Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

\# Bruns & Herzog → Cohen-Macaulay rings
- 1st Part

\# Affine algebra

X

Derived Category

Gelfand & Manin

E. Riehl's : Cat theory in context

Dwyer & Spalinski

$R \to$ ring (possibly non comm.)

R-complexes :

$$\cdots \to M_{i+1} \xrightarrow{\partial} M_i \xrightarrow{\partial} M_{i-1} \to \cdots \qquad \partial^2 = 0$$

$$\mathrm{im}(\partial_{i+1}) \subset \ker(\partial_i)$$

$$H_i(M) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

$$H(M) = (H_i(M))_{i \in \mathbb{Z}}$$

$\mathcal{C}(R)$ = category of R-complexes (morphisms as usual)

If $f : M \to N$, we get an induced map $H(f) : H(M) \to H(N)$.

Def^n f is a quasi isomorphism (or weak equivalence) if $H(f)$ is bijective. (Automatically iso.)

$W :=$ collection of weak equivalences in $\mathcal{C}(R)$

$D(R) := \mathcal{C}(R)[W^{-1}]$. (or $W^{-1}\mathcal{C}(R)$)

- Key property : W has the 2-out-of-6 property:

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,

- Key property: W has the 2-out-of-6 property:
  Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,
  if gf and $hg \in W$, then f, g, h, hgf are in W.

Ex: $\Rightarrow$ 2-out-of-3 property
If f, g, fg are defined and 2 are in W, then so is the third.

Concretely: $\mathcal{C}(R) \rightsquigarrow K(R) \rightsquigarrow D(R)$.
$\uparrow$ homotopy category

$M, N \to$ R-complexes

$\operatorname{Hom}_R(M, N) :=$ Hom-complex of abelian groups
(when R is comm. this is an R-complex)

$\operatorname{Hom}_R(M, N)_n :=$ Maps of degree n from M to N (no compatibility!)

$$\begin{tikzcd}
\cdots \arrow[r] & M_i \arrow[r] \arrow[dl] & M_{i-1} \arrow[r] \arrow[dl] & \cdots \\
N_{i+n} \arrow[r] & N_{i+n-1} \arrow[r] & \cdots &
\end{tikzcd}$$

$$= \prod_{i \in \mathbb{Z}} \operatorname{Hom}_R(M_i, N_{i+n})$$

$$\partial : \operatorname{Hom}_R(M, N)_{n+1} \longrightarrow \operatorname{Hom}_R(M, N)$$

$$\partial(f) = \partial^N f - (-1)^{n+1} f \partial^M.$$

Check: $\partial^2 = 0$.

Observe: $Z_0(\operatorname{Hom}_R(M, N)) = \operatorname{Hom}_{\mathcal{C}}(M, N)$.

Def. $f, g \in \operatorname{Hom}_{\mathcal{C}}(M, N)$ are homotopic if
$f - g \in B_0(\operatorname{Hom}_R(M, N))$, i.e.,
$f - g = \partial h$ for some $h \in \operatorname{Hom}_R(M, N)$.

$K(R) := \mathcal{C}(R)/_{\text{homotopy relation.}}$

Object = R-complexes

$\mathrm{Hom}_K(M,N) = H_0(\mathrm{Hom}_R(M,N))$.

- $f \sim g$ in $\mathcal{C}$ $\Rightarrow$ $f = g$ in $K(R)$.

$\Rightarrow H(f) = H(g)$

Defn. M an R-complex.

ΣM (or $M[1]$) is the R-complex

$$(\Sigma M)_i = M_{i-1}$$

with $\partial^{\Sigma M} = -\partial^M$.

$\mathrm{Proj}\, R :=$ Projective R-modules

$$\begin{tikzcd}
K(R) \\
K(\mathrm{Proj}\, R) \arrow[u, phantom, "\cup"] \arrow[r, shift right, "q"'] & D(R) \arrow[l, dashed, shift right, "p"'] \arrow[from=ul, "q"]
\end{tikzcd}$$

q = localisation

$\exists p : D(R) \to K(\mathrm{Proj}\, R)$, a full and faithful embedding

"projective resolutions"

left adjoint to q.

$$\mathrm{Hom}_K(pM, N) = \mathrm{Hom}_D(M, qN)$$

$f : M \to N$ morphism

$\mathrm{cone}(f) := N \oplus \Sigma M$ with differential

$$\begin{matrix} N_{i+1} \\ \oplus \\ M_i \end{matrix} \to \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \qquad \partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$$0 \to N \hookrightarrow \mathrm{cone}(f) \to \Sigma M \to 0.$$

f is w.e. $\Leftrightarrow$ $H(\mathrm{cone}(f)) = 0$.

