

Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

Bruns & Herzog \rightarrow Cohen-Macaulay rings
- 1st Part

Affine algebra



Derived Category

$R \rightarrow$ ring (possibly noncomm.)

R -complexes :

$$\dots \rightarrow M_{i+1} \xrightarrow{\delta} M_i \xrightarrow{\delta} M_{i-1} \rightarrow \dots \quad \delta^2 = 0$$

$$\text{im}(\delta_{i+1}) \subset \text{ker}(\delta_i)$$

$$H_i(M) = \text{ker}(\delta_i) / \text{im}(\delta_{i+1})$$

$$H(M) = (H_i(M))_{i \in \mathbb{Z}}$$

$C(R) =$ category of R -complexes
(morphisms as usual)

If $f: M \rightarrow N$, we get an induced map

$$H(f): H(M) \rightarrow H(N).$$

Defn: f is a quasiisomorphism (or weak equivalence)

if $H(f)$ is bijective.
(Automatically iso.)

$W :=$ collection of weak equivalences in $C(R)$

$$D(R) := C(R)[W^{-1}] \quad (\text{or } W^{-1}C(R))$$

- Key property: W has the 2-out-of-6 property:
i.e. ... composable morphisms $\dots \xrightarrow{f} \xrightarrow{g} \xrightarrow{h} \dots$

- Key property: W has the $2\text{-out-of-}3$ property:
 Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,
 if gf and $hg \in W$, then f, g, h, hgf
 are in W .

Ex: $\Rightarrow 2\text{-out-of-}3$ property

If f, g, fg are defined and 2 are
 in W , then so is the third.

Concretely:

$$C(R) \rightsquigarrow K(R) \rightsquigarrow D(R).$$

\uparrow
homotopy category

$M, N \rightarrow R\text{-complexes}$

$\text{Hom}_R(M, N) :=$ Hom-complex of abelian groups
 (when R is comm this is
 an R -complex)

$\text{Hom}_R(M, N)_n :=$ Maps of degree n from
 $M \rightarrow N$ (no compatibility!)

$$\dots \rightarrow M_i \rightarrow M_{i+1} \rightarrow \dots = \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i, N_{i+n})$$

\swarrow \searrow
 $\dots \rightarrow N_{i+n} \rightarrow N_{i+n-1} \rightarrow \dots$

$$\partial: \text{Hom}_R(M, N)_{n+1} \rightarrow \text{Hom}_R(M, N)_n$$

$$\partial(f) = \partial^n f - (-)^{n+1} f \partial^m.$$

$$\text{Check: } \partial^2 = 0.$$

Observe: $Z_0(\text{Hom}_R(M, N)) = \text{Hom}_e(M, N).$

Def. $f, g \in \text{Hom}_e(M, N)$ are homotopic if

$$f-g \in B_0(\text{Hom}_R(M, N)), \text{ i.e.,}$$

$$f-g = \partial h \quad \text{for some } h \in \text{Hom}_R(M, N).$$

$K(R) := \mathcal{C}(R)/\text{homotopy relation.}$

Object = R -complexes

$$\text{Hom}_K(M, N) = H_0(\text{Hom}_R(M, N)).$$

- $f \sim g$ in $\mathcal{C} \Rightarrow f = g$ in $K(R)$.

$\Rightarrow H(f) = H(g)$

Defn. M an R -complex.

$\sum M$ (or $M[i]$) is the R -complex

$$(\sum M)_i = M_{i-1}$$

with $\partial^{\sum M} = -\partial^M$.

$\text{Proj } R := \text{Projective } R\text{-modules}$

$$\begin{array}{ccc} K(R) & & \\ \downarrow & \nearrow \text{localisation} & \\ K(\text{Proj } R) & \xleftarrow[-f-]{} & D(R) \\ \downarrow & & \end{array}$$

$\exists p : D(R) \rightarrow K(\text{Proj } R)$, a full and faithful embedding
"projective resolutions"

left adjoint to q .

$$\text{Hom}_K(pM, N) = \text{Hom}_D(M, qN)$$

$f : M \rightarrow N$ morphism

$\text{cone}(f) := N \oplus \sum M$ with differential

$$\begin{matrix} N_i & \xrightarrow{+} & N_i \\ \oplus & & \oplus \\ M_i & \rightarrow & M_{i-1} \end{matrix}$$

$$\partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$$0 \rightarrow N \hookrightarrow \text{cone}(f) \rightarrow \sum M \rightarrow 0.$$

f is w.e. $\Leftrightarrow H(\text{cone}(f)) = 0$.

Image of P ?

K-projectives.

P an R -complex is K-projective if given any solid diagram

$$\begin{array}{ccc} & \overset{Z}{\nearrow} & M \\ P & \xrightarrow{\alpha} & N \\ & \pi \downarrow & \end{array} \quad \text{w.e.}, \quad \exists \text{ lift } Z.$$

FACT: $p: D(R) \xrightarrow{\sim} K\text{-Proj}(R) \subseteq K(\text{Proj } R)$.

\hookleftarrow morphism up here are homotopy

- $\text{Hom}_e(R, M) = Z_0(M)$

$$\begin{array}{ccc} 0 & \xrightarrow{\circ} & M_0 \\ R & \xrightarrow{\circ} & M_0 \xrightarrow{\circ} 0 \\ & \downarrow & \downarrow \\ & \circ & M_{-1} \end{array}$$

Using this,

check: R is K-projective.

(use:
surjective + w.e.
 $Z(M) \rightarrow Z(R)$ onto)

- $(P_\lambda)_\lambda$ family of K-projectives

Then, $\bigoplus P_\lambda$ is also K-projective.

Conversely closed under direct summands.

- K-projectives are closed under suspensions.

$$\dots \xrightarrow{\circ} P_{i+1} \xrightarrow{\circ} P_i \xrightarrow{\circ} P_{i-1} \xrightarrow{\circ} \dots$$

is K-proj, if P_i projective $\forall i$.

Ex: $0 \rightarrow P' \xrightarrow{f} P \xrightarrow{g} P'' \rightarrow 0$ exact seq.
of complexes.

Then, if P' and P'' are K-projective, so is P .

If P', P, P'' are complexes of projectives, then
any two being K-projective \Rightarrow third is K-proj.

any two being K-projective \Rightarrow third is K-proj.

Corollary. Any bounded complex of projectives is K-projective.

Proof. P. : $0 \rightarrow P_b \rightarrow \dots \rightarrow P_a \rightarrow 0$, each P_i proj.

Induce on $b-a$.

$b-a=0$ done earlier.

$$0 \rightarrow P_{\leq b-1} \rightarrow P \rightarrow \sum^b P_b \rightarrow 0. \quad \blacksquare$$

" "

$$0 \rightarrow P_{b-1} \rightarrow \dots \rightarrow P_a \rightarrow 0$$

Next: Any complex of projectives with $P_i = 0 \forall i > a$ is K-projective.

$$P. : \dots \rightarrow P_{a+1} \rightarrow P_a \rightarrow 0$$

$P = \underset{n \geq a}{\operatorname{colim}} P_{\leq n}$, each $P_{\leq n}$ is projective since bounded.

$$0 \rightarrow \bigoplus_n P_{\leq n} \xrightarrow{1-\delta} \bigoplus_n P_{\leq n} \rightarrow P \rightarrow 0.$$

\downarrow K-proj \downarrow projectives

Use 2-out-of-3.

Do directly...

$$\begin{array}{ccc} M & & \\ \downarrow \pi & \swarrow \text{v.e.} & \\ N & & \end{array}$$

$$P \text{ is K-projective} \Leftrightarrow \operatorname{Hom}_K(P, M) \cong \operatorname{Hom}_K(P, N).$$

Lecture 2 (11-01-2023)

11 January 2023 13:24

R ring.

$$D(R) \simeq k\text{Proj}(R)$$

Recall: $P \in \mathcal{C}(\text{Proj } R)$ is K -projective if

$$\begin{array}{ccc} & X & \\ P & \xrightarrow{\sim} & Y \\ & \downarrow \varepsilon & \end{array}$$

Example $P \in \mathcal{C}(\text{Proj } R)$ with $P_i = 0$ for all $i < 0$.

$$\dots \rightarrow P_{n-1} \rightarrow P_n \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

Sketch. Construct lifting one step at a time.

Suppose $\tilde{\alpha}: P_{\leq n} \rightarrow X$ is a lifting.

Want $\tilde{\alpha}: P_{\leq n+1} \rightarrow X$ compatibly.

Let $s \in P_{n+1}$

We must have

$$-\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$$

$$-\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$$

$$\begin{array}{ccc} & X & \\ \tilde{\alpha} & \nearrow & \downarrow \varepsilon \simeq \\ & \varepsilon & \end{array}$$

Check that the above can be solved.

This uses three things: ε surjective $\Rightarrow \varepsilon$ surjective on boundaries

, $H(E)$ iso $\Rightarrow \varepsilon$ surjective on cycles

, $\ker(\varepsilon)$ is acyclic.

\Rightarrow Every module has a K -projective resolution.

$$F: \dots \rightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M$$

Can avoid choices
by taking generating
set to be $M, \ker \varepsilon,$
 $\ker \partial_1, \dots$

$$F \xrightarrow{\varepsilon} M.$$

Defn. A K -projective resolution of $M \in \mathcal{C}(R)$ is a morphism

$$E: P \rightarrow M \text{ s.t.}$$

① E is a quasi iso,

(Not insisting
surjective.)

② P is K -projective.

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This makes it functorial!

Thm $\forall M \in \mathcal{C}(R)$, \exists surjective K -projective resolution:

$$P \xrightarrow{\sim} M$$

\downarrow
 $K\text{-proj.}$

Defn. An R -complex F is semi-free if F admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \dots \subseteq \bigcup_{n \geq 0} F(n) = F$$

s.t. ① $F(n) \subseteq F$ is a subcomplex

② $\frac{F(n+1)}{F(n)}$ graded free module with $\partial = 0$,
i.e. $\partial(F(n+1)) \subseteq F(n)$

Fact: semi-free $\Rightarrow K\text{-proj.}$

Example. $\dots \rightarrow F_{a+1} \rightarrow F_a \rightarrow 0$

$$F(n) = F_{\leq n} = \dots \rightarrow F_n \rightarrow \dots \rightarrow F_a \rightarrow 0$$

$$\frac{F(n+1)}{F(n)} = \dots \rightarrow F(n+1) \rightarrow 0 \rightarrow \dots$$

$$= \sum^{n+1} F_{n+1}.$$

Thm. Each $M \in \mathcal{C}(R)$ has a surjective semi-free resolution

$$F \xrightarrow{\sim} M$$

\uparrow
 $K\text{-projective}$

Corollary. Every K -projective is a retract of a semi-free.

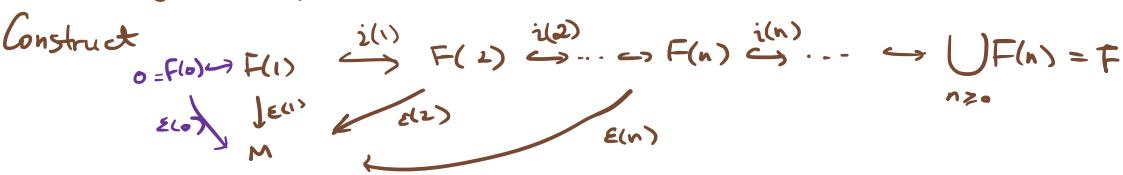
Proof. P is K -projective:

$$\begin{array}{ccc} & F & \\ P & \xrightarrow{\quad i \quad} & F \\ \dashrightarrow & & \downarrow z_1 \\ P & \xrightarrow{\quad id \quad} & P \end{array}$$

□

Sketch (Baby ver of Quillen's "small object argument".)

Construct $\dots \rightarrow F(0) \hookrightarrow F(1) \xrightarrow{i(1)} F(2) \hookrightarrow \dots \hookrightarrow F(n) \xrightarrow{i(n)} \dots \hookrightarrow \bigcup F(n) = F$



s.t. ① $F(n+1)/F(n)$ is graded free with $\partial = 0$,

② $\varepsilon(1)$ is surjective on homology.

(In turn, each $\varepsilon(n)$ is surjective on homology.)

③ $\ker(H(E(n))) \subseteq H(F(n))$ maps to 0 under $H(i(n))$.

This does the job. [Something 0 in column is 0 at finite stage.]

Why is ε surjective?

Remark. $\varepsilon: X \rightarrow Y$ s.t.

$Z(\varepsilon)$ surjective + $H(\varepsilon)$ bijective.

Then, ε is surjective.

Indeed, we have c.c.e.s:

$$\begin{array}{ccccccc} 0 & \rightarrow & B(x) & \rightarrow & Z(x) & \rightarrow & H(x) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & B(y) & \rightarrow & Z(y) & \rightarrow & H(y) \rightarrow 0 \end{array}$$

Snake lemma
 $B(x) \rightarrow B(y)$
is epi

$$\begin{array}{ccccccc} 0 & \rightarrow & Z(x) & \rightarrow & x & \rightarrow & \sum B(x) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & Z(y) & \rightarrow & y & \rightarrow & \sum B(y) \rightarrow 0 \end{array}$$

$x \rightarrow y$ is surj

Construction of $F(n)$, $\varepsilon(n)$:

$\varepsilon(1): F(1) \rightarrow M$ free cover of cycles.

$\underline{\text{o diff}}$ ($\varepsilon(1)$ morphism since mapping on cycles)

Say we have constructed $\varepsilon(n): F(n) \rightarrow M$.

Choose cycles $(z_n)_n \subseteq F(n)$ that map
to a generating set of $\ker(H(\varepsilon(n)))$

Pick w_x s.t. $\partial(w_x) = \varepsilon(n)(z_x)$.

