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MA 406

General Topology

Notes By: Aryaman Maithani

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Lecture 1 (07-01-2021)

07 January 2021 15:04

Def^n. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties: (Topology)

(1) ϕ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(Open set)

Any $U \in \mathcal{T}$ is called an open set of X w.r.t. $\mathcal{T}$.
The pair $(X, \mathcal{T})$ or just the set X is called a topological space. (abuse of notat^n.)

Can reconcile the above with open sets in $\mathbb{R}$, or in general, any metric space X. That can be seen as a motivation for the definition.

Examples

(1) $X = \{a, b, c\}$

$\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ Can be seen (fairly easily) that this is a topology

$\mathcal{T}_2 = \{\phi, X\}$ trivial (pun intended, cf. next example)

(2) If X is any set, the collection of all subsets of X is a topology on X, it is called the discrete topology. ($\mathcal{T} = \mathcal{P}(X)$, that is) (Discrete topology)

The collection $\{\phi, X\}$ is also a topology on X called the indiscrete topology or trivial topology. (Indiscrete topology
Trivial topology)

(3) Let X be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\phi\}.$$

(Finite complement topology)

Then, $\mathcal{T}_f$ is a topology on X, called the finite complement topology on X.

- $\phi \in \mathcal{T}_f$ is clear. $X \in \mathcal{T}_f$ since $|X \setminus X| = 0 < \infty$.
- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be sets in $\mathcal{T}_f$. WLOG, $U_\alpha \neq \phi$ $\forall \alpha$.

Note $$X \setminus \left(\bigcup_\alpha U_\alpha\right) = X \cap \left(\bigcup_\alpha U_\alpha\right)^c$$
$$= \bigcap_\alpha (U_\alpha^c)$$

Note that each U_α^c is finite. $(U_\alpha \neq \phi)$

Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to $\bigcup_{i=1}^{n} (U_i^c)$ and conclude as earlier.

(Here, if some U_i were ϕ, then so would be the intersection.)

(If X is finite, the $\mathcal{T}_f = \mathcal{P}(X)$. Thus, we get discrete.)

(4) Let X be a set.

Let $\mathcal{T}_c$ be the collection of subsets such that $X \setminus U$ is either countable or all of X.

Called the co-countable topology.

(Generalising the previous)

(Cocountable topology
Co-countable topology)

Def^n Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X.

If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and that $\mathcal{T}$ is coarser than $\mathcal{T}'$!

If $\mathcal{T}' \supsetneq \mathcal{T}$, then the above is strictly finer and strictly coarser, respectively. (Finer, coarser, strictly finer, strictly coarser)

(The above gives us a way to compare two topologies)

EXAMPLE We have the usual topology on $\mathbb{R}$. ← strictly coarser than this
We also have the discrete topology on $\mathbb{R}$.

Def^n. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that (Basis)

(1) for each $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Note that in the above, $\mathcal{B}$ is just some collection of subsets of X satisfying (1) & (2). No topology is mentioned so far.

EXAMPLES

(1) $X = \mathbb{R}^2$, $\mathcal{B}$ is the collection of all discs w/o boundary.
(2) ——"—— ——"—— -rectangles ——"——
(3) Any X. The singletons form a basis.

We now get a topology out of a basis:

Def^n. If $\mathcal{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows: (Topology generated)

A subset U of X is said to be open if for every $x \in U$, there exists $B \in \mathcal{B}$ s.t.
$x \in B \subset U$.

$$x \in B \subset U.$$

(By "open" in above, we mean element of *T*. Same thing for what we see in the proof below.)

EXAMPLES (1) & (2) $\to$ gives standard topology on $\mathbb{R}^2$

(3) $\to$ gives discrete topology on X.

We still have to show that it is topology.

Proof.

- $\phi \in \mathcal{T}$ vacuously

  $X \in \mathcal{T}$ since given any $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.

  $B \subset X$ is by definition.

- Let $\{U_\alpha\}_{\alpha \in \Lambda \neq \phi}$ be open. Let $U := \bigcup_\alpha U_\alpha$.

  Fix $\alpha_0 \in \Lambda$.

  Let $x \in U$ be arbitrary. Then, $x \in U_{\alpha_0} \leftarrow$ open

  $\therefore \exists B \in \mathcal{B}$ s.t. $x \in B \subset U_{\alpha_0} \subset U$.

  $\therefore U \in \mathcal{T}$.

- Let U_1 and U_2 be open. Put $U := U_1 \cap U_2$.

  Let $x \in U$.

  Then $x \in U_1$ and $x \in U_2$

  $\Downarrow$ $\Downarrow$

  $\exists B_1 \in \mathcal{B}$ $\exists B_2 \in \mathcal{B}$

  s.t. $x \in B_1 \subset U_1$ s.t. $x \in B_2 \subset U_2$

  $$\therefore \underbrace{x \in B_1 \cap B_2}_{\Downarrow} \subset U_1 \cap U_2$$

  $$\exists B_3 \in \mathcal{B} \text{ s.t. } x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U.$$

  $\Rightarrow U \in \mathcal{T}$.

By induction, any finite intersection is in $\mathcal{T}$. $\blacksquare$

$n \geq 3$ $\left(\bigcap_{i=1}^{n} U_i = U_n \cap \left(\bigcap_{i=1}^{n-1} U_i \right) \right.$.

Lecture 2 (11-01-2021)

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Lemma 1 Let $\mathcal{B}$ be a basis and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then, $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Note that $\emptyset$ is empty union.

Proof. Given $\{U_\alpha\} \subset \mathcal{B}$, it is clear that $\bigcup U_\alpha \in \mathcal{T}$ since $\mathcal{T}$ is a topology and U_α are open. (By defn.)

Conversely, let $U \in \mathcal{T}$. Given any $x \in U$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. (By defn of $\mathcal{T}$.)

Thus,
$$\bigcup_{x \in U} B_x = U.$$
∎

($\subseteq$) since $B_x \subset U$

($\supseteq$) Each $x \in U$ is in B_x.

(Note that if $U = \emptyset$, the last union is the empty union!)

[The above gives us a way of extracting a basis $\mathcal{B}$ if we are already given a topology $\mathcal{T}$. Namely, pick any subcollection $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{T}$ is precisely the collection of all unions of elements of $\mathcal{B}$.]

Lemma 2. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$. (recall this means $\mathcal{T} \subset \mathcal{T}'$)

(ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \in B$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and $B \in \mathcal{B}$ be arbitrary.
Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.
Thus, $B \in \mathcal{T}'$. (by (i))

Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.
Thus, $B \in \mathcal{T}'$. (by (i))
Since B is open in $\mathcal{T}'$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.
(Def^n of top. generated.)

(ii) $\Rightarrow$ (i) Suppose $U \in \mathcal{T}$. We show that $U \in \mathcal{T}'$.
Let $x \in U$. By def^n of $\mathcal{T}$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
By (ii), $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B \subset U$.

Since x was arbit, we see that $U \in \mathcal{T}'$. (By def^n of $\mathcal{T}'$)
Thus, $\mathcal{T} \subset \mathcal{T}'$. $\boxtimes$

<u>Lemma 3</u>. Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X s.t. for each open set $U \subset X$ and each $x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.
Then $\mathcal{C}$ is a basis for the topology.

<u>Proof</u>. • Showing $\mathcal{C}$ is a basis.
(i) Given any $x \in X$, X is an open set containing x.
Thus, by hypothesis, $\exists C \in \mathcal{C}$ s.t. $x \in C$.

(ii) Let $C_1, C_2 \in \mathcal{C}$ s.t. $x \in C_1 \cap C_2$.
Note that C_1, C_2 are open and hence, $C_1 \cap C_2$ is open.
By hypothesis $\exists C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$.

Thus, $\mathcal{C}$ satisfies both properties of a topology.

• $\mathcal{C}$ generates the topology.
Let $\mathcal{T}$ denote the topology of X. Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$.

Let $U \in \mathcal{T}'$, then U is some union of elements of $\mathcal{C}$.

But elements of $\mathcal{C}$ are elements of $\mathcal{T}$ and thus, $U \in \mathcal{T}$.
($\mathcal{T}$ is topo.)

Thus, $\mathcal{T}' \subset \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. For each $x \in U$, $\exists C_x \in \mathcal{C}$ s.t.
$x \in C_x \subset U$.

As earlier,
$$U = \bigcup_{x \in U} C_x \in \mathcal{T}'$$

Thus, $\mathcal{T} \subset \mathcal{T}'$. ∎

Def^n. Let $\mathcal{B}$ be the collection of all bounded intervals.
That is,
$$\mathcal{B} = \{ (a, b) : -\infty < a < b < \infty \}.$$

$\mathcal{B}$ is a basis and the topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$. (Standard topology on \Bbb R)

If $\mathcal{B}'$ is the collection of all half open intervals of the form $[a, b)$, then $\mathcal{B}'$ is also a basis and the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$. (Lower limit topology on \Bbb R)

Lemma 4. The lower limit topology is strictly finer than the standard topology.

Proof. Let $\mathcal{T}$ denote the standard topology and $\mathcal{T}'$ the lower limit.

- $\mathcal{T} \subseteq \mathcal{T}'$. Let (a, b) be an arbit. basis element and let $x \in (a, b)$.
  Then, $[x, b)$ is a basis element for $\mathcal{T}'$ & $x \in [x, b) \subset (a, b)$.

Thus, $\mathcal{T} \subseteq \mathcal{T}'$, by Lemma 2.

- $\mathcal{T}' \neq \mathcal{T}$. Note that $[0, 1) \in \mathcal{T}'$.
  But given $0 \in [0, 1)$, there is no $(a,b) \ni 0$
  s.t. $(a, b) \subset [0, 1)$. ∎

Def^n A subbasis $\mathcal{S}$ for a topology is a collection of subsets of X whose union is X. (Subbasis, sub basis)

(Note that no topology given so far. Similar to what we saw for basis.)

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

We need to show that the topology defined above is actually a basis.

Let $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{S}$. We show $\mathcal{B}$ is a basis. (This suffices. Why? Lemma 1.)

(i) Let $x \in X$. Then, $\exists\, S \in \mathcal{S}$ s.t. $x \in S$. $\left(\because \bigcup_{S \in \mathcal{S}} S = X\right)$
But $S \in \mathcal{B}$.

(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$.
But note that $B_1 \cap B_2 \in \mathcal{B}$. (Why?)

Thus, both the conditions are satisfied.

Remark: The standard topology of $\mathbb{R}$ is also called the order topology on $\mathbb{R}$, because of the order relation of $\mathbb{R}$.

(We will see this in general, later.)

Def^n Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.

(Product topology)

(Open in the respective topologies, i.e.)