Image of p?

K-projectives.

P an R-complex is K-projective if given any solid diagram

$$\begin{tikzcd} & M \arrow[d, "\pi", "?"'] & \text{w.e.} \\ P \arrow[r, "\alpha"'] \arrow[ur, dashed, "\tilde{\alpha}"] & N & \end{tikzcd}, \quad \exists \text{ lift } \tilde{\alpha}.$$

FACT: $p: D(R) \xrightarrow{\sim} K\text{-}\mathrm{Proj}(R) \subseteq K(\mathrm{Proj}\ R)$.

morphisms here are up to homotopy

- $\mathrm{Hom}_{\mathcal{C}}(R, M) = Z_0(M)$

$$\begin{tikzcd} 0 \arrow[d] & \\ R \arrow[r] \arrow[d] & M_0 \arrow[d, "\partial"] \\ 0 \arrow[r] & M_{-1} \end{tikzcd}$$

Using this, check: R is K-projective.

Use: surjective + w.e. $\Downarrow$ $Z(M) \twoheadrightarrow Z(N)$ onto

- $(P_\lambda)_\lambda$ family of K-projectives

  Then, $\oplus P_\lambda$ is also K-projective.

  Conversely closed under direct summands.

- K-projectives are closed under suspensions.

  $\cdots \to \ldots \xrightarrow{\partial} P_{i+1} \xrightarrow{\partial} P_i \xrightarrow{\partial} P_{i-1} \xrightarrow{\partial} \cdots$

  is K-proj, if P_i projective $\forall i$.

Ex: $0 \to P' \xrightarrow{f} P \xrightarrow{g} P'' \to 0$ exact seq. of complexes.

Then, if P and P'' are K-projective, so is P'.

If P', P, P'' are complexes of projectives, then any two being K-projective $\Rightarrow$ third is K-proj.

any two being K-projective $\Rightarrow$ third is K-proj.

Corollary. Any bounded complex of projectives is K-projective.

Proof. $P_\cdot$: $0 \to P_b \dashrightarrow P_a \to 0$, each P_i proj.

Induce on $b-a$.

$b-a=0$ done earlier.

$$0 \to P_{\leq b-1} \to P \to \Sigma^b P_b \to 0. \quad \blacksquare$$

$$P_{\leq b-1} = 0 \to P_{b-1} \to \cdots \to P_a \to 0$$

Next: Any complex of projectives with $P_i = 0$ $\forall i \ll 0$ is K-projective.

$$P_\cdot : \quad \cdots \to P_{a+1} \to P_a \to 0$$

$P = \varinjlim_{n \geq a} P_{\leq n}$, each $P_{\leq n}$ is projective since bounded.

$$0 \to \bigoplus_n P_{\leq n} \xrightarrow{1-\delta} \bigoplus_n P_{\leq n} \to P \to 0.$$

K-proj

projectives

Use 2-out-of-3.

Do directly...

$$\begin{array}{c} M \\ \pi \downarrow \text{ w.e.} \\ N \end{array}$$

P is K-projective

$\Updownarrow$

$$\mathrm{Hom}_K(P, M) \cong \mathrm{Hom}_K(P, N).$$

Lecture 2 (11-01-2023)

11 January 2023 13:24

R ring.

$D(R) \simeq k\mathrm{Proj}(R)$.

Recall: $P \in \mathcal{C}(\mathrm{Proj}\ R)$ is K-projective if

$$\begin{tikzcd} & X \arrow[d, "\simeq"] \\ P \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Example $P \in \mathcal{C}(\mathrm{Proj}\ R)$ with $P_i = 0$ for all $i \ll 0$.

$$\cdots \to P_{a-1} \to P_a \to 0 \to 0 \to \cdots$$

Sketch. Construct lifting one step at a time.

Suppose $\tilde{\alpha} : P_{\leq n} \to X$ is a lifting.

