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MA 526

Commutative Algebra

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Contents

Notation

1. $\mathcal{N}(R)$ denotes the nilradical of R.
2. $\mathcal{J}(R)$ denotes the Jacobson radical of R.
3. $\operatorname{Spec}(R)$ denotes the set of prime ideals of R.
4. $\operatorname{mSpec}(R)$ denotes the set of maximal ideals of R.
5. $N \leq M$ is read as "N is a submodule of M."
6. $I \trianglelefteq R$ is read as "I is an ideal of R."
7. For an integral domain R, $Q(R)$ denotes its field of fractions.
8. k denotes a field. If k is algebraically closed, we write this as $\mathsf{k} = \overline{\mathsf{k}}$.
9. When we say that "M is a finite R-module," we mean that "M is a finitely generated R-module."

Lecture 1. Associated primes of ideals and modules

Definition 1.1. Suppose M, N are R-submodules of some R-module M'. Then,

$$M :_R N := \{r \in R \mid rN \subset M\}.$$

Definition 1.2. Let M be an R-module and $0 \neq x \in M$. If $\mathfrak{p} = 0 :_R x$ is a prime in R, then we say that $\mathfrak{p}$ is an associated prime of M.

$$\mathrm{Ass}_R(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = 0 :_R x \text{ for some } x \in M \setminus \{0\}\}.$$

Definition 1.3. The elements of $\mathrm{Ass}(M)$ which are not minimal in $\mathrm{Ass}(M)$ are called embedded primes.

Definition 1.4. Fix $x \in X$. The map $\mu_x : R \to M$ defined by $r \mapsto rx$ is called the homothety by x.

Note that $\ker \mu_x = 0 :_R x$.

Proposition 1.5. A prime $\mathfrak{p}$ is an associated prime of M iff $R/\mathfrak{p}$ is isomorphic to a submodule of M.

Definition 1.6. $a \in R$ is a zero divisor on M if $ax = 0$ for some $0 \neq x \in M$.

$$\mathcal{Z}(M) := \{a \in R \mid a \text{ is a zero divisor on } M\}.$$

If a is not a zero divisor, then μ_a is called a non zero divisor on M or M-regular.

Note that a is a zero divisor iff μ_a is not injective.

Proposition 1.7. Let R be Noetherian and $M \neq 0$ finitely generated R-module. Then,

1. the maximal elements among $\{(0 : x) \mid x \neq 0\}$ are prime. In particular, $\mathrm{Ass}\, M \neq \emptyset$.
2. $\mathcal{Z}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.

Example 1.8. Let $R = \mathsf{k}[x, y]$ for a field k and put $I = \langle x^2, xy\rangle$. Then, $\operatorname{Ass}(R/I) = \{\langle x\rangle, \langle x, y\rangle\}$. Note that $\langle x\rangle$ is not maximal among the annihilators.

Proposition 1.9. Let $S \subset R$ be a multiplicatively closed set. Then,

1. $\operatorname{Ass}_{S^{-1}R}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}(M),\ \mathfrak{p} \cap S = \varnothing\}$.
2. $\mathfrak{p} \in \operatorname{Ass}_R(M) \iff \mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$.

Definition 1.10. $\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$.

Proposition 1.11. If M is finitely generated, then $\operatorname{Supp}(M) = V(\operatorname{ann}(M))$.

Proposition 1.12. If $0 \to N \to M \to L \to 0$ is exact, then $\operatorname{Supp} M = \operatorname{Supp} N \cup \operatorname{Supp} L$.

Proposition 1.13. Let L, K be f.g. R-modules. Then, $\operatorname{Supp}(K \otimes_R L) = \operatorname{Supp} L \cap \operatorname{Supp} K$. In particular, $\operatorname{Supp}(M/IM) = \operatorname{Supp} M \cap V(I)$.

Proposition 1.14. $\operatorname{Ass}(M) \subset \operatorname{Supp}(M)$.

Note that if R is Noetherian and $I \trianglelefteq R$ is an ideal, then $\operatorname{Ass}(R/I) \subset \operatorname{Supp}(R/I) = V(I)$.

Assume that R and M are Noetherian from now.

Proposition 1.15. $\operatorname{Ass} M$ and $\operatorname{Supp} M$ have the same set of minimal primes.

Remark 1.16. Note that $\mathfrak{p}$ is a minimal prime over $\mathfrak{p}^n$. That is, it is a minimal element of $V(\mathfrak{p}^n) = \operatorname{Supp}(R/\mathfrak{p}^n)$ and hence, an element of $\operatorname{Ass}(M/\mathfrak{p}^n)$.

Note that $V(\mathfrak{p}^n) = \operatorname{Supp}(R/\mathfrak{p}^n)$ is true because of the Noetherian assumption.

Theorem 1.17. 1. There exists a sequence of R-submodules of M

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that $M_{i+1}/M_i \cong R/\mathfrak{p}_i$ for $\mathfrak{p}_i \in \operatorname{Spec}(R)$.

2. Given any sequence as above, we have

$$\operatorname{Ass} M \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subset \operatorname{Supp} M.$$

In particular, $\operatorname{Ass} M$ is always finite and hence, the set of minimal primes over any ideal is finite.

Definition 1.18. Let $N \leq M$ be a submodule such that $\operatorname{Ass}(M/N) = \{\mathfrak{p}\}$. Then, M is called $\mathfrak{p}$-primary.

Definition 1.19. Let M be a module such that $\operatorname{Ass} M = \{\mathfrak{p}\}$. Then, M is called $\mathfrak{p}$-coprimary.

Example 1.20. If $\mathfrak{m} \subset R$ is maximal, then $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for all $n \geq 1$.
If $\mathfrak{p} \subset R$ is prime, then $\mathfrak{p}^n$ need not be $\mathfrak{p}$-primary.

