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MA 5106

Introduction to Fourier Analysis

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Lecture 1 (06-01-2021)

Preliminaries

- Rectangle in $\mathbb{R}^d$: $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$. closed
- Cube in $\mathbb{R}^d$: $Q = [a_1, b_1] \times \cdots \times [a_d, b_d]$ where $b_1 - a_1 = \cdots = b_d - a_d$.
- Volume of R: $|R| = \prod_{i=1}^{r} (b_i - a_i)$
- Exterior measure of $E \subseteq \mathbb{R}^d$: (Exterior measure)

$$m_*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j,\ Q_j \text{ are cubes} \right\}$$

Observations:

(1) Any singleton has exterior measure 0.

(2) Exterior measure of any (closed/open) rectangles is equal to its volume.

(3) $m_*(\mathbb{R}^d) = \infty$.

(4) $m_*(\text{Cantor set}) = 0$.

Properties:

(1) $E \subseteq F \Rightarrow m_*(E) \leq m_*(F)$

(2) $m_*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m_*(E_j)$ (equality needn't hold even if disjoint)

Measurable set

(Measurable set, Lebesgue measurable set)

Def^n A set $E \subseteq \mathbb{R}^d$ is called (Lebesgue) measurable if for every $\epsilon > 0$, $\exists$ an open set O with $O \supseteq E$ s.t. $m_*(O \setminus E) = 0$.

If $E \subseteq \mathbb{R}^d$ is measurable, then (Lebesgue) measure of E is denoted by $m(E)$ and defined as

$$m(E) = m_*(E).$$

(Lebesgue measure)

Examples of measurable sets

(1) Any open set is measurable.

(2) E s.t. $m_*(E) = 0 \Rightarrow E$ is measurable

(3) Countable union of measurable sets are measurable.

(4) Complement of a meas. set is meas.

(5) Any closed set. Any countable intersection of meas. sets.

Thm. (1) Let $E_1, E_2, \ldots$ be disjoint measurable sets. Then,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

(2) $m(E+h) = m(E)$ $\forall$ measurable $E \subseteq \mathbb{R}^d$, $\forall h \in \mathbb{R}^d$

$(E+h := \{ y+h \mid y \in E \})$

"$E+h$ is also measurable" is implicit. Similar for next ones.

(3) $m(cE) = c^d\, m(E)$, $\quad c>0$

$(cE := \{cy \mid y \in E\})$

(4) $m(-E) = m(E)$. $\quad (-E := \{-y \mid y \in E\})$

Def^n. $f: \mathbb{R}^d \to [-\infty, \infty]$ is said to be measurable if for any $a>0$,

$$f^{-1}([-\infty, a)) \subseteq \mathbb{R}^d$$

is measurable. (Measurable function)

Examples

(1) Any continuous function is measurable.

(2) If f is measurable and $p: \mathbb{R} \to \mathbb{R}$ is measurable, then $p \circ f$ is measurable.

(3) If $\{f_n\}_n$ is a sequence of measurable functions, then the functions

$$\sup f_n, \quad \inf f_n, \quad \limsup f_n, \quad \liminf f_n.$$

are all measurable.

(4) Limit (Pointwise) of a sequence of measurable functions is measurable.

(5) If f, g are measurable, then so are $f \pm g$, $f \cdot g$.

Ex. Characteristic function. Let $E \subseteq \mathbb{R}^d$.

Ex. Characteristic function. Let $E \subseteq \mathbb{R}^d$.
Define

$$\chi_E(x) := \begin{cases} 1 & ; \text{ if } x \in E \\ 0 & ; \text{ if } x \notin E \end{cases}$$

Then, χ_E is a measurable $f^n \iff E$ is measurable.

Note $f^{-1}([-\infty, a)) = \begin{cases} E^c & ; \quad 0 < a \leq 1 \\ \mathbb{R}^d & : \quad 1 < a \end{cases}$

Thus, χ_E is a meas. $f^n \iff E^c$ is meas $\iff E$ is.

Def^n. A function $f: \mathbb{R}^d \to \mathbb{R}$ is said to be (Simple function)
simple if

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}. \qquad \begin{matrix} (a_k \in \mathbb{R} \text{ constants}) \\ (\exists\, m(E_k) < \infty) \end{matrix}$$

Thm. Let f be a non-negative measurable function on $\mathbb{R}^d$.
Then, $\exists$ an increasing seq. of non-neg simple functions $\{\varphi_k\}_k$
s.t.

$$\lim_{k \to \infty} \varphi_k = f \qquad \text{pointwise.}$$

$$(\phi_k(x) \leq \phi_{k+1}(x) \quad \forall x)$$

Integration

(Integration)

(1) Let f be a simple function.

