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MA 406

General Topology

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Lecture 1 (07-01-2021)

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Def^n. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) ϕ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any <u>finite</u> subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Any $U \in \mathcal{T}$ is called an open set of X w.r.t. $\mathcal{T}$. The pair $(X, \mathcal{T})$ or just the set X is called a topological space.

Can reconcile the above with open sets in $\mathbb{R}$, or in general, any metric space X. That can be seen as a motivation for the definition.

<u>Examples</u>

(1) $X = \{a, b, c\}$

$\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ → (Can be seen (fairly easily) that this is a topology

$\mathcal{T}_2 = \{\phi, X\}$

→ trivial (pun intended, cf. next example)

(2) If X is any set, the collection of all subsets of X is a topology on X, it is called the discrete topology. ($\mathcal{T} = \mathcal{P}(X)$, that is)

The collection $\{\phi, X\}$ is also a topology on X called the indiscrete topology or trivial topology.

(3) Let X be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\emptyset\}.$$

Then, $\mathcal{T}_f$ is a topology on X, called the finite complement topology on X.

- $\emptyset \in \mathcal{T}_f$ is clear. $X \in \mathcal{T}_f$ since $|X \setminus X| = 0 < \infty$.
- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be sets in $\mathcal{T}_f$. WLOG, $U_\alpha \neq \emptyset$ $\forall \alpha$.

Note $$X \setminus \left(\bigcup_\alpha U_\alpha \right) = X \cap \left(\bigcup_\alpha U_\alpha \right)^c$$
$$= \bigcap_\alpha (U_\alpha^c)$$

Note that each U_α^c is finite. $(U_\alpha \neq \emptyset)$
Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to $\bigcup_{i=1}^{n} (U_i^c)$ and conclude as earlier.

(Here, if some U_i were $\emptyset$, then so would be the intersection.)

(If X is finite, the $\mathcal{T}_f = \mathcal{P}(X)$. Thus, we get discrete.)

(4) Let X be a set.
Let $\mathcal{T}_c$ be the collection of subsets such that $X \setminus U$ is either countable or all of X.
Called the co-countable topology.
(Generalising the previous)

Def^n Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X.
If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and that $\mathcal{T}$ is coarser than $\mathcal{T}'$.
If $\mathcal{T}' \supsetneq \mathcal{T}$, then the above is strictly finer and strictly ...

coarser, respectively.

(The above gives us a way to compare two topologies)

EXAMPLE We have the usual topology on $\mathbb{R}$. ← strictly coarser than this
We also have the discrete topology on $\mathbb{R}$.

Def^n. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

(1) for each $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Note that in the above, $\mathcal{B}$ is just some collection of subsets of X satisfying (1) & (2). No topology is mentioned so far.

EXAMPLES

(1) $X = \mathbb{R}^2$, $\mathcal{B}$ is the collection of all discs w/o boundary.
(2) ———— " ———————— " - rectangles —"—
(3) Any X. The singletons form a basis.

We now get a topology out of a basis:

Def^n. If $\mathcal{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows:

A subset U of X is said to be open if for every $x \in U$, there exists $B \in \mathcal{B}$ s.t.

$$x \in B \subset U.$$

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(By "open" in above, we mean element of ***T***. Same thing for what we see in the proof below.)

EXAMPLES (1) & (2) $\rightarrow$ gives standard topology on $\mathbb{R}^2$

(3) $\rightarrow$ gives discrete topology on X.

We still have to show that it is topology.

Proof.

- $\phi \in \mathcal{T}$ vacuously

  $X \in \mathcal{T}$ since given any $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.

  $B \subset X$ is by definition.

- Let $\{U_\alpha\}_{\alpha \in \Lambda \neq \phi}$ be open. Let $U := \bigcup_\alpha U_\alpha$.

  Fix $\alpha_0 \in \Lambda$.