Want $\tilde{\alpha} : P_{\leq n+1} \to X$ compatibly.

Let $s \in P_{n+1}$

We must have

$$\begin{tikzcd} & & X \arrow[d, "\varepsilon"', "\simeq"] \\ \cdots \to P_{n+1} \arrow[r] & (P_n \to P_{n-1} \to \cdots \to P_a \to 0) \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

- $\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$
- $\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$

Check that the above can be solved.

This uses three things:
- ε surjective $\Rightarrow \varepsilon$ surjective on boundaries
- $H(\varepsilon)$ iso $\Rightarrow \varepsilon$ surjective on cycles
- $\ker(\varepsilon)$ is acyclic.

$\Rightarrow$ Every module has a K-projective resolution.

$$\begin{tikzcd} F: \cdots F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[dr, "\varepsilon"] & \\ & \ker \partial_1 \arrow[ur] & & \ker \varepsilon \arrow[ur] & & M \end{tikzcd}$$

(Can avoid choices by taking generating set to be $M, \ker\varepsilon, \ker\partial_1, \ldots$)

$$F \xrightarrow{\varepsilon} M.$$

Defn: A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism

$$\varepsilon : P_\cdot \longrightarrow M \quad \text{s.t.}$$

① ε is a quasi iso,

② $P_\cdot$ is K-projective.

(Not insisting surjective.)

② $P_\cdot$ is K-projective.

This makes it functorial!

Thm $\forall\, M_\cdot \in \mathcal{C}(R)$, $\exists$ surjective K-projective resolution:

$$\underset{\text{K-proj.}}{P_\cdot} \xrightarrow{\sim} M_\cdot$$

Def^n. An R-complex F is semi-free if F admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \cdots \subseteq \bigcup_{n \geq 0} F(n) = F$$

s.t. ① $F(n) \subseteq F$ is a subcomplex,

② $F(n+1)/F(n)$ graded free module with $\partial = 0$,

i.e. $\partial(F(n+1)) \subseteq F(n)$

Fact/Ex. Semifree $\Rightarrow$ K-proj

Example. $\cdots \to F_{a+1} \to F_a \to 0$

$$F(n) = F_{\leq n} = 0 \to F_n \to \cdots \to F_a \to 0$$

$$\frac{F(n+1)}{F(n)} = \cdots \to 0 \to F(n+1) \to 0 \to \cdots$$

$$= \Sigma^{n+1} F_{n+1}.$$

Thm. Each $M_\cdot \in \mathcal{C}(R)$ has a surjective semi-free resolution

$$\underset{\text{K-projective}}{F_\cdot} \xrightarrow{\sim} M_\cdot.$$

Corollary. Every K-projective is a retract of a semifree.

Proof. P is K-projective:

$$\begin{tikzcd} & F \arrow[d, "\wr"] \\ P \arrow[ur, dashed] \arrow[r, equal, "id"'] & P \end{tikzcd}.$$

∎

Sketch (Baby ver of Quillen's "small object argument"):

Construct $0 = F(0) \hookrightarrow F(1) \overset{i(1)}{\hookrightarrow} F(2) \overset{i(2)}{\hookrightarrow} \cdots \hookrightarrow F(n) \overset{i(n)}{\hookrightarrow} \cdots \hookrightarrow \bigcup F(n) = F$

Construct

$$\begin{tikzcd}
0 = F(0) \arrow[r, hook] \arrow[dr, "\varepsilon(0)"'] & F(1) \arrow[r, hook, "i(1)"] \arrow[d, "\varepsilon(1)"] & F(2) \arrow[r, hook, "i(2)"] \arrow[dl, "\varepsilon(2)"'] & \cdots \arrow[r, hook] & F(n) \arrow[r, hook, "i(n)"] \arrow[dlll, bend left, "\varepsilon(n)"] & \cdots \arrow[r, hook] & \bigcup_{n \geq 0} F(n) = F \\
& M & & & & &
\end{tikzcd}$$

s.t. ① $F(n+1)/F(n)$ is graded free with $\partial = 0$,

② $\varepsilon(1)$ is surjective on homology.

(In turn, each $\varepsilon(n)$ is surjective on homology.)

