

Lecture 1 (03-01-2022)

03 January 2022 13:58

- Texts:
- Real and Complex Analysis - Rudin
 - Complex Analysis - Lang

Topics:

- Review of basic C analysis, Harmonic functions, ...;
- Maximum modulus theorem, ...,
- Runge's, Mittag-Leffler, Weierstrass theorems,
- Riemann mapping theorem,
- Analytic continuation,
- Little / Big Picard's theorem,

(If time) Introduction to Several Complex variables.

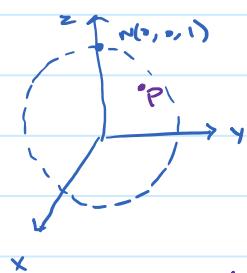
Evaluation (tentative !!!):

- Presentation 10 - 15 %
- Assignments 10 - 15 %
- Midsem, Endsem
- 1/2 Quizzes maybe

Riemann sphere / The Extended Complex plane.

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

The stereographic projection is a function $\Phi: S^2 \xrightarrow{\subset \mathbb{R}^3} \hat{\mathbb{C}}$.



$$P \in S^2, P \neq N.$$

Define the stereographic projection of $P(x, y, z) \mapsto$
as follows:

Join N to P. Extend it. It hits the
(equatorial) plane $z=0$ at some point $(x, y, 0)$.

$$P \mapsto x + iy \text{ is the map.}$$

Stereographic projection

Analytically the line is:

Analytically, the line is:

$$t(x, y, z) + (1-t)(0, 0, 1).$$

We need $tz + 1-t = 0$ or $t = \frac{1}{1-z}$.

$$\therefore x = \frac{x}{1-z} \quad \text{and} \quad y = \frac{y}{1-z} \quad (\text{note: } z \neq 1)$$

Finally, $N \mapsto \infty$.

(E.g.: Under the above map, $(0, 0, -1) \mapsto (0, 0)$ or $0+0z$.)

To sum it up: Define $\theta: S^2 \rightarrow \hat{\mathbb{C}}$ by

$$\theta(x, y, z) = \begin{cases} \frac{x}{1-z} + \frac{zy}{1-z} & ; \quad z \neq 1, \\ \infty & ; \quad \text{else.} \end{cases}$$

Check: θ is a bijection.

To see that it is onto, let $z = x+zy \in \mathbb{C}$ be arbit.

Check that

$$P(x, y, z) := \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2} \right)$$

maps to z .

(As usual, $|z| = \sqrt{x^2+y^2}$.)

Q. What happens to P above as $|z| \rightarrow \infty$?

Evidently $P \rightarrow N(0, 0, 1)$.

→ Using the above, we can define a topology on $\hat{\mathbb{C}}$.

In fact, we now define a metric on $\hat{\mathbb{C}}$ as follows:

For $w, z \in \hat{\mathbb{C}}$, define the distance between w and z to be the length of the straight line segment joining $\theta^{-1}(w)$ and $\theta^{-1}(z)$, i.e.,

$$d(w, z) := \| \theta^{-1}(w) - \theta^{-1}(z) \|_{\infty}$$

$$= \frac{\sqrt{2} |w-z|}{\sqrt{1+|z|^2} \sqrt{1+|w|^2}}$$

) after calculations
(both $z, w \neq \infty$)

If $w = \infty$ and $z \neq \infty$, we get $d(z, \infty) = \frac{\sqrt{2}}{\sqrt{1+|z|^2}}$.

Fix $z \in \hat{\mathbb{C}}$, $r > 0$.

$$B_d(z, r) := \{w \in \hat{\mathbb{C}} : d(z, w) < r\}.$$

Describe the above set when $z = \infty$

Describe the open sets in $\hat{\mathbb{C}}$.

Defn. A domain in \mathbb{C} is an open connected subset of \mathbb{C} .
Domain (Nonempty!)

Remark. Automatically path-connected. Ω will usually denote a domain

Defn. Let $\Omega \subseteq \mathbb{C}$ be a domain, $z_0 \in \Omega$, let $f: \Omega \rightarrow \mathbb{C}$.
We say that f is (complex) differentiable at z_0 if

Complex differentiable

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists (and is finite).

The above quantity is then denoted $f'(z_0)$ and called the (complex) derivative of f at z_0 .

Defn. $\Omega \subseteq \mathbb{C}$ domain, $f: \Omega \rightarrow \mathbb{C}$ function.
 f is said to be complex analytic/holomorphic on Ω
if f is complex differentiable at each point of Ω .

Complex analytic, holomorphic

$$\Theta(\Omega) := \{ f: \Omega \rightarrow \mathbb{C} \text{ is holomorphic}\}.$$

"script O"

\rightarrow R-valued \nwarrow

Obs. Let $f \in \Omega(\Omega)$. Write $f(z) = u(z) + i v(z)$

$$\begin{aligned} &= u(x+iy) + i v(x+iy) \\ &= u(x, y) + i v(x, y) \end{aligned}$$

↑ R-valued
↑ treat $\Omega \subseteq \mathbb{R}^2$

Fix $z_0 = x_0 + iy_0 \in \Omega$.

$$\begin{aligned} f'(z_0) &= \lim_{\substack{\mathbb{C} \ni h \rightarrow 0}} \frac{f'(z_0+h) - f(z_0)}{h} \quad \text{exists.} \\ &= \lim_{\substack{\mathbb{R} \ni h \rightarrow 0}} \left\{ \left[\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} \right] + i \left[\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \right] \right\} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Similarly let $h \rightarrow 0$ along $i\mathbb{R}$.

$$\text{Then, } f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0).$$

In particular, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ on Ω .

\hookrightarrow Cauchy-Riemann equations \hookrightarrow

Defn. Define the operators

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \\ &\equiv 0 \end{aligned}$$

) if $f \in \Omega(\Omega)$

Check : If f is complex diff. at z_0 , it is also real differentiable as a function $\Omega \xrightarrow{S^1 \times \mathbb{R}^2} \mathbb{R}^n$.

Lecture 2 (06-01-2022)

06 January 2022 14:01

Integration :

Integration

Let Ω be a domain in \mathbb{C} , and $\gamma: [a, b] \rightarrow \Omega$ is piecewise - C^1 . For any $f \in C^0(\Omega)$,

$$\int_{\gamma} f := \int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

$(f: \Omega \rightarrow \mathbb{C})$

Index of a point wrt. a path:

For $\gamma: [a, b] \rightarrow \mathbb{C}$ is piecewise C^1 . Assume γ is closed, i.e., $\gamma(a) = \gamma(b)$. Let $\Omega := \mathbb{C} \setminus \text{im}(\gamma)$.

Then, Ω has possibly many connected components, out of which exactly one is unbounded.

Let $z_0 \in \Omega$. We define

$$\begin{aligned} \text{Ind}_{\gamma}(z_0) &:= \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\xi - z_0} d\xi \\ &= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0} dt. \end{aligned}$$

\hookrightarrow well-defined since $z \notin \text{im}(\gamma)$.

Properties:

- (1) Ind_{γ} is an integer-valued function on Ω .
- (2) Thus, Ind_{γ} is constant on the connected components of Ω .
- (3) $\text{Ind}_{\gamma} = 0$ on the unbounded component.

$\Omega = \mathbb{C} \setminus \gamma$

Propn. (Cauchy's Theorem)

Cauchy's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $f: \Omega \rightarrow \mathbb{C}$ be continuous.
TFAE:

(i) $\int_{\gamma} f = 0$ for every closed γ in Ω .

(ii) $\exists F \in \Theta(\Omega)$ such that $F' = f$ on Ω .

Consequently, $f \in \Theta(\Omega)$ (since once differentiable is always differentiable)

Example. Let γ be ... in \mathbb{C} .

If $a \notin \text{im}(\gamma)$, then evaluate

$$I_n := \int_{\gamma} (z-a)^n dz \quad \text{for } n \in \mathbb{Z}.$$

If $n \neq -1$, we have an antiderivative for the integrand
on $\mathbb{C} \setminus \{a\}$. $\therefore I_n = 0$.

If $n = -1$, then we simply have $I_n = 2\pi i \text{Ind}_{\gamma}(a)$.

Defn. Path homotopy

Path homotopy

Given $\gamma_0, \gamma_1: [0, 1] \rightarrow \Omega$ two closed paths in Ω based at ∞_0 .

A path homotopy between γ_0 and γ_1 is a function

$$H: [0, 1] \times [0, 1] \rightarrow \Omega$$

s.t. ① H is continuous,

② $H(s, 0) = \gamma_0(s) \quad \forall s \in [0, 1]$,

③ $H(s, 1) = \gamma_1(s) \quad \forall s \in [0, 1]$

④ $H(0, t) = \infty_0 = H(1, t) \quad \forall t \in [0, 1]$.

Recall: $\gamma_0 \sim \gamma_1$, path-homotopic, null-homotopic ($\gamma \sim 0$).
(equiv. rel'n)

EXAMPLES. (i) $\Omega = \mathbb{C}$.

Any two loops are homotopic.

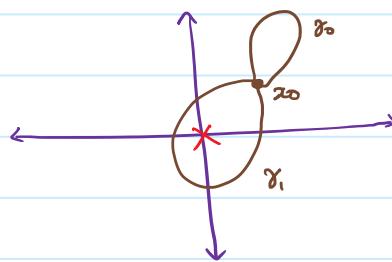
Indeed, $H(s, t) = (1-t)\gamma_0(s) + t\gamma_1(s)$ does the job.

$$\textcircled{2} \quad \Omega = \mathbb{C} \setminus \{\gamma_0\}.$$

$$\gamma_0 = 1t i.$$

The drawn loops

are not homotopic.



Theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let γ_0, γ_1 be loops based at the same point with $\gamma_0 \sim \gamma_1$. Then,

$$\int_{\gamma_0} f = \int_{\gamma_1} f \quad \text{for all } f \in \Omega(\Omega).$$

EXAMPLE. The paths $\gamma_1, \gamma_2 : [0, 1] \rightarrow \Omega = \mathbb{C} \setminus \{\gamma_0\}$ defined as

$$\gamma_1(t) := \frac{1}{\gamma_0(t)} := e^{-2\pi i t}$$

cannot be path homotopic since $f = (z \mapsto \frac{1}{z}) \in \Omega(\Omega)$ and

$$\int_{\gamma_0} f = 2\pi i \neq -2\pi i = \int_{\gamma_1} f.$$

Corollary

Let Ω be a domain and γ be a loop in Ω with $\gamma \sim 0$. Then,

$$\int_{\gamma} f = 0 \quad \forall f \in \Omega(\Omega).$$

Def'

An open set $\Omega \subseteq \mathbb{C}$ is said to be simply-connected if Ω is connected and $\gamma \sim 0$ for every loop γ in Ω .

Simply-connected, simply connected

(Non-)EXAMPLES

• \mathbb{C} , $D(0, 1)$, convex sets, star-shaped domains, $\mathbb{C} \setminus \{z_0\}$

\hookrightarrow s-c

• $\mathbb{C} \setminus \{\gamma_0\}$, $D(0, r) \setminus \{\gamma_0\}$, $D(0, a) \setminus D(0, b)$ for $a > b > 0$

↳ no sc.