Proposition 1.21. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of R, then $\mathfrak{q}R_\mathfrak{p}$ is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal.

Proof. Note that $(R/\mathfrak{q})_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}$ as $\mathbb{R}_\mathfrak{p}$-modules. By Proposition 1.9, we see that

$$\begin{aligned} \mathfrak{a}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) &\iff \mathfrak{a}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}\left((R/\mathfrak{q})_\mathfrak{p}\right) \\ &\iff \mathfrak{a} \in \operatorname{Ass}_R(R/\mathfrak{q}) = \{\mathfrak{p}\} \\ &\iff \mathfrak{a} = \mathfrak{p} \end{aligned}$$

and hence, $\mathfrak{q}R_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$-primary. □

Definition 1.22. For $a \in R$, define $\mu_a : M \to M$ as $x \mapsto ax$.

Definition 1.23.

$$\begin{aligned} \operatorname{nil}(M) &:= \{a \in R \mid \mu_a \text{ is nilpotent}\} \\ &= \{a \in R \mid a^n M = 0 \text{ for some } n\} \\ &= \sqrt{\operatorname{ann}(M)} \end{aligned}$$

Proposition 1.24. If $\operatorname{Ass}(M) = \{\mathfrak{p}\}$, then $\mathcal{Z}(M) = \operatorname{nil}(M) = \sqrt{\operatorname{ann}(M)}$.

Theorem 1.25. $|\operatorname{Ass} M| = 1 \iff \mathcal{Z}(M) = \operatorname{nil}(M)$.
If either condition holds, we have $\operatorname{Ass} M = \{\sqrt{\operatorname{ann}(M)}\}$.

Corollary 1.26. If $N \leq M$ is $\mathfrak{p}$-primary, then $\operatorname{Ass}(M/N) = \{\sqrt{\operatorname{ann}(M/N)}\}$.

Corollary 1.27. I is $\mathfrak{p}$-primary implies $\mathfrak{p} = \sqrt{I}$.

Remark 1.28. Note that if $\sqrt{I}$ is prime, it does not imply that I is $\sqrt{I}$-primary.

Corollary 1.29. I is $\mathfrak{p}$-primary iff $\bigcup_{\mathfrak{p} \in \operatorname{Ass}(R/I)} \mathfrak{p} = \mathcal{Z}(R/I) = \operatorname{nil}(R/I) = I$.

Proposition 1.30. If N_1 and N_2 are $\mathfrak{p}$-primary, so is $N_1 \cap N_2$.

Definition 1.31. A submodule $N \leq M$ is called reducible if $N = N_1 \cap N_2$ with $N_1 \neq N \neq N_2$. It is called irreducible otherwise.

Proposition 1.32. Prime ideals are irreducible.

Theorem 1.33. Proper irreducible submodules are primary.

Theorem 1.34. Any proper submodule can be written as an intersection of finitely many irreducible submodules.

Corollary 1.35. Let R be a Noetherian ring and M a Noetherian R-module. If $N \lneq M$ is a proper R-submodule, then N can be written as

$$N = N_1 \cap \cdots \cap N_r,$$

where $N_1, \ldots, N_r$ are primary submodules.

The above is called a primary decomposition of N.

Definition 1.36. A primary decomposition is called minimal if $\mathrm{Ass}(M/N_i) \neq \mathrm{Ass}(M/N_j)$ for $i \neq j$.

It is called irredundant if N_i can be removed.

Theorem 1.37. If $N = N_1 \cap \cdots \cap N_r$ is an irredundant primary decomposition and $\mathrm{Ass}(M/N_i) = \{\mathfrak{p}_i\}$, then $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Theorem 1.38. If $\mathfrak{p}$ is a minimal associated prime of M/N, then the $\mathfrak{p}$-primary component of N is $\varphi_{\mathfrak{p}}^{-1}(N\mathfrak{p})$, where $\varphi_{\mathfrak{p}} : M \to M_{\mathfrak{p}}$ is the natural map $x \mapsto \frac{x}{1}$.

In particular, the component corresponding to the minimal prime is uniquely determined.

Lecture 2. Artinian rings and Artinian modules

We now drop the assumption from the previous chapter of rings and modules being Noetherian.

Definition 2.1. An R-module M is called Artinian if every descending chain of R-submodules of M stabilises.

R is said to be an Artinian ring if R is Artinian as an R-module.

Proposition 2.2. Let k be a field and V a k-module, i.e., a k-vector space. Then, V is Artinian iff V is finite dimensional iff V is Noetherian.

Proposition 2.3. Let R be an Artinian ring.

1. If I is an ideal of R, then R/I is an Artinian ring.
2. If R is an integral domain, then R is a field.
3. More generally, every non zero divisor of R is a unit.
4. If $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\mathfrak{p}$ is maximal. That is, $\operatorname{Spec}(R) = \operatorname{mSpec}(R)$.
   Thus, $\mathcal{N}(R) = \mathcal{J}(R)$.
5. R has finitely many maximal ideals. (It may have infinitely many ideals, however.)
6. If $I \trianglelefteq R$, then $\operatorname{Ass}(R/I) = \operatorname{Supp}(R/I) = V(I)$.
7. If $N = \mathcal{N}(R)$, then there exists k such that $N^k = 0$.
8. Let $0 \to N \to M \to L \to 0$ be an exact sequence. Then M is Artinian iff N and L are Artinian.
   In particular, $\bigoplus_{i=1}^n M_i$ is Artinian iff each M_i is.
9. If M is a finitely generated R-module, then M is an Artinian R-module and $R/\operatorname{ann}(M)$ is an Artinian ring.