  Let $x \in U$ be arbitrary. Then, $x \in U_{\alpha_0} \leftarrow$ open

  $\therefore \exists B \in \mathcal{B}$ s.t. $x \in B \subset U_{\alpha_0} \subset U$.

  $\therefore U \in \mathcal{T}$.

- Let U_1 and U_2 be open. Put $U := U_1 \cap U_2$.

  Let $x \in U$.

  Then $x \in U_1$ and $x \in U_2$

  $\Downarrow$ $\Downarrow$

  $\exists B_1 \in \mathcal{B}$ $\exists B_2 \in \mathcal{B}$

  s.t. $x \in B_1 \subset U_1$ s.t. $x \in B_2 \subset U_2$

  $$\therefore \underbrace{x \in B_1 \cap B_2}_{\Downarrow} \subset U_1 \cap U_2$$

  $$\exists B_3 \in \mathcal{B} \text{ s.t. } x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U.$$

  $\Rightarrow U \in \mathcal{T}$.

By induction, any finite intersection is in $\mathcal{T}$. $\blacksquare$

$$n \geq 3 \left(\bigcap_{i=1}^{n} U_i = U_n \cap \left(\bigcap_{i=1}^{n-1} U_i \right). \right.$$

Lecture 2 (11-01-2021)

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Lemma 1 Let $\mathcal{B}$ be a basis and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then, $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Note that $\emptyset$ is empty union.

Proof. Given $\{U_\alpha\} \subset \mathcal{B}$, it is clear that $\bigcup U_\alpha \in \mathcal{T}$ since $\mathcal{T}$ is a topology and U_α are open. (By def^n)

Conversely, let $U \in \mathcal{T}$. Given any $x \in U$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. (By def^n of $\mathcal{T}$.)

Thus,
$$\bigcup_{x \in U} B_x = U. \qquad \blacksquare$$

($\subseteq$) since $B_x \subset U$

($\supseteq$) Each $x \in U$ is in B_x.

(Note that if $U = \emptyset$, the last union is the empty union!)

[The above gives us a way of extracting a basis $\mathcal{B}$ if we are already given a topology $\mathcal{T}$.
Namely, pick any subcollection $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{T}$ is precisely the collection of all unions of elements of $\mathcal{B}$.]

Lemma 2. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$. (recall this means $\mathcal{T} \subset \mathcal{T}'$)

(ii) for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \in B$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and $B \in \mathcal{B}$ be arbitrary.
Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.

Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.

Thus, $B \in \mathcal{T}'$. (by (i))

Since B is open in $\mathcal{T}'$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

(Def^n of top. generated.)

(ii) $\Rightarrow$ (i) Suppose $U \in \mathcal{T}$. We show that $U \in \mathcal{T}'$.

Let $x \in U$. By def^n of $\mathcal{T}$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

By (ii), $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B \subset U$.

Since x was arbit, we see that $U \in \mathcal{T}'$. (By def^n of $\mathcal{T}'$)

Thus, $\mathcal{T} \subset \mathcal{T}'$. ☒

Lemma 3. Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X s.t. for each open set $U \subset X$ and each $x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.

Then $\mathcal{C}$ is a basis for the topology.

Proof. • Showing $\mathcal{C}$ is a basis.

(i) Given any $x \in X$, X is an open set containing x.

Thus, by hypothesis, $\exists C \in \mathcal{C}$ s.t. $x \in C$.

(ii) Let $C_1, C_2 \in \mathcal{C}$ s.t. $x \in C_1 \cap C_2$.

Note that C_1, C_2 are open and hence, $C_1 \cap C_2$ is open.

By hypothesis, $\exists C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$.

Thus, $\mathcal{C}$ satisfies both properties of a topology.

• $\mathcal{C}$ generates the topology.

Let $\mathcal{T}$ denote the topology of X. Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$.

Let $U \in \mathcal{T}'$, then U is some union of elements of $\mathcal{C}$.
But elements of $\mathcal{C}$ are elements of $\mathcal{T}$ and thus, $U \in \mathcal{T}$.
($\mathcal{T}$ is topo.)

