Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

\# Bruns & Herzog → Cohen-Macaulay rings
- 1st Part

\# Affine algebra

X

Derived Category

- Gelfand & Manin
- E. Riehl's : Cat theory in context
- Dwyer & Spalinski

$R \to$ ring (possibly non comm.)

R-complexes :

$$\cdots \to M_{i+1} \xrightarrow{\partial} M_i \xrightarrow{\partial} M_{i-1} \to \cdots \qquad \partial^2 = 0$$

$$\mathrm{im}(\partial_{i+1}) \subset \ker(\partial_i)$$

$$H_i(M) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

$$H(M) = (H_i(M))_{i \in \mathbb{Z}}$$

$\mathcal{C}(R)$ = category of R-complexes (morphisms as usual)

If $f : M \to N$, we get an induced map $H(f) : H(M) \to H(N)$.

Def^n f is a quasi isomorphism (or weak equivalence) if $H(f)$ is bijective. (Automatically iso.)

$W :=$ collection of weak equivalences in $\mathcal{C}(R)$

$D(R) := \mathcal{C}(R)[W^{-1}]$. (or $W^{-1}\mathcal{C}(R)$)

- Key property : W has the 2-out-of-6 property :

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,

- Key property: W has the 2-out-of-6 property:
  Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,
  if gf and $hg \in W$, then f, g, h, hgf are in W.

Ex: $\Rightarrow$ 2-out-of-3 property
If f, g, fg are defined and 2 are in W, then so is the third.

Concretely: $\quad \mathcal{C}(R) \rightsquigarrow K(R) \rightsquigarrow D(R)$.
$\uparrow$ homotopy category

$M, N \to$ R-complexes

$\operatorname{Hom}_R(M, N)$:= Hom-complex of abelian groups
(when R is comm. this is an R-complex)

$\operatorname{Hom}_R(M,N)_n$:= Maps of degree n from M to N (no compatibility!)

$$\begin{tikzcd} \cdots \arrow[r] & M_i \arrow[r] \arrow[dl] & M_{i-1} \arrow[r] \arrow[dl] & \cdots \\ \cdots \arrow[r] & N_{i+n} \arrow[r] & N_{i+n-1} \arrow[r] & \cdots \end{tikzcd} \quad = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_R(M_i, N_{i+n})$$

$$\partial : \operatorname{Hom}_R(M,N)_{n+1} \longrightarrow \operatorname{Hom}_R(M,N)$$

$$\partial(f) = \partial^N f - (-1)^{n+1} f \partial^M.$$

Check: $\partial^2 = 0$.

Observe: $Z_0(\operatorname{Hom}_R(M,N)) = \operatorname{Hom}_{\mathcal{C}}(M,N)$.

Def. $f, g \in \operatorname{Hom}_{\mathcal{C}}(M,N)$ are homotopic if
$f - g \in B_0(\operatorname{Hom}_R(M,N))$, i.e.,
$f - g = \partial h$ for some $h \in \operatorname{Hom}_R(M,N)_1$

$K(R) := \mathcal{C}(R) / \text{homotopy relation}.$

Object = R-complexes

$\mathrm{Hom}_K(M,N) = H_0(\mathrm{Hom}_R(M,N)).$

- $f \sim g$ in $\mathcal{C}$ $\Rightarrow$ $f = g$ in $K(R)$.

  $\Rightarrow H(f) = H(g)$

Def^n M an R-complex.

ΣM (or $M[1]$) is the R-complex

$$(\Sigma M)_i = M_{i-1}$$

with $\partial^{\Sigma M} = -\partial^M$.

$\mathrm{Proj}\, R :=$ Projective R-modules

$$\begin{tikzcd}
K(R) \arrow[dr, "q = \text{localisation}"] & \\
\cup \\
K(\mathrm{Proj}\, R) \arrow[r, "q"'] & D(R) \arrow[l, dashed, bend right, "p"']
\end{tikzcd}$$

$\exists p : D(R) \to K(\mathrm{Proj}\, R)$, a full and faithful embedding

"projective resolutions"

left adjoint to q.

$$\mathrm{Hom}_K(pM, N) = \mathrm{Hom}_D(M, qN)$$

$f : M \to N$ morphism

$\mathrm{cone}(f) := N \oplus \Sigma M$ with differential

$$\begin{matrix} N_{i+1} \\ \oplus \\ M_i \end{matrix} \longrightarrow \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \qquad \partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$$0 \to N \hookrightarrow \mathrm{cone}(f) \to \Sigma M \to 0.$$

f is w.e. $\iff$ $H(\mathrm{cone}(f)) = 0$.

Image of p?

K-projectives.

P an R-complex is K-projective if given any solid diagram

$$\begin{tikzcd} & M \arrow[d, "\pi", "?"'] & \text{w.e.} \\ P \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & N \end{tikzcd}$$

, $\exists$ lift $\tilde{\alpha}$.

FACT: $p : D(R) \xrightarrow{\sim} K\text{-}\mathrm{Proj}(R) \subseteq K(\mathrm{Proj}\ R)$.

morphisms here are up to homotopy

- $\mathrm{Hom}_{\mathcal{C}}(R, M) = Z_0(M)$

$$\begin{tikzcd} 0 \arrow[d] & \\ R \arrow[r] \arrow[d] & M_0 \arrow[d, "\partial"] \\ 0 \arrow[r] & M_{-1} \end{tikzcd}$$

Using this,

check : R is K-projective.

Use:
surjective + w.e.
$\Downarrow$
$Z(M) \to Z(N)$ onto

- $(P_\lambda)_\lambda$ family of K-projectives

  Then, $\bigoplus P_\lambda$ is also K-projective.

  Conversely closed under direct summands.

- K-projectives are closed under suspensions.

  $$\cdots \xrightarrow{\partial} P_{i+1} \xrightarrow{\partial} P_i \xrightarrow{\partial} P_{i-1} \xrightarrow{\partial} \cdots$$

  is K-proj, if P_i projective $\forall i$.

Ex: $0 \to P' \xrightarrow{f} P \xrightarrow{g} P'' \to 0$ exact seq. of complexes.

Then, if P' and P'' are K-projective, so is P.

If P', P, P'' are complexes of projectives, then any two being K-projective $\Rightarrow$ third is K-proj.

any two being K-projective $\Rightarrow$ third is K-proj.

Corollary. Any bounded complex of projectives is K-projective.