Proposition 2.4. Let M be an R-module and $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \operatorname{mSpec} R$ are maximal ideals such that $\mathfrak{m}_1 \cdots \mathfrak{m}_n M = 0$. Then,
M is Noetherian $\iff$ M is Artinian.

Note that the maximal ideals above need not be distinct. Moreover, R is not assumed to be Artinian.

Proposition 2.5. Let R be an Artinian ring. Then, R is Noetherian ring.

Proposition 2.6. Let R be a Noetherian ring with $\operatorname{Spec} R = \operatorname{mSpec} R$. Then, R is an Artinian ring.

Proposition 2.7. If R is Artinian and M an Artinian R-module, then M is a Noetherian R-module. In particular, M is finitely generated.

Theorem 2.8. Let R be an Artinian ring. Then, there exist uniquely determined Artinian local rings $R_1, \ldots, R_n$ such that

$$R \cong R_1 \times \cdots \times R_n.$$

Definition 2.9. An R-module $M \neq 0$ is called simple if the only R-submodules of M are 0 and M.

Proposition 2.10. An R-module M is simple iff $M \cong R/\mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{mSpec} R$. The isomorphism is as R-modules. In particular, M is cyclic.

Lemma 2.11. A simple module is both Noetherian and Artinian.

Definition 2.12. Let M be an R-module. A series of the form

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

is called a composition series if M_{i+1}/M_i is simple for each i. n is called the length of this composition series.

Note that a composition series has finite length, by definition.

Theorem 2.13. M has a composition series $\iff$ M is Artinian and Noetherian.

Definition 2.14. Let $M \neq 0$ be an R-module. Define

$$l_R(M) := \min\{n \mid M \text{ has a composition series of length } n\}.$$

$l_R(M) = \infty$ if the set on the right is empty. $l_R(M)$ is called the length of M over R.

Note that if $R = \mathsf{k}$ is a field, then the length of M is simply the dimension.

Definition 2.15. If $l_R(M) < \infty$, then M is called a finite length module.

Proposition 2.16. M is a finite length module iff M is Artinian and Noetherian.

Proposition 2.17. Let R be a Noetherian ring and M a finite length R-module. Then, $\mathrm{Ass}(M) \subset \mathrm{mSpec}(R)$.

Proposition 2.18. Let M be a finite length module and $N \leq M$. Then, N also has finite length and $l_R(N) \leq l_R(M)$ with equality iff $N = M$.

Theorem 2.19 (Jordan-Hölder)**.** Every composition series of a finite length module M has the same length.

Now, if

$$\begin{aligned} 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M, \\ 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = M \end{aligned}$$

are two composition series of M, then there exists a permutation $\sigma \in S_n$ such that

$$M_i / M_{i-1} \cong N_{\pi(i)} / N_{\pi(i)-1}$$

for all $1 \leq i \leq n$. In other words, the quotients that appear are the same.

Proposition 2.20. Let M be a finite length module. Any series of the form

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

is actually a composition series.

Proposition 2.21. Let $0 \to N \to M \to L \to 0$ be an exact sequence. Then, $l_R(M) = l_R(N) + l_R(L)$.

Note that M is finite length iff N and L both are.

Proposition 2.22. If R is Noetherian and M a finite length R-module, then $\mathrm{Ass}(M) \subset \mathrm{mSpec}(R)$.

Lecture 3. Integral Extensions of Rings

Definition 3.1. Let $R \subset S$ be non-zero commutative rings. An element $s \in S$ is called integral over R if there exists a monic polynomial $f(x) \in R[x]$ such that $f(s) = 0$.

Let
$$T = \{s \in S \mid s \text{ is integral over } R\}.$$
T is called the integral closure of R in S.

If R is an integral domain and $S = Q(R)$, then T is called the normalisation of R. R is called normal or integrally closed if $T = R$.

Recall that if R is an integral domain, then $Q(R)$ denotes the field of fractions of R.

We shall shortly show that T is a subring of S.

Theorem 3.2. If R is a UFD, then R is integrally closed. In other words, UFDs are normal.

The converse is not true.

Theorem 3.3 (Cayley-Hamilton)**.** Let $I \trianglelefteq R$ be an ideal and M a finitely generated R-module. Let $\varphi : M \to M$ be an R-endomorphism such that $\varphi(M) \subset IM$. Then, φ satisfies a monic polynomial of the form
$$x^n + a_1 x^{n-1} + \cdots + a_n$$
with $a_1, \ldots, a_n \in I$.

Corollary 3.4 (Nakayama)**.** Suppose M is finitely generated over R and $I \trianglelefteq R$ is such that $M = IM$. Then, there exists $a \in I$ such that $(1 + a)M = 0$.
In particular, if $I \subset \mathcal{J}(R)$, then $M = 0$.

Corollary 3.5. If $\varphi : M \to M$ is a surjective R-linear map, then φ is an isomorphism. (M is finitely generated as usual.)

Corollary 3.6. Suppose $M \cong R^n$. Then, any set of n generators is linearly independent.

Corollary 3.7. Let R be a non-zero commutative ring. Then, $R^n \cong R^m$ (as R-modules) iff

$n = m$.

Theorem 3.8. Let $R \subset S$ be non-zero commutative rings and $s \in S$. TFAE:

1. s is integral over R.
2. $R[s]$ is a finitely generated as an R-module.
3. There exists a subring T such that $R[s] \subset T \subset S$ and T is a finitely generated R-module.

Theorem 3.9. Let $R \subset S$ be a ring extension and $T = \overline{R}^S$ the integral closure of R in S. Then, T is a subring of S.

Proposition 3.10. If $R \subset T$ and $T \subset S$ are integral extensions, then so is $R \subset S$.

Corollary 3.11. If T is the integral closure of R in S, then the integral closure of T in S is T.