③ $\ker(H(\varepsilon(n))) \subseteq H(F(n))$ maps to 0 under $H(i(n))$.

This does the job. ["Something 0 in colim is 0 at finite stage."]

Why is ε surjective?

Remark. $\varepsilon : X \to Y$ s.t.

$Z(\varepsilon)$ surjective + $H(\varepsilon)$ bijective.

Then, ε is surjective.

Indeed, we have s.e.s.es:

$$\begin{tikzcd}
0 \arrow[r] & B(X) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & H(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B(Y) \arrow[r] & Z(Y) \arrow[r] & H(Y) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & Z(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \Sigma B(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] & Y \arrow[r] & \Sigma B(Y) \arrow[r] & 0
\end{tikzcd}$$

Snake lemma

$\Downarrow$

$B(X) \to B(Y)$ is epi

$\Downarrow$

$X \to Y$ is surj

Construction of $F(n), \varepsilon(n)$:

$\varepsilon(1) : F(1) \to M$ free cover of cycles.

0 diffs

($\varepsilon(1)$ morphism since mapping on cycles)

Say we have constructed $\varepsilon(n) : F(n) \longrightarrow M$.

Choose cycles $(z_\lambda)_\lambda \subseteq F(n)$ that map to a generating set of $\ker(H(\varepsilon(n)))$

Pick w_λ s.t. $\partial(w_\lambda) = \varepsilon(n)(z_\lambda)$.

Set $F(n+1) = F(n) \oplus R e_\lambda$ $\qquad \deg(e_\lambda) = \deg(z_\lambda) + 1$.

with $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_x) = z_x.$$

Define $\varepsilon(n+1) : F(n+1) \longrightarrow M$

$$\varepsilon(n+1)|_{F(n)} = \varepsilon(n),$$

$$\varepsilon(n+1)(e_x) = \omega_x.$$

$\boxtimes$

Remarks: ① As before the above construction can be made functorial by avoiding choices (consider all choices!).

② Depending on what we wish to do with the resolution, there are other constructions.

Given a module, we have the graded homology module $H(M) = \langle H_i(M) \rangle_{i \in \mathbb{Z}}$.

(Recall: for us, a graded module is a collection of modules.)

If $P_\cdot \xrightarrow{\sim} H(M)$ is a projective resol^n, one can "perturb" the differentials of $P_\cdot$ to construct a K-projective resolution of M.

(Adam's resolution, Cartan/Eilenberg resolution)

Exercise. $P_\cdot$ K-proj $\Rightarrow$ P_i projective $\forall i$.

Converse of above NOT true.

Example (Dold's): Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the complex

$$P_\cdot : \quad \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} \cdots$$

One way of seeing that the above is not K-projective is to do the following exercise and note that $P_\cdot$ is acyclic but not contractible.

is to do the following exercise and [illegible]
$P_\bullet$ is acyclic but not contractible.

Exercise. If P is K-proj and $H(P) = 0$, then P is contractible, i.e., $\mathrm{id}_P \sim 0$.
(or: P is the mapping cone of some id.)

We saw we have an inclusion

$$\mathrm{KProj}(R) \hookrightarrow K(\mathrm{Proj}\ R).$$

FACT. Let R be comm. Noetherian.
The above inclusion is an equality iff R is regular.

Examples of reg. rings: $\mathbb{Z}$, $k[x_1, \dots, x_n]$.

Derived functors

Let $M_\bullet$ be an R-complex.

FACT. If $P_\bullet \xrightarrow{\sim} M_\bullet$ and $Q_\bullet \xrightarrow{\sim} M_\bullet$ are K-projective resolutions, then $P_\bullet \simeq Q_\bullet$ in $K(\mathrm{Proj}\ R)$.

Given any $N_\bullet \in \mathcal{C}(R)$, set

$$\mathrm{RHom}_R(M_\bullet, N_\bullet) := \mathrm{Hom}_R(P_\bullet, N_\bullet),$$

where $P_\bullet$ is a K-proj. res^n of $M_\bullet$.

The object on the right is defined in the homotopy category of abelian groups, i.e.,
$\mathrm{RHom}_R(-, N_\bullet)$ is a functor
$\mathcal{D}(R) \to K(\mathbb{Z})$.