Proof. $P_\cdot : 0 \to P_b \to \cdots \to P_a \to 0$, each P_i proj.

Induce on $b-a$.

$b-a=0$ done earlier.

$$0 \to P_{\le b-1} \to P \to \Sigma^b P_b \to 0. \quad \blacksquare$$

$$P_{\le b-1} = 0 \to P_{b-1} \to \cdots \to P_a \to 0$$

Next: Any complex of projectives with $P_i = 0 \ \forall i \ll 0$ is K-projective.

$$P_\cdot : \quad \cdots \to P_{a+1} \to P_a \to 0$$

$P = \underset{n \geq a}{\underrightarrow{\operatorname{colim}}}\, P_{\le n}$, each $P_{\le n}$ is projective since bounded.

$$0 \to \bigoplus_n P_{\le n} \xrightarrow{1-\delta} \bigoplus_n P_{\le n} \to P \to 0.$$

K-proj

projectives

Use 2-out-of-3.

Do directly...

- $M \xrightarrow[\text{w.e.}]{\pi} N$

  P is K-projective

  $\Updownarrow$

  $\operatorname{Hom}_K(P, M) \cong \operatorname{Hom}_K(P, N)$.

Lecture 2 (11-01-2023)

11 January 2023 13:24

R ring.

$D(R) \simeq K\mathrm{Proj}(R)$.

Recall: $P \in \mathcal{C}(\mathrm{Proj}\ R)$ is K-projective if

$$\begin{tikzcd} & X \arrow[d, "\simeq"] \\ P \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Example $P \in \mathcal{C}(\mathrm{Proj}\ R)$ with $P_i = 0$ for all $i \ll 0$.

$$\cdots \to P_{a-1} \to P_a \to 0 \to 0 \to \cdots$$

Sketch. Construct liftings one step at a time.

Suppose $\tilde{\alpha} : P_{\le n} \to X$ is a lifting.

Want $\tilde{\alpha}: P_{\le n+1} \to X$ compatibly.

Let $s \in P_{n+1}$

We must have

- $\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$
- $\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$

$$\begin{tikzcd} & & & X \arrow[d, "\varepsilon"', "\simeq"] \\ \cdots \arrow[r] & P_{n+1} \arrow[r] & (P_n \to P_{n-1} \to \cdots \to P_a \to 0) \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

Check that the above can be solved.

This uses three things:
- ε surjective $\Rightarrow$ ε surjective on boundaries
- $H(\varepsilon)$ iso $\Rightarrow$ ε surjective on cycles
- $\ker(\varepsilon)$ is acyclic.

$\Rightarrow$ Every module has a K-projective resolution.

$$\begin{tikzcd} F: \cdots & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[dr, "\varepsilon"] & \\ & & \ker \partial_1 \arrow[ur] & & \ker \varepsilon \arrow[ur] & & M \end{tikzcd}$$

(Can avoid choices by taking generating set to be M, $\ker\varepsilon$, $\ker\partial_1, \ldots$)

$$F \xrightarrow{\varepsilon} M.$$

Def^n A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism $\varepsilon : P_\cdot \to M$ s.t.

① ε is a quasi iso, (Not insisting surjective.)

② $P_\cdot$ is K-projective.

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This makes it functorial!

Thm $\forall M_\cdot \in \mathcal{C}(R)$, $\exists$ surjective K-projective resolution:

$$P_\cdot \xrightarrow{\sim} M_\cdot$$

K-proj.

Defn. An R-complex F is semi-free if F admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \cdots \subseteq \bigcup_{n \geq 0} F(n) = F$$

s.t. ① $F(n) \subseteq F$ is a subcomplex,

② $F(n+1)/F(n)$ graded free module with $\partial = 0$,

i.e. $\partial(F(n+1)) \subseteq F(n)$

Fact/Ex. Semifree $\Rightarrow$ K-proj.

Example. $\cdots \to F_{a+1} \to F_a \to 0$

$$F(n) = F_{\leq n} = 0 \to F_n \to \cdots \to F_a \to 0$$

$$\frac{F(n+1)}{F(n)} = \cdots \to 0 \to F(n+1) \to 0 \to \cdots$$

$$= \Sigma^{n+1} F_{n+1}.$$

Thm. Each $M_\cdot \in \mathcal{C}(R)$ has a surjective semi-free resolution

$$F_\cdot \xrightarrow{\sim} M_\cdot.$$

K-projective

Corollary. Every K-projective is a retract of a semifree.

Proof. P is K-projective:

$$\begin{tikzcd} & F \arrow[d, "\wr"] \\ P \arrow[ur, dashed] \arrow[r, equal, "id"'] & P \end{tikzcd}$$

$\square$

Sketch (Baby ver of Quillen's "small object argument"):

Construct $0 = F(0) \hookrightarrow F(1) \xrightarrow{i(1)} F(2) \xrightarrow{i(2)} \cdots \hookrightarrow F(n) \xrightarrow{i(n)} \cdots \hookrightarrow \bigcup F(n) = F$

Construct

$$0 = F(0) \hookrightarrow F(1) \xrightarrow{i(1)} F(2) \overset{i(2)}{\hookrightarrow} \cdots \hookrightarrow F(n) \overset{i(n)}{\hookrightarrow} \cdots \hookrightarrow \bigcup_{n\geq 0} F(n) = F$$

with $\varepsilon(0): F(0) \to M$, $\varepsilon(1): F(1) \to M$, $\varepsilon(2)$, $\varepsilon(n)$.

s.t. ① $F(n+1)/F(n)$ is graded free with $\partial = 0$,

② $\varepsilon(1)$ is surjective on homology.

(In turn, each $\varepsilon(n)$ is surjective on homology.)

③ $\ker(H(\varepsilon(n))) \subseteq H(F(n))$ maps to 0 under $H(i(n))$.

This does the job. ["Something 0 in colim is 0 at finite stage."]

Why is ε surjective?

Remark. $\varepsilon: X \to Y$ s.t.

$Z(\varepsilon)$ surjective + $H(\varepsilon)$ bijective.

Then, ε is surjective.