Symbolically, if $T = \overline{R}^S$, then $\overline{T}^S = T$.

Note that if $R \subset S$ is a ring extension and $I \trianglelefteq S$ is an ideal, then $I^c = R \cap I$ is an ideal of R. (Called the contraction.) Also, one has the natural inclusion and projection maps as

$$R \overset{i}{\hookrightarrow} S \overset{\pi}{\twoheadrightarrow} S/I.$$

Then, $I^c = \ker(\pi \circ i)$ and hence, R/I^c is isomorphic to a subring of S/I. We denote this inclusion by writing $R/I^c \hookrightarrow S/I$.

Proposition 3.12. If $R \subset S$ is an integral extension, then so is $R/I^c \hookrightarrow S/I$.

Definition 3.13. Suppose $\varphi : R \to S$ is a ring map. Then, φ is called integral if $\varphi(R) \subset S$ is an integral extension.

Proposition 3.14. Let $U \subset R$ be a multiplicatively closed subset and let $R \subset S$ be an integral extension. Then, $U^{-1}R \subset U^{-1}S$ is an integral extension.

Proposition 3.15. Let R be an integral domain. TFAE:

1. R is integrally closed (normal).
2. $R_{\mathfrak{p}}$ is integrally closed for all $\mathfrak{p} \in \operatorname{Spec}(R)$.
3. $R_{\mathfrak{m}}$ is integrally closed for all $\mathfrak{m} \in \operatorname{mSpec}(R)$.

Lemma 3.16. Let $R \subset S$ be an integral extension of integral domains.
Then, R is a field $\iff$ S is a field.

Corollary 3.17. Let $R \subset S$ be rings (not necessarily domains) and $\mathfrak{q} \in \operatorname{Spec} S$. Define $\mathfrak{p} := R \cap \mathfrak{q}$.
Then, $\mathfrak{p} \in \operatorname{mSpec} R \iff \mathfrak{q} \in \operatorname{mSpec} S$.

In particular, given an integral extension, the contraction of a maximal ideal is maximal.

Definition 3.18. Let $R \subset S$ be rings. Suppose $Q \in \operatorname{Spec} S$ and $P \in \operatorname{Spec} R$. Q is said to lie over P if $Q^c = Q \cap R = P$.

Theorem 3.19 (Lying over theorem)**.** Let $R \subset S$ be an integral extension of rings and $\mathfrak{p} \in \operatorname{Spec} R$. Then, there exists $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.

In other words: Given an integral extension, there is always a prime lying over a prime.

Theorem 3.20 (Going up theorem)**.** Let $R \subset S$ be an integral extension. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} R$ with $\mathfrak{p}_1 \subset \mathfrak{p}_2$ and $\mathfrak{q}_1 \in \operatorname{Spec} S$ be such that $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$. Then, there exists $\mathfrak{q}_2 \in \operatorname{Spec} S$ such that $\mathfrak{q}_1 \subset \mathfrak{q}_2$ and $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$.

$$\begin{array}{ccccc} S & \quad & \mathfrak{q}_1 & \subset & \exists \mathfrak{q}_2 \\ \text{int}\,\big| & & \big| & & \vdots \\ R & & \mathfrak{p}_1 & \subset & \mathfrak{p}_2 \end{array}$$

In fact, inductively, we see that any chain above can be "completed."

$$\begin{array}{ccccccccccccccc} S & \quad & \mathfrak{q}_1 & \subset & \mathfrak{q}_2 & \subset & \cdots & \subset & \mathfrak{q}_m & \subset & \exists \mathfrak{q}_{m+1} & \subset & \cdots & \subset & \exists \mathfrak{q}_n \\ \text{int}\,\big| & & \big| & & \big| & & & & \big| & & \vdots & & & & \vdots \\ R & & \mathfrak{p}_1 & \subset & \mathfrak{p}_2 & \subset & \cdots & \subset & \mathfrak{p}_m & \subset & \mathfrak{p}_{m+1} & \subset & \cdots & \subset & \mathfrak{p}_n \end{array}$$

Proposition 3.21 (Incompatibility (INC))**.** Let $R \subset S$ be an integral extension of rings. Let $Q_1, Q_2 \in \operatorname{Spec} S$ lie over $P \in \operatorname{Spec} R$. If Q_1 and Q_2 are distinct, then they are incomparable. That is, $Q_1 \neq Q_2 \implies Q_1 \not\subset Q_2$ and $Q_2 \not\subset Q_1$.

$$\begin{tikzcd}
S \arrow[d, dash, "\text{int}"'] & Q_1 \arrow[dr, dash] & & Q_2 \arrow[dl, dash] \\
R & & P &
\end{tikzcd}$$

Lemma 3.22. Let $f : R \to S$ be any ring homomorphism and $P \in \operatorname{Spec} R$. TFAE:

1. $P^{ec} = f^{-1}(f(P)S) = P$, and
2. $\exists Q \in \operatorname{Spec} S$ such that $Q^c = P$. That is, a prime lies over P.

Note that the above is a general fact, no assumptions of integral extensions.

Theorem 3.23 (Going down theorem)**.** Let R be a normal domain, S an integral domain and $R \subset S$ be an integral extension.

