Lecture 1 (30-07-2021)

30 July 2021 13:51

§ Introduction.

Consider the set $\mathcal{C}^0[a, b]$.
It can be given a metric d given by

$$d(f, g) := \sup_{t \in [a,b]} |f(t) - g(t)| \qquad \forall f, g \in \mathcal{C}^0[a,b]$$

(Can be verified that (i) $d(f,g) = 0 \iff f = g$,
(ii) $d(f, g) \geq 0$,
(iii) $d(f, g) = d(g, f)$,
(iv) $d(f, h) \leq d(f, g) + d(g, h)$.)

Indeed, d induces a metric on $\mathcal{C}^0[a, b]$.
Thus, we can talk about convergence of sequences of functions.
(This gives uniform convergence.)

- BUT... there is another structure as well: a linear structure

Indeed, if $f, g \in \mathcal{C}^0$ and $\alpha \in \mathbb{R}$, then

$$\alpha f + g \in \mathcal{C}^0.$$

(Historically: Integral operators defined on $\mathcal{C}^0[a, b]$ were exploited.
This led to axiomatisation.)

Normed Linear Spaces

Def: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
Given a vector space X over $\mathbb{F}$, a norm is a function $\|\cdot\| : X \longrightarrow \mathbb{R}$ s.t. $\forall x, y \in X$ and $\forall \alpha \in \mathbb{F}$ we have:

(i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$. (Positive definite)

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$. (Homogenity)

(iii) $\|x + y\| \leq \|x\| + \|y\|$. (Triangle inequality)

$(X, \|\cdot\|)$ is called a normed linear space (NLS).

To any NLS X, we can associated a metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) := \|x - y\| \quad \forall\, x, y \in X.$$

It is easy to see that d is indeed a metric

Thus, X is now endowed with a metric topology.

Ex. $\big|\, \|x\| - \|y\| \,\big| \leq \|x \pm y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in X$.

Convergence of sequence in this context becomes:

Defn. A sequence (x_n) in an NLS X is said to converge to $x \in X$ if

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

We denote this by $\displaystyle\lim_{n \to \infty} x_n = x$.

Propn. Let everything have obvious meanings. $(x, y \in V,\ \alpha \in \mathbb{F}.)$ Then:

(i) The limit of (x_n), if it exists, is unique.

(ii) If $x_n \to x$, then (x_n) is bounded, i.e., $\exists\, r \geq 0$ s.t. $\|x_n\| < r \quad \forall n$.

(iii) $x_n \to x \implies \|x_n\| \to \|x\|$.

(Use $\big|\, \|x_n\| - \|x\| \,\big| \leq \|x_n - x\| \to 0$.)

(Use $\big|\|x_n\| - \|x\|\big| \le \|x_n - x\| \to 0$.)

(iv) $x_n \to y$ and $y_n \to y \Rightarrow (x_n + y_n) \to (x+y)$.

$\left(\|x_n + y_n - x - y\| \le \|x_n - x\| + \|y_n - y\| \to 0. \right)$

"+ and · are continuous"

(v) $x_n \to x$ and $\alpha_n \to \alpha \Rightarrow \alpha_n x_n \to \alpha x$.

$$\begin{aligned}\Big(\|\alpha_n x_n - \alpha x\| &\le \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\| \\ &= \underbrace{|\alpha_n|}_{\text{bounded}} \|x_n - x\| + |\alpha_n - \alpha| \underbrace{\|x\|}_{\text{constant}} \to 0. \Big)\end{aligned}$$

Def^n. The sequence (x_n) in an NLS X is said to be a *Cauchy sequence* if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.
$$\|x_n - x_m\| < \epsilon \quad \forall\ m, n \ge N.$$

Prop^n. In an NLS X, each convergent sequence is Cauchy.

Def^n. For $1 \le p \le \infty$, its *conjugate exponent* q is given by
$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma. For $p \in (1, \infty)$, the following Young's inequality holds for $a, b \ge 0$:
$$a^{1/p} b^{1/q} \le \frac{a}{p} + \frac{b}{q}.$$

Proof. For $t \ge 1$, define $f(t) := \lambda(t-1) - t^{\lambda} + 1$ for some fixed $\lambda \in (0,1)$.

Then, $f'(t) = \lambda - \lambda t^{\lambda - 1} = \lambda(1 - t^{\lambda-1}) \ge 0$.

Thus, $f \uparrow$ on $[1, \infty)$ and $f(1) = 0$.

$$\therefore f(t) \ge 0 \quad \forall\ t \ge 1.$$

Thus, $t^{\lambda} \le \lambda(t-1) + 1 \quad \forall\ t \ge 1.$

The result now follows by taking $t = a/b$ and $\lambda = 1/p$. ☒
(we can assume $a \geq b \neq 0$.)

Lemma. (Hölder's inequality)
Let $p \in (1, \infty)$ and q be its conjugate.
Then, $\{x_i, y_i\}_{i=1}^N \subseteq \mathbb{R}$, we have

$$\sum_{i=1}^N |x_i y_i| \leq \left(\sum_{i=1}^N |x_i|^p \right)^{1/p} \left(\sum_{i=1}^N |y_i|^q \right)^{1/q}.$$

Lemma. (Minkowski's inequality)
Let $1 \leq p < \infty$, $x, y \in \mathbb{R}^N$.
Then,

$$\|x+y\|_p := \left(\sum_{j=1}^N |x_j + y_j|^p \right)^{1/p} \leq \|x\|_p + \|y\|_p.$$

Proof. $p=1$ is direct. For $p>1$, use Hölder.

Lecture 2 (03-08-2021)

03 August 2021 14:04

Q. Can every metric defined on a vector space be obtained from a norm?

No. Check that discrete metric cannot be obtained.

Another: let S be the space of all complex sequences

Define
$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1+|x_j-y_j|}.$$

Check this does not work.

Note: If d is induced by $\|\cdot\|$, we can recover $\|\cdot\|$ by $\|x\| := d(x,0)$. Check homogeneity is not satisfied.

Q. When is a metric induced by a norm?

Ans. d should satisfy: (i) $d(x+a, y+a) = d(x,y)$ $\forall x,y \in X$,
(ii) $d(\alpha x, \alpha y) = |\alpha|\, d(x,y)$ $\forall x,y \in X\ \forall \alpha \in \mathbb{F}$.

(Check! Define $\|x\| := d(x,0)$ and verify.)

Defn. A sequence $\{x_n\} \subseteq X$ converges to $x \in X$ if $\|x_n - x\| \longrightarrow 0$.

A series $\sum_{j=1}^{\infty} x_j$ converges to $x \in X$ if the sequence $\{S_n\}$ of partial sums converges to x. $\left(S_n = \sum_{j=1}^{n} x_j\right)$

A series is said to be absolutely convergent if
$$\sum_{j=1}^{\infty} \|x_j\| < \infty.$$
↑ This is in $\mathbb{R}$.

- Examples of NLS.

(1) $X = \mathcal{C}([a,b])$ with $-\infty < a < b < \infty$.
Define $\|x\| := \max_{t \in [a,b]} |x(t)|$.

(2) $X = \mathcal{C}^k([a,b])$ where
$$\|x\|_{k,\infty} = \max_{0 \le j \le k} \max_{t \in [a,b]} |x^{(j)}(t)|.$$

3. l_p spaces. For $1 \le p < \infty$, define (l^p spaces, lp spaces)
$$l_p = \left\{ x = \{x_j\}_{j=1}^{\infty} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}$$
↳ p^{th} summable sequences

Check l_p is a vspace with standard $+$ and $\cdot$.
(Show: subspace of $\mathbb{R}^{\mathbb{N}}$.)

For $x \in l_p$, define
$$\|x\|_p := \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}.$$

Claim: $\|\cdot\|_p$ is a norm.

Only triangle inequality is non-trivial.
Follows from the Minkowski inequality by taking limit $N \to \infty$.

For $p = \infty$:

$$\ell_\infty := \{ \{x_j\}_{j=1}^\infty : \sup_j \|x_j\| < \infty \}.$$

$$\|x\|_\infty = \sup_j \|x_j\|.$$

(Check!)

$c = \{ x \in \ell_\infty : \{x_j\} \text{ converges} \}.$
$c_0 = \{ x \in c : x_j \to 0 \}$
$c_{00} = \{ x \in c_0 : \text{all but finitely many } x_j \text{ are } 0 \}.$

Note: $c_{00} \subseteq \ell_p \quad \forall p.$

5. $\Omega \subseteq \mathbb{R}^d$ measurable, μ is the Lebesgue measure on $\mathbb{R}^d$.
For $1 \le p < \infty$, define

$$L^p(\Omega) := \left\{ f : \int_\Omega |f|^p < \infty \right\} / \sim$$

where $f \sim g$ if $f = g$ μ a.e.

for $f \in L^p(\Omega)$, define

$$\|f\|_p := \left(\int_\Omega |f|^p \right)^{1/p}.$$

For $p = \infty$: $L^\infty(\Omega)$ is the space of essentially bounded functions

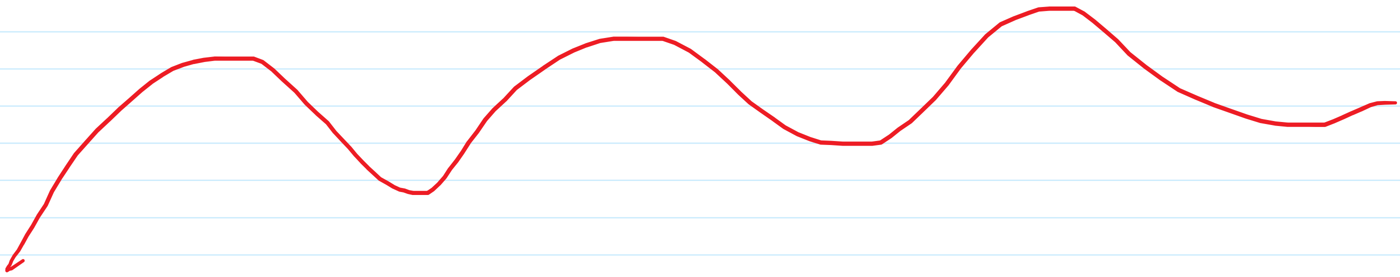
$$\|f\|_\infty := \inf \{ C \geq 0 : |f(x)| \le C \text{ for almost every } x. \}$$

$(L^\infty(\Omega) := \{ f : \|f\|_\infty < \infty \}.)$

Alter: $\|f\|_\infty := \inf_{g : g \sim f} \sup_{x \in \Omega} |g(x)|.$
($g = f$ a.e.)

Def^n (Banach space)
A normed linear space X over the field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) is called a Banach space if it is complete under norm topology.

Examples. (i) $\mathbb{R}$ or $\mathbb{C}$ with $|\cdot|$ norm is complete.
(ii) So are $\mathbb{R}^n$ or $\mathbb{C}^n$ for $n \in \mathbb{N}$.
(iii) ℓ^p is complete.
(Take a Cauchy sequence in ℓ^p (x_n).
Each $x_n^{(i)}$ is a Cauchy sequence.
Form the limit sequence.)



Lecture 3 (06-08-2021)

06 August 2021 14:08

Recall what it means for $T : X \to Y$ to be linear.
If X and Y are NLS, then we can talk about continuity.

Defn. T is continuous at $x_0 \in X$ if for every sequence $(x_n)_n$ with $x_n \to x_0$, we have $T(x_n) \to T(x_0)$.
The above is equivalent to the ε-δ definition.

Thm. Let X and Y be NLS and let $T : X \to Y$ be a linear operator. TFAE:

(i) T is continuous.
(ii) T is continuous at $0 \in X$.
(iii) $\exists K > 0$ s.t. $\|Tx\| \le K\|x\|$ $\forall x$.
(iv) If $B = \{x \in X : \|x\| \le 1\}$, then $T(B)$ is bounded.
(v) T is uniformly continuous.

Proof. (i) $\Rightarrow$ (ii) trivial.

(ii) $\Rightarrow$ (iii) Take $\varepsilon = 1$. $\exists \delta > 0$ s.t. $\|x\| < 2\delta \Rightarrow \|Tx\| < 1$.

Now given $0 \neq x' \in X$, define $x := \delta \dfrac{x'}{\|x'\|}$.

Then, $\|x\| = \delta < 2\delta$. Thus, $\|Tx\| < 1$.

But $Tx = T\left(\delta \dfrac{x'}{\|x'\|}\right) = \dfrac{\delta}{\|x'\|} T(x')$

and so, $\dfrac{\delta}{\|x'\|} T(x') < 1$.

$$\Rightarrow T(x') < \frac{\|x'\|}{\delta} \quad \forall x' \in X \setminus \{0\}.$$

Take $K = 1/\delta$ and verify it holds for $x' = 0$ as well.

(iii) $\Rightarrow$ (iv) For $x \in B$, we have $\|T(x)\| \le K$.

(ii) $\Rightarrow$ (i) Fix $x \in X$.

If $x_n \to x$, then $x_n - x \to 0$, then $T(x_n - x) \to T(0) = 0$.

$\therefore\ T(x_n) \longrightarrow T(x)$. ∎

(iii) $\Rightarrow$ (ii) Let $x_n \to x$. Then, $\| T(x_n) \| \leq K \|x_n\| \to 0$.

$1 \Leftrightarrow 2$, $4 \Rightarrow 3$ (iv) $\Rightarrow$ (iii) Let $T(B)$ be bounded by K.

Then, for $x \neq 0$, $\frac{x}{\|x\|}$ has norm 1 and thus

$$T\left(\frac{x}{\|x\|}\right) \leq K \quad \text{or} \quad T(x) \leq K\|x\|.$$

True for $x = 0$ as well.

(v) $\Rightarrow$ (i) trivial

(iii) $\Rightarrow$ (v). For $x, x' \in X$, we have

$$\|T(x) - T(x')\| = \|T(x - x')\| \leq K \|x - x'\|.$$

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2K}$. ∎

The above suggests putting a norm on the operator $T: X \to Y$.

Def^n. Assume that $T: X \to Y$ is bounded, i.e., $\exists K > 0$ s.t.

$$\|Tx\|_Y \leq K \|x\|_X \quad \forall x \in X.$$

The inf over all $K > 0$ satisfying the above gives us the operator norm of T.

Said differently,

$$\|T\| := \sup_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X}.$$

The space of all bounded (or continuous) linear operators, say $BL(X, Y)$ is a normed linear space with above norm.

(Check!)

Note that we can also rewrite the above as

$$\|T\| := \sup \|Tx\|_Y.$$

$\|x\|_X = 1$

Theorem If X and Y are NLS and Y is complete, then $BL(X,Y)$ is complete.

Proof. Let $\{T_n\}_n \subseteq BL(X,Y)$ be Cauchy.
Then, for each $x \in X$, we have

$$\|T_n(x) - T_m(x)\|_Y \leq \|T_n - T_m\| \ \|x\|_X.$$

$\therefore \{T_n(x)\}_n$ is Cauchy $\forall n$.

Since Y is complete, we can define

$$T(x) := \lim_n T_n(x).$$

That T is continuous is clear (uniform convergence).

TS: T is linear.

Let $x, x' \in X$ and $\alpha \in \mathbb{F}$.

Then,
$$T(x + \alpha x') = \lim_n T_n(x + \alpha x')$$
$$= \lim_n T_n(x) + \alpha \lim_n T_n(x')$$
$$= T(x) + \alpha T(x'). \qquad \blacksquare$$

When $Y = X$, $BL(X, X)$ is denoted as $BL(X)$.

X Banach $\Rightarrow$ $BL(X)$ is a Banach space.

Moreover, we talk about composition.

If $T_1, T_2 \in BL(X)$, then $T_1 \circ T_2 \in BL(X)$.

Denote $T_1 \circ T_2$ by $T_1 T_2$.

We also have $\|T_1 T_2\| \leq \|T_1\| \ \|T_2\|$ and $\|id\| = 1$.

This makes $BL(X)$ a Banach algebra.

Lecture 4 (10-08-2021)

10 August 2021 14:01

Recall, for $T \in BL(X, Y)$, the norm is defined as

$$\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

If Y is Banach, so is $BL(X, Y)$.
If $Y = X$ is complete, then $BL(X, X)$ is a Banach Algebra.

$$(T_1 T_2)(x) := T_1(T_2(x)) \quad \text{and}$$
$$\|T_1 T_2\| \leq \|T_1\| \|T_2\|,$$
$$\|id\| = 1.$$

Example. (1) Let $\mathcal{M}_N = M_N(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
$\qquad\qquad\quad \| \; BL(\mathbb{F}^N, \mathbb{F}^N)$

On X, define norm for $p \in [1, \infty)$ and $x = (x_1, \ldots, x_N)$

$$\|x\|_{\ell_p^N} := \left(\sum_{j=1}^{N} |x_j|^p \right)^{1/p} \qquad (\text{for } p \neq \infty)$$

and for $p = \infty$, $\|x\|_{\ell_\infty^N} = \max_{1 \leq j \leq N} |x_j|$.

If $A \in \mathcal{M}_n$, then

$$\|A\|_{p,N} := \max_{\|x\|_{\ell_p^N} = 1} \|Ax\|_{\ell_p^N}$$

- Special cases : $p = 1, 2, \infty$.

(i) Show that if $p = 1$, then

$\|A\|_{1,N}$ is the max abs column sum of the matrix A.

(2) For $p = \infty$, it is row.

(3) $p = 2$:

$$\|A\|_{2,N} := \sup_{0 \neq x} \frac{\|Ax\|_{\ell_2^N}}{\|x\|_{\ell_2^N}} = \sqrt{\rho(A^* A)}$$

spectral radius of T

where $\rho(T) = \max_{\lambda \in \sigma(T)} |\lambda|$,

$\sigma(T) = \{\lambda : \lambda$ is an eval$\}$.

spectrum of T

(Note: $A^* A$ is positive semi-definite and thus, $\sqrt{\;}$ is meaningful.)

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ be the eigenvalues counted with multiplicity.
Let $\{z_j\}_{j=1}^N$ be the corresponding orthogonal eigenbasis.

(To recall why positive:
Let $x \neq 0$ and $\lambda \in \mathbb{R}$ be s.t. $A^* A x = \lambda x$.
Then, $\lambda = \frac{x^* A^* A x}{x^* x} = \frac{\|Ax\|_2}{\|x\|_2} \geq 0$.)