Indeed, we have s.e.s.es:

$$\begin{array}{ccccccccc} 0 & \to & B(X) & \to & Z(X) & \to & H(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B(Y) & \to & Z(Y) & \to & H(Y) & \to & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \to & Z(X) & \to & X & \to & \Sigma B(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z(Y) & \to & Y & \to & \Sigma B(Y) & \to & 0 \end{array}$$

Snake lemma
$\Downarrow$
$B(X) \to B(Y)$ is epi
$\Downarrow$
$X \to Y$ is surj

Construction of $F(n)$, $\varepsilon(n)$:

$\varepsilon(1): F(1) \to M$ free cover of cycles.

0 diffs

($\varepsilon(1)$ morphism since mapping on cycles)

Say we have constructed $\varepsilon(n): F(n) \longrightarrow M$.

Choose cycles $(z_\lambda)_\lambda \subseteq F(n)$ that map to a generating set of $\ker(H(\varepsilon(n)))$

Pick w_λ s.t. $\partial(w_\lambda) = \varepsilon(n)(z_\lambda)$.

Set $F(n+1) = F(n) \oplus Re_\lambda$ $\qquad \deg(e_\lambda) = \deg(z_\lambda)+1$.

with $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_\lambda) = z_\lambda.$$

Define $\varepsilon(n+1) : F(n+1) \longrightarrow M$

$$\varepsilon(n+1)|_{F(n)} = \varepsilon(n),$$

$$\varepsilon(n+1)(e_\lambda) = \omega_\lambda.$$

$\boxtimes$

Remarks. ① As before the above construction can be made functorial by avoiding choices (consider all choices!).

② Depending on what we wish to do with the resolution, there are other constructions.

Given a module, we have the graded homology module $H(M) = \langle H_i(M)\rangle_{i\in\mathbb{Z}}$.

(Recall: for us, a graded module is a collection of modules.)

If $P_\cdot \xrightarrow{\sim} H(M)$ is a projective res^n, one can "perturb" the differentials of $P_\cdot$ to construct a K-projective resolution of M.

(Adam's resolution, Cartan/Eilenberg resolution)

Exercise. $P_\cdot$ K-proj $\Rightarrow$ P_i projective $\forall i$.

Converse of above NOT true.

Example (Dold's): Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the complex

$$P_\cdot : \quad \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} \cdots$$

One way of seeing that the above is not K-projective is to do the following exercise and note that $P_\cdot$ is acyclic but not contractible.

is to do the following exercise and ...

$P_\cdot$ is acyclic but not contractible.

Exercise. If P is K-proj and $H(P) = 0$, then P is contractible, i.e., $\mathrm{id}_P \sim 0$.

(or: P is the mapping cone of some id_C.)

We saw we have an inclusion

$$K\mathrm{Proj}(R) \hookrightarrow K(\mathrm{Proj}\ R).$$

FACT. Let R be comm. Noetherian.
The above inclusion is an equality iff R is regular.

Examples of reg. rings: $\mathbb{Z}$, $k[x_1, \dots, x_n]$.

Derived functors

Let $M_\cdot$ be an R-complex.

FACT. If $P_\cdot \xrightarrow{\sim} M_\cdot$ and $Q_\cdot \xrightarrow{\sim} M_\cdot$ are K-projective resolutions, then $P_\cdot \simeq Q_\cdot$ in $K(\mathrm{Proj}\ R)$.

Given any $N_\cdot \in \mathcal{C}(R)$, set

$$\mathrm{RHom}_R(M_\cdot, N_\cdot) := \mathrm{Hom}_R(P_\cdot, N_\cdot),$$

where $P_\cdot$ is a K-proj. res^n of $M_\cdot$

The object on the right is defined in the homotopy category of abelian groups, i.e.,

$\mathrm{RHom}_R(-, N_\cdot)$ is a functor

$\mathcal{C}(R) \to K(\mathbb{Z})$.

$\mathbf{R}\mathrm{Hom}_R(-, N.)$ is a functor

$$\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$$

(If R is comm, then $\longrightarrow K(R)$.)

Define $\mathrm{Ext}^i_R(M., N.) := H^i(\mathbf{R}\mathrm{Hom}_R(M., N.))$
$= H_{-i}(\mathrm{Hom}_R(P., N.))$

$\mathrm{Ext}^0_R(M, N) = H_0(\mathrm{Hom}_R(P., N.))$
$=$ morphisms $P. \to N.$, up to homotopy

$\mathrm{Ext}^i_R(M, N) =$ — " — $P. \to \Sigma^i N.$, — " —

If $Q \xrightarrow{\sim} N$ is a K-proj resol^n, then
$\mathrm{Hom}_R(P., Q.) \cong \mathrm{Hom}_R(P., N.)$.
quasi-iso

Tensors Let $M.$ be a chain complex of <u>right</u> R-modules.

Let $N. \in \mathcal{C}(R)$.

$M. \otimes N.$ is a complex of $\mathbb{Z}$-modules defined by

$$(M. \otimes N.)_i = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-1)^{|m|} m \otimes \partial n$$

FACT. If $X. \xrightarrow{\sim} Y.$ is a quasi iso,
then $P. \otimes_R X. \xrightarrow{\sim} P. \otimes Y.$ for any K-proj $P.$

Defn. $M_\cdot \otimes_R^L N_\cdot := P_\cdot \otimes_R N_\cdot$ where $P_\cdot \xrightarrow{\sim} M_\cdot$ is a K-proj. resol^n.

$$\mathrm{Tor}_i^R(M_\cdot, N_\cdot) = H_i(P_\cdot \otimes_R N_\cdot).$$

$X \xrightarrow{\sim} Y$ quasi iso $\Rightarrow \mathrm{Tor}_i^R(M_\cdot, X_\cdot) = \mathrm{Tor}_i^R(M_\cdot, Y_\cdot)$.

Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \to$ comm. Noetherian ring

$M \to$ R-module

$r \in R$ is a zerodivisor on M if $r \cdot m = 0$ for some $m \neq 0$.

nzd = "not a zero divisor"

$$Z_R(M) = \{ r \in R : r \text{ is a z.d. on } M \}$$

$$= \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

(M need not be finite. Union need not be)

Fix R, M. Let $\underline{x} = x_1, \dots, x_n$ be a sequence in R.

$\underline{x}$ is weakly M-regular or a weakly regular sequence on M

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \dots, x_i)M} \quad \text{for } 0 \leq i \leq n-1.$$

$\underline{x}$ is M-regular (or ...) if further $M/(\underline{x})M \neq 0$.

Ex. $R = k[x_1, \dots, x_n]$.
$\underline{x} := x_1, \dots, x_n$ is a regular sequence on R.