Given $P_0, P_1 \in \operatorname{Spec} R$ with $P_0 \supset P_1$ and a prime $Q_0 \in \operatorname{Spec} S$ lying over P_0, there exists a prime $Q_1 \in \operatorname{Spec} S$ lying over P_1 with $Q_0 \supset Q_1$.

$$\begin{tikzcd}
S \arrow[d, dash, "\text{int}"'] & \mathfrak{q}_1 \arrow[d, dash] & \supset & \exists \mathfrak{q}_2 \arrow[d, dash, dotted] \\
R & \mathfrak{p}_1 & \supset & \mathfrak{p}_2
\end{tikzcd}$$

In fact, inductively, we see that any chain above can be "completed."

$$\begin{tikzcd}
S \arrow[d, dash, "\text{int}"'] & \mathfrak{q}_1 \arrow[d, dash] & \supset & \mathfrak{q}_2 \arrow[d, dash] & \supset & \cdots & \supset & \mathfrak{q}_m \arrow[d, dash] & \supset & \exists \mathfrak{q}_{m+1} \arrow[d, dash, dashed] & \supset & \cdots & \supset & \exists \mathfrak{q}_n \arrow[d, dash, dashed] \\
R & \mathfrak{p}_1 & \supset & \mathfrak{p}_2 & \supset & \cdots & \supset & \mathfrak{p}_m & \supset & \mathfrak{p}_{m+1} & \supset & \cdots & \supset & \mathfrak{p}_n
\end{tikzcd}$$

Theorem 3.24. Let R be a Noetherian normal domain with quotient field K. Let $K \subset L$ be a separable extension. Then, $\overline{R}^L$ is a finite R-module. In particular, it is a Noetherian ring.

Lecture 4. Dimension Theory of Affine Algebra over Fields

Lemma 4.1 (Artin-Tate Lemma)**.** Let $R \subset S \subset T$ be rings. Suppose

1. R is Noetherian,
2. T is a finitely generated S module,
3. T is a finitely generated R algebra.

$$\begin{array}{c} R[t_1, \ldots, t_s] = T = St_1' + \cdots + St_k' \\ \Big| \\ S \\ \Big| \\ R \end{array}$$

Then, S is a finitely generated R-algebra. In other words, there exist $s_1, \ldots, s_n \in S$ such that $S = R[s_1, \ldots, s_n]$.
In particular, S is Noetherian.

Definition 4.2. Let k be a field. An affine k-algebra is a ring of the form $R = \mathsf{k}[x_1, \ldots, x_n]/I$ for some ideal $I \trianglelefteq \mathsf{k}[x_1, \ldots, x_n]$.

Lemma 4.3 (Zariski)**.** Let k be any field and $R = \mathsf{k}[x_1, \ldots, x_n]/I$ be an affine k-algebra which is also a field. (That is, I is maximal.)
Then, R is an algebraic extension of k.

Corollary 4.4. Let $\varphi : R \to S$ be a ring homomorphism, where R and S are affine k-algebras. Let $\mathfrak{m} \in \operatorname{mSpec}(S)$. Then, $\varphi^{-1}(\mathfrak{m}) \in \operatorname{mSpec}(R)$.

(We had used the fact that if we have an algebraic extension $K \subset F$ of fields and an integral domain R such that $K \subset R \subset F$, then R is a field.)

Theorem 4.5 (Weak Nullstellensatz)**.** If k is algebraically closed, then maximal ideals $\mathfrak{m} \in \operatorname{mSpec} \mathsf{k}[x_1, \ldots, x_n]$ are precisely those of the form $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$ for some $(a_1, \ldots, a_n) \in \mathsf{k}^n$.

Corollary 4.6 (Criterion for solvability)**.** Let $p_1(x_1, \ldots, x_n), \ldots, p_s(x_1, \ldots, x_n)$ be polynomials in $\mathsf{k}[x_1, \ldots, x_n]$. Then, the polynomials have a common solution iff the ideal generated by them is not the whole ring.

Remark 4.7. In fact, one need not assume $s < \infty$ in the above.

Definition 4.8. Given a field k, $\mathbb{A}^n_{\mathsf{k}}$ denotes the affine n-space over k. It is simply the set k^n without any vector space structure.

Given any ideal $I \trianglelefteq \mathsf{k}[x_1, \ldots, x_n]$, we define the zero set of I as

$$\mathcal{Z}(I) = \{\underline{a} \in \mathbb{A}^n_{\mathsf{k}} : f(\underline{a}) = 0 \text{ for all } f \in I\} \subset \mathbb{A}^n_{\mathsf{k}}.$$

A subset of $\mathbb{A}^n_{\mathsf{k}}$ which is the zero set of some ideal is called an algebraic set.

Given an algebraic set $X \subset \mathbb{A}^n_{\mathsf{k}}$, we define the ideal of X as

$$\mathcal{I}(X) = \{f \in \mathsf{k}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in X\} \subset \mathsf{k}[x_1, \ldots, x_n].$$

Remark 4.9. An ideal of an algebraic set is always a radical ideal.

Theorem 4.10 (Strong Nullstellensatz)**.** If k is algebraically closed and $I \trianglelefteq \mathsf{k}[x_1, \ldots, x_n] = S$ an ideal, then $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$.

In particular, there is a bijection

$$\{\text{radical ideals in } S\} \leftrightarrow \{\text{algebraic subsets in } \mathbb{A}^n_{\mathsf{k}}\}.$$

Definition 4.11. Given a polynomial $f \in \mathsf{k}[x_1, \ldots, x_n]$, we can write

$$f = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} a_\alpha x^\alpha.$$

If $a_\alpha \neq 0$, we say that x_α is a term of f.

Writing $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha|$ denotes the maximum of $\alpha_1, \ldots, \alpha_n$.

Proposition 4.12. Let k be any field. Let $f \in S = \mathsf{k}[x_1, \ldots, x_n]$ be a non-constant polynomial. Let

$$N > \max\{|\alpha| : \alpha \in (\mathbb{N} \cup \{0\})^n,\ x^\alpha \text{ is a term of } f\}.$$

Without loss of generality, we may assume that x_n appears non-trivially in some term of f. Define the map $\Phi : S \to S$ by identity on k and

$$x_i \mapsto \begin{cases} x_i - x_n^{N^i} & i \neq n, \\ x_n & i = n. \end{cases}$$

Then, Φ is an automorphism such that $\Phi(f)$ is monic in x_n, up to a constant. That is,

$$\Phi(f) = cx_n^r + g_1 x_n^{r-1} + \cdots + g_n,$$

where $0 \neq c \in \mathsf{k}$ and $g_1, \ldots, g_n \in \mathsf{k}[x_1, \ldots, x_{n-1}]$.