Set $F(n+1) = F(n) \oplus R\mathbb{Q}_x$ $\deg(\varepsilon_{n+1}) = \deg(z_x) + 1$.

with $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_n) = z_n.$$

Define $\varepsilon^{(n+1)} : F^{(n+1)} \rightarrow M$
 $\varepsilon^{(n+1)}|_{f(n)} = \varepsilon^{(n)},$
 $\varepsilon^{(n+1)}(e_n) = \omega.$

□

- Remarks. ① As before, the above construction can be made functorial by avoiding choices (consider all choices!).
② Depending on what we wish to do with the resolution, there are other constructions.

Given a module, we have the graded homology module $H(M) = \langle H_i(M) \rangle_{i \in \mathbb{Z}}.$

(Recall: for us, a graded module is a collection of modules.)

If $P_\cdot \xrightarrow{\sim} H(M)$ is a projective resolution, one can "perturb" the differentials of P_\cdot to construct a K -projective resolution of M .

(Adam's resolution, Gorenstein/Gitlenberg resolution)

Exercise. P_\cdot K -proj $\Rightarrow P_i$ projective $\forall i$.

Converse of above NOT true.

Example (Dold's): Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the complex

$$P_\cdot : \dots \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} \dots$$

One way of seeing that the above is not K -projective is to do the following exercise and note that P_\cdot is acyclic but not contractible.

is to do the following exercise now:

P is acyclic but not contractible.

Exercise. If P is K -proj and $H(P) = 0$, then P is contractible, i.e., $\text{id}_P \sim 0$.
(or: P is the mapping cone of some idc.)

We saw we have an inclusion

$$K\text{Proj}(R) \hookrightarrow K(\text{Proj } R).$$

FACT. Let R be comm. Noetherian.

The above inclusion is an equality iff R is regular.

Examples of reg. rings: \mathbb{Z} , $k[x_1, \dots, x_n]$.

Derived functors

Let M_\cdot be an R -complex.

FACT. If $P_\cdot \xrightarrow{\sim} M_\cdot$ and $Q_\cdot \xrightarrow{\sim} M_\cdot$ are K -projective resolutions, then $P_\cdot \cong Q_\cdot$ in $K(\text{Proj } R)$.

Given any $N_\cdot \in \mathcal{C}(R)$, set

$$R\text{Hom}_R(M_\cdot, N_\cdot) := \text{Hom}_R(P_\cdot, N_\cdot),$$

where P_\cdot is a K -proj. resl" of M_\cdot .

The object on the right is defined in the homotopy category of abelian groups, i.e.,

$R\text{Hom}_R(-, N_\cdot)$ is a functor
 $\mathcal{C}(R) \rightarrow K(\mathbb{Z}).$

$R\text{Hom}_R(-, N)$ is a functor
 $\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$
 $(\exists R \text{ is comm, then } \rightarrow K(R).)$

Define $\text{Ext}_R^i(M, N) := H^i(R\text{Hom}_R(M, N))$
 $= H_{-i}(\text{Hom}_R(P, N))$

$\text{Ext}_R^0(M, N) = H_0(\text{Hom}_R(P, N))$
 $= \text{morphisms } P \rightarrow N, \text{ up to homotopy}$

$\text{Ext}_R^i(M, N) = - \rightarrowtail P \rightarrow \sum^i N, \rightarrowtail$

If $Q \xrightarrow{\sim} N$ is a K -proj "rel", then

$\text{Hom}_R(P, Q) \cong \text{Hom}_R(P, N).$
 $\uparrow \text{ quasi-iso}$

Tensors Let $M.$ be a chain complex of
right R -modules.

Let $N. \in \mathcal{C}(R).$

$M. \otimes N.$ is a complex of \mathbb{Z} -modules defined
 by

$$(M. \otimes N.)_i = \bigoplus_{j \in \mathbb{Z}} M_j \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-)^{|m|} m \otimes \partial n$$

FACT. If $X. \xrightarrow{\sim} Y.$ is a quasiiso,

then $P. \otimes_R X. \xrightarrow{\sim} P. \otimes Y.$ for any K -proj $P.$

Defn. $M \otimes_R^L N := P \otimes_R N$. where

$P \xrightarrow{\sim} M$ is a
 K -proj. resolⁿ.

$$\overline{\text{Tor}}_i^R(M, N) = H_i(P \otimes_R N).$$

$$X \xrightarrow{\sim} Y \text{ quasi iso} \Rightarrow \overline{\text{Tor}}_i^R(M, X) = \overline{\text{Tor}}_i^R(M, Y).$$

Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \rightarrow$ comm. Noetherian ring

$M \rightarrow R\text{-module}$

$r \in R$ is a zero divisor on M if $r \cdot m = 0$ for some $m \neq 0$.
 nzd = "not a zero divisor"

$$Z_R(M) = \{r \in R : r \text{ is a z.d. on } M\}$$

$$= \bigcup_{p \in \text{Ass}(M)} p.$$

(M need not be finite.
 Union need not be.)

Fix R, M . Let $\underline{x} = x_1, \dots, x_n$ be a sequence in R .

\underline{x} is weakly M -regular or a weakly regular sequence on M

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \dots, x_i)M} \text{ for } 0 \leq i \leq n-1.$$

\underline{x} is M -regular (or ...) if further $M/(x)M \neq 0$.

Ex. $R = k[x_1, \dots, x_n]$.
 $\underline{x} := x_1, \dots, x_n$ is a regular sequence on R .

Koszul Complexes. Given $r \in R$,

$$K(r; R) = 0 \rightarrow R \xrightarrow{r} R \rightarrow 0.$$

$\uparrow \deg 1 \quad \uparrow \deg 0$

$H_1(K(r; R)) = 0 \Leftrightarrow r \text{ is nzd on } R$.

Given $\underline{x} = x_1, \dots, x_n$, we define

$$K(\underline{x}; R) = \bigoplus_{i=1}^n K(x_i; R).$$

$$\begin{matrix} & \pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \\ x_n \end{bmatrix} \\ \text{K}(\underline{x}; R) & \sim \cdots \rightarrow 0 \rightarrow R \xrightarrow{\binom{x_1}{x_2}} \cdots \rightarrow R \xrightarrow{\binom{x_1}{x_2} \cdots \binom{x_1}{x_n}} R \rightarrow 0 \end{matrix}$$

$$K(\underline{x}; R) = 0 \rightarrow R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \\ x_n \end{bmatrix}} R'' \rightarrow \dots \rightarrow R^{\binom{n}{2}} \xrightarrow{\cdot(x_1 \dots x_n)} R^n \rightarrow R \rightarrow 0$$

$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mapsto \sum r_i x_i$

Now, given $M \in \mathcal{C}(R)$,

$$K(\underline{x}; M) := K(\underline{x}, R) \otimes M.$$

\hookrightarrow Koszul complex on \underline{x} with coefficients in M .

$$H_i(\underline{x}; M) := H_i(K(\underline{x}; M)). \rightarrow \text{Koszul homology}$$

If M is simply an R -module (viewed in degree 0),

then

$$K(\underline{x}; M) :$$

$$0 \rightarrow M \rightarrow M'' \rightarrow \dots \rightarrow M^n \rightarrow M \rightarrow 0$$

"same" differentials

$$H_0(\underline{x}; M) = M / \underline{x}M,$$

$$\begin{aligned} H_n(\underline{x}; M) &= \{m \in M : x_i m = 0 \ \forall i\} \\ &= (0 :_M (\underline{x})). \end{aligned}$$

$$\begin{aligned} ① \quad K(\underline{x}; M) &= K(x_1; R) \otimes_R K(x_2; R) \otimes_R \dots \otimes_R K(x_n; R) \otimes_R M \\ &= K(x_1; R) \otimes K(x_{\geq 2}, M) \\ &= K(x_1; K(x_{\geq 2}, M)) \end{aligned}$$

$$② \quad x, y \in \mathcal{C}(R) \rightsquigarrow x \otimes_R y \xrightarrow{\sim} y \otimes_R x \text{ as } R\text{-complexes.}$$

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

$$\therefore K(\underline{x}; R) \cong K(\underline{x}^\sigma; R) \quad \text{for any } \sigma \in S_n.$$

$$\Rightarrow K(\underline{x}; M) \cong K(\underline{x}^\sigma; M) \quad \text{---} \quad \text{H} \quad \text{---}$$

(Can apply this to Obs ①.)

2nd Perspective: "Koszul complexes are iterative"
mapping cones.

$f: X \rightarrow Y$ morphism of complexes

$$\text{cone}(f) = (Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & -\partial^X \end{pmatrix}).$$

$$\text{s.e.s. : } 0 \longrightarrow Y \longrightarrow \text{cone}(f) \longrightarrow \Sigma X \longrightarrow 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$

↓

$\begin{pmatrix} y \\ x \end{pmatrix} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow{H_i(f)} H_i(Y) \longrightarrow H_i(\text{cone}(f)) \rightarrow H_i(\Sigma X) \rightarrow \dots$$

\downarrow

$H_{i-1}(X)$

connecting map

Consider: $x \in R$

$$f: R \xrightarrow{x} R$$

$1 \mapsto x$

$$\text{cone}(R \xrightarrow{x} R) = (R \oplus R, \begin{pmatrix} \circ & x \\ 0 & \circ \end{pmatrix})$$

$\overset{\text{deg } 0}{\uparrow} \quad \overset{\text{deg } 1}{\uparrow}$

$= K(x; R).$

Ditto: If $x \in R$ and $M \in \mathcal{C}(R)$ *no complex, not necessarily in deg 0*

$$\text{cone}(M \xrightarrow{x} M) = K(x; M)$$

$$\underline{x} = x_1, x_2, \dots, x_n$$

$$K(\underline{x}; M) = K(x_1; K(x_{\geq 2}; M)) \\ = \text{cone}\left(K(x_{\geq n}; M) \xrightarrow{x_1} K(x_{\geq 2}; M)\right)$$

on homology

iterate
:

$$H_i(x_{\geq 2}; M) \xrightarrow{\pm x_1} H_i(x_{\geq 2}; M) \rightarrow H_i(\underline{x}; M) \\ \downarrow \\ H_{i-1}(x_{\geq 2}; M) \\ \downarrow \pm x_1 \\ \vdots$$

↓ s.e.s.

$$0 \rightarrow \frac{H_i(x_{\geq 2}; M)}{x_1 H_i(x_{\geq 2}; M)} \rightarrow H_i(\underline{x}; M) \rightarrow (0 : \frac{x_1}{H_{i-1}(x_{\geq 2}; M)}) \rightarrow 0$$

$M \rightarrow R\text{-module}$

$$K(\underline{x}; M) \rightsquigarrow H_0(\underline{x}; M) = \frac{M}{\underline{x}M}$$

$$\text{So, } K(\underline{x}; M) \rightarrow \frac{M}{\underline{x}M} \text{ is a w.e. quasi iso} \\ \Leftrightarrow H_i(\underline{x}; M) = 0 \quad \forall i > 0.$$

Defn. \underline{x} is Koszi-regular on M (or...) if

$$H_i(\underline{x}; M) = 0 \quad \forall i \geq 1.$$

Lemma. When \underline{x} is weakly M -eq, $(M \rightarrow R\text{-mod})$

(weakly-reg) $K(\underline{x}; M) \rightarrow M / (\underline{x}M)$ is a w.eq.
 \Rightarrow Koszi-reg

Proof. $n=1$: $0 \rightarrow M \xrightarrow{\underline{x}} M \rightarrow 0$.

$H_1(\underline{x}; M) = 0 \Leftrightarrow \underline{x}$ nad on M

$n \geq 2$: $K(\underline{x}; M) = K(x_n; K(x_{\leq n}; M))$.

By induction,

$$K(x_{\leq n}; M) \xrightarrow{\sim} \frac{M}{(x_{\leq n})M}.$$

Now, $0 \rightarrow R \xrightarrow{x_n} R \rightarrow 0$
is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x}; M) = K(x_n; R) \otimes_R K(x_{\leq n}; M)$$

$$\cong K(x_n; R) \otimes_R \frac{M}{(x_{\leq n})M}$$

Now note that x_n is a nzd
on $\frac{M}{(x_{\leq n})M}$ and we
are done. \square

Instead of semifree,
can use l.e.c. of homology
and induction.

$$\begin{aligned} &\text{Semifree Lemma} \\ &\Rightarrow \left(\begin{array}{l} M \cong N \text{ quasi} \\ \Downarrow \\ K(\underline{x}; M) \cong K(\underline{x}; N) \text{ quasi} \end{array} \right) \end{aligned}$$

Note: ① \underline{x} Koszti-reg $\Rightarrow \underline{x}^\sigma$ is Koszti-reg $\forall \sigma \in S_n$.

② Not true for weakly regular. \rightarrow

Theorem. Say $\underline{x} \subseteq J(R)$ and M f.g. R -module.

TFAE:

- 1) \underline{x} is M -regular. (\equiv weakly M -reg. by NAK.)
- 2) \underline{x} is Koszti M -regular, i.e., $H_i(\underline{x}; M) = 0 \ \forall i \geq 1$.
- 3) $H_1(\underline{x}; M) = 0$.

In particular, take R local and x_i nonunits.

Proof. ① \Rightarrow ② \Rightarrow ③ is clear.

Only need to prove ③ \Rightarrow ①.

Already saw for $n=1$.

Induction:

$$K(\underline{x}; M) = K(x_n; K(x_{\leq n}; M)).$$

I.e.s.

$$0 \rightarrow \frac{H_i(x_{\leq n}; M)}{x_n H_i(x_{\leq n}; M)} \rightarrow H_i(\underline{x}; M) \rightarrow (0 : \frac{x_n}{H_{i-1}(x_{\leq n}; M)}) \rightarrow 0. \quad (*)$$

Put $i=1$ to get $\frac{H_1(x_{\leq n}; M)}{x_n H_1(x_{\leq n}; M)} = 0$

$$\xrightarrow{\text{NAK}} H_1(x_{\leq n}; M) = 0.$$

(Note: K₁-homology modules are f.g.
when M is f.g.)

