

# $\mathbb{R}$ eal Analysis

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## §1. Sets and stuff

## §2. Topology

1. Let  $X$  be a metric space and let  $U \subset X$ . Define the *boundary* of  $U$  as

$$\partial U = \bar{U} \cap \overline{(U^c)}.$$

Show that  $\partial U = U \setminus U^\circ$ .

2. Prove or disprove that

$$(\partial U)^\circ = \emptyset$$

for any subset  $U$  of any metric space  $X$ .

**HIDDEN:** Disprove it. Even in the case that  $X = \mathbb{R}^n$ .

3. Let  $(X, d)$  be a metric space and  $x \in X$ . Let  $\delta > 0$ . Define the following sets:

$$B_\delta(x) := \{y \in X \mid d(x, y) < \delta\},$$

$$C_\delta(x) := \{y \in X \mid d(x, y) \leq \delta\}.$$

Show that  $\overline{B_\delta(x)} \subset C_\delta(x)$ .

Can this inclusion be proper?

**HIDDEN:** Not if you stay in  $\mathbb{R}^n$ . Think about other spaces.

4. **Topological Nim**

You and your friend want to play Topological Nim. Here's how it works:

Let  $X$  be your favourite compact metric space and  $r > 0$  your favourite (positive) real number.

Each player removes an open disk of radius  $r$  from the space on their turn (only the center of the disk must not have been removed in a prior move), until one player—the winner—removes what remains of the space on his turn.

Show that no matter what moves are played, the game stops after a finite number of moves. (In other words, there is no infinite sequence of legal moves.)

**Bonus:** Fix  $n \in \mathbb{N}$  and  $r > 0$ . Assuming optimal play, who will win the game if

$$X = S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$$

with the standard metric?

(The answer will depend on  $r$ .)

Credits: <https://puzzling.stackexchange.com/questions/99859/>

5. Show that every open set  $U$  in  $\mathbb{R}$  can be written as a disjoint union of open intervals. Moreover, show that this set of open intervals is at most countable.  
**HIDDEN:** First part: Consider an equivalence relation  $\sim$  on  $U$  where  $x \sim y$  iff  $[x, y] \subset U$ .  
Second part: Each open interval contains a rational.
6. Let  $I \subset \mathbb{R}$  be such that every  $x \in I$  is an isolated point.  
Show that  $I$  is at most countable.
7. Let  $K$  be a compact subset of  $\mathbb{R}^n$ . Fix a constant  $r > 0$ .  
Show that there exists a finite collection of points  $x_1, \dots, x_k \in K$  such that the collection of open balls  $\{B(x_i, 2r)\}_{i=1}^k$  forms an open cover of  $K$  while  $B(x_i, r)$  are mutually disjoint.

### §3. Continuity and derivatives

1. Prove or disprove:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable. If  $f'(x_0) > 0$  for some  $x_0 \in \mathbb{R}$ , then there exists an interval  $I$  containing  $x_0$  such that  $f$  is increasing on  $I$ .

**HIDDEN:** Prove.

2. Prove or disprove:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable. If  $f'(x_0) > 0$  for some  $x_0 \in \mathbb{R}$ , then there exists an interval  $I$  containing  $x_0$  such that  $f$  is increasing on  $I$ .

**HIDDEN:** Disprove.

3. Let  $\pi_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the first projection map, that is,

$$\pi_1(x, y) = x.$$

Show that  $\pi_1$  is an *open map*, that is,  $\pi_1(U)$  is open in  $\mathbb{R}$  if  $U$  is open in  $\mathbb{R}^2$ . Is it a closed map?

**HIDDEN:** No.

4. **Pasting lemma.**

Let  $X$  be a metric space and  $\{U_\alpha\}_{\alpha \in I}$  be an open cover of  $X$ .

Let  $Y$  be an arbitrary metric space. Suppose that for each  $\alpha \in I$ , we have a continuous function

$$f_\alpha : U_\alpha \rightarrow Y.$$

Moreover, assume that whenever  $x \in U_\alpha \cap U_\beta$ , then  $f_\alpha(x) = f_\beta(x)$ . (That is, the functions agree on their common domains.)

Show the following:

- (a) There exists a unique function  $f : X \rightarrow Y$  such that

$$f|_{U_\alpha} = f_\alpha \quad \text{for all } \alpha \in I.$$

(What the above means is that: for all  $\alpha \in I$ , for all  $x \in U_\alpha$ ,  $f(x) = f_\alpha(x)$ .)

- (b) The above function  $f$  is continuous.

5. Show that the above is not true if we replace “open” with “closed.”  
(In particular, observe very carefully where you used open-ness of  $U_\alpha$ .)
6. Show that the above becomes true once again after replacing “open” with “closed” if we further impose that  $I$  be finite.