Note that $\mathcal{B}$ is a basis because:

(i) $X \times Y$ is itself a basis element

(ii) $U \times V,\ U' \times V' \in \mathcal{B} \Rightarrow (U \times V) \cap (U' \times V') = \underbrace{(U \cap U')}_{} \times \underbrace{(V \cap V')}_{} \in \mathcal{B}$

intersection of open sets

Note $\mathcal{B}$ itself won't be the topology. (In general)

Thm 5. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}_x$ on X, and $\mathcal{C}$ for $\mathcal{T}_y$ on Y, then the collection

$$\mathcal{D} = \{ B \times C : B \in \mathcal{B},\ C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We check that the hypotheses of Lemma 3 are satisfied.

Let $W \subset X \times Y$ be open and $(x, y) \in W$.
Then, by def^n of prod. top., $\exists\, U \in \mathcal{T}_x,\ V \in \mathcal{T}_y$ s.t.

$$(x, y) \in U \times V \subset W.$$

Since $\mathcal{B}$ is a basis for $\mathcal{T}_x$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

$\|^{ly}$ $\exists\, C \in \mathcal{C}$ s.t. $y \in C \subset V$.

$\Rightarrow (x, y) \in B \times C \subset U \times V \subset W$. ☒

$\uparrow$

$\mathcal{D}$

Lecture 3 (14-01-2021)

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By last lecture's discussion, we know that

$$\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$$

is a basis for the product topology on $\mathbb{R}^2$.
This is called the standard topology on $\mathbb{R}^2$.

Def^n. Given any two sets X and Y, we have the two projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given as

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y \quad \forall (x, y) \in X \times Y.$$

(Projections)

Note that $\pi_1^{-1}(U) = U \times Y$ for any $U \subseteq X$ and similarly $\pi_2^{-1}(V) = X \times V$ for any $V \subseteq Y$.

Thm 1. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \subseteq X \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \subseteq Y \text{ open}\}. \quad \subseteq \mathcal{P}(X \times Y)$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}_p$ denote the product topology on $X \times Y$.
Let $\mathcal{T}_s$ ——"—— topology generated by $\mathcal{S}$.

Note that any element of $\mathcal{S}$ is of the form $U \times Y$ or $X \times V$.
($U \subseteq X$ open, $V \subseteq Y$ open)

Thus, $\mathcal{S} \subseteq \mathcal{T}_p$ since both the above are actually basis elements. Since $\mathcal{T}_p$ is a topology, it is closed under arbitrary unions of finite intersections. Thus, $\mathcal{T}_s \subseteq \mathcal{T}_p$.

On the other hand, consider any arbitrary basis elt. of $\mathcal{T}_p$.
It is of the form $U \times V$. $U \subseteq X$, $V \subseteq Y$ open.
Note now

$$U \times V = \underbrace{\pi_1^{-1}(U)}_{\in \mathcal{S}} \cap \underbrace{\pi_2^{-1}(V)}_{\mathcal{S}} \in \mathcal{T}_S$$

Thus, $U \times V \subset \mathcal{T}_S$. Since $\mathcal{T}_S$ is a topology, arbitrary union of basis elements of $\mathcal{T}_p$ is in $\mathcal{T}_S$.
Thus, $\mathcal{T}_p \subseteq \mathcal{T}_S$. ∎

Def^n. Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.
With this topology, $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$.

(Subspace topology)

(We will often just say "Y is a subspace of X" if it is clear.)

We now check that $\mathcal{T}_Y$ is actually a topology.

Check. (i) Since $\phi \in \mathcal{T}$, we get $\phi = \phi \cap Y \in \mathcal{T}_Y$.
$\underset{\subseteq X}{Y} \quad Y = Y \cap X \in \mathcal{T}_Y$.

(ii, iii) Let $\{\mathcal{U}_i\}_{i \in I} \subseteq \mathcal{T}_Y$. Then, we have $\{\mathcal{U}_i'\}_{i \in I} \subseteq \mathcal{T}$ s.t.

$$\mathcal{U}_i' \cap Y = \mathcal{U}_i \qquad \forall i \in I.$$

Then,
$$\bigcup_{i \in I} \mathcal{U}_i = \bigcup_{i \in I} (\mathcal{U}_i' \cap Y) = \underbrace{\left(\bigcup_{i \in I} \mathcal{U}_i'\right)}_{\in \mathcal{T}} \cap Y$$

and similarly,
$$\bigcap_{i \in I} \mathcal{U}_i = \underbrace{\left(\bigcap_{i \in I} \mathcal{U}_i'\right)}_{\in} \cap Y.$$ ∎

and similarly, $\bigsqcup_{i \in I} \mathcal{U}_i = \left(\bigsqcup_{i \in I} \mathcal{U}_i \right) \cap Y$. ⊠

$\mathcal{T}$ if $|I| < \infty$

Lemma 2. If $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, then the collection

$$\mathcal{B}_Y = \{ B \cap Y : B \in \mathcal{B} \}$$

is a basis for $(Y, \mathcal{T}_Y)$, i.e., the subspace topology on Y.

Proof. We use Lemma 3 from Lecture 2.
Using that, it suffices to show that for any $\mathcal{U} \in \mathcal{T}_Y$ and any $x \in \mathcal{U}$, $\exists B \in \mathcal{B}_Y$ s.t. $x \in B \subset \mathcal{U}$.

To this end, let x, $\mathcal{U}$ be as given. Then,
$\mathcal{U} = \mathcal{U}' \cap Y$ for some $\mathcal{U}' \in \mathcal{T}$.
Clearly, $x \in \mathcal{U}'$.
Then, $\exists B' \in \mathcal{B}$ s.t. $x \in B' \subset \mathcal{U}'$. ($\mathcal{B}$ is a basis for $\mathcal{T}$.)

Then, $B = B' \cap Y \in \mathcal{B}_Y$ and
$x \in B \subset \mathcal{U}$. ⊠

Lemma 3. Let Y be a subspace of X. If $\mathcal{U}$ is open in Y and Y is open in X, then $\mathcal{U}$ is open in X.

Proof. $\mathcal{U} = \mathcal{U}' \cap Y$ for some $\mathcal{U}' \subset X$ open.
Since $\mathcal{U}'$ and Y are open in X, so is $\mathcal{U} = \mathcal{U}' \cap Y$. ⊠
(finite intersection of open sets.)

EXAMPLES. (i) If $Y = [0, 1] \subset X = \mathbb{R}$ in subspace topology, it has a basis given as
$$\{ (a, b) \cap Y : a < b \in \mathbb{R} \}.$$

More explicitly, here we have

$$(a, b) \cap Y = \begin{cases} (a, b) & a \in Y \ni b \\ [0, b) & a \notin Y \ni b \\ (a, 1] & a \in Y \not\ni b \\ \phi \text{ or } Y & a \notin Y \not\ni b \end{cases}$$

(2) Consider $Y = [0, 1) \cup \{2\} \subseteq \mathbb{R}$.

Note that

$$\{2\} = (1.5, 2.5) \cap Y.$$

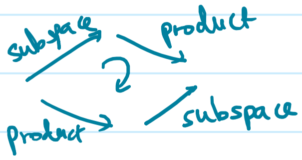
Thus, $\{2\}$ is open in Y. (Was not open in $\mathbb{R}$!)

Similarly, $[0, 1)$ is open in Y but not $\mathbb{R}$.

Thm 4. If A is a subspace of X and B of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Note that the above tells us that the following two ways of topolog-ising $A \times B$ are the same:

- consider A and B as spaces by themselves and give $A \times B$ the product topology
- consider the topological space $X \times Y$ in product topology. Note that $A \times B$ is a <u>subset</u> of $X \times Y$ and hence, can be given the subspace topology.



Proof. Note the following:

typical basis elt of $X \times Y$

$$\{ (U \times V) \cap (A \times B) : U \subseteq X, V \subseteq Y \text{ open} \}$$

basis for subspace topology on $A \times B$ by Lemma 2

$= \{ (U \cap A) \times (V \cap B) : U \subseteq X, V \subseteq Y \text{ open} \}$

$U \cap A$, $V \cap B$: a general open set in the subspace topology of $A \subseteq X$ or $B \subseteq Y$

basis for prod top. on $A \times B$

Thus, both the topologies have a common basis.

Def^n. A subset of a topological space is said to be closed if its complement is open.

(Closed set)

Examples. (1) $[a, b] \subseteq \mathbb{R}$ is closed because

$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open.

(2) $[0, \infty) \times [0, \infty) \subseteq \mathbb{R}^2$ is closed because

$\mathbb{R}^2 \setminus [0, \infty)^2 = \big((-\infty, 0) \times \mathbb{R} \big) \cup \big(\mathbb{R} \times (-\infty, 0) \big)$ is open.

(3) In the discrete topology, every set is open and hence, every set is closed.

(4) Consider $Y = [-1, 0] \cup (2, 3) \subseteq \mathbb{R}$.

Both $\underset{(-2,1) \cap Y}{\underset{\shortparallel}{[-1, 0]}}$ and $(2, 3)$ are open in Y.

Since they are complements of each other (in Y), we have that both the sets are closed as well, in Y.

Thm S. Let X be a topological space. Then,

(i) $\emptyset$ and X are closed,

(ii) arbitary intersection of closed sets is closed,

(iii) finite union of closed sets is closed.

Proof. $X \setminus \emptyset = X$, $X \setminus X = \emptyset$.

Proof. $X \setminus \phi = X, \quad X \setminus X = \phi.$

$$X \setminus \left(\bigcap_{i \in I} C_i \right) = \bigcup_{i \in I} (X \setminus C_i).$$

$$X \setminus \left(\bigcup_{i=1}^{n} C_i \right) = \bigcap_{i \in I} (X \setminus C_i).$$

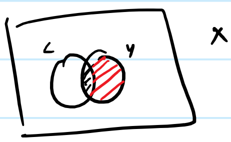
Conclude. ∎

Remark. The above is also a way to define a topology.

Thm 6. Let Y be a subspace of X and $A \subseteq Y$. Then, A is closed in Y iff A equals the intersection of a closed set (in X) with Y.

Proof. $(\Leftarrow)$ Suppose $A = C \cap Y$ for some closed set $C \subseteq X$.
Then,

$$Y \setminus A = \underbrace{\underset{\text{open in } X}{(X \setminus C)} \cap Y}_{\text{open in } Y}$$



$\therefore$ A is closed in Y.

$(\Rightarrow)$ Suppose A is closed in Y.
Then, $Y \setminus A$ is open in Y. Thus, $\exists\, \mathcal{U} \subseteq X$ open s.t.

$$Y \setminus A = \mathcal{U} \cap Y$$

Then,

$$\begin{aligned} \underset{A}{\underset{\parallel}{Y \setminus (Y \setminus A)}} &= Y \setminus (\mathcal{U} \cap Y) \\ &= Y \cap (\mathcal{U} \cap Y)^c \\ &= Y \cap (\mathcal{U}^c \cup Y^c) \\ &= (Y \cap \mathcal{U}^c) \cap (Y \cup Y^c) \end{aligned}$$

The $(\ )^c$ is complement in X.

$$\Rightarrow \quad A = Y \cap \mathcal{U}^c$$

Since $\mathcal{U}^c \subseteq X$ is closed, we are done.