For $x \in X$, $\|Ax\|^2_{\ell_2^N} = \langle Ax, Ax \rangle$
$= \langle x, A^* A x \rangle$.

Write $x = \sum_{j=1}^N \alpha_j z_j$ with $\alpha_j = \langle x, z_j \rangle$.

Then, $A^* A x = \sum_{j=1}^N \alpha_j \lambda_j z_j$.

Thus, $\|Ax\|^2_{\ell_2^N} = \langle x, A^* A x \rangle$
$= \sum_{j=1}^N \lambda_j |\alpha_j|^2$.

$$= \sum_{j=1}^{N} \lambda_j |\alpha_j|^2.$$

As λ_1 is the max evalue, then

$$\|Ax\|^2_{\ell_2^N} \leq \lambda_1 \sum_{j=1}^{N} |\alpha_j|^2 = \lambda_1 \|x\|^2_{\ell_2^N}.$$

$$\therefore \|A\|^2_{2,N} \leq \lambda_1 = \rho(A^*A).$$

For equality, simply take $x = z_1$. Then, $\|x\|_{\ell_2^N} = 1$ and $\|Ax\|_{\ell_2^N} = \lambda_1 = \rho(A^*A)$. ∎

There is a norm called Frobenius norm on A not induced by any ℓ_p^N norm:

$$\|A\|_F := \sqrt{\operatorname{trace}(A^*A)} = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|^2 \right)^{1/2}.$$

$$\|Ax\|_{\ell_2^N} \overset{CS}{\leq} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|^2 \right)^{1/2} \left(\sum_{j=1}^{N} |x_j|^2 \right)^{1/2}$$

$$= \|A\|_F \, \|x\|_{\ell_2^N}.$$

$$\Rightarrow \|A\|_{2,N} \leq \|A\|_F.$$

It is not induced by any vector norm as $\|I\|_F = \sqrt{N} \neq 1$ for $N \geq 2$.

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(2) Let $A = [a_{ij}]$ be an infinite matrix of scalars s.t. the i^{th} component of Ax is given by

$$\sum_{j=1}^{\infty} a_{ij} x_j. \qquad (A \text{ defines a map } \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}.)$$

$$\sum_{j=1}^{\infty} a_{ij} x_j.$$ (A defines a map $\mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$.)

(1) Assume that $\alpha = \sup_j \sum_{i=1}^{\infty} |a_{ij}| < \infty$.

Then, $A \in BL(\ell_1)$.

Proof. A is linear is clear.

$$\|Ax\|_{\ell_1} = \sum_{i=1}^{\infty} |(Ax)_i|$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|\,|x_j|$$

$$\leq \alpha \|x\|_{\ell^1}.$$

$\Rightarrow$ $Ax \in \ell_1$ and $\|A\|_{\ell_1} \leq \alpha$.

In fact, equality holds.

(ii) Similarly, if $\alpha = \sup_i \sum_{j=1}^{\infty} |\alpha_{ij}| < \infty$, then we do for ℓ_∞.

(3) $X = C^0[a,b]$, $\|f\|_\infty = \max_{t \in [a,b]} |f(t)|$.

Define $T: X \to X$ by $(Tf)(x) := \int_a^x f$.

Clearly, T is linear.

Moreover, for $x \in [a,b]$

$$|Tf(x)| \leq \int_a^x |f| \leq \int_a^b |f| \leq (b-a)\|f\|_\infty.$$

$\Rightarrow$ $\|Tf\|_\infty \leq (b-a)\|f\|_\infty$.

$\Rightarrow \ \|T\|_\infty \leq b-a.$

Moreover, taking $f = 1$ shows $\|T\|_\infty = b-a$

Linear but not bounded: $X = C^1[a,b]$ with sup norm,
$Y = C^0[0,1]$ with sup norm.

Define $A : X \longrightarrow Y$ by $Af = f'$.
A is linear clearly.
But A is not bounded. Take $f_n(t) := t^n$.
Then, $\|f_n\|_\infty = 1$ but $\|Af_n\| = n \to \infty$.

But if $X = C'[a,b]$ is equipped with the norm
$$\|x\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty,$$
then the operator becomes bounded.
Indeed,
$$\|Ax\|_\infty = \|x'\|_\infty \leq \|x\|_\infty + \|x'\|_\infty$$
$$= \|x\|_{1,\infty}.$$

Prop^n. Let $\mathcal{B}$ be a Banach algebra, and let $\{A_n\}_n$ and $\{B_n\}$, A, B, be in $\mathcal{B}$.
Then,
(i) $\|A^k\| \longrightarrow \|A\|^k$ for $k = 0, 1, \ldots$ with $A^0 = I$.
(ii) If $A_n \longrightarrow A$ and $B_n \longrightarrow B$ in $\mathcal{B}$ as $n \to \infty$,
then $A_n B_n \longrightarrow AB$.

(Simply use that $\|AB\| \leq \|A\| \, \|B\| \ \forall A, B \in \mathcal{B}$.)

Prop^n. Let X be a Banach space and let

$$F(A) := \sum_{j=1}^{\infty} a_j A^j,$$

$a_j \in \mathbb{F}$, $A \in BL(X)$ with $\sum_{j=1}^{\infty} |a_j| |z|^j < \infty$ $\forall z \in \mathbb{F}$ with $|z| < r$ for some $r > 0$.

Then, for $A \in BL(X)$ with $\|A\| < r$, $F(A) \in BL(X)$.

Proof. Since $\|A\| < r$, then

$$\sum_{j=1}^{N} \|a_j A^j\| \leq \sum_{j=1}^{N} |a_j| \, \|A^j\| \leq \sum_{j=1}^{N} |a_j| \, \|A\|^j$$
$$\leq \sum_{j=1}^{N} |a_j| \, r^j$$
$$\leq \sum_{j=1}^{\infty} |a_j| \, r^j < \infty.$$

Let $N \to \infty$ to get $\sum_{j=1} \|a_j A^j\| < \infty$.

Since $BL(X)$ is Banach, we are done. $\blacksquare$

Application. We can talk about e^A for $A \in BL(X)$.

Lecture 5 (13-08-2021)

13 August 2021 13:55

Def^n. The map $f : Y \longrightarrow Y$ has a fixed point $u^* \in Y$ if $f(u^*) = u^*$.

Thm. (Banach Fixed Point Theorem)

Let X be a Banach space over $\mathbb{F}$.

Let $Y \subseteq X$ be a nonempty closed subset. (Not necessarily a subspace.)

Let $f : Y \longrightarrow Y$ be a map s.t. $\exists \alpha < 1$ and $\|f(u) - f(v)\| \leq \alpha \|x - y\|$.

Then: (i) $\exists ! \, u^* \in Y$ s.t. $f(u^*) = u^*$. (Existence and uniqueness)

(ii) Given any $u^{(0)} \in Y$, the iterative procedure
(Convergence of iterative method)

$$u^{(n+1)} = f(u^{(n)}) \quad \text{for } n = 0, 1, \ldots$$

gives a sequence which converges to the unique fixed point.

(iii) For $n = 0, 1, \ldots$ the following a priori estimate
(Error estimates)

$$\|u^{(n)} - u^*\| \leq \frac{\alpha^n}{1-\alpha} \|u^{(1)} - u^{(0)}\|$$

and the a posteriori estimate

$$\|u^{(n+1)} - u^*\| \leq \frac{\alpha}{1-\alpha} \|u^{(n+1)} - u^{(n)}\| \quad \text{hold.}$$

(iv) For $n = 0, 1, 2, \ldots$
(Rate of convergence) $\|u^{(n+1)} - u^*\| \leq \alpha \|u^{(n)} - u^*\|$.

Proof. Let $\{u^{(n)}\}_n$ be as defined. (Can pick $u^{(0)}$ since $Y \neq \emptyset$.)

Proof. Let $\{u^{(n)}\}_n$ be as defined. (Can pick $u^{(0)}$ since $Y \neq \emptyset$.)

Then,

$$\|u^{(n+1)} - u^{(n)}\| = \|\mathcal{F}(u^{(n)}) - \mathcal{F}(u^{(n-1)})\|$$
$$\leq \alpha \|u^{(n)} - u^{(n-1)}\|$$
$$\vdots$$
$$\leq \alpha^n \|u^{(1)} - u^{(0)}\|.$$

For $n \geq 0$ and $m \geq 1$, note

$$\|u^{(n)} - u^{(n+m)}\| \leq \|u^{(n)} - u^{(n+1)}\| + \cdots + \|u^{(n+m-1)} - u^{(n+m)}\|$$
$$\leq \frac{\alpha^n}{1-\alpha} \|u^{(1)} - u^{(0)}\| \xrightarrow{n\to\infty} 0.$$

$\therefore$ $\{u^{(n)}\}_n$ is Cauchy and has a limit $u^* \in X$.
Since Y is closed, $u^* \in Y$.

Now, $\mathcal{F}(u^{(n-1)}) = u^{(n)}$.

Taking $n \to \infty$ and continuity (take $\delta = \varepsilon$.) of $\mathcal{F}$ gives
$\mathcal{F}(u^*) = u^*$.

The error estimates and rate of convergence follow easily.

Uniqueness: If $v^* \in Y$ is a fixed point, then

$$\|v^* - u^*\| = \|\mathcal{F}(v^*) - \mathcal{F}(u^*)\|$$
$$\leq \alpha \|v^* - u^*\|.$$

Since $\alpha < 1$, it follows $\|v^* - u^*\| = 0$.
Thus, $v^* = u^*$. ∎

Lecture 6 (17-08-2021)

17 August 2021 14:00

Finite Dimensional Normed Linear Spaces and Riesz' Lemma

Def^n. Let X be a vector space and $Y \subseteq X$ be convex. $f : Y \longrightarrow \mathbb{R}$ is called *convex* if

$$f(\lambda u + (1-\lambda) v) \le \lambda f(u) + (1-\lambda) f(v)$$

for $\lambda \in [0,1]$ and $u, v \in Y$.

Example. ① $(X, \|\cdot\|) \to$ NLS

Then $\|\cdot\|$ is continuous and convex.

Given any convex $Y \subseteq X$, the restriction is also convex.

In particular, take $Y \subseteq X$ to be a subspace.

② Note that balls are convex.

Def^n. A subset M of a nls X is said to be *sequentially compact* (or simply *compact*) if each sequence $\{u_n\}_n$ in M has a convergent subsequence (which converges in M).

Example. (1) Consider $M \subseteq \mathbb{F}^N$ with $\|\cdot\|_{\ell^N_\infty}$ norm.

M is compact $\Leftrightarrow$ M is closed + bounded.

(2) (Arzelà - Ascoli theorem)

$X = \mathcal{C}[a,b]$ with sup norm.

Suppose $M \subseteq X$ satisfies:

(i) M is bounded, i.e., $\exists\, r > 0$ s.t. $\|u\|_\infty \le r$ $\forall u \in M$.

(ii) M is equicontinuous, i.e.,

$\forall\, \epsilon > 0$, $\exists\, \delta = \delta(\epsilon) > 0$ s.t.

$\forall t, s \in J$, $\forall u \in M$: $|t - s| < \delta \Rightarrow |u(t) - u(s)| < \epsilon$

Then, M is a compact subset of X.

Recall:

Thm. Let $f : M \to \mathbb{R}$ be continuous, where $M \subseteq X$ is nonempty and compact. Then, f has a minimum and maximum on M.

Proof. Let $S = f(M)$. Put $\alpha := \sup(S)$.

Then, $\exists$ a sequence $\{u_n\}_n$ in M s.t. $f(u_n) \to \alpha$.

By compactness, $\exists$ subsequence $u_{n_j} \to u \in M$.

Then,
$$f(u) = \lim_j f(u_{n_j}) = \lim_n f(u_n) = \alpha.$$

In particular, $\alpha < \infty$ and is attained.

Same for min. ∎

Also, recall:

Thm. If $f : M \to Y$ is continuous, then it is uniformly continuous. ($M \subseteq X$ compact, Y nls)

- Finite dimensional Banach spaces.

Def. Let X be a linear space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are said to be *equivalent* if $\exists\, C_1, C_2 > 0$ such that

$$C_1 \cdot \|u\|_1 \leq \|u\|_2 \leq C_2 \cdot \|u\|_1$$

for all $u \in X$.

Consequence: Both generate the same topology.

Def. Let $A \in BL(X)$ be a bijection, where X is a NLS.

We say that A is invertible if A^{-1} is continuous.

Thm. Any two norms on a finite dimensional linear space X are equivalent.

Proof. If $X = \{0\}$, nothing to prove.

Assume $\dim(X) = N > 0$.

Let $\{e_j\}_{j=1}^{N}$ be a basis of X.

Define the max norm $\|\cdot\|_\infty$ on X by

$$\left\| \sum_{j=1}^{N} \alpha_j e_j \right\|_\infty := \max_{1 \leq j \leq N} |\alpha_j|.$$

We show that any norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_\infty$.

First note

$$\|u\| = \left\| \sum_{j=1}^{N} \alpha_j e_j \right\| \leq \sum_{j=1}^{N} \| \alpha_j e_j \|$$

$$\leq C_2 \|u\|_\infty,$$

where $C_2 := \sum_{j=1}^{N} \|e_j\|$.

Need to construct C_1 now.

Set $M = \{ \alpha \in \mathbb{F}^N : |\alpha|_\infty = 1 \}$.

Note that M is compact in $\mathbb{F}^N$.

Define $f : \mathbb{F}^N \to \mathbb{R}$ by

$$f(\alpha_1, \ldots, \alpha_N) := \left\| \sum_{j=1}^{N} \alpha_j e_j \right\|.$$

$$|f(\alpha_1, \ldots, \alpha_N) - f(\beta_1, \ldots, \beta_N)|$$

$$= \left| \sum_{j=1}^{N} (\alpha_j - \beta_j) e_j \right\|$$

$$\leq |\alpha - \beta|_\infty \sum_{j=1}^{N} \|e_j\|.$$

Thus, f is continuous. By compactness of M, f attains its minimum C_1 on M.

Check $\|u\| \geq C_1 \|u\|_\infty \quad \forall u \in X$. ▢

Prop. Let $\{u_n\}_n$ be a sequence in a fdls X.

$u_n \to u$ in X $\iff$ each component sequence (wrt some fixed basis) converges appropriately.

Cor. Each finite dimensional NLS (over $\mathbb{R}$ or $\mathbb{C}$) is a Banach space.

Def^n. Two NLS X and Y are said to be (linearly) homeomorphic if there exists a LINEAR isomorphism $F: X \to Y$ which is a topological homeomorphism.

Def^n. X, Y are called isometric if such a function F above is norm-preserving, i.e.,

$$\|x-y\|_X = \|F(x) - F(y)\|_Y.$$

Exercise. Every N-dimensional NLS is (linearly) homeomorphic to each other.

The same is not true in an infinite dimensional space.

Indeed, take $X = \mathcal{C}[0,1]$ with $\|\cdot\|_{\ell^1}$ and $\|\cdot\|_{\ell^\infty}$.

Set $\{f_n\} \subset \mathcal{C}[0,1]$ as

$$f_n(t) := \begin{cases} 1-nt & ; \quad 0 \leq t \leq 1/n, \\ 0 & ; \quad \frac{1}{n} \leq t \leq 1. \end{cases}$$

$\cdot \|f_n\|_\infty = 1 \quad \forall n$ but $\|f_n\|_{\ell^1} = \frac{1}{2n} \to 0$ as $n \to \infty$.

Thus, $\|\cdot\|_\infty \not\sim \|\cdot\|_{\ell^1}$

Lemma. (Riesz' Lemma)

Let X be a NLS and $Y \subset X$ be a closed and proper subspace. Then, for every $\varepsilon > 0$, $\exists\, u = u(\varepsilon) \in X$ s.t.

$$\|u\| = 1 \quad \text{and} \quad \|u - v\| \geq 1 - \varepsilon \quad \forall\, v \in Y.$$

Proof. Since $Y \subsetneq X$, fix $v \in X \setminus Y$.

Let

$$\delta := \inf_{y \in Y} \|v - y\|.$$

$\delta > 0$ since Y is closed. Let $\varepsilon \in (0,1)$ be arbitrary. Then, $\exists\, y_0 \in Y$ s.t.

$$\delta \leq \|v - y_0\| \leq \frac{\delta}{1-\varepsilon}.$$

Put $u := \dfrac{v - y_0}{\|v - y_0\|}$. Then $\|u\| = 1$.

Let $z \in Y$ be arbitrary. Then,

$$\|u - z\| = \frac{\|v - \underbrace{(y_0 + \|v - y_0\| z)}_{=:\, y_1 \in Y}\|}{\|v - y_0\|}$$

$$= \frac{\|v - y_1\|}{\|v - y_0\|}.$$

By def^n of δ, we have $\|v - y_1\| \geq \delta$.

Thus,

$$\|u - z\| \geq \frac{\|v - y_1\|}{\|v - y_0\|} \geq \frac{\delta}{\|v - y_0\|} \geq \frac{\delta}{\frac{\delta}{1-\varepsilon}} = 1 - \varepsilon.$$

Since $z \in Y$ was arbitrary, we are done. ∎

Cor. If X is a NLS such that closed unit ball is compact, then X is finite dimensional.

Proof. Suppose $\dim(X) = \infty$.

Pick $u_1 \in X$ with $\|u_1\| = 1$.

Then, $\langle u_1 \rangle \neq X$. Pick u_2 s.t. $\|u_2\| = 1$ and $\|u_1 - u_2\| \geq 1/2$

closed as well

Again, $\langle u_1, u_2 \rangle$ is closed and proper.

We can find u_3 with $\|u_3\| = 1$ and

$$\|u_1 - u_3\| \geq 1/2, \quad \|u_2 - u_3\| \geq 1/2.$$

Similarly, we can find $u_4, u_5, \ldots$

s.t. $\|u_n\| = 1 \ \forall n$ and

$$\|u_n - u_m\| \geq 1/2 \quad \forall n \neq m.$$

Thus, no subsequence of $\{u_n\}_n$ is Cauchy.

$\therefore$ unit ball is not compact. ∎

Example. Consider ℓ_2, the space of square summable sequences.

Define $e_j = (0, \ldots, 0, 1, 0, \ldots)$, with the 1 in position j.

Then, $\|e_j\|_2 = 1 \ \forall j$ but $\|e_j - e_k\| = \sqrt{2}$ for $j \neq k$.

