

Fourier Inversion for L^1 Functions

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- 1 Recap
- 2 Notations and Setup
- 3 Proof of the Main Theorem
- 4 The Stronger Theorem

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for those $x \in \mathbb{R}^n$ for which (\star) holds.

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We will actually prove the result for a broader class of approximate identities.

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for all $x \in \text{Leb}(f)$.

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It is now clear that proving the Main Theorem will show that (\star) holds for $x \in \text{Leb}(f)$.

Some final notation

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$$\int_{B(x, r)} 1 = V_n r^n$$

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Using this, we first show that $I_2(t) \xrightarrow{t \rightarrow 0} 0$.

$$l_2(t)$$

$$I_2(t) = \left| \int_{|u| \geq \delta} [f(x-u) - f(x)] \varphi_t(u) \, du \right|$$

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With these notations, we do some more calculations.

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Integrate by parts

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The Green Integral

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This completes the proof.

The Stronger Theorem

- 1 Recap
- 2 Notations and Setup
- 3 Proof of the Main Theorem
- 4 The Stronger Theorem

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Concluding Remark

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Reference: *Introduction to Fourier Analysis on Euclidean Spaces* by Stein and Weiss