Remark. A set can be both open and closed. For example, $\emptyset$ and X.

A less trivial example: Take $X = [0,1] \cup [2,3]$.

Then, $A = [0,1] \subset X$ is both open & closed.

Lecture 4 (18-01-2021)

18 January 2021 15:24

Def^n. Given a topological space X and $A \subset X$, we define:

(Interior) The interior of A as the union of all open sets contained in A.

Notation: int A or $\mathring{A}$. $(\phi \subset A)$

(Closure) The closure of A as the intersection of all closed sets containing A.

Notation: $cl(A)$ or $\bar{A}$. $(A \subset X)$

Remark. $\mathring{A}$ is an open set and $\bar{A}$ is a closed set. Further,

$$\mathring{A} \subset A \subset \bar{A}.$$

A is open iff $A = \mathring{A}$.

A is closed iff $A = \bar{A}$.

Def^n. Let $x \in X$. A neighbourhood of x is any set A such that there is an open set $U \subset X$ with

$$x \in U \subseteq A.$$

(Neighbourhood)

(That is, a neighbourhood is any set containing an open set containing the point. This is different from the def^n in Munkres!)

Thm 1. Let A be a subset of a topological space X.
Then, $x \in \bar{A}$ iff every neighbourhood U of x intersects A.

Proof. $(\Leftarrow)$ $x \notin \bar{A} \Rightarrow U = X \setminus \bar{A}$ is a nbd of x not intersecting A.

$(\Rightarrow)$ Suppose $\exists$ nbd C of x s.t. $C \cap A = \phi$.

Let U be open s.t. $x \in U \subseteq C$. (Def^n of nbd.)

Then, $X \setminus U$ is a closed set s.t. $A \subset X \setminus U$.

$\Rightarrow \bar{A} \subset X \setminus U$ (Why? $\because \bar{A}$ is the inter. of all closed sets cont. A.)

$\Rightarrow \bar{A} \cap U = \phi$

$\Rightarrow \quad \bar{A} \cap \mathcal{U} = \phi$

$\Rightarrow \quad x \notin \bar{A}.$ ☒

Examples. ① $X = \mathbb{R}$ and $A = (0,1]$. Then, $\bar{A} = [0,1]$.

However, if $X = (0,1] = A$, then $\bar{A} = A$.

② $X = \mathbb{R}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, $\bar{B} = B \cup \{0\}$.

③ $C = \{0\} \cup (1, 2)$. Then, $\bar{C} = \{0\} \cup [1,2]$

④ $\bar{\mathbb{Q}} = \mathbb{R}$

⑤ $\bar{\mathbb{N}} = \mathbb{N}$

⑥ $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{0\} = [0, \infty)$.

Def^n. Let X be a top. space and $A \subset X$. (Limit point)

A point $x \in X$ is said to be a limit point of A if every neighbourhood of x intersects A in some point other than x.

Notation : A'

Examples

	Subset of $\mathbb{R}$	Set of limit points
①	$(1, 2]$	$[1, 2]$
②	$\{\frac{1}{n} : n \in \mathbb{N}\}$	$\{0\}$
③	$\{0\} \cup (1,2)$	$[1,2]$
④	$\mathbb{Q}$	$\mathbb{R}$
⑤	$\mathbb{N}$	ϕ
⑥	$\mathbb{R}_+$	$\overline{\mathbb{R}_+}$

Thm 2. $\bar{A} = A \cup A'$. (Proof at the end)

Corollary 3. A is closed iff $A' \subset A$.

Proof. A is closed $\Leftrightarrow A = \bar{A} \underset{\text{Thm 2.}}{\Leftrightarrow} A' \subset A$.

Def^n. (Order relation or Simple order)

A relation C on set A is called an order relation (or a simple order) if it has the following properties:

(1) (Comparability) For every $x, y \in A$, $x \neq y \Rightarrow x C y$ or $y C x$.

(2) (Non reflexity) $\nexists\, x \in A$ s.t. $x C x$

(3) (Transitivity) $x C y$ and $y C z \Rightarrow x C z$.

A set with a simple order is called an ordered set.

Example. Usual '$<$' on $\mathbb{R}$ is a simple order.

Def^n. If X is a set and '$<$' a simple order relation. Then, we define "$x \leq y$" as "$x < y$ or $x = y$."

Let $A \subset X$. An element $a \in A$ is said to be the smallest element of A if

$$a \leq x \quad \forall\, x \in A.$$

Similarly, we define the largest element.

(We have used "the" since uniqueness is simple to check. Existence, however, is not guaranteed. $\mathbb{R}$ has no largest or smallest element. Neither does $(0, 1)$.)

Def^n. If $(X, <)$ is an ordered set, then for $a, b \in X$, we define the intervals

$$(a, b) := \{ x \in X : a < x < b \},$$
$$(a, b] := \{ x \in X : a < x \leq b \},$$
$$[a, b) := \{ x \in X : a < x < b \},$$
$$[a, b] := \{ x \in X : a < x < b \}.$$

(Intervals)

Def^n. (Order topology)

Let $(X, <)$ be an ordered set. Let $\mathcal{B}$ be the collection

Let $(X, <)$ be an ordered set. Let $\mathcal{B}$ be the collection of sets of the form:

(1) All (a, b) for $a, b \in X$,

(2) All $[a_0, b)$ for $b \in X$ where $a_0 \in X$ is the smallest element of X, if any.

(3) All $(a, b_0]$ for $a \in X$ where $b_0 \in X$ is the largest element of X, if any.

Then, $\mathcal{B}$ is a basis (check) and the topology generated is called the order topology on X.

Example. The standard topology on $\mathbb{R}$ is the order topology derived from the usual order on $\mathbb{R}$.

Def^n (Dictionary order)

Suppose that $(A, <_A)$ and $(B, <_B)$ are two ordered sets. We can define $<$ on $A \times B$ by

(we will denote elements of $A \times B$ by $a \times b$ instead of (a, b).)

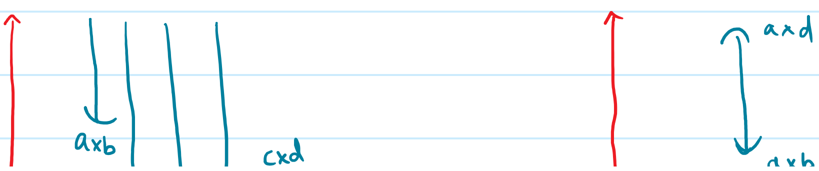
$$a_1 \times b_1 < a_2 \times b_2.$$

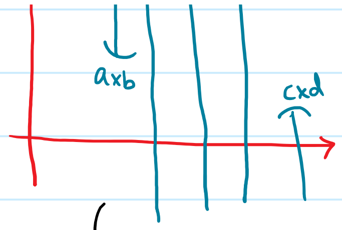
if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 < b_2$.

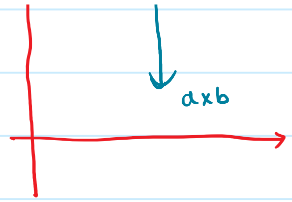
$<$ is a simple order on $A \times B$, called the dictionary order on $A \times B$.

Example $\mathbb{R} \times \mathbb{R}$ can be given an order topology in this dict. order. A basis will be

$\{(a \times b,\ c \times d)\}$ where $a < c$ or $a = c$ & $b < d$.







$x \times y \in (a \times b, c \times d)$ iff
- $x = a$ and $b < y$ or
- $a < x < c$ and $y \in \mathbb{R}$
- $x = c$ and $y < d$

Remark If $Y = [0, 1) \cup \{2\}$, then $\{2\}$ is NOT open in the order topology.

Note that any basis element containing is of the form $B = (a, 2]$ with $a \in Y$.

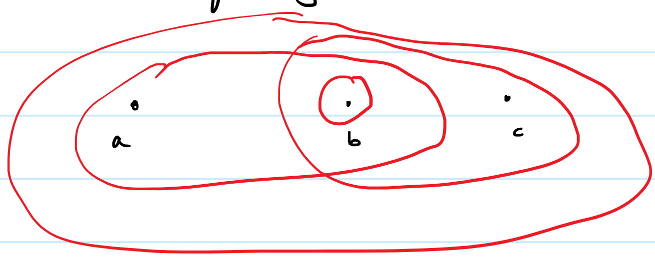
This means that $0 \le a < 1$ and hence, $\frac{a+1}{2} \in B$.

Thus, it always contains a point distinct from 2.

This shows that subspace and order topologies do not "commute"

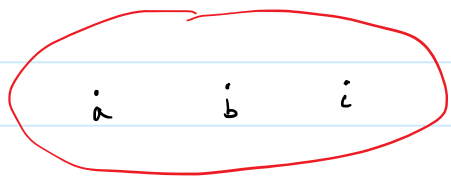
Remark. Singletons in $\mathbb{R}$ (or $\mathbb{R}^n$) are closed. This need not be true in general.

Consider the following topologies



$X = \{a, b, c\}$

$\mathcal{T} = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}$



$X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X\}$

$\{b\}$ is not closed in either of the above since $\{a, c\}$ is not open.

These spaces are not "nice". In fact, in the above spaces, a convergent sequence may have multiple limits. (Haven't defined this yet, though!)
We restrict ourselves to "nicer" spaces.

Def^n. A topological space X is called Hausdorff if for every distinct $x_1, x_2 \in X$, there exist neighbourhoods U_1, U_2 of x_1, x_2, respectively such that

$$U_1 \cap U_2 = \emptyset.$$

Thm 4. Every finite set in a Hausdorff space is closed.

Proof. It suffices to show the statement for singletons since finite unions of closed sets is closed.

Let $x_0 \in X$ be arbitrary. We show $\{x_0\}$ is closed.

Clearly, $\{x_0\} \subset \overline{\{x_0\}}$. Now, consider $y \in \{x_0\}^c$.

That is, $y \neq x_0$. By Hausdorffness, $\exists\, U_1, U_2$ nbds s.t.

$$x \in U_1, \quad y \in U_2 \quad \text{and} \quad U_1 \cap U_2 = \emptyset.$$

Thus, $U_2 \cap \{x_0\} = \emptyset$. Thus, $y \notin \overline{\{x_0\}}$. ∎

(Thm 1)

Proof of Thm 2. $\bar{A} = A \cup A'$.

($\subseteq$) Let $x \in \bar{A}$. Suppose $x \notin A$. We show $x \in A'$.

Let U be an arbit. nbd of x.

By Thm 1, $U \cap A \neq \emptyset$.

By assumption, $x \notin U \cap A$.

Thus, $x \in A'$, by def^n of A'.

($\supseteq$) $A \subset \bar{A}$ is clear. $A' \subset \bar{A}$ is also clear by def^n of A' and Thm 1. ∎

Lecture 5 (21-01-2021)

21 January 2021 15:36

Thm 1. Let X be a Hausdorff space, $A \subset X$, and $x \in X$. Then, $x \in \bar{A} \Leftrightarrow$ every nbd of x contains infinitely many points of A.