Thus, no convergent subsequence. ∎

Lecture 7 (20-08-2021)

20 August 2021 14:03

Dual spaces and Hahn-Banach Theorem

Defn. ① Given a linear space X over $\mathbb{F}$, a (linear) functional is a linear operator $f : X \to \mathbb{F}$.

② Given an NLS X, a bounded (linear) functional is a bounded linear operator $f : X \to \mathbb{F}$.

Can replace "bounded" with "continuous".

In other words, $\exists c > 0$ s.t.

$$|f(x)| \leq c\|x\| \quad \forall x \in X.$$

Remark. Since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ for us, $\mathbb{F}$ is a Banach space and thus, $BL(X, \mathbb{F})$ is a Banach space w.r.t. sup norm.

$$\|f\| = \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} = \sup_{0 < \|x\| \leq 1} |f(x)| = \sup_{\|x\|=1} |f(x)|.$$

Examples.

(i) Let $X = (\mathcal{C}[a,b], \|\cdot\|_\infty)$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \int_a^b x(t)\,dt.$$

Then, f is linear. Moreover,

$$|f(x)| \leq \int_a^b |x(t)|\,dt \leq \|x\|_\infty (b-a).$$

$$\Rightarrow \|f\| \leq b-a.$$

In fact, equality holds, as witnessed by $x \equiv 1$.

(ii) Fix $t_0 \in [a,b]$. Define $g : X \xrightarrow{= \mathcal{C}[a,b]} \mathbb{R}$ by

(ii) Fix $t_0 \in [a,b]$. Define $g: X \to \mathbb{R}$ by

$$g(x) = x(t_0).$$

Then, $|g(x)| = |x(t_0)| \le \|x\|$.

$\therefore \|g\| \le 1$. Again $x \equiv 1$ shows $\|g\| = 1$.

(iii) $X = \ell_2$. Fix $\alpha = \{\alpha_j\}_{j=1}^{\infty} \in \ell_2$.

Define $f: X \to \mathbb{R}$ by

$$f(x) = f(\{x_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \alpha_j x_j.$$

this converges, by Cauchy Schwarz.

f is clearly linear.

By Cauchy Schwarz, we have

$$|f(x)| \le \sum_{j=1}^{\infty} |\alpha_j x_j| \le \left(\sum |\alpha_j|^2\right)^{1/2} \left(\sum |x_j|^2\right)^{1/2}$$
$$= \|\alpha\|_2 \; \|x\|_2.$$

$\therefore \|f\| \le \|\alpha\|_2$.

Taking $x = \overline{\alpha}$ shows $\|f\| = \|\alpha\|_2$.

- Given a vector space X, then the set of all linear functionals on X is itself a vector space. This is the algebraic dual space.

Def^n The space of all continuous linear functionals on an NLS X constitutes a normed linear space X^*, called the dual space of X.

The norm is defined as

$$\|f\| := \sup_{\substack{\|x\|=1 \\ x \in X}} |f(x)|.$$

Remark. As noted earlier, X^* is complete since $\mathbb{F}$ is so.

Q. How rich are the dual spaces?

Suppose $x \neq y \in X$. Does there exist $f \in X^*$ s.t. $f(x) \neq f(y)$?

Thm. (Hahn-Banach extension theorem)

Setup

(i) Let V be a real vector space, and W a subspace of V.

(ii) $p: V \longrightarrow \mathbb{R}$ is a sublinear functional, i.e.,

$$p(\alpha u) = \alpha\, p(u) \quad \text{and} \quad p(u+v) \leq p(u) + p(v)$$
$$\text{for } \alpha > 0 \text{ and } u, v \in V.$$

(iii) $f: W \to \mathbb{R}$ is a linear functional such that
$$f(u) \leq p(u) \quad \forall\, u \in W.$$

Then, f can be extended to a linear functional $F: V \to \mathbb{R}$ such that $F(u) \leq p(u) \quad \forall\, u \in V$.

Proof next class. (Note: No assumption of V being an NLS above!)

Consequences:

Corollary Let f be a bounded linear functional on $W \subseteq V$. (NLS)
Then, $\exists$ a bounded linear functional F on V and has the same norm.

That is, $F(w) = f(w) \quad \forall\, w \in W$ and

$$\sup_{\substack{\|v\|=1 \\ v \in V}} |F(v)| = \sup_{\substack{\|w\|=1 \\ w \in W}} |f(w)|.$$

Proof. Let $C := \|f\|_{W^*}$.

Choose $p(v) := C\|v\|$ and apply HBT. ∎

Cor. Let V be a NLS and $v_0 \neq 0$ be a nontrivial element in V. Then, there exists a bounded linear functional F s.t.

$$\|F\|_{V^*} = 1 \quad \text{and} \quad F(v_0) = \|v_0\|.$$

Proof. Put $W = \operatorname{span}\{v_0\}$. Define f on W by

$$f(\alpha v_0) = \alpha \|v_0\| \qquad \text{for } \alpha \in \mathbb{R}.$$

Then, $\|f\|_{W^*} = 1$.

By HBT, $\exists$ bounded functional F on V such that

$$F(v_0) = \|v_0\| \quad \text{with} \quad \|F\|_{V^*} = 1.$$

In particular, if $V \neq 0$, then $V^* \neq 0$.

Corollary. If $v \in V$ is such that $f(v) = 0 \ \forall f \in V^*$, then $v = 0$. ∎

Corollary. If $u, v \in V$ are such that $u \neq v$, then $\exists f \in V^*$ s.t. $f(u) \neq f(v)$. ∎

Corollary. Let V be a NLS and $v \in V$. Then,

$$\|v\| = \sup_{\substack{f \in V^* \\ \|f\| \leq 1}} |f(v)|.$$

Application to separation of hyperplanes:

Def^n V: NLS.

The set $H := \{u \in V : f(u) = \alpha\}$ (where $f \in V^*$ and $\alpha \in \mathbb{R}$ are fixed) is called a closed hyperplane.

Prop^n. Let $W \leq V$. Let $u_0 \in V$ be s.t.

$$\text{dist}(u_0, W) = \inf_{w \in W} \|u_0 - w\| > 0.$$

Then, $\exists f \in V^*$ s.t.

$$f(u) = 0 \quad \forall u \in W \quad \text{with}$$
$$\|f\|_{V^*} = 1 \quad \text{and} \quad f(u_0) = \text{dist}(u_0, W).$$

Proof. Let $W_0 = \text{span}\{u_0\} + W$.

Then, $u \in W_0 \iff u = \lambda u_0 + w$ for $\lambda \in \mathbb{R}$ and $w \in W$.

Moreover, this representation is unique.

Define $f(u) := \lambda \, \text{dist}(u_0, W) \quad \forall u \in W_0$.

Then, $f : W_0 \to \mathbb{F}$ is linear and

$$|f(u)| \leq \|u\| \quad \forall u \in W_0.$$

Thus, by earlier Corollary, $\exists F : V \to \mathbb{F}$ s.t. $F|_{W_0} = f$ and

$$|F(v)| \leq \|v\| \quad \forall v \in V.$$

Fix $\varepsilon > 0$. $\exists v \in W$ s.t.

$$\|u_0 - v\| < \text{dist}(u_0, W) + \varepsilon.$$

As $F = f$ on W_0, it follows that

$$F(u_0 - v) = \text{dist}(u_0, W) \quad \text{and thus,}$$

$$\frac{F(u_0 - v)}{\|u_0 - v\|} \geq \frac{\text{dist}(u_0, W)}{\text{dist}(u_0, W) + \varepsilon}.$$

Letting $\varepsilon \to 0$ gives $\|F\|_{V^*} = 1$.

Lecture 8 (24-08-2021)

24 August 2021 14:12

Thm (Hahn–Banach)

Assume that

(i) W is a linear subspace of a real linear space V,

(ii) $p: V \to \mathbb{R}$ is a sublinear functional, i.e.,

$$\left.\begin{aligned} p(u+v) &\le p(u) + p(v), \\ p(\alpha u) &= \alpha p(u), \end{aligned}\right\} \begin{aligned} &\forall\, u, v \in V \\ &\alpha \ge 0 \end{aligned}$$

(iii) $f: W \to \mathbb{R}$ is a linear functional such that

$$f \le p \quad \text{on } W.$$

Then, f can be extended to a linear functional $F: V \to \mathbb{R}$ such that $F \le p$ on V.

Proof.

Step 1. Assume $V = W + \operatorname{span}\{v\}$ for some $v \notin W$.

Define $F: V \to \mathbb{R}$ by

$$F(w + \lambda v) := f(w) + c\lambda \qquad \text{for } w \in W \text{ and } \lambda \in \mathbb{R}.$$

Where we fix any $c \in \mathbb{R}$ satisfying

$$\sup_{w \in W} \left(f(w) - p(w - v) \right) \le c \le \inf_{w \in W} \left(p(w+v) - f(w) \right)$$

First, assuming existence of such a c, we show that F does the job.

Clearly, F is linear. Remains to show

$$F(w + \lambda v) \le p(w + \lambda v).$$

- $\lambda = 0$. Then, it is clear.
- $\lambda > 0$. We have

$$\begin{aligned} c &\le p(\lambda^{-1} w + v) - f(\lambda^{-1} w) \\ &= p(\lambda^{-1}(w + \lambda v)) - f(\lambda^{-1} w) \end{aligned}$$

$$= \lambda^{-1}\{p(w+\lambda v) - f(w)\}$$

$$\Rightarrow \quad \underset{\substack{\| \\ F(w+\lambda v)}}{\lambda c + f(w)} \leq p(w+\lambda v)$$

✓

- $\lambda < 0$:

$$c \geq f(-\lambda^{-1}w) - p(-\lambda^{-1}w - v)$$
$$= -\lambda^{-1}(f(w) - p(w+\lambda v))$$
$$\Rightarrow \quad -c\lambda \geq f(w) - p(w+\lambda v)$$
$$\Rightarrow \quad p(w+\lambda v) \geq f(w) + c\lambda = F(w+\lambda v)$$

✓

Now, we have to show existence of such a c.
For $u, w \in W$, we note

$$f(u) + f(w) = f(u+w) \leq p(u+w)$$
$$\leq p(u-v) + p(v+w)$$

$$\Rightarrow \quad f(u) - p(u-v) \leq p(v+w) - f(w).$$

The above is true for all $u, w \in W$. Take inf and sup. ☒

Step II. Zorn's Lemma. Assume $W \leq V$ is arbitrary.

Define $\Sigma = \{(W', F') : W \leq W' \leq V$ and $F' : W' \to \mathbb{R}$ extends f and $F' \leq p.\}$

Define $(W', F') \leq (W'', F'')$ if $W' \leq W''$ and $F''|_{W'} = F'$.

$\Sigma \neq \phi$ since $(W, f) \in \Sigma$.

$\leq$ defines a partial order on Σ.

Given any chain, the usual union trick works.

Thus, Σ has a maximal element. If that is not all of V, use Step I to extend and get a contradiction. ☒

Extension to complex spaces:

Thm. Let V be a NLS over $\mathbb{C}$ and W be a linear subspace of V. Let $f: W \to \mathbb{C}$ be a linear functional such that $\exists\, \alpha > 0$ with

$$|f(w)| \leq \alpha \|w\| \quad \forall\, w \in W.$$

Then, f can be extended to a linear continuous functional $F: V \to \mathbb{C}$ s.t.

$$|F(v)| \leq \alpha \|v\| \quad \forall\, v \in V.$$

Proof. Define $h(w) := \mathrm{Re}\, f(w)$ for $w \in W$.

Then,

$$\begin{aligned} f(w) &= \mathrm{Re}\, f(w) + i\, \mathrm{Im}\, f(w) \\ &= \mathrm{Re}\, f(w) + i\, \mathrm{Im}\, (-i)\, f(iw) \\ &= \mathrm{Re}\, f(w) - i\, \mathrm{Re}\, f(iw) \\ &= h(w) - i\, h(iw) \qquad \forall\, w \in W. \end{aligned}$$

$$\left(\begin{aligned} &z = x + iy \\ &\mathrm{Im}\,(-iz) \\ &= \mathrm{Im}(-ix + y) \\ &= -x = -\mathrm{Re}(z) \end{aligned} \right)$$

Moreover, $|h(w)| \leq \alpha \|w\| \quad \forall w \in W$.

Regarding V as an $\mathbb{R}$-nls, $\exists\, H: V \to \mathbb{R}$ continuous#

s.t. $H|_W = h$ and $|H(v)| \leq \alpha \|v\| \quad \forall\, v \in V$.

Now, set $F(v) = H(v) - i\, H(iv) \quad \forall\, v \in V$.

Check: $F|_W = f$.

Moreover, F is linear.

Claim. $|F(v)| \leq \alpha \|v\| \quad \forall\, v \in V$.

Proof. Take $v \in V$ and write $F(v) = re^{i\theta}$ for $r \geq 0$.

Then,

$$\begin{aligned} |F(v)| = r &= \mathrm{Re}\,(e^{-i\theta} F(v)) \\ &= \mathrm{Re}\, F(e^{-i\theta} v) \\ &= H(e^{-i\theta} v) \leq \alpha \|e^{-i\theta} v\| = \alpha \|v\|. \ \blacksquare \end{aligned}$$

#**Remark.** The "continuous" above is due to a corollary of HBT that we saw last class.

Is this extension unique?

Not necessarily.

① Take $V = \mathbb{R}^2$ and $W = \mathbb{R} \times \{0\} \subseteq V$.

Let $f : W \to \mathbb{R}$ be $(y_1, 0) \mapsto y_1$.

Then, $F_1, F_2 : V \to \mathbb{R}$ defined by $(x_1, x_2) \mapsto x_1 \pm x_2$ both extend f.

Both of these preserve the ℓ_1 norm.

② Let $V = \mathcal{C}[0,1]$ with sup norm.

$W = \{x \in V : x \text{ is constant}\}$

Define $f : W \to \mathbb{R}$ by $w \mapsto w(0)$.

Now, for $t \in [0,1]$, define $F_t : V \to \mathbb{R}$ by $v \mapsto v(t)$.

Then, each F_t is a norm preserving extension since $\|F_t\| = 1 = \|f\|$.

There are infinitely many distinct extensions.

Can we impose some condition on V to attain unique extension?

Def^n. A NLS V is said to be strictly convex if

$$\left\| \frac{u+v}{2} \right\| < 1 \quad \text{whenever } \|u\| = \|v\| = 1 \text{ with } u \neq v.$$

Examples. On X : N dim NLS with norm $\|\cdot\|_{\ell_p^N}$.

When $p = 1$ or ∞, it is not strictly convex.

For $1 < p < \infty$, it is strictly convex.

To check this, use Minkowski, which says that

$$\left(\sum_{j=1}^{N} |u_j + v_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{N} |u_j|^p \right)^{1/p} + \left(\sum_{j=1}^{N} |v_j|^p \right)^{1/p}$$

and equality holds iff $\exists \lambda \in \mathbb{R}$ s.t.

$u_j = \lambda v_j$ for all $j = 1, \dots, N$.

Thm. Let X be a NLS and X^* be its dual. Then, every bounded linear functional on every subspace of X has a unique norm preserving extension to X $\Leftrightarrow$ X^* is strictly convex.

Lecture 10 (27-08-2021)

27 August 2021 14:00

Dual spaces of C[a, b] and L^p

F. Riesz (1909) showed that any continuous linear functional

$$f : \mathcal{C}[a,b] \to \mathbb{R}$$

could be written uniquely as

$$f = u \longmapsto \int_a^b u(x)\, d\rho(x)$$

where ρ is a function of bounded variation in $[a,b]$ provided it satisfies some additional properties

Def^n A function $\rho : [a,b] \to \mathbb{R}$ is said to be of bounded variation on $[a,b]$ if its total variation $V(\rho)$ on $[a,b]$ is finite, where

$$V(\rho) = \sup_{\text{all partitions}} \sum_{i=1}^{n} |\rho(t_j) - \rho(t_{j-1})|.$$

Let $BV[a,b]$ be the set of all functions of bounded variations on $[a,b]$. It is a vector space with norm

$$\|\rho\|_{BV[a,b]} := |\rho(a)| + V(\rho).$$

$BV[a,b]$ is a NLS.

For any $u \in \mathcal{C}[a,b]$ and $\rho \in BV[a,b]$, we can associate a Riemann-Stieltjes integral

$$\int_a^b u(t)\, d\rho(t) \quad \text{as follows:}$$

Let Π_n be any partition of $[a,b]$ given as
$$a = t_0 < t_1 < \cdots < t_n = b \quad \text{and}$$
$$h(\Pi_n) := \max_{1 \le i \le n} |t_i - t_{i-1}|.$$

Define
$$s(\Pi_n) := \sum_{j=1}^{n} u(t_j)\big(\rho(t_j) - \rho(t_{j-1})\big).$$

There exists $I \in \mathbb{R}$ such that:
$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall \Pi_n : h(\Pi_n) < \delta \Rightarrow |s(\Pi_n) - I| < \varepsilon.$$

This number I (necessarily unique) is called the Riemann–Stieltjes integral and is denoted by
$$\int_a^b u(t)\, d\rho(t).$$

($\rho(t) = t$ gives the ordinary Riemann-integral.)

Moreover, as $\rho \in BV[a,b]$, then

$$(10.1) \qquad \left| \int_a^b u(t)\, d\rho(t) \right| \le \max_{a \le t \le b} |u(t)|\, V(\rho) = \|u\|_\infty V(\rho).$$

For each $\rho \in BV[a,b]$, define $f_\rho : C[0,1] \to \mathbb{R}$ by
$$f_\rho(u) := \int_a^b u(t)\, d\rho(t).$$

Clearly, f_ρ is linear.

By (10.1), f_ρ is continuous as well.

Thus, $f_\rho \in (\mathcal{C}[a,b])^*$.

Note that $f_{\rho_1} = f_{\rho_2} \Leftarrow \rho_1 - \rho_2 \equiv$ constant.

We wish to make $\rho \mapsto f_\rho$ injective.

To this end, we define the subspace $NBV[a,b] \subseteq BV[a,b]$ by

$$NBV[a,b] = \{ \rho \in BV[a,b] : \rho(a) = 0 \text{ and } \rho \text{ is right continuous} \}.$$

This space is known as Normalised Bounded Variation.