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}, \qquad (E_k \text{ measurable } \& \; m(E_k) < \infty)$$

$$\int_{\mathbb{R}^d} f := \sum_{k=1}^{N} a_k \, m(E_k).$$

(Has to be checked that this is independent of $((a_k),(E_k),N)$.)

Example. $\int_{\mathbb{R}} \chi_{[0,1]} = 1.$

(2) Let f be a bounded measurable function with

$$m(\operatorname{supp} f) < \infty \quad \text{where}$$

$\operatorname{supp} f = \{x : f(x) \neq 0\}.$ $\longrightarrow$ will be measurable since f is

Then, $\exists \{\varphi_n\}_n$ of simple functions s.t. $\varphi_n \leq M$ and

$$\varphi_n \longrightarrow f \quad \text{a.e.}$$

(i.e., the set of points x for which $\varphi_n(x) \not\to f(x)$ is of measure zero.)

and $\operatorname{supp} \varphi_n \subseteq \operatorname{supp} f.$

Then,
$$\int_{\mathbb{R}^d} f := \lim_{n\to\infty} \underbrace{\int_{\mathbb{R}^d} \varphi_n}_{\text{this defined by (1)}}.$$

(Again, independent of this $\{\varphi_n\}_n$.)

(3) Assume $f \geq 0$.

$$\int_{\mathbb{R}^d} f := \sup \left\{ \int_{\mathbb{R}^n} g \;:\; 0 \leq g \leq f,\ g \text{ is bounded, measurable with } m(\operatorname{supp} g) < \infty \right\}$$

$$\int_E f := \underbrace{\int_{\mathbb{R}^d} f \cdot \chi_E}_{\text{this is defined earlier}} \qquad (E \subseteq \mathbb{R}^d \text{ is measurable})$$

this is defined earlier
note $f \cdot \chi_E$ is measurable and ≥ 0.

Def^n. $f \geq 0$ is integrable if $\int_{\mathbb{R}^d} f < \infty$. (Integrable)

Now, if $f : \mathbb{R}^d \to \mathbb{R}$ is any function, we can write

$$f(x) = f^+(x) - f^-(x)$$

$$f^+(x) := \max\{f(x), 0\}, \qquad f^-(x) := \max\{-f(x), 0\}.$$

Note that $f^+, f^- \geq 0$.

Def^n. f is integrable if $\int_{\mathbb{R}^d} |f| < \infty$ and

Can be extended $\int_{\mathbb{R}^d} f$ to $:= f : \mathbb{R}^d \to \mathbb{C}$ componentwise

$$\int_{\mathbb{R}^d} f := \int_{\mathbb{R}^d} f^+ \quad \int_{\mathbb{R}^d} f^-$$

EXAMPLE. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1 ; & x \in \mathbb{Q} \cap [0,1] \\ 0 ; & x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

Then, f is not Riemann integrable on $[0,1]$.
However,

$$\int_{[0,1]} f = \int_{\mathbb{Q} \cap [0,1]} f + \int_{[0,1] \setminus \mathbb{Q}} f$$

$= \quad 0$

Thm. Let f be Riemann integrable on $[a, b]$. Then, f is measurable and both the integrals (Riemann & Lebesgue) coincide.

Lecture 2 (08-01-2021)

08 January 2021 09:24

Recap. $f \geq 0$

1. $f = \sum a_i \chi_{E_i}$, then $\int_{\mathbb{R}^d} f := \sum a_i \, m(E_i)$

2. $m(\text{supp } f) < \infty$, then $\exists \{\varphi_n\}$ simple s.t. $\varphi_n \longrightarrow f$ a.e.
(f bounded)

$$\int_{\mathbb{R}^d} f := \lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi_n$$

3. $$\int_{\mathbb{R}^d} f \, dx := \sup \left\{ \int_{\mathbb{R}^d} g \; : \; \underset{g \text{ bounded}}{0 \leq g \leq f} \; \& \; m(\text{supp } g) < \infty \right\}$$

Properties.