Proof. ($\Leftarrow$) Trivial since infinitely many points imply one point apart from x.

($\Rightarrow$) Let x be a limit point.

For the sake of contradiction, let $N \supseteq U \ni x$ (open) be a nbd of x s.t. $A \cap (U \setminus \{x\}) = \{x_1, \ldots, x_n\}$ is finite.

Note $\{x_1, \ldots, x_n\}$ is closed since X is Hausdorff.

Thus, $V := U \cap (X \setminus \{x_1, \ldots, x_n\})$ is a nbd of x.

But $V \cap (A \setminus \{x\}) = \phi$. $\rightarrow\leftarrow$ ☒

Note this makes sense even if $x \notin A$.

Recall from tutorial:

(1) Order top. is Hausdorff.

(2) Product of Hausdorff spaces is Hausdorff.

(3) Subspace of Hausdorff spaces is Hausdorff

Def^n Continuous functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A function $f: X \to Y$ is said to be **continuous** if $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_Y$.

In other words, inverse image of open sets (in Y) is open (in X).

Remark. By our earlier discussions, it is easily to see that it suffices to check that inverse images of basis (or subbasis) elements are open.

Recall $$f^{-1}\left(\bigcup_{\alpha\in\Lambda} B_\alpha\right) = \bigcup_{\alpha\in\Lambda} f^{-1}(B_\alpha)$$
$$f^{-1}\left(\bigcap_{\alpha\in\Lambda} B_\alpha\right) = \bigcap_{\alpha\in\Lambda} f^{-1}(B_\alpha)$$

Examples. (i) $f : \mathbb{R} \to \mathbb{R}_\ell$, $f(x) := x$ is not continuous since the topology of $\mathbb{R}_\ell$ is strictly finer.

(ii) $g : \mathbb{R}_\ell \to \mathbb{R}$, $g(x) := x$ is continuous.

Thm 2. Let X and Y be top. spaces and $f : X \to Y$.
TFAE
(i) f is continuous.
(ii) For every $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.
(iii) $f^{-1}(B)$ is closed for every closed $B \subset Y$.

Proof. (i) $\Rightarrow$ (ii)

Let $y \in f(\bar{A})$. Then, $y = f(x)$ for some $x \in \bar{A}$.
We show $x \in \overline{f(A)}$.
Let V be any open nbd. of y. (Want to show $V \cap f(A) \neq \phi$)
Then, $f^{-1}(V)$ is an open nbd of x.
Then, $A \cap f^{-1}(V) \neq \phi$. Let $x' \in A \cap f^{-1}(V)$.
Then, $f(x') \in f(A) \cap f(f^{-1}(V))$
$\Rightarrow f(x') \in f(A) \cap V$ (*)
Thus, $f(A) \cap V \neq \phi$, as desired.
Since any nbd contains an open nbd, we are done.

(ii) $\Rightarrow$ (iii) Let $B \subset Y$ be closed.
Put $A = f^{-1}(B)$. To show: A is closed.
A is closed $\Leftrightarrow A = \bar{A} \Leftrightarrow \bar{A} \subset A$.

$x \in \bar{A} \Rightarrow f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$ (*)

$\Rightarrow x \in f^{-1}(B)$

$\Rightarrow x \in A$.

(*) $f(f^{-1}(B)) \subset B$, in general. Equality if f onto.

(iii) $\Rightarrow$ (i) Obvious since $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$. ☒

Def^n. Let X and Y be topological spaces and $f: X \to Y$ be a bijection. f is said to be a homeomorphism if f and f^{-1} are both continuous.

X and Y are said to be homeomorphic if there exists a homeomorphism $f: X \to Y$.

(Homeomorphism, homeomorphic)

A homeomorphism can also be defined as a bijection $f: X \to Y$ s.t. $f(U)$ is open in Y iff U is open in X.

Thus, f is not only a bijection of X and Y but also of $\mathcal{T}_X$ and $\mathcal{T}_Y$.

Def^n. Let $f: X \to Y$ be an injective continuous function. Let $Z = f(X)$ be the image of X in the supspace topology. Then, the restriction

$f': X \to Z$ is a bijection.

If f' is a homeomorphism, then we say that $f: X \to Y$ is a topological imbedding or an imbedding of X in Y.

(Imbedding)

Example (i) $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) := 2x + 4$ is a homeomorphism.

(ii) $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined $f(x) := \tan x$ is a homeomorphism.

(iii) $g : \mathbb{R}_\ell \to \mathbb{R}$ defined $g(x) := x$ is bijective and continuous but not a homeomorphism.

(iv) Let $S^1 := \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ be in subspace topology of $\mathbb{R}^2$.
Let $f : [0,1) \to S^1$ be defined by

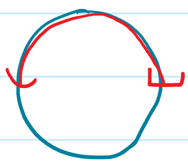
$$f(t) = (\cos 2\pi t, \ \sin 2\pi t).$$

The f is bijective and continuous but f^{-1} is not continuous. To see the last part, consider

$$U = [0, 1/2) \in [0,1).$$

U is open but $f(U) \to$ top arc of S^1

not open in S^1



note that $1 \times 0 \in$ top arc but no basis elt around that point.

Thm 3. Let X, Y, and Z be topological spaces.
If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof. Use $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. ∎

Def^n Box topology, Product Topology

Let J be an indexing set and $\{X_\alpha\}_{\alpha \in J}$ a collection of topological spaces.
Let us consider a basis for a topology on the Cartesian product

$$\prod_{\alpha \in J} X_\alpha ,$$

the collection of all set of the form

$$\prod_{\alpha \in J} U_\alpha ,$$

where each U_α is open in X_α. The topology induced is called the box topology.

Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \longrightarrow X_\beta$ be the projection map

$$\pi_\beta \left((x_\alpha)_{\alpha \in J} \right) = x_\beta .$$

Let $S_\beta = \{ \pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta \}$ and let

$$S = \bigcup_{\beta \in J} S_\beta .$$

Then S is a subbasis for a topology on $\prod_{\alpha \in J} X_\alpha$.
The topology generated is called the product topology.

Remark. ① A typical basis elt. for prod. topology is

$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

[$\beta_1, \dots, \beta_n$ p-wise distinct]

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \begin{cases} U_{\beta_i} ; & \alpha = \beta_i \\ \dots \end{cases}$$

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \begin{cases} U_{\beta_i} & ; \alpha = \beta_i \\ X_\alpha & ; \text{else} \end{cases}$$

② If J is finite, both box and product coincide.

③ In general, box topology is finer than product.
If $|J| = \infty$, then it can be strictly finer.

(If each $X_\alpha = \mathbb{R}$, then strictly finer.
If each $X_\alpha = \{0\}$, then not.
If each X_α is in indsicrete topology, then not.)

Lecture 6 (25-01-2021)

25 January 2021 15:37

Thm 1. The box topology is finer than the product topology.

Proof. Every basis element of prod. topology is also one of box. ☒

Remarks (1) For finite products, the two are the same.

(2) If we simply refer to the product space, we shall mean the product topology, by default.

Thm 2. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then, f is continuous iff each f_α is continuous

Proof. Note that $\pi_\beta : \prod X_\alpha \to X_\beta$ is continuous $\forall \beta$ since each $\pi_\beta^{-1}(U_\beta)$ is a subbasis element

($\Rightarrow$) Now, suppose that $f : A \to \prod X_\alpha$ is continuous.

So, $f_\alpha = \pi_\alpha \circ f$ is continuous $\forall \alpha$.

($\Leftarrow$) Conversely, suppose each f_α is continuous.

It suffices to show that inverse images of subbasis elements are open.

A typical subbasis elt is $\pi_\alpha^{-1}(U_\alpha)$ for U_α open in X_α.

But $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha)$ ☒

open since f_α is continuous.

Remark. Above not true for box topology.

Take

$$f: \mathbb{R} \longrightarrow \prod_{i=1}^{\infty} \mathbb{R} \quad \text{given by}$$

$$t \longmapsto (t, t, \dots) \quad \text{is not continuous in box.}$$

Consider the open set $U = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$.

Def^n. A metric d on a set X is a function

$$d: X \times X \longrightarrow \mathbb{R} \quad \text{satisfying}$$

(1) $d(x, y) \geq 0$ and $d(x,y) = 0 \iff x = y$.

(2) $d(x, y) = d(y, x)$

(3) $d(x, z) \leq d(x, y) + d(y, z)$

For a metric d on X, the number $d(x, y)$ is called the distance between x and y in metric d.

Given $\varepsilon > 0$, the set

$$B_d(x, \varepsilon) = \{ y \in X \mid d(x, y) < \varepsilon \}$$

is called the ε-ball centered at x.

We often write $B(x, \varepsilon)$ if d is understood.

The collection $\mathcal{B} = \{ B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0 \}$ is a basis and the topology induced is called the metric topology on X.

A topological space is called metrisable if there exists a metric on X which induces the given topology on X.

Examples. (1) Given a set X, define

$$d(x, y) = \begin{cases} 1 & ; x \neq y, \\ 0 & ; x = y. \end{cases}$$

This d is a metric and the topology induced is the discrete topology, since $B(x, 1) = \{x\}$.
(Thus, singletons are open and thus, every set is.)

(2) Standard topology on $\mathbb{R}$ is induced by
$$d(x, y) := |x - y|.$$

Note
$$(a, b) = B_d(x, \varepsilon) \quad \text{for} \quad x = \frac{a+b}{2} \text{ and}$$
$(a < b)$
$$\varepsilon = \frac{b-a}{2}.$$

(3) On $\mathbb{R}^n$, we have the Euclidean metric given by
$$d(x, y) = \|x - y\| = \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{1/2}.$$
Another example is the square metric
$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Both the metrics induce the same topology, which is the same as the usual product topology.

Thm 3. Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \to Y$ a function. Then,
f is continuous $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

Proof. Exercise. ☒

Defn. (Sequence and convergence)
Let X be a set. A sequence $(x_n)_{n=1}^{\infty}$ is a function $\mathbb{N} \to X$. $(n \mapsto x_n)$

It is said to converge to $x \in X$ if for every nbd U of x, $\exists n_0 \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq n_0$.
It is said to converge or be convergent if it converges to some $x \in X$.

Lemma. Let X be a topological space and $A \subset X$.
If $\exists$ a seq. $(x_n)_{n=1}^{\infty} \subset A$ which converges to $x \in X$, then $x \in \bar{A}$.
The converse is true if X is metrisable.

Proof. ($\Rightarrow$) Let $(x_n)_{n=1}^{\infty}$ and x be as in Lemma. Let U be an arbitrary nbd of x. We show $U \cap A \neq \phi$ to conclude.

By def^n of convergence, $\exists n_0 \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq n_0$.
Thus, $\phi \neq U \cap A$ since $x_{n_0+1} \in U \cap A$.