<u>Koszul Complexes</u>. Given $r \in R$,

$$K(r; R) = 0 \to \underset{\text{deg } 1}{R} \xrightarrow{r} \underset{\text{deg } 0}{R} \to 0.$$

$H_1(K(r;R)) = 0 \iff r$ is nzd on R.

Given $\underline{x} = x_1, \dots, x_n$, we define

$$K(\underline{x}; R) = \bigotimes_{i=1}^{n} K(x_i; R).$$

$$K(\underline{x}; R) = 0 \to R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \dots \to R^{\binom{n}{2}} \to R^n \xrightarrow{(x_1 \cdots x_n)} R \to 0$$

$$K(\underline{x};R) = 0 \to R \xrightarrow{\pm\begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{\cdot(x_1 \cdots x_n)} R \to 0$$

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mapsto \sum r_i x_i$$

Now, given $M \in \mathcal{C}(R)$,

$$K(\underline{x};M) := K(\underline{x},R) \otimes M.$$

↳ Koszul complex on $\underline{x}$ with coefficients in M.

$$H_i(\underline{x};M) := H_i(K(\underline{x};M)).$$ → Koszul homology

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If M is simply an R-module (viewed in degree 0), then

$K(\underline{x};M)$:

$$0 \to M \to M^n \to \cdots \to M^n \to M \to 0$$

"same" differentials

$$H_0(\underline{x};M) = M/\underline{x}M,$$

$$H_n(\underline{x};M) = \{m \in M : x_i m = 0 \ \forall i\}$$
$$= (0 :_M (\underline{x})).$$

---

① $$K(\underline{x};M) = K(x_1;R) \otimes_R K(x_2;R) \otimes_R \cdots \otimes K(x_n;R) \otimes M$$
$$= K(x_1;R) \otimes K(x_{\geq 2}, M)$$
$$= K(x_1 ; K(x_{\geq 2}, M))$$

② $X, Y \in \mathcal{C}(R) \rightsquigarrow X \otimes_R Y \xrightarrow{\sim} Y \otimes_R X$ as R-complexes.

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

$\therefore$ $K(\underline{x}; R) \cong K(\underline{x}^\sigma; R)$ for any $\sigma \in S_n$.

$\Rightarrow$ $K(\underline{x}; M) \cong K(\underline{x}^\sigma; M)$ — " —.

(Can apply this to Obs ①.)

2nd Perspective: "Koszul complexes are iterative mapping cones."

$f: X \to Y$ morphism of complexes

$$\mathrm{Cone}(f) = \left(Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ & -\partial^X \end{pmatrix}\right).$$

s.e.s.: $0 \to Y \to \mathrm{Cone}(f) \to \Sigma X \to 0.$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$

$\begin{pmatrix} y \\ x \end{pmatrix} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow[H_i(f)]{} H_i(Y) \to H_i(\mathrm{Cone}(f)) \to \underset{\substack{\cong \\ H_{i-1}(X)}}{H_i(\Sigma X)} \to \cdots$$

connecting map

Consider: $x \in R$

$f: R \xrightarrow{x} R$

$1 \mapsto x$

$$\mathrm{Cone}(R \xrightarrow{x} R) = (\underset{\deg 0}{R} \oplus \underset{\deg 1}{R}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix})$$

$= K(x; R)$.

Ditto: If $x \in R$, and $M \in \mathcal{C}(R)$ → complex, not necessarily in deg 0

$$\mathrm{Cone}(M \xrightarrow{x} M) = K(x; M)$$

$\underline{x} = x_1, x_2, \ldots, x_n$

$$K(\underline{x}; M) = K(x_1 ; K(x_{\geq 2}; M))$$
$$= \mathrm{Cone}\left(K(x_{\geq 2}; M) \xrightarrow{x_1} K(x_{\geq 2}; M)\right)$$

on homology

iterate
⋮

$$H_i(x_{\geq 2}; M) \xrightarrow{\pm x_1} H_i(x_{\geq 2}; M) \to H_i(\underline{x}; M)$$
$$\downarrow$$
$$H_{i-1}(x_{\geq 2}; M)$$
$$\downarrow \pm x_1$$
$$\vdots$$

s.e.s.

$$0 \to \frac{H_i(x_{\geq 2}; M)}{x_1 H_i(x_{\geq 2}; M)} \to H_i(\underline{x}; M) \to (0 :_{H_{i-1}(x_{\geq 2}; M)} x_1) \to 0$$

$M \to$ R-module

$$K(\underline{x}; M) \rightsquigarrow H_0(x; M) = \frac{M}{\underline{x}M}$$

So, $K(\underline{x}, M) \to \dfrac{M}{\underline{x}M}$ is a w.e. quasi iso

$\Leftrightarrow H_i(\underline{x}; M) = 0 \ \forall i > 0.$

Defn. $\underline{x}$ is Koszul-regular on M (or...) if

$$H_i(\underline{x}; M) = 0 \ \forall i \geq 1.$$

Lemma. When $\underline{x}$ is weakly M-reg, ($M \to$ R-mod)

(weakly-reg $\Rightarrow$ Koszul-reg)

$K(\underline{x}, M) \twoheadrightarrow M/(\underline{x})M$ is a w.eq.

Proof. $n=1$: $0 \to M \xrightarrow{x} M \to 0.$

$H_1(x; M) = 0 \iff x$ nzd on M

$n \geq 2$: $K(\underline{x}; M) = K(x_n ; K(x_{<n}; M)).$

By induction,

$$K(x_{<n}; M) \xrightarrow{\sim} \frac{M}{(x_{<n})M}.$$

Now, $0 \to R \xrightarrow{x_n} R \to 0$ is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x}; M) = K(x_n; R) \otimes_R K(x_{<n}; M)$$

$$\simeq K(x_n; R) \otimes_R \frac{M}{(x_{<n})M}$$

now note that x_n is a nzd on $M/(x_{<n})M$ and we are done. ∎

Instead of semifree, can use l.e.s. of homology and induction.

(Semifree business $\Rightarrow$ ($M \simeq N$ quasi $\Downarrow$ $K(\underline{x}; M) \simeq K(\underline{x}; N)$ quasi))

Note: ① $\underline{x}$ Koszul-reg $\Rightarrow$ $\underline{x}^\sigma$ is Koszul-reg $\forall \sigma \in S_n$.