$\xrightarrow{\text{induction}}$ x_1, \dots, x_{n-1} is M -reg. — (1)

Moreover, (*) now tells us

$$(0 : \frac{x_n}{H_0(x_{\leq n}; M)}) = 0.$$

$$\text{ku} \left(\frac{M}{x_{\leq n} M} \xrightarrow{x_n} \frac{M}{x_{\leq n} M} \right).$$

$\therefore x_n$ is nzd on $\frac{M}{(x_{\leq n})M}$. — (2)

① & ② finish. \square

Corollary. $\mathfrak{g} \subseteq J(R)$, M f.g., the property of \underline{x} being regular is not dependent on the order of x_i .

(Permutation of regular is regular.)

$$R = k[x, y, z]$$

$x, y(1-n), z(1-n)$ reg |
 $y(1-n), z(1-n), x$ NOT |

Lemma. If $\underline{x} = x_1, \dots, x_n \subseteq R$, M an R -module.

TFAE:

① \underline{x} is M-Koszul-regular.

② \underline{x}^a is M-Koszul-regular for some $a \geq 1$.

Proof. Suffices to prove:

$x_1, \boxed{x_2, \dots, x_n}$ is M-KR

$\Leftrightarrow x_1^a, \boxed{x_2, \dots, x_n}$ is M-KR for some $a \geq 1$.
(all)

x_1, x_2, \dots, x_n KR

$\Rightarrow K(x_1; K(x_{\geq 2}; M)) \xrightarrow{\sim} K\left(x_1; \frac{M}{(x_{\geq 2})M}\right)$.

Replacing M by $M/\underline{(x_{\geq 2})M}$ we are reduced
to $n=1$.

But

x is reg on M

$\Leftrightarrow x$ is nzd on M

$\Leftrightarrow x^a$ is nzd on M for some $a \geq 1$

$\Leftrightarrow x^a$ is reg on M — n —.

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$ and M f.g. R -module.

Let $i \geq 0$ be s.t. $H_i(\underline{x}; M) = 0$.

Then, $H_j(\underline{x}; M) = 0 \quad \forall j \geq i$.

HW.

Lecture 4 (23-01-2023)

Monday, January 23, 2023 1:19 PM

$R \rightarrow$ commutative Noetherian

Given complexes, $M, N \in \mathcal{C}(R)$.

$$R\text{Hom}_R(M, N) := \text{Hom}_R(pM, N)$$

$pM \xrightarrow{\sim} M$ is a K-proj res^r

$$\text{Ext}_R^+(M, N) := H^i(R\text{Hom}_R(pM, N)).$$

Given $M, N, P \in \mathcal{C}(R)$, we have a morphism

$$\Theta : R\text{Hom}_R(M, N) \otimes_R^L P \longrightarrow R\text{Hom}_R(M, N \otimes_R^L P).$$

Lemma. Θ is a w.e. when P is perfect, i.e,

$$P \simeq (0 \rightarrow P_1 \rightarrow \dots \rightarrow P_n \rightarrow 0),$$

each P_i f.g. R -module.

$$\begin{aligned} & \text{Hom}_R(pM, N) \otimes_R pP \longrightarrow \text{Hom}_R(pM, N \otimes_R pP) \\ & f \otimes x \longmapsto m \mapsto (-)^{\binom{lm}{m}} f(m) \otimes x. \end{aligned} \quad \left. \begin{array}{l} \text{in} \\ \mathcal{C}(R) \end{array} \right\}$$

check this is a morphism.

Proof. Replace P with $0 \rightarrow P_1 \rightarrow \dots \rightarrow P_n \rightarrow 0$.

"Things true for R free" Then it reduces to checking for one f.g. projective P .

"for perfect." But that can be reduced to f.g. free.

That reduces to R. Obstruction \Rightarrow

Rees' Lemma:

Setup. $\underline{x} = x_1, \dots, x_c$ " $\underline{x} \subseteq \mathbb{R}$ finite subset.

$N \rightsquigarrow R\text{-module}$ s.t. $\exists N = 0$.

$M \rightarrow R$ -module s.t. x is K

$M \rightarrow R\text{-module}$ s.t. \exists is Koszi-regular on M .

$$G^{\text{re}} \times (\underline{x}; N) \xrightarrow{\sim} M_{\mathcal{M}_M}$$

Lemma: Then, $R\text{Hom}_R(N, M/\varphi M) \xrightarrow{\sim} R\text{Hom}_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

In particular, $\text{Ext}_R^*(N, \frac{M}{\varphi M}) \cong \text{Ext}_R^*(N, M) \otimes_R 1^*(\varSigma R^c)$.

$$\Lambda^*(\Sigma R^c) : 0 \rightarrow R \xrightarrow{\circ} R^c \xrightarrow{\circ} \dots \xrightarrow{\circ} R \xrightarrow{\binom{c}{2}} R^c \xrightarrow{\circ} R \xrightarrow{\circ} 0$$

$\hookdownarrow \text{def}_c \quad \quad \quad \circlearrowleft$

$$\begin{aligned}
 \text{Proof. } R\text{Hom}_R(N, M/\underline{\gamma}M) &\simeq R\text{Hom}_R(N, K(\underline{x}; M)) \\
 &\simeq R\text{Hom}_R(N, M \overset{L}{\otimes}_R K(\underline{x}; R)) \\
 &\simeq R\text{Hom}_R(N, M) \overset{L}{\otimes}_R K(\underline{x}; R) \quad \text{perfect}
 \end{aligned}$$

$$\text{Since } \underline{x}^N = 0, \quad \underline{x}^T \underline{x}^N_{B^*}(n, m) = 0.$$

$$\text{Altér: } R\text{Hom}_R(N, M) \cong \text{Hom}_R(N, I) \quad \left. \begin{array}{l} \text{where } M \hookrightarrow I \text{ is} \\ \text{an injective reln.} \end{array} \right\}$$

$$\text{Now, } \underline{x} \cdot \text{Home}_k(N, I) = 0.$$

$$\cong \text{Hom}_R(N, I) \otimes_R K(\underline{x}; R)$$

$$\cong K(x; \text{Hom}_R(n, \mathbb{Z}))$$

$$\cong K\left(\underline{0}; \text{Hom}_R(N, I)\right)$$

$\xrightarrow{\text{length } c}$

$$\cong \text{Hom}_R(N, I) \otimes_R K(Q; R)$$

$$\begin{aligned} &\supset \text{Hom}_R(N, I) \otimes_R K(\Omega; R) \\ &= \text{Hom}_R(N, I) \otimes_R \Lambda^*(\Sigma R^c). \end{aligned}$$

Last statement means:

$$\begin{aligned} \text{Ext}_R^n\left(N, \frac{M}{\underline{\chi} M}\right) &\cong \left(\text{Ext}_R(N, M) \otimes_R \Lambda(\Sigma R^c)\right)^n \\ &= \bigoplus_i \text{Ext}_R^i(N, M) \otimes_R \left(N(\Sigma R^c)\right)^{n-i} \\ &= \bigoplus_i \text{Ext}_R^i(N, M) \otimes_R R^{\binom{c}{i-n}}. \end{aligned}$$

$V \rightarrow \mathbb{Z}$ -graded object

View V as having both an upper grading and lower grading via $V^i = V_{-i}$.

Notation:

$$\begin{aligned} \sup \text{V}^* &= \sup \{i : V^i \neq 0\}, \\ \inf \text{V}^* &= \inf \{i : V^i \neq 0\}. \\ \sup \text{V}_* &= \sup \{i : V_i \neq 0\} \dots \\ &= -\inf \text{V}^* \end{aligned}$$

Corollary. $\rho := \inf \text{Ext}_R^*(N, M) = \inf \text{Ext}_R^*(N, M/\underline{\chi} M) + c.$

$$\text{Ext}_R^\rho(N, M) = \text{Ext}_R^{\rho-c}(N, M/\underline{\chi} M)$$

Defn. Fix $I \subseteq R$ ideal. $M \in \mathcal{C}(R)$.

$$\text{depth}_R(I, M) := \inf \text{Ext}^*(R/I, M).$$

I -depth of M \uparrow

Properties. ① $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ exact sequence
of complexes on . . .

Properties. ① $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ exact sequences
of complexes or modules).

Then, $\text{depth}_R(I, M) \geq \min \{\text{depth}_R(I, L), \text{depth}_R(I, N)\}$.

Similarly relations with other two using I.e.s.

$$\rightarrow \text{Ext}_R^i(R/I, L) \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \text{Ext}_R^i(R/I, N)$$

$$\rightarrow \text{Ext}_R^{i+1}(R/I, L) \rightarrow \dots$$

Thm (depth).

② Let $\underline{x} = x_1, \dots, x_c$ be a gen. set for the ideal I .

$$\text{depth}_R(I, M) = c - \sup H_{\infty}(\underline{x}; M) \quad \forall M \in \mathcal{C}(R).$$

Will prove when M is an R -module. (recall Koszul homology)

Koszul complexes revisited

Giving x_1, \dots, x_c in R

\leftrightarrow Giving a map $f: F \rightarrow \mathbb{C}$, where F is a free mod of rank c and chosen basis

$K(f) := (\Lambda^*(\mathbb{C}^F), \partial)$, where ∂ is

$$e_{i_1} \wedge \dots \wedge e_{i_m} \mapsto \sum_{j=1}^n (-1)^{j-1} f(e_{ij}) x_{i_1} \dots x_{i_{j-1}} \hat{x}_{ij} x_{i_{j+1}} \dots x_{i_n}$$

Lemma: Fix $\underline{x} = x_1, \dots, x_c \subseteq R$ for any $y \in (x)$, we have

$$\begin{aligned} K(\underbrace{\underline{x}, y}_{c+1 \text{ Seq.}} ; M) &\cong K(\underline{x}, 0 ; M) \\ &\cong K(\underline{x} ; M) \otimes (0 \rightarrow R \rightarrow R \rightarrow 0). \end{aligned}$$

Proof.

$$\begin{array}{ccccc} R^{c+1} & \xrightarrow{[x_1 \dots x_c \ y]} & R & & \uparrow \text{deg } 0 \\ \downarrow \cong & \xrightarrow{2} & \parallel & & \\ \left[\begin{array}{c|c} \vdots & \vdots \\ \hline r_1 & x_1 \\ \vdots & \vdots \\ r_c & x_c \end{array} \right] & & & & y = \sum r_i x_i \end{array}$$

$$\begin{array}{c} \left[\begin{array}{c|ccccc} & x_1 & & & & \\ \hline & & x_2 & & & \\ & & & x_3 & & \\ & & & & \ddots & \\ & & & & & x_n \end{array} \right] \xrightarrow[R^{\text{can}}]{\cong} \begin{array}{c} 2 \\ \xrightarrow{\quad [x_1 \cdots x_n] \quad} \\ R \end{array} \end{array} \quad \text{if } y = \sum r_i x_i$$

In particular,

$$\sup H_{*}(\underline{x}, y; M) = 1 + \sup H_{*}(\underline{x}; M)$$

Thus,

$$c + 1 - \sup H_{*}(\underline{x}, y; M) = c - \sup H_{*}(\underline{x}; M).$$

Corollary. $c - \sup H_{*}(\underline{x}, y; M)$ is independent of a

generating set of I . (If $\underline{x}, \underline{y}$ generate, look at $\underline{x}, \underline{y}$ & $\underline{y}, \underline{x}$.)

Proof of Thm (Depth). When M is a module

$$\begin{aligned} \text{depth}(I, M) = 0 &\Leftrightarrow \text{Hom}(R/I, M) \neq 0 \\ &\Leftrightarrow I \subseteq \text{Zdu}_R(M) \\ &\Leftrightarrow H_c(\underline{x}; M) \neq 0 \\ &\Leftrightarrow \sup H_{*}(\underline{x}; M) = c. \end{aligned} \quad \left. \begin{array}{l} \text{Actually,} \\ \left\{ \begin{array}{l} H_c(\underline{x}; M) \\ \text{Hom}(R/I, M) \end{array} \right. \end{array} \right\} \forall i$$

Can assume $\text{depth}_R(I, M) \geq 1$, i.e., $\exists y \in I$ _{nzd on M}

Then, in particular, y is Kosei-reg on M .

$$\text{By Reg, } \because \text{depth}_R(R/I) = 0 \quad \inf \text{Ext}_R^{*}(R/I, M/yM) = \inf \text{Ext}_R^{*}(R/I, M) - 1.$$

$$\therefore \text{depth}_R(I, M) = 1 + \text{depth}_R(I, \frac{M}{yM})$$

induction now applies

$$= 1 + \left(c - \sup H_{*}(\underline{x}; \frac{M}{yM}) \right)$$

$$\sup H_{*}(\underline{x}; M/yM)$$

$$\begin{aligned}
 & H^*(\underline{x}; M/y_M) \\
 \left\{ \begin{aligned} & \vdash H^*(\underline{x}; K(y; M)) \\ & \vdash H^*(K(\underline{x}, y; M)) \\ & \vdash H^*(K(\underline{x}, o; M)) \end{aligned} \right. \\
 & = C + 1 - \sup H^*(\underline{x}, o; M) \\
 & = C - \sup H^*(\underline{x}; M). \quad \square
 \end{aligned}$$

Ex. Show $\text{depth}_R(I, M)$ is the length of longest irregular in I .

Lecture 5 (25-01-2023)

Wednesday, January 25, 2023 1:19 PM

$R \rightarrow$ comm. noetherian ring

$I \subseteq R$ ideal

$M \rightarrow R\text{-complex}$

$$\text{depth}(I, M) = \inf \text{Ext}_R^*(R/I, M) \quad \xrightarrow{\quad I = (x_1, \dots, x_c) \quad}$$

$$= c - \sup H_*(\underline{x}; M)$$

When M is a module, $\text{depth}(I, M) =$ length of any maximal
 M -Koszul-regular sequence in I

(Maybe $IM \neq M$ needed.)