($\Leftarrow$) Assume d metrises X and $x \in \bar{A}$.
For each $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \cap A \neq \phi$.
For each $n \in \mathbb{N}$, pick $x_n \in B(x, \frac{1}{n})$. (Need some choice.)
Then, $d(x, x_n) < \frac{1}{n} \to 0$ and thus,

$$x_n \to x.$$ ☒

(Note: An easy check that convergence of sequences in metric space coincides.)

Def^n. A space X is said to have a countable basis at x if there is a countable collection $\mathcal{B}$ of open nbds of x s.t. each nbd of x contains an element $\mathcal{B}$. A space that has a countable basis at each $x \in X$ is said to be first countable.

Eg. $\mathbb{R}$, $\mathbb{R}^n$, take $\{ B(x, \frac{1}{n}) \mid n \in \mathbb{N} \}$ at each x.

Lemma $4\frac{1}{2}$. The converse of lemma 4 holds even if X is first countable. More generally, if $x \in \bar{A}$, then only countable basis at x is required.

Lecture 7 (28-01-2021)

28 January 2021 15:34

Thm 1. Let X, Y be topological spaces and $f: X \to Y$ be continuous.
(Suppose $x_n \to x$ in X. Then, $f(x_n) \to f(x)$ in Y.) $(*)$

If X is metrisable, then $(*)$ implies continuity.

(That is, if $f(x_n) \to f(x)$ for every convergent subsequence $x_n \to x$ for every $x \in X$, then f is continuous.)

Proof. Let $x_n \to x$ in X.

Let U be an arbitrary neighbourhood of $f(x)$.

Then, $f^{-1}(U)$ is a nbd of x.

Thus, $\exists N \in \mathbb{N}$ s.t. $x_n \in f^{-1}(U)$ $\forall n \geq N$.

Thus, $f(x_n) \in U$ $\forall n \geq N$ proving that $f(x_n) \to f(x)$.

Now, suppose that X is Hausdorff.

Assume that $(*)$ is satisfied.

It suffices to show $f(\bar{A}) \subset \overline{f(A)}$.

Let $A \subset X$ be arbitrary and let $y \in f(\bar{A})$.

Then, $y = f(x)$ for some $x \in \bar{A}$.

Thus, $\exists (x_n) \subseteq A$ s.t. $x_n \to x$. (Lemma 4 from Lec 6, X is metrisable.)

By our condition, $f(x_n) \to f(x)$ and $f(x_n) \in f(A)$.

Thus, $y = f(x) \in \overline{f(A)}$. (In general.) ∎

Remark. As in Lec 6, the "metrisable" can be relaxed to first countability.

Thm 2. If X is a topological space and $f, g : X \to \mathbb{R}$ are continuous, then $f \pm g$, $f \cdot g$ are continuous.

If $g(x) \neq 0$ $\forall x \in X$, then f/g is continuous.

Proof. $+, -, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

$x \mapsto 1/x$ is continuous $\mathbb{R}\setminus\{0\} \to \mathbb{R}$.

Since $f \times g : X \to \mathbb{R} \times \mathbb{R}$ is continuous, we are done. ∎

Defn. $\mathbb{R}^\omega := \prod_{n\in\mathbb{N}} X_n$ where $X_n = \mathbb{R}$ for all $n \in \mathbb{N}$.

Example (1) Let $f: \mathbb{R} \to \mathbb{R}^\omega$ be defined by
$f(t) := (t, t, \dots)$.

Then, (a) f is continuous if $\mathbb{R}^\omega$ is equipped with prod topology.

Thm 2 of Lec 6

(b) f is NOT continuous if $\mathbb{R}^\omega$ has box topology.

Note $B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$
$= \prod_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n})$ is a basis elt. of box topology.

Thus, Thm 2 of Lec 6 is not true

Then, $f^{-1}(B) = \{0\}$ is not open in $\mathbb{R}$.

(2) Again, take $\mathbb{R}^\omega$ in box topology.

Let $A = \{(x_1, x_2, \dots) \mid x_i > 0 \ \forall i\}$.

Then, $\underline{0} \in \bar{A}$. (Here $\underline{0} = (0, 0, \dots) \in \mathbb{R}^\omega$.)

std basis elt. around it: $B = (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) \times \cdots$

Then, $\underline{x} = (\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}, \dots) \in B \cap \bar{A}$.

Claim. $\nexists\ (x_n) \subset A$ s.t. $\underline{x_n} \to \underline{0}$.

Proof. Assume not. Let $(\underline{x_n}) \subset A$ be s.t. $\underline{x_n} \to 0$.

Note that $\underline{x_n} = (x_{1n}, x_{2n}, \dots)$ where $x_{in} > 0\ \forall i$.

Define
$B = (-y_{11}, y_{11}) \times (-y_{22}, y_{22}) \times \cdots$.

Clearly, $\underline{0} \in B$ but $y_n \notin B\ \forall n$.

Thus, $y_n \not\to 0$.

Cor 3. $\mathbb{R}^\omega$ in box topology is not metrisable.
(Lemma 4 from Lec 6.)

Lemma 4. Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the metric

$$\rho(x,y) = \max\{|x_i - y_i| : 1 \le i \le n\}.$$

Then, ρ is a metric which induces the standard (product) topology on $\mathbb{R}^n$.

(The proof that ρ is indeed a metric is omitted.)

Proof. Let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a std. basis elt of prod. topology. Let $x = (x_1, \ldots, x_n) \in B$.

For each $i = 1, \ldots, n$, pick $\epsilon_i^{>0}$ s.t. $(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$.

Then, put $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$.

Then, $x \in B_d(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$

$\subseteq (x_1 - \epsilon_1, x_1 + \epsilon_1) \times \cdots \times (x_n - \epsilon_n, x_n + \epsilon_n)$

$\subseteq B.$

Conversely, each ϵ ball in the metric topology is a basis element of the product topology. ⊠

Def^n. (Connected, separation)

Let X be a topological space.

A *separation* of X is a pair U, V of non-empty disjoint open subsets of X such that $U \cup V = X$.

X is said to be *connected* if no separation exists.

Lemma 5. A space X is connected iff the only clopen (closed as well as open) subsets of X are $\emptyset$ and X.

Proof. $(\Rightarrow)$ Let U be a clopen set s.t. $\emptyset \ne U \ne X$.

Then, $V = U^c$ is also clopen and nonempty.

Then $X = U \cup V$. $\rightarrow\leftarrow$

$(\Leftarrow)$ Suppose X is not connected. Let U, V be a separation, then $\emptyset \neq U \neq X$ and $U = V^c$ is clopen. ☒

(Ex.) Let Y be a subspace of X and $A \subset Y$.
Then $\bar{A} \cap Y$ is the closure of A in Y.

Thm 6. A pair of disjoint non-empty sets A and B whose union is Y is a separation of Y iff neither contains a limit point of the other.

Proof. ($\Rightarrow$) Then, $A = \underbrace{cl_Y(A)}_{\text{closure in } Y} = \bar{A} \cap Y$. (in X)

Claim. $\bar{A} \cap B = \emptyset$

Proof. $A = \bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B)$
$= A \cup (\bar{A} \cap B)$

Thus, $\bar{A} \cap B \subset A$.

Thus, $(\bar{A} \cap B) \cap A = \bar{A} \cap B$
$= \bar{A} \cap (B \cap A) = \emptyset$

Similarly, $A \cap \bar{B} = \emptyset$, as desired.

($\Leftarrow$) We only need to show that A and B are open in Y.
Equivalently, it suffices to show that A and B are closed in Y.
We know $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.
Thus, $\bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B)$
$= A \cup \emptyset = A$.

Thus, $cl_Y(A) = A$. Thus, A is closed in Y.
$\quad\quad$ "ly, B — " — . ☒

EXAMPLES

(1) Any set in indiscrete topology is connected.

(2) $Y = (0,5) \cup (5,7) \subseteq \mathbb{R}$ is not connected.

$(0,5), (5,7)$ form a separation.

(3) $Y = (0,5] \cup (5,7) = (0,7)$.

$(0,5], (5,7)$ does NOT form a separation.

Note $(0,5]$ contains the limit point 5 of $(5,7)$.

Aliter: $(0,5]$ is not open in Y.

Later, we shall see that intervals in $\mathbb{R}$ are connected.

(4) $\mathbb{Q}$ is not connected. Let $I = (\sqrt{2}, \infty) \subset \mathbb{R}$.

$I \cap \mathbb{Q}$ is clearly open in $\mathbb{Q}$ since I is open in $\mathbb{R}$.

Now, $\mathbb{Q} \setminus (I \cap \mathbb{Q}) = (\mathbb{R} \setminus I) \cap \mathbb{Q}$

$= (-\infty, \sqrt{2}] \cap \mathbb{Q}$ $\quad \sqrt{2} \notin \mathbb{Q}$

$= (-\infty, \sqrt{2}) \cap \mathbb{Q}$

also open. ∎

(5) Let $A = \mathbb{R} \times \{0\}$ and $B = \{(x,y) : x > 0, y = \frac{1}{x}\}$.

Put $Y = A \cup B \subseteq \mathbb{R}^2$ in subspace topology.

Then, A and B are closed in $\mathbb{R}^2$ and hence, in Y.

Since $A \cap B = \emptyset$, we are done. $(A \neq \emptyset \neq B)$

Lecture 8 (01-02-2021)

01 February 2021 20:58

Lemma 1. If the sets C and D form a separation of X and if Y is a connected subspace of X, then $Y \subseteq C$ or $Y \subseteq D$.

Proof. Note that $(Y \cap C) \cup (Y \cap D) = Y$ and $(Y \cap C) \cap (Y \cap D) = \phi$ with $Y \cap C$ and $Y \cap D$ open in Y. Thus, one must be empty. $Y \cap D = \phi \Rightarrow Y \subset C$ and $Y \cap C = \phi \Rightarrow Y \subseteq D$. ☒

Proof. Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of connected spaces.

Pick $p \in \bigcap A_\alpha$.

Put $Y = \bigcup A_\alpha$. Suppose, for the sake of contradiction, that $Y = C \cup D$ is a separation.

WLOG, $p \in C$. $(\therefore p \notin D)$

Now, given any $\alpha \in I$, we must have $A_\alpha \subset C$, by the previous theorem.

Thus, $A_\alpha \subset C$ $\forall \alpha$. Thus, $Y \subset C$ and hence, $D = \phi$. $\rightarrow\leftarrow$

Thm 3. If $A \subset X$ is connected and $B \subset X$ is such that $A \subset B \subset \bar{A}$, then B is connected.

In particular, $\bar{A}$ is connected.

Proof. Suppose $B = C \cup D$ is a separation.

Then, $A \subset C$ wlog. (A is connected.)

Thus, $\bar{A} \subset \bar{C}$. Moreover, $\bar{C} \cap D = \phi$, since (C, D) form a separation. Thus, $\bar{A} \cap D = \phi$.