② Not true for weakly regular.

Theorem. Say $\underline{x} \subseteq J(R)$ and $M \overset{\neq 0}{}$ f.g. R-module.

TFAE:

1) $\underline{x}$ is M-regular. ($\equiv$ weakly M-reg, by NAK.)

2) $\underline{x}$ is Koszul M-regular, i.e., $H_i(\underline{x}; M) = 0$ $\forall i \geq 1$.

3) $H_1(\underline{x}; M) = 0$.

In particular, take R local and x_i nonunits.

Proof. ① $\Rightarrow$ ② $\Rightarrow$ ③ is clear.

Only need to prove ③ $\Rightarrow$ ①.

Already saw for $n=1$.

Induction:

$$K(\underline{x}; M) = K(x_n; K(x_{<n}; M)).$$

l.e.s.

$$0 \to \frac{H_i(\underline{x}_{<n};M)}{x_n H_i(\underline{x}_{<n};M)} \to H_i(\underline{x};M) \to (0 :_{H_{i-1}(\underline{x}_{<n};M)} x_n) \to 0. \quad (*)$$

Put $i=1$ to get $\dfrac{H_1(\underline{x}_{<n};M)}{x_n H_1(\underline{x}_{<n};M)} = 0$

$$\xRightarrow{\text{NAK}} H_1(\underline{x}_{<n};M) = 0.$$

(Note: Koszul homology modules are f.g. when M is f.g.)

$$\xRightarrow{\text{induction}} x_1, \ldots, x_{n-1} \text{ is } M\text{-reg.} \quad — (1)$$

Moreover, $(*)$ now tells us

$$(0 :_{H_0(\underline{x}_{<n};M)} x_n) = 0.$$

$$\| $$

$$\ker\left(\frac{M}{\underline{x}_{<n} M} \xrightarrow{x_n} \frac{M}{\underline{x}_{<n} M}\right).$$

$\therefore x_n$ is nzd on $M/(\underline{x}_{<n})M$. — (2)

(1) & (2) finish. ∎

Corollary. $I \subseteq J(R)$, M f.g., the property of $\underline{x}$ being regular is not dependent on the order of x_i.

(Permutation of regular is regular.)

$R = k[x, y, z]$

$x, \; y(1-x), \; z(1-x)$ reg

$y(1-x), \; z(1-x), \; x$ NOT

Lemma. If $\underline{x} = x_1, \ldots, x_n \subseteq R$, M an R-module.

TFAE:

① $\underline{x}$ is M-Koszul-regular.

② $\underline{x}^{\underline{a}}$ is M-Koszul-regular for some $\underline{a} \geq \underline{1}$.

Proof. Suffices to prove:

$x_1, \boxed{x_2, \ldots, x_n}$ is M-KR

$\Leftrightarrow$ $x_1^a, \boxed{x_2, \ldots, x_n}$ is M-KR for some (/all) $a \geq 1$.

$x_1, x_2, \ldots, x_n$ KR

$\Rightarrow K(x_1 ; K(x_{\geq 2} ; M)) \xrightarrow{\sim} K\left(x_1 ; \dfrac{M}{(x_{\geq 2})M}\right)$.

Replacing M by $M/(x_{\geq 2})M$ we are reduced to $n = 1$.

But

x is reg on M

$\Leftrightarrow$ x is nzd on M

$\Leftrightarrow$ x^a is nzd on M for some $a \geq 1$

$\Leftrightarrow$ x^a is reg on M —— " ——.

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$ and M f.g. R-module.

Let $i \geq 0$ be s.t. $H_i(\underline{x} ; M) = 0$.

Then, $H_j(\underline{x} ; M) = 0 \quad \forall\, j \geq i$.

HW.

Lecture 4 (23-01-2023)

Monday, January 23, 2023 1:19 PM

$R \to$ commutative Noetherian

Given complexes, $M, N \in \mathcal{C}(R)$.

$$RHom_R(M,N) := Hom_R(pM, N)$$

$pM \xrightarrow{\sim} M$ is a K-proj res^n

$$Ext^*_R(M,N) := H^i(RHom_R(pM,N)).$$

Given $M, N, P \in \mathcal{C}(R)$, we have a morphism

$$\theta : RHom_R(M,N) \otimes^L_R P \longrightarrow RHom_R(M, N \otimes^L_R P).$$

Lemma. θ is a w.e. when P is perfect, i.e.,

$$P \simeq (0 \to P_0 \to \cdots \to P_n \to 0),$$

each P_i f.g. R-module.

$$Hom_R(pM, N) \otimes_R pP \longrightarrow Hom_R(pM, N \otimes_R pP)$$

$$f \otimes x \longmapsto m \mapsto (-1)^{|m||x|} f(m) \otimes x.$$

in $\mathcal{C}(R)$

Check this is a morphism.

Proof. Replace P with $0 \to P_0 \to \cdots \to P_n \to 0$.

"Things true for $R \rightsquigarrow$ true for perfect."

Then it reduces to checking for one f.g. projective P.

But that can be reduced to f.g. free.

That reduces to R. Obvious. ∎

Rees' Lemma

Setup. $\underline{x} = x_1, \dots, x_c \subset R$ finite subset.

$N \rightsquigarrow R$-module s.t. $\underline{x} N = 0$.

$M \rightsquigarrow R$-module s.t. $\underline{x}$ is Koszul-regular on M.

i.e. $K(\underline{x}; M) \xrightarrow{\sim} M/\underline{x}M$

Lemma. Then, $\mathrm{RHom}_R(N, M/\underline{x}M) \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

In particular, $\mathrm{Ext}^*_R\left(N, \frac{M}{\underline{x}M}\right) \cong \mathrm{Ext}^*_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

$\Lambda^*(\Sigma R^c) : 0 \to \underset{\deg c}{R} \xrightarrow{0} R^c \xrightarrow{0} \cdots \xrightarrow{0} R^{\binom{c}{2}} \xrightarrow{0} R^c \to \underset{\deg 0}{R} \to 0$

Proof.

$$\begin{aligned}
\mathrm{RHom}_R(N, M/\underline{x}M) &\simeq \mathrm{RHom}_R(N, K(\underline{x}; M)) \\
&\simeq \mathrm{RHom}_R(N, M \otimes^L_R K(\underline{x}; R)) \\
&\simeq \mathrm{RHom}_R(N, M) \otimes^L_R K(\underline{x}; R)
\end{aligned}$$

perfect

Since $\underline{x} N = 0$, $\underline{x}\, \mathrm{Ext}^*_R(N, M) = 0$.