Corollary. Let Ω be a sc domain in \mathbb{C} and let γ be a loop in Ω . Then,

$$\int_{\gamma} f = 0 \quad \forall f \in \Theta(\Omega).$$

Cor. Let Ω be a sc domain in \mathbb{C} . Let $f \in \Theta(\Omega)$. Then, $\exists F \in \Theta(\Omega)$ s.t. $F' = f$ on Ω .

Cor. Let Ω be a sc domain in \mathbb{C} and let $f \in \Theta(\Omega)$ be s.t. $f(z) \neq 0 \quad \forall z \in \Omega$. Then, $\exists g \in \Theta(\Omega)$ s.t.

Analytic branch of logarithm

$$f = \exp \circ g.$$

(g is an analytic branch of logarithm of f .)

Proof. Since $f \neq 0$, $\frac{f'}{f} \in \Theta(\Omega)$.

$\therefore \Omega$ is sc, $\exists h \in \Theta(\Omega)$ s.t. $h' = \frac{f'}{f}$.

Let $\tilde{g} := \exp \circ h$

Then, $\tilde{g}' \neq 0 \quad \therefore \frac{f}{\tilde{g}} \in \Theta(\Omega)$.

$$\tilde{g}^2 \left(\frac{f}{\tilde{g}} \right)' = f' \tilde{g} - f \tilde{g}'$$

$$= f' \tilde{g} - f (\exp \circ h) \cdot h'$$

$$= f' \tilde{g} - f \tilde{g} h' = \tilde{g} \cdot (f' - fh') = 0.$$

$$\therefore \frac{f}{\tilde{g}} = c \neq 0.$$

$$\Rightarrow f = c \cdot \tilde{g} = c \cdot \exp \circ h = \exp \left(h + c \right)$$

=: g. 5

Lecture 3 (10-01-2022)

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Maximum Principle

① Let $\Omega \subseteq \mathbb{C}$ be a domain, and $f \in \mathcal{O}(\Omega)$.

Let $a \in \Omega$ such that $\exists r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$.

Then,

$$|f(a)| \leq \max_{0 \leq \theta \leq 2\pi} |f(a + re^{i\theta})|.$$

Moreover, equality holds iff f is constant.

② Let Ω be a bounded open set in \mathbb{C} . (Maximum Modulus Theorem)
Let $f \in C^0(\overline{\Omega}) \cap \mathcal{O}(\Omega)$. Then,

Maximum Modulus Theorem

$$|f(z)| \leq \max_{\partial\Omega} |f| \quad \forall z \in \Omega.$$

In words, $|f|$ attains its maximum on the boundary.

Equivalently:

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

EXAMPLE: $H := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Define $f(z) = \exp(-z^2)$ on \overline{H} .
 $f \in \mathcal{O}(H) \cap C^0(\overline{H})$.

Note that $|f(z)| \leq 1$ for $z \in \mathbb{R} = \partial H$.

But

$$|f(iy)| = e^{y^2} \text{ grows rapidly on } i\mathbb{R}.$$

Thus, MMT need not hold if Ω is unbounded.

Now, we wish to formulate a similar theorem for unbounded.

- Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f: \Omega \rightarrow \mathbb{C}$.
For $a \in \bar{\Omega}$, define

$$\limsup_{\Omega \ni z \rightarrow a} f(z) = \lim_{r \rightarrow 0^+} \sup \left\{ |f(z)| : z \in \Omega \cap D(a, r) \right\}.$$

↓
this limit exists in $[0, \infty]$.

If a is the point at infinity, $D(a, r)$ is the neighbourhood of a in the metric d on the extended complex plane.

The extended boundary of Ω in $\mathbb{C} \cup \{\infty\}$ is denoted by $\partial_\infty \Omega$.

Note:

$$\partial_\infty \Omega = \begin{cases} \partial \Omega & ; \Omega \text{ is bounded,} \\ \partial \Omega \cup \{\infty\} & ; \text{else.} \end{cases}$$

- ③ MMT: Let Ω be a domain in \mathbb{C} , $f \in \Theta(\Omega)$.

(Not necessarily bounded!)

Suppose that $\limsup_{z \rightarrow a} |f(z)| \leq M$ for all $a \in \partial_\infty \Omega$.

Then,

$$|f| \leq M \text{ on } \Omega.$$

Proof. Let $\delta > 0$, and define $S := \{z \in \Omega : |f(z)| > M + \delta\}$.

We will show that $S = \emptyset$ and we done.

Note that $|f|$ is continuous and thus, S is open.

We claim that S is bounded in \mathbb{C} .

If Ω is bounded, then done.

Suppose Ω is unbounded. Then, $\infty \in \partial_\infty \Omega$.

But $\limsup_{\Omega \ni z \rightarrow \infty} |f(z)| < M$.

$\therefore \exists R > 0$ s.t. $|f(z)| < M + \delta$ for all $|z| \geq R, z \in \Omega$.

(Check: $D(\infty, r)$ is the complement of a compact set.)

$\therefore S \subset D(0, R)$.

Applying MMT (2) to $f|_S$, we see that

$$|f(z)| \leq \max_{\partial S} |f| \quad \forall z \in S.$$

But by defⁿ of S , it follows that $|f| = M + \delta$ on ∂S .

$$\therefore |f(z)| \leq M + \delta \quad \forall z \in S.$$

But by defⁿ of S , we have $|f(z)| > M + \delta$ for $z \in S$.
 $\therefore S = \emptyset$. \blacksquare

Remark. In our previous example, we have $\limsup_{H \ni z \rightarrow \infty} f(z) = \infty$.

Thus, this MMT did not apply!

Generalisations of MMT to unbounded domains.

Phragmén-Lindelöf Theorems.

Phragmén-Lindelöf, Phragmen-Lindelof

Liouville's Theorem: Bounded + entire \Rightarrow constant

Also, recall the following exercise (using Cauchy's estimate, for example):

If $f \in \Theta(\mathbb{C})$ and $|f(z)| \leq 1 + |z|^{\frac{1}{3}}$, then f is constant.

\hookrightarrow "Generalisation" of Liouville.

Similarly, we generalise MMT.

(Phragmén-Lindelöf)

Theorem A. Let $\Omega \subseteq \mathbb{C}$ be simply-connected, and $f \in \Theta(\Omega)$. Fix $M > 0$.
Let $\partial_\infty \Omega = I \cup \bar{I}$ be such that

(1) $\limsup_{\Omega \ni z \rightarrow a} |f(z)| \leq M \quad \text{for all } a \in I, \text{ and}$

(D) $\exists \phi \in \Theta(\Omega)$, nonvanishing and bounded on Ω such that

$$\limsup_{\Omega \ni z \rightarrow a} |f(z)(\phi(z))'| \leq M$$

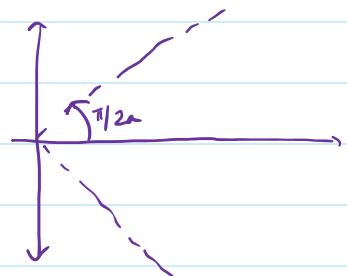
for all $a \in \mathbb{I}$ and for all $\eta > 0$.

Then, $|f| \leq M$ on Ω .

EXAMPLE. Fix $a > b_2$. Let $S\Omega = \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2a}\}$.

Let $f \in \Theta(\Omega)$ be s.t.

(a) $\overline{\lim}_{z \rightarrow z \in \partial \Omega} |f(z)| \leq M$, and



(b) $|f(z)| \leq A \exp(|z|^b)$ for $|z| \gg 1$,

using above theorem where A and b are positive constants such that $b < a$. Then, $|f(z)| \leq M \forall z \in \Omega$.

Clearly, Ω is s.c.

Now, we find $\phi \in \Theta(\Omega)$ as in P-L.

Consider $\phi(z) = \exp(-|z|^c)$, where $c > 0$ is chosen later.

Note that this is holo on Ω .

Also, $\phi(z) \neq 0 \forall z \in \Omega$

$$|\phi(z)| = |\exp(-|z|^c)| \quad \begin{matrix} \downarrow \\ z = r e^{i\theta}, \quad |\theta| < \pi/2a. \end{matrix}$$

$$= |\exp(-r^c e^{i\theta c})|$$

$$= \exp(-r^c \cos(c\theta)) \leq 1.$$

if $c < a$, then $\cos(c\theta) > 0$.

Thus, ϕ is bdd.

Take $I = \partial\Omega$ and $\text{II} = \{\infty\}$.

Now, fix $\eta > 0$ and for $z = re^{i\theta} \in \Omega$.
For large $|z|$, we have

$$\begin{aligned}|f(z)\phi(z)^\eta| &\leq A \exp(|z|^b) |\exp(-z^c)|^\eta \\&= A \exp(r^b - \eta r^c \cos(c\theta)) \quad \delta := \inf_{0 \leq \theta < \pi/2} \cos(c\theta). \\&\leq A \exp(r^b - \eta r^c \delta).\end{aligned}$$

The above goes to 0 if $c > b$.

Thus, we can choose any $c \in (b, a)$.
we are now done. □

Proof of P-L. Since Ω is sc and ϕ nonvanishing, ϕ has an analytic log, and ϕ^η makes sense as a holo. function.

Let $K > 0$ be s.t. $|\phi| \leq K$ on Ω .

Consider $g_n(z) := g(z) := \frac{f(z)\phi(z)^\eta}{K^n}$. $g \in \mathcal{O}(\Omega)$.

Note : $|g(z)| \leq |f(z)|$.

Thus,

$$\limsup_{z \rightarrow z \in I} |g(z)| \leq M.$$

On II :

$$\limsup_{z \rightarrow z \in \text{II}} \left| \frac{f(z)\phi(z)^\eta}{K^n} \right| \leq \frac{M}{K^n}.$$

Now, HMT (3) from earlier applied to g , we get that

$$|g_1| \leq \max\left(M, \frac{M}{K^\eta}\right) \quad \text{on } \Omega.$$

$$\Rightarrow |f(z)| \leq |\phi(z)|^{-\eta} \max(MK^\eta, M) \quad \forall z \in \Omega, \forall \eta > 0.$$

Fix $z \in \Omega$ and let $\eta \rightarrow 0^+$ to conclude. \square

Lecture 4 (13-01-2022)

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(Phragmén-Lindelöf)

Theorem B. Fix reals $a < b$, and $B > 0$.

Let $\Omega = \{z \in \mathbb{C} : a < \operatorname{Re}(z) < b\}$, and $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega})$.

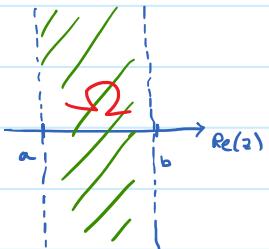
Assume that :

$$|f| < B \quad \text{on } \Omega,$$

$$|f| \leq 1 \quad \text{on } \partial\Omega.$$

Then,

$$|f| \leq 1 \quad \text{on } \bar{\Omega}.$$



Remark: Note that the above is a type of MMT.