$\left\{ \begin{array}{l} \text{If } M \text{ f.g. and} \\ I \subseteq \text{Jac}(R) \end{array} \right.$

= length of any maximal
 M -reg. seq in I

Observation. $\underline{x} = x_1, \dots, x_c$
 $\underline{y} = y_1, \dots, y_d$

$$\sup H_*(\underline{x}, \underline{y}; M) \leq \sup H_*(\underline{x}; M) + d.$$

(Can use I.e.s. to see this.)

$$k(\underline{x}, \underline{y}; M) \cong k(\underline{x}; k(\underline{y}; M))$$

This implies

$$I \subseteq J \Rightarrow \text{depth}_R(I, M) \leq \text{depth}_R(J, M).$$

$$\text{Also, } \sqrt{I} = \sqrt{J} \Rightarrow \text{depth}_R(I, M) = \text{depth}_R(J, M).$$

Today. (R, \mathfrak{m}, k) is a local ring:

- R is commutative Noetherian,
- \mathfrak{m} is the unique maximal ideal of R ,

$$- k = R/\mathfrak{m}.$$

In this case, $\text{depth}(M) = \text{m-depth of } M$
 $= \inf_R \text{Ext}_R^*(k, M).$

It suffices to compute the above using $\underline{x} = x_1, \dots, x_c$
 s.t. $\sqrt{(\underline{x})} = \mathfrak{m}.$

Thus we can take c minimal as $c = \dim(R)$
 (Then, \underline{x} is a system of parameters.)

\therefore Can compute using $\dim(R)$ elements.

Ausland-Buchsbaum Equality

- $F \rightarrow$ an R -complex

F has finite flat dimension if

$$F \cong (0 \rightarrow F_0 \rightarrow \dots \rightarrow F_a \rightarrow 0)$$

with F_i flat.

We write $\text{flat dim}_R F < \infty$.

Examples • Flat modules.

- Perfect complexes.
- Koszul complexes.

If $\text{flat dim}_R F < \infty$, then $\text{Tor}_i(-, F) = 0 \quad \forall |i| > 0$
 on $\text{Mod } R$.

$$\begin{aligned}\text{Tor}_i^R(M, F) &= H_i(M \otimes_R (0 \rightarrow F_b \rightarrow \dots \rightarrow F_a \rightarrow 0)) \\ &= 0 \quad \text{for } i \notin [a, b].\end{aligned}$$

In fact, the above characterizes flat dim_R F < ∞.

Theorem (AB equality) (R, m, k) local.

F → finite flat dimension. Then,

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R M - \underbrace{\sup H_k(k \otimes_R^L F)}_{\text{Tor}_k^R(k, F)},$$

for A \cong N R-complex M.

Specialise: ① M = R.

$$\text{depth}_R(F) = \text{depth}_R(R) - \sup H_k(k \otimes_R^L F).$$

Now, let N be a f.g. R-module.

Such an N has a minimal free resolution.

$$\begin{array}{ccccccc} \cdots & R^{b_2} & \xrightarrow{\partial_2} & R^{b_1} & \xrightarrow{\partial_1} & R^{b_0} & \xrightarrow{\varepsilon \text{ minimally}} N \\ & \downarrow & \nearrow & \downarrow \min & \nearrow & \downarrow & \\ & \ker \partial_2 & \subseteq \eta_{Rb_2} & \ker \varepsilon & \subseteq \eta_{R^{b_0}} & \text{by minimality} & \end{array}$$

This gives a complex

$$G: (\dots \rightarrow R^{b_2} \xrightarrow{\partial_2} R^{b_1} \xrightarrow{\partial_1} R^{b_0} \rightarrow 0) \xrightarrow{\sim} N$$

$\partial G \subseteq \eta G$. G turns out to be unique up to isomorphism of complexes.

up to isomorphism of complexes.

"The" minimal free resolution of N .

$$\text{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes G)_i.$$

$H_i = \text{everything}$ since $\partial \otimes k = 0$

$$\therefore \text{Tor}_i^R(k, N) = 0 \iff G_i = 0.$$

$\therefore \text{flat dim}_R N < \infty \iff N \text{ has a finite free resolution, i.e., } N \text{ is perfect.}$

$$\sup \text{Tor}_*^R(k, N) = \text{length of } G$$

$$=: \text{proj dim}_R N.$$

② If N is a f.g. R -module with $\text{proj dim}_R N < \infty$,

then

$$\text{proj dim}_R(N) + \text{depth}_R(N) = \text{depth}(R).$$

(Classical AB Equality.)

Corollary. $\text{proj dim}_R N < \infty \Rightarrow \text{depth}(N) \leq \text{depth}(R)$.

Furthermore, equality if
False without assumption of $\text{proj dim} < \infty$
 N is projective.

Note we had:

$$\text{depth}_R(F) = \text{depth}_R(R) - \sup H_*(k \otimes^L F)$$

$$\text{depth}_R(M \otimes^L F) = \text{depth}_R(M) - \sup H_*(k \otimes^L F)$$

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

Subtract:

$$\text{depth}_R(F) - \text{depth}_R(M \otimes_R^L F) = \text{depth}(R) - \text{depth}(M).$$

Note: Some terms above may be ∞ .

When $H_i(\Sigma; M) = 0 \forall i$, then

$$\sup H_*(\Sigma; M) = -\infty.$$

Proof of AB Equality.

$$R\text{Hom}_R(k, M \otimes_R^L F) \xleftarrow{\cong} \underbrace{R\text{Hom}_R(k, M)}_{\text{quasi iso because } k \text{ is a f.g. module}} \otimes_R^L F$$

$\therefore \exists G \rightarrowtail k$ where each G_i is finite free
And flat dim $F < \infty$.

More generally:

$$\text{Hom}_R(N, M) \otimes_R \cong \text{Hom}_R(N, M \otimes_R^L F)$$

N f.g. R -mod, F flat

Key observation:

$R\text{Hom}_R(k, M) \cong$ complexes of k -vector spaces
(take injective resolution of M)

$$R\text{Hom}_R(k, M) \otimes_R^L F \cong R\text{Hom}_R(k, M) \otimes_R^L (k \otimes_R^L F)$$

$$\therefore \text{Ext}_R^*(k, M \otimes_R^L F) = \text{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^L F). \quad \square$$

Observation: Let M be an R -complex.

Suppose $s := \sup H_*(M)$ is finite.

Then, $\text{depth}_R M \geq -s$

Then, $\operatorname{depth}_R M \geq -s$

with equality iff $\operatorname{depth}_R H_s(M) = 0$.

Note. For an R -module M , $\operatorname{depth} M = 0 \Leftrightarrow \inf \operatorname{Ext}(k, M) = 0$

$\Leftrightarrow \operatorname{Hom}(k, M) \neq 0$

$\Leftrightarrow k \hookrightarrow M$

$\Leftrightarrow \eta \in \operatorname{Ass}_R M$.

→ One proof: M as above.

$$\operatorname{Ext}_R^{-s}(N, M) = \operatorname{Hom}_R(N, H_s(M))$$

N any R -module.

- key. $M \cong M'$ with $M'_i = 0 \forall i > s$.

$$\dots \rightarrow M_{s+1} \xrightarrow{\partial} M_s \rightarrow M_{s-1} \rightarrow \dots = M.$$

$$\downarrow \circ \quad \downarrow \quad \quad \quad \parallel \quad \quad \quad \parallel \quad \quad \quad \downarrow \circ$$

$$\dots \rightarrow 0 \rightarrow \frac{M_s}{\partial(M_{s+1})} \rightarrow M_{s-1} \rightarrow \dots = M.$$

\therefore Can assume $M_i = 0$ for $i \geq s+1$.

In particular,

$$0 \rightarrow \sum^s H_s(M) \hookrightarrow M \rightarrow M'' \rightarrow 0.$$

\uparrow
iso on homology
in degrees $\leq s-1$

$$H_i(M'') = 0 \quad i \geq s.$$

Let $\Sigma = x_1, \dots, x_n$ gen set for η .

Then,

Then,

$$H_{i+1}(\underline{x}; M'') \rightarrow H_i(\underline{x}, \sum^s H_s(M)) \rightarrow H_i(\underline{x}; M) \rightarrow H_i(\underline{x}; M'')$$

— (*)

$$H_j(M'') = 0 \quad \forall j \geq s.$$

$$\text{So, } M'' \cong M''' \quad \text{with} \quad M_j''' = 0 \quad \text{for } j \geq s.$$

$$K(\underline{x}; M''')_j = 0 \quad \text{for } j \geq s+n+1.$$

$$\text{Thus, } H_j(\underline{x}; M''') = 0 \quad \forall j \geq s+n+1.$$

$$\Rightarrow H_i(\underline{x}; \sum^s H_s(M)) = 0 \quad \forall i \geq s+n+1.$$

Put $i \geq n+s$ in (*) :

$$H_j(\underline{x}; M) = 0 \quad \forall j \geq n+s+1$$

$$\Rightarrow \sup H_k(\underline{x}; M) \leq n+s$$

$$\Rightarrow -s \leq n - \sup H_k(\underline{x}; M) = \text{depth } M. \quad \square$$

Moreover,

$$H_{n+s}(\underline{x}; M) \cong H_{n+s}(\underline{x}; \sum^s H_s(M))$$

$$\cong H_n(\underline{x}, H_s(M))$$

The above is nonzero iff $\text{depth } H_s(M) = 0$. \square

① flat $\dim_R F < \infty$. Then $\forall M$

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_k(k \otimes_R^L F).$$

② $s = \sup H_k(M)$ is finite.

Then, $\text{depth}_R(M) \geq -s$.

Equality $\Leftrightarrow \text{depth}(H_s(M)) = 0$.

Application. Say

$F = 0 \rightarrow F_d \rightarrow \dots \rightarrow F_0 \rightarrow 0$
is a finite free complex s.t.
 $\hookrightarrow \text{minimal}$
 $\partial F \subseteq \eta F$

$$0 < \text{length}_R H_*(F) < \infty.$$

(all $H_i(F)$ are finite length and
at least one is non-zero.)

Then, for any M ,

$$\text{depth}_R M = d - \sup H_*(F \otimes_R M).$$

Thus such an F is depth sensitive.

When $\eta M \neq M$, one can check some $H_*(F \otimes_R M) \neq 0$.

Then, one gets

$$d \geq \text{depth}_R(M).$$

Note: Over any local ring, $\exists M$ s.t. $\eta M \neq M$ and
 M need not be f.g. $\text{depth}_R M = \dim R$.

In particular, $d \geq \dim(R)$. \rightarrow New Intersection

Theorem

Hochster ('70)
André (2016)
Bhatt (2021)

Proof. Assume $s = \sup H_*(F \otimes_R M)$.

Take any prime $p \neq \eta$.

$$H_i(F \otimes_R M)_p \cong H_i(F_p \otimes_{R_p} M_p)$$

$$H_i(F_p) = 0 \quad (\because \text{length } H_i(F) < \infty)$$

i.e. $F_p = 0$ in $D(R_p)$.

$$\therefore H_i(F_p \otimes_{R_p} M_p) = 0$$

Thus, $H_i(F \otimes_R M)$ is η -power torsion.

(I.e., each $a \in H_i(F \otimes_R M)$ is killed by some η^n .)

$$\therefore \text{depth } H_s(F \otimes_R M) = 0.$$

Thus, by previous result,

$$\text{depth}_R(F \otimes_R M) = -s$$

|| AB

$$\text{depth}(M) = \sup H(k \otimes_R^L F)$$

$$\Rightarrow \text{depth}(M) = \sup \underset{d}{\underbrace{H}}_k(k \otimes_R^L F) - s. \quad \text{D}$$

Lecture 6 (30-01-2023)

Monday, January 30, 2023 1:26 PM

Recap. $I \subseteq R$ Comm. Noe
 M an R -complex.

$$\text{depth}_R(I, M) = \inf \text{Ext}_R^*(R/I, M).$$

. Choose any fin gen set $\underline{x} = x_1, \dots, x_c$ of I .

$$\text{depth}_{\underline{x}}(I, M) = c - \sup H_{\underline{x}}(\underline{x}; M).$$

(Focus on the case $H_{\underline{x}}(M)$ bounded.)

$(R, M_R, k) \rightarrow \text{local}$

$$\text{depth}_R M := \text{depth}_R(M_R, M).$$

$$\text{depth}_R(M) \geq -\sup H_{\underline{x}}(M). \quad \text{--- (1)}$$

$$\text{Equality} \Leftrightarrow m \in \text{Ass}(H_s(M)) \\ s = \sup H_{\underline{x}}(M).$$

Exercise 1.

Suppose $M = 0 \rightarrow M_b \rightarrow \dots \rightarrow M_a \rightarrow 0$

$$\text{depth}_R M \geq \inf \{ \text{depth}(M_i) - i : a \leq i \leq b \}.$$

$$\text{depth}_R M \geq \inf \{ \text{depth } H_i(M) - i : \inf H_s(M) \leq i \leq \sup H_{\underline{x}}(M) \}. \quad \text{--- (2)}$$

(Note (2) \Rightarrow (1).)

Setup. $H_{\underline{x}}(M)$ bounded. $\underline{x} = x_1, \dots, x_c$. $(R \text{ not necessarily local.})$

$$\sup H_{\underline{x}}(M) \stackrel{(i)}{\leq} \sup H_{\underline{x}}(\underline{x}; M) \stackrel{(ii)}{\leq} \sup H_{\underline{x}}(M) + c$$

Lemma 2. (a) Inequality (ii) always holds.

$$\text{Equality iff } \text{depth}_R(\underline{x}; H_s(M)) = 0 \quad s = \sup H_{\underline{x}}(M)$$

(b) (i) holds if $\underline{x} \subseteq \mathfrak{J}(R)$, each $H_i(M)$ is f.g.