1. $\int (af + bg) = a \int f + b \int g \qquad \forall a, b \in \mathbb{C}$

2. $E \cap F = \emptyset$ and E, F measurable, then

$$\int_{E \cup F} f = \int_E f + \int_F f$$

3. $\left| \int f \right| \leq \int |f|$

4. $f \geq 0$ and $\int_{\mathbb{R}^d} f = 0 \Rightarrow f = 0$ a.e.

If $f = 0$ a.e., then $\int f = 0$.

5. $\int_{\mathbb{R}^d} |f| < \infty \Rightarrow |f| < \infty$ a.e.

Suppose $f_n \to f$ pointwise.

$$\lim_{n\to\infty} \int f_n \stackrel{?}{=} \int f$$

(We know above is true if uniform conv. & f_n Riemann integ.)

Thm. (Monotone Convergence Theorem) (Monotone Convergence Theorem)

Let $\{f_n\}_n$ be a sequence of non-negative measurable functions, converging pointwise to f and $f_n \le f_{n+1}$.
Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.$$

Thm. (Dominated Convergence Theorem) (Dominated Convergence Theorem)
(DCT)

Let $\{f_n\}_n$ be a sequence of measurable functions such that

$$f_n \to f \quad \text{a.e.}$$

Assume further that $\exists$ an integrable function g s.t.

$$|f_n(x)| \le g(x).$$

Then,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.$$

Thm.

- $$\int_{\mathbb{R}^d} f(x-h)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$$

- $$\int_{\mathbb{R}^d} f(-x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$$

- $$\int_{\mathbb{R}^d} f(cx)\,dx = \frac{1}{c^d}\int_{\mathbb{R}^d} f(x)\,dx \quad ; \quad c>0$$

Thm. (Fubini's Theorem) (Fubini's Theorem)

(a) Let f be a non-negative measurable function on

$$\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^{d_1+d_2}.$$

Then,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy \right) dx$$

$$\| \qquad\qquad \|$$

$$\int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,d(x,y)$$

(b) Let f be integrable on $\mathbb{R}^{d_1+d_2}$ (i.e., $\int_{\mathbb{R}^{d_1+d_2}} |f| < \infty$).

Then,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy \right) dx$$

$$\| \qquad\qquad \|$$

$$\int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,d(x,y)$$

To use (b), we need to check if $\int_{\mathbb{R}^{d_1+d_2}} |f| < \infty$. However,

since $|f| \geq 0$, we can compute the above integral using (a).

Def^n. $\cdot \mathcal{L}^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} \;\middle|\; f \text{ is meas. and } \int_{\mathbb{R}^d} |f|^p < \infty \right\}$.

$1 \leq p < \infty$

(L^p spaces, Lp spaces)

· Normed linear space (NLS): $(X, \|\cdot\|)$

$X \to$ vector space over $\mathbb{R}$ or $\mathbb{C}$

and $\|\cdot\| : X \to [0, \infty)$ s.t.

(i) $\|x\| = 0 \iff x = 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in \mathbb{F}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

Any nls is a metric space with $d_X(x, y) = \|x - y\|$.

· $\mathcal{L}^p(\mathbb{R}^d)$ is a vector space, easy to see.
(linear space)

Moreover, defining $\|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \right)^{1/p}$

· $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

· $\|\alpha f\|_p = |\alpha| \, \|f\|_p$

· $\|f\|_p = 0 \Rightarrow \int |f|^p = 0 \to |f|^p = 0$ a.e.

- $\|f\|_p = 0 \Rightarrow \int_{\mathbb{R}^d} |f|^p = 0 \rightarrow |f|^p = 0$ a.e.

$$\Downarrow$$

$f = 0$ a.e

not necessarily 0

In fact, $\mathcal{L}^p(\Omega)$ is actually classes of functions where $f \sim g \Leftrightarrow f = g$ a.e.

Then, $\mathcal{L}^p(\mathbb{R}^d)$ is an NLS.

- $\mathcal{L}^\infty(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C}$ meas. f^n which are bounded a.e.$\}$

$$\|f\|_\infty := \text{ess sup } |f|$$

$$\therefore |f(x)| \leq \|f\|_\infty \quad \text{a.e.}$$

- An NLS $(X, \|\cdot\|)$ is called a Banach space if X is complete as a metric space.

- $\mathcal{L}^p(\mathbb{R}^d)$ is a Banach space for $1 \leq p \leq \infty$.