Idea: Introduce a typical multiplicative factor g_ε with $\lim_{\varepsilon \rightarrow 0} g_\varepsilon = 1$, such that $|fg_\varepsilon| < M$ on the boundary of a BOUNDED subdomain Ω_ε of Ω . Then, apply usual MMT on Ω_ε . Moreover, pick the family $\{\Omega_\varepsilon\}_{\varepsilon > 0}$ nicely enough to cover all of Ω . Then take $\varepsilon \rightarrow 0$.

Proof. For each $\varepsilon > 0$, define $g_\varepsilon : \bar{\Omega} \rightarrow \mathbb{C}$ by

$$g_\varepsilon(z) := \frac{1}{1 + \varepsilon(z - a)}.$$

denominator is 0 if

$$z = a - \frac{1}{\varepsilon} \notin \bar{\Omega}.$$

For $z \in \bar{\Omega}$, we have :

$$\begin{aligned} |f(z)g_\varepsilon(z)| &\leq \frac{1}{|1 + \varepsilon(z - a)|} \leq \frac{1}{|\operatorname{Re}(1 + \varepsilon(z - a))|} \\ &= \frac{1}{|1 + \varepsilon(\operatorname{Re}(z) - a)|} \end{aligned}$$

$$|1 + \varepsilon(\operatorname{Re}(z) - a)|$$

$$\leq 1.$$

For $z = x + iy \in \bar{\Omega}$, we have:

Note: $|1 + \varepsilon(z - a)| \leq B$ on $\bar{\Omega}$ by continuity.

$$\begin{aligned} |f(z)g_\varepsilon(z)| &\leq \frac{B}{|1 + \varepsilon(z - a)|} \leq \frac{B}{|\operatorname{Im}(1 + \varepsilon(z - a))|} \\ &= \frac{B}{|\varepsilon y|}. \quad (*) \end{aligned}$$

Now, define $\Omega_\varepsilon := \{z \in \bar{\Omega} : -B/\varepsilon < y < B/\varepsilon\}$.

We have proven above that $|fg_\varepsilon| \leq 1$ on $\partial\Omega_\varepsilon$.

By the usual MMT, we have $|fg_\varepsilon| \leq 1$ on Ω_ε .

But also, by $(*)$, we see that $|fg_\varepsilon(z)| \leq 1$ if $y > \frac{B}{\varepsilon}$ as well!

Thus, for all z and for all $\varepsilon > 0$, we have the inequality. Fix z and let $\varepsilon \rightarrow 0^+$ to conclude. \square

Theorem C. Fix reals $a < b$, and $B > 0$.

Let $\Omega = \{z \in \mathbb{C} : a < \operatorname{Re}(z) < b\}$, and $f \in \mathcal{O}(\Omega) \cap C(\bar{\Omega}) \setminus \{0\}$.

Assume that $|f| < B$. Define $M: [a, b] \rightarrow [0, \infty)$ by

$$M(x) := \sup_{y \in \mathbb{R}} |f(x + iy)|.$$

Then $\log \circ M$ is a convex function on (a, b) .

Remarks: (i) For $a \leq x < v < y \leq b$:

$$M(v) \stackrel{(y-x)}{\leq} M(x) \stackrel{(y-v)}{\leq} M(v) \cdot \frac{(v-x)}{M(x)}.$$

$$M(v) \stackrel{(y-x)}{\leq} M(a) \stackrel{(y-v)}{\cdot} M(y) \stackrel{(v-x)}{\cdot}$$

(ii) Since $\log \circ M$ is convex on (a, b) , we get

$$M(x) \leq \max \{ M(a), M(b) \} \quad \forall x \in [a, b].$$

In particular, if $M(a) = M(b)$, then we get Theorem B.

(iii) By continuity, we have $|f| \leq B$ on $\partial\Omega$. Thus, $\sup_{\partial\Omega} |f| < \infty$.

By the above, we get

$$|f| \leq \sup_{\partial\Omega} |f| \quad \text{on } \Omega.$$

Proof. Suffices to show that

$$M(v) \stackrel{b-a}{\leq} M(a) \stackrel{b-v}{\cdot} M(b) \stackrel{v-a}{\cdot}$$

for $v \in (a, b)$.

What if $M(a)$ or $M(b) = 0$?

Take care of this separately.

Consider the entire function g defined as

$$g(z) := M(a)^{\frac{b-z}{b-a}} \cdot M(b)^{\frac{z-a}{b-a}}.$$

($\lambda^z := \exp(z \log \lambda)$)

Also note that g is nonvanishing.

$$\cdot |g(z)| = M(a)^{\frac{b-z}{b-a}} \cdot M(b)^{\frac{z-a}{b-a}}.$$

The above is continuous as a function of z and is non-vanishing. Then, $\exists c > 0$ s.t. $\frac{1}{|g|} \leq c$ on $\bar{\Omega}$.

$|g(z)|$

Now, consider $\frac{f}{g} \in \Omega(\Omega) \cap \ell^1(\Omega)$.

$$\left| \frac{f(a+iy)}{g(a+iy)} \right| = \left| \frac{f(a+iy)}{M(a)} \right| \leq 1 \quad \forall y \in \mathbb{R}.$$

$\Rightarrow \left| \frac{f}{g} \right| \leq 1 \text{ on } \partial\Omega.$

Moreover, $\left| \frac{f}{g} \right| \leq CB \text{ on } \Omega$

Thus, by Theorem B, we have $\left| \frac{f}{g} \right| \leq 1 \text{ on } \Omega \text{ or}$

$$|f| \leq |g| \text{ on } \Omega.$$

Expanding out, we get

$$|f(x+iy)|^{b-a} \leq M(a)^{b-x} M(b)^{x-a}$$

$\forall x \in (a, b)$

$\forall y \in \mathbb{R}.$

Take sup over $y \in \mathbb{R}$ and we are done. \square

Consequences of MMT.

Schwarz Lemma: Let $f \in \Omega(D(0,1))$ such that $f(0) = 0$ and $|f| \leq 1$.

Then,

(a) $|f'(0)| \leq 1$ and

(b) $|f(z)| \leq |z| \quad \forall z \in D(0,1).$

Moreover if equality holds either in (a) or for some $z \neq 0$ in (b), then $\exists \lambda \in S^1$ s.t. $f(z) = \lambda z$.

Sketch. Define $g: D(0,1) \rightarrow \mathbb{C}$ by $g(z) := \frac{f(z)}{z}$ holomorphically. Fix $r \in (0,1)$. On $|z|=r$, we have

$$|g(z)| = \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}.$$

$$\therefore |g| \leq \frac{1}{r} \text{ on } D(0, r) \text{ by MMT.}$$

Let $r \rightarrow 1$ appropriately to get $|g| \leq 1$ on $D(0, 1)$.
This gives (a) and (b). Equality in MMT $\Rightarrow \frac{f(z)}{z}$ is const. \blacksquare

Lecture 5 (17-01-2022)

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- $\mathbb{D} := D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.
 - $\text{Aut}(\mathbb{D}) := \{f : \mathbb{D} \rightarrow \mathbb{D} \mid f \text{ is bijective, } \{f^{-1} \in \Theta(\mathbb{D})\}\}$.
- \hookrightarrow group under composition
 $\text{Aut}(\mathbb{D})$

Automorphisms of \mathbb{D} fixing the origin: Automorphisms of the disc

Theorem. If $f \in \text{Aut}(\mathbb{D})$ and $f(0) = 0$, then f is a rotation, i.e.,
 $\exists \lambda \in \partial \mathbb{D}$ s.t. $f(z) = \lambda z \quad \forall z \in \mathbb{D}$.

Proof. Let $f \in \text{Aut}(\mathbb{D})$ with $f(0) = 0$.

By Schwarz, $|f'(0)| \leq 1$ and $|f(z)| \leq |z| + z$.

Moreover, $f' \in \text{Aut}(\mathbb{D})$ also fixes the origin.

By Schwarz again, $|f'^{-1}(0)| \leq 1$ and $|f^{-1}(z)| \leq |z| + z$.

Thus, it follows that $|f(z)| = |z| + z$ and hence, $\exists \lambda \in \mathbb{S}'$
 s.t. $f = (z \mapsto \lambda z)$. □

Möbius transforms:

Möbius, Mobius

Let $\alpha \in \mathbb{D}$, and consider $z \mapsto \frac{z - \alpha}{1 - \bar{\alpha}z}$, $z \in \mathbb{D}$.

Note that Ψ_α makes sense on $\mathbb{C} \setminus \{\bar{\alpha}\} \supseteq \mathbb{D}$.

Moreover, Ψ_α is holomorphic on \mathbb{D} , i.e., $\Psi_\alpha \in \Theta(\mathbb{D})$.

$\Psi_\alpha(\alpha) = 0$.

$\Psi_\alpha(\mathbb{D}) = ?$

Check: $|\Psi_\alpha(e^{it})| \leq 1$ for $t \in \mathbb{R}$.

Thus, by MMT $\Psi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$.

Also, $\Psi_\alpha : \mathbb{D} \rightarrow \mathbb{D}$ has inverse as $\Psi_{-\alpha}$.

Thus, $\Psi_\alpha \in \text{Aut}(\mathbb{D}) \quad \forall \alpha \in \mathbb{D}$.

Theorem. $\text{Aut}(\mathbb{D}) = \{\lambda \Psi_\alpha : \lambda \in \partial \mathbb{D}, \alpha \in \mathbb{D}\}$.

Proof:

(2) is clear.

(\Leftarrow) Let $f \in \text{Aut}(\mathbb{D})$.

Put $\alpha := f(0)$.

Then, $(f \circ \varphi_{-\alpha}) \in \text{Aut}(\mathbb{D})$ and $(f \circ \varphi_{-\alpha})(0) = 0$.

Thus, $f \circ \varphi_{-\alpha}$ is r_λ (rotation by λ) for some $\lambda \in \partial \mathbb{D}$.

$\Rightarrow f = r_\lambda \circ \varphi_\lambda$. □

Let $\alpha, \beta \in \mathbb{D}$. Let $f: \mathbb{D} \rightarrow \overline{\mathbb{D}}$ be holomorphic and $f(\alpha) = \beta$. Among all such f , what is the maximum possible value of $|f'(\alpha)|$?

$$\begin{array}{ccc}
 \mathbb{D} & \xrightarrow{f} & \overline{\mathbb{D}} \\
 \psi_{-\alpha} \uparrow & \Downarrow & \downarrow \psi_\beta \\
 \mathbb{D} & \xrightarrow{g} & \overline{\mathbb{D}}
 \end{array}
 \quad g := \psi_\beta \circ f \circ \psi_{-\alpha}.$$

$g(0) = 0$.

By Schwarz, we have $|g'(0)| \leq 1$.