(b) (i) holds if $\underline{x} \subseteq \mathfrak{z}(R)$, each $H_i(M)$ is f.g.
 Equality holds iff \underline{x} is $H_S(M)$ -regular.

Proof (a) $H(\underline{x}; M) = H(x_1; k(x_2, \dots, x_c; M))$.

Reduce to $c=1$. $x := x_1$.

In this case, we have

$$H_{i+1}(M) \xrightarrow{\cong} H_i(M) \rightarrow H_{i+1}(x; M) \rightarrow H_i(M) \xrightarrow{\cong} H_i(M)$$

for $i \geq 3$, we get $H_{i+1}(x; M) = 0$.
 \therefore (ii) follows.

Moreover, $H_{i+1}(x; M) \neq 0 \Leftrightarrow x$ is a zd on $H_i(M)$.

$$(b) H_i(M) \neq 0 \stackrel{\text{NAK}}{\Rightarrow} H_i(x; M) \neq 0 \\ \stackrel{\cong}{\Rightarrow} \frac{H_i(M)}{xH_i(M)}$$

$$\therefore \sup H_x(x; M) \geq \sup H_x(M).$$

Moreover,

$$0 \rightarrow H_{i+1}(x; M) \rightarrow H_S(M) \xrightarrow{\cong} H_S(M). \quad \text{B}$$

Corollary 3. $\operatorname{depth}_R(\underline{x}; M) \geq -\sup H_x(M)$.

Equality $\Leftrightarrow \operatorname{depth}(\underline{x}, H_S(M)) = 0$
 $\Leftrightarrow (\underline{x}) \subseteq \mathfrak{p} \in \operatorname{Ass} H_S(M)$.

Propⁿ 4. (R, \mathfrak{m}, k) local.

Let M be any bounded complex.

Then, for any $I \subset R$,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

In particular, if M is a f.g. module

$$\operatorname{depth}_R(M) \leq \inf \left\{ \dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M \right\} \\ \leq \dim_R(M). \quad \hookrightarrow \sup \left\{ \dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M \right\}$$

Proof. Say $I = (y_1, \dots, y_c)$ and $d = \dim(R/I)$.

... \therefore they form a SGP in R/I

Proof. Say $I = (y_1, \dots, y_c)$ and $d := \dim(R/I)$.
Let $x_1, \dots, x_d \in R$ s.t. they form a S.O.P in R/I .

Then, $\sqrt{(y, x)} = \sqrt{M_R}$.

Apply Lemma (2)(b) to $k(y; M)$ to get

$$\sup H_k(y; M) \leq \sup H_k(x; k(y; M)) \\ \sup H_k(x, y; M).$$

$$\therefore d + c - \underbrace{\sup H_k(y; M)}_{\sup H_k(x, y; M)} \geq d + c - \underbrace{\sup H_k(x; k(y; M))}_{\sup H_k(x; M)}$$

$$\Rightarrow d + \text{depth}(I, M) \geq \text{depth } M. \quad \blacksquare$$

Local case:

AB Equality. F an R -complex with $\text{flatdim}_R F < \infty$.

Then, for any R -complex M ,

$$\text{depth}(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_k(k \otimes_R^L F)$$

$$\text{depth}_R M \geq -\sup H_k(M) = \leq$$

with equality iff $\text{depth}(H_S(M)) = 0$.

Propn. $R \rightarrow$ comm. Noe.

$I \subseteq R$ ideal.

$$\text{depth}(I, M) = \inf \left\{ \text{depth}_{R_p} M_p : p \in V(I) \right\}.$$

Proof. $I \subset p \Rightarrow \text{depth}_R(I, M) \leq \text{depth}_R(p, M) \leq \text{depth}_{R_p}(pR_p, M_p)$

\downarrow

check using
Koszul

This gives \leq .

We now construct p achieving $\text{depth}_R(I, M)$.

Let $T = (x_1, \dots, x_c)$. Let $s := \sup H_k(T; M)$.

We now construct p achieving $\text{depth}_R(\mathbb{I}, M)$.

Let $\mathbb{I} = (x_1, \dots, x_c)$. Let $s := \sup H_s(\underline{x}; M)$.

Pick $\mathfrak{p} \in \text{Ass } H_s(\underline{x}; M)$. (Maybe even minimal.)

Then, $\text{depth}_{R_p} H_s(\underline{x}; M)_{\mathfrak{p}} = 0$.

Consider $K(\underline{x}; M)_{\mathfrak{p}} = K(\underline{x}; M_p)$.

$$\sup H_{\mathfrak{p}}(K(\underline{x}; M_p)) = \sup H_{\mathfrak{p}}(\underline{x}; M_p) = s$$

$$\text{depth}_{R_p} K(\underline{x}; M_p) = -s \quad \begin{matrix} \leftarrow \\ \backslash \end{matrix}$$

$$\text{depth}_{R_p}(K(\underline{x}; R_p) \otimes M_p) \\ \parallel AB$$

$$\text{depth}(M_p) = \sup H_{\mathfrak{p}}(k(p) \otimes_{R_p} K(\underline{x}; R_p))$$

$$\geq \text{depth}(M_p) - c$$

$$\Rightarrow -s \geq \text{depth}(M_p) - c$$

$$\therefore c - s \geq \text{depth}(M_p)$$

$$\text{depth}_R(\mathbb{I}, M)$$

Remark. The proof says

$$\text{depth}_R(\mathbb{I}, M) = \text{depth}_{R_p}(M_p)$$

$$\forall \mathfrak{p} \in \text{Ann } H_s(\underline{x}; M)$$

Thm. $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$ local map.

(i.e., $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$)

Let M be an R -complex, N an S -module s.t.

N is flat as an R -module.

Then,

$$\text{depth}_S(N \otimes_R M) = \text{depth}_R(M) + \text{depth}_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N).$$

Corollary When $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$ is flat. Then $\binom{m=R}{n=S}$ gives

$$\text{depth}_S(S) = \text{depth}_R(R) + \text{depth}_{S/\mathfrak{m}_RS}(S/\mathfrak{m}_RS).$$

↳ "fiber"

$$R \xrightarrow{\quad} S \xrightarrow{\quad} \frac{S}{\mathfrak{m}_R} S$$

$$\begin{array}{ccc} \text{Spec}(R/\mathfrak{m}_RS) & \hookrightarrow & \text{Spec } S \\ \downarrow & & \downarrow \\ \{\mathfrak{m}_R\} & \hookrightarrow & \text{Spec}(R) \end{array}$$

Under the same hypothesis, we have

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_RS).$$

Proof. $\underline{x} = x_1, \dots, x_c \in R$ s.t. $\mathfrak{m}_R = (\underline{x})$.

Pick $\underline{y} = y_1, \dots, y_d \in S$ s.t. $\underline{y}(\frac{S}{\mathfrak{m}_RS})$ is the

maximal ideal of $\frac{S}{\mathfrak{m}_RS}$
(i.e., $\mathfrak{m}_S/\mathfrak{m}_RS$).

Then, $(\underline{x}S, \underline{y}) = \mathfrak{m}_S$.

$$K(\underline{x}S, \underline{y}; N \otimes_R M) \cong K(\underline{y}; N) \otimes_R K(\underline{x}; M)$$

\downarrow assoc. of \otimes

N flat over $R \Rightarrow K(\underline{y}; N)$ has fin flat dim k

$$\begin{aligned} \text{depth}_R K(\underline{x}, \underline{y}; N \otimes_R M) &= \text{depth}_R K(\underline{x}; M) \\ &\quad - \sup H_{\underline{x}}(k \otimes K(\underline{y}; N)) \end{aligned}$$

$(k = R/\mathfrak{m}_R)$

Note. $(\underline{x}, \underline{y}) \cdot H_{\underline{x}}(\underline{x}, \underline{y}; N \otimes_R M) = 0$

$$\begin{aligned} \mathfrak{m}_S H_{\underline{x}}(\underline{x}, \underline{y}; N \otimes_R M) &= 0 \\ \Rightarrow \mathfrak{m}_R H_{\underline{x}}(\underline{x}, \underline{y}; N \otimes_R M) &= 0 \end{aligned}$$

Similarly $\mathfrak{m}_R H_{\underline{x}}(\underline{x}; M) = 0$.

$$-\sup H_X(x, y; N \otimes_R M) = -\sup H_X(x, M)$$

$-\sup H_X(y; \frac{N}{M \otimes_R N})$

Add $c+d$ ↗ $\text{depth}_S(N \otimes_R M)$

Exercise R local. M f.g. R-module

$$\begin{aligned} \text{depth } R - \dim_R M &\leq \text{grade}_R M \leq \text{codim}_R M \\ &\leq \dim R - \dim M \leq \text{pdim}_R M. \end{aligned}$$

↓ use intersection thm

height(ann M)
" $\inf \{\dim R_p : p \supseteq \text{ann } M\}$

Recall. New Intersection Theorem (P. Roberts '86)

$$0 \rightarrow F_d \rightarrow \dots \rightarrow F_0 \rightarrow 0 \quad \text{finite free}$$

$$0 < \text{length } H_X(F) < \infty.$$

Then, $d \geq \dim \dim(R)$. \$8\$

"Simple consequence" of AB + Existence of big CM modules

Corollary. (Intersection theorem)

R local.
M $\neq 0$ f.g. R-module s.t. $\text{pdim}_R M < \infty$.

Then, for any f.g. R-module N,

$$\dim_R(N) - \dim_R(M \otimes_R N) \leq \text{pdim}_R(M).$$

Inspired by : R regular local (e.g. $k[[x_1, \dots, x_n]]$)
(Serre)
M, N any finite R-module.

(Serre) M, N any finite R -module.

$$\dim_R N - \dim_R (M \otimes_R N) \leq \dim R - \dim M.$$

Lecture 7 (01-02-2023)

Wednesday, February 1, 2023 1:24 PM

$R \rightarrow$ Commutative noetherian.

$I \subseteq R$ ideal

Invariance of domain

Let $R \rightarrow S$ finite map. (S is a finitely R -module.)

Let M be an S -complex.

Then,

$$\text{depth}_R(I, M) = \text{depth}_S(IS, M).$$

[Only need S noetherian for this.]

↳ Obvious using Koszul.

$$I = (x)$$

$$\begin{aligned} K(x; M) &= K(x; R) \otimes_R R \\ &= (K(x; R) \otimes_R S) \otimes_S M \\ &= K(x; S) \otimes_S M. \end{aligned}$$

$$\dim_R M = \dim_S M. \quad [\text{finiteness needed here.}]$$

Say $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$ is finite and local.

$$\begin{aligned} \text{depth}_R(M) &= \text{depth}(\mathfrak{m}_R, M) \\ &= \text{depth}(\mathfrak{m}_R S, M) \\ &= \text{depth}(\mathfrak{m}_S, M) \\ &= \text{depth}_S(M). \end{aligned}$$

↗ finiteness $\Rightarrow \sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$

Special case. R, M an R -module.

Take $S = R/\text{ann}_R(M)$.

Recall. (R, \mathfrak{m}_R) local, $M \stackrel{\neq 0}{\sim}$ a f.g. R -module.

$$1. \text{ if } \dim(M) \stackrel{\textcircled{1}}{\leq} \text{grade}(M) \stackrel{\textcircled{2}}{\leq} \text{codim}(M) \stackrel{\textcircled{3}}{\leq} \dim R - \dim M$$

$$\text{depth}(R) - \dim(M) \stackrel{\textcircled{1}}{\leq} \text{grade}(M) \stackrel{\textcircled{2}}{\leq} \text{codim}(M) \stackrel{\textcircled{3}}{\leq} \dim R - \dim M$$

$\in \text{pdimp } M.$

$$\text{depth}(M) \leq \text{depth}_R(I, M) + \dim(R/I).$$

Using invariance of domain, can tighten the above to

$$\text{depth}_R M \leq \text{depth}_R(I, M) + \underbrace{\dim(M/I M)}_{= \dim(R/I + \text{ann } M)}$$

Example. ① $R = \frac{k[x, y]}{(x^2, xy)}$

$$m_R = (x, y).$$

$$\dim(R) = 1 : \sqrt{(x^2, xy)} = (x)$$

$$\therefore \dim R = \dim \frac{k[x, y]}{(x^2)} = \dim k[x] = 1.$$

$$\text{depth}(R) = 0, \text{ i.e., } \text{Hom}_R(k, R) \neq 0$$

i.e., $(0 : m_R) \xleftarrow{\text{socle}} \neq 0.$

Taking $M = R$ shows
 $\text{depth } R - \dim M < \text{grade}(M)$ is strict.

② Take same R .

$$M = R/m. \text{ Then, } \text{grade } M < \text{codim } M.$$

③ $k[x, y, z]/(x \cap (y, z)) = R.$



$\text{Spec } R = V(x) \cup V(y, z)$ Pick $M = R/\mathfrak{p}$ where \mathfrak{p} is a min't prime
but $\dim(R/\mathfrak{p}) < \dim(R).$

$$\mathfrak{p} = (y, z).$$

④ Say $\text{pdimp } M < \infty.$

$$\dim R - \dim M \leq \text{pdimp } M$$

↑ ↗ AB

Now, take $R = k[x_1, y_1]$ and $M = \frac{R}{(x_1^a, y_1^b)}$.

Then, LHS of (5) is 0 but RHS is not.

Defⁿ: (R, M) local, $M \xrightarrow{f_0} f.g.$ over R .

$$\text{Cmd}_R(M) := \dim_R M - \text{depth}_R M.$$

 Cohen-Macaulay defect

If $\text{Gnd}_R(M) = 0$, i.e., $\dim_R(M) = \text{depth}_R(M)$, then

M is Cohen-Macaulay (CM).

M is maximal Cohen-Macaulay (MCM) if $\text{depth}(M) = \dim(R)$.