Thm. Let $-\infty < a < b < \infty$.

Then, $f \in X^* = (\mathcal{C}[a,b])^*$ iff

$\exists \rho \in NBV[a,b]$ s.t. $f = f_\rho$.

In addition,

$$\|f\|_{X^*} = V(\rho).$$

Thus, $\mathcal{C}[a,b]^*$ is isometrically isomorphic to $NBV[a,b]$.

Proof. Let $f = f_\rho$. We have shown

$$|f(u)| \leq V(\rho) \|u\|_\infty \quad \forall u \in X.$$

$$\therefore \|f\|_{X^*} \leq V(\rho) \qquad (10.2)$$

To show: reverse inequality

Let Y denote the space of all bounded functions from $[a,b]$ to $\mathbb{R}$. This is a NLS with sup norm.

Moreover, X is a linear subspace of Y.

By Hahn-Banach, we can extend f to $F: Y \to \mathbb{R}$ continuous s.t. $\|F\|_{Y^*} = \|f\|_{X^*}$.

Claim. $V(\rho) \leq \|F\|_{Y^*}$.

Proof. Set $\rho(t) := F(v_t) \quad \forall t \in [a, b)$ with

$$v_t := \begin{cases} 0 \; ; & t = a, \\ 1 \; ; & a < t \leq b. \end{cases}$$

Claim. $\rho \in BV[a, b]$, $V(\rho) \leq \|f\|_{X^*}$.
$\rho(0) = 0$.

Consider $\Pi_n = \{a = t_0 < \cdots < t_n = b\}$ and observe that with $\alpha_j = \operatorname{sgn}(\rho(t_j) - \rho(t_{j-1}))$, we have

$$\sum_{j=1}^{n} |\rho(t_j) - \rho(t_{j-1})| = \sum_{j=1}^{n} \alpha_j \left(\rho(t_j) - \rho(t_{j-1}) \right).$$

$$\therefore \sum_{j=1}^{n} |\rho(t_j) - \rho(t_{j-1})| = F(u),$$ with
$$u = \sum_{j=1}^{n} \alpha_j \left(v_{t_j} - v_{t_{j-1}} \right).$$

Dual of L^p Spaces

Let Ω be an open set in $\mathbb{R}^d$. (Can replace "open" with "measurable".)
Recall, for $1 \leq p \leq \infty$, we have

$$\mathcal{L}^p(\Omega) := \{ f : \|f\|_{L^p(\Omega)} := \int_\Omega |f|^p < \infty \}.$$

(Modulo = a.e.)

f are measurable.

For $p = \infty$, we have

$$\|f\|_{L^\infty(\Omega)} = \inf \{ M > 0 : \mu(\{ |f| > M \}) = 0 \}.$$

$$\mathcal{L}^p(\Omega) := \{ f : \|f\|_\infty < \infty \}.$$

Recall:

Lemma. (Hölder) Let $p \in [1, \infty)$ with q s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$. Then,

$$\|uv\|_1 \leq \|u\|_p \|v\|_q.$$

(Minkowski) For $p \in [1, \infty]$, we have

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Thus, $L^p(\Omega)$ is a NLS. In fact, it is Banach.

Property 1. For $1 \leq q \leq p$, we have (not assuming conjugates)

$$L^p(\Omega) \subseteq L^q(\Omega), \quad \text{provided } \mu(\Omega) < \infty.$$

In fact, this embedding is continuous.

Sketch. For $u \in \mathcal{L}^p(\Omega)$, check

$$\int_\Omega |u|^q \leq \left(\int_\Omega (|u|^q)^{p/q} \right)^{q/p} \left(\int_\Omega 1 \right)^{1 - q/p}$$

$$= \left(\int_\Omega |u|^p \right)^{q/p} (\mu(\Omega))^{1 - q/p}$$

$$\Rightarrow \quad \|u\|_{L^q(\Omega)} \le (\mu(\Omega))^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(\Omega)}. \quad \blacksquare$$

Riesz Representation Theorem (1910):

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Then, the dual space of $L^p(\Omega)$ is isometrically isomorphic to $L^q(\Omega)$.

(Today we only shown inclusion. Onto later.)

Proof. Let $v \in L^q(\Omega)$. Define $T_v : L^p(\Omega) \to \mathbb{R}$ by

$$T_v(u) := \int_\Omega u v \qquad \forall u \in L^p(\Omega).$$

Check: T_v is linear.

By Hölder, T_v is continuous with $\|T_v\|_{(L^p)^*} \le \|v\|_{L^q}$.

Thus, $T_v \in (L^p(\Omega))^*$.

To show: $\|v\|_{L^q} \le \|T_v\|_{(L^p)^*}$.

Choose $u(x) := \begin{cases} |v(x)|^{q-2} v(x) & ; v(x) \neq 0, \\ 0 & ; v(x) = 0. \end{cases}$

Then, $|u|^p = |v|^{(q-1)p} = |v|^q$ and thus, $u \in L^p(\Omega)$.

Further, $T_v(u) = \int_\Omega |v|^q$. Thus, we are done. $\blacksquare$

Thus, we have shown $v \mapsto Tv$ is an isometry.

(It is also linear.) ∎

Lecture 11 (31-08-2021)

31 August 2021 13:58

Banach-Steinhaus Theorem
(Uniform boundedness principle)

In 1922, Hahn proved "If $T_n \in BL(X, Y)$ (where X is Banach and Y is any NLS) satisfy:

$$\forall u \in X, \exists M_u \geq 0, \forall n \geq 1: \quad |T_n(u)| \leq M_u.$$

Then, $\{\|T_n\|_{BL(X,Y)}\}_n$ is bounded."

Used the "method of gliding bump".

Recall:

Thm. (Baire Category Theorem)

- Let (X, d) be a complete metric space, and let $\{V_n\}_{n=1}^{\infty}$ be a collection of open dense sets. Then,

$$\bigcap_{n=1}^{\infty} V_n \quad \text{is also dense in } X.$$

- Equivalently, a (nonempty) complete metric space cannot be a countable union of nowhere dense sets.

  ↳ closure has empty interior

Def^n A countable intersection of open sets in a topological space is called a G_δ set.

Cor. Let W be a subset of a Banach space X, and let W be of the first category (i.e., can be written as a countable union of nowhere dense subsets of X).
Then, $\exists u \in X$ s.t. $u \notin W$.
Moreover, W^c is of second category in X.

Consequence: (Weierstrass)

There exists a nowhere differentiable continuous function $f : [0, 1) \to \mathbb{R}$.

<u>Sketch</u>. Define

$$W := \{ f \in \mathcal{C}[0,1] : \exists x^* \in [0,1) \text{ s.t. right hand derivative } f'_+(x^*) \text{ exists} \}.$$

<u>Claim</u>: W is of 1^{st} category.

Assume Claim. Then, $X - W$ is nonempty. This proves the result.

<u>Proof of Claim</u>.

Define $W_n := \{ f \in \mathcal{C}[0,1] : \exists x^* \in [0,1)$ with $|f(x^*+h) - f(x^*)| \le nh \quad \forall h \in [0,1)$ with $x^*+h \le 1 \}$.

Then, $W \subseteq \bigcup_{n=1}^{\infty} W_n$.

To show: each W_n is nowhere dense in $\mathcal{C}[0,1]$.

① Each W_n is closed.

Let $\{f_j\}_{j=1}^{\infty} \subseteq W_n$ have a limit in X.

Then, $\exists x_j \in [0,1]$ s.t.

$$(*) \quad |f_j(x_j + h) - f_j(x_j)| \le nh \quad \forall h \in [0,1] \text{ with } x_j + h \le 1.$$

By HBT, $\exists$ convergent subseq. of $\{x_j\}_j$. Call it $\{x_j\}_j$ again.

Then, $x_j \to x^*$.

Let $j \to \infty$ in $(*)$ to get $|f(x^* - h) - f(x^*)| \le nh$.

$\therefore f \in W_n$. ∎

② $\text{int}(W_n) = \phi \quad \forall n$.

Let $f \in W_n$ and $\varepsilon > 0$ be given.

$\exists$ piecewise linear continuous $g : [0,1] \to \mathbb{R}$ s.t.

$$\|f - g\|_\infty = \max_{0 \le x \le 1} |f(x) - g(x)| < \varepsilon$$

and $|g_+'(x)| > n \quad \forall\, x \in [0,1]$.

$\therefore f \notin \operatorname{int}(W_n)$.

Uniform Boundedness Theorem

Let $\{T_i\}_{i \in I}$ be a family in $BL(X, Y)$, where X is a Banach space, Y is an NLS, and I is an arbitrary indexing set.

Further, let

$$\sup_{i \in I} \|T_i x\|_Y < \infty \quad \forall x \in X. \qquad (11.1)$$

Then,

$$\sup_{i \in I} \|T_i\|_{BL(X,Y)} < \infty, \qquad (11.2)$$

i.e., $\exists\, c > 0$ s.t.

$$\|T_i x\|_Y \le c\|x\|_X \quad \forall x \in X\ \forall i \in I.$$

Proof. For every $n \ge 1$, define

$$X_n := \{x \in X : \|T_i x\| \le n \quad \forall i \in I\}. \qquad (11.3)$$

Clearly, each X_n is closed.

Moreover,

$$X = \bigcup_{n=1}^{\infty} X_n,$$

$$X = \bigcup_{n=1}^{\infty} X_n,$$

by (11.1).

By BCT, $\exists\, n_0 \geq 1$ s.t. $\operatorname{int}(X_{n_0}) \neq \emptyset$.

Pick $x^* \in X_{n_0}$ and $r > 0$ s.t.

$$B_r(x^*) \subseteq X_{n_0}.$$

By definition (11.3), it follows that

$$\|T_i(x^* + rz)\|_Y \leq n_0 \quad \forall\, i \in I \quad \forall z \in B_1(0).$$

This implies $\forall z \in B_1(0)$,

$$r\,\|T_i(z)\|_Y - \|T_i(x^*)\|_Y \leq \|r\,T_i(z) + T_i(x^*)\| \leq n_0.$$

$$\Rightarrow \|T_i(z)\|_Y \leq \frac{1}{r}\Big(\underbrace{n_0 + \|T_i(x^*)\|_Y}_{=:\,c}\Big) \quad \forall i \in I \quad \forall z \in B_1(0).$$

Taking $\sup\limits_{z \in B_1(0)}$ shows

$$\|T_i\|_{BL(X,Y)} \leq C \quad \forall\, i \in I. \qquad \blacksquare$$

Some consequences:

Corollary. Let $\{T_n\}_n$ be a sequence in $BL(X, Y)$ $\begin{pmatrix} X \to \text{Banach} \\ Y \to \text{NLS} \end{pmatrix}$ and $T : X \to Y$ be defined by

$$T(x) := \lim_{n \to \infty} T_n(x) \qquad \forall\, x \in X.$$

(limit exist $\forall x$.)

Then, the following hold:

(i) $\sup_n \|T_n\|_{BL(X,Y)} < \infty$,

(ii) $T \in BL(X,Y)$,

(iii) $\|T\|_{BL(X,Y)} \leq \liminf_{n \to \infty} \|T_n\|_{BL(X,Y)}$.

Proof. (i) follows directly from UBP.

(ii) Linearity of T is trivial.
Letting $n \to \infty$ gives T is bounded.

(iii) $\forall x \in X$, we have

$$\|T_n x\|_Y < \|T_n\|_{BL(X,Y)} \|x\|_X.$$

Taking $\liminf_{n \to \infty}$ gives

$$\|Tx\|_Y \leq \left(\liminf_{n \to \infty} \|T_n\|_{BL(X,Y)}\right) \|x\|_X. \qquad 习$$

Cor. Let V be a Banach space and $Z \subseteq V$.
Further suppose that $\forall f \in V^*$, the set
$$f(Z) := \{ f(z) : z \in Z \}$$ is bounded in $\mathbb{F}$.
Then, Z is bounded.

Proof. Apply UBP with $X = V^*$ and $Y = \mathbb{F}$ and $I = Z$.
For $z \in Z$, define $T_z : V^* \to \mathbb{F}$ by
$$T_z(f) = f(z).$$

(Note T_z is indeed linear. Furthermore,
$$\|T_z(f)\| = |f(z)| \leq \|f\| \cdot \|z\|.$$
$\therefore \|T_z\| \leq \|z\|$ and hence, T_z is bounded)

By assumption, $\sup_{z \in Z} |T_z(f)| < \infty \quad \forall f \in V^*$.

UBP
"

Thus, BST says that $\exists c > 0$ s.t.

$$|f(z)| \leq c \, \|f\|_{V^*} \quad \forall f \in V^* \quad \forall z \in Z.$$

Hence, $\forall z \in Z$, we have

$$\|z\|_V \underset{\text{HBT}}{=} \sup_{0 \neq f \in V^*} \frac{\|f(z)\|}{\|f\|_{V^*}} \leq c. \qquad \blacksquare$$

Dual corollary: Let V be a Banach space and $Z^* \subseteq V^*$. Assume that $\forall x \in V$, the set

$$x(Z^*) = \{ f(x) : f \in Z^* \} \text{ is bounded.}$$

Then, Z^* is bounded.

Proof. Exercise.

Applications:

(1) Space of polynomials. Let X be the space of all polynomials with norm

$$\left\| \sum \alpha_i x^i \right\| = \max_i |\alpha_i|.$$

X is a NLS. Is it complete?

No. We shall apply BST.

"Construct a sequence of BL operators which are p-wise bounded but not uniformly."

(2) $\exists$ real valued continuous functions whose Fourier series diverges at a point.

Lecture 12 (03-09-2021)

03 September 2021 14:14

Open Mapping Theorem (Banach Schauder Theorem)

Theorem. If a ~~continuous~~ linear operator between two normed linear spaces is an open map, then it is surjective.

takes open sets to open sets

Prop^n 12.1. Interior of a proper subspace of a normed linear is empty.

Proof. Let W be a subspace of a NLS V s.t. $W^\circ \neq \phi$.
Pick $w \in W^\circ$. Then, $w \in W$ and $\exists r > 0$ s.t. $B_r(w) \subseteq W$.
Let $v \in V \setminus \{0\}$ be arbitrary.
Then,
$$w + \frac{r}{2\|v\|} v \in B_r(w) \subseteq W.$$
$$\therefore \quad w + \frac{r}{2\|v\|} v \in W.$$
Thus, $v \in W$, as desired. ∎

Cor 11.2. A Banach space can not have a countably infinite (Hamel) basis.

(Use BCT and Prop^n 12.1. If $\{v_1, v_2, \dots\}$ is a basis, consider $V_n = \text{span}\{v_1, \dots, v_n\}$. Then, $V = \bigcup V_n$.)

Example. The space $\mathcal{P}$ of polynomials is not Banach since $\{1, t, t^2, \dots\}$ is a basis.

Def^n. A sequence $\{u_n\}_{n=1}^{\infty}$ in an infinite dimensional NLS V is said to be a Schauder basis of V if for every $v \in V$, $\exists$ unique sequence $(\alpha_n)_{n=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ s.t.

$$v = \sum_{i=1}^{\infty} \alpha_j v_j.$$

Observations.

(1) If $\{v_i\}_{i=1}^{\infty}$ is a Schauder basis, then it is linearly independent.

(2) If $\{v_i\}_{i=1}^{\infty}$ is a SB, then span $\{v_i\}_{i=1}^{\infty}$ is dense in V.

(3) Any countable Hamel basis is also a SB.

Def^n. An NLS is separable if it has a countable dense subset.

Examples.
- ℓ_p, $1 \leq p \lneq \infty$,
- $C(\bar{\Omega})$ with $\|\cdot\|_{L^p}$ $1 \leq p \leq \infty$ as $P(\bar{\Omega})$ is dense in $C(\bar{\Omega})$.
- $\mathcal{L}^p(\bar{\Omega})$ for $1 \leq p \lneq \infty$ is sep. since $C(\bar{\Omega})$ is dense in $\mathcal{L}^p(\bar{\Omega})$.

Prop^n 12.3. Let X and Y be two NLS and $T : X \to Y$ be a linear operator. If T is an open map, then T is surjective.

Proof. If T is open, then the range $R(T)$ is open. Thus, $(R(T))^{\circ} = R(T) \neq \phi$. By Prop^n 11.2., it follows that T is onto. ∎

The "converse" is true if X and Y are Banach and T is continuous.

Thm 12.4. (Open mapping theorem)

Let $T : V \to W$ be a continuous linear operator, where V and W are Banach. Further, assume that T is onto. Then, T is an open map.

Some preliminary propositions:

Let $A, B \subseteq X$. $\rightarrow$ vector space

Define

$$A + B := \{x + y : x \in A, y \in B\},$$
$$\lambda A := \{\lambda x : x \in A\}.$$

If A is convex, then $A + A = 2A$.

Proof. $2A \subset A + A$ is trivial (and true in general).

Let $z \in A + A$ be arbitrary. Write $z = x + y$ for $x, y \in A$.

Then, $z = 2\underbrace{\left(\frac{x+y}{2}\right)}_{\in A, \text{ by convexity}}$. ∎

Lemma 12.5. Let $T \in BL(V, W)$, where V and W are Banach spaces. Further, assume T is onto. Then,

$$\exists r > 0 \text{ s.t.}$$
$$B_W(0, r) \subseteq T(B_V(0, 1)).$$

Assuming the above, let us prove the open mapping theorem 12.4.

Proof (of 12.4 using 12.5)

Let $U \subseteq V$ be an arbitrary open set. TS: $T(U)$ is open.

Let $w \in T(U)$. There exist $u \in U$ s.t. $T(u) = w$.

Since U is open, can find $r > 0$ s.t.

$$B_V(u, r) \subset U.$$

By translation, we get that $U - u$ contains

$B_r(0, r)$ ↳ translation

Thus, $\frac{1}{r}(U - u)$ contains $B_r(0, 1)$.

By 12.5., $T\left(\frac{1}{r}(U-u)\right) \supseteq B_W(0, \varepsilon)$ for some $\varepsilon > 0$.

Using linearity, we can scale and translate back to get that $T(U)$ contains an open ball around w. Since $w \in T(U)$ was arbitrary, $T(U)$ is open. ∎

Cor 11.6. If $T \in BL(V, W)$ is bijective and V, W Banach, then T^{-1} is continuous.