Using chain rule, we have:

$$g'(0) = \psi'_\beta(f(\psi_{-\alpha}(0))) \cdot f'(\psi_{-\alpha}(0)) \cdot \psi'_{-\alpha}(0)$$

$$= \psi'_\beta(f(\alpha)) \cdot f'(\alpha) \cdot \psi'_{-\alpha}(0)$$

$$= \psi'_\beta(\beta) \cdot f'(\alpha) \cdot \psi'_{-\alpha}(0)$$

$$= \frac{1 - \bar{\beta}\beta}{(1 - \bar{\beta}\beta)^2} \cdot f'(\alpha) \cdot \frac{1 - (\alpha \bar{\alpha})}{1^2}$$

$$\psi_\beta(z) := \frac{z - \beta}{1 - \bar{\beta}z}$$

$$\Rightarrow \psi'_\beta(z) = \frac{1 - \bar{\beta}z - (z - \beta)(-\bar{\beta})}{(1 - \bar{\beta}z)^2}$$

$$= \frac{1 - |\alpha|^2}{1 - |\beta|^2} \cdot f'(\alpha)$$

$$\therefore |f'(\alpha)| \leq \frac{1 - |\beta|^2}{1 - |\alpha|^2}$$

Note that equality is possible. For example, $f = \psi_{-\beta} \circ \varphi_\alpha$. In fact, it happens iff $\exists \lambda \in S^1$ s.t.

$$f = \varphi_{-\beta} \circ r_\gamma \circ \varphi_\alpha.$$

Ex. Calculate $\text{Aut}(\mathbb{D} \setminus \{0\})$.

Towards the Riemann-Mapping Theorem

$$\Theta(\Omega) \subseteq C^\circ(\Omega; \mathbb{C}).$$

Want to make this a metric space.

let us consider $\Omega = \mathbb{D}$.

There is a sequence $\{K_n\}$ of compact sets in \mathbb{C} s.t.:

$$(1) \quad \mathbb{D} = \bigcup_{n=1}^{\infty} K_n^\circ,$$

$$(2) \quad K_n \subset K_{n+1}^\circ \quad \text{for all } n \in \mathbb{N},$$

$$(3) \quad \text{for each compact } K \subset \mathbb{D}, \quad \exists n \in \mathbb{N} \text{ s.t. } K \subseteq K_n.$$

One can take $K_n := D(0, 1 - \frac{1}{n})$, for example.

Claim. One can do the above for any open $\Omega \subseteq \mathbb{C}$.

Given any open $\Omega \subseteq \mathbb{C}$, \exists a sequence $\{K_n\}_n$ of compact subset of \mathbb{C} s.t.

(Compact exhaustion)

$$(1) \quad \Omega = \bigcup_{n=1}^{\infty} K_n^\circ,$$

$$(2) \quad K_n \subset K_{n+1}^\circ \quad \forall n \in \mathbb{N},$$

$$(3) \quad \text{for any compact } K \subset \Omega, \quad \exists n \in \mathbb{N} \text{ s.t. } K \subseteq K_n.$$

Proof. For each $n \in \mathbb{N}$, let

$$K_n := \overline{D(0, n)} \cap \{z \in \Omega : \text{dist}(z, \partial\Omega) \geq \chi_n^2\}.$$

Check that K_n satisfies (1) - (3). ◻

Using the above, we define a metric on $C^0(\Omega; \mathbb{C})$.

Fix some $\{K_n\}_n$ as given by compact exhaustion.

Let $f, g \in C^0(\Omega; \mathbb{C})$.

Define

$$p_n(f, g) := \sup_{z \in K_n} |f(z) - g(z)|.$$

Finally, define

$$p(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(f, g)}{1 + p_n(f, g)}.$$

Ex $(C^0(\Omega; \mathbb{C}), p)$ is a metric space.

② A sequence $\{f_k\}_{k \geq 1}$ converges to f in $(C^0(\Omega; \mathbb{C}), p)$ iff $f_k \rightarrow f$ uniformly on compact subsets of Ω .

What are open sets in $(C^0(\Omega; \mathbb{C}), p)$?

↙ This ex. shows that the topology does not depend on $\{K_n\}_{n \geq 1}$.

Lecture 6 (20-01-2022)

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$$\Omega(\Omega) \subseteq \ell^0(\Omega; \mathbb{C})$$

metric space

↪ subspace topology

Prop. $\Omega(\Omega)$ is closed in $\ell^0(\Omega; \mathbb{C})$.

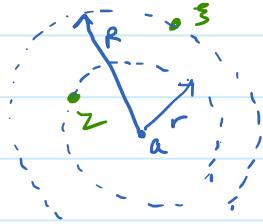
That is, if $(f_n)_n \in \Omega(\Omega)^{*}$ and $f_n \rightarrow f$ in $\ell^0(\Omega; \mathbb{C})$, then $f \in \Omega(\Omega)$.

Moreover, $f_n^{(k)} \rightarrow f^{(k)}$ in $\Omega(\Omega)$ for all $k \geq 1$.

Proof. To show $f \in \Omega(\Omega)$, we may assume Ω is a disc and use Morera's theorem and that $\int_T f_n = 0$ for every triangle $T \subseteq \Omega$.

We now show that $f_n^{(k)} \rightarrow f^{(k)}$ uniformly on compact subsets of Ω . Suffices to prove it for $k=1$ and use induction.

$$(f_n' - f')(z) = \frac{1}{2\pi i} \int_{|\xi - a|=R} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2} d\xi$$



for all $z \in \overline{D(a, r)}$.

$$[D(a, r) \subsetneq D(a, R) \subseteq \Omega]$$

$$\Rightarrow |f_n'(z) - f'(z)| \leq \frac{1}{2\pi} \int_{|\xi - a|=R} \frac{|f_n(\xi) - f(\xi)|}{|\xi - z|^2} d\xi$$

$$|\xi - a| = R$$

$$\leq \frac{1}{(R-r)^2} \left(\sup_{\partial D(a, R)} |f_n - f| \right)$$

↓
0 as $n \rightarrow \infty$.

$$\Rightarrow |f_n'(z) - f'(z)| \rightarrow 0 \quad \text{uniformly for } z \in \overline{D(a, r)}$$

Thus, $f_i \rightarrow f$ uniformly on closed discs.

Now, given any arbitrary $K \subseteq \mathbb{R}$, we can cover it by finitely many closed discs contained in Ω . \square

Normal Families.

Normal family

Defn. Let $\Omega \subseteq \mathbb{C}$ be a domain, and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$.

\mathcal{F} is said to be normal if for every sequence $(f_n)_n \in \mathcal{F}^{\mathbb{N}}$, it is possible to extract a subsequence $(f_{n_k})_k$ such that either

(a) $(f_{n_k})_k$ converges uniformly on compact subsets of Ω , or

(b) given any pair of compact sets $K \subset \Omega$, $L \subset \mathbb{C}$,
 $\exists k_0 = k_0(K, L) \in \mathbb{N}$ s.t.

$$f_{n_k}(K) \cap L = \emptyset \quad \forall k \geq k_0.$$

$(f_{n_k} \rightarrow \infty \text{ uniformly on compact subsets of } \Omega.)$

EXAMPLES. (i) $\Omega_1 = D(0, 1)$.

$$\mathcal{F}_1 = \left\{ z \mapsto z^n : n \in \mathbb{N} \right\}. \xrightarrow{\text{normal because of (a)}}$$

(ii) $\Omega_2 = \{z \in \mathbb{C} : |z| > 1\}$.

$$\mathcal{F}_2 = \{z \mapsto z^n : n \in \mathbb{N}\}. \xrightarrow{\text{normal because of (b)}}$$

(iii) $\Omega_3 = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$.

$$\mathcal{F}_3 = \{z \mapsto z^n : n \in \mathbb{N}\}.$$

The above is not normal. Consider the

sequence $f_n = (z \mapsto z^n) \in \mathcal{F}_3$.

Consider $K = \overline{D(1, \epsilon)}$ for some small $\epsilon > 0$

s.t. $K \subset \Omega_3$.

Consider $K \cap \Omega_1$ and $K \cap \Omega_2$ to see f_z is

NOT NORMAL.

(iv) Let $\Omega \subseteq \mathbb{C}$ be a domain.

$\mathcal{F} = \{z \mapsto z^n : n \in \mathbb{N}\}$ is NOT NORMAL
if $\partial D(0, 1) \subseteq \Omega$.

REMARKS. (i) If (a) is true and $f_{n_k} \rightarrow f$, then $f \in \Theta(\Omega)$.
(ii) However, f above need not be in \mathcal{F} .

Theorem (Montel's Theorem)

Montel's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $\mathcal{F} \subseteq \Theta(\Omega)$ be locally uniformly bounded on Ω , i.e., for all compact $K \subseteq \Omega$, $\exists M = M(K) > 0$ such that

$$|f(z)| \leq M \quad \forall f \in \mathcal{F}, \forall z \in K.$$

Then, \mathcal{F} is a normal family.

In fact, \mathcal{F} is normal and satisfying (a) of the def'.

EXAMPLE. Let $\Omega \subseteq \mathbb{C}$ be a domain.

Then, given any subset $\mathcal{F} \subseteq \{f \in \Theta(\Omega) : f(\Omega) \subseteq D(0, 1)\}$, Montel's theorem asserts that \mathcal{F} is normal!

Recall:

Theorem (Arzelà - Ascoli Theorem)

Let $\mathcal{F} \subseteq C^0(\Omega; \mathbb{C})$.

in $(C^0(\Omega; \mathbb{C}), \|\cdot\|)$

Every sequence in \mathcal{F} admits a convergent subsequence iff :

(i) \mathcal{F} is pointwise bounded, i.e., $\exists M: \Omega \rightarrow [0, \infty)$ s.t.

$$|f(z)| \leq M(z) \quad \forall z \in \Omega, \text{ and}$$

(ii) \mathcal{F} is equicontinuous at each point of Ω .

Proof of Montel's Theorem: Let \mathcal{F} be as given.

It suffices to show that \mathcal{F} is

Hour of - more - more. In , -- as given.

It suffices to show that \mathcal{F} is equicontinuous at each $z_0 \in \Omega$.

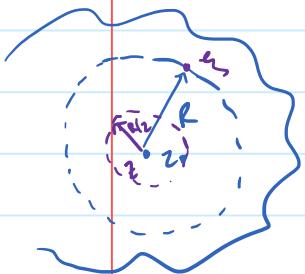
That is : $\forall z_0 \in \Omega \quad \forall \epsilon > 0 \quad \exists \delta = \delta(z_0, \epsilon) > 0$ s.t.

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

$\forall z \in \Omega \quad \forall f \in \mathcal{F}$.