Theorem 4.13 (Noetherian Normalisation Lemma)**.** Let $R = \mathsf{k}[\theta_1, \ldots, \theta_n]$ be an affine k-algebra. Then, there exist algebraically independent elements $z_1, \ldots, z_d \in R$ such that $\mathsf{k}[x_1, \ldots, x_n] \subset R$ is an integral extension.

$$\begin{array}{c} R \\ \Big|\,\text{integral} \\ \mathsf{k}[z_1, \ldots, z_d] = S. \end{array}$$

In particular, R is a finite S-module.

Corollary 4.14. Let R be an affine k-algebra and $I \trianglelefteq R$ an ideal. Then

$$\sqrt{I} = \bigcap_{\mathfrak{m} : I \subset \mathfrak{m} \in \mathrm{mSpec}(R)} \mathfrak{m}.$$

In particular, $\mathcal{N}(R) = \mathcal{J}(R)$.

Definition 4.15. A saturated chain of prime ideals is a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

of prime ideals such that no prime ideal can be inserted strictly in between anywhere above. (In other words, there exists no $i \in \{0, \ldots, n-1\}$ and no $\mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{p}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_{i+1}$.)

The length of the above chain is n. The Krull dimension of R is defined as

$$\dim(R) = \sup\{n : \exists \text{ a saturated chain of length } n\}.$$

$\dim(R)$ may be ∞ even if R is Noetherian.

Definition 4.16. Given a prime ideal $\mathfrak{p} \trianglelefteq R$, the height of $\mathfrak{p}$ is defined as

$$\text{height}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}).$$

Example 4.17. Here are some examples.

1. If R is Artinian, then $\dim(R) = 0$. In particular, $\dim(\mathsf{k}) = 0$.
2. $\dim(\mathbb{Z}) = 1$.
3. $\dim(\mathsf{k}[X]) = 1$.
4. In general, if R is a PID and not a field, then $\dim(R) = 1$.

Proposition 4.18. Let $R \subset S$ be an integral extension of rings. Then,

1. $\dim(R) = \dim(S)$.
2. If $I \triangleleft S$ is a proper ideal, then $\dim(S/I) = \dim(R/I \cap R)$.
3. Suppose S is integral and R normal. Let $Q \in \text{Spec}(S)$. Then, $\dim(S_Q) = \dim(R_{Q \cap R})$.

Theorem 4.19. Let R be an affine algebra over a field k. Let $z_1, \ldots, z_d \in R$ be such that $S = \mathsf{k}[z_1, \ldots, z_d] \subset R$ is an integral extension. (Exists by NNL.)
Then,

1. $\dim(R) = d = \dim(\mathsf{k}[z_1, \ldots, z_d])$.
2. Any maximal saturated chain of prime ideals in R has length d.

Remark 4.20. The above shows that the d in Noetherian Normalisation Lemma is determined uniquely. Moreover, it shows that the dimension of polynomial ring in d variables over a field is d.

Lecture 5. Graded Rings and Graded Modules

Definition 5.1. Let R be any commutative ring. (In particular, it is an additive subgroup.) Let $\{R_n : n = 0, 1, \dots\}$ be a sequence of additive subgroups with the properties that

$$R = \bigoplus_{n \geq 0} R_n \quad \text{and} \quad R_n R_m \subset R_{n+m} \text{ for all } n, m \geq 0.$$

Then, R is called a graded ring with grading $(R_n)_n$. Any $x \in R_n$ is called homogeneous of degree n.

Remark 5.2. Note that for $n = m = 0$, we have $R_0 R_0 \subset R_0$. Thus, R_0 is closed under multiplication as well. Moreover, writing $1 = r_0 + \cdots + r_n$ for $r_i \in R_i$ and noting that $1^2 = 1$ gives that $n = 0$ and $1 \in R_0$.

Thus, R_0 is actually a subring of R. In particular, R is an R_0-module. Moreover, $R_0 R_n \subset R_n$ gives us that each R_n is an R_0-module.

Example 5.3. Consider $R = \mathsf{k}[X_1, \dots, X_r]$ and let R_n be the k vector space generated by monomials of degree n. Then, R is a graded ring with grading $(R_n)_n$.

Definition 5.4. Let R be a graded ring with grading $(R_n)_{n \geq 0}$. An R-module M is called graded if there exists a sequence of submodules $(M_n)_{n \geq 0}$ such that

$$M = \bigoplus_{n \in \mathbb{Z}} M_n \quad \text{and} \quad R_m N_n \subset M_{n+m} \text{ for all } m, n \geq 0.$$

Note that in the above we have used $\mathbb{N}_0$ for grading. However, we may use $\mathbb{N}$ or $\mathbb{Z}$ as well. In fact, one may use any monoid or even semigroup.

Definition 5.5. Let R be a graded ring and M a graded R-module. A submodule $N \leq M$ is called a graded submodule if N is generated by homogeneous elements of M.

Theorem 5.6 (Characterisation of graded submodules)**.** Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring and $M = \bigoplus_{n \geq 0} M_n$ a graded R-module and $N \leq M$ a submodule. TFAE:

1. N is a graded R-submodule of M, that is, N is generated by homogeneous elements.
2. $N = \bigoplus_{n \geq 0} (N \cap M_n)$.