Thus, $\mathcal{T}' \subset \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. For each $x \in U$, $\exists C_x \in \mathcal{C}$ s.t. $x \in C_x \subset U$.

As earlier,
$$U = \bigcup_{x \in U} C_x \in \mathcal{T}'.$$

Thus, $\mathcal{T} \subset \mathcal{T}'$. ∎

Def^n. Let $\mathcal{B}$ be the collection of all bounded intervals.
That is,
$$\mathcal{B} = \{ (a, b) : -\infty < a < b < \infty \}.$$

$\mathcal{B}$ is a basis and the topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$.

If $\mathcal{B}'$ is the collection of all half open intervals of the form $[a, b)$, then $\mathcal{B}'$ is also a basis and the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$.

Lemma 4. The lower limit topology is strictly finer than the standard topology.

Proof. Let $\mathcal{T}$ denote the standard topology and $\mathcal{T}'$ the lower limit.

- $\mathcal{T} \subseteq \mathcal{T}'$. Let (a, b) be an arbit. basis element and let $x \in (a, b)$.

Then, $[x, b)$ is a basis element for $\mathcal{T}'$ &

$x \in [x, b) \subset (a, b)$.

Thus, $\mathcal{T} \subseteq \mathcal{T}'$, by Lemma 2.

- $\mathcal{T}' \neq \mathcal{T}$. Note that $[0, 1) \in \mathcal{T}'$.
  But given $0 \in [0, 1)$, there is no $(a,b) \ni 0$
  s.t. $(a, b) \subset [0, 1)$. ∎

Def^n. A **subbasis** $\mathcal{S}$ **for a topology** is a collection of subsets of X whose union is X.

(Note that no topology given so far. Similar to what we saw for basis.)

The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

We need to show that the topology defined above is actually a basis.

Let $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{S}$. We show $\mathcal{B}$ is a basis. (This suffices. Why? Lemma 1.)

(i) Let $x \in X$. Then, $\exists\, S \in \mathcal{S}$ s.t. $x \in S$. $\left(\because \bigcup_{S \in \mathcal{S}} S = X\right)$
But $S \in \mathcal{B}$.

(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$.
But note that $B_1 \cap B_2 \in \mathcal{B}$. (Why?)

Thus, both the conditions are satisfied.

Remark The standard topology of $\mathbb{R}$ is also called the **order topology**

on $\mathbb{K}$, because of the order relation of $\mathbb{R}$.
(We will see this in general, later.)

Def^n Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.

(Open in the respective topologies, i.e.)

Note that $\mathcal{B}$ is a basis because:

(i) $X \times Y$ is itself a basis element

(ii) $U \times V$, $U' \times V' \in \mathcal{B} \Rightarrow (U \times V) \cap (U' \times V') = \underbrace{(U \cap U')}_{} \times \underbrace{(V \cap V')}_{} \in \mathcal{B}$

intersection of open sets

Note $\mathcal{B}$ itself won't be the topology. (In general)

Thm 5. If $\mathcal{B}$ is a basis for a topology $\mathcal{J}_X$ on X, and $\mathcal{C}$ for $\mathcal{J}_Y$ on Y, then the collection

$$\mathcal{D} = \{ B \times C \ : \ B \in \mathcal{B}, \ C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We check that the hypotheses of Lemma 3 are satisfied.

Let $W \subset X \times Y$ be open and $(x, y) \in W$.
Then, by def^n of prod. top., $\exists\, U \in \mathcal{J}_X$, $V \in \mathcal{J}_Y$ s.t.

$$(x, y) \in U \times V \subset W.$$

Since $\mathcal{B}$ is a basis for $\mathcal{T}_X$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
$||^{ly}$ $\exists C \in \mathcal{C}$ s.t. $y \in C \subset V$.

$$\Rightarrow (x, y) \in \underset{\substack{\uparrow \\ \mathcal{D}}}{B \times C} \subset U \times V \subset W. \qquad \blacksquare$$