(Isomorphism + continuous $\Rightarrow$ isomorphism + homeomorphism)

Cor 11.7 (Well-posedness of linear operator)

Let $T \in BL(X, Y)$, with X, Y Banach spaces.

TFAE:

(i) The problem (*) is well-posed:

Given $y \in Y$, find $x \in X$ s.t. $Tx = y$. (*)

Well-posed means: $\exists!$ unique sol^n and it depends continuously on data.

(ii) For each $y \in Y$, $T(x) = y$ has a sol^n and $Tz = 0 \Rightarrow z = 0$

Completeness of X and Y cannot be omitted.

Lecture 13 (07-09-2021)

07 September 2021 14:00

Open mapping theorem and closed graph theorem

Proof of 12.5:

Step 1. Claim: $\exists\, r > 0$ s.t. $B_W(0, 2r) \subseteq \overline{T(B_V(0,1))}$.

Proof. Set $W_n := n\,\overline{T(B_V(0,1))}$.

Each W_n is closed. Since T is linear and onto, we have

$$W = \bigcup_{n=1}^{\infty} W_n.$$

Since W is Banach, BCT gives $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\operatorname{int}(W_{n_0}) \neq \emptyset.$$

Note $W_{n_0} = n_0\,\overline{T(B_V(0,1))}$ and thus

$$\left(\overline{T(B_V(0,1))}\right)^{\circ} \neq \emptyset.$$

$\therefore\ \exists\, y_0 \in W$ and $r > 0$ s.t.

$$B_W(y_0, 4r) \subseteq \overline{T(B_V(0,1))}.$$

Note that by "symmetry", we also have $-y_0 \in \overline{T(B_V(0,1))}$.

Note that any element of $B_W(y_0, 4r)$ is of the form $y_0 + z$ for some $z \in B_W(0, 4r)$.

Also, note any $z \in B_W(0,4r)$ can be written as

$$z = (y_0 + z) + (-y_0).$$

Thus,

$$B_W(0, 4r) \subset \overline{T(B_V(0,1))} + \overline{T(B_V(0,1))}$$

$$\parallel \quad \text{by convexity}$$

$$2\,\overline{T(B_V(0,1))}$$

Thus, $B_W(0, 4r) \subset 2\,\overline{T(B_V(0,1))}$ or

$$B_W(0, 2r) \subset \overline{T(B_V(0,1))}.$$

This finishes the first step.

Step 2 To show: $B_W(0, r) \subset T(B_V(0,1))$.

Let $y \in B_W(0, r)$. To show: $\exists x \in B_V(0,1)$ s.t. $T(x) = y$.

By step 1: Choosing $\epsilon = r/2$, we find $z_1 \in V$ s.t. $\|z_1\|_V < 1/2$ and $\|y - T(z_1)\|_W < \epsilon$.

(We showed $B_W(0, 2r) \subset \overline{T(B_V(0,1))}$
Thus, $B_W(0, r) \subset \overline{T(B_V(0, 1/2))}$. Since $y \in B_W(0, r)$, we can $z \in B_V(0, 1/2)$ as above.)

Now, apply the result to $y - T(z_1)$ to get $z_2 \in B_V(0, 1/4)$ s.t.

$$\|y - T(z_1) - T(z_2)\| < r/2^2.$$

Keep proceeding to get $z_1, z_2, z_3, \ldots$ s.t.

$$\|z_n\| < 1/2^n \quad \text{and} \quad \|y - T(z_1 + \cdots + z_n)\| < r/2^n.$$

Since $\sum \|z_n\| < \infty$, Banachness gives us that $\sum z_n$ converges to say, z.
Thus,

$$\|y - T(z)\| = \lim_n \left\|y - T\left(\sum_{k=1}^n z_k\right)\right\|$$

$$\leq \lim_n r/2^n = 0.$$

Moreover $\|z\| \leq \sum_{n=1}^{\infty} 1/2^n = 1$.

This finishes the proof. ∎

Def^n. By the graph $G(T)$ of the operator

$V, W \to$ NLS

$$T : \text{domain}(T) \xrightarrow{\subseteq V} W,$$ we mean

$$G(T) := \{(v, T(v)) : v \in \text{dom}(T)\} \subset V \times W.$$

The operator is called closed if $G(T)$ is closed in $V \times W$.
(Not assuming linear.)

Note: If T is continuous and $\text{dom}(T)$ is closed, then $G(T)$ is closed.

Theorem 13.1. (Closed graph theorem)

Let V and W be Banach spaces. Further, let

$$T : D(T) \xrightarrow{\subset V} W$$ be a linear mapping

closed subspace

s.t. $G(T)$ is closed. Then, T is continuous.

Proof. Define the norm on $V \times W$ as
$$\|(v, w)\|_{V \times W} := \|v\|_V + \|w\|_W.$$
(This make $V \times W$ a Banach space with product topology.)

Note that $D(T)$ and $G(T)$ are Banach themselves in the restricted norm.

Define
$$S : G(T) \longrightarrow D(T) \quad \text{by}$$
$$S(u, T(u)) := u.$$

Clearly, S is linear, bounded, and a bijection.

Thus, by OMT, S^{-1} is also continuous. But the inverse is simply

$$p(u) = (u, T(u)).$$

$\therefore \exists c$ s.t.

$$\underbrace{\|p(u)\|}_{\|u\| + \|T(u)\|} \leq c\|u\| \Rightarrow \|T(u)\| \leq (c-1)\|u\|. \blacksquare$$

<u>Example</u> of <u>a closed linear operator which is not bounded</u>

$V = (\mathcal{C}[0,1], \|\cdot\|)$. Define

$$T: D(T) \to V \text{ by } Tu = u',$$

where $D(T) = \mathcal{C}'[0,1]$.

We know T is not bounded.

<u>Claim</u>. T is closed.

<u>Proof</u>. Suppose $\{u_n\} \subset D(T)$ is s.t. $u_n \to v \in V$ and $u_n' \to w$.

Then,
$$\int_0^t w = \int_0^t \lim u_n' = \lim \int_0^t u_n' = \lim [u_n(t) - u_n(0)]$$
$$= v(t) - v(0).$$

$\therefore v \in D(T)$ with $T(v) = v' = w$. $\blacksquare$

(This shows that $D(T)$ is not closed.)

Conversely, boundedness does not imply closed.

Let V be a space with $D(T) \subsetneq V$ a dense proper subspace and define $T: D(T) \hookrightarrow V$ to be inclusion.

Some interesting results:

Some interesting results:

Let $A : D(A) \xrightarrow{} Y$ ($D(A) \subset X$) be a bounded linear operator with X, Y NLS. Then:

(i) If $D(A) \subset X$ is closed, then $A \subset X$ is closed.

(ii) If $A \subset X$ is closed and Y is complete, then $D(A) \subset X$ is closed.

Lecture 14 (21-09-2021)

21 September 2021 14:06

Duality, weak convergence

Recall: Given an NLS X, we have its dual X^*.
Moreover, X^* is Banach even if X is not.

Def^n. (Duality pairing)
This is a bilinear map $\langle \cdot, \cdot \rangle : X^* \times X \longrightarrow \mathbb{F}$.
$\forall f \in X^* \setminus \{0\}$, $\exists u \in X$ s.t. $f(u) \neq 0$,
$\forall u \in X^* \setminus \{0\}$, $\exists f \in X^*$ s.t. $f(u) \neq 0$.

Example. ① $\langle f, u \rangle := f(u)$.

② $C_c^\infty(\Omega) \to C^\infty$ and compact support.
$\quad \searrow$ Can't put a norm

$C_c^\infty(\Omega)^*$ is a huge space.

$$C_c^\infty(\Omega) \subset L^1(\Omega) \subset \cdots \subset (C_c^\infty(\Omega))^*.$$

For every $f \in (C_c^\infty(\Omega))^*$, define its generalised partial derivative $\partial_j f$ as

$$\langle \partial_j f, \varphi \rangle := - \left\langle f, \frac{\partial \varphi}{\partial x_j} \right\rangle \quad \forall \varphi \in C_c^\infty(\Omega).$$

$$\left((\partial_j f)(\varphi) := -f(\partial_j \varphi). \right)$$

§ 14.1 WEAK CONVERGENCE

Recall the classical Weierstrass problem

" let $F : M \xrightarrow{\subseteq X} \mathbb{R}$ be continuous, where X is Banach and $\emptyset \neq M \subseteq X$ is compact.

$$\exists\, u \in M \text{ s.t. } F(u) = \min_{v \in M} F(v). \text{"} \qquad (14.1)$$

Unfortunately, in infinite dim. Banach spaces, closed and bounded balls are not compact. Thus, we cannot use the above result for such balls.

Recall:

Defn. (Strong Convergence)

A sequence $(v_n)_n$ in a NLS X is said to be strongly (norm) convergent if $\exists\, v \in X$ s.t.

$$\lim_{n \to \infty} \| v_n - v \|_X = 0.$$

Denoted: $v_n \to v$ in X.

The above is the usual convergence. We now weaken it.

Defn. A sequence $(v_n)_n$ in a NLS X is said to be weakly convergent if $\exists\, v \in X$ s.t. for all $f \in X^*$:

$$\lim_{n \to \infty} f(v_n) = f(v). \qquad \left(\langle f, v_n \rangle \to \langle f, v \rangle \right)$$

We denote this as: $v_n \rightharpoonup v$ in X.

(Note the single-sided arrow.)

We also call v a weak limit of $(v_n)_n$.

can replace with "the"

Connection to topological structure:

Basic nbd system at any $v_0 \in X$ is defined as:

$\{ U_{r,I} \}$ where

$$U_{r,I} := \{ v \in X : |f(v - v_0)| < r, \ \forall f \in I \},$$

where $r > 0$ and $I \subseteq X^*$ is finite.

In fact, this is the coarsest topology on X^* wrt which every element of X^* is continuous.
In particular, this is coarser than the topology generated by $\|\cdot\|_X$.

Exercise: Use Hahn-Banach extension to show that the above space is Hausdorff.
Similar to this:

Prop^n. Weak limit is unique (if it exists).

Proof. Suppose $v_n \rightharpoonup v$ in X and $v_n \rightharpoonup w$ in X.
Then, $f(v-w) = 0 \quad \forall f \in X^*$.
$\therefore \quad v = w$. ■ (Had seen X^* separates points.)

Prop^n. If $v_n \rightharpoonup v$, then $\{\|v_n\|\}_n$ is bounded.

Proof. For each f, let C_f denote an upper bound of $\{|f(v_n)|\}_n$.
Define
$$J : X \to X^{**}$$
$$J(v)(f) = f(v).$$
Bounded linear map. Injective

Write $J(v)$ as J_v.

Now,
$$|J_{v_n}(f)| = |f(v_n)| \le C_f.$$
As X^* is complete, UBP gives that $\exists C$ s.t.
$$\|J_{v_n}\| \le C.$$
Moreover, $\|J_{v_n}\| = \|v_n\|$. Thus, we are done ■

Exercise. $\|v\| \le \liminf_{n\to\infty} \|v_n\|$.

Example. (Weak convergence $\not\Rightarrow$ Strong Convergence)
Consider $X = \ell_2$. We have $\ell_2^* \cong \ell_2$.
Let $\{e_i\}_{i=1}^{\infty} \subseteq \ell_2$ be the usual sequences.

For $x \in \ell_2^*$ with $x = \{x_j\}_{j=1}^{\infty}$, we have
$\langle a, e_n \rangle = x_n \to 0$.
Thus $e_n \rightharpoonup 0$ in ℓ_2. But $\|e_n\| = 1 \Rightarrow e_n \not\to 0$.

Thm 14.2. Let X be a NLS. Then,
(i) $u_n \to u$ in $X \Rightarrow u_n \rightharpoonup u$ in X.
(ii) Converse holds if X is finite dimensional.

Proof. (i) True since any $f \in X^*$ is continuous
(ii) Pick a basis $\{v_i\}_{i=1}^N$ of X and let $\{f_i\}_{i=1}^N$ be the corresponding dual basis of X^*.
$(f_i(v_j) = \delta_{ij}.)$

Let $u_n \rightharpoonup u$.
Write $u_n = \sum_{j=1}^N \alpha_j^{(n)} v_j$ and $u = \sum_{j=1}^N \alpha_j v_j$.

By def^n, $f_j(u_n) \xrightarrow{n\to\infty} f_j(u)$ $\forall j \in \{1, \dots, N\}$.
Thus, $\alpha_j^{(n)} \xrightarrow{n\to\infty} \alpha_j$ $\forall j \in \{1, \dots, N\}$.

In turn,
$$\|u_n - u_n\| \leq \sum_{j=1}^N \|\alpha_j^{(n)} - \alpha\| \, \|v_j\| \to 0. \qquad \blacksquare$$

- Weak topology is strictly coarser if X is inf. dim.
  Weak closed $\Rightarrow$ Strongly closed.
  $\not\Leftarrow$

Prop^n 14.3. Let M be a nonempty closed and CONVEX subset of a Banach space V. Then, M is weakly closed.

Proof. Sufficient to assume V is a $\mathbb{R}$-Banach space.
Let $M \subset V$ be closed and convex.
We show $V \setminus M$ is weakly open. Pick $v_0 \in V \setminus M$.

By Hahn-Banach, $\exists f \in V^* \ \exists \alpha \in \mathbb{R}$ s.t.

$$f(v_0) < \alpha < f(v) \quad \forall v \in M.$$

Now, $L = \{ v \in V : f(v) < \alpha \}$ is a weakly open nbd of v_0 which does not intersect M. ∎

Def^n ① $f : M \overset{\subseteq V}{\longrightarrow} \mathbb{R}$, V: NLS.

f is weakly (sequentially) continuous if

$$u_n \rightharpoonup u \text{ in } M \Rightarrow f(u_n) \longrightarrow f(u).$$

(Again, weak cont. $\not\Rightarrow$ strong cont. / $\Leftarrow$)

② $U \subseteq V$ is weakly compact if every sequence in U has a weakly convergent subsequence in U.

Generalisation of Weierstrass: not necessarily linear (codomain is $\mathbb{R}$)

Every weakly continuous functional on a weakly compact set has a minimum (as well as a maximum).

Proof. Let $f : U \overset{\subseteq V}{\longrightarrow} \mathbb{R}$ be weakly cont. & U weakly compact.

Let $\alpha := \inf_{v \in V} f(v)$. $\quad (\alpha \in [-\infty, \infty))$

To show: $\exists u \in U$ s.t. $f(u) = \alpha$.

By def^n of inf, $\exists (u_n)_n$ s.t. $f(u_n) \to \alpha$.

By compactness, weakly conv. subsequence.

WLOG assume $u_n \rightharpoonup u$ in U.

Then, $f(u_n) \to f(u)$. $\therefore f(u) = \alpha$. ∎

Lecture 15 (24-09-2021)

24 September 2021 14:06

Weak convergence

Defn. Let $\{T_n\}$ be a sequence in $BL(X,Y)$.
Let $T: X \to Y$ be a linear operator. (Not necessarily bounded for (ii) & (iii).)
We say that

(i) (UOC) T_n converges to T uniformly if
$$\|T_n - T\|_{BL(X,Y)} \to 0.$$
(For this UOC, we assume that $T \in BL(X,Y)$ as well)

(ii) (SOC) T_n converges to T strongly if
$$\|T_n u - Tu\|_Y \to 0$$
$\forall u \in X$.

(iii) (WOC) T_n converges to T weakly if
$$|f(T_n u) - f(Tu)| \to 0$$
$\forall u \in X$ and $f \in Y^*$.

Lemma 15.1. UOC $\Rightarrow$ SOC $\Rightarrow$ WOC.

Proof. $\forall f \in Y^*$ and $\forall u \in X$, note that
$$|f(T_n u) - f(Tu)| \le \|f\| \, \|T_n u - Tu\| \quad \text{and}$$
$$\|T_n u - Tu\| \le \|T_n - T\| \cdot \|u\|.$$
∎

The converses don't hold.

Examples. ① SOC $\not\Rightarrow$ UOC

Fix $1 < p < \infty$. Define $T_n : \ell_p \to \ell_p$ by
$$T_n(\underset{\sim}{u}) = (u_1, u_2, \dots, u_n, 0, 0, \dots).$$
Fix $u \in \ell_p$. Note that
$$\|T_n \underset{\sim}{u} - Iu\| = \|(0, \dots, 0, -u_{n+1}, -u_{n+2}, \dots)\|$$

$$\|T_n \underset{\sim}{u} - \underbrace{I}_{\text{identity operator}} u\| = \|(0,\dots,0,-u_{n+1},-u_{n+2},\dots)\|$$

$$= \left(\sum_{k=n+1}^{\infty} |u_k|^p\right)^{1/p} \longrightarrow 0.$$

$\therefore\ T_n \longrightarrow I$ strongly.

However,

$$\|(T_n - I)(\underset{\sim}{e}_{n+1})\| = \|\underset{\sim}{e}_{n+1}\| = 1.$$

$$\therefore\ \|T_n - I\| \geq 1 \quad \forall n \in \mathbb{N}.$$

② WOC $\not\Rightarrow$ SOC.

Define $T_n : \ell_2 \to \ell_2$ by $T_n(\underset{\sim}{u}) = (0,\dots,\underset{\substack{\uparrow\\ n}}{0}, u_1, u_2, \dots)$.

Then, $\|T_n \underset{\sim}{e}_1 - T_m \underset{\sim}{e}_1\| = \sqrt{2}$ for $n \neq m$.

$\therefore\ (T_n \underset{\sim}{e}_1)_n$ is not convergent.

In particular, T_n does not strongly converge to any T.

However, it weakly converges to 0.

Indeed, let $f \in \ell_2^* = \ell_2$. Then, f is of the form

$$f(\underset{\sim}{x}) = \sum_{i=1}^{\infty} x_i \overline{y_i} \quad \text{for some } \underset{\sim}{y} \in \ell_2.$$

Then,

$$|f(T_n \underset{\sim}{x}) - f(0)| = \left|\sum_{i=1}^{\infty} x_i \overline{y_{n+i}}\right|$$

$$\leq \|\underset{\sim}{x}\|_2 \left(\sum_{i=1}^{\infty} |y_{n+i}|^2\right)^{1/2} \longrightarrow 0. \ \blacksquare$$

Defn Let X be an NLS. Let $(f_n)_n$ be a sequence in X^*. We say that $f_n \xrightarrow{*} f$ if $f_n(x) \longrightarrow f(x)$ $\forall x \in X$.

weak* convergence

Note: Weak* topology on X^* is coarser than the weak topology, in view of $J_x \in X^{**}$.

