The Rank Conjectures

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§1. Introduction

Let G be a group acting on a topological space X. For $x \in X$, we define the subgroup $G_x := \{g \in G : g(x) = x\}$, called the isotropy subgroup at the point x. We say the action of G is free if $G_x = \{e\}$ for all $x \in X$, and almost free if G_x is finite group for all $x \in X$.

The theme of the rank conjectures will be as follows: Given a group G acting (almost) freely on X, what can we say about the "ranks" of X and G.

Given a space X, and a field k, we define

$$\operatorname{rank}_k H_*(X; k) := \sum_{i=0}^{\infty} \dim_k H_i(X; k).$$

In our examples, X will either be a manifold or a finite-dimensional CW complex, so the sum above will be finite. As a beginning example, we note

$$\operatorname{rank} H_*((S^n)^k; \mathbb{Z}/p) = 2^k.$$

§2. Carlsson's rank conjecture

We first look at the action of elementary abelian p-groups on product of spheres. Recall that for p a positive prime, an elementary abelian p-group is a group of the form $(\mathbb{Z}/p)^r$ for some $r \geqslant 0$. This r is uniquely determined, and is called the rank of the group.

Note that if G acts on a space X, then functoriality of homology gives an action of G on $H_k(X)$. More generally, an action on homology (and cohomology) with coefficients.

The following is from [Car82].

Theorem 2.1. Suppose $(\mathbb{Z}/p)^r$ acts freely on $(S^n)^k$, with trivial action on integral homology. Then, $k \geqslant r$.

A similar theorem on these lines is the following, see [Car86, Theorem I.1].

Theorem 2.2. Suppose $G = (\mathbb{Z}/p)^r$ acts freely on a finite complex X which is homotopy equivalent to $(S^n)^k$, and suppose that the action of G on $H_n(X;\mathbb{Z}/p)$ is trivial. Then, $k \geqslant r$.

The above theorems have certain restrictions on the action: Not only must it be free, it must also be trivial on certain homologies. Moreover, the space to which it applies is not a general product of spheres, that would be a space of the form $S^{n_1} \times \cdots \times S^{n_k}$. A conjecture to this end would be the following, appearing in [Car87].

Conjecture 2.3 (Carlsson)**.** If $(\mathbb{Z}/p)^r$ acts freely on a CW-complex X, then

$$\operatorname{rank} H_*(X;\mathbb{Z}/p) \geqslant 2^r.$$

A purely algebraic generalisation to the above would be the following.

Conjecture 2.4. Let k be a field of positive characteristic p, $G = (\mathbb{Z}/p)^r$, and kG be the corresponding group algebra. If F is a bounded complex of free kG-modules of finite rank and $H_*(F) \neq 0$, then $\operatorname{rank}_k H_*(F) \geqslant 2^r$.

Carlsson proved this conjecture for $p = 2$ and $r \leqslant 3$ [Car87, Theorem 2].
The above conjecture would imply Carlsson's conjecture: In the case that G acts freely (and cellularly?) on X, the chain complex that computes the cellular homology has the additional structure of being a complex of free kG-modules. $\Big($Roughly: the i-th module in the complex is a free k-module, being indexed by the i-cells: $\bigoplus_{e^i_\alpha} k$. By the G-action, we can further refine this as

$$\bigoplus_{\mathcal{O}^i_\beta} \bigoplus_{e^i_\alpha \in \mathcal{O}_\beta} k,$$

where $\mathcal{O}^i_\beta$ ranges over the G-orbits of the i-cells. Since the action is free, each orbit has size $|G|$. So, the inner term is isomorphic (at least as a k-vector space) to kG.$\Big)$

However, the algebraic version is false for all p odd and $r \geqslant 8$. Iyengar and Walker [IW18] gave a counterexample. However, they remark that they do not know whether their complex comes from a space with a free G-action.

§3. Toral rank conjecture

This section is taken from [FOT08].

Now, we will consider the actions of Lie groups on manifolds. Specifically, the action of the r-torus $\mathbb{T}^r := (S^1)^r$. The rank of a Lie group will be its dimension as a manifold.

Definition 3.1. The toral rank of a space X, denoted $\mathrm{rk}(X)$, is the largest integer r such that a torus $\mathbb{T}^r$ acts almost freely on X.

Example 3.2. Let X be the wedge of more than one sphere (of possibly different dimensions). We claim that $\mathrm{rk}(X) = 0$.

Indeed, consider the "wedge point" $p \in X$. p is the only point such that $X \setminus \{p\}$ is disconnected. Consequently, every homeomorphism of X must fix p. Thus, if G is an infinite group acting on X, then G_p will be infinite.

Example 3.3. Recall the (free) Hopf action of S^1 on $S^3 \subseteq \mathbb{C}^2$:

$$e^{\iota\theta} : (z, w) \mapsto (e^{\iota\theta} z, e^{\iota\theta} w).$$

Similarly, S^1 also acts freely on $S^1 \times S^2$ by

$$e^{\iota\theta} : (z, p) \mapsto (e^{\iota\theta} z, p).$$

In each space, we can select an S^1-orbit, and glue S^3 and $S^1 \times S^2$ along these orbits. Call this space Y. Evidently, $\mathrm{rk}(Y) \geqslant 1$.

However, one can check that Y and $S^2 \vee S^3 \vee S^3$ are homotopy equivalent. Thus, the toral rank is *not* a homotopy invariant.