$R\mathrm{Hom}_R(-, N.)$ is a functor

$$\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$$

(If R is comm, then $\longrightarrow K(R)$.)

Define $\mathrm{Ext}^i_R(M., N.) := H^i(R\mathrm{Hom}_R(M., N.))$
$= H_{-i}(\mathrm{Hom}_R(P., N.))$

$\mathrm{Ext}^0_R(M, N) = H_0(\mathrm{Hom}_R(P., N.))$
$=$ morphisms $P. \to N.$, up to homotopy

$\mathrm{Ext}^i_R(M, N) =$ — " — $P. \to \Sigma^i N.$, — " —

If $Q \xrightarrow{\sim} N$ is a K-proj resol^n, then
$\mathrm{Hom}_R(P., Q.) \simeq \mathrm{Hom}_R(P., N.)$.
quasi-iso

Tensors Let $M.$ be a chain complex of right R-modules.

Let $N. \in \mathcal{C}(R)$.

$M. \otimes N.$ is a complex of $\mathbb{Z}$-modules defined by

$$(M. \otimes N.)_i = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-1)^{|m|} m \otimes \partial n$$

FACT. If $X. \xrightarrow{\sim} Y.$ is a quasi iso,
then $P. \otimes_R X. \xrightarrow{\sim} P. \otimes Y.$ for any K-proj $P.$

Defn. $M_\cdot \otimes_R^{\mathbb{L}} N_\cdot := P_\cdot \otimes_R N_\cdot$ where $P_\cdot \xrightarrow{\sim} M_\cdot$ is a K-proj. resolⁿ.

$$\mathrm{Tor}_i^R(M_\cdot, N_\cdot) = H_i(P_\cdot \otimes_R N_\cdot).$$

$X \xrightarrow{\sim} Y$ quasi iso $\Rightarrow \mathrm{Tor}_i^R(M_\cdot, X_\cdot) = \mathrm{Tor}_i^R(M_\cdot, Y_\cdot)$.

Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \to$ comm. Noetherian ring

$M \to$ R-module

$r \in R$ is a zerodivisor on M if $r \cdot m = 0$ for some $m \neq 0$.

nzd = "not a zero divisor"

$$Z_R(M) = \{ r \in R : r \text{ is a z.d. on } M \}$$

$$= \bigcup_{p \in Ass(M)} p.$$

(M need not be finite. Union need not be)

Fix R, M. Let $\underline{x} = x_1, \dots, x_n$ be a sequence in R.

$\underline{x}$ is weakly M-regular or a weakly regular sequence on M

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \dots, x_i)M} \quad \text{for } 0 \leq i \leq n-1.$$

$\underline{x}$ is M-regular (or ...) if further $M/(\underline{x})M \neq 0$.

Ex. $R = k[x_1, \dots, x_n]$.
$\underline{x} := x_1, \dots, x_n$ is a regular sequence on R.

<u>Koszul Complexes</u>. Given $r \in R$,

$$K(r; R) = 0 \to \underset{\text{deg } 1}{R} \xrightarrow{r} \underset{\text{deg } 0}{R} \to 0.$$

$H_1(K(r;R)) = 0 \iff r$ is nzd on R.

Given $\underline{x} = x_1, \dots, x_n$, we define

$$K(\underline{x}; R) = \bigotimes_{i=1}^{n} K(x_i; R).$$

$$K(\underline{x}; R) = 0 \to R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \dots \to R^{\binom{n}{2}} \to R^n \xrightarrow{(x_1 \cdots x_n)} R \to 0$$

$$K(\underline{x};R) = 0 \to R \xrightarrow{\pm\begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{(x_1 \cdots x_n)} R \to 0$$

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mapsto \sum r_i x_i$$

Now, given $M \in \mathcal{C}(R)$,

$$K(\underline{x};M) := K(\underline{x},R) \otimes M.$$

↳ Koszul complex on $\underline{x}$ with coefficients in M.

$$H_i(\underline{x};M) := H_i(K(\underline{x};M)). \quad \to \text{Koszul homology}$$

If M is simply an R-module (viewed in degree 0), then

$K(\underline{x};M)$:

$$0 \to M \to M^n \to \cdots \to M^n \to M \to 0$$

"same" differentials

$$H_0(\underline{x};M) = M/\underline{x}M,$$

$$H_n(\underline{x};M) = \{m \in M : x_i m = 0 \ \forall i\}$$

$$= (0 :_M (\underline{x})).$$

---

① $K(\underline{x};M) = K(x_1;R) \otimes_R K(x_2;R) \otimes_R \cdots \otimes K(x_n;R) \otimes M$

$$= K(x_1;R) \otimes K(x_{\geq 2}, M)$$

$$= K(x_1; K(x_{\geq 2}, M))$$

② $X, Y \in \mathcal{C}(R) \rightsquigarrow X \otimes_R Y \xrightarrow{\sim} Y \otimes_R X$ as R-complexes.