3. If $y \in N$ and $y = y_0 + \cdots + y_n$, where $y_i \in M_i$, then $y_i \in N$.

The second point says that N can be considered a graded R-submodule by itself. The third says that if we write an element of N as a sum of homogeneous elements of different degrees, then each such element must be in N itself.

Example 5.7. Consider $R = M = \mathsf{k}[x]$, with the usual grading as in the previous example. Then, the submodule $I = \langle x \rangle$ is a graded R-submodule of M since it is generated by a homogeneous element.

On the other hand, the submodule $J = \langle x - 1 \rangle$ is *not* graded because we have $x - 1 \in J$ and $x, 1$ are homogeneous of different degrees but $1 \notin J$.

Theorem 5.8 (Characterisation of Noetherian graded rings)**.** Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring. TFAE:

1. R is Noetherian.
2. R_0 is Noetherian and R is a finitely generated R_0-algebra, i.e., $R = R_0[r_1, \ldots, r_n]$.

Definition 5.9. The sum $R_+ = \bigoplus_{i \geq 1} R_i$ is an ideal of R, called the irrelevant ideal.

Definition 5.10. Let R be a commutative ring and $F = \{I_n\}_{n \geq 0}$ a filtration of ideals as

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

satisfying $I_n I_m \subset I_{n+m}$.

Let t be an indeterminate. We define the Rees ring of F as

$$\mathcal{R}(F) = \bigoplus_{n \geq 0} I_n t^n \subset R[t].$$

$\mathcal{R}(F)$ is a graded ring with grading $(I_n t^n)_{n \geq 0}$.

The fact that it is graded follows from the condition that $I_n I_m \subset I_{n+m}$.
A special case of the above is when we take $I_n = I^n$ for some fixed ideal $I \trianglelefteq R$.

Definition 5.11. The Rees ring of an ideal I is defined as

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n = \left\{ \sum a_i t^i : a_i \in I^i \right\}.$$

(Convention: $I^0 = R$.)

In the following, we have the following notation: Let R be a ring, $I \trianglelefteq R$ and ideal and M and R-module.

Definition 5.12. Let $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ be a filtration of submodules. Then this filtration is called an *I-filtration* if

$$IM_n \subset M_{n+1} \text{ for all } n \in \mathbb{N}.$$

$\bigoplus_{n\geq 0} M_n t^n \subset M[t]$ is a graded $\mathcal{R}(I)$-module since $I^n M_m \subset M_{n+m}$ for all $n, m \geq 0$.

The filtration is called *I-stable* if

$$IM_n = M_{n+1} \text{ for } n \gg 0.$$

That is, there exists $N \in \mathbb{N}$ such that $IM_n = M_{n+1}$ for all $n > N$.

Example 5.13. $\{I^n M\}_{n\geq 0}$ is an I-stable filtration.

Definition 5.14. Given filtrations $F = \{I_n\}_{n\geq 0}$ of ideals of R and $G = \{M_n\}_{n\geq 0}$ of R-submodules of M, we say that G is *F-compatible* if $I_n M_m \subset M_{n+m}$ for all $n, m \geq 0$.

Example 5.15. $\{I^n M\}_{n\geq 0}$ is $\{I^n\}_{n\geq 0}$-compatible.

Theorem 5.16. Suppose R is Noetherian, $I \trianglelefteq R$ an ideal, M a finitely generated R-module and G a filtration of R-submodules of M which is $\{I^n\}_{n\geq 0}$-compatible. Then,

$$\mathcal{R}(G) \text{ is a finitely generated } \mathcal{R}(I)\text{-module} \iff G \text{ is } I\text{-stable}.$$

The next two theorems (ARL and KIT) were obtained as corollaries.

Theorem 5.17 (Generalised Artin-Rees Lemma)**.** Suppose R is Noetherian, $I \trianglelefteq R$ an ideal, M a finitely generated R-module, $N \leq M$ an R-submodule, and $\{M_n\}_{n\geq 0}$ is an I-stable filtration. Then, $\{N \cap M_n\}_{n\geq 0}$ is I-stable.

In other words, if $IM_n = M_{n+1}$ for $n \gg 0$, then $I(M_n \cap N) = M_{n+1} \cap N$ for $n \gg 0$.

Corollary 5.18 (Artin-Rees Lemma)**.** Suppose R is Noetherian, $I \trianglelefteq R$ an ideal, M a finitely generated R-module, $N \leq M$ an R-submodule. Then,

$$N \cap I^n M = I^{n-k}(N \cap I^k M) \text{ for } n \gg 0.$$

Theorem 5.19 (Krull Intersection Theorem)**.** Let R be a Noetherian ring, $I \trianglelefteq R$ an ideal, and M a finitely generated R-module. Define the submodule $N \leq M$ as

$$N := \bigcap_{n \geq 1} I^n M.$$

Then,

1. There exists $a \in I$ such that $(1 + a)N = 0$.
2. If $I \subset \mathcal{J}(R)$, then $N = \bigcap_{n \geq 1} I^n M = 0$.
3. If R is a Noetherian domain and $I \neq R$, then $\bigcap_{n \geq 1} I^n = 0$.
4. If R is local and $I \neq R$, then $N = \bigcap_{n \geq 1} I^n M = 0$.

Example 5.20. KIT need not be true if R is not Noetherian. Consider the ring $R = \mathcal{C}^\infty(\mathbb{R})$, the ring of smooth real valued functions defined on $\mathbb{R}$ with pointwise operations. This ring is not Noetherian. We construct a maximal ideal such that the intersection of its powers is not zero.

Consider the map $\varphi : R \to \mathbb{R}$ given by $f \mapsto f(0)$. This is a surjective ring homomorphism. Since $\mathbb{R}$ is a field, we see that $m := \ker \varphi$ is maximal.

Let $x \in R$ denote the identity function.

