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MA 839

Advanced Commutative Algebra

Notes By: Aryaman Maithani

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Lecture 0 (07-01-2021)

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A Quick Intro.

Setup: A ring is commutative with 1.

Let M be an R-module.

Observation: ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$),
we get an R-linear map $R \twoheadrightarrow M$, $a \mapsto ax$ which is onto.

Then, $M \cong R/I$ where I is the kernel.
In this case, $I = \mathrm{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R.

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$
$= \{ax + by \mid (a, b) \in R^{\oplus 2}\}$

Then, we get an onto R-linear map $\underbrace{Re_1 \oplus Re_2}_{\{e_1, e_2\} \text{ is a basis}} \xrightarrow{\varphi} M$
$e_1 \mapsto x$
$e_2 \mapsto y$ } extend this

this lets us extend the map

In particular, $M \cong R^2/\ker \varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle\ \rangle} \oplus \frac{R}{\langle\ \rangle} ?$$

This has a positive answer: ① R is a field
② R is a PID

CAUTION: We won't include fields as PID.
That is, when we say "PID", we exclude fields !!

③ Suppose M is a finitely generated (f.g.) R-module.

(That is, suppose $M = \langle x_1, \dots, x_n \rangle$.)

way to get this

Then, M is a quotient of $R^{\oplus n}$.

Define $R^{\oplus n} \overset{\varphi}{\twoheadrightarrow} M$ by $e_i \longmapsto x_i$.

$M \cong R/\ker \varphi$.

④ In general, consider a free module with "M as basis", call it $F(M)$. Then $F(M)$ maps onto M.

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$, then $F(A)$ maps onto M.

Thus, M can be written as a quotient of a free-module.

To Summarise: If M is an R-module, then M can be written as a quotient of a free R-module. Moreover, if M is f.g., then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R-module mapping onto M with kernel K. That is, $\varphi : F \twoheadrightarrow M$ is onto R-linear and $\ker \varphi = K$.

Now, $\exists$ a free R-module G and an onto map $\psi : G \to K$

We capture this in the following diagram

$\leftarrow$ compose ψ with $K \hookrightarrow F$

$$\begin{tikzcd} G \arrow[rr, dashed, "\psi"] \arrow[dr, "\psi"'] & & F \arrow[r, "\varphi"] & \underbrace{M \longrightarrow 0} \\ & K \arrow[ur] & & \end{tikzcd}$$

means φ is onto

$$\begin{tikzcd}
G \arrow[dr, "\psi"'] & & F \arrow[r] & M \arrow[r] & 0 \\
& K \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

means φ is onto

means the map $K \to F$ is 1-1

Note that $\operatorname{im} \psi = K = \ker \varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \longrightarrow 0$.

① φ is onto and $\ker \varphi = \operatorname{im} \psi$.
② G and F are free R-modules.

Note that we can repeat the above process with K instead of F.

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \varphi$, $\varphi_1 := \psi$.

$$\begin{tikzcd}
& K_1 \arrow[dr] & & & & & \\
F_2 \arrow[ur, "\varphi_2"] \arrow[rr, "\varphi_2"'] & & F_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\varphi_1"'] & & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
& & & K_0 \arrow[ur] \arrow[dr] & & & \\
& & & & 0 & &
\end{tikzcd}$$

Thus, we get free modules $\{F_n, \varphi_n : F_n \to F_{n-1}\}$ such that $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$ written as

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

F_i s are free, φ_0 is onto & $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F_\bullet : \quad \cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow \underbrace{F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0}$$ as an

R free resolution of M.

$\operatorname{im} \varphi_1 = K$, this is not exact here.
φ_1 not onto (nec.)

Q: ① If M is f.g.:
Can we get F_is so that $\text{rank}(F_i) < \infty$ $\forall i$.

② If yes, are $\text{rank}(F_i)$s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_is s.t. the maps are "described nicely"?

Q. If two modules have "isomorphic" free resolutions, are they isomorphic?