Thm. (Hölder's Theorem) (Hölder's Theorem, Holder's Theorem)

$\|fg\|_1 \leq \|f\|_p \|g\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

$(p \geq 1, \quad p = \infty \Rightarrow \frac{1}{p} = 0)$

Result. (Using Hahn-Banach Theorem)

$$\|f\|_p = \sup \left\{ \left| \int_{\mathbb{R}^d} fg \right| : \|g\|_q < 1 \right\}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Convolution (Convolution)

Def^n. Let f, g be integrable functions on $\mathbb{R}^d$ $(f, g \in \mathcal{L}^1(\mathbb{R}^d))$.
Then, convolution of f and g is defined as

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy.$$

Q. Does RHS exist? Yes, for almost every $x \in \mathbb{R}^d$.

Proof. Note

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)|\, |g(x-y)|\, dy \right) dx$$

Fubini (a)

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)|\, |g(x-y)|\, dx \right) dy$$

$$= \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |g(x-y)|\, dx \right) dy$$

translation

$$= \int_{\mathbb{R}^d} |f(y)| \underbrace{\left(\int_{\mathbb{R}^d} |g(x)|\, dx \right)}_{\text{constant}} dy$$

$$= \left[\int_{\mathbb{R}^d} |f| \right] \left[\int_{\mathbb{R}^d} |g| \right] < \infty \quad \text{since } f, g \in \mathcal{L}^1$$

$$\Rightarrow x \mapsto \int_{\mathbb{R}^d} |f(y)|\, |g(x-y)|\, dy \quad \text{is finite a.e.} \quad \blacksquare$$

Thus, $(f*g)(x)$ exists for almost every x.

Also, $\|f*g\|_1 \le \|f\|_1 \|g\|_1$, by the above.

Thm. Let $p \in [1,\infty)$. If $f \in \mathcal{L}^p(\mathbb{R}^d)$ and $g \in \mathcal{L}^1(\mathbb{R}^d)$, then
$$f*g \in \mathcal{L}^p(\mathbb{R}^d)$$
and
$$\|f*g\| \le \|f\|_p \|g\|_1.$$

- $$(f*g)(x) = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy \qquad y \mapsto x-z$$
$$= \int_{\mathbb{R}^d} f(x-z)\, g(z)\, dz$$
$$= (g*f)(x)$$

- Convolution can be defined on any measurable group $(G, \cdot)$.

  $f, g \in \mathcal{L}^1(G)$, then
$$(f*g)(x) = \int_G f(y)\, g(xy^{-1})\, dy$$

Can define convolution on $S^1 = \mathbb{T} \cong [0, 2\pi]/\sim$.

- $f*(g*h) = (f*g)*h$

- $(f+g)*h = f*h + g*h$

Now, $\text{supp } f = \overline{\{x : f(x) \neq 0\}}$.

Thm. Let $C_c(\mathbb{R}^d)$ be the set of continuous func $f : \mathbb{R}^d \to \mathbb{C}$ with compact support. (C_c(\Bbb R^d))

Obs. $C_c(\mathbb{R}^d) \subseteq \mathcal{L}^p(\mathbb{R}^d)$.

(well, technically $\mathcal{L}^p$ is equiv. classes but note that if cont. f^n are equal a.e., then they are equal.)

Proof. $f \in C_c(\mathbb{R}^d)$

$$\Rightarrow \int_{\mathbb{R}^d} \|f\|^p = \int_{\text{supp } f} \|f\|^p \leq \|f\|_\infty \int_{\text{supp}} 1 = \|f\|_\infty \, m(\text{supp } f) < \infty.$$

Thm. 1. $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$. Here, $1 < p \nleq \infty$.

2. $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$. Here, $1 < p \nleq \infty$.

$\hookrightarrow$ inf. differentiable

Def^n: (Approximate identity in $\mathcal{L}^1(\mathbb{R}^d)$) (Approximate identity)

A sequence $\{k_n\}_n$ in $\mathcal{L}^1(\mathbb{R}^d)$ is called approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$ if

(1) $k_n \geq 0 \quad \forall n \in \mathbb{N}$

(2) $\int_{\mathbb{R}^d} k_n = 1 \quad \forall n \in \mathbb{N}$

(3) For any $\delta > 0$,

$$\lim_{n \to \infty} \int_{\|x\| > \delta} k_n = 0.$$

Thm. Let $\{k_n\}_n$ be an approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$. Let $f \in \mathcal{L}^1(\mathbb{R}^d)$.