(Thm 6, last lec)

form a separation. (Thm 6, last lec.) Thus, $\bar{A} \cap D = \phi$.

$$\underset{\cup\!\!\shortparallel}{\bar{A} \cap D} = \phi, \quad B \cap D = D$$

Thus, $D = \phi$. ∎

Thm 4. Let $f : X \to Y$ be continuous. If X is connected, then $f(X)$ is connected.

Proof. Put $Z = f(X)$. Then, we get a function $f : X \to Z$.
Moreover, this new f is still continuous. (Z in subspace topology.)

(If $U \subseteq Z$ is open, then $U = V \cap Z$ for V open in Y.
Then, $f^{-1}(U) = f^{-1}(V \cap Z) = f^{-1}(V) \cap f^{-1}(Z) = f^{-1}(V) \cap X$
$= f^{-1}(V) \to$ open.)

We now look at the surjective map $f : X \to Z$.
Suppose $Z = A \cup B$ is a separation.

$$\text{Then, } \underset{X}{\underset{\shortparallel}{f^{-1}(Z)}} = f^{-1}(A) \cup f^{-1}(B) \quad \text{and} \quad f^{-1}(A) \cap f^{-1}(B) = \phi.$$

non-empty and open

$\to\leftarrow$

Thm 5. Cartesian product of finitely many connected spaces is connected.

Proof. Since $X_1 \times \cdots \times X_n \simeq (X_1 \times \cdots \times X_{n-1}) \times X_n$ for $n \geq 3$, it suffices to prove for $n = 2$.

Let X and Y be connected, we say $X \times Y$ is connected.
Fix a point $(a, b) \in X \times Y$. Let $x \in X$ be arbit.
Consider the connected sets $\{x\} \times Y$ $(\simeq Y)$ and $X \times \{b\}$ $(\simeq X)$.
Moreover, the slices have (x, b) in common. Thus,

$$T_x = (\{x\} \times Y) \cup (X \times \{b\})$$

is connected for each $x \in X$.
However, note that $(a, b) \in T_x \quad \forall x \in X$.

Thus, $\bigcup_{x \in X} T_x$ is connected. But $X \times Y = \bigcup_{x \in X} T_x$, as desired. ☒

Def^n. (Path, path-connected)

Let X be a topological space and $x, y \in X$.
A path from x to y in X is a function $f: [0,1] \to X$ s.t. $f(0) = x$ and $f(1) = y$.
X is said to be path-connected if for any $x, y \in X$, there exists a path from x to y.

(Usually we may take $[a,b]$ instead of $[0,1]$.)

Fact: Intervals in $\mathbb{R}$ are connected. (Recall from $\mathbb{R}$ Analysis.)

Thm 6. Any path connected space is connected.

Proof. Suppose X is path-connected and $X = A \cup B$ is a sep.
Pick $x \in A$ and $y \in B$. By hypothesis, $\exists f: [0,1] \to X$ s.t. $f(0) = x$ & $f(1) = y$.
But $[0,1]$ is connected and thus, so is $f([0,1])$.
Thus, by Lemma 1, $f([0,1]) \subset A$ or $f([0,1]) \subset B$. $\rightarrow\leftarrow$

Examples. (1) The unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| < 1\} \subseteq \mathbb{R}^n$ is path-connected. (The straight line path works.)

(2) $\mathbb{R}^n \setminus \{\underline{0}\}$ is path-connected if $n > 1$.

Proof. Let $x, y \in \mathbb{R}^n \setminus \{\underline{0}\}$. If $\underline{0}$ does not lie on the line seg. joining x and y, take that line seg.
Else, pick z not on line and join x to z and z to y. ∎

If $n=1$, then $\mathbb{R}\setminus\{0\} = (-\infty, 0)\cup(0,\infty)$ is not even connected, let alone path-connected.

(3) For $n \geq 2$, define $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \subseteq \mathbb{R}^n$.
It is path-connected. To see this, define

$$g : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \text{ by}$$
$$x \mapsto x/\|x\|.$$

Then, g is continuous and maps $\mathbb{R}^n\setminus\{0\}$ <u>onto</u> S^{n-1}.
The image of a path-connected space is path-connected and hence, S^{n-1} is path-connected.
(Ex.)

(Ex.) Continuous image of path connected space is path-connected.

Proof. Let $g : X \to Z$ be continuous and onto.
Pick $z_1, z_2 \in Z$. Then, $\exists x_1, x_2 \in X$ s.t. $x_1 \mapsto z_1$ & $x_2 \mapsto z_2$.
Now, $\exists \gamma : [0,1] \to X$ s.t. $x_1 \xrightarrow{\gamma} x_2$.

Then, $g\circ\gamma : [0,1] \to Z$ is continuous

and $(g\circ\gamma)(0) = z_1$ & $(g\circ\gamma)(1) = z_2$. ☒

Lecture 9 (08-02-2021)

08 February 2021 15:32

(4) Let $S = \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$.

The set $\bar{S}$ is called the topologist's sine curve.

(Topologist's sine curve)

$\bar{S} = S \cup \{0\} \times [-1, 1]$.

Note that S is connected, being the image of a connected set $(0, 1]$ under a continuous map $x \mapsto (x, \sin\frac{1}{x})$.

As seen, this implies $\bar{S}$ is connected.

Claim. However, $\bar{S}$ is not path-connected.

Proof. Suppose not. Let $f : [a, c] \to \bar{S}$ be a path from $(0, 0)$ to $(1, \sin 1)$.

Let $D = f^{-1}(\{0\} \times [-1, 1])$. $D \subset [a, c]$ is closed.

Thus, $b = \sup D \in D$.

$\therefore f : [b, c] \to \bar{S}$ has the property that $f(b) \in \{0\} \times [-1, 1]$ but $f(x) \in S$ for $x > b$.

WLOG, $[b, c] = [0, 1]$. Write $f(t) = (x(t), y(t))$.

Claim. $\exists (t_n) \subset (0, 1]$ s.t. $t_n \to 0$ and $y(t_n) = (-1)^n$.

Proof. For $n \in \mathbb{N}$, $x(1/n) > 0$. Thus, we can choose u_n s.t. $0 < u_n < x(1/n)$ and $\sin(1/u_n) = (-1)^n$.

By IVT, $\exists t_n$ s.t. $0 < t_n < 1/n$ and $x(t_n) = u_n$.

Thus, $y(t_n) = \sin(1/x(t_n)) = \sin(1/u_n) = (-1)^n$.

$0 < t_n < 1/n \Rightarrow t_n \to 0$. ☒

Thus, $t_n \to 0$ and $y(t_n)$ does not converge. Thus, y is not continuous. Therefore, f is not continuous. $\rightarrow\leftarrow$ ☒

Defn. (Connected components)

Given X, define the equivalence relation $x \sim y$ if $\exists$ a

connected subset of X containing x and y.
The equivalence classes are called the **components** or **connected components**.

Remark. Reflexive: $\{x\}$ is connected.
Sym.: Trivial.
Transitive: let $x \sim y$ and $y \sim z$. $\exists A, B \subset X$ connected s.t. $x, y \in A$ and $y, z \in B$. Then, $A \cup B$ is connected since $y \in A \cap B$. But $x, z \in A \cup B$. $\therefore x \sim z$.

Thm 1. The components of X are connected disjoint subsets of X whose union is X, s.t. each connected subset of X intersects only one of them.

Proof. The part about being disjoint and union being X follows because $\sim$ was an equiv. relation.
Now, suppose A is a connected set s.t. A intersects the components C_1 and C_2. Let $x_1 \in A \cap C_1$ and $x_2 \in A \cap C_2$.
But then, $x_1, x_2 \in A$ and hence, $x_1 \sim x_2$. $\therefore C_1 = C_2$.
This proves the second part.
We just have to prove that each component C is connected.
Fix $x_0 \in C$. $\forall x \in C$, $x_0 \sim x$. $\therefore \exists A_x$ s.t. $x, x_0 \in A_x$ and A_x connected. By the earlier part, $A_x \subset C$.
$\therefore A_x \subset C \quad \forall x$

$$\Rightarrow C = \bigcup_{x \in C} A_x. \qquad \text{But } \bigcap_{x \in C} A_x \ni x_0.$$

$$\therefore C = \bigcup_{x \in C} A_x \text{ is connected.} \qquad \blacksquare$$

Def^n. (Cover, open cover) A collection $\mathcal{U}$ of subsets of X is said to be a **cover** of X if $\bigcup_{U \in \mathcal{U}} U = X$.

If each $U \in \mathcal{U}$ is open, then $\mathcal{U}$ is said to be an open cover of X.

Def^n. (Compact) X is said to be compact if every open cover (of X) has a finite sub-cover.

Examples (1) $\mathbb{R} \to$ not compact

$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2)$$ but no finite subcover since $\mathbb{R}$ is not bounded.

(2) $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

Let $\mathcal{U}$ be an open cover of K.

$\exists\, U_0 \in \mathcal{U}$ s.t. $0 \in U_0$.

Thus, $\exists N$ s.t. $K \cap [0, \frac{1}{N}) \subset U_0$.

Now, for $k = 1, \dots, N$, choose $U_k \in \mathcal{U}$ s.t. $\frac{1}{k} \in U_k$.

Then, $K \subset U_0 \cup U_1 \cup \dots \cup U_N$. ☒

(3) $(0, 1]$ not compact. $(0,1] = \bigcup_{n \geq 2} (\frac{1}{n}, 1]$.

Def^n. If Y is a subspace of X, and $\mathcal{C}$ a collection of subsets of X, then $\mathcal{C}$ is said to cover Y if
$$Y \subseteq \bigcup_{C \in \mathcal{C}} C.$$

Lemma 2. Let Y be a subspace of X.
Then, Y is compact (in subspace topology) iff every covering of Y by sets open in X contains a finite subcollection covering Y. ☒

Lemma 3. Every closed subspace of a compact space is compact. ∎

Thm 4. Every compact subspace of a Hausdorff space is closed.

Proof. Let $Y \subseteq X$ be closed, where $X \leftarrow$ Hausdorff.
We prove that $X \setminus Y$ is open.
Let $x_0 \in X \setminus Y$. For each $y \in Y$, $\exists$ disjoint open nbds U_y and V_y of x_0 and y, resp. The collection

$$\{V_y : y \in Y\}$$

covers Y. Thus, $\exists\, y_1, \dots, y_n \in Y$ s.t. $Y \subset V_{y_1} \cup \dots \cup V_{y_n}$.
(Y is compact.)

Then, $U_{y_1} \cap \dots \cap U_{y_n}$ is an open nbd of x contained in $(V_{y_1} \cup \dots \cup V_{y_n})^c \supset Y^c$.

$\therefore Y \setminus X$ is open.

Remark. The above proof shows the following:
If X is Hausdorff, $Y \subseteq X$ is compact, and $x_0 \notin Y$, then $\exists$ disjoint open sets U and V of X containing x_0 and Y, resp.