Alter: $\mathrm{RHom}_R(N, M) \simeq \mathrm{Hom}_R(N, I)$ where $M \xrightarrow{\simeq} I$ is an injective resol^n.

Now, $\underline{x} \cdot \mathrm{Hom}_R(N, I) = 0$.

$\therefore$

$$\begin{aligned}
&\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{x}; R) \\
&\simeq K(\underline{x}; \mathrm{Hom}_R(N, I)) \\
&\simeq K(\underset{\text{length } c}{\underline{0}}; \mathrm{Hom}_R(N, I)) \\
&\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{0}; R)
\end{aligned}$$

$$\cong \mathrm{Hom}_R(N, I) \otimes_R K(\underline{x}; R)$$

$$= \mathrm{Hom}_R(N, I) \otimes_R \Lambda^*(\Sigma R^c).$$

Last statement means:

$$\mathrm{Ext}^n_R\left(N, \frac{M}{\underline{x}M}\right) \cong \left(\mathrm{Ext}_R(N, M) \otimes_R \Lambda(\Sigma R^c)\right)^n$$

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R \left(\Lambda(\Sigma R^c)\right)^{n-i}$$ } think

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R R^{\binom{c}{i-n}}.$$

$V \to \mathbb{Z}$-graded object

View V as having both an upper grading and lower grading via $V^i = V_{-i}$.

Notation:

$$\sup V^* = \sup\{i : V^i \neq 0\}.$$

$$\inf V^* = \inf\{i : V^i \neq 0\}.$$

$$\sup V_* = \sup\{i : V_i \neq 0\} \quad \ldots$$

$$= -\inf V^*$$

Corollary. $\rho := \inf \mathrm{Ext}^*_R(N, M) = \inf \mathrm{Ext}^*_R(N, M/\underline{x}M) + c.$

$$\mathrm{Ext}^{\rho}_R(N, M) = \mathrm{Ext}^{\rho-c}_R(N, M/\underline{x}M)$$

Defn. Fix $I \subseteq R$ ideal. $M \in \mathcal{C}(R)$.

$$\mathrm{depth}_R(I, M) := \inf \mathrm{Ext}^*(R/I, M).$$

I-depth of M

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences of complexes on . .

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences (of complexes or modules).

Then, $\operatorname{depth}_R(I, M) \geq \min\{\operatorname{depth}_R(I,L), \operatorname{depth}_R(I,N)\}$.

Similarly relations with other twos using l.e.s.

$$\to \operatorname{Ext}^i_R(R/I, L) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, N)$$
$$\to \operatorname{Ext}^{i+1}_R(R/I, L) \to \cdots$$

Thm (depth).
② Let $\underline{x} = x_1, \dots, x_c$ be a gen. set for the ideal I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M) \qquad \forall M \in \mathcal{C}(R).$$

(recall Koszul homology)

Will prove when M is an R-module.

Koszul complexes revisited

Giving $x_1, \dots, x_c$ in R

$\leftrightarrow$ Giving a map $f : F \to C$, where F is a free mod of rank c and chosen basis

$K(f) := (\Lambda^*(\Sigma F), \partial)$, where ∂ is

$$e_{i_1} \wedge \cdots \wedge e_{i_n} \longmapsto \sum_{j=1}^{n} (-1)^{j-1} f(e_{i_j})\, x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_n}.$$

Lemma. Fix $\underline{x} = x_1, \dots, x_c \subseteq R$. For any $y \in (\underline{x})$, we have

$$K(\underbrace{\underline{x}, y}_{c+1 \text{ seq.}}; M) \cong K(\underline{x}, 0; M)$$
$$\cong K(\underline{x}; M) \otimes (0 \to R \to \underset{\substack{\uparrow \\ \deg 0}}{R} \to 0).$$

Proof.

$$R^{c+1} \xrightarrow{[x_1 \; \cdots \; x_c \;\; y]} R$$

$$\uparrow \cong \qquad \circlearrowleft \qquad ||$$

$y = \sum r_i x_i$

$$\begin{array}{ccc} \left[\begin{smallmatrix} \ddots & & -x_1 \\ & & \vdots \\ & & -x_c \\ & & 1 \end{smallmatrix}\right] \Big\uparrow \cong & & \Vert \\ R^{c+1} & \xrightarrow[{[x_1 \cdots x_c \ 0]}]{} & R \end{array} \qquad y = \sum r_i x_i$$

In particular,

$$\sup H_*(\underline{x}, y; M) = 1 + \sup H_*(\underline{x}; M)$$

Thus,

$$c+1 - \sup H_*(\underline{x}, y; M) = c - \sup H_*(\underline{x}; M).$$

Corollary. $c - \sup_* H(\underline{x}, y; M)$ is independent of a generating set of I. (If $\underline{x}$, $\underline{y}$ generate, look at $\underline{x}, \underline{y}$ & $\underline{y}, \underline{x}$.)

Proof of Thm (Depth) When M is a module

$$\begin{aligned} \operatorname{depth}(I, M) = 0 &\iff \operatorname{Hom}(R/I, M) \neq 0 \\ &\iff I \subseteq \operatorname{Zdv}_R(M) \\ &\iff H_c(\underline{x}; M) \neq 0 \\ &\iff \sup H_*(\underline{x}; M) = c. \end{aligned}$$

Actually, $H_c(\underline{x}; M) \cong \operatorname{Hom}(R/I, M)$

Can assume $\operatorname{depth}_R(I, M) \geq 1$, i.e., $\exists y \in I$ nzd on M

Then, in particular, y is Koszul-reg on M.

By Rees, $\because y(R/I) = 0$

$$\inf \operatorname{Ext}_R^*(R/I, M/yM) = \inf \operatorname{Ext}_R^*(R/I, M) - 1.$$

$$\therefore \operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R\left(I, \frac{M}{yM}\right)$$

induction now applies

$$= 1 + \left(c - \sup H_*\left(\underline{x}; \frac{M}{yM}\right)\right)$$

$H_*(\underline{x}; M/yM)$

$($ $yM))$

$$H_*(\underline{x}; M/yM) \cong H_*(\underline{x}; K(y;M)) \cong H_*(K(\underline{x}, y; M)) \cong H_*(K(\underline{x}, 0; M))$$

$$= c + 1 - \sup H_*(\underline{x}, 0; M)$$

$$= c - \sup H_*(\underline{x}; M).$$

$\boxtimes$

Ex. Show $\operatorname{depth}_R(I, M)$ is the length of longest ___ M-regular in I.