Let $\overline{D(z_0, R)} \subseteq \Omega$. Then, $\exists M > 0$ s.t.

$$|f(z)| \leq M \quad \forall z \in \overline{D(z_0, R)} \quad \forall f \in \mathcal{F}$$



for $z \in D(z_0, R)$:

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - z_0| = R} \frac{f(\xi)}{\xi - z} d\xi \quad \forall z \in D(z_0, R)$$

$$|f(z_0) - f(z)| \leq \frac{1}{2\pi} \left| \int_{|\xi - z_0| = R} f(\xi) \left(\frac{1}{\xi - z_0} - \frac{1}{\xi - z} \right) d\xi \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\xi - z_0| = R} \frac{f(\xi)(z_0 - z)}{(\xi - z_0)(\xi - z)} d\xi \right|$$

$$\stackrel{\text{(we took } z \in D(z_0, R/2))}{\leq} \frac{1}{2\pi} \cdot (2\pi R) \cdot \frac{M \cdot |z_0 - z|}{R \cdot R/2}.$$

Thus, for all $f \in \mathcal{F}$ and for all $z \in D(z_0, R/2)$, we have

$$|f(z) - f(z_0)| \leq \left(\frac{2M}{R} \right) \cdot |z - z_0|.$$

Equivcontinuity follows.

Lecture 7 (24-01-2022)

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EXAMPLE. Montel's Theorem fails on \mathbb{R} .

Indeed, consider the family $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$, where $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f_n(x) := \sin(nx)$.

Clearly, \mathcal{F} is locally uniformly bounded as $|f_n(x)| \leq 1 \quad \forall x \in \mathbb{R} \quad \forall n \in \mathbb{N}$.

However, given any $\delta > 0$, pick n s.t. $x = \frac{\pi}{2n} < \delta$.

$$\text{Then, } |f_n(x) - f_n(0)| = |\sin\left(\frac{\pi}{2}\right)| = 1.$$

Thus, no δ exists for $\epsilon = 1$.

Thus, \mathcal{F} is not equicontinuous.

Theorem (Hurwitz's Theorem)

Hurwitz's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \Theta(\Omega)^{\mathbb{N}}$, $f_n \rightarrow f$ in $\Theta(\Omega)$.

Suppose that $\exists a \in \Omega$, $r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$ such that f has no zeroes on $\partial D(a,r)$.

Then, $\exists N \in \mathbb{N}$ such that f and f_n have the same number of zeroes in $D(a,r)$ for all $n \geq N$.
Counting multiplicities

Remark. Note that if f is not identically zero, one can find $a \in \Omega, r > 0$ as stated. In fact, for any $a \in \Omega$, we can find an $r > 0$ since zeroes are isolated!

Proof. Since $f \neq 0$ on $\partial D(a,r)$, $\min_{\partial D(a,r)} |f| =: \delta > 0$.

Since $f_n \rightarrow f$ uniformly on compact subsets of Ω , it follows that $\exists N \in \mathbb{N}$ s.t.

$$|f_n(z) - f(z)| < \frac{\delta}{2} \quad \forall z \in \partial D(a,r) \quad \forall n \geq N.$$

$$\text{Thus, } |f_n(z) - f(z)| < |f(z)| \quad \forall z \in \partial(a, r) \text{ and } n \gg 0.$$

Now, by Rouché's theorem, we are done. \square

Corollary 1. Let Ω be a domain in \mathbb{C} , $f_n \in \Omega(\Omega)$ $\forall n$,
 $f_n \rightarrow f$ in $\Omega(\Omega)$.

Suppose that each f_n is non-vanishing on Ω .

Then, either $f = 0$ or f is also non-vanishing.

Corollary 2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \Omega(\Omega)^{\mathbb{N}}$, $f_n \rightarrow f$ in $\Omega(\Omega)$.

Suppose that each f_n is injective on Ω , then f is injective on Ω .

Riemann Mapping Theorem

Theorem (RMT). Let $\Omega \subsetneq \mathbb{C}$ be simply-connected.

Then, Ω is biholomorphic to $D(0, 1)$.

Remark. ① \mathbb{C} cannot be biholo. to $D(0, 1)$, by Liouville.

② If Ω is biholo. to $D(0, 1)$, then Ω is homeomorphic to $D(0, 1)$ and thus, simply-connected.

Question. Is this Riemann map unique?

$$f: \Omega \rightarrow D(0, 1)$$

No. These will precisely "differ" by $\text{Aut}(D)$.

Proof of RMT. Let $\Omega \subsetneq \mathbb{C}$ be as specified.

Fix $p \in \Omega$.

Let

$$\mathcal{F} = \{f \in \Omega(\Omega) : f(p)=0, f \text{ is injective}, f(\Omega) \subseteq D(0, 1)\}.$$

If we can find $f \in \mathcal{F}$ such that $f(0) = D(0, 1)$, then we are done since f is also holomorphic.

Steps: (I) $\mathcal{F} \neq \emptyset$

(II) $\sup_{f \in \mathcal{F}} |f'(p)| = |f''(p)|$ for some $f \in \mathcal{F}$.

(III) f_0 (as above) is onto.

Motivation: Suppose we a compact exhaustion $(k_n)_{n \in \mathbb{N}}$ of Ω with $p \in k_n \forall n$.

By choosing f as in (II), we get a function which "starts out fastest" at p . Then, $\bigcup_{n=1}^{\infty} f_0(k_n) = D(0, 1)$ is likely.

(I) To show: $\mathcal{F} \neq \emptyset$.

(a) If Ω is bounded, then $z \mapsto \frac{z-p}{M}$ works for an appropriate $M \gg 0$.

(b) As $\Omega \subsetneq \mathbb{C}$, pick $Q \in \mathbb{C} \setminus \Omega$.

Let $\phi(z) := z - Q$ is nonvanishing on Ω .

As Ω is simply-connected, \exists a holomorphic square root of ϕ .

$\exists h \in \mathcal{O}(\Omega)$ s.t. $(h(z))^2 = \phi(z) \quad \forall z \in \Omega$.

Note that since ϕ is injective, we get

$h(z_1) \neq h(z_2)$ and $h(z_1) \neq -h(z_2)$

for $z_1 \neq z_2 \in \Omega$.

In particular, h is nonconstant on Ω .

Thus, h is an open map.

Let $b \in h(\Omega)$. Then, $D(b, r) \subseteq h(\Omega)$ for some $r > 0$.

Then, $D(-b, r) \cap h(\Omega) = \emptyset$ by earlier observation.

For $z \in \Omega$, define $f(z) := \frac{z-b}{r}$.

$$2(b + h(z))$$

Then, $f \in O(\mathcal{O})$ and $|f(z)| \leq \frac{1}{2}$.

Clearly, f is injective.

$f(p) = 0$ not guaranteed but just compose with appropriate Möbius transform.

Then, $f \neq \emptyset$.

Lecture 8 (27-01-2022)

27 January 2022 14:00

(II) To show: $\exists g \in \mathcal{F}$ s.t. $\sup_{f \in \mathcal{F}} |f'(p)| = |g'(p)|$.

Since $\mathcal{F} \neq \emptyset$, $\lambda := \sup_{f \in \mathcal{F}} |f'(p)| > 0$.

(Injective $\Rightarrow f'$ never vanishing in (Analysis!))

Thus, $\exists (f_n)_{n \in \mathbb{N}} \in \mathcal{F}^*$ s.t. $|f'_n(p)| \rightarrow \lambda$ as $n \rightarrow \infty$.
 ($\lambda = \infty$ is not ruled out yet.)

Note that Montel's theorem ensures that \mathcal{F} is a normal family.

Thus, we may assume $(f_n)_n$ itself converges to g in $\Omega(\Omega)$. Then, $f'_n \rightarrow g'$ in $\Omega(\Omega)$.

In particular,

$$|g'(p)| = \lambda. \quad (\text{Also shows that } \lambda < \infty (!))$$

Now, we show that $g \in \mathcal{F}$ to conclude!

- As $g := \lim_n f_n$, it follows that $g(p) = 0$.
- As $f_n(\Omega) \subseteq D(0, 1)$, we have $g(\Omega) \subseteq \overline{D(0, 1)}$.
 (WTS: $g(\Omega) \subseteq D(0, 1)$)
 By MMT, if $g(\Omega) \cap 2D(0, 1) \neq \emptyset$, then g is constant.
 But $g(p) = 0$, thus it can't happen.

Thus, $g(\Omega) \subseteq D(0, 1)$.

- It follows from corollary 2 of Hurwitz's theorem that g is injective on Ω (g is not constant since $|g'(p)| = \lambda \neq 0$).

(III) We show that $g(\Omega) = D(0, 1)$.

Suppose not. Then, $g(\Omega) \not\subseteq D(0, 1)$. Pick $a \in D(0, 1) \setminus g(\Omega)$.

We construct $s \in \mathcal{F}$ s.t. $|s'(p)| > |g'(p)|$, giving us the desired contradiction.

Define $p = \psi_a \circ g$.

$$p(z) = \frac{g(z) - a}{1 - \bar{a}g(z)} ; z \in \Omega.$$

$$p(p) = -a.$$

- $p \in \Theta(\Omega)$, $p(\Omega) \subseteq D(0, 1)$.
- p never vanishes on Ω .

As Ω is simply-connected, it follows that $\exists h \in \Theta(\Omega)$ s.t. $p(z) = (h(z))^2 \forall z \in \Omega$.

Then, $h(\Omega) \subseteq D(0, 1)$. $(h(p))^2 = -a$.

Let $s := \psi_{h(p)} \circ h : \Omega \rightarrow D(0, 1)$.

g injective $\Rightarrow p$ injective $\Rightarrow h$ injective $\Rightarrow s$ injective.

- $s \in \Theta(\Omega)$,
- $s(p) = 0$,
- $s(\Omega) \subseteq D(0, 1)$,
- s injective.

Thus, $s \in \mathcal{F}$.

$$s(z) = (\psi_{h(p)} \circ h)(z) = \frac{h(z) - h(p)}{1 - \overline{h(p)}h(z)}.$$

$$s'(z) = \frac{h'(z) (1 - \overline{h(p)} h(z)) - (h(z) - h(p)) (-\overline{h(p)} h'(z))}{(1 - \overline{h(p)} h(z))^2}$$

$$\therefore s'(p) = \frac{h'(p)}{|1 - h(p)|^2}.$$

$$(h(z))^2 = g(z) = (\gamma_a \circ g)(z)$$

$$= \frac{g(z) - a}{1 - \bar{a}g(z)}.$$

$$\Rightarrow 2h(z)h'(z) = \frac{1}{(1 - \bar{a}g(z))^2} (g'(z)(1 - \bar{a}g(z)) - (g(z) - a)(-\bar{a}g'(z)))$$

$$\Rightarrow 2h(p)h'(p) = g'(p)(1 - |\alpha|^2).$$

($g(p) = 0$)

$$\therefore s'(p) = \frac{(1 - |\alpha|^2) g''(p)}{2h(p) (1 - |h(p)|^2)} \quad \left((h(p))^2 = -a \right)$$

$$= \frac{(1 - |\alpha|^2) g'(p)}{2h(p) (1 - |\alpha|)}$$

$$= \frac{1 + |\alpha|}{2h(p)} g'(p).$$

$$\Rightarrow \frac{|s'(p)|}{|g'(p)|} = \frac{1 + |\alpha|}{2\sqrt{|\alpha|}} > 1. \quad \begin{matrix} \text{as } GM \\ \alpha \in D(0,1). \end{matrix}$$

This is the desired contradiction! \square

Remark. The only property of simple-connectedness that we used was that every nonvanishing function has a holomorphic square root.