It is also considered CM and MCM.

$\hookrightarrow \text{depth } M \geq \dim M$ for $M = 0$.
 " " " " ∞ - " ∞ One can define \dim , etc for complexes as well.

(R, m_R) local. $M^{(n)}$ f.g. module.

$$\operatorname{depth}_R(M) \leq \inf \{\dim R/p : p \in \operatorname{Ass}_R M\} \leq \dim_R M.$$

Corollary. $M \text{ MCM} \Rightarrow \text{Ann}_R M = \text{Min}_R M$
 and $\dim(R/p) = \dim(M)$
 $\forall p \in \text{Min}(M).$

Geometrically: $\text{Supp}(M)$ has no "embedded components" and all components have the same dimension.

Prop. (R, \mathfrak{m}) local, M f.g. and CM.

$$\textcircled{1} \quad \text{depth}_R(I, M) = \dim M - \dim(M/IM) \quad \forall I \subseteq R$$

\textcircled{2} Given $\underline{x} = x_1, \dots, x_c \in \mathfrak{m}$, then

$$\underline{x} \text{ is } M\text{-regular} \iff \dim(M) - \dim(M/\underline{x}M) = c.$$

Note \iff is always true.

(\Leftarrow) Apply \textcircled{1} to $I = (\underline{x})$.

The hypothesis gives

$$\text{depth}_R(\underline{x}, M) = c$$

$$\Leftrightarrow H_i(\underline{x}; M) = 0 \quad \forall i \geq 1$$

$\Rightarrow \underline{x}$ is regular. \(\blacksquare\)

\textcircled{3}: \underline{x} is part of an SOP for $M \rightsquigarrow_{SOP \text{ for}} R/\text{ann}(M)$
 $\Leftrightarrow \underline{x}$ is M -regular.

Thm. $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$ local map.

ok M f.g. R -module, N f.g. S -module, flat over R .

Then,

$$\text{Cnd}_S(N \otimes_R M) = \text{Cnd}_R(M) + \text{Cnd}_{(S/\mathfrak{m}_R S)}(N/\mathfrak{m}_R N).$$

Proof: We saw the above for depth instead of Cnd.
 Same holds for dim. \(\blacksquare\)

Corollary. Under same hypothesis,

$$N \otimes_R M \text{ is CM/S} \iff M \text{ CM/R} + N/\mathfrak{m}_R N \text{ CM/}_{S/\mathfrak{m}_R S}$$

Special case: $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_R)$ flat-local.

S CM $\iff R$ and $S/\mathfrak{m}_R S$ are CM.



Defn: A local ring R is CM if it is so as a module over itself.

Key Consequence: For any f.g. R -module M ,

$$\text{grade}_R(M) = \text{codim}_R(M) = \dim(R) - \dim(M).$$

Exercise: (R, \mathfrak{m}) M f.g.

$$M \text{ CM} \Rightarrow M_p \text{ CM over } R_p \quad \forall p \in \text{Supp } M.$$

Defn: $R \rightarrow$ comm. no.e.

$M \rightarrow$ f.g. R -module

Then, M is CM if $M_p \text{ CM}/R_p$ for all $p \in \text{Supp}_R M$.

Equivalently, for all $\mathfrak{m} \in \text{Max}(\text{Supp}_R M)$.

R is CM if ...

Examples: $\cdot k[x_1, \dots, x_c]$ is CM. (How?)

$\cdot k[x_1, \dots, x_c]$ is CM. $\because \frac{\dim = c}{\text{depth} = c}$ since x_1, \dots, x_c .

$\cdot R \text{ CM} \Leftrightarrow \Lambda^I R$ is CM for some ($=$ all) I .

\hookrightarrow completion wrt I

If R local,

$(R, \mathfrak{m}) \rightarrow \Lambda^I R$ "is flat and local."

Check $S/\mathfrak{m}_R S = R/\mathfrak{m}_R \rightarrow$ field (CM)

$\therefore \Lambda^I R$ is CM $\Leftrightarrow R$ is CM.

Thm: (R, \mathfrak{m}) CM local.

M f.g. R module s.t. $\text{pd}_{R, \mathfrak{m}} M < \infty$

Then,

M is CM $\Leftrightarrow \text{pd}_{R, \mathfrak{m}} M = \text{grade}_R M$.

Proof. $\operatorname{pd}_{\text{R}} M = \operatorname{grade} N$



$$\operatorname{depth} R - \operatorname{depth} M = \dim R - \dim M$$



$$\operatorname{depth} M = \dim M.$$

□

Lecture 8 (06-02-2023)

Monday, February 6, 2023 1:25 PM

Regular Rings

For now, (R, \mathfrak{m}, k) is a comm. noetherian local ring.

Recall: $\text{depth}(R) \stackrel{\textcircled{1}}{\leq} \dim(R) \stackrel{\textcircled{2}}{\leq} \text{edim}(R).$
 $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$

When equality holds in ①, then R is Cohen-Macaulay.
(By defn.)

② holds by Krull's height theorem.

Notation: (embedding) codepth of R is

$$\text{codepth}(R) = \text{edim}(R) - \text{depth}(R).$$

Def: R is regular if $\text{codepth}(R) = 0$.

Exercise: R is regular $\Leftrightarrow \mathfrak{m}$ is generated by a regular sequence

$$\Leftrightarrow \text{edim}(R) = \dim(R)$$

Example:

① $k[[x_1, \dots, x_n]]$ with k a field is regular
since \mathfrak{m} is gen by a reg. seq.

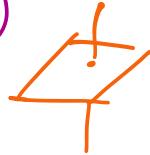
② $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ ——————.

③ $\mathbb{Z}_{(p)}$ similar.

④ Regular \Rightarrow CM. (Non CM \Rightarrow Non reg.)

$$\frac{k[x, y, z]}{(xz, yz)}$$

not reg.



⑤ $R = \frac{k[x, y]}{(xy)}$. codepth = 1 > 0.
NOT regular.

Flat Maps

Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \eta)$ be a local flat extension.

Recall

S is CM $\Leftrightarrow R$ is CM and $S/\mathfrak{m}S$ are CM.

Analog fails for regularity. (\Leftarrow) does hold.

Comida $R \xrightarrow{\varphi} k[[x^2]] \hookrightarrow k[[x]]$ (flat ext since S is free)
 \swarrow $S/\mathfrak{m}S = k[[x]]/(x^2)$
 regular \searrow not regular
 codepth = 1

Also, if R is CM, then R_p is CM $\forall p$.

Question. [30s, Krull, Zariski, ...]

Does R regular imply R_p regular $\forall p \in \text{Spec } R$?

Homological characterisation of regularity. (Used to prove!)

Recall: for a f.g. R -module N , a minimal free resolution over (R, \mathfrak{m}, k) is a free resolution

$$F \xrightarrow{\sim} M$$

with $\partial(F) \subseteq \eta F$.

Recall/Prove. Minimal resolutions exist and are unique up to isomorphism of complexes.

The i th Betti number of M is

$$\begin{aligned}\beta_i^R(M) &= \text{rank}_R(F_i) \\ &= \text{rank}_k(\text{Tor}_i^R(M, k)) \\ &= \text{rank}_k(\text{Ext}_R^i(M, k)).\end{aligned}$$

Example. ① $R = k[[x_1, \dots, x_n]]$. $f \in \eta \setminus \{0\}$.

$$M := R/(f).$$

Then the min'l free resⁿ of M is

$$0 \rightarrow R \xrightarrow{f} R \rightarrow 0.$$

↪ Koszul

$$\beta_i(M) = \begin{cases} 1 & ; i=0, \\ 0 & ; \text{else} \end{cases}$$

The min'l resⁿ of k is $K(x)$.

$$\beta_i(k) = \binom{n}{i} \quad \text{for } i \geq 0.$$

$$\textcircled{2} \quad R = \frac{k[[x, y]]}{(xy)}.$$

$M = R/(k)$ has min'l free resⁿ:

$$\dots \xrightarrow{\cdot x} R^3 \xrightarrow{\cdot x} R^2 \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \rightarrow 0$$

$$\therefore \beta_i(M) = 1 \quad \forall i.$$

$\text{Re}l^R \text{ for } k: \quad R = R/(x,y)$

$$\text{repeat } \rightarrow R \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x,y)} R \rightarrow 0$$

$$\beta_i(M) = \begin{cases} 1 & j; i=0 \\ 2 & ; i \geq 1 \end{cases}$$

FACT. $0 \neq M$ f.g.:

$$\text{projdim}_R(M) = \sup_{\parallel} \{ i \geq 0 : \beta_i^R(M) \neq 0 \}.$$

length of min'l free rel^n
 $\inf_{\parallel} (\text{length}(\text{free root}))$

Theorem. [Auslander - Buchsbaum - Serre, '50s]
 (ABS)

(R, m, k) local. TFAE:

① R is regular.

② $\text{projdim}_R(M) < \infty$ for all f.g. M .

③ $\text{projdim}_R(k) < \infty$.

Proof. ① \Rightarrow ② $\underline{x_1, \dots, x_d}$
 Let \underline{x} be a min'l gen set for m . (Hence reg. seq.)
 Then, $K(\underline{x})$ is a min'l free rel^n for k .

$$\beta_i^R(M) = \text{rank}_k(\text{Tor}_i^R(M, k))$$

$$= \text{rank}_k H_i(M \otimes_R K(\underline{x}))$$

$$= 0 \quad \text{for } i > d.$$

$$\therefore \text{projdim}_R(M) < \infty.$$

② \Rightarrow ③. —

$\textcircled{2} \Rightarrow \textcircled{3}$. —

$\textcircled{3} \Rightarrow \textcircled{1}$. Some's proof of ABS Theorem relies on:

Lemma: (R, m, k) local.

Then, $\beta_i^R(k) \geq \binom{\text{edim}(R)}{i}$ for $i \geq 0$.

Proof of Lemma. Let $F \rightarrow k$ be a mil rel.
 $(\because \beta_i^R(k) = \text{rk}_k F_i)$

Let $x = x_0, \dots, x_r$ be a

min gen set for m .

$$\begin{array}{ccc} k & := & K(x) \xrightarrow{\quad} k \\ & & \uparrow \cong \\ & \dashrightarrow & \uparrow \\ & \varphi \quad \text{a map} & \downarrow F \\ & \text{from complexes} & (\because \text{perfect}) \end{array}$$

Claim: φ_i is a split injection. (Then it is clear.)

$$\left[\begin{array}{l} \varphi_i: K_i \rightarrow F_i \text{ is split injective} \\ \Leftrightarrow k \otimes_R K_i \rightarrow k \otimes_R F_i \\ \text{NAK} \quad \text{is an injection.} \end{array} \right]$$

By induction, we show φ_i is split inj.

$$i=0 \vee R \cong R$$

$i > 0$: Let $a \in K_i$ with $\varphi_i(a) \in m F_i$.
 $(\text{we are using the NAK result.})$

WTS: $a \in m K_i$.

$$\therefore \partial(\varphi_i(a)) \in m^2 F_{i-1}$$

$$\stackrel{!}{=} \varphi_i(\partial a)$$

By induction,

$$\partial^* a \in m^2 K_{i-1}$$

$$\begin{array}{ccccc} a \in K_i & \xrightarrow{\quad} & K_{i-1} & \xrightarrow{\quad} & \\ \downarrow & & \downarrow & & \\ F_i & \xrightarrow{\quad} & F_{i-1} & \xrightarrow{\quad} & \\ \varphi(a) \in m F_i & \xrightarrow{\quad} & \partial(\varphi(a)) & \xrightarrow{\quad} & m^2 F_{i-1} \\ \partial(F) \subseteq m F & \xrightarrow{\quad} & \cap & & \end{array}$$

Notice that by def of ∂^* and
since x is a min gen set for m ,

Notice that \mathfrak{m} is a zero-dimensional ideal since \mathfrak{X} is a minimal set for \mathfrak{m} ,

(*) $\Rightarrow \mathfrak{a} \in \mathfrak{m} K_i$. This does it. \blacksquare

Back to: $\text{projdim}_R(k) < \infty \Rightarrow R$ is regular.

From Serre's inequality:

$$\text{projdim}_R k \geq \text{edim } R.$$

$\parallel \rightsquigarrow$ By AB Equality, since $\text{projdim } k < \infty$.

$$\text{depth}(R)$$

But $\text{depth}(R) \leq \text{edim}(R)$ always true. \blacksquare

The above now solves the localisation problem.

Corollary. R regular $\Rightarrow R_p$ is regular for all $p \in \text{Spec } R$.

Proof. R is reg.

$$\Rightarrow \text{projdim}_R R/\mathfrak{p} < \infty$$

$$\Rightarrow \text{projdim}_{R_p}(k(p)) < \infty \Rightarrow R_p \text{ is reg. } \blacksquare$$

Propⁿ. $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{m}_S)$ local flat.

① R and S/\mathfrak{m}_S reg $\Rightarrow S$ is reg.

② S is regular $\Rightarrow R$ is regular.

Proof. ① $S = S/\mathfrak{m}_S \xrightarrow{\text{flat}}$
 $\text{depth } S = \text{depth}(R) + \text{depth}(\bar{S})$) regular hypothesis
 $= \text{edim}(R) + \text{edim}(\bar{S})$

There is always a right exact sequence

more - \oplus \cup

$$\frac{m_R}{m_R^2} \otimes k \rightarrow \frac{m_S}{m_S^2} \rightarrow \frac{m_{\bar{S}}}{m_{\bar{S}}^2} \rightarrow 0$$
$$l = S/m_S$$

$$\therefore \text{cdim } S \leq \text{cdim } R + \text{cdim } (\bar{S}) = \text{depth}(S).$$

② Let $F \xrightarrow{F} k$ min'l, since φ flat AND local,

$$F \otimes_R S \rightarrow \bar{S} \quad \text{min'l.}$$

$$\beta_i^R(k) = \beta_i^S(\bar{S}) = 0 \quad \text{for } i > 0.$$

$\therefore \text{projdim}_R k < \infty.$

$\Rightarrow R$ is reg. □

Lecture 9 (08-02-2023)

Wednesday, February 8, 2023 1:26 PM

Recall: R local is regular if $\operatorname{codepth}(R) = 0$,
i.e., \mathfrak{m} generated by a regular sequence.

