{\begin{pmatrix} -3x^2 \\ 2y \end{pmatrix}} R^2 \longrightarrow \Omega_{R/K} \longrightarrow 0$$

happens to be 1-1.

$R \rightsquigarrow$ K-algebra. $I \to$ R-module.

An (comm algebra) extension of R by I is an exact sequence of K-modules:

$$0 \to I \xrightarrow{i} E \xrightarrow[p]{} R \to 0, \quad \text{where}$$

- p is a map of K-algebras, and
- $e \cdot i(x) = p(e) \cdot x$ for $x \in I$, $e \in E$.

In particular, $i(x) \cdot i(y) = 0$.

Equivalently: $I \subseteq E$ ideal s.t. $I^2 = 0$ and induced R-algebra structure on $I = I/I^2$ is the one we started with.

"Square-zero deformation of R by I."

R and R-module M. Want diagram

$$0 \to M \xrightarrow[i]{} E \xrightarrow[p]{} R \to 0 \quad \text{exact}$$

s.t. p map of k-alg and $i(M)^2 = 0$

Two such extensions are equivalent if $\exists\, \varphi$ map of k-alg.

$$\begin{array}{ccccccccc} 0 & \to & M & \to & E & \to & R & \to & 0 \\ & & \| & & \downarrow \varphi & & \| & & \\ 0 & \to & M & \to & E' & \to & R & \to & 0 . \end{array}$$

($\Rightarrow \varphi$ iso)

$\mathrm{Exalgcom}(R/K;\, M)$ = Equiv. classes of commutative algebra extensions of R by M.

Trivial extension: $R \ltimes M$.

Underlying K-module: $R \oplus M$. (Happens to be R-mod.)

Multiplication: $(r, m)\cdot(s, n) = (rs,\ rn + ms)$.

$$0 \to M \to R \ltimes M \to R \to 0$$

$(r, m) \longmapsto r$

$m \longmapsto (0, m)$

An extension

$$0 \to M \to E \xrightarrow{p} R \to 0 \quad \text{is split}$$

if $\exists\, k$-algebra map $s: R \to E$ s.t. $ps = \mathrm{id}$.

Ex An extension is split $\Leftrightarrow$ equivalent to the trivial extension.

Splitting need not be unique.

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & M \arrow[d] \\
 & E \arrow[d, "p"] \\
S \arrow[ur, bend left=50, "s_1"] \arrow[ur, bend left=20, "s_2"'] \arrow[r, "\varphi"'] & R \arrow[d] \\
 & 0
\end{tikzcd}$$

extension

Let s_1 and s_2 be two liftings (k-alg. maps).

Then, $s_1 - s_2 : S \longrightarrow M$

$\hookrightarrow$ S-module via φ.

<u>Check</u> : $s_1 - s_2$ is a k-linear derivation

<u>Conversely</u>. If $s : S \to E$ is a lifting and $d : S \to M$ is a k-linear der, then $s + d$ is also a lifting.