Example. $H = \{z \in C : \operatorname{Im} z > 0\}.$

$$\pi: H \rightarrow \mathbb{C}^+ \cup \{0\} \quad z \mapsto e^{iz} \quad \text{is a } 1:1 \text{ map.}$$

Then, $H \subseteq \mathbb{C}$ is a simply-connected domain.

We have an explicit biholomorphism $f: H \rightarrow D(0,1)$
given by $z \mapsto \frac{z-i}{z+i}$.



Next up: Weierstrass Factorisation Theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{C}$. Then, f is either identically zero on Ω or $Z(f) := \{z \in \Omega : f(z) = 0\}$ is discrete in Ω .

Q. Let $A \subseteq \mathbb{C}$ be discrete. Can we find $f \in \mathcal{O}(\Omega)$ such that $Z(f) = A$?

Note: A must be countable. If finite, consider polynomials.

Now, assume $(a_n)_{n \in \mathbb{N}}$ is an enumeration of A .

Naive guess: $f(z) = (z - a_1)(z - a_2) \cdots (z - a_n) \cdots$.

How to make sense of f ?

Another attempt: Construct $f_1, f_2, \dots \in \mathcal{O}(\mathbb{C})$ s.t. $Z(f_n) = \{a_n\}$
and put $f = \prod_n f_n$.

need to make
sense of infinite products.

Infinite Products

Infinite products

Def. Suppose that $(u_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Define the sequence $(p_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$
by

$$p_n := (1+u_1) \cdots (1+u_n).$$

If $\lim_{n \rightarrow \infty} p_n = p$ exists (in \mathbb{C}), then we write

$$p = \prod_{n=1}^{\infty} (1+u_n).$$

p_n are called the partial products of the infinite product $\prod_{n=1}^{\infty} (1+u_n)$.

In this case, we say that $\prod_{n=1}^{\infty} (1+u_n)$ converges (to p).

- Suppose that $z_n \neq 0 \ \forall n$. Assume $z := \prod_{n=1}^{\infty} z_n$ exists and $z \neq 0$.

Let $p_n := z_1 \dots z_n$. Then, $\lim_{n \rightarrow \infty} (z_n) = \lim_{n \rightarrow \infty} \left(\frac{p_{n+1}}{p_n} \right) = \frac{\lim_{n \rightarrow \infty} p_{n+1}}{\lim_{n \rightarrow \infty} p_n} = \frac{z}{z} = 1$.

(Each p_n is nonzero and $p_n \rightarrow z \neq 0$.)

Lemma

Let $u_1, \dots, u_N \in \mathbb{C}$. Define $p_n := \prod_{n=1}^N (1+u_n)$, $p_N^* := \prod_{n=1}^N (1+|u_n|)$.

Then,

$$(i) \quad p_N^* \leq \exp(|u_1| + \dots + |u_N|),$$

$$(ii) \quad |p_N - 1| \leq p_N^* - 1.$$

Lecture 9 (31-01-2022)

31 January 2022 14:03

Theorem

Let X be a metric space. Let $u_n: X \rightarrow \mathbb{C}$ be a sequence of functions such that $\sum_{n=1}^{\infty} |u_n|$ converges uniformly to a bounded function. (Say, bounded by $M > 0$.)

Then, (1) $\prod_{n=1}^{\infty} (1 + u_n)$ converges uniformly on X .

Define $f(x) := \prod_{n=1}^{\infty} (1 + u_n(x))$ for $x \in X$.

(2) For $x_0 \in X$: $f(x_0) = 0 \Leftrightarrow u_M(x_0) = -1$ for some $M \in \mathbb{N}$.

(3) For every permutation $\sigma \in S_N$, the infinite product

(Rearrangement) $\prod_{k=1}^{\infty} (1 + u_{\sigma(k)}(x))$ converges to $f(x)$, for all $x \in X$.

Proof. (1) Let $p_N(x) := \prod_{n=1}^N (1 + u_n(x))$, $x \in X$.

We will show that $(p_N)_{N=1}^{\infty}$ is uniformly Cauchy on X .

For $M > N$, note

$$\begin{aligned}
 |p_M(x) - p_N(x)| &= \left| p_N(x) \cdot \prod_{n=N+1}^M (1 + u_n(x)) - p_N(x) \right| \\
 &= |p_N(x)| \cdot \left| \prod_{n=N+1}^M (1 + u_n(x)) - 1 \right| \quad \text{last bc's last lemma} \\
 &\leq |p_N(x)| \left[\prod_{n=N+1}^M (1 + |u_n(x)|) - 1 \right] \quad \text{--} \\
 &\leq |p_N(x)| \left[\exp \left(\sum_{n=N+1}^M |u_n(x)| \right) - 1 \right] \\
 &\quad \hookrightarrow \text{this term is uniformly Cauchy since } \sum |u_n| \text{ converges uniformly}
 \end{aligned}$$

Cauchy since $\sum f_n$ converges uniformly

$$\leq \exp(M) \cdot (\text{small}). \quad \checkmark$$

(2) Let f denote the limit. Let $x \in \mathbb{C}$ be s.t. $p_n(x) \neq 0 \forall n$.

From the above, given $\epsilon = \frac{\epsilon}{4}$, we can get N_0 s.t.

$$|p_M(x) - p_{N_0}(x)| < 2|p_{N_0}(x)| \in \forall M > N_0.$$

Then, $|f(x)| \geq (1 - 2\epsilon) |p_{N_0}(x)|$.

In particular, $f(x) \neq 0$.

Thus, $f(x) = 0 \Rightarrow p_n(x) = 0 \text{ for some } n$

$\Rightarrow 1 + u_n(x) = 0 \text{ for some } n$

$\Rightarrow u_n(x) = -1 \text{ for some } n$. \checkmark

(3) Exercise. \square

Theorem: let Ω be a domain in \mathbb{C} . Let $(f_n)_n \in \Theta(\Omega)^{\mathbb{N}}$ be such that no f_n is identically zero.

Suppose that $\sum_{n=1}^{\infty} |1 - f_n|$ converges uniformly on compact subsets of Ω .

(1) Then, $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of Ω .

Consequently $f := \prod_{n=1}^{\infty} f_n$ is holomorphic.

(2) Let $a \in \Omega$. If $f(a) = 0$, then $f_n(a) = 0$ for some n .

Moreover, this is true for only finitely many n .

Lastly,

$$\text{ord}_f(a) = \sum_{n=1}^{\infty} \text{ord}_{f_n}(a).$$

multiplicity

this is only nonzero for finitely many.

Proof

(1) follows from earlier by taking $u_n := f_n - 1$.

(2) Each f_n has countably many zeros. By (2) of earlier thm,
 $Z(f) \subseteq \bigcup_{n=1}^{\infty} Z(f_n)$.

$\therefore Z(f)$ is countable. $\therefore f \neq 0$ on Ω .

$\therefore Z(f)$ is discrete in Ω . Let $a \in \Omega$ be s.t. $f(a) = 0$.

Pick $r > 0$ s.t. $f(z) \neq 0$ for $z \in D(a, r) \setminus \{a\}$.

Consequently:

- each f_n is nonzero on $D(a, r) \setminus \{a\}$,

- $f_n(a) = 0$ for some $n \in \mathbb{N}$.

As $\sum |1-f_n|$ converges uniformly ^{on $\{a\}$} , we have $f_n(a) \rightarrow 1$.
 $\therefore f_n(a) = 0$ only for finitely many n .

To conclude: $A := \{n \in \mathbb{N} : f_n(a) = 0\}$ is a finite nonempty subset of \mathbb{N} .

Write $f(z) := \prod_{n \in A} f_n(z)$

$$= \prod_{n \in A} f_n(z) \underbrace{\prod_{n \notin A} f_n(z)}$$

rearrangement

holomorphic, nonvanishing on $D(a, r)$

$$\therefore \operatorname{ord}_f(a) = \sum_{n \in A} \operatorname{ord}_{f_n}(a).$$

□

Lecture 10 (03-02-2022)

03 February 2022 14:00

If we can find $g_k \in \mathcal{O}(\Omega)$ for $k \in \mathbb{N}$ s.t.

(i) g_{k_∞} has no zeroes on Ω , and

(ii) $\sum_{k=1}^{\infty} |1 - (z - z_k)g_k(z)|$ converges uniformly on compact ...,

$$\text{then } z \mapsto \prod_{k=1}^{\infty} (z - z_k)g_k(z) \in \mathcal{O}(\Omega)$$

and the zeroes are precisely $\{z_k\}_{k=1}^{\infty}$.

Given: $g_k = \exp(h_k)$ for some $h_k \in \mathcal{O}(\Omega)$.
 $(\because \text{we want } g_k \neq 0)$

Elementary Factors:

Weierstrass elementary factors

Defn. $E_0(z) = 1 - z \quad \text{for } z \in \mathbb{C}$.

For $p \in \mathbb{N}$, define

$$E_p(z) := (1-z) \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right).$$

These functions are called (Weierstrass) Elementary factors.

Below, we have $p \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$.

- Each E_p vanishes precisely at 1.
- 1 is a simple zero (order = 1) for each E_p .
- $E_p(0) = 1$

- For $|z| < 1$,

$$E_p(z) = (1-z) \exp\left(\sum_{k=1}^p \frac{z^k}{k}\right)$$

$$= (1-z) \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k}\right) \exp\left(-\sum_{k=p+1}^{\infty} \frac{z^k}{k}\right)$$

Heuristic!

~~now~~ $(\sum_{k=1}^{\infty} z^k) - (\sum_{k=p+1}^{\infty} \bar{z}^k)$
 branch of \log of $z \mapsto \frac{1}{1-z}$
 on $\Delta(0,1)$

$$= (1-z) \cdot \frac{1}{1-z} \cdot \exp\left(-\sum_{k=p+1}^{\infty} \frac{z^k}{k}\right)$$

$$= \exp\left(-\sum_{k=p+1}^{\infty} \frac{z^k}{k}\right)$$

$$= 1 - \frac{z^{p+1}}{p+1} + \text{higher order}$$

HAND-WAAY! Thus, if p is large, we expect $E_p \approx 1$.