Definition 3.4. The rational toral rank of a space X, $\mathrm{rk}_0(X)$, is the maximum of $\mathrm{rk}(Y)$ for all finite CW complexes Y in the rational homotopy type of X.

Tautologically, the rational toral rank is a homotopy invariant.

Recall that X and Y are said to have the same rational homotopy type if there is a finite sequence of maps

$$X \to X_1 \leftarrow \cdots \leftarrow X_n \to Y$$

such that each map is an isomorphism on rational homology.

Definition 3.5. A space X is said to be nilpotent if $\pi_1(X)$ is a nilpotent group and acts nilpotently on $\pi_n(X)$ for $n \geqslant 2$.

If X is a nilpotent space with finite-dimensional rational cohomology, then X is said to be rationally elliptic if $\sum_{n \geqslant 2} \mathrm{rank}(\pi_n(X) \otimes \mathbb{Q}) < \infty$.

For a rationally elliptic space, the homotopy Euler characteristic is defined by

$$\chi_\pi(X) := \operatorname{rank} \pi_{\text{even}}(X) - \operatorname{rank} \pi_{\text{odd}}(X).$$

Example 3.6. Spheres are rationally elliptic spaces. Indeed, it is clear that they are nilpotent spaces. For S^1, this follows since the higher homotopy groups are zero. For the higher spheres, this follows since π_1 is trivial.

Serre computed the rational homotopy groups of spheres as:

$$\pi_i(S^{2a-1}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i = 2a-1, \\ 0 & \text{otherwise.} \end{cases}$$
$$\pi_i(S^{2a}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i \in \{2a, 4a-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\chi_\pi(\text{odd sphere}) = -1$.

For some of these spaces, we have some idea of the rational toral rank.

Theorem 3.7. If M is a nilpotent rationally elliptic space, then $\operatorname{rk}_0(M) \leqslant -\chi_\pi(M)$.

If G is a compact connected Lie group, then $\operatorname{rk}_0(G) = \operatorname{rank}(G)$. More generally, if K is a compact connected subgroup, then $\operatorname{rk}_0(G/K) = \operatorname{rank}(G) - \operatorname{rank}(K)$.

The above calculations involve using minimal models and existence of maximal torus for one end of the bound.

Example 3.8. Let X be an odd sphere. Then, the above theorem tells us $\operatorname{rk}_0(X) \leqslant 1$. Since X does admit a free S^1 action, we get $\operatorname{rk}_0(\text{odd sphere}) = 1$.

Example 3.9. Since $\mathbb{T}^r$ is a Lie group of rank r, we have $\operatorname{rk}_0(\mathbb{T}^r) = r$. As noted before, $\operatorname{rank} H^*(\mathbb{T}^r; \mathbb{Q}) = 2^r = 2^{\operatorname{rk}_0(\mathbb{T}^r)}$.

Remark 3.10. More generally, if G is a compact Lie group, then $2^{\operatorname{rank} G} = \operatorname{rank} H^*(G; \mathbb{Q})$.

Conjecture 3.11 (Toral Rank Conjecture (TRC))**.** Let X be a nilpotent finite CW complex. Then,

$$\operatorname{rank} H^*(X;\mathbb{Q}) \geqslant 2^{\mathrm{rk}_0(X)}.$$

The above is open in general; we look at some cases for which it is proven.

Theorem 3.12. The TRC is true for a product of odd-dimensional spheres.

Proof. Suppose $\mathbb{T}^r$ acts almost freely on $X = S^{n_1} \times \cdots \times S^{n_p}$. Then, $r \leqslant -\chi_\pi(X) = p$. Moreover, $2^p = \operatorname{rank} H^*(X;\mathbb{Q})$. So, $2^r \leqslant \operatorname{rank} H^*(X;\mathbb{Q})$. □

Theorem 3.13. If G is a compact connected Lie group, and $K \subseteq G$ a compact connected subgroup, then the TRC is true for G/K.

Sketch. Putting together the previous results, it suffices to show that

$$\operatorname{rank} H^*(G) \leqslant (\operatorname{rank} H^*(G/K)) \cdot (\operatorname{rank} H^*(K)).$$

This follows from the Serre spectral sequence $H^*(G/K) \otimes H^*(K) \Rightarrow H^*(G)$. □

A weaker form has been proved by Allday and Puppe [AP06].

Theorem 3.14. If a torus $\mathbb{T}^r$ acts almost freely on a compact nilpotent manifold M, then

$$\dim H^*(M;\mathbb{Q}) \geqslant 2r.$$

§4. Total Rank Conjecture

For simplicity, we state the local versions.

Conjecture 4.1 (Buchsbaum-Eisenbud-Horrocks (BEH))**.** If R is a noetherian local ring, and M a nonzero R-module of finite length having a finite free resolution

$$0 \to F_d \to \cdots \to F_0 \to M \to 0,$$

then

$$\operatorname{rank} F_i \geqslant \binom{\dim(R)}{i}.$$

A slight weakening of the above gives the total rank conjecture.

Conjecture 4.2 (Total Rank Conjecture (TRC))**.** With R, M, F_* as above, we have

$$\operatorname{rank} F_* \geqslant 2^{\dim(R)},$$

where $\operatorname{rank} F_* = \sum_i \operatorname{rank}(F_i)$.

BEH remains open. However, the TRC has been proven ([VW23]) for all noetherian rings that contain a field! This had been proven earlier when the characteristic of R was an odd prime (or if R satisfied some other technical condition).

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