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MA 503

Functional Analysis

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§1. Definitions

§§1.1. Banach spaces

$\mathbb{F}$ will either be $\mathbb{R}$ or $\mathbb{C}$ throughout, with the usual Euclidean topology. All vector spaces considered will be over $\mathbb{F}$.

Definition 1.1.1. A topological vector space V is a vector space with a topology such that

1. V is Hausdorff;
2. the map $(v, w) \mapsto v + w$ is continuous;
3. the map $(\alpha, v) \mapsto \alpha v$ is continuous.

Definition 1.1.2. A norm on a vector space V is a function $\|\cdot\| : V \to [0, \infty)$ such that

1. $\|x\| = 0$ iff $x = \mathbf{0}$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $(\alpha, x) \in \mathbb{F} \times V$;
3. (Triangle Inequality) $\|x + y\| \leqslant \|x\| + \|y\|$ for every $x, y \in V$.

The pair $(V, \|\cdot\|)$ is called a normed linear space (NLS).

Proposition 1.1.3 (Reverse Triangle Inequality)**.** Let V be an NLS. $|\|x\| - \|y\|| \leqslant \|x - y\|$ for all $x, y \in V$.

The norm induces a metric on V defined by $(x, y) \mapsto d(x, y)$. This gives V a topology. Check that this also makes V a topological vector space, and that the norm map is continuous. Since this V is a metric space, it makes sense to talk about completeness.

Definition 1.1.4. A normed linear space is said to be a Banach space if it is complete.

Definition 1.1.5. A Banach algebra A is an (associative unital) $\mathbb{F}$-algebra that is also a Banach space satisfying

1. $\|xy\| \leqslant \|x\|\|y\|$ for all $x, y \in A$;
2. $\|\mathbf{1}\| = 1$.

The first condition ensures that the multiplication map is continuous.

Definition 1.1.6. For $p \in [1, \infty]$, we define $p^* \in [1, \infty]$ to be the unique number satisfying

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

p^* is called the conjugate exponent to p.

§§1.2. Continuous Linear Transformations

Definition 1.2.1. Let V and W be normed linear spaces.

A linear map $T : V \to W$ is said to be continuous if T is continuous with respect to the norm topologies on V and W. The space of all continuous functions from V to W is denoted by $\mathcal{L}(V, W)$. We also define $\mathcal{L}(V) := \mathcal{L}(V, V)$. This is an $\mathbb{F}$-algebra.

T is said to be bounded if there exists $C \geqslant 0$ such that $\|Tx\|_W \leqslant C\|x\|_V$ for all $x \in V$.

Definition 1.2.2. $T \in \mathcal{L}(V, W)$ is said to be an isomorphism if there exists $S \in \mathcal{L}(W, V)$ such that $T \circ S$ and $S \circ T$ are the appropriate identity maps.

Equivalently: T is bijective, linear, continuous with T^{-1} also continuous.

Definition 1.2.3. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are said to be equivalent if either of two equivalent conditions holds:

1. both induce the same topology (the identity map is a homeomorphism, interpreted correctly);
2. there exist constants $c, C > 0$ such that for all $x \in V$, we have

$$c\|x\|_1 \leqslant \|x\|_2 \leqslant C\|x\|_1.$$

Definition 1.2.4. Let V and W be NLS, and $T : V \to W$ be a continuous linear transform. Let B be the closed unit ball in V. The norm of T, denoted $\|T\|$, is given by

$$\|T\| := \sup_{x \in B} \|Tx\|_W.$$

This makes $\mathcal{L}(V, W)$ into a normed linear space.

Definition 1.2.5. Let V be a normed linear space. The dual space of V is defined by

$$V^* := \mathcal{L}(V, \mathbb{F}).$$

Note that V^* is a Banach space since $\mathbb{F}$ is so.

§§1.3. Hahn-Banach theorems

Definition 1.3.1. Let V be a vector space over $\mathbb{F}$. A sublinear functional is a map $p : V \to \mathbb{R}$ such that

1. $p(\alpha x) = \alpha p(x)$ for all $\alpha \in (0, \infty)$ and all $x \in V$
(this condition holds iff $p(\alpha x) \leqslant \alpha p(x)$ for all $\alpha > 0$ and all $x \in V$);
2. $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in V$.

Definition 1.3.2. Let V be an NLS, and $x \in V$. Define $J_x : V^* \to \mathbb{R}$ by

$$J_x(f) := f(x).$$

It follows that $J_x \in (V^*)^* = V^{**}$. (In fact, $\|J_x\|_{V^{**}} = \|x\|_V$.)

Define $J : V \to V^{**}$ by $x \mapsto J_x$. This is an isometry.

A Banach space V is said to be reflexive if the canonical embedding $J : V \to V^{**}$ is surjective.

Since V is a Banach space, J above being surjective is equivalent to J being an isomorphism.

§§1.4. Baire's Theorem and Applications

Definition 1.4.1. A subset of a metric space is a G_δ set if is a countable intersection of open sets.

Definition 1.4.2. Let $f : X \to Y$ be a function. The graph of f, denoted $G(f)$, is defined as

$$G(f) := \{(x, f(x)) : x \in X\} \subseteq X \times Y.$$

If X and Y are topological spaces, f is continuous, and Y is Hausdorff, then $G(f)$ is closed in the product space $X \times Y$.

§§1.5. Weak and Weak* Topologies

Definition 1.5.1. Let V be an NLS. The weak topology on V is the coarsest (smallest) such that every element of V^* is continuous.

If $(x_n)_n$ converges to x in the weak topology, we write this as $x_n \rightharpoonup x$.

Note that this is coarser than the norm topology since every element of V^* *is* (by definition