More precisely:

Lemma For every $p \geq 0$,

$$|1 - E_p(z)| \leq |z|^{p+1} \quad \text{if } |z| \leq 1.$$

Proof. Fix $p \geq 0$.

$$\text{Write } E_p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n.$$

$$\Rightarrow E'_p(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

$(p=0 \text{ is clear})$

$(\text{This expansion is valid on } \mathbb{C} \text{ since } E_p \text{ is entire.})$
 $0 = E_p(1) = 1 + \sum_{n=1}^{\infty} a_n$

$$\text{OR}, \quad E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right)$$

$$\Rightarrow E'_p(z) = -\exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right)$$

$$+ (1-z) \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right) (1+z+\dots+z^{p-1})$$

$$= \exp\left(z + \dots + \frac{z^p}{p}\right) \left[(-1) + (1-z) \left(\frac{1-z^p}{1-z}\right)\right]$$

$$= \exp\left(\dots\right) \left[(-1) + (1-z^p) \right]$$

$$= -z^p \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right).$$

$\therefore E_p'$ has a zero of order p at the origin.
 Thus, $a_1 = \dots = a_p = 0$.

$$\text{Thus, } E_p(z) = 1 + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \dots$$

Also, equating

$$-z^p \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right) = \sum_{n=p+1}^{\infty} a_n z^{n-1}$$

shows us that $a_n \in \mathbb{R} \quad \forall n$ and $a_n \leq 0 \quad \forall n \geq p+1$.
 Coefficients here are +ve

$$\begin{aligned} \text{For } |z| \leq 1: \quad |E_p(z) - 1| &= \left| \sum_{n=p+1}^{\infty} a_n z^n \right| \\ &= |z|^{p+1} \left| \sum_{n=p+1}^{\infty} a_n z^{n-p-1} \right| \\ &\leq |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n| \quad \because a_n \leq 0 \quad \forall n \geq p+1 \\ &= -|z|^{p+1} \sum_{n=p+1}^{\infty} a_n \\ &= -|z|^{p+1} (E_p(1) - 1) \\ &= |z|^{p+1}. \end{aligned}$$

Remark. The function $z \mapsto E_p(\frac{z}{a})$ has a simple zero at $z = a$ (and no other zeros).

(Weierstrass Product Theorem)

Theorem Let $(a_n)_{n \geq 1} \in \mathbb{C}^\times$ be such that $a_n \neq 0 \quad \forall n \geq 1$ and $|a_n| \rightarrow \infty$ as $n \rightarrow \infty$.

(Note: the sequence need not consist of distinct points.)

However, $|a_n| \rightarrow \infty$ forces that no point appears inf often.)

IF $(p_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$ is such that

$$\sum_{n=1}^{\infty} \left(\frac{1}{r} \right)^{p_n + 1} < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty$$

for every $r > 0$, THEN:

(i) $\prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{a_n} \right)$ converges in $\Theta(\mathbb{C})$.

Write f for the above function.

(ii) $f \in \Theta(\mathbb{C})$ and $Z(f) = \{a_n : n \in \mathbb{N}\}$.

(iii) The multiplicity of any zero is precisely the number of times that it appears in the sequence.

Remarks: (1) Since $|a_n| \rightarrow \infty$ as $n \rightarrow \infty$, for every $r > 0$, $\exists N_0 = N_0(r) \in \mathbb{N}$ s.t. $|a_n| > 2r$ for all $n \geq N_0$.

Thus,

$$\left(\frac{r}{|a_n|} \right) < \frac{1}{2} \quad \forall n \geq N_0.$$

In turn,

$$\left(\frac{r}{|a_n|} \right)^{p_n+1} < \left(\frac{1}{2} \right)^{p_n+1} \quad \forall n \geq N_0.$$

Thus, $p_n = n-1$ ALWAYS works for any $(a_n)_n$ with $|a_n| \rightarrow \infty$.

(2) Suppose that $\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$.

Then, $p_n \equiv 0$ works!

$$\text{In this case, } f(z) = E_0 \left(\frac{z}{a_n} \right)$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n} \right) \text{ works.}$$

(3) If $\sum \frac{1}{|a_n|} = \infty$ but $\sum \frac{1}{|a_n|^2} < \infty$, then $p_n = 1$ works.

$$\begin{aligned}\therefore f(z) &= \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right) \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n}\right).\end{aligned}$$

(4) To create a zero of order k at the origin, simply multiply with z^k .

Thus, given the theorem and the remarks, we have completely answered the desired question on \mathbb{C} .

Proof of the theorem. Let $(p_n)_n$ be as given.

We wish to use the theorem from last lecture. Will show

$$\sum_{n=1}^{\infty} \left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \text{ converges uniformly on compact } \subseteq \mathbb{C}.$$

Suppose this to prove for the compact sets $\overline{D(0, r)}$ for all $r > 0$.
By the earlier lemma,

$$\left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \leq \left|\frac{z}{a_n}\right|^{p_n+1} \quad \text{for } |z| \leq |a_n|.$$

Fix $r > 0$. Then, for $n \gg 0$, $r < |a_n|$.

Then, for $z \in \overline{D(0, r)}$ and $M > N \gg 0$, we have:

$$\begin{aligned}\sum_{n=N}^M \left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| &\stackrel{|z/a_n| \leq 1}{\leq} \sum_{n=N}^M \left|\frac{z}{a_n}\right|^{p_n+1} \\ &\leq \sum_{n=r}^{\infty} \left|\frac{r}{a_n}\right|^{p_n+1} \xrightarrow{|z| \leq r} 0.\end{aligned}$$

Thus, we are done. \blacksquare

Lecture 11 (07-02-2022)

07 February 2022 14:03

EXAMPLE : Construct $f \in \Theta(\mathbb{C})$ with

(i) simple zeroes at \mathbb{Z} ,

(ii) zeroes of order 2 at $\pm i\sqrt{n}$ for $n \in \mathbb{N}$,

no other zeroes.

Let us first construct one with (i).

Note : $\sum \frac{1}{n^2} < \infty$. Can take $p_n = 1$.

Can take

$$f_1(z) = z \cdot \prod_{n=1}^{\infty} E_1\left(\frac{z}{n}\right) \cdot \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right).$$

For (ii) : Note $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^3 < \infty$. Can take $p_n = 2$.

Thus, can take $f_2(z) = \prod_{n=1}^{\infty} E_2\left(\frac{z}{i\sqrt{n}}\right) \cdot \prod_{n=1}^{\infty} E_2\left(-\frac{z}{i\sqrt{n}}\right)$.

f_2 satisfies (ii).

The final desired function is $f = f_1 f_2$.

Weierstrass Factorisation Theorem

Theorem : Let $f \in \Theta(\mathbb{C}) \setminus \{0\}$ and let $(a_n)_{n \geq 1}$ be the nonzero zeroes of f , listed with multiplicity. Suppose f has a zero at the origin of order $m \geq 0$.

Then, $\exists g \in \Theta(\mathbb{C})$ and $(p_n)_n \in \mathbb{N}^\mathbb{N}$ such that

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof. Since zeroes are isolated, $|a_n| \rightarrow \infty$.

$\dots, -1, 0, 1, \dots, p_n+1$

Proof.

Since zeroes are isolated, $|a_n| \rightarrow \infty$.

As discussed last time, $\exists (p_n)_n$ s.t. $\sum \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \quad \forall r > 0$.

(e.g.: $p_n = n-1$)

Thus, $h(z) = z^m \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$ is holomorphic on C and has SAME zeroes as f (with mult.).

Thus, f/h is entire and nonvanishing. $\therefore \exists g \in \mathcal{O}(C)$ s.t.

$$\frac{f}{h} = \exp(g).$$

2

Theorem.

Let $\Omega \subsetneq C \cup \{\infty\}$ be an open set.

Suppose $A \subset \Omega$ has no limit points in Ω .

Let $m: A \rightarrow \mathbb{N}$ be any function.

Then, $\exists f \in \mathcal{O}(\Omega)$ such that $I(f) = A$, and f has a zero of multiplicity $m(x)$ for every $x \in A$.

Proof. It suffices to prove the theorem in the special case where:

Ω is a deleted neighbourhood of ∞ and $\infty \notin \bar{A}$.

Justification. $\Omega = C \setminus K$ for some compact $K \subseteq C$.

Let Ω_1 and A_1 be as in the hypothesis of the theorem.

Fix $\infty \neq a \in \Omega_1 \setminus A_1$. Define

$$T(z) = \frac{1}{z-a}.$$

$$z \neq a$$

T is a linear fractional transformation from \hat{C} onto itself.

T is a homeomorphism of Ω_1 onto $T(\Omega_1) =: \Omega$.

Define $A := T(A_1)$. Then, A has no limit points in Ω .

Now, Ω and A satisfy the requirements of the special case.

Now, if theorem holds for special case, we can translate it back.

Now, we prove the theorem for the special case.

If $A = \{a_1, \dots, a_n\}$, take

$$f(z) := (z - a_1)^{m_1} \cdots (z - a_n)^{m_n}$$

$A = \{a_1, \dots, a_n\}$, take

$$f(z) := \frac{(z - a_1)^{m_1} \cdots (z - a_n)^{m_n}}{(z - b)^{m_1 + \dots + m_n}}$$

for some $b \in \mathbb{C} \setminus \Omega$.

Suppose $|A| = \infty$. Let $(z_n)_n$ be an enumeration with the multiplicity taken care of.

For each n , $\exists w_n \in \mathbb{C} \setminus \Omega$ such that

$$|w_n - z_n| = \text{dist}(z_n, \mathbb{C} \setminus \Omega).$$

\hookrightarrow this is a nonempty compact set.

Lecture 12 (10-02-2022)

10 February 2022 13:52

Recall: Had reduced theorem to special case.

We now prove it for the special case:

$$\Omega = \mathbb{C} \setminus K' \text{ for } K' \neq \emptyset \text{ compact, } \left(\begin{array}{l} \text{if } \Omega = \mathbb{C}, \\ \text{we already know.} \end{array} \right)$$

$$\infty \notin \bar{\Omega}.$$

Had done it for finite A.

$(z_n)_{n \geq 1}$: enumeration of A, with multiplicities.

$(w_n)_{n \geq 1}$: satisfy $\text{dist}(z_n, \mathbb{C} \setminus \Omega) = |z_n - w_n|$.

\hookrightarrow lie in $\mathbb{C} \setminus \Omega$

If $|z_n - w_n| \rightarrow 0$, then \exists subsequence s.t. $|z_{n_k} - w_{n_k}| \geq \delta > 0$.

But A is bounded. $\exists (z_{n_k})$ s.t. $z_{n_k} \rightarrow z_0 \in \mathbb{C} \setminus \Omega$.

But then $|z_{n_k} - w_{n_k}| \rightarrow 0$. $\rightarrow \leftarrow$

Thus, $|z_n - w_n| \rightarrow 0$.

Note that if $\frac{a-b}{z-b} \in \Omega$, then $z \mapsto E_p\left(\frac{a-b}{z-b}\right)$ is holomorphic on Ω and has a simple zero at a.

Claim: $z \mapsto \prod_{n=1}^{\infty} E_p\left(\frac{z_n - w_n}{z - w_n}\right)$ converges in $\Omega(\Omega)$.

From the claim, everything follows.

Proof. Suffices to show that

$$z \mapsto \sum_{n=1}^{\infty} \left| 1 - E_p\left(\frac{z_n - w_n}{z - w_n}\right) \right| \text{ converges in } \Omega(\Omega).$$

Fix $K \subseteq \Omega$. Then, $\text{dist}(K, \mathbb{C} \setminus \Omega) =: \delta > 0$.