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

$\therefore\ K(\underset{\sim}{x}; R) \cong K(\underset{\sim}{x}^\sigma; R)$ for any $\sigma \in S_n$.

$\Rightarrow K(\underset{\sim}{x}; M) \cong K(\underset{\sim}{x}^\sigma; M)$ — " —.

(Can apply this to Obs ①.)

<u>2nd Perspective</u>: "Koszul complexes are iterative mapping cones."

$f: X \to Y$ morphism of complexes

$$\mathrm{Cone}(f) = \left(Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ & -\partial^X \end{pmatrix}\right).$$

s.e.s. : $0 \to Y \to \mathrm{Cone}(f) \to \Sigma X \to 0.$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$

$\begin{pmatrix} y \\ x \end{pmatrix} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow[H_i(f)]{} H_i(Y) \to H_i(\mathrm{Cone}(f)) \to H_i(\Sigma X) \to \cdots$$

$H_i(f)$ = connecting map

$H_i(\Sigma X) \cong H_{i-1}(X)$

<u>Consider</u>: $x \in R$

$f: R \xrightarrow{x} R$

$1 \mapsto x$

$$\mathrm{Cone}(R \xrightarrow{x} R) = \left(\underset{\deg 0}{R} \oplus \underset{\deg 1}{R}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = K(x; R).$$

<u>Ditto</u>: If $x \in R$, and $M \in \mathscr{C}(R)$ — complex, not necessarily in deg 0

$$\mathrm{Cone}(M \xrightarrow{x} M) = K(x; M)$$

$\underline{x} = x_1, x_2, \ldots, x_n$

$$K(\underline{x}; M) = K(x_1 ; K(x_{\geq 2}; M))$$

$$= \text{Cone}\left(K(x_{\geq 2}; M) \xrightarrow{x_1} K(x_{\geq 2}; M)\right)$$

on homology

iterate

$$H_i(x_{\geq 2}; M) \xrightarrow{\pm x_1} H_i(x_{\geq 2}; M) \to H_i(\underline{x}; M) \to H_{i-1}(x_{\geq 2}; M) \xrightarrow{\pm x_1} \cdots$$

s.e.s.

$$0 \to \frac{H_i(x_{\geq 2}; M)}{x_1 H_i(x_{\geq 2}; M)} \to H_i(\underline{x}; M) \to (0 :_{H_{i-1}(x_{\geq 2}; M)} x_1) \to 0$$

$M \to R$-module

$$K(\underline{x}; M) \rightsquigarrow H_0(x; M) = \frac{M}{\underline{x}M}$$

So, $K(\underline{x}, M) \to \dfrac{M}{\underline{x}M}$ is a w.e. quasi iso

$\Leftrightarrow H_i(\underline{x}; M) = 0 \quad \forall i > 0.$

Defn. $\underline{x}$ is Koszi-regular on M (or...) if

$$H_i(\underline{x}; M) = 0 \ \forall\, i \geq 1.$$

Lemma. When $\underline{x}$ is weakly M-reg, $(M \to R\text{-mod})$

(weakly-reg $\Rightarrow$ Koszi-reg)

$$K(\underline{x}, M) \twoheadrightarrow M/(\underline{x})M \quad \text{is a w.eq.}$$

Proof. $n=1$: $0 \to M \xrightarrow{x} M \to 0.$

$$H_1(x; M) = 0 \Leftrightarrow x \text{ nzd on } M$$

$n \geq 2$: $K(\underline{x}; M) = K(x_n ; K(x_{<n}; M)).$

By induction,

$$K(x_{<n}; M) \xrightarrow{\sim} \frac{M}{(x_{<n})M}.$$

Now, $0 \to R \xrightarrow{x_n} R \to 0$ is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x}; M) = K(x_n; R) \otimes_R K(x_{<n}; M)$$

$$\simeq K(x_n; R) \otimes_R \frac{M}{(x_{<n})M}$$

now note that x_n is a nzd on $M/(x_{<n})M$ and we are done. ☒

Instead of semifree, can use l.e.c. of homology and induction.