Lecture 5 (25-01-2023)

Wednesday, January 25, 2023 1:19 PM

$R \to$ comm. noetherian ring

$I \subseteq R$ ideal

$M \to$ R-complex

$$\operatorname{depth}(I, M) = \inf \operatorname{Ext}^*_R(R/I, M)$$

$I = (x_1, \ldots, x_c)$

$$= c - \sup H_*(\underline{x}; M)$$

- When M is a module, $\operatorname{depth}(I, M)$ = length of any maximal M-Koszul-regular sequence in I

(Maybe $IM \neq M$ needed?)

If M f.g. and $I \subseteq \operatorname{Jac}(R)$

= length of any maximal M-reg. seq in I

<u>Observation</u>. $\underline{x} = x_1, \ldots, x_c$

$\underline{y} = y_1, \ldots, y_d$

$$\sup H_*(\underline{x}, \underline{y}; M) \le \sup H_*(\underline{x}; M) + d.$$

(Can use l.e.s. to see this.)

$K(\underline{x}, \underline{y}; M) \cong K(\underline{x}; K(\underline{y}; M))$

This implies

$I \subseteq J \Rightarrow \operatorname{depth}_R(I, M) \le \operatorname{depth}_R(J, M)$.

Also, $\sqrt{I} = \sqrt{J} \Rightarrow \operatorname{depth}_R(I, M) = \operatorname{depth}_R(J, M)$.

<u>Today</u>. $(R, \mathfrak{m}, k)$ is a local ring:

- R is commutative Noetherian,
- $\mathfrak{m}$ is the unique maximal ideal of R,

- $k = R/\mathfrak{m}$.

In this case, $\operatorname{depth}(M) = \mathfrak{m}\text{-depth of } M$

$$= \inf \operatorname{Ext}^*_R(k, M).$$

It suffices to compute the above using $\underline{x} = x_1, \dots, x_c$

s.t. $\sqrt{(\underline{x})} = \mathfrak{m}$.

Thus we can take c minimal as $c = \dim(R)$.

(Then, $\underline{x}$ is a system of parameters.)

$\therefore$ Can compute using $\dim(R)$ elements.

×

Ausland-Buchsbaum Equality

- $F \to$ an R-complex

F has finite flat dimension if

$$F \simeq (0 \to F_b \to \cdots \to F_a \to 0)$$

with F_i flat.

We write $\operatorname{flat\,dim}_R F < \infty$.

Examples
- Flat modules.
- Perfect complexes.
- Koszul complexes.

If $\operatorname{flat\,dim}_R F < \infty$, then $\operatorname{Tor}_i(-, F) = 0 \quad \forall |i| \gg 0$

on $\operatorname{Mod} R$.

$$\operatorname{Tor}_i^R(M, F) = H_i\left(M \otimes_R (0 \to F_b \to \cdots \to F_a \to 0)\right)$$

$$= 0 \qquad \text{for } i \notin [a, b].$$

In fact, the above characterises flat $\dim_R F < \infty$.

Theorem (AB equality) $(R, \mathfrak{m}, k)$ local.

$F \to$ finite flat dimension. Then,

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R M - \underbrace{\sup H_*(k \otimes_R^L F)}_{\operatorname{Tor}_*^R(k, F)}.$$

for ANY R-complex M.

Specialise: ① $M = R$.

$$\operatorname{depth}_R(F) = \operatorname{depth}_R(R) - \sup H_*(k \otimes_R^L F).$$

Now, let N be a f.g. R-module.

Such an N has a minimal free resolution.

$$\begin{tikzcd}
\cdots R^{b_2} \arrow[r, "\partial_2"] \arrow[d] & R^{b_1} \arrow[r, "\partial_1"] \arrow[d, "\min"] & R^{b_0} \arrow[r, two heads, "\varepsilon"', "\text{minimally}"] & N \\
\ker \partial_1 \subseteq \mathfrak{m} R^{b_1} \arrow[ur] & \ker \varepsilon \subseteq \mathfrak{m} R^{b_0}, \text{ by minimality} \arrow[ur] & &
\end{tikzcd}$$

This gives a complex

$$G : (\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to 0)$$
$$\downarrow \simeq$$
$$N$$

$\partial G \subseteq \mathfrak{m} G$. G turns out to be unique up to isomorphism of complexes.

up to isomorphism of complexes.

"The" minimal free resolution of N.

$$\mathrm{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes G)_i.$$

H_i = everything since $\partial \otimes k = 0$

$$\therefore \quad \mathrm{Tor}_i^R(k, N) = 0 \iff G_i = 0.$$

$\therefore$ flat $\dim_R N < \infty \iff N$ has a finite free resolution, i.e., N is perfect.

$$\sup \mathrm{Tor}_*^R(k, N) = \text{length of } G$$

$$=: \text{proj} \dim_R N.$$

② If N is a f.g. R-module with $\text{proj}\dim_R N < \infty$, then

$$\text{proj}\dim_R(N) + \text{depth}_R(N) = \text{depth}_R(R).$$

(Classical AB Equality.)

Corollary. $\text{proj}\dim_R N < \infty \Rightarrow \text{depth}(N) \leq \text{depth}(R)$.

Furthermore, equality iff N is projective.

False without assumption of proj dim $< \infty$

Note we had:

$$\text{depth}_R(F) = \text{depth}_R(R) - \sup H_*(k \otimes_R^L F)$$

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

Subtract:

$$\operatorname{depth}_R(F) - \operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}(R) - \operatorname{depth}(M).$$

Note: Some terms above may be ∞.
when $H_i(\underline{x}; M) = 0 \ \forall i$, then
$\sup H_*(\underline{x}; M) = -\infty$.

Proof of AB Equality

$$\operatorname{RHom}_R(k, M \otimes_R^L F) \xleftarrow{\simeq} \operatorname{RHom}_R(k, M) \otimes_R^L F$$

quasi iso because k is a f.g. module
$\therefore \exists\, G \xrightarrow{\simeq} k$ where each G_i is finite free
AND flat dim $F < \infty$.