Lecture 10 (10-02-2021)

10 February 2021 16:05

Thm 1. The continuous image of a compact set is compact. ☒

Thm 2. Let $f: X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

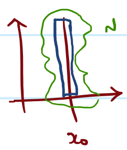
Proof. We need to show f^{-1} is continuous.
Let $K \subseteq X$ be closed. Then, K is closed, since X is compact.
Thus, $f(K) \subseteq Y$ is compact. Then, $f(K)$ is closed, since Y is Hausdorff. Thus, f is a closed map and hence, f^{-1} is continuous. ☒

Thm 3. The cartesian product of finitely many compact spaces is compact.

Proof. As for the case of connectedness, it suffices to prove for product of two spaces.

Proof. Let Y be a compact space.

Step 1. Suppose that $x_0 \in X$ and N is an open set in $X \times Y$ containing the "slice" $\{x_0\} \times Y$. We show that $\exists$ nbd W of x_0 in X s.t. $W \times Y \subseteq N \times Y$.
↳ called a "tube"



$N = \bigcup_{i \in J} U_i \times V_i$

First, we cover $\{x_0\} \times Y$ by $\{U_i \times V_i\}$ → open sets, basis elements.
By compactness, we can cover by finitely many, $i = 1, \dots, n$.
WLOG, assume that $(U_i \times V_i) \cap (\{x_0\} \times Y) \neq \emptyset$ for $i = 1, \dots, n$.
Then, $W = U_1 \cap \dots \cap U_n$ is a neighbourhood of x_0.
Then, $W \times Y \subseteq N$ is the desired tube.

Now, assume X is also compact.

Step 2. Let $\mathcal{A}$ be an open covering of $X \times Y$.
Given $x_0 \in X$, $\{x_0\} \times Y$ is compact and hence covered by finitely many $A_1, \ldots, A_n \in \mathcal{A}$. Then,

$$N = A_1 \cup \cdots \cup A_n \text{ is an open set}$$

containing $\{x_0\} \times Y$.
By step 1, $\exists$ tube $W \times Y$ s.t. $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.

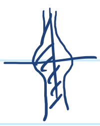
Thus, for each $x \in X$, $\exists W_x$ s.t. $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$. By compactness of X, X is covered by finitely many $W_{x_1}, \ldots, W_{x_n}$. Each corresponding tube is covered by finitely many elements of $\mathcal{A}$. ∎

Thm 4 (Tube Lemma) Let Y be compact and $x_0 \in X$. Let $N \subseteq X \times Y$ be open such that $\{x_0\} \times Y \subseteq N$. Then, $\exists$ open $W \subseteq X$ s.t. $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.

Proof. Step 1 of earlier. ∎

Remark. Compactness of Y is needed.
Take $X = Y = \mathbb{R}$ and $0 \in X$ and

$$N = \left\{ (x, y) \in \mathbb{R}^2 : |x| < \frac{1}{1+y^2} \right\}.$$



No tube exists!

Lecture 11 (11-02-2021)

11 February 2021 15:36

Def^n. A collection $\mathcal{C}$ of subsets of X is said to satisfy the finite intersection condition if for every finite subcollection $\{C_1, \dots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \dots \cap C_n$ is non-empty.

Thm 1. Let X be a topological space. Then X is compact iff every collection $\mathcal{C}$ of closed sets in X satisfying the finite intersection condition satisfies $\bigcap_{C \in \mathcal{C}} C \neq \phi$.

Proof. For $\mathcal{C}$, define $\mathcal{U}_{\mathcal{C}} = \{X \setminus C : C \in \mathcal{C}\}$.

$\mathcal{C} \to$ closed sets, $\mathcal{U}_{\mathcal{C}} \to$ open sets

$\bigcap_{C \in \mathcal{C}} C = \phi \iff \mathcal{U}_{\mathcal{C}}$ is an open cover.

$\mathcal{C}$ has finite inter. property $\iff$ no finite subcollection of $\mathcal{U}_{\mathcal{C}}$ covers X.

Conclude! ☒

Cor 2. If X is compact and $X \supset C_1 \supset C_2 \supset \cdots$ with $C_n \neq \phi$ for all $n \in \mathbb{N}$, then $\bigcap_{n \geq 1} C_n \neq \phi$.

Proof. $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ satisfies finite intersection property. ☒

Def^n. (Limit point compact) A space X is said to be limit point compact if every infinite subset of X has a limit point.

Thm 3. Compactness $\Rightarrow$ Limit point compactness.

Proof. Let X be compact and $A \subset X$ be s.t. A has no limit point. We show A is finite.
Since $A' = \phi$, $\bar{A} = A$; i.e., A is closed.
For each $a \in A$, we can choose an open nbd U_a of a that does not intersect $A \setminus \{a\}$.
(Since a is not a lt. point of A.)
Note that $\{U_a : a \in A\}$ cover A. Since A is closed in X, A is compact. Thus, it has a finite subcover.
But $|U_a \cap A| = 1$ $\forall a \in A$, we see that A is finite. ☒

Remark. The converse of the above is not true.
Consider any set Y with two points and give it the indiscrete topology.
" $\{\bar{0}, \bar{1}\}$
Consider $X = \mathbb{N} \times Y$ in product topology.
Then, any non-empty subset of X has a lt. point.
(A basis of X is $\{\{n\} \times Y : n \in \mathbb{N}\}$. Thus, given any $\emptyset \neq A \subseteq X$, pick $(n, x) \in A$. Then, $(n, 1-x)$ is in any nbd of (n, x).)
However, X is NOT compact. We have
$$X = \bigcup_{n \in \mathbb{N}} \{n\} \times Y.$$ ☒

Defn. (Sequentially compact) A space X is said to be **sequentially compact** if every sequence has a convergent subsequence.

Thm 4. Let X be a metrisable space. TFAE:
(i) X is compact.
(ii) X is limit point compact.
(iii) X is sequentially compact.

Proof. (i) $\Rightarrow$ (ii) Previous theorem.

(ii) $\Rightarrow$ (iii)

Let (a_n) be a sequence in X. If (a_n) has a constant subsequence, we are done. Thus, assume not. Then, $A = \{a_n : n \in \mathbb{N}\}$ is infinite and hence, has a limit point x.

Pick an element in $A \cap B(x,1)$. It is of the form a_{k_1} for some $k_1 \in \mathbb{N}$.

Assume we have chosen $k_1 < k_2 < \dots < k_n$ s.t.

$$a_{k_i} \in B(x, 1/i) \cap A \quad \forall\, i = 1, \dots, n.$$

Now, $B(x, 1/n+1) \cap A$ is infinite.

Thus, can choose $k_{n+1} > k_n$ s.t. $a_{k_{n+1}} \in B(x, \frac{1}{n+1}) \cap A$.

Then, $a_{n_k} \longrightarrow x$ as $k \longrightarrow \infty$.

(iii) $\Rightarrow$ (i) ① We show the following statement:

Let $\mathcal{A}$ be an open cover of X. Then $\exists \delta > 0$ s.t. for each subset of X having diameter less than δ, there is an element of $\mathcal{A}$ containing it.

$\rightarrow$ Let $\mathcal{A}$ be an open cover for which no such δ exists. Taking $\delta = 1/n$, we get sets B_n such that $\operatorname{diam}(B_n) < \delta$ and $B_n \not\subset A \quad \forall\, A \in \mathcal{A}$.

Choose an $x_n \in B_n \quad \forall n$. Then (x_n) has a convergent subsequence (x_{n_k}). Let $x \in X$ be the limit. Choose $\epsilon > 0$ s.t. $B(x, \epsilon) \subset A$.

Eventually, $x_{n_k} \in B(x, \epsilon/2)$.

By choose k large enough, $B_{n_k} \subset B(x, \epsilon) \subset A$. $\rightarrow\leftarrow$

② We show the following:

For every $\epsilon > 0$, $\exists$ a finite covering of X by ϵ-balls.

$\rightarrow$ Assume $\exists \epsilon > 0$ for which the above does not hold.

Choose $x_1 \in X$ arbitrarily. Then, $B(x_1, \epsilon) \neq X$.

Choose $x_2 \in X - B(x_1, \epsilon)$.

Inductively, we can choose

$$x_{n+1} \in X - \bigcup_{i=1}^{n} B(x_i, \epsilon).$$

Then, $(x_n)_{n=1}^{\infty}$ has a convergent subsequence (x_{n_k}).

But $d(x_i, x_j) \geq \epsilon \quad \forall \ i \neq j$. $\therefore$ No subseq can conv. $\Rightarrow\Leftarrow$

③ Now, we show X is compact.

Let $\mathcal{A}$ be an open cover of X. Choose $\delta > 0$ such that ① holds.

By ②, X can be covered by finitely many $\delta/3$-balls. Since each have diameter $\leq 2\delta/3$, each of them lie in an element of $\mathcal{A}$. Choose one such element for each ball (of which there are finitely many). These elements form a finite subcover. $\blacksquare$

Remark. δ is called the Lebesgue number of $\mathcal{A}$.

Lecture 12 (18-02-2021)

18 February 2021 15:36

Def^n: Let $p: X \to Y$ be a surjective function.
The map is said to be a quotient map if any
$U \subseteq Y$ is open iff $p^{-1}(U) \subseteq X$ is open.
Can replace "open" with "closed" since $p^{-1}(U^c) = (p^{-1}(U))^c$.

Remarks: (1) A quotient map is continuous.

(quotient map)

(2) It need not be bijective.

(3) A homeomorphism is a quotient map.

(4) Quotient + Injective $\Rightarrow$ Homeomorphism

(5) Surj. + open map $\Rightarrow$ Quotient ($\Leftarrow$) not true!

(6) Surj. + closed map $\Rightarrow$ Quotient ($\Leftarrow$) not true!

Def^n. A subset $C \subseteq X$ is said to be saturated w.r.t. p if
$p^{-1}(\{y\}) \cap C \neq \phi \Rightarrow p^{-1}(\{y\}) \subseteq C$.

(Saturated)

(That is, if C contains one pre-image, it contains all.)

Remark. Thus, p is a quotient map iff p is a continuous surjection that maps open saturated sets to open sets.
(Or "closed" instead of "open"!)

Example ① Let $X = [0,1] \cup [2,3] \subseteq \mathbb{R}$ and $Y = [0,2] \subseteq \mathbb{R}$.
Define $p: X \to Y$ by

$$p(x) = \begin{cases} x & ; x \in [0,1] \\ x-1 & ; x \in [2,3] \end{cases}$$

p is continuous and surjective. Moreover, it is closed because X is compact and Y Hausdorff.
Thus, p is a quotient map.
(Not homeomorphism since $p(1) = p(2)$.)
However, p is NOT open. $[0,1] \subset X$ is open but

$p([0,1]) = [0,1] \subset Y$ is NOT open.

(Note that $[0,1]$ is NOT saturated since $p^{-1}(\{1\}) \cap [0,1] \neq \emptyset$ but $p^{-1}(\{1\}) \not\subset [0,1]$.)