We proved Auslander - Buchsbaum Theorem:

(R, \mathfrak{m}, k) TFAE:

① R is regular,

② $\operatorname{projdim}_R M < \infty$ for all f.g. M

③ $\operatorname{projdim}_R(k) < \infty$.

① \Rightarrow ② \Rightarrow ③ was ok.

③ \Rightarrow ①: showed $\operatorname{projdim}_R(k) \geq \operatorname{edim}(R)$ and then AB Equality.

Sketch of second proof.

Theorem [Nagata]. Let (R, \mathfrak{m}, k) be local and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be u.zd.
(not necessarily regular)
Set $\bar{R} = R/(x)$.

Then, for any f.s. R/\mathfrak{m} -module M , we have

$$\operatorname{Tor}_*^R(M, k) \cong \operatorname{Tor}_*^{\bar{R}}(M, k) \otimes_{\bar{R}} \Lambda(\sum k).$$

$$= \circ \rightarrow \bar{k} \rightarrow \bar{k} \rightarrow \circ$$

In particular, $\beta_i^R(M) = \beta_i^{\bar{R}}(M) + \beta_{i-1}^{\bar{R}}(M)$.

In fact, one can show the min'l R -free res' of M

has the form

$$\cdots \rightarrow G_3 \xrightarrow{\begin{pmatrix} \alpha_3 & x \\ \beta_3 & -\alpha_2 \end{pmatrix}} G_2 \xrightarrow{\begin{pmatrix} \alpha_2 & x \\ \beta_2 & -\alpha_1 \end{pmatrix}} G_1 \xrightarrow{\begin{pmatrix} \alpha_1 & x \\ \beta_1 & -\alpha_0 \end{pmatrix}} G_0$$

multiplication
by x

$G_i \rightarrow \text{free}$

and $\cdots \rightarrow \bar{G}_3 \xrightarrow{\alpha_3} \bar{G}_2 \xrightarrow{\alpha_2} \cdots$ is the
 min'l \bar{R} -resol' of M .
 (Stronger result but not proving this.)

Example. $R = k[[x,y]]/(xy)$ (non regular)

Minimal R -free resol' of k :

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & xy \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \rightarrow 0$$

Consider $y-x^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$ (mod on R).

By row/col operations, the min'l resol' is iso to,

$$\cdots \xrightarrow{\begin{pmatrix} x & y-x^2 \\ -y & \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & y-x^2 \\ -x & \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y-x^2 \end{pmatrix}} R \rightarrow 0$$

Now, going mod $y-x^2$ [using the result]
 we get

$$\cdots \rightarrow \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{y} \bar{R} \xrightarrow{x} \bar{R} \rightarrow 0$$

$\bar{R} = k[[x,y]]$

$$\dots \rightarrow \bar{R} \rightarrow R$$

$$= \dots \rightarrow \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{x^2} \bar{R} \xrightarrow{x} \bar{R} \rightarrow 0$$

$$\begin{aligned}\bar{R} &= \frac{k[[x, y]]}{(xy, y-x^2)} \\ &\cong \frac{k[[x]]}{(x^2)}\end{aligned}$$

Proof of Nagata's Thm.

If $a \in R$ is nzd, and set $\bar{R} = R/a$.
 $M \rightsquigarrow \bar{R}$ -module.

There is a long exact sequence

$$\begin{array}{ccccccc} x & \hookrightarrow & \text{Tor}_{n-1}^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_n^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_{n+1}^{\bar{R}}(M, k) \\ & & \curvearrowright & & \curvearrowright & & \end{array}$$

$$\begin{array}{ccccccc} x & \hookrightarrow & \text{Tor}_{n-2}^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_{n-1}^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_{n+1}^{\bar{R}}(M, k) \\ & & \curvearrowright & & \curvearrowright & & \end{array}$$

Specialising to our setting, we wish to show
 that each x is 0.

One can compute x in the following way:

$$\bar{F} \xrightarrow{\sim} M \text{ mod } (\bar{R} - \text{red})$$

Lift this to a sequence of free R -modules:
(not saying complex!)

$$\dots \rightarrow F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \rightarrow \dots$$

∂^2 may not be 0 but
- 2 - - - when

∂^2 may not be 0 even

$$\partial^2 = x \Theta \text{ where}$$

$$\Theta = \{ \Theta_i : F_i \rightarrow F_{i-2} \}$$

is a chain map

and the following diagram commutes: (up to sign)

$$\begin{array}{ccc}
 F_i \otimes_R k & \xrightarrow{\Theta_i \otimes_R k} & F_{i-2} \otimes_R k \\
 \downarrow \cong & & \downarrow \cong \\
 \widetilde{F}_i \otimes_{\bar{R}} k & & \\
 \downarrow \cong & & \\
 \mathrm{Tor}_i^{\bar{R}}(M, k) & \xrightarrow{x} & \mathrm{Tor}_{i-2}^{\bar{R}}(M, k)
 \end{array}
 \quad (*)$$

Now, since x is linear and $\partial^2 = x \Theta$, we have that $\Theta(F) \subseteq \eta F$.

\therefore the top map in $(*)$ is zero. $\therefore x = 0$. \blacksquare

Second proof of $\textcircled{3} \Rightarrow \textcircled{1}$ in ABT.

Given $\mathrm{projdim}_R(k) < \infty$. WTS: R is regular.

Induct on $d := \mathrm{depth}(R)$.

$d=0$: AB Equality gives $\mathrm{projdim}_R(k) = 0$.

$\therefore R = k$ is a field. R is reg.

$d>0$: By prime avoidance, $\exists x \in m \setminus \eta^2$ nzd.

$\therefore \mathrm{codeth}(R) = \mathrm{codeth}(R/x)$.

By Nagata's thm,

$$\mathrm{ordim}_n k = \mathrm{projdim}_R k - 1 < \infty.$$

by Nagata's Thm,

$$\text{projdim}_{R/x} k = \text{projdim}_R k - 1 < \infty.$$

But R/x has smaller depth. \(\square\)

(R, \mathfrak{m}, k) local

Propn. $x \in \mathfrak{m}$ nzd.

R is CM $\Leftrightarrow R/x$ is CM.

$(\text{CMD}(R) = \text{CMD}(R/\mathfrak{x}).)$

$(\text{CMD} = \text{dim} - \text{depth.})$

Proposition. $x \in \mathfrak{m}$ nzd.

① Suppose R is regular.

R/x is regular $\Leftrightarrow x \notin \mathfrak{m}^2$.

② If R/x is regular, then R is regular.
(Hence, $x \notin \mathfrak{m}^2$.)

Proof.

$$\text{codepth}(R/\mathfrak{x}) = \begin{cases} \text{codepth}(R) & x \notin \mathfrak{m}^2, \\ \text{codepth}(R) + 1 & x \in \mathfrak{m}^2. \end{cases}$$
\(\square\)

Global Setting

$R \rightarrow$ comm. noetherian, not necessarily local
or of finite Krull dim.

Dfn. R is regular if R_p is a regular local ring
for all $p \in \text{Spec}(R)$

$(\Leftrightarrow \forall \mathfrak{m} \in \text{max Spec}(R)).$

$(\Leftrightarrow \forall m \in \text{mSpec}(R)).$

(Since regularity localizes for local rings, the above makes sense.)

Exercise. R is regular $\Leftrightarrow R[x]$ is regular
 $\Leftrightarrow R[[x]]$ is regular

Example ① $k[x_1, \dots, x_n]$ is regular. ($k = \text{field}$)

② $\mathbb{Z}[x_1, \dots, x_n] \dashrightarrow$ \mathbb{Z} is regular since every localization is a field or a DVR.

③ Nagata's example. (Infinite Krull dimension but regular)

Thm. [Bass - Murthy '60s].

$R \rightarrow \text{comm noetherian}$
 $M \rightarrow \text{f.g.}$

$\text{projdim}_R M < \infty \Leftrightarrow \text{projdim}_{R_p} M_p < \infty$
for all $p \in \text{Spec } R$.

Corollary. $R \rightarrow \text{comm noe.}$

R is regular $\Leftrightarrow \text{projdim}_R M < \infty$
for all M f.g.

Proof.

$(\Rightarrow) -$

(\Leftarrow) Let $F \xrightarrow{\sim} M$ be a free res^f of

M . F_i f.g. for all $i \geq 0$.

For $n \geq 0$, define

For $n \geq 0$, define

$$D_n := \{p \in \text{Spec } R : \text{proj dim}_{R_p} M_p \leq n\}$$
$$= \{p \in \text{Spec } R : \text{im}(\partial_n^F)_p \text{ is free over } R_p\}.$$

Since $\text{im}(\partial_n^F)$ is f.g., the above set is
open in $\text{Spec}(R)$. "free locus is open"

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$$

$$\text{and } \text{Spec } R = \bigcup_{n \geq 0} D_n.$$

Since R is noetherian, $\text{Spec } R$ is a
noe. top space.

$$\therefore \text{Spec } R = D_n \text{ for some } n.$$

$\Rightarrow \text{im}(\partial_n^F)$ is locally free.

$\Rightarrow \text{im}(\partial_n^F)$ is projective.

Now, $0 \rightarrow \text{im}(\partial_n^F) \hookrightarrow F_n \rightarrow \dots \rightarrow F_0 \rightarrow 0$

is a proj. resl.

□

Lecture 10 (13-02-2023)

13 February 2023 13:26

$R \rightarrow$ comm noetherian

$$\varphi: F_1 \longrightarrow F_0 \quad \text{free modules of finite rank}$$

$$\begin{matrix} 2^{11} & 2^{11} \\ R^s & \xrightarrow{(a_s)} R^r \end{matrix}$$

$I_c(\varphi) =$ ideal generated by $c \times c$ minors of (a_{ij}) .

$$I_0(R) = R \supseteq I_1(R) = (a_{11}, \dots, a_{rs}) \supseteq \dots \supseteq I_i(R) \supseteq I_{i+1}(R) = 0$$

$i = \min(r, s)$

M f.g. R -module

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \rightarrow 0$$

$\hookdownarrow \text{rank } r$

finite free presentations

$$G_1 \xrightarrow{\psi} G_0 \longrightarrow M \rightarrow 0$$

$\hookdownarrow \text{rank } s$

Then, $I_{r-c}(\varphi) = I_{s-c}(\psi)$ for all c .
 (Exercise.)

Def. $\text{Fitt}_c^R(M) := I_{r-c}(\varphi)$, c^{th} fitting ideal of M .

We get $\text{Fitt}_0(M) \subseteq \text{Fitt}_1(M) \subseteq \dots$ eventually R .

$\text{Fitt}_0(M) \neq 0$ only if $r \leq s$

$\text{Fitt}_c(M) = R$ for $c > r$.

Examples.

① $R = k$ a field, \vee some vector space (f.g.) of rank n .

$0 \rightarrow k^n$ is a presentation

$$\text{Fitt}_c(V) = k \Leftrightarrow c = n.$$

② R a PID.

$$0 \rightarrow R^s \xrightarrow{\quad} R^r \rightarrow M \rightarrow 0$$

$\begin{pmatrix} d_1 & & \\ \ddots & & \\ 0 & & d_r \end{pmatrix}$

(can assume $s \leq r$)

↓ can put in normal form

$$\text{Fitt}_0(M) \neq 0 \Leftrightarrow r = s \Leftrightarrow M \text{ is torsion.}$$

In this case, $\text{Fitt}_0(M) = (d_1 \dots d_r)$.

Properties.

$$R^s \xrightarrow{\varphi} R^r \rightarrow M \rightarrow 0.$$

① If $R \rightarrow S$ is any map of rings.

$$\text{Fitt}_c^S(S \otimes_R M) = S \cdot \text{Fitt}_c^R(M).$$

$$② (\text{ann}_R(M))^r \subseteq \text{Fitt}_0(M) \subseteq \text{ann}_R(M).$$

Proof. Pick an $(r \times r)$ -minor ' a ' in \varPhi .

WTP: $a \cdot R^r \subseteq \text{im}(\varPhi)$

Can assume

$$\varPhi = \left(\begin{array}{c|c} \diagdown & | \\ \diagup & | \\ \vdots & | \\ \hline & r \times r \end{array} \right)_{r \times s} = \alpha$$

$\det = a$

Take the $s \times r$ matrix

$$\left(\begin{array}{c|c} \text{Signed} & \\ \text{co-factor matrix} & \\ \hline & 0 \\ & r \times r \\ & s \times r \end{array} \right) = \beta$$

Then, $A\beta = \begin{pmatrix} a & \dots & 0 \\ 0 & \dots & a \end{pmatrix}.$

$\therefore \alpha R^n \subseteq \text{im } \varphi.$ proves $\text{Fitt}_0 \subseteq \text{ann}.$

Next: $\text{ann}(M) \subseteq \text{Fitt}_0(M).$

Fix $a_1, \dots, a_r \in \text{ann}(M).$

$$\begin{array}{ccc} & R^r & \\ \swarrow & \downarrow & \\ R^s & \xrightarrow{\varphi} & R^r \\ & (a_1, \dots, a_r) & \end{array}$$

Apply $\Lambda^r(\cdot)$ to the above to get

$$\begin{array}{ccc} & R & \\ \searrow & \downarrow & \\ R^s & \xrightarrow{\Lambda^r(\varphi)} & R \\ & (a_1, \dots, a_r) & \end{array}$$

□

(3) Fix $c \geq 0$ and $p \in \text{Spec}(R).$

TFAE:

i) $\text{Fitt}_c(M) \notin p.$

ii) $\text{im}(\varphi)_p$ contains a free summand of R_p^r of rank $\geq r-c.$

iii) $\nu_{R_p}(M_p) \geq c.$

$\underbrace{\quad}_{\text{min}\# \text{ gen}}$

Sketch. Can assume (R, p, k) local. $m := p.$

$\text{Fitt}_c(M) \notin m \Leftrightarrow \text{Fitt}_c(R) = 0 \Leftrightarrow \text{Fitt}_c(k \otimes_R M) \neq 0$

$\nu_n(M) \leq c \Leftrightarrow \nu_n(k \otimes_R M) \leq c \stackrel{\text{NAK}}{\Leftrightarrow} \text{first example}$

(4) Fix $c \geq 0$ and $p \in \text{Spec } R.$ TFAE

i) $\text{Fitt}_{c-1}(M)_p = 0$ and $\text{Fitt}_c(M)_p = R_p.$

ii) $\text{im}(\varphi)_p$ is a free summand of R_p^r of rank $r-c.$

$$\textcircled{iii} \quad M_p \cong (R_p)^c.$$

Deduce from ③.