For $z \in K$:

$$\left| \frac{z_n - w_n}{z - w_n} \right| \leq \frac{|z_n - w_n|}{\delta} \rightarrow 0.$$

$$\therefore \left| \frac{z_n - w_n}{z - w_n} \right| \leq \frac{1}{2} \quad \forall n \gg 0.$$

$$\therefore \left| 1 - E_n \left(\frac{z_n - w_n}{z - w_n} \right) \right| \leq \left(\frac{1}{2} \right)^n \quad \forall n > 0.$$

EXAMPLE. Consider $f(z) = \sin(\pi z)$, $z \in \mathbb{C}$.

$f \in \mathcal{O}(\mathbb{C})$ and $Z(f) = \mathbb{Z}$.

$$\{a_n\}_{n=1}^{\infty} \cup \{0\}$$

$$\sum_{n=1}^{\infty} \frac{1}{\tan^2} < \infty.$$

By Weierstrass factorisation, $\exists h \in \mathcal{O}(z)$ with $Z(h) = \mathbb{Z}$.

Then, $\frac{f}{h} = \exp \circ g$ for some $g \in \mathcal{O}(\mathbb{C})$.

One explicit construction of h is:

$$h(z) = z \cdot \prod_{n=1}^{\infty} E_i \left(\frac{z}{n} \right) \cdot \prod_{n=1}^{\infty} E_i \left(-\frac{z}{n} \right)$$

$$= z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n} \right) \exp \left(\frac{z}{n} \right) \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) \exp \left(-\frac{z}{n} \right).$$

) abs. conv.

$$= z \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{n} \right) \exp \left(\frac{z}{n} \right) \left(1 + \frac{z}{n} \right) \exp \left(-\frac{z}{n} \right) \right]$$

$$= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right).$$

$$\therefore \sin(\pi z) = z \exp(g(z)) \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right), \quad z \in \mathbb{C}.$$

Remark. Using some more analysis, one can determine g somewhat.
(log derivative?) We don't do it here.

Harmonic Functions:

Def. Let $S \subseteq \mathbb{C}$ be open.

Let $u: \Omega \rightarrow \mathbb{R}$ be C^2 .

u is said to be harmonic on Ω if

$$\Delta u := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0.$$

Laplacian operator

Harmonic function, Laplacian operator

We define two more operators:

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{i\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right).$$

CALCULATION:

$$\begin{aligned} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} (u) &= \frac{1}{4} \left(\frac{\partial}{\partial x} - \frac{i\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y} \right) u \\ &= \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + i \frac{\partial^2}{\partial x \partial y} - i \frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2} \right) u. \end{aligned}$$

If $u \in C^2(\Omega)$, then

$$\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} (u) = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = \frac{1}{4} \Delta u.$$

mixed partials cancel!

For harmonic u , $\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} u = 0$.

EXAMPLES. Harmonic functions:

(1) $u(x,y) = ax + by + c$,

(2) $u(x,y) = 2xy$,

(3) $u(x,y) = x^3 - 3x^2y$,

(4) if $f \in \mathcal{O}(\Omega)$, then $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are harmonic, by the Cauchy-Riemann equations.

NON harmonic:

(5) $u(x,y) = x^2 + y^2$.

LAPLACIAN IN POLAR: Define $u(z) = \log(|z|)$.

Write $z = re^{i\theta}$, $|z| = r$.

$$x = r \cos\theta, \quad y = r \sin\theta.$$

(Exercise) $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$

Then, $\Delta u = \frac{\partial^2}{\partial r^2} \log(r) + \frac{1}{r} \frac{\partial}{\partial r} \log(r)$

$$= -\frac{1}{r^2} + \frac{1}{r} \cdot \frac{1}{r} = 0.$$

Question: Let $\Omega \subseteq \mathbb{C}$ be a domain.

Let $u: \Omega \rightarrow \mathbb{R}$ be harmonic on Ω .

Does there exist another harmonic $v: \Omega \rightarrow \mathbb{R}$ s.t.

$$f := u + iv \in \mathcal{D}(\Omega).$$

In such a case, v is said to be a **harmonic conjugate** of u .

Observation: Let $\Omega \subseteq \mathbb{C}$ be a domain.

Suppose u is harmonic on Ω .

Define $g: \Omega \rightarrow \mathbb{C}$ by

$$g(z) := \frac{\partial u}{\partial x}(z) - i \frac{\partial u}{\partial y}(z).$$

Note that $\operatorname{Re}(g) = u_x$, $\operatorname{Im}(g) = -u_y$.

Since $u \in C^1(\Omega)$, the above two have continuous (first) partials on Ω .

Moreover, $\Delta u = 0 \Rightarrow g$ satisfies the CR equations on Ω .

Conclusion: g is holomorphic. (For any domain Ω , and any harmonic u)

Now, if v is a harmonic conjugate of u , then CR equations tell us:

$$\nabla \varphi = (-u_y, u_x).$$

↑
(ANNOT ALWAYS SOLVE THIS!)

EXAMPLE. Let $\Omega = \mathbb{C} \setminus \{0\}$.

Define $u: \Omega \rightarrow \mathbb{R}$ by $u(z) = \log|z|$ or
 $u(x, y) = \frac{1}{2} \log(x^2 + y^2)$.

$\Delta u = 0$. Suppose $\exists \varphi: \Omega \rightarrow \mathbb{R}$ harmonic s.t.

$$\nabla \varphi = (-u_y, u_x).$$

Then, $(\nabla \varphi)(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$.

→ ← (Integrate along unit circle!)

Mitler: Then, $f = \log|z| + i\varphi$ is holomorphic.

Why is this a contradiction?

Lecture 13 (14-02-2022)

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- Let $u: \Omega \rightarrow \mathbb{R}$ be harmonic with Ω a domain.
If $v_1, v_2: \Omega \rightarrow \mathbb{R}$ are harmonic conjugates of u , then

$$i(v_1 - v_2) = (u + iv_1) - (u + iv_2) \in \mathcal{O}(\Omega).$$

But $i(v_1 - v_2)$ is purely imaginary valued. Thus, $v_1 = v_2 + c$
for some constant $c \in \mathbb{R}$.

- Last time, we saw that not every harmonic function has a harmonic conjugate.

Let $\Omega \subseteq \mathbb{C}$ be a domain, $u: \Omega \rightarrow \mathbb{R}$ be harmonic. Define

$$g: \Omega \rightarrow \mathbb{C} \quad \text{by}$$

$$g := u_x - iu_y.$$

Then, g is holomorphic on Ω .

SUPPOSE f is an antiderivative of g .

$$\text{Let } f = \tilde{u} + i\tilde{v}.$$

$$\text{Then, } f' = \tilde{u}_x + i\tilde{v}_x = \tilde{u}_x - i\tilde{u}_y.$$

$$\therefore \tilde{u} \stackrel{g}{=} u + c.$$

Thus, \tilde{v} is a harmonic conjugate of u !

Thus, u has a harmonic conjugate whenever g has an antiderivative. As g is holomorphic, this does happen whenever Ω is simply-connected.

Consequences:

- Let Ω be an open set in \mathbb{C} , $u: \Omega \rightarrow \mathbb{R}$ be harmonic.

Let $a \in \Omega$. Then, $\exists r > 0$ s.t. $\overline{D(a, r)} \subseteq \Omega$.

As the disc is simply-connected, $\exists f \in \Theta(D(a, r))$ s.t.

$\operatorname{Re}(f) = u$. Thus, u is C^∞ -smooth on $D(a, r)$. In turn,

u is C^∞ -smooth on Ω . (In fact, real analytic!)

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\partial D(a, r)} \frac{f(\xi)}{\xi - a} d\xi \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} r/e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta. \end{aligned}$$

Taking the real part gives:

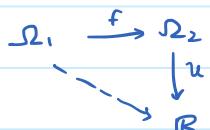
$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

Then, u satisfies the mean value property.

Prop. Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be domains and $f: \Omega_1 \rightarrow \Omega_2$ be holomorphic.

Let $u: \Omega_2 \rightarrow \mathbb{R}$ be harmonic.

Then, $u \circ f$ is also harmonic.



Proof. For any $a \in \Omega_2$, we can find a disc $D(a, r_a)$ compactly contained in Ω_2 and $g_a \in \Theta(D(a, r_a))$ s.t. $\operatorname{Re}(g_a) = u|_{D(a, r_a)}$.

Now, given $b \in \Omega_1$, set $a := f(b)$.

By continuity, $\exists \delta > 0$ s.t. $f(D(b, \delta)) \subseteq D(a, r_a)$.

Thus, $\operatorname{Re}(g_a \circ f) = u \circ f$ is harmonic on $D(b, \delta)$. □

Def. Let $\Omega \subseteq \mathbb{C}$ be open, and $u: \Omega \rightarrow \mathbb{R}$.

u has the mean value property if whenever $D(a, \delta) \subset \Omega$, then

u has the mean value property if whenever $D(a, \delta) \subset \Omega$, then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + \delta e^{i\theta}) d\theta.$$

↳ compactly contained

We showed: harmonic \Rightarrow MUP.

We will show: MUP \Rightarrow harmonic.

Maximum Principle

Thm. Let $\Omega \subseteq \mathbb{C}$ be a domain, $u: \Omega \rightarrow \mathbb{R}$ have the MUP on Ω .

If $\exists p_0 \in \Omega$ s.t.

$$u(p_0) = \sup_{z \in \Omega} u(z),$$

then u is constant.

Proof. Let $E = \{\xi \in \Omega : u(\xi) = \sup_{z \in \Omega} u(z)\}$.

$E \neq \emptyset$ as $p_0 \in E$. E is clearly closed (in Ω).

We show E is open and thus, $E = \Omega$ as Ω is connected.

Let $p \in E$. Pick $R > 0$ s.t. $D(p, R) \subset \Omega$.

Fix $r \in (0, R)$. By MUP, we have

$$u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(p + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u(p) d\theta = u(p).$$

Thus, $u(p + re^{i\theta}) = u(p) \quad \forall \theta \in [0, 2\pi]$.

Thus, u is constant on $D(p, R)$. $\therefore D(p, R) \subset E$. \blacksquare

Similarly, we have the minimum principle

Thm.

(Global version)

Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, and $u: \Omega \rightarrow \mathbb{R}$ have MUP.
Suppose $u \in C^0(\bar{\Omega})$. Then,

$$\max_{\bar{\Omega}} u = \max_{\partial \Omega} u \quad \text{and}$$

$$\min_{\bar{\Omega}} u = \min_{\partial \Omega} u.$$

Proof.

$\max_{\bar{\Omega}} u$ attained somewhere. If interior, then constant... □

Corollary

Let $\Omega \subseteq \mathbb{C}$ be a bounded domain in \mathbb{C} .

Suppose $u_1, u_2 \in C^0(\bar{\Omega})$ are s.t. u_1, u_2 have the MUP on Ω .

If $u_1|_{\partial \Omega} = u_2|_{\partial \Omega}$, then $u_1 \equiv u_2$.

Proof.

$u_1 - u_2$ has the MUP and is 0 on $\partial \Omega$... □