(Semifree business $\Rightarrow$ ($M \simeq N$ quasi $\Downarrow$ $K(\underline{x}; M) \simeq K(\underline{x}; N)$ quasi))

Note: ① $\underline{x}$ Koszi-reg $\Rightarrow$ $\underline{x}^\sigma$ is Koszi-reg $\forall \sigma \in S_n$.

② Not true for weakly regular.

Theorem. Say $\underline{x} \subseteq J(R)$ and $M \neq 0$ f.g. R-module.

TFAE:

1) $\underline{x}$ is M-regular. ($\equiv$ weakly M-reg, by NAK.)

2) $\underline{x}$ is Koszi M-regular, i.e., $H_i(\underline{x}; M) = 0$ $\forall i \geq 1$.

3) $H_1(\underline{x}; M) = 0$.

In particular, take R local and x_i nonunits.

Proof. ① $\Rightarrow$ ② $\Rightarrow$ ③ is clear.

Only need to prove ③ $\Rightarrow$ ①.

Already saw for $n=1$.

Induction:

$$K(\underline{x}; M) = K(x_n; K(x_{<n}; M)).$$

l.e.s.

$$0 \to \frac{H_i(x_{<n};M)}{x_n H_i(x_{<n};M)} \to H_i(\underline{x};M) \to (0 :_{H_{i-1}(x_{<n};M)} x_n) \to 0. \qquad (*)$$

Put $i=1$ to get $\dfrac{H_1(x_{<n};M)}{x_n H_1(x_{<n};M)} = 0$

$$\xRightarrow{\text{NAK}} H_1(x_{<n};M) = 0.$$

(Note: Koszul homology modules are f.g. when M is f.g.)

$\xRightarrow{\text{induction}}$ $x_1, \ldots, x_{n-1}$ is M-reg. — (1)

Moreover, $(*)$ now tells us

$$(0 :_{H_0(x_{<n};M)} x_n) = 0.$$

$$\| $$

$$\ker\left(\frac{M}{x_{<n}M} \xrightarrow{x_n} \frac{M}{x_{<n}M}\right).$$

$\therefore$ x_n is nzd on $M/(x_{<n})M$. — (2)

(1) & (2) finish. ∎

Corollary. $\mathfrak{J} \subseteq J(R)$, M f.g., the property of $\underline{x}$ being regular is not dependent on the order of x_i.

(Permutation of regular is regular.)

$R = k[x, y, z]$

x, $y(1-x)$, $z(1-x)$ reg

$y(1-x)$, $z(1-x)$, x NOT

Lemma. If $\underline{x} = x_1, \ldots, x_n \subseteq R$, M an R-module.

TFAE:

① $\underline{x}$ is M-Koszul-regular.

② $\underline{x}^{\underline{a}}$ is M-Koszul-regular for some $\underline{a} \geq \underline{1}$.

Proof. Suffices to prove:

$x_1, \boxed{x_2, \ldots, x_n}$ is M-KR

$\iff x_1^a, \boxed{x_2, \ldots, x_n}$ is M-KR for some $a \geq 1$. (/all)

$x_1, x_2, \ldots, x_n$ KR

$\Rightarrow K(x_1 ; K(x_{\geq 2} ; M)) \xrightarrow{\sim} K\left(x_1 ; \dfrac{M}{(x_{\geq 2})M}\right).$

Replacing M by $M/(x_{\geq 2})M$ we are reduced to $n = 1$.

But

x is reg on M

$\iff x$ is nzd on M

$\iff x^a$ is nzd on M for some $a \geq 1$

$\iff x^a$ is reg on M ——"——.

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$ and M f.g. R-module.

Let $i \geq 0$ be s.t. $H_i(\underline{x} ; M) = 0$.

Then, $H_j(\underline{x} ; M) = 0 \quad \forall j \geq i.$

HW.