⑤ $c \geq 0$. M is projective of rank c .

$$\Leftrightarrow \text{Fitt}_{c-1}(M) = 0 \text{ and } \text{Fitt}_c(M) = R.$$

Hilbert - Birch Theorem

$R \rightarrow$ comm noetherian

Given $I \subseteq R$ with free resolution

$$0 \rightarrow R^n \xrightarrow{\varphi} R^{n+1} \rightarrow I \rightarrow 0.$$

Then, \exists nzd $a \in R$ s.t. $I = a \cdot \text{In}(\varphi)$.

Moreover, if I is projective then I is principal.

If $\text{projdim}(I) = 1$, then $\text{depth}(\text{In}(\varphi), R) \geq 2$.

Conversely, if $\varphi: R^n \rightarrow R^{n+1}$ is -matrix s.t. $\text{depth}(\text{In}(\varphi), R) \geq 2$,

$$\text{then } 0 \rightarrow R^n \rightarrow R^{n+1} \rightarrow \text{In}(\varphi) \rightarrow 0$$

is a free resolution.

Regular Local Rings are VDFs.

Thm. $M \rightarrow$ projective module

If M has a finite free resⁿ, then it must have a free resⁿ of length 1.

$$\text{Say } 0 \rightarrow F_1 \rightarrow F_2 \rightarrow \dots \rightarrow F_n \rightarrow 0.$$

$\downarrow \cong$

M projective $\rightarrow \exists (I_i)$ proj $\forall i \geq 1 \dots R$

S-, Projective + FFR \Rightarrow stably free.
 \hookrightarrow finite free resolution

Corollary. $I \subseteq R$ proj + FFR $\Rightarrow I$ principal.

Proof. $0 \rightarrow R^n \rightarrow R^{n+1} \rightarrow I \rightarrow 0.$
 I proj $\Rightarrow I = (a).$ □

Thm. R comm. noe. domain s.t. each f.g. R -module has FFR. Then, R is a UFD.

Corollary. Regular local rings are UFDs.
 \hookrightarrow not true otherwise

\rightarrow Proof. Suppose R is local. (\because Regular.)

- Induction on $\dim R$.
 - $\dim R \leq 1$ is clear.

Pick $w \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Then, R/w is also RLC and hence a domain.

$\therefore w$ is prime.

Suffice to verify R_w is a UFD.

Note $\dim R_w < \dim R$.

Pick $p \in \text{Spec}(R_w)$ with $\text{ht } p = 1$.

Suffice to show p is principal.

Note. p has a FFR. (For p comes from R .)

It now suffices to prove it is projective.

(Earlier corollary.)

This can be tested locally. But localisations
are RLRs of $\dim < \dim R$.

\therefore UFD.

$\therefore p$ is locally free
and hence projective.

For general R , again fix p of ht = 1.

WANT: p is principal.

Since p has ffr, suffices to prove projective.

This is local. \square

$R \rightarrow \text{PID}$, M torsion module.

$$0 \rightarrow R^r \rightarrow R^r \rightarrow M \rightarrow 0$$
$$\begin{pmatrix} d_1 & 0 \\ 0 & \ddots & d_r \end{pmatrix} \quad d_i \neq 0$$

$$\text{Fit}_0(M) = (\pi_{d_i}) \subseteq \text{ann}_R(M).$$

$$\text{length}_R(M) = \text{length}(R/\text{Fit}_0(M))$$

Lecture 11 (15-02-2023)

Wednesday, February 15, 2023

1:18 PM

Complete Intersections

Cohen Structure Theory

(R, \mathfrak{m}, k) local

- R is equicharacteristic if R contains a field (as a subring).
 - $\mathbb{Z} \rightarrow R$ structure map.
If $\text{char } R = p > 0$ prime, then $\text{char } k = p$.
 - When $\text{char } k = 0$, $\mathbb{Q} \hookrightarrow R$.
- Mixed characteristic : $\mathbb{Z} \hookrightarrow R$ and $\text{char } k = p > 0$.

When R is \mathfrak{m} -adically complete :

- If R equicharacteristic, then R contains a copy of k as a subring.
- If R mixed char., then R contains a complete DVR \mathcal{O}

s.t. $\mathcal{O} \hookrightarrow R \rightarrow k$

field of fractions

$$R = k[[x_1, \dots, x_n]] / I \quad (\text{equi. case})$$

$$R = \mathcal{O}[[x_1, \dots, x_n]] / I.$$

Thus, R is of the form S/I , where S is a regular local ring.

We call this a Cohen presentation of R .
 $(S/I, S \text{ where } S \text{ is an RLR.})$

Say $R = S/I$ is a Cohen presentation.

$I \subseteq \mathfrak{m}_S$. Suppose $I \neq \mathfrak{m}_S^2$. Pick $x \in I \setminus \mathfrak{m}_S^2$.

$$\begin{array}{ccc} S & \longrightarrow & R \\ & \searrow & \nearrow \\ & S/xS & \hookrightarrow \text{regular local} \end{array}$$

Thus, we can assume that $I \subseteq \mathfrak{m}_S^2$.

Equivalently, $\text{edim } R = \text{edim } S$.

$$\begin{array}{ccc} \text{rank}_k \left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \right) & & \text{rank}_k \left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2} \right) \\ " & & " \\ \text{rank}_k \left(\frac{\mathfrak{m}_S}{I + \mathfrak{m}_S^2} \right) & & \end{array}$$

In this case, we say that S/I is a minimal Cohen presentation.

Dfn. (R, \mathfrak{m}, k) local is said to be a complete intersection (c.i.) if in some Cohen presentation

$$\widehat{R} = S/I$$

\mathfrak{m} -adic

$\kappa = -/\mathbb{I}$
 m-adic
 Completion of R

and I can be generated by a regular sequence in S .

[Not every c.i. is a quotient of a regular ring.]

Intrinsic characterisation?

Let $K^R := \text{Koszul complex on a minimal gen for } m_R$.
 (Well defined up to iso of complexes
 (with multiplicative structure).)

$$- \hat{R} \otimes_R K^R = K^{\hat{R}} \quad (\because m_{\hat{R}} = m_R)$$

- $R \rightarrow \hat{R}$ induces

$$K^R \longrightarrow \hat{R} \otimes_R K^R = K^{\hat{R}}.$$

Then, $H(K^R) = H(\hat{R} \otimes_R K^R)$
 || flatness of $R \rightarrow \hat{R}$
 $\hat{R} \otimes_R H(K^R)$

Since $m_H(K^R) = 0$, the map

$$H(K^R) \longrightarrow \hat{R} \otimes_R H(K^R)$$

is an iso.

- Thus $K^R \xrightarrow{\sim} K^{\hat{R}}$.
 quasi

Say, $R = S/I$ is a minimal Cohen presentation.
 $(I \subseteq M_S^2)$

$$\underline{\text{Lemma.}} \quad \text{rank}_k \left(H_i(R) \right) = \beta_i^s(R).$$

In particular, the RHS is independent of the min'l
column presentation.

Proof. Let K^S → Koszul complex on S .

S regular, so $K^S \xrightarrow{\sim} k$.

$$\text{Since } I \subseteq \mathfrak{m}_S^2,$$

$$I \subseteq \eta_s^L, \quad R \otimes_s K^s = K^R. \quad \left(\begin{array}{l} \text{min'l gen set for } \eta_s \\ \text{gives a min'l gen set for } \eta_R. \end{array} \right)$$

$$\therefore H_k(K^R) = H_*(R \otimes_{\mathbb{Z}} K^S) \xrightarrow{\quad} \because K^S \cong k$$

$$= \text{Tor}_{k_*}^S(R, k).$$

四

$R = S/I \rightarrow \text{min'l Cohen presentation}$

$$\operatorname{ht} I \leq \nu_s(I)$$

↗ # gens of I as an S -module
(Kunz)

$\operatorname{depth}_S(I_S) =$, since S reg.

Recall: I is generated by a regular seq iff equality holds.

Thus, I is gen by regular sequence

$$\Leftrightarrow V_S(I) = \dim(S) - \dim(S/I) \quad \text{5 regulär}$$

$$\begin{aligned}\beta_1^S(R)'' &= e \dim(S) - \dim(R) \\ \beta_1'' &= e \dim(R) - \dim(R)\end{aligned} \quad \begin{matrix} \curvearrowleft \\ \because \text{min'l} \end{matrix}$$

$$\text{rank}_R H_i(K^R)$$

Therefore, if $R = S/I$ min'l Cohen presentation, then

I is gen. by a regular sequence

$$\Leftrightarrow \text{rank}_k H_1(k^R) = \text{edim}(R) - \dim(R).$$

Corollary. R is a local ring.

R is c.i. iff

$$\text{rank}_k H_1(k^R) = \text{edim}(R) - \dim(R).$$

Proof. All the numbers above are same for R and \hat{R} . \blacksquare

Remark. Say $\hat{R} = S/I$ is some Cohen presentation where $I = \langle \text{reg. seq.} \rangle$.

Then, we can reduce it to a min'l Cohen presentation where ideal $= \langle \text{reg. seq.} \rangle$.

Proof.

$$S \longrightarrow \hat{R}$$

$$\downarrow \quad \quad \quad \uparrow$$

$$S/\pi S = S'$$

$x \in I \setminus \mathfrak{m}_S^2$.

Then, $x \in \mathfrak{m}_{S'}I$ and so can be extended to a min gen set for I .
(x, y)

Then, y is a regular seq...

Theorem. R c.i. $\Rightarrow R_p$ c.i. $\forall p \in \text{Spec } R$.

Nontrivial!!!

This is clear if R itself has a Cohen presentation.

$R = S/I$, I gen. by reg. seq.

$p \in \text{Spec } R \subseteq \text{Spec } S$. $R_p \cong S_p/I S_p \dots$

$$(R, \mathfrak{m}, k): R \longrightarrow \hat{R}^n$$

↓ ↓?

$$R_p \longrightarrow \hat{R}_p^{n_{R_p}}$$

↑ not a localisation of \hat{R}^n .

Pick $q \in \text{Spec } \hat{R}$ s.t. $q \cap R = p$.

$$\begin{array}{ccc} R & \longrightarrow & \hat{R} \\ \downarrow & & \downarrow \\ R_p & \longrightarrow & (\hat{R})_q \\ & \uparrow & \\ & \text{flat + local} & \end{array}$$

Theorem. $\varphi: R \rightarrow S$ flat local map.

Then, S c.i. $\Leftrightarrow R$ and $S/\mathfrak{m}_R S$ are c.i.

Fact. R c.i. $\Leftrightarrow R[\![x_1, \dots, x_n]\!]$ c.i.

(R, \mathfrak{m}, k) local

$R = S/I$ min'l Cohen presentation (say it exists)

Write $\mathfrak{m} = (x_1, \dots, x_n)$ with $n = \text{edim}(I)$.

$I = (f_1, \dots, f_c)$, min'l gen set.

$$\mathfrak{m}_R = \underline{\underline{x}} \cdot R.$$

$$K^R : \longrightarrow \overset{n}{\oplus} R e_i \longrightarrow R \rightarrow 0$$

$$K^R : \dots \longrightarrow \bigoplus_{i=1}^n R e_i \longrightarrow R \rightarrow 0$$

$$\partial e_i = x_i$$

Since $I \subseteq M_s^2$, can write

$$f_j = \sum_{i=1}^n s_{ij} x_i, \text{ where } s_{ij} \in \mathcal{M}.$$

$$\text{Let } z_j = \sum_{i=1}^n \bar{s}_{ij} e_i \in \bigoplus_{i=1}^n R e_i.$$

$$\text{Note } \partial(z_j) = \sum_{i=1}^n \bar{s}_{ij} \bar{x}_i = \bar{f}_j = 0.$$

$\therefore z_1, \dots, z_c$ are cycles.

Claim. The classes $[z_1], \dots, [z_c] \in H$, form a k -basis. \square

Tate's results. (R, \mathcal{M}, k)

$$R^n \xrightarrow{f} R \quad \begin{matrix} \text{im}(f) = \mathcal{M} \\ n = \text{edim}(R) \end{matrix}$$

$$K^R = (\Lambda^*(\sum R^n), \partial).$$

$\leadsto K^R$ is a dg (= differential graded) R -algebra.
(Even graded commutative.)

$\leadsto H(K^R)$ is a graded-commutative k -algebra.

Universal property gives:

$$\chi^R : \Lambda_* \left(\sum H_*(K^R) \right) \longrightarrow H(K^R)$$

\sum

map of k -algebras

Theorem (Tate) R is c.i. $\Leftrightarrow \chi^R$ is an isomorphism.

(Asmno) $\Leftrightarrow \Lambda^2 H_1(K^R) \longrightarrow H_2(K^R).$