[More generally:
$\operatorname{Hom}_R(N, M) \otimes R \xrightarrow{\simeq} \operatorname{Hom}_R(N, M \otimes_R F)$
N f.g. R-mod, F flat]

Key observation:

$\operatorname{RHom}_R(k, M) \simeq$ complexes of k-vector spaces
(take injective resolution of M)

$$\operatorname{RHom}_R(k, M) \otimes_R^L F \simeq \operatorname{RHom}_R(k, M) \otimes_k^L (k \otimes_R^L F)$$

$$\therefore \operatorname{Ext}_R^*(k, M \otimes_R^L F) = \operatorname{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^L F).$$ ☒

Observation: Let M be an R-complex.
Suppose $s := \sup H_*(M)$ is finite.
Then, $\operatorname{depth}_R M \geq -s$

Then, $\operatorname{depth}_R M \geq -s$

with equality iff $\operatorname{depth}_R H_s(M)$.

Note. For an R-module M, $\operatorname{depth} M = 0 \Leftrightarrow \inf \operatorname{Ext}^{i}(k, M) = 0$

$\Leftrightarrow \operatorname{Hom}(k, M) \neq 0$

$\Leftrightarrow k \hookrightarrow M$

$\Leftrightarrow \mathfrak{m} \in \operatorname{Ass}_R M$.

One proof. M as above.

$$\operatorname{Ext}_R^{-s}(N, M) = \operatorname{Hom}_R(N, H_s(M))$$

N any R-module.

- key. $M \simeq M'$ with $M_i' = 0 \ \forall\ i > s$.

$$\begin{array}{ccccccccccc}
\cdots \to & M_{s+1} & \xrightarrow{\partial} & M_s & \to & M_{s-1} & \to \cdots & & = & M \\
\downarrow 0 & 0 \downarrow & & \downarrow & & \| & & \| & & \downarrow 2 \\
\cdots \to & 0 & \to & \dfrac{M_s}{\partial(M_{s+1})} & \to & M_{s-1} & \to \cdots & & = & M'
\end{array}$$

$\therefore$ Can assume $M_i = 0$ for $i \geq s+1$.

In particular,

$$0 \to \Sigma^s H_s(M) \hookrightarrow M \to M'' \to 0.$$

iso on homology in degrees $\leq s-1$

$H_i(M'') = 0 \quad i \geq s$.

Let $\underline{x} = x_1, \ldots, x_n$ gen set for $\mathfrak{m}$.

Then,

Then,

$$H_{i+1}(\underline{x}; M'') \to H_i(\underline{x}, \Sigma^* H_s(M)) \to H_i(\underline{x}; M) \to H_i(\underline{x}; M'') \quad \text{--- } (*)$$

$H_j(M'') = 0 \ \forall j \geq s.$

So, $M'' \simeq M'''$ with $M'''_j = 0$ for $j \geq s$.

$K(\underline{x}; M''')_j = 0$ for $j \geq s+n+1$.

Thus, $H_j(\underline{x}; M''') = 0 \quad \forall j \geq s+n+1.$

$$\Rightarrow H_i(\underline{x}; \Sigma^s H_s(M)) = 0 \quad \forall j \geq s+n+1.$$

Put $i \geq n+s$ in $(*)$:

$$H_j(\underline{x}; M) = 0 \quad \forall j \geq n+s+1$$

$$\Rightarrow \sup H_*(\underline{x}; M) \leq n+s$$

$$\Rightarrow -s \leq n - \sup H_*(\underline{x}; M) = \operatorname{depth} M.$$

Moreover,

$$H_{n+s}(\underline{x}; M) \cong H_{n+s}(\underline{x}; \Sigma^s H_s(M))$$
$$\cong H_n(\underline{x}, H_s(M))$$

The above is nonzero iff $\operatorname{depth} H_s(M) = 0$. ☒

---

① $\operatorname{flat\,dim}_R F < \infty$. Then $\forall M$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F).$$

② $s = \sup H_*(M)$ is finite.

Then, $\operatorname{depth}_R(M) \geq -s$.

Equality $\Leftrightarrow$ $\operatorname{depth}(H_s(M)) = 0$.

<u>Application</u>. Say

$$F = \quad 0 \to F_d \to \cdots \to F_0 \to 0$$

↳ minimal, $\partial F \subseteq \mathfrak{m} F$

is a finite free complex s.t.

$$0 < \operatorname{length}_R H_*(F) < \infty.$$

(All $H_i(F)$ are finite length and at least one is nonzero.)

Then, for any M,

$$\operatorname{depth}_R M = d - \sup H_*(F \otimes_R M).$$

Thus, such an F is depth sensitive.

When $\mathfrak{m}M \neq M$, one can check some $H_*(F \otimes_R M) \neq 0$.

Then, one gets

$$d \geq \operatorname{depth}_R(M).$$

<u>Note</u>: Over any local ring, $\exists M$ s.t. $\mathfrak{m}M \neq M$ and $\operatorname{depth}_R M = \dim R$.

M need not be f.g.

In particular, $d \geq \dim(R)$. $\longrightarrow$ New Intersection Theorem

Hochster ('70)
André (2016)
Bhatt (2021)

Proof. Assume $s = \sup H_*(F \otimes_R M)$.

Take any prime $\mathfrak{p} \neq \mathfrak{m}$.

$$H_i(F \otimes_R M)_{\mathfrak{p}} \cong H_i(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$$

$$H_i(F_{\mathfrak{p}}) = 0 \quad (\because \text{length } H_i(F) < \infty)$$

i.e. $F_{\mathfrak{p}} \simeq 0$ in $D(R_{\mathfrak{p}})$.

$$\therefore H_*(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$$

Thus, $H_i(F \otimes_R M)$ is $\mathfrak{m}$-power torsion.

(I.e., each $a \in H_i(F \otimes_R M)$ is killed by some $\mathfrak{m}^n$.)

$\therefore$ depth $H_s(F \otimes_R M) = 0$.

Thus, by previous result,

$$\text{depth}_R(F \otimes_R M) = -s$$

$$\| \text{ AB}$$

$$\text{depth}(M) - \sup H(k \otimes^L_R F)$$

$$\Rightarrow \text{depth}(M) = \underbrace{\sup H_*(k \otimes^L_R F)}_{d} - s. \qquad \blacksquare$$