Morphisms of Schemes: Chevalley's Theorem

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Notations

1. X and Y will denote topological spaces.
2. U, V, W will denote open subsets of the ambient topological space.
3. By a cover $\{U_i\}$ of U, we mean that $U = \bigcup_i U_i$. In particular, $U_i \subset U$ for all i.
4. A will denote a commutative ring with 1. (All our rings will be of this form!)
5. $\operatorname{Spec} A$ will denote the set of prime ideals of A.
6. Given $S \subset A$, $\langle S \rangle$ will denote the ideal generated by S.
7. Given $f \in A$, A_f will denote the localisation of A at the multiplicative set $\{1, f, f^2, \ldots\}$.

Definition 1 (Presheaf)

Let X be a topological space. A presheaf (of rings) $\mathscr{F}$ on X is the following collection of data:

1. For each open set $U \subset X$, we are given a ring $\mathscr{F}(U)$.
2. For open sets $U \subset V \subset X$, we have a ring map $\mathrm{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$, called the restriction map.

The above data is required to satisfy the following conditions:

1. $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$ for all open $U \subset X$.
2. If $U \subset V \subset W$ are open sets, then the following diagram commutes

$$\begin{tikzcd} \mathscr{F}(W) \arrow[r, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & \mathscr{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ & \mathscr{F}(U) \end{tikzcd}.$$

Definition 2 (Sheaf)

Let X be a topological space. A sheaf (of rings) $\mathscr{F}$ on X is a presheaf $\mathscr{F}$ on X satisfying the following:
Given an open set $U \subset X$, an open cover $\{U_i\}$ of U, and elements $f_i \in \mathscr{F}(U_i)$, there exists a unique $f \in \mathscr{F}(U)$ such that

$$\operatorname{res}_{U,U_i}(f) = f_i$$

for all i.

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Given elements on patches, we can glue them uniquely.

Definition 4 (Ringed space)

A ringed space is a tuple $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf on X.

Definition 5 (Morphism of ringed spaces)

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces. A morphism $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is the following data:

1. A continuous map $\pi : X \to Y$.
2. For every open $V \subset Y$, we have a ring map
$$\mathscr{O}_Y(V) \to \mathscr{O}_X(\pi^{-1}(V)).$$

Moreover, the "obvious diagrams" must commute.

Zariski topology

Goal: Turn $\operatorname{Spec} A$ into a ringed space. First, we need a topology.

Definition 6 (Distinguished and Vanishing sets)

Let A be a ring, and $f \in A$. Define

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}.$$

Given a subset $S \subset A$, define

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} A : S \subset \mathfrak{p}\}.$$

(Check: $D(f) = \operatorname{Spec} A \setminus V(f)$.)

Simple check 1: Given $S \subset A$, we have $V(S) = V(\langle S \rangle)$.
Simple check 2: If $D(g) \subset D(f)$, then f is invertible in A_g. Thus, there is a natural map $A_f \to A_g$.

Zariski topology

Definition 7 (Zariski topology)

Let A be a ring. Then, the collection

$$\{V(I) : I \subset A \text{ is an ideal}\}$$

describes a topology on $\operatorname{Spec} A$ by denoting the collection of *closed* subsets. This is called the Zariski topology on $\operatorname{Spec} A$.

Proposition 8 (A basis for the Zariski topology)

The collection $\{D(f) : f \in A\}$ forms a basis for the above topology.

A Helper Example

Let k be a field. We denote $\operatorname{Spec} k[x]$ by $\mathbb{A}^1_k$.

Since $k[x]$ is a PID, the prime ideals are $\langle 0 \rangle$ and the maximal ideals.

The set $\{\langle 0 \rangle\}$ is dense in $\mathbb{A}^1_k$.

The closed sets are given precisely as:

1. The empty set.
2. The whole space.
3. Sets containing finitely many maximal ideals.

In particular, maximal ideals are *closed points*, i.e., $\{\mathfrak{m}\}$ is closed. Consequently, $\{\mathfrak{m}\}$ is not dense in $\mathbb{A}^1_k$.

To conclude, the only closed singleton subset of $\mathbb{A}^1_k$ is $\{\langle 0 \rangle\}$.

We now describe a sheaf $\mathscr{O}_{\operatorname{Spec} A}$. However, we shall cheat a bit. We only define the objects and arrows on the level of basis elements. One must check that this does indeed a sheaf on the whole space.