Q. If $T_n \to T$ (SOC or WOC), then is $T \in BL(X,Y)$?

No.

Example. Consider $X = c_{00} \subseteq \ell_2$.

↳ eventually 0 sequences

Define $T_n : \ell_2 \longrightarrow \ell_2$ by

$$T_n(\underset{\sim}{u}) = (u_1, 2u_2, \ldots, nu_n, u_{n+1}, u_{n+2}, \ldots).$$

Then, $T_n \in BL(\ell_2, \ell_2)$

$$\left(\|T_n(\underset{\sim}{u})\|^2 = \|\underset{\sim}{u}\|^2 + |u_2|^2 + 2|u_3|^2 + \cdots + (n-1)|u_n|^2 \le \|u\|^2 + (n-1)(|u_2|^2 + \cdots + |u_n|^2) \le n\|u\|^2. \right)$$

Let $S_n = T_n\big|_X$. Then, $S_n \in BL(X, \ell_2)$.

However, note that S_n converges strongly to the following operator: $S : c_{00} \longrightarrow \ell_2$ defined by

$$S(\underset{\sim}{u}) = (u_1, 2u_2, 3u_3, \ldots).$$

Indeed, for a fixed $\underset{\sim}{u} \in X^{=c_{00}}$, we have

$$\|(S_n - S)(\underset{\sim}{u})\| = (0, \ldots, 0, (n+1)u_{n+1}, \ldots)$$

for $n >> 0$, the above is 0 since $\underset{\sim}{u} \in c_{00}$.

However, $S : X \to \ell_2$ is <u>NOT</u> bounded.

We have $\|\underset{\sim}{e}_n\| = 1 \ \forall n$ but

$$\|S(e_n)\| = n \longrightarrow \infty.$$

<u>Note</u>: The example was for $BL(c_{00}, \ell_2)$. c_{00} is NOT complete. See next result.

Thm 15.2. Let $(T_n)_n$ be a seq. in $BL(X,Y)$, where X is Banach and Y is an NLS. If $T_n \longrightarrow T$ strongly, then $T \in BL(X,Y)$.

Proof. We had seen this as a corollary of the Banach-Steinhaus theorem (UBP).

Thm 15.3. A sequence $(T_n)_n \in BL(X,Y)^{\mathbb{N}}$ (where X and Y are Banach) is strongly convergent iff
(i) $\left(\|T_n\|_{BL(X,Y)}\right)_n$ is bounded, and
(ii) $(T_n u)_n$ is Cauchy (in Y) for $u \in M$ for some dense $M \subseteq X$.

Proof. $(\Rightarrow)$ Assume $T_n \longrightarrow T$ strongly.
Then, $\sup_n \|T_n u\| < \infty$ $\forall n$. Use UBP.
This proves (i). (ii) is obvious.

$(\Leftarrow)$ We first define $T : X \to Y$ as
$$T(x) := \lim_n T_n(x).$$

To check that this is well-defined, we need to show that $(T_n(x))_n$ is Cauchy in Y $\forall x \in X$.
(Since Y is Banach)

Let $\varepsilon > 0$ be given. Choose $u \in M$ s.t.
$\|u - x\| < \varepsilon$.

Then,
$$\|T_n(x) - T_m(x)\| \leq \|T_n(x) - T_n(u)\| + \|T_n(u) - T_m(u)\| + \|T_m(u) - T_m(x)\|$$

$C := \sup \|T_n\| < \infty$

$$\leq \|T_n\| \, \|x - u\| + \|T_n(u) - T_m(u)\| + \|T_m\| \, \|x - u\|$$
$$\leq 2C\varepsilon + \|T_n(u) - T_m(u)\|$$

↳ can be made
$< C\varepsilon$
for $n, m \geq N$.

By construction, $T_n \to T$ strongly. ☒

Lecture 16 (28-09-2021)

28 September 2021 14:09

Reflexivity and Separability

Recall that a metric space is separable if it has a countable dense subset.

As an application HBET, we saw

$$\|v\|_X = \sup_{0 \neq f \in X^*} \frac{|\langle f, v\rangle|}{\|f\|_{X^*}}$$

(16.1)
$$= \sup_{\|f\|_{X^*} \leq 1} |\langle f, v\rangle| \qquad (\langle f, v\rangle := f(v))$$

$$= \max_{\|f\|_{X^*} \leq 1} |\langle f, v\rangle|$$

The above suggests that there is a relation between X and X^{**}.

§16.1. Reflexivity

For $v \in X$, define

$$J_v(f) := f(v) \qquad \forall f \in X^*.$$

Then, $\|J_v(f)\| \leq \|v\| \, \|f\|$. Clearly, J_v is linear.

Thus, $J_v \in X^{**}$ $\forall v \in X$.

In fact, by (16.1), we have $\|J_v\|_{X^{**}} = \|v\|_X$.

Moreover $J_{v+\alpha w} = J_v + \alpha J_w$.

To summarise, we have a map
$$J : X \longrightarrow X^{**}$$
defined as $v \longmapsto Jv$.

The above is a linear isometry.

Q. When is the above canonical map J <u>onto</u>?

Def^n. A Banach space X is said to be reflexive if the canonical map J is surjective.

Then, we can identify X with X^{**}.

Recall: We had define $\|f\|_{X^*} = \sup_{\|v\|_X \leq 1} |\langle f, v\rangle|$.

Is the sup attained? It is attained for reflexive Banach spaces.

This deep result was proved by R.C. James (1957, Annals of Math) with additional condition of separability (of X^*?).
The general result was proven in 1964 (TAMS).

Recall that X^* is the topological dual, which coincides with alg. dual when $\dim(X) < \infty$. In such a case, we already know $X \cong (X^*)^*$.

Prop^n 16.1. A f.d.v.s. X is reflexive.

Lemma 16.2. Every closed subspace of a reflexive B-space is reflexive.

Proof. Let $X \to$ refl. B-space & $Y \leq X$ be closed.
Then, Y is Banach.
For $v^* \in X^*$, let $\tilde{v}^*$ denote the restriction of v^* to Y.

Then, $v_R^* \in Y^*$. Under this, we have

$$" X^* \subseteq Y^* ." \qquad (16.2)$$

(Note that $v^* \mapsto v_R^*$ is NOT injective in general. For example, if $Y = 0$.)

Similarly, we have $(Y^*)^* \subseteq (X^*)^*$. $\qquad (16.3)$

Now, we show that the map $J: Y \to Y^{**}$ is onto.
Pick $w^{**} \in Y^{**}$.
Then, $w^{**} \in X^{**}$ under (16.3).
By reflexivity of X, $\exists\, w \in X$ s.t.

$$w^{**}(v^*) = v^*(w) \qquad \forall v^* \in X^*.$$

In particular,

$$w^{**}(v^*) = v^*(w) \qquad \forall v^* \in Y^*.$$

(Note: every functional on Y is indeed a restriction of a functional on X, by HBET.)

Claim: $w \in Y$.

Proof. Suppose not. Then, $\mathrm{dist}(w, Y) > 0$.
By separation of convex sets, $\exists\, v^* \in X^*$ s.t.

$$v_R^* = 0 \quad \text{and} \quad v^*(w) \neq 0.$$

But then, $0 = w^{**}(v_R^*) = v^*(w) \neq 0$. $\to\leftarrow$

Thus, we are done. ∎

Thm 16.3. Each bounded sequence $(u_n)_n$ in a reflexive B-space has a weakly convergent subsequence.

Proof. Assume $X \neq \{0\}$.
Case 1. X^* is separable.

Let $(f_k)_{k\geq 1}$ be a dense subset of X^*.

Note that $(f_1(u_n))_n$ is bounded in $\mathbb{F}$.

Thus, it has a convergent subsequence, say $(f_1(u_{1n}))_n$.

Now, $(f_2(u_{1n}))_n$ is bounded in $\mathbb{F}$.

Thus, . . . call it $(f_2(u_{2n}))_n$.

$\vdots$

$$\begin{array}{llll} f_1(u_{11}) & f_1(u_{12}) & f_1(u_{13}) & \cdots \to a_1 \\ f_2(u_{21}) & f_2(u_{22}) & f_2(u_{23}) & \cdots \to a_2 \\ f_3(u_{31}) & f_3(u_{32}) & f_3(u_{33}) & \cdots \to a_3 \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Now, define $w_n := u_{nn}$. This is a subsequence of $(u_n)_n$.

Note that for a fixed k, we have

$$\lim_{n\to\infty} f_k(w_n) = a_k,$$

since $(w_n)_n$ is a subsequence of $(u_{kn})_n$.

(Ignoring the first k terms, at least.)

<u>Claim</u>: $(f(w_n))_n$ converges for every $f \in V^*$.

<u>Proof</u>. We show that the sequence is Cauchy.

Let $\varepsilon > 0$ be given. Then, by denseness, $\exists k \in \mathbb{N}$ s.t.

$C := \sup_n \|u_n\| < \infty$.

$$\|f_k - f\|_{X^*} < \frac{\varepsilon}{4C}$$

Then,

$$\begin{aligned} \|f(w_m) - f(w_n)\| &\leq \|f(w_m) - f_k(w_m)\| + \|f_k(w_m) - f_k(w_n)\| \\ &\quad + \|f_k(w_n) - f(w_n)\| \\ &\leq \|f - f_k\|\|w_m\| + \|f_k(w_m) - f_k(w_n)\| \\ &\quad + \|f_k - f\|\,\|w_n\| \\ &\leq \frac{\varepsilon}{2} + \|f_k(w_m) - f_k(w_n)\| \end{aligned}$$

↳ can be made $< \frac{\varepsilon}{2}$.

Let $J_v \in X^{**}$ be the eval. map. We have shown that

$J_{w_n}(f)$ converges for all $f \in V^*$.

Define $G \in X^{**}$ by

$$G(f) := \lim_{n\to\infty} J_{w_n}(f).$$

Then, it is easy to check that G is indeed bounded. (UBP.)

But since X is reflexive, we have $G = J_w$ for some $w \in X$.

Thus, we have $f(w_n) \to f(w)$ $\forall f \in V^*$.

$\therefore w_n \rightharpoonup w$.

(ii) Don't assume X^* sep.

Let $Y = \overline{\operatorname{span}\{u_1, u_2, u_3, \dots\}}$.

Then, Y is a separable closed subspace of X.

finite $\mathbb{Q}$-linear combinations of u_i is dense (or $\mathbb{Q} + i\mathbb{Q}$)

Being closed, it is reflexive (by Lemma 16.2).

Thus, Y^* is separable (by Prop^n 16.4).

By Case 1, we are now done. $\blacksquare$

Prop^n 16.4. Let X be a B-space over $\mathbb{F}$. Then,

(i) X^* separable $\Rightarrow$ X separable,

(ii) X separable + reflexive $\Rightarrow$ X^* separable

Proof. (i) Let $(f_n)_n$ be dense in X^*.

Recall that $\|f_n\|_{X^*} = \sup_{\|u\|=1} |f_n(u)|$.

Thus, for each $n \in \mathbb{N}$, we can pick $u_n \in X$ with $\|u_n\| = 1$ s.t. $|f_n(u_n)| \geq \frac{1}{2}\|f_n\|_{X^*}$.

Define $Y := \operatorname{span}\{u_1, u_2, \dots\}$.

FTSOC, assume $Y \neq X$. Then, by HBT, $\exists f \in X^*$ s.t. $f \neq 0$ and $f|_Y = 0$.

Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|f - f_n\|_{X^*} &\geq |(f - f_n)(u)| \\ &= |f_n(u)| \\ &\geq \frac{1}{2} \|f_n\|_{X^*} \\ &\geq \frac{1}{2}\left(\|f\|_{X^*} - \|f - f_n\|_{X^*}\right) \end{aligned}$$

$$\Rightarrow \quad \|f - f_n\|_{X^*} \geq \frac{1}{3} \|f\|_{X^*} \quad \forall n \in \mathbb{N}.$$

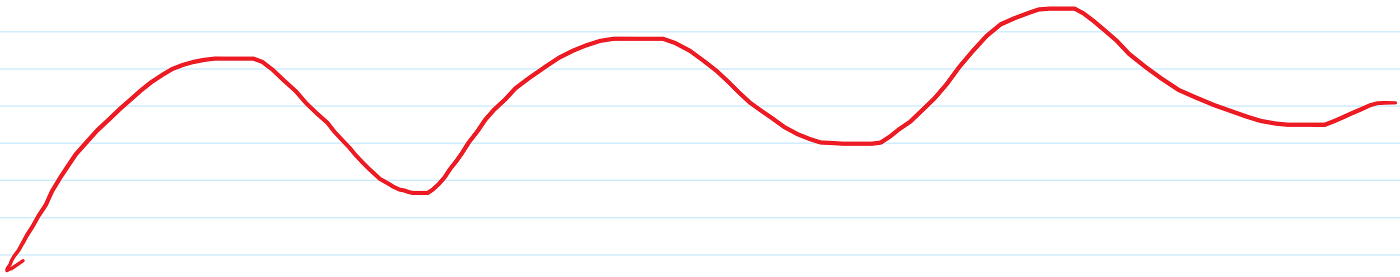
positive

This contradicts density!

(ii) Use (i) and the fact that $(X^*)^* \cong X$. ⊟

Lecture 17 (01-10-2021)

01 October 2021 14:00



Lecture 18 (05-10-2021)

05 October 2021 14:01

Hilbert spaces

Def^n An inner product on a linear space V is a mapping
$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$$
satisfying:

(i) $\langle v, v \rangle \geq 0 \quad \forall v \in V$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$, (Positive definite)

(ii) $\forall u, v, w \in V, \ \forall \alpha \in \mathbb{F}$:
$$\langle u + \alpha v, w \rangle = \langle u, w \rangle + \alpha \langle v, w \rangle$$
(linearity in first variable),

(iii) $\forall u, v \in V$:
$$\langle u, v \rangle = \overline{\langle v, u \rangle}.$$
(Conjugate symmetry or Hermitian)

Then, $(V, \langle \cdot, \cdot \rangle)$ (or simply V) is called an inner product space (IPS).

Note: $\forall u, v, w \in V, \ \forall \alpha \in \mathbb{F}$, we have
$$\begin{aligned}\langle u, v + \alpha w \rangle &= \overline{\langle v + \alpha w, u \rangle} \\ &= \overline{\langle v, u \rangle + \alpha \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \bar{\alpha}\, \overline{\langle w, u \rangle} = \langle u, v \rangle + \bar{\alpha} \langle u, w \rangle.\end{aligned}$$

Given an inner product, we get a norm (induced by the inner product) defined as
$$\|v\| := \langle v, v \rangle^{1/2}.$$

Clearly, $\|\cdot\|$ satisfies the def. & homogenity.
Need to check triangle inequality.

Prop^n 18.1. (Cauchy-Schwarz)

For $u, v \in V$, we have

For $u, v \in V$, we have
$$|\langle u, v\rangle| \leq \|u\| \, \|v\|. \qquad (18.1)$$

Proof. (18.1) is trivially true if $v = 0$.
Assume $v \neq 0$. For every $\alpha \in \mathbb{F}$, note

$$0 \leq \|u - \alpha v\|^2 = \langle u - \alpha v, u - \alpha v\rangle$$
$$= \|u\|^2 - \alpha\langle v, u\rangle - \bar{\alpha}\langle v, u\rangle + |\alpha|^2 \|v\|^2$$
$$= \|u\|^2 - \bar{\alpha}\langle v, u\rangle - \alpha\left(\langle u, v\rangle - \bar{\alpha}\, \|v\|^2\right).$$

Choose $\bar{\alpha} = \dfrac{\langle u, v\rangle}{\|v\|^2}$. Then, the quantity in brackets is 0.

$$\therefore \quad 0 \leq \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2}. \qquad \boxtimes$$

Remark. The above product shows that equality holds iff $\{u, v\}$ is lin dep.

Prop^n 18.2. $\|u + v\| \leq \|u\| + \|v\|$.

Proof. Note: $\|u + v\|^2 = \langle u + v, u + v\rangle$
$$= \|u\|^2 + 2\mathrm{Re}\,\langle u, v\rangle + \|v\|^2$$
$$\leq \|u\|^2 + 2\,|\langle u, v\rangle|^2 + \|v\|^2$$
$$\leq \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2 \qquad \text{C-S}$$
$$= (\|u\| + \|v\|)^2. \qquad \boxtimes$$

Prop^n 18.3. (Parallelogram law)
The parallelogram law holds for an IPS V. That is,
$$\|u + v\|^2 + \|u - v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) \quad \forall\, u, v \in V.$$

Proof $\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle + \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$$

$$= 2(\langle u,u\rangle + \langle v,v\rangle) = 2(\|u\|^2 + \|v\|^2). \blacksquare$$

Prop^n 18.4. (Polarisation identity)

$V \to$ IPS, $\forall u, v \in V$, we have:

$$\langle u, v\rangle = \frac{1}{4}\Big(\|u+v\|^2 - \|u-v\|^2 + \underbrace{i\|u+iv\|^2 - i\|u-iv\|^2}_{\text{Ignore these terms if } V \text{ is a } \mathbb{R} \text{ IPS.}}\Big).$$

Remark: If $\|\cdot\|$ satisfies $\|^{gram}$ law, then defining $\langle\cdot,\cdot\rangle$ using polar identity gives an inner product that induces $\|\cdot\|$.

Example: ① $V = \mathbb{F}^n$, define $\langle x, y\rangle := \sum_{j=1}^{n} x_j \bar{y}_j$.

② $\ell_2 = \{ \underset{\sim}{x} = (x_n)_n : (\sum_{n=1}^{\infty} |x_n|^2)^{1/2} < \infty \}$.

Then, $\langle \underset{\sim}{x}, \underset{\sim}{y}\rangle := \sum_{n=1}^{\infty} x_n \bar{y}_n$ is well-defined

Check: $\langle\cdot,\cdot\rangle$ is an inn. prod.

Also, $\langle\cdot,\cdot\rangle$ induces the ℓ_2 norm.

③ NON-EXAMPLE. ℓ_p for $p \neq 2$ is not an IPS.

It does not satisfy the $\|^{gram}$ law.