*Remark.* The above lemma for closed sets makes it especially easy to directly verify the continuity of “piece-wise” defined functions which agree on the intersections. A particular easy case is when the sets have empty intersection. (cf. 9)

7. Give a counterexample if we further drop “closed” completely, even if  $I$  is finite. (In fact, you can give one with  $X = \mathbb{R}$  and  $|I| = 2$ .)
8. Given an example of a continuous bijection  $f : X \rightarrow Y$  such that  $f^{-1} : Y \rightarrow X$  is not continuous.
9. Justify that the following is an example for the above question:  
 $f : [0, 1] \cup (2, 3] \rightarrow [0, 2]$  defined by

$$f(x) := \begin{cases} x & x \in [0, 1] \\ x - 1 & x \in (2, 3] \end{cases}.$$

10. Let  $f : X \rightarrow Y$  be a function between metric spaces.
  - (a)  $f$  is said to be *open continuous* if  $f^{-1}(U)$  is open in  $X$  whenever  $U$  is open in  $Y$ .
  - (b)  $f$  is said to be *closed continuous* if  $f^{-1}(U)$  is closed in  $X$  whenever  $U$  is closed in  $Y$ .

Show that  $f$  is continuous iff  $f$  is open continuous iff  $f$  is closed continuous.

11. Let  $K$  be a compact metric space and  $Y$  an arbitrary metric space. Assume that  $f : K \rightarrow Y$  is a continuous bijection.
  - (a) Let  $C \subset K$  be closed. Show that  $C$  is compact.
  - (b) Show that  $f(C)$  is compact.
  - (c) Show that  $f(C)$  is closed.

Conclude that  $f^{-1} : Y \rightarrow K$  is continuous.

12. The following question appeared on a test:
 

Given an example of a continuous bijection  $f : X \rightarrow Y$  such that  $f^{-1} : Y \rightarrow X$  is not continuous.

The lazy TA sees that a student has started their answer as

The following is example:

Let  $f : S^1 \rightarrow S^1$  be defined as...

The TA sees that and marks it wrong straight away. Was the TA justified (mathematically, not morally) in doing so? Why?

13. Let  $I \subset \mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be continuous. We know that if  $I$  is compact, then  $f$  is uniformly continuous.

Show that if  $I$  is not compact, then one can always construct a continuous  $f$  which is *not* uniformly continuous.

14. Let  $I \subset \mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be continuous. We know that if  $I$  is compact, then  $f$  is bounded and it achieves (both) its bounds.

Show that if  $I$  is not compact, then one can always construct:

- (a) a continuous  $f$  which is not bounded,
- (b) a continuous  $f$  which is bounded but fails to achieve one (or both) of its bounds.

15. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous such that

$$\lim_{x \rightarrow \infty} f(x) \text{ and } \lim_{x \rightarrow -\infty} f(x)$$

both exist and are finite.

Show that  $f$  is bounded.

16. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function such that  $\lim_{x \rightarrow \infty} f(x)$  exists and is finite.

Prove or disprove:

$$\lim_{x \rightarrow \infty} f'(x) = 0.$$

**HIDDEN:** The limit need not exist.

17. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function such that  $\lim_{x \rightarrow \infty} f(x)$  exists and is finite. Further assume that  $f'$  is uniformly continuous.

Prove or disprove:

$$\lim_{x \rightarrow \infty} f'(x) = 0.$$

**HIDDEN:** Prove.

18. Suppose  $f$  is continuous on  $[0, 1]$  with  $f(0) = f(1) = 0$ . For all  $x \in (0, 1)$ , there exists  $h > 0$  with  $0 \leq x - h < x < x + h \leq 1$  such that  $f(x) =$

$$\frac{f(x+h) + f(x-h)}{2}.$$

Show that  $f(x) = 0$  for all  $x \in [0, 1]$ .

(Note that given any  $x$ , the above only says that there's a *particular*  $h$  with the given property.)



## §4. Integration

1. Does there exist a function  $f : [0, 1] \rightarrow \mathbb{R}$  such that it takes only a finitely many values and is Riemann Integrable on  $[0, 1]$  but is not locally constant?

**HIDDEN:** Yes. Find/show the existence of one.

## §5. Sequence and series of functions

### 1. (Non-)converse of Weierstrass M-test

Construct an example of a family  $(f_n)_{n \in \mathbb{N}}$  of functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\sum f_n$  converges uniformly but  $\sum M_n$  does not, where  $M_n := \sup_{x \in \mathbb{R}} |f_n(x)|$ .

**HIDDEN:** Consider  $f_n$  such that  $f_n$  takes value  $1/n$  at  $n$  and 0 otherwise.

2. Recall that if  $f : K \rightarrow \mathbb{R}$  is a continuous function and  $K$  is compact, then there exists a sequence  $(P_n)_{n \in \mathbb{N}}$  of polynomials such that  $P_n \rightarrow f$  uniformly on  $K$ . Show that this need not be true if  $K$  is not compact.

**HIDDEN:** Consider  $K = \mathbb{R}$  and  $f = \exp$ .

3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Show that there exists a sequence  $(P_n)_{n \in \mathbb{N}}$  of polynomials such that  $P_n \rightarrow f$  **pointwise** on  $\mathbb{R}$ .