Definition 9 (Structure sheaf)

Let A be a ring. Given $f \in A$, we define

$$\mathscr{O}_{\operatorname{Spec} A}(D(f)) := A_f.$$

Given $D(g) \subset D(f)$, the restriction map is the natural map $A_f \to A_g$.
This is called the structure sheaf on $\operatorname{Spec} A$.

Schemes

Definition 10 (Affine scheme)

An affine scheme is a ringed space which is isomorphic to some $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$.

Definition 11 (Scheme)

A scheme is a ringed space $(X, \mathscr{O}_X)$ such that every $p \in X$ has an open neighbourhood U such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

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A scheme can be covered by affine opens.

In fact, (it follows that) the affine opens form a basis for X.

Morphisms of affine schemes

Let $\pi^\sharp : A \to B$ a map of rings. This induces a map $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ given by $\mathfrak{p} \mapsto (\pi^\sharp)^{-1}(\mathfrak{p})$. This is continuous.

Moreover, this also induces a morphism of ringed spaces. More explicitly, given $g \in B$, we have the map

$$\begin{tikzcd}
\mathscr{O}_{\operatorname{Spec} B}(D(g)) \arrow[r] \arrow[d, equal] & \mathscr{O}_{\operatorname{Spec} A}(\pi^{-1}(D(g))) \arrow[r, equal] & \mathscr{O}_{\operatorname{Spec} A}(D(\pi^\sharp g)) \arrow[d, equal] \\
B_g \arrow[rr] & & A_{\pi^\sharp g}
\end{tikzcd}.$$

The above is a morphism of affine schemes. That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

Definition 13 (Morphism of schemes)

A morphism of schemes $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces that "locally looks like" a morphism of affine schemes.
More precisely, for each choice of affine open sets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, such that $\pi(\operatorname{Spec} A) \subset \operatorname{Spec} B$, the restricted morphism is one of affine schemes.

Some definitions

Definition 14 (Compact morphism)

A morphism $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of schemes is compact if the preimage of any compact open subset is compact.

Definition 15 (Finite type morphism)

A *compact* morphism $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of schemes is of finite type if for every affine open $\operatorname{Spec} B \subset Y$, $\pi^{-1}(\operatorname{Spec} B)$ can be covered by affine open subsets $\operatorname{Spec} A_i$, so that each A_i is a finitely generated B-algebra.

Definition 16 (Noetherian schemes)

A scheme $(X, \mathscr{O}_X)$ is said to be Noetherian if X can be covered by finitely many affine opens $\operatorname{Spec} A_i$ such that each A_i is a Noetherian ring.

Definition 17 (Locally closed set)

A subset of a topological space X is said to be locally closed if it is the intersection of an open subset and a closed subset.

Definition 18 (Constructible set)

A subset of a topological space X is said to be constructible if it can be written as a finite disjoint union of locally closed sets.

Example 19 (Simple example)

$X \subset X$ is a constructible subset. $\{\langle 0 \rangle\} \subset \mathbb{A}^1_k$ is not.

Caution 20

What we call "compact" is usually called *quasicompact*.
The definition of "constructible set" above is not the standard one. However, for Noetherian topological spaces (whatever those are), the two are equivalent.

Theorem 21 (Chevalley)

If $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a finite type morphism of Noetherian schemes, then the image of any constructible set is constructible. In particular, the image of π is constructible.

A consequence

Corollary 22 (Nullstellensatz)

Let $k \subset K$ be a field extension. Suppose K is a finitely generated k-algebra. Then, K is a finite extension of k.

Proof.

Let K be generated by $x_1, \ldots, x_n$, as a k-algebra. It suffices to show that each x_i is algebraic over k. Suppose some x_i is not. Then, we have an inclusion of rings $k[x_i] \hookrightarrow K$, and $k[x_i]$ is isomorphic to the polynomial ring over k.
This corresponds to a dominant morphism $\pi : \operatorname{Spec} K \to \mathbb{A}_k^1$.
Since $\operatorname{Spec} K$ is a singleton, so is the image of π. By dominance of π (and the Helper example), the image is $\{\langle 0 \rangle\}$. But this is not constructible (Simple example). This contradicts Chevalley's Theorem. $\square$