Indeed, take $\underset{\sim}{x} = (1, 1, 0, 0, \dots)$ &

$\underset{\sim}{y} = (1, -1, 0, 0, \dots)$.

Clearly, $\underset{\sim}{x}, \underset{\sim}{y} \in \ell_p$. Now,

$$\|\underset{\sim}{x}+\underset{\sim}{y}\|_p^2 + \|\underset{\sim}{x}-\underset{\sim}{y}\|_p^2 = 2^2+2^2 = 8.$$

$$2\left(\|\underset{\sim}{x}\|_p^2 + \|\underset{\sim}{y}\|_p^2\right) = 2\left(2^{2/p}+2^{2/p}\right) = 4\cdot 2^{2/p}.$$

Thus, $\|$ gram law $\Leftrightarrow 2^{2/p} = 2 \Leftrightarrow p = 2$.

(4) Fix $-\infty < a < b < \infty$.
On $\mathcal{C}[a,b]$, define $\langle u, v\rangle := \int_a^b uv$.

This defines an inn. prod. but the induced norm is not complete.

On $C^1[a,b]$, we can define

$$\langle u, v\rangle_1 := \int_a^b (uv + u'v').$$

Then, $\|\cdot\|_1$ is again not complete.

(5) On $L^2(\Omega)$, we can define

$$\langle u, v\rangle := \int_\Omega u\bar{v}.$$

Def^n A complete inner product space is called a Hilbert space.

Exercise. Every Hilbert space is reflexive. In fact, it is uniformly convex.

Let H be a Hilbert space. For $y \in H$, set

$$f_y(x) := \langle x, y\rangle \quad \forall x \in H.$$

Then, f_y is a linear functional.

Moreover, $\|f_y(x)\| \le |\langle x, y\rangle| \le \|x\|\,\|y\|$. Thus, f_y is cts.

In fact, $\|f_y(y)\| = \|y\|\,\|y\|$. $\therefore$ $\|f_y\|_{H^*} = \|y\|_H$.

Suppose $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in H.

Then,

$$\langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle.$$

Hint: $|\langle x_n, y_n\rangle - \langle x, y\rangle| \le |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$

$\le \|x_n\|\,\|y_n - y\| + |f_y(x_n - x)| \to 0.$

Exercise: Show that for a sequence $(x_n)_n$ in an IPS X, the conditions : $x_n \longrightarrow x$ and $\|x_n\| \longrightarrow \|x\|$ imply : $x_n \longrightarrow x$ in X.

Hint: $\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle$

$= \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2$

$\longrightarrow 2\|x\|^2 - 2\langle x, x\rangle = 0.$

Lecture 19 (08-10-2021)

08 October 2021 14:05

On Riesz Representation Theorem

H will be a Hilbert space with IP $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$.

H^* will denote its dual, as usual.

Q. Now that H has additional structure (inner product), can we say something more?

Example. We had seen that $\|\cdot\|$ on ℓ_2 is induced by

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \sum_{i=1}^{\infty} x_i y_i .$$

(Assume everything in $\mathbb{R}$)

Note that we can define $f_{\underset{\sim}{y}} : \ell_2 \to \mathbb{R}$ by

$$f_{\underset{\sim}{y}}(x) := \langle x, y \rangle .$$

Then, $f_{\underset{\sim}{y}}$ is linear, continuous, and $\|f_{\underset{\sim}{y}}\|_{\ell_2^*} = \|y\|_{\ell_2}$.

Thus, $f_{\underset{\sim}{y}} \in \ell_2^*$.

In fact, we had seen that any element of ℓ_2^* is of this form.

Similarly, given any Hilbert space H, we can define (for $y \in H$)

$$f_y : H \to \mathbb{F} \quad \text{by}$$

$$f_y(x) = \langle x, y \rangle . \qquad \forall x \in H$$

Moreover, $f_y \in H^*$ and $\|y\|_H = \|f_y\|_{H^*}$.

Q. Are there more elements in H^*?

Thm 19.1. (Riesz Representation Theorem)

Thm 19.1. (Riesz Representation Theorem)
Let $f^* \in H^*$. Then, there exists a unique $y \in H$ s.t. $f^* = f_y$.

Proof. Uniqueness is clear since $y \mapsto f_y$ is an isometry.
Define $\phi : H \longrightarrow H^*$ by $\phi(y) := f_y$.
Need to show ϕ is onto.
Since ϕ is an isometry and H is complete, it follows that $\phi(H)$ is closed. Thus, to finish the proof, it suffices to prove that $\phi(H)$ is dense.
Consider a linear functional φ^{**} on H^* which vanishes on $\phi(H)$. Since H is reflexive, $\varphi^{**} = J(x)$ for some $x \in H$. We have

$$J(x)\big(\phi(y)\big) = 0 \qquad \forall y \in H.$$

In other words,
$$f_y(x) = 0 \qquad \forall y \in H \quad \text{or}$$
$$\langle x, y \rangle = 0 \qquad \forall y \in H.$$

But this means that $x = 0$. In turn, $\varphi^{**} = 0$.
This means that any functional vanishing on $\phi(H)$, vanishes on H^*. Thus, $\phi(H)$ is dense in H^*. ∎

Remark. The map $y \longmapsto f_y$ is an isometry of H onto H^*.
If $\mathbb{F} = \mathbb{R}$, then this map is linear. If $\mathbb{F} = \mathbb{C}$, then it is CONJUGATE linear.
Thus, for real H, $H \cong H^*$ via Riesz.

An application: RRT can be reformulated as:
Given $f \in H^*$, $\exists!\, u \in H$ s.t. u solves

$$f(v) = \langle u, v \rangle_H \qquad \forall v \in H.$$

Can we replace the inner product by a bilinear form and have similar results? (Assume over $\mathbb{R}$.)

Given a bilinear form $a: H \times H \to \mathbb{R}$ and $f \in H^* \cong H$, find $u \in H$ s.t.

$$a(u, v) = f(v) \quad \forall v \in H.$$

Under what conditions on $a(\cdot,\cdot)$ does the problem above have a unique solution?

Crude version of Lax Milgram Lemma:

Thm 19.2. Given a bilinear form $a: H \times H \to \mathbb{R}$ and linear form $f: H \to \mathbb{R}$, assume that

(i) $a(\cdot, \cdot)$ is bounded, i.e., $\exists M > 0$ s.t.

(19.1) $$a(u, v) \leq M \|u\| \|v\| \quad \forall u, v \in H.$$

(ii) $a(\cdot, \cdot)$ is symmetric, i.e., for all $u, v \in H$,

(19.2) $$a(u, v) = a(v, u)$$

(iii) $a(\cdot, \cdot)$ is H-elliptic or coercive, i.e., $\exists \alpha_0 > 0$ s.t.

(19.3) $$a(u, u) \geq \alpha_0 \|u\|^2 \quad \forall u \in H.$$

(iv) $f \in H^*$.

Then, $\exists! u \in H$ s.t. $f(v) = a(u, v) \quad \forall v \in H.$ (19.4)

Moreover, the dependence $f \mapsto u$ is continuous. We will show

$$\|u\|_H \leq \frac{M}{\alpha_0} \|f\|_{H^*}. \qquad (19.5)$$

Sketch Define $((u, v)) := a(u,v)$.

Check: $((\cdot, \cdot))$ is an inner product on H.

This induces the norm $|||\cdot|||$ on H.

Check: $\|\cdot\|$ and $|||\cdot|||$ are equivalent. (More precisely: $\sqrt{\alpha_0}\,\|u\| \le |||u||| \le \sqrt{M}\|u\|$)

Thus, they induce the same topology. In particular, they give the same dual space.

Thus, given $f \in H^*$, we can apply RRT to get the unique u. We have

$$|||u||| = |||f|||_{H^*}.$$

Use this to get (19.5) ☒

Question 2: Define $J(v) := \frac{1}{2} a(v, v) - f(v)$.

Find $u \in H$ s.t.

$$\min_{v \in H} J(v) = J(u). \qquad (19.6)$$

Thm 19.3. Assume that $a(\cdot,\cdot)$ is symmetric, bounded, coercive and $f \in H^*$.

Then, (19.4) and (19.6) are equivalent.

Proof. ($\Leftarrow$) Assume $J(u) = \min_{v \in H} J(v)$.

Fix $v \in H$. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(t) := J(u+tv).$$

For $t \neq 0$, note

$$\begin{aligned}\Phi(t) - \Phi(0) &= J(u+tv) - J(u) \\ &= \tfrac{1}{2} a(u+tv, u+tv) - f(u+tv) \\ &\quad - \tfrac{1}{2} a(u,u) + f(u)\end{aligned}$$

Linearity /

$$- \frac{1}{2} a(u,u)] + f(u)$$

linearity

$$= \frac{t}{2}\left(a(u,v) + a(v,u)\right) + \frac{t^2}{2} a(v,v) - t f(v)$$

symmetry

$$= t\left(a(u,v) - f(v)\right) + \frac{t^2}{2} a(v,u)$$

Then, $\dfrac{\phi(t) - \phi(0)}{t} = a(u,v) - f(v) + \dfrac{t}{2} a(v,u)$.

$\therefore$ $\phi'(0)$ exists and equal $a(u,v) - f(v)$.

But 0 is a point of minimum for ϕ.

$\therefore$ $\phi'(0) = 0$ or $a(u,v) = f(v)$.

As v was arbitrary, we have shown that

$$a(v,u) = f(v) \qquad \forall v \in H.$$

$(\Longrightarrow)$ Assume $u \in H$ is s.t. $f(v) = a(v,u)$.

Then, given $v \in H$, we have

$$J(v) - J(u) = \frac{1}{2} a(v,v) - f(v) - \frac{1}{2} a(u,u) + f(u)$$

$$= \frac{1}{2} a(v,v) - \frac{1}{2} a(u,u) + f(u-v)$$

$$= \frac{1}{2} a(v,v) - \frac{1}{2} a(u,u) + a(u-v,u)$$

$$= \frac{1}{2} a(v,v) - \frac{1}{2} a(u,u) + a(u,u) - a(v,u)$$

$$= \frac{1}{2} a(v,v) + \frac{1}{2} a(u,u) - a(v,u)$$

$$= \frac{1}{2}\left(a(v,v) + a(u,u) - a(v,u) - a(u,v)\right)$$

$$= \frac{1}{2}\left(a(v,v-u) + a(u,u-v)\right)$$

$$= \frac{1}{2}\Big(a(v, v-u) - a(u, v-u)\Big)$$

$$= \frac{1}{2}\, a(v-u, v-u) \geq 0. \qquad \blacksquare$$

Lecture 20 (22-10-2021)

22 October 2021 13:55

Def^n. Let H be an IPS. A subset $S \subseteq H$ is said to be orthonormal if for all $x, y \in S$, we have

$$\langle x, y \rangle = \begin{cases} 0 & ; \ x \neq y, \\ 1 & ; \ x = y. \end{cases}$$

Ex. An orthonormal set is linearly independent.
(Not necessarily countable.)

Also, recall Gram-Schmidt orthogonalisation:

Prop^n. Let H be an IPS. Let $\{x_1, \dots, x_N\} \subseteq H$ be linearly independent. Then, $\exists$ an orthonormal set of vectors $\{y_1, \dots, y_N\}$ s.t.

$$\operatorname{span}\{x_1, \dots, x_k\} = \operatorname{span}\{x_1, \dots, x_k\}$$

for all $1 \leq k \leq N$.

Examples. ① $\ell_2 \rightarrow \{e_i\}_{i=1}^{\infty}$ is an orthonormal set.

② $H = \ell_2^N \ (= \mathbb{R}^N)$.

Let A be an invertible $n \times n$ matrix.

Let $(x_1, \dots, x_n)$ be columns of A.

$$\downarrow \text{GS, standard inner product}$$

$(y_1, \dots, y_n)$

Put $Q = [y_1 \ \cdots \ y_n]$.

Then, $A = QR$ for an upper Δ matrix R.

Moreover, Q is an orthogonal matrix.

③ Let $\mathcal{P}_n \subseteq \mathcal{L}^2[0,1]$ be the space of polynomials of deg $\leq n$.

Let $f \in \mathcal{L}^2[0,1]$.

To find: $p \in \mathcal{P}_n$ s.t. $\|f - p\|_2$ is minimised.

$$p(t) = \sum_{j=0}^{n} \alpha_j t^j, \quad \text{need to find } \alpha_j.$$

Solve:
$$\underbrace{\begin{bmatrix} \langle p_0, p_0\rangle & \langle p_0, p_1\rangle & \cdots \\ \vdots & \ddots & \\ & \langle p_i, p_j\rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{=A} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \langle f, p_0\rangle \\ \langle f, p_1\rangle \\ \vdots \\ \langle f, p_n\rangle \end{bmatrix}, \quad \text{where } p_j(t) = t^j.$$

Let $H = \mathbb{F}^N$ with standard inner product.

Let $\{e_1, \ldots, e_M\}$ be an orthonormal set $(M \leq N)$.

Then, we can decompose
$$H = V_M \oplus V_M^{\perp}, \quad \text{where } V = \operatorname{span}\{e_1, \ldots, e_M\}.$$

Given $y \in V_M$, we can write $y = \sum_{j=1}^{M} \alpha_j e_j$.

Moreover, α_j can be calculated as
$$\alpha_j = \langle y, e_j\rangle.$$

Moreover, $\|y\|^2 = \sum_{j=1}^{M} |\alpha_j|^2$.

Propⁿ: (Bessel's Inequality (Finite))

Let $\{e_1, \ldots, e_n\}$ be a finite orthonormal set in an IPS H.

Then, for any $x \in H$, we have
$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

$$\sum_{j=1} |\langle x, e_j\rangle|^2 \leq \|x\|^2.$$

Further, $$x - \sum_{j=1}^{n} \langle x, e_j\rangle e_j \in \{e_1, \ldots, e_n\}^{\perp}.$$

Prop^n. Let $\{e_i : i \in I\}$ be an orthonormal set in a Hilbert space H.
Then, for any $x \in H$, the set $S = \{i \in I : \langle e_i, x\rangle \neq 0\}$ is at most countable.

Proof. Set $S_n := \left\{ i \in I : |\langle x, e_i\rangle| > \frac{\|x\|^2}{n} \right\}$.

By the previous prop^n, $|S_n| \leq n-1$.

Since $S = \bigcup_{n=1}^{\infty} S_n$, it follows that S is countable.

Prop^n. (Bessel's Inequality)
Let $\{e_i\}_{i \in I}$ be orthonormal in a Hilbert space H.
Let $x \in H$. Then,
$$\sum_{i \in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

Proof. Let $S = \bigcup_{n \geq 1} S_n$ be as before.
If $|S| < \infty$ then we are done by earlier.
Assume $S = \{e_1, e_2, \ldots\}$. (Note that desired sum does not depend on order.)
By earlier, for all finite $n \in \mathbb{N}$, we have
$$\sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \leq \|x\|^2.$$

Let $n \longrightarrow \infty$ to conclude. ∎

Thm. (Riesz-Fischer)

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal set in a Hilbert space $\mathcal{H}$.
For a sequence $\{\alpha_j\}_{j=1}^{\infty}$ of scalars, TFAE:

(i) $\exists\, x \in H$ s.t. $\langle x, e_j \rangle = \alpha_j \quad \forall j \in \mathbb{N}$,

(ii) $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$,

(iii) $\sum_{j=1}^{\infty} \alpha_j e_j$ converges in H.

Proof. (ii) $\Leftrightarrow$ (iii) is true since $\mathcal{H}$ is Banach.

(i) $\Rightarrow$ (ii) by Bessel.

(iii) $\Rightarrow$ (ii) Use continuity of $\langle \cdot, \cdot \rangle$.

Indeed, let $x := \sum_{j=1}^{\infty} \alpha_j e_j$.

$$\langle x, e_k \rangle = \left\langle \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j e_j,\ e_k \right\rangle$$

$$= \lim_{\substack{n\to\infty \\ n \geq k+1}} \left\langle \sum_{j=1}^{n} \alpha_j e_j, e_k \right\rangle$$

$$= \lim_{\substack{n\to\infty \\ n \geq k+1}} \alpha_k = \alpha_k .$$

$\boxtimes$

Remark. Is the x in part (i) unique?
That is, if $\langle x, e_j \rangle = \alpha_j \ \forall j$, then is

$$x = \sum \alpha_j e_j \ ?$$

No. Can take any orthonormal countably inf. set $\{e_1, e_2, \ldots\}$.
Consider $\{e_2, e_3, \ldots\}$.

Then, $\langle e_1, e_j \rangle = 0 = \langle 0, e_j \rangle \quad \forall j \geq 2$.

Lecture 21 (24-10-2021)

24 October 2021 17:01

Complete Orthonormal Sets and Applications

In last class, we saw the following:
$\mathcal{H} \to$ Hilbert Space, $\{u_n\}_{n=1}^{\infty}$ orthonormal
Given $(\alpha_n)_n \in \ell_2$, $\exists\, x \in \mathcal{H}$ s.t. $\langle x, u_n\rangle = \alpha_n$.
We also saw: x need not be unique.

Q) When is this x unique?

Def^n An orthonormal set in a H. space is said to be complete if it is maximal (w.r.t. inclusion) among orthonormal sets.
(Not necessarily countable.

Thm 21.1. Let $\{e_i : i \in I\}$ be an orthonormal set in a H. space $\mathcal{H}$. TFAE:

(i) $\{e_i : i \in I\}$ is complete.

(ii) (Fourier Expansion) For $x \in \mathcal{H}$, we have

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n,$$

where $\{e_j\}_{j=1}^{\infty} = \{ e_i : \langle x, e_i\rangle \neq 0\}$.
("∞ could be finite".)

(iii) (Parseval's identity) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2$ for $x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$

(iv) $x \in \mathcal{H}$ and $\langle x, e_j\rangle = 0 \;\; \forall j \in I$
$$\Downarrow$$
$$x = 0.$$

Proof. (i) $\Rightarrow$ (ii)

We had seen $S = \{e_j : \langle x, e_j \rangle \neq 0\}$ is at most countable.

By Bessel's,
$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

$\therefore$ $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ converges as we saw earlier.

Let $y \in \mathcal{H}$ be the limit. Then, $y - x \perp e_n \ \forall e_n \in S$.