Claim 1. $\mathfrak{m} = (x)$.

Proof. $\supset$ is clear since $x(0) = 0$.

($\subset$) Let $f \in \mathfrak{m}$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) := \begin{cases} f(t)/t & t \neq 0, \\ f'(0) & t = 0. \end{cases}$$

Then, one sees that g is smooth. Clearly, g is infinitely differentiable on $\mathbb{R} \setminus \{0\}$. For 0, one may use L'Hôpital's rule to successively compute the derivatives (and show their existence).

Clearly, $xg = f$ and hence, $f \in (x)$, as desired. $\square$

In particular, we have $\mathfrak{m}^n = (x^n)$. Note that x^n is the function $t \mapsto t^n$. We now identify this ideal.

Claim 2. $(x^n) = \{f \in R : f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0\}$.

Proof. As before $\subset$ is clear since $(x^n)' = nx^{n-1}$.

($\supset$) The idea is as before. Let $g \in R$ be such that $f(0) = \cdots = f^{(n-1)}(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) := \begin{cases} f(t)/t^n & t \neq 0, \\ f^{(n)}(0)/n! & t = 0. \end{cases}$$

As earlier, we get $g \in R$ and $f = x^n g \in (x^n)$. □

Claim 3. $\bigcap_{n \geq 1} \mathfrak{m}^n \neq 0$.

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then, $f \in R$ and $f^{(n)}(0) = 0$ for all n. However, f is not the zero function. Thus, $0 \neq f \in \bigcap_{n \geq 1} \mathfrak{m}^n$, as desired. □

Definition 5.21. Given an ideal $I \trianglelefteq R$, we define the ideal I^* as

$$I^* = \bigoplus_{n \geq 0} (I \cap R_n).$$

Note that $I^* \subset I$.

Recall that $I = I^* \iff I$ is generated by homogeneous elements. (Characterisation of graded submodules.)

Proposition 5.22. Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring.

1. If $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\mathfrak{p}^* \in \operatorname{Spec}(R)$.
2. Any minimal prime of R is a graded ideal.
3. Maximal graded ideals of R are precisely of the of the form $\mathfrak{m} \oplus R_1 \oplus R_2 \oplus \cdots$ for $\mathfrak{m} \in \operatorname{mSpec}(R)$.
4. If R is Noetherian and $M = \bigoplus_{n \geq 0} M_n$ graded, then $\mathfrak{p} \in \operatorname{Ass}_R(M)$ is graded and

$\mathfrak{p} = \mathrm{ann}_R(x)$ for some *homogeneous* element $x \in M$.

5. If the non-zero *homogeneous* elements of R are non-zero-divisors, then R is an integral domain.

6. Let $\mathfrak{p} \neq R$ be a graded ideal. If for all *homogeneous* elements $a, b \in R$ the following holds:

$$ab \in \mathfrak{p} \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{p},$$

then $\mathfrak{p}$ is a prime ideal.

Lecture 6. Dimension Theory of Finite Modules over Noetherian Local Rings

Recall that by "finite module," we mean "finitely generated modules."

In the following, R will denote a Noetherian ring and M a finitely generated R-module.

Definition 6.1. The Krull dimension of $M \neq 0$ is defined as $\dim(M) := \dim(R/\operatorname{ann}(M))$. If $M = 0$, we define $\dim(M) = -1$.

(Recall that we had already defined the Krull dimension of a ring in Definition 4.15.)

Caution: Note that if R is a field and V a finite dimensional vector space over R, then $\dim(V)$ will not be the usual vector space dimension. If $V \neq 0$, then (Krull) $\dim(V) = 0$ since $R/\operatorname{ann}(M) = R$ and $\dim(\textbf{field}) = 0$. On the other hand, if $V = 0$, then (Krull) $\dim(V) = -1$, by the above convention.

Definition 6.2. The Chevalley dimension of $M \neq 0$ is defined as

$$c(M) := \inf\{n : \exists x_1, \ldots, x_n \text{ s.t. } l_R(M/(x_1, \ldots, x_n)M) < \infty\}.$$

For $M = 0$, we define $c(M) := -1$.

Note that if $M \neq 0$ and M has finite length, then $c(M) = 0$.

Definition 6.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module. The $\mathfrak{m}$-adic filtration of R is

$$\mathcal{F} : \quad R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots .$$

$\mathcal{R}(\mathfrak{m}) := \mathcal{R}(\mathcal{F}) = \bigoplus_{n=0}^{\infty} \mathfrak{m}^n t^n$.

Define $\mathcal{G}$ as the following filtration of M :

$$\mathcal{G} : \quad M \supset \mathfrak{m}M \supset \mathfrak{m}^2 M \supset \cdots .$$

$\mathcal{R}(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \mathfrak{m}^n M t^n$.

Since $\mathcal{G}$ is $\mathfrak{m}$-stable, we see that $\mathcal{R}(\mathcal{G})$ is a finitely generated module over the Noetherian ring $\mathcal{R}(\mathfrak{m})$.

We define the graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ and the $\mathrm{gr}_{\mathfrak{m}}(R)$-graded module $\mathrm{gr}_{\mathfrak{m}}(M)$ as

$$\mathrm{gr}_{\mathfrak{m}}(R) := \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathcal{R}(\mathfrak{m})/\mathfrak{m}\mathcal{R}(\mathfrak{m}),$$
$$\mathrm{gr}_{\mathfrak{m}}(M) := \bigoplus_{n=0}^{\infty} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M = \mathcal{R}(\mathcal{G})/\mathfrak{m}\mathcal{R}(\mathcal{G}).$$

Then, $\mathrm{gr}_{\mathfrak{m}}(M)$ is a finite $\mathrm{gr}_{\mathfrak{m}}(R)$-module.