Then,

$$f * K_n \longrightarrow f \quad \text{in} \quad \mathcal{L}^1 \quad \text{as} \quad n \to \infty.$$

That is,

$$\lim_{n \to \infty} \| f * K_n - f \| = 0.$$

Remark. $(\mathcal{L}^1(\mathbb{R}^d), *)$ does not have an identity.
That is, $\nexists\, g \in \mathcal{L}^1(\mathbb{R}^d)\ \forall f \in \mathcal{L}^1(\mathbb{R}^d)\ (f * g = f.)$

We prove the theorem in the next class. Before that, we have the following lemma.

(Lemma) Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then, the map $y \mapsto T_y f$ is a continuous function $\mathbb{R}^d \to \mathcal{L}^1(\mathbb{R}^d)$, where

$$T_y f(x) := f(x - y).$$

That is, for any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|y_1 - y_2\|_2 < \delta \Rightarrow \|T_{y_1} f - T_{y_2} f\|_1 < \epsilon.$

Proof. Let $g \in C_c(\mathbb{R}^d)$. Then

$$\|T_{y_1} g - T_{y_2} g\|_1 = \int_{\mathbb{R}^d} |T_{y_1} g(x) - T_{y_2} g(x)|\, dx$$

$$= \int_{\mathbb{R}^d} | g(x - y_1) - g(x - y_2)|\, dx$$

$$= \int_{\mathbb{R}^d} |g(x + y_2 - y_1) - g(x)|\, dx$$

let $K = \operatorname{supp} g$
compact

$$= \int_{K \cup (K + y_2 - y_1)} |g(x + y_2 - y_1) - g(x)|\, dx$$

$\because g$ is continuous, can choose $\delta > 0$ s.t.

$\|y_1 - y_2\| < \delta \Rightarrow |g(x + y_2 - y_1) - g(x)| < \epsilon / m(K \cup (K + y_2 - y_1))$

$\cup$

$\|y_1 - y_2\| < \delta \Rightarrow |g(x + y_2 - y_1) - g(x)| < \epsilon / m(K \cup (K + y_2 - y_1))$

$< \varepsilon$ if $\|y_1 - y_2\| < \delta$.

Now, use the fact that $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^1(\mathbb{R}^d)$.
Note the following to conclude:

$$\|T_{y_1}f - T_{y_2}f\|_1 \leq \|T_{y_1}f - T_{y_1}g\|_1 + \|T_{y_1}g - T_{y_2}g\|_1 + \|T_{y_2}g - T_{y_2}f\|_1$$

$$= \|T_{y_1}(f-g)\| + \|T_{y_1}g - T_{y_2}g\| + \|T_{y_2}(g-f)\|$$

$$= \|f-g\| + \|T_{y_1}g - T_{y_2}g\| + \|f-g\|$$

can be made $< \varepsilon$.

Lecture 3 (13-01-2021)

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Recall:

A sequence $\{k_n\}_n$ in $\mathcal{L}^1(\mathbb{R}^d)$ is called approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$ if

(1) $k_n \geq 0 \quad \forall n \in \mathbb{N}$

(2) $\int_{\mathbb{R}^d} k_n = 1 \quad \forall n \in \mathbb{N}$

(3) For any $\delta > 0$,

$$\lim_{n \to \infty} \int_{\|x\| > \delta} k_n = 0.$$

Thm. Let $\{k_n\}_n$ be an approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$. Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then,

$$f * k_n \to f \quad \text{in } \mathcal{L}^1 \quad \text{as } n \to \infty.$$

That is,

$$\lim_{n \to \infty} \|f * k_n - f\| = 0.$$

To prove that, we had seen the following lemma.

Lemma. Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then, the map $y \mapsto T_y f$ is a continuous function $\mathbb{R}^d \to \mathcal{L}^1(\mathbb{R}^d)$, where

$$T_y f(x) := f(x - y).$$

Remark. In fact, if $f \in \mathcal{L}^p(\mathbb{R}^d)$, $1 < p < \infty$, then

$y \mapsto T_y f$ is continuous as a f^n $\mathbb{R}^d \to \mathcal{L}^p(\mathbb{R}^d)$.

Proof (of Thm). $f \in \mathcal{L}^1(\mathbb{R}^d)$ and $\{k_n\}_n$ is approximate identity in $\mathcal{L}^1$.

$$\| f * k_n - f \|_1 = \int_{\mathbb{R}^d} |(f * k_n)(x) - f(x)| \, dx$$

$$= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y) \, k_n(y) \, dy - \underbrace{\int_{\mathbb{R}^d} f(x) \, k_n(y) \, dy}_{\left(\because \int_{\mathbb{R}^d} k_n = 1\right)} \right| dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} | f(x-y) - f(x) | \, k_n(y) \, dy \, dx \qquad (k_n \geq 0)$$

interchanging, we'll finally show it's finite (Fubini)

$$= \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} | T_y f(x) - f(x) | \, dx \right\} k_n(y) \, dy$$

$$= \int_{\mathbb{R}^d} \| T_y f - f \|_1 \, k_n(y) \, dy$$

$\because$ $y \mapsto T_y f$ is continuous, for every $\epsilon > 0$, $\exists \delta > 0$ s.t. $\| T_y f - f \|_1 < \epsilon$ if $\|y\| < \delta$.

$$= \int_{|y| < \delta} \| T_y f - f \|_1 \, k_n(y) \, dy + \int_{|y| \geq \delta} \| T_y f - f \|_1 \, k_n(y) \, dy$$

$\| T_y f \| = \| f \|$

$$\leq \frac{\epsilon}{2} \int_{|y| < \delta} k_n(y) \, dy + 2 \| f \|_1 \int_{|y| \geq \delta} k_n(y) \, dy$$

$\int_{\mathbb{R}^d} k_n = 1$

By def^n of approx. id.

By our approx. id.

$\int_{|y|\geq\delta} k_n \to 0$

$\int_{\mathbb{R}^d} k_n = 1$

$$\leq \frac{\varepsilon}{2}\cdot 1 + 2\|f\|_1 \frac{\varepsilon}{4\|f\|_1} \qquad \forall n \geq N$$

$$= \varepsilon$$

$\therefore\ f * k_n \to f$ in $\mathcal{L}^1$.

Remark. We shall see later that $(\mathcal{L}^1(\mathbb{R}^d), *)$ does not have an identity but it has (many) approximate identities.

Construction of an Approximate Identity

Let $\varphi \geq 0$ be an integrable function.

(That is, $\int_{\mathbb{R}^d} \varphi < \infty$. That is $\varphi \in \mathcal{L}^1(\mathbb{R}^d)$)

Suppose $\int_{\mathbb{R}^d} \varphi = 1$.

For each $n \in \mathbb{N}$, let $\varphi_n(t) := n^d \varphi(nt)$, $t \in \mathbb{R}^d$.

(nt is the usual scalar multiplication in the space $\mathbb{R}^d$.)

Then, $\{\varphi_n\}_n$ is an approximate identity in $\mathcal{L}^1(\mathbb{R}^d)$.

Check. (i) $\varphi_n \geq 0 \quad \forall n \in \mathbb{N}$ is obvious.

(ii) $\int_{\mathbb{R}^d} \varphi_n = n^d \int_{\mathbb{R}^d} \varphi(nt)\, dt$

$$\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \dots$$

$$= n^d \int_{\mathbb{R}^d} \varphi(y) \frac{dy}{n^d} = \int_{\mathbb{R}^d} \varphi(y)\, dy = 1.$$

(iii) Fix $\delta > 0$.

$$\int_{|t| \geq \delta} \varphi_n = n^d \int_{|t| \geq \delta} \varphi(nt)\, dt$$

$$= \int_{|y| \geq n\delta} \varphi(y)\, dy \xrightarrow{n \to \infty} 0$$

EXAMPLE. For $d=1$, $\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ works.

Obser. $f \in \mathcal{L}^1$, $g \in \mathcal{L}^p \Rightarrow f * g \in \mathcal{L}^p$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Q. Let $g \in \mathcal{L}^p$ and $\{k_n\}_n$ an approx. id. in $\mathcal{L}^1$. $(1 \leq p < \infty)$
Is it true that

$$\|g * k_n - g\|_p \to 0 \,?$$

(That is, $\lim_{n \to \infty} g * k_n = g$?)

Yes! Will show later.

Thm. (Minkowski's integral inequality)
Given two measure spaces (X, μ) and (Y, ν) with σ-finite

Given two measure spaces (X, μ) and (Y, ν) with σ-finite measures :

$$\left(\int_X \left| \int_Y f(x, y)\, d\nu(y) \right|^p d\mu(x) \right)^{1/p} \le \int_Y \left(\int_X |f(x,y)|^p d\mu(x) \right)^{1/p} d\nu(y)$$

Thm. Let $\{K_n\}_n$ be an approximate identity in $\mathcal{L}^1(\mathbb{R}^d)$. Then, $f \in \mathcal{L}^p(\mathbb{R}^d)$. Then,

$$\|f * K_n - f\|_p \to 0.$$

Proof.

$$\|f * K_n - f\|_p = \left(\int_{\mathbb{R}^d} |(f * K_n)(x) - f(x)|^p dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [f(x-y) - f(x)) K_n(y)] dy \right|^p dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [T_y f(x) - f(x)) K_n(y)] dy \right|^p dx \right)^{1/p}$$

Minkowski

$$\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |T_y f(x) - f(x)|^p \cdot |K_n(y)|^p dx \right)^{1/p} dy$$

$$= \int_{\mathbb{R}^d} K_n(y) \left\{ \int_{\mathbb{R}^d} |T_y f(x) - f(x)|^p dx \right\}^{1/p} dy$$

$$= \int_{\mathbb{R}^d} \|T_y f - f\|_p \, K_n(y)\, dy$$

Now we are in the same position as earlier. ∎

($y \mapsto T_y f$ is continuous as $\mathbb{R}^d \to \mathcal{L}^p(\mathbb{R}^d)$ since $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$, for $1 \leq p < \infty$.)

Thm. Let f be a continuous function with compact support. (Then, $f \in \mathcal{L}^\infty$.) Let $\{K_n\}_n$ be an approximate identity in $\mathcal{L}^1$.
Then,

$$\| f * K_n - f \|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

(Need to prove that for $f \in C_c(\mathbb{R}^d)$, $y \mapsto T_y f$ is cts as $\mathbb{R}^d \to \mathcal{L}^\infty(\mathbb{R}^d)$)

- Let $\mathbb{T} := S^1 = \mathbb{R}/\mathbb{Z}$. $\qquad x \mapsto e^{2\pi i x}$.

  We will do analysis on $\mathbb{T}$. (Torus)
  Then, we identify f with a function on $\mathbb{R}$ which is periodic with period 1.

- $\mathcal{L}^p(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} \text{ measurable} \mid \int_0^1 |f|^p < \infty \}$.

  $$\| f \|_{\mathcal{L}^p(\mathbb{T})} := \left(\int_0^1 |f|^p \right)^{1/p}.$$

  for emphasis. Sometimes we will simply write $\|f\|_p$.

- $\mathcal{L}^p(\mathbb{T}) \subseteq \mathcal{L}^q(\mathbb{T})$ if $q \leq p$

  (note the reversal)

  $(\mathcal{L}^1(\mathbb{T}) \supseteq \mathcal{L}^2(\mathbb{T}) \supseteq \cdots)$

  true for any finite measure space.
  Not true in $\mathbb{R}$.

Proof. We show for 1 and 2 using Cauchy Schwarz.

$$\int_0^1 |f(t)|\, dt = \int_0^1 |f(t)| \cdot 1\, dt$$

$$\leq \left(\int_0^1 |f(t)|^2\right)^{1/2} \left(\int_0^1 1\, dt\right)^{1/2}$$

$$= \|f\|_2 < \infty$$

Thus $f \in \mathcal{L}^1$ and $\|f\|_1 \leq \|f\|_2$.

· In general, we use Hölder.

$$\int_0^1 |f|^q = \int_0^1 \left(|f(t)|^p\right)^{q/p} \cdot 1\, dt$$

$\alpha = p/q$

$$\leq \left(\int_0^1 \left(\left(|f(t)|^p\right)^{q/p}\right)^{p/q}\right)^{q/p} \cdot 1$$

$\int fg \leq \left(\int f^\alpha\right)^{1/\alpha} \left(\int g^\beta\right)^{1/\beta}$

(Hölder) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$= \left(\int_0^1 |f|^p\right)^{q/p}$$

$$\Rightarrow \left(\int_0^1 |f|^q\right)^{1/q} \leq \left(\int_0^1 |f|^p\right)^{1/p}$$

$\therefore f \in \mathcal{L}^q$ ∎

$$\mathcal{L}^1(\mathbb{T}) \supseteq \mathcal{L}^p(\mathbb{T}) \quad \forall\, p \geq 1.$$

Thus, we only do Fourier Analysis for $\mathcal{L}^1$, which takes care of all.