Moreover, for $e_j \notin S$, we have
$$\langle y - x, e_j \rangle = \langle y, e_j \rangle - \langle x, e_j \rangle = 0$$

$\therefore$ $y - x \perp \{e_j\}_{j \in J}$. $\therefore$ $y - x = 0$ by maximality.

(Else, we can add $z = \frac{y-x}{\|y-x\|}$.)

Above also indicates how (i) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (iii) Given $x \in \mathcal{H}$ and $\{e_n\}_{n=1}^{\infty}$ as before, we have

$$\langle x, x \rangle = \left\langle \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n, \sum_{m=1}^{\infty} \langle x, e_m \rangle e_m \right\rangle$$

$$= \lim_{N \to \infty} \left\langle \sum_{n=1}^{N} \langle x, e_n \rangle e_n, \sum_{m=1}^{\infty} \langle x, e_m \rangle e_m \right\rangle$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \left\langle \langle x, e_n \rangle e_n, \sum_{m=1}^{\infty} \langle x, e_m \rangle e_m \right\rangle$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{M \to \infty} \left\langle \langle x, e_n \rangle e_n, \sum_{m=1}^{M} \langle x, e_m \rangle e_m \right\rangle$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \langle x, e_n \rangle \overline{\langle x, e_n \rangle}$$

$$= \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2.$$

$n=1$

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i) Suppose not complete. Pick $z \notin \{e_j\}_j$ s.t. $\{e_j\}_j \cup \{z\}$ is orthonormal. Then, $\langle z, e_j \rangle = 0 \ \forall j$ but $\|z\| = 1$ and thus $z \neq 0$. ☒

Remark. If $\mathcal{H}$ (→ H space) has a (infinite) countable o.normal set, then $\mathcal{H}$ is isometrically isomorphic to ℓ_2. $\left(x \overset{\simeq}{\mapsto} (n \mapsto \langle x, e_n \rangle) \right)$

Adjoints.

Defn. Given a Hilbert space $\mathcal{H}$ and $A \in BL(\mathcal{H})$, define the **adjoint** of A as the mapping $A^* : \mathcal{H} \to \mathcal{H}$ s.t.

$$\langle x, A^* y \rangle = \langle Ax, y \rangle \qquad \forall x, y \in \mathcal{H}.$$

Remark. Why does such an A^* exist (and is unique)?

Fix $y \in \mathcal{H}$. Then,

$x \mapsto \langle Ax, y \rangle$ is a bounded linear map.

Riesz Rep: ↳ Thus, $\exists! z \in \mathcal{H}$ s.t. $\langle x, z \rangle = \langle Ax, y \rangle \ \forall x \in \mathcal{H}$.

Define $A^* y := z$.

Propn 21.2. Let $A, A_1, A_2 \in BL(H)$ and $\alpha \in \mathbb{F}$. Then,

(i) $\|A\| = \|A^*\|$,

(ii) $\|A^* A\| = \|A\|^2$,

(iii) $(A^*)^* = A$,

(iv) $(A_1 + A_2)^* = A_1^* + A_2^*$,

(v) $(A_1 A_2)^* = A_2^* A_1^*$,

(vi) $(\alpha A)^* = \bar{\alpha} A^*$.

Proof. (iii) $\langle x, A^* y \rangle = \langle Ax, y \rangle \quad \forall x, y$

$\Rightarrow \langle A^* y, x \rangle = \langle y, Ax \rangle \quad \forall x, y$

$\Rightarrow (A^*)^* = A$.

(i) $\|A\| = \sup_{\|x\| \le 1} \|Ax\|$

$\|x\| = \sup_{\|y\| \le 1} |\langle x, y \rangle|$

(i) $\|A\| = \sup_{\|x\|\le 1} \|Ax\|$

$\|x\| = \sup_{\|y\|\le 1} |\langle x, y\rangle|$

$= \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} |\langle Ax, y\rangle|$

$= \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} \langle x, A^* y\rangle \le \|A^*\|.$

Use (iii) to finish.

(ii) $\|A^2 x\|^2 = \langle Ax, Ax\rangle = \langle x, A^* Ax\rangle$

$\le \|A^* Ax\| \|x\| \le \|A^* A\| \|x\|^2.$

$\therefore \|A^2\|$

Lecture 22 (26-10-2021)

26 October 2021 14:02

Compact operators

Def^n. A bounded linear operator $T: V \to W$ is called compact if for all bounded $M \subseteq V$, the set $T(M)$ is relatively compact, i.e., $\overline{T(M)}$ is compact.

Remark. Since T is linear, it suffices to only check that $\overline{T(B_1(0))}$ is compact.

Obs. $T \in BL(V,W)$ is compact iff for every bdd sequence $(u_n)_n$ in V, the sequence $(Tu_n)_n$ in W has a convergent subsequence.

Proof ($\Rightarrow$) clear, take $M = \{u_n : n \in \mathbb{N}\}$ and use the fact compact $\Rightarrow$ seq. compact in metric spaces.

($\Leftarrow$) Let M be bdd.

To show: $\overline{T(M)}$ is compact.

Since W is a metric space, suffices to show that $\overline{T(M)}$ is seq. compact.

Let $(w_n)_n$ be a seq. in $\overline{T(M)}$.

For each n, we can find $u_n \in M$ s.t.

$$\| Tu_n - w_n \| \le 1/2^n.$$

By hypothesis, $(Tu_n)_n$ has a convergent subseq. Easy to see that the corresp. subseq. of $(w_n)_n$ also converges to that.

Ex. ① If $T \in BL(V,W)$ and $\dim T(V) < \infty$, then T is compact.

② —————— " —————— $\dim(V) < \infty$, —————— " —————— .

③ $id_V : V \to V$ is compact $\iff$ $\dim(V) < \infty$.

④ If we drop "bounded" from hypothesis, compact still implies bounded. (Just assume linear.)

⑤ Let $V \to$ Banach.
$T: V \to V$ compact. If T^{-1} exists and is bdd, then $\dim(V) < \infty$.

⑥ Let X, Y, Z be B-space. Let $S \in BL(X,Y)$ and $T \in BL(Y,Z)$.
If one of them is compact, then so is $T \circ S$.

Example. ① Let $\hat{i} : (C^1[0,1], \|\cdot\|_{1,\infty}) \hookrightarrow (C[0,1], \|\cdot\|_\infty)$ be the natural inclusion. Then, $\hat{i}$ is compact.

Proof. For $f \in C^1[0,1]$, we have $\|f\|_{1,\infty} = \max(\|f\|_\infty, \|f'\|_\infty)$.
Let $(f_n)_n$ be bdd in $C^1[0,1]$, by K.
Then,
$$\|\hat{i}(f_n)\| = \|f_n\|_\infty \leq \|f_n\|_{1,\infty} < K.$$

Thus, $(f_n)_n$ is uniformly bdd. We now show that the family is equicontinuous and appeal to Arzela-Ascoli.

For $x, y \in [0,1]$, $n \in \mathbb{N}$, note that
$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n'(?)|\,|x-y| \qquad \text{by MVT} \\ &\leq K\,|x-y|. \end{aligned}$$

Thus, for $\varepsilon > 0$, taking $\delta = \varepsilon/K$ does the job. ∎

② For $k \in C([0,1]\times[0,1])$ with sup norm, and for $f \in C[0,1]$, define
$$T(f)(x) := \int_0^1 k(x,t)\, f(t)\, dt.$$

Then, $T(f)$ is a continuous function and $\|T(f)\|_\infty \leq k\|f\|_\infty$.

$\therefore T : (C[0,1], \|\cdot\|_\infty) \longrightarrow (C[0,1], \|\cdot\|_\infty)$ is a BL op.

on V.

Claim: T is compact.

Again, we use Arzela-Ascoli.

- Equibounded: If $\|f\|_\infty \le c$, then $\|Tf\|_\infty \le Kc$.

- Equicontinuous. Let $\|f\|_\infty \le c$. Then, for $x, y \in [0,1]$, we have

$$|(Tf)(x) - (Tf)(y)| \le \int_0^1 |K(x,t) - K(y,t)||f(t)|dt.$$

Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |K(x,t) - K(y,t)| < \frac{\varepsilon}{c}$.

Then, for $|x-y| < \delta$, we have

$$|(Tf)(x) - (Tf)(y)| \le \int_0^1 \frac{\varepsilon}{c} |f(t)|\, dt < \varepsilon. \qquad \blacksquare$$

Defn. $\mathcal{K}(V, W) \longrightarrow$ set of all compact linear operators from V to W.

Thm. 22.1. Let $V \to$ NLS, $W \to$ B-space.

If $(T_n)_n$ in $\mathcal{K}(V, W)$. If $T_n \longrightarrow T$ in operator norm, then $T \in \mathcal{K}(V, W)$.

Proof. Let $B = B_1(0)$. We show $\overline{T(B)}$ is compact.

Since W is a complete metric space, it suffices to show that for every $\varepsilon > 0$, $\exists$ a finite cover of $T(B)$ by a finite number of balls of radius ε.

Choose N sufficiently large so that $\|T_N - T\| < \varepsilon/2$.

By compactness of T_N, we have

$$T_N(B) \subseteq \bigcup_{k=1}^{m} B_{\varepsilon/2}(w_k).$$

Then $T(B) \subseteq \bigcup^{m} B_\varepsilon(w_k)$. $\blacksquare$

Then, $$T(B) \subseteq \bigcup_{k=1}^{m} B_{\varepsilon/2}(w_k).$$ ■

Remark Easy to check $K(V,W)$ is a subspace of $BL(V,W)$.
We have shown that it is a closed subspace when W is compact.

Example. • We cannot relax the condition in thm to "strong convergence".
Indeed, take $V = W = \ell_2$ and let
$$T_n(\underset{\sim}{x}) = (x_1, \dots, x_n, 0, 0, \dots).$$
Then, $T_n \longrightarrow id_{\ell_2}$ strongly.
Each T_n is compact since $\dim T_n(V) < \infty$.
But id_{ℓ_2} is not since $\dim(\ell_2) = \infty$.

• Define $T : \ell_2 \longrightarrow \ell_2$ by
$$T(\underset{\sim}{x}) = \left(\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right)$$

Claim: T is compact

Proof. Define $T_n \in K(\ell_2, \ell_2)$ by
$$T_n(\underset{\sim}{x}) = \left(\frac{x_1}{1}, \dots, \frac{x_n}{n}, 0, 0, \dots\right).$$
$\dim T_n(\ell_2) < \infty$ & thus, $\ell_2 \in K(\ell_2, \ell_2)$.

Moreover, $\|T_n - T\|_{BL} = \frac{1}{n+1} \longrightarrow 0.$

$\therefore T$ is compact, by the theorem.

Prop^n 23.2. Let $T \in K(V,W)$, with V Banach & W Hilbert.
Then, T is a (uniform) limit of finite rank operators.

Proof. Let $B = B_1(0)$. Then, $K = \overline{T(B)}$ is compact, by hypothesis. Let $\varepsilon > 0$. Then, $\exists$ finite index set I and $\{w_i\}_{i \in I} \subseteq W$ s.t.

$$K \subseteq \bigcup_{i \in I} B_\varepsilon(w_i).$$

Let $G = \operatorname{span}\{w_i\}_{i \in I}$. Then, $\dim(G) < \infty$.
Let $P : W \to G$ be the orthogonal projection (G is closed). Then, $P \circ T$ is of finite rank. Note $\|P\| = 1$.

Let $x \in B$. Then, $\exists i_0 \in I$ s.t.

$$\|Tx - w_{i_0}\| < \varepsilon.$$

Then, $\|(P \circ T)(x) - w_{i_0}\| = \|P(T(x)) - P(w_{i_0})\|$
$= \|P(Tx - w_{i_0})\| \leq \|Tx - w_{i_0}\| < \varepsilon.$

$$\therefore \|(P \circ T)(x) - T(x)\| \leq 2\varepsilon.$$

$$\therefore \|P \circ T - T\| \leq 2\varepsilon.$$

Spectrum of Compact Operators

Now, assume V is a <u>complex</u> Banach space.

Defn. Let $T \in BL(V)$. Then, the spectrum of T is the set

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}.$$

in the category of NLS

The resolvent of T is the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

We show $\rho(T)$ is open and hence $\sigma(T)$ is closed.

Prop^n 2.23. Let $A \in BL(V)$.

(i) If $k \in \mathbb{F}$ and $\|A\| < |k|$, then $k \in \rho(A)$ and $\|(A - kI)^{-1}\|_{BL(V)} \leq \dfrac{1}{|k| - \|A\|}$.

(ii) Let A be invertible. If $\|B - A\| < \dfrac{1}{\|A^{-1}\|}$, then B is invertible. Further, if $\|B - A\| \leq \dfrac{\varepsilon}{\|A^{-1}\|}$ with $\varepsilon \in (0,1)$, then $\|B^{-1} - A^{-1}\| \leq \|A^{-1}\|^2 \dfrac{\|B - A\|}{1-\varepsilon}$.

Lecture 23 (29-10-2021)

29 October 2021 14:05

Let $T : D(T) \longrightarrow V$, $D(T) \subseteq V$, be a linear operator (possibly not bounded) with $V \neq \{0\}$ a complex NLS.

Def^n A regular value $\lambda \in \mathbb{C}$ of T is a scalar s.t.

(i) $R_\lambda(T) = (T - \lambda I)^{-1}$ exists,

(ii) $R_\lambda(T)$ is bounded,

(iii) $R_\lambda(T)$ is defined on a set which is dense in X.

The set of all such regular values is called the resolvent set of T, denoted $\rho(T)$.

$$\sigma(T) = \mathbb{C} - \rho(T).$$

spectrum

Classification of spectrum:

(1) Point spectrum $\sigma_p(T)$: λ s.t. $R_\lambda(T)$ does not exist.
$\lambda \in \sigma_p(T)$ is called an eigenvalue of T.

(2) Continuous spectrum $\sigma_c(T)$: λ s.t. $R_\lambda(T)$ exists (and defined on a dense subset of V) but is not bounded.

Remark: Note that T itself was not assumed to be bounded.

(3) Residual spectrum $\sigma_r(T)$: λ s.t. domain of $R_\lambda(T)$ is not dense in V.

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \quad \text{and} \quad \mathbb{C} = \sigma(T) \sqcup \rho(T)$$

Eigenvalue Problem: $\lambda \in \sigma(T)$ is called an eigenvalue of T if
$(T - \lambda I)u = 0$ for some $u \neq 0$.

↳ eigenvector corresponding to λ

In this case, $\dim \mathcal{N}(T-\lambda I)$ is called the geometric multiplicity of λ.

Examples. Spectral but not eigenvalue:

① $V = \ell_2$ (complex)

Define the right shift $T: \ell_2 \longrightarrow \ell_2$ by

$$T(\underset{\sim}{u}) = (0, u_1, u_2, \ldots).$$

T is 1-1 but $\mathrm{im}(T)$ is not dense. Thus, $0 \in \sigma(T) \setminus \sigma_p(T)$.

We have $\underbrace{R_0(T)}_{= T^{-1}}$ defined on $\mathrm{im}(T) = \{ \underset{\sim}{v} \in \ell_2 : v_1 = 0 \}$.

↳ not dense

Prop^n 23.1. Let V be a complex B-space.
Let $T: V \longrightarrow V$ and $\lambda \in \rho(T)$. Assume further that
(i) T is closed or (ii) T is bounded.
Then, $R_\lambda(T)$ is defined on the whole space.

Proof Use closed graph theorem to see that $\mathrm{im}(T) = \overline{\mathrm{im}(T)} = V$. ∎

Lemma 23.2. Let $T: V \longrightarrow V$ be compact linear with V Banach.
Then, $\dim \mathcal{N}(T - \lambda I) < \infty \quad \forall \lambda \neq 0$.

Proof: Suffices to prove that the unit ball in $\mathcal{N}(T-\lambda I)$ is (sequentially) compact. Let $(u_n)_n$ be a sequence in it.
Then, $u_n = \frac{1}{\lambda} T(u_n)$. Since T is compact, we get a convergent subsequence, as desired. ∎

Corollary 23.3. $\dim \mathcal{N}(T-\lambda I)^n < \infty \quad \forall n \geq 1$. (Same notation as above.)

Corollary 23.3. $\dim \mathcal{N}(T - \lambda I)^n < \infty \quad \forall n \geq 1.$ (Same notation as above.)

Thm 23.4. Let $T \in K(V)$, V Banach.
Then, for $\lambda \neq 0$, $R(T_\lambda) = \operatorname{im}(T - \lambda I)$ is closed.

Thm 23.5. Let $T \in K(V)$, V Banach.
Then, the set of eigenvalues of T is countable.
The only possible accumulation point is 0.

Proof. Suffices to prove that
$E_t := \{\lambda \in \sigma_p(T) : |\lambda| \geq t\}$ is finite for all $t > 0$.

Suppose not. Then, $|E_{t_0}| = \infty$ for some $t_0 > 0$.
Pick $\lambda_1, \lambda_2, \dots$ distinct in E_{t_0}.
Let $u_1, u_2, \dots$ be corresp. e-vects.
Note that these are lin. indep. Set $V_n := \operatorname{span}\{u_1, \dots, u_n\}$.
Every $u \in V_n$ has a unique rep.

$$u = \sum_{j=1}^{n} \alpha_j u_j.$$

Then,

$$\begin{aligned}(T - \lambda_n I) u &= \sum_{j=1}^{n} (T - \lambda_n I) u_j \\ &= \sum_{j=1}^{n} (\lambda_j - \lambda_n) u_j \\ &= \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) u_j \in V_{n-1}.\end{aligned}$$

By Riesz, we can get $(v_n)_n$ s.t.

$$v_n \in V_n, \quad \|v_n\| = 1, \quad \|v_n - u\| \geq 1/2 \quad \forall u \in V_{n-1}.$$

Claim: $\|Tv_n - Tv_m\| \geq \frac{1}{2} t_0 \quad \forall n > m.$

Note that the claim finishes the proof, since it contradicts compactness of T.

<u>Proof</u> Fix $n > m$. Define $\bar{u}$ by

$$Tv_n - Tv_m = \lambda_n u_n - \bar{u}.$$

Note $\bar{u} = (\lambda_n I - T) v_n + T v_m \in V_{n-1}$.

$$\therefore \|Tv_n - Tv_m\| = \lambda_n \left\| u_n - \frac{\bar{u}}{\lambda_n} \right\| \geq \frac{\lambda_n}{2} \geq \frac{t_0}{2}.$$