② Let $A = [0,1) \cup [2,3] \subseteq X$.

Define $q: A \to Y$ by $q = p|_A$.

Then, q is a bijection and thus, every subset is saturated. However, $[2,3]$ is open in X but $q([2,3])$ is not open in Y. (Note q is continuous!)

Def^n. Let X be a topological space A a set. Let $p: X \to A$ be a surjective function. Then, there exists a unique topology on A which q a quotient map. This is called the quotient topology on A.

(Quotient topology)

Proof.

Let $\mathcal{T} = \{U \subset A : p^{-1}(U) \text{ is open in } X\}$.

T.S.T. $\mathcal{T}$ is a topology.

① $\emptyset = p^{-1}(\emptyset)$ and $A = p^{-1}(X)$. $\therefore \emptyset, A \in \mathcal{T}$.

② $p^{-1}\left(\bigcup_{\alpha \in \Lambda} U_\alpha\right) = \bigcup_{\alpha \in \Lambda} p^{-1}(U_\alpha)$ and

③ $p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$ Show

closure under finite intersec. and arbitrary union.

Uniqueness is clear. That $p: X \to A$ is a quotient map is also clear.

Def^n. Let X be a topological space and X^* a partition of X. Let $p: X \to X^*$ be the natural projection map. (This is surjective) The space X^*

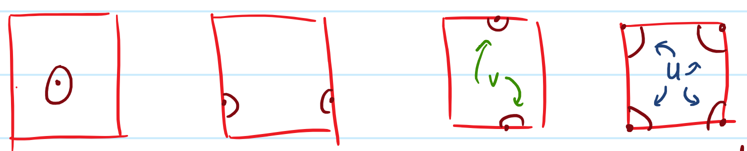
with the quotient topology induced by p is called a quotient space of X.

Recall that partitions of a set are equivalent to (no pun intended) an equivalence relation X.
A subset $\mathcal{U} \subseteq X^*$ is a collection of equivalence classes and $p^{-1}(U) \subset X$ is simply the union of those equivalence classes.

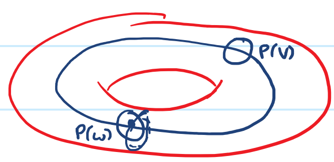
Example. ③ Let $X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ be the closed unit disc in $\mathbb{R}^2$.
Let $X^* = \{\{(x,y)\} : x^2+y^2 < 1\} \cup \{S^1\}$.
(All $\{(x,y)\}$ for $(x,y) \in X^\circ$ and S^1.)

④ $X = [0,1] \times [0,1]$.

$$\begin{aligned} X^* = &\{\{(x,y)\} : (x,y) \in \mathring{X}\} \cup \\ &\{\{(0,y),(1,y)\} : 0 < y < 1\} \cup \\ &\{\{(x,0),(x,1)\} : 0 < x < 1\} \cup \\ &\{\{(0,0),(1,0),(1,1),(0,1)\}\}. \end{aligned}$$



some saturated sets



Thm. Let $p : X \to Y$ be a quotient map.
Let A be a subspace which is saturated w.r.t. p.
Let $q : A \to p(A)$ be the restriction $p|_A$.
Then,

(1) If A is either open or closed, then q is

a quotient map.

(2) If p is either open or closed, then q is a quotient map.

Proof. Step 1.

Claim. (a) $V \subset p(A) \Rightarrow q^{-1}(V) = p^{-1}(V)$ and

(b) $p(U \cap A) = p(U) \cap p(A)$ if $U \in X$.

Proof. (a)($\subseteq$) Let $x \in q^{-1}(V)$.

Then $q(x) \in V \subset p(A)$.

$\therefore$ $q(x) \in p(A)$. Thus, $\underset{\overset{\shortparallel}{q(a)}}{q(x) = p(a)}$ for some $a \in A$.

Since A is saturated, $x \in p^{-1}(\{p(a)\}) \subset A$.

($\supseteq$) Let $x \in p^{-1}(V)$. Then $p(x) \in V \subset p(A)$. Same argument again.

(b) $p(U \cap A) \subset p(U) \cap p(A)$ is true in general.

($\supseteq$) Let $y \in p(U) \cap p(A)$.

Then, $y = p(u) = p(a)$ for some $a \in A$ and $u \in U$.

$a \in p^{-1}\{y\} \cap A$.

$\therefore$ $p^{-1}\{y\} \subset A$ and hence, $u \in A$.

$\therefore u \in U \cap A$.

Step 2. ① Suppose A is open in X.

Claim. q is a quotient map.

Proof. Let $V \subseteq p(A)$.

Suppose $q^{-1}(V)$ is open in A.) A is open in X.

Then, $q^{-1}(V)$ is open in X.) (a) of Step 1.

Then, $p^{-1}(V)$ is open in X.

Then, V is open in Y since p is quotient.

In particular, V is open in $p(A)$. ∎

② Suppose p is open.

Claim. q is a quotient map.

Proof. Let $V \subseteq p(A)$ be s.t. $q^{-1}(V)$ is open (in A).

Then, $p^{-1}(V) = q^{-1}(V)$ is open.

$p^{-1}(V)$ is open in A and hence,

$$p^{-1}(V) = U \cap A \quad \text{for } U \subseteq X \text{ open.}$$

$\Rightarrow p(p^{-1}(V)) = p(U \cap A)$

onto $\Rightarrow V = p(U) \cap p(A)$ — (b) from Step 1.

But U is open and p is an open map.

Thus, V is open in $p(A)$.

Thus, q is a quotient map.

Step 3. Do the same as prev. step by replacing "open" with "closed". ∎

Lecture 13

21 February 2021 11:19

Prop^n 1. Let $p : X \to Y$, $q : Y \to Z$ be quotient maps.
Then $q \circ p : X \to Z$ is a quotient map.

Proof. Clearly, $q \circ p$ is surjective and continuous.
Let $U \subseteq Z$ be s.t. $(q \circ p)^{-1}(U)$ is open.
That is, $p^{-1}(q^{-1}(U))$ is open.
Since p is quotient, $q^{-1}(U)$ is open. Since q is quotient, U is open. $\boxtimes$

Thm 2. Let $p : X \to Y$ be a quotient map. Let Z be a topological space. Let $g : X \to Z$ be a map such that g is constant on each $p^{-1}(\{y\})$ for $y \in Y$.

In other words, $p(x) = p(x') \Rightarrow g(x) = g(x')$.
Let us refer to this as "g respects p."

Then, g induces a map $f : Y \to Z$ s.t.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

$f \circ p = g$ and

(i) f is continuous iff g is continuous.

(ii) f is a quotient map iff g is a quotient map.

Proof. Since g respects p and p is onto, we get a unique well-defined map $f : Y \to Z$ defined by
$$f(p(x)) = g(x).$$
(Since each $y \in Y$ is of the form $p(x)$ and if $p(x) = p(x')$, then $g(x) = g(x')$.)

(i) If f is continuous, then $g = f \circ p$ is continuous, being the composition of continuous maps.
Conversely, suppose g is continuous.
Let $U \subseteq Z$ be open.
Then, $p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U) = g^{-1}(U)$ is open since g is cont. But p is quotient. Thus, $f^{-1}(U)$ is open. Hence, f is continuous.

(ii) If f is quotient, then $g = f \circ p$ is, by Prop^n 1
Conversely, let g be a quotient map.
$\therefore$ g is onto and hence, so is f.
Also, g is continuous and hence, so is f, by (i).
Now, let $U \subseteq Z$ be s.t. $f^{-1}(U)$ is open.
Ts: U is open.
Note $f^{-1}(U)$ open $\xRightarrow{p \text{ cont.}}$ $p^{-1}(f^{-1}(U))$ is open
$\Rightarrow$ $g^{-1}(U)$ is open) $g = f \circ p$
$\Rightarrow$ U is open) g quotient ∎

Cor 3. Let $g: X \to Z$ be a surjective continuous map
Let X^* be the partition induced by the equivalence relation $\sim$ on X given by
$$x \sim x' \iff g(x) = g(x').$$

$$\left(X^* = \{ g^{-1}(\{z\}) : z \in Z \}. \right)$$

Consider X^* with the quotient topology induced by the natural $p: X \twoheadrightarrow X^*$.
(a) g induces a bijective continuous map $f: X^* \to Z$, which is a homeomorphism iff g is a quotient map.
(b) If Z is Hausdorff, then so is X^*.

Proof. Here, $Y = X^*$ and p is the canonical map.

By construction, f is bijective. (It was already onto, it is 1-1, since X^* is precisely the partition based on fibers of g.)

By the previous theorem, $f: X^* \to Z$ is continuous and bijective.

(a) Now, f is a homeo $\Leftrightarrow$ f is quotient Thm 2 (ii)

$\Updownarrow$

g is quotient

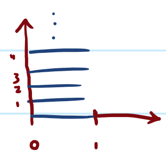
(b) Let Z be Hausdorff.

Let $x, y \in X^*$ be s.t. $x \neq y$. Since f is 1-1, $f(x) \neq f(y)$ in Z.

$\therefore \exists\, U \ni f(x),\ V \ni f(y)$ open s.t. $U \cap V = \phi$.

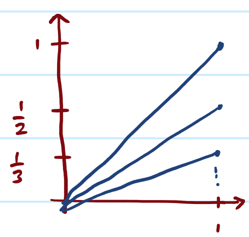
Then, $f^{-1}(U) \cap f^{-1}(V) = \phi$ and are nbds of x and y, resp.

Example. Let $X = \bigcup_{n \in \mathbb{N}} \left([0,1] \times \{n\} \right) \subseteq \mathbb{R}^2$



be in subspace topology and let

$$Z = \left\{ \left(x, \frac{x}{n} \right) : x \in [0,1],\ n \in \mathbb{N} \right\} \subseteq \mathbb{R}^2$$



also be in subspace top.

Define $g: X \to Z$ by

$$g(x, n) = \left(x, \frac{x}{n} \right).$$

Now, if $z \in Z \setminus \{(0,0)\}$, then $g^{-1}(\{z\})$ is a singleton. But if $z = (0, 0) \in Z$, then

$$g^{-1}(\{z\}) = \{(0, n) : n \in \mathbb{N}\}.$$

Now, take $X^* = \{g^{-1}(\{z\}) : z \in Z\}$. give quotient topology

By the earlier, we have a bijective continuous map

$$f : X^* \longrightarrow Z.$$

Q. Is f a homeomorphism?

A. No.

Consider the set $A = \left\{ \left(\frac{1}{n}, n\right) \in X : n \in \mathbb{Z} \right\}$.

A is closed since $A' = \phi$. Moreover A is saturated w.r.t. g. However,

$$g(A) = \left\{ \left(\frac{1}{n}, \frac{1}{n^2}\right) : n \in \mathbb{N} \right\}$$

does have a limit point outside $g(A)$. Thus, $g(A)$ is not closed. Thus, g is not a quotient map and hence, f is not a homeo. ∎