Local Cohomology and Depth

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Let R be a noetherian ring, $I \subseteq R$ an ideal, and M an arbitrary R-module. We have surjections

$$\cdots \twoheadrightarrow R/I^2 \twoheadrightarrow R/I,$$

giving us an *inverse* limit system.

In turn, this gives us a *direct* limit system

$$\cdots \to \operatorname{Ext}_R^i(R/I^t, M) \to \operatorname{Ext}_R^i(R/I^{t+1}, M) \to \cdots,$$

for all $i \geqslant 0$.

Considering the colimit (over t), we get the i-th local cohomology module of M with support in I:

$$H_I^i(M) = \varinjlim_t \operatorname{Ext}_R^i(R/I^t, M).$$

Observation 1. When $i = 0$, then $\operatorname{Ext}_R^i(R/I^t, M) = \operatorname{Hom}_R(R/I^t, M)$. This can be identified with the submodule of M consisting of elements killed by I^t. The colimit can then be identified with the union. We get

$$H_I^0(M) = \{x \in M : x \text{ is killed by some power of } I\}.$$

Observation 2. Every element of $H_I^i(M)$ is killed by a power of I.

Proof. Every element is in the image of $\operatorname{Ext}_R^i(R/I^t, M)$ for some t, and I^t kills this Ext. $\square$

Note that if we have a short exact sequence of modules

$$0 \to A \to B \to C \to 0,$$

then for each t, we get a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Ext}^0_R(R/I^t, A) \to \operatorname{Ext}^0_R(R/I^t, B) \to \operatorname{Ext}^0_R(R/I^t, C) \to \operatorname{Ext}^1_R(R/I^t, A) \to \operatorname{Ext}^1_R(R/I^t, B) \to \cdots \\ \cdots \to \operatorname{Ext}^i_R(R/I^t, A) \to \operatorname{Ext}^i_R(R/I^t, B) \to \operatorname{Ext}^i_R(R/I^t, C) \to \operatorname{Ext}^{i+1}_R(R/I^t, A) \to \cdots. \end{aligned}$$

Now, we also have arrows between varying t. Since colimits preserve exactness, we get a long exact sequence as

$$\begin{gathered} 0 \to H^0_I(A) \to H^0_I(B) \to H^0_I(C) \to H^1_I(A) \to \cdots \\ \cdots \to H^i_I(A) \to H^i_I(B) \to H^i_I(C) \to H^{i+1}_I(A) \to \cdots. \end{gathered}$$

Theorem 3. Let $(R, \mathfrak{m})$ be a local ring, and M be a nonzero finitely generated R-module. The least value of i such that $H^i_{\mathfrak{m}}(M) \neq 0$ is the depth of M.

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be a maximal M-sequence. By induction on d, we will show that $H^i_{\mathfrak{m}}(M) = 0$ if $i < d$ and $H^d_{\mathfrak{m}}(M) \neq 0$.

$d = 0$: This means that every element of $\mathfrak{m}$ is a zerodivisor on M. By prime avoidance,[1] $\mathfrak{m} \in \operatorname{Ass}(M)$ and hence, there is some nonzero element $x \in M$ annihilated by $\mathfrak{m}$. Thus, $x \in H^0_{\mathfrak{m}}(M)$ is nonzero.

$d > 0$: The short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ with $x = x_1$ yields the following exact sequence

$$H^{i-1}_{\mathfrak{m}}(M/xM) \to H^i_{\mathfrak{m}}(M) \xrightarrow{x} H^i_{\mathfrak{m}}(M).$$

If $i < d$, the induction hypothesis shows that the leftmost module above vanishes. Thus, x is a nonzerodivisor on $H^i_{\mathfrak{m}}(M)$. But every element of this module is killed by a power of $x \in \mathfrak{m}$. Thus, $H^i_{\mathfrak{m}}(M) = 0$.
If $i = d$, we use the following part of the exact sequence:

$$H^{d-1}_{\mathfrak{m}}(M) \to H^{d-1}_{\mathfrak{m}}(M/xM) \to H^d_{\mathfrak{m}}(M).$$

We have already concluded that the leftmost module is zero. By induction, the middle module is nonzero. Thus, the rightmost module is nonzero since a nonzero module injects into it. $\square$

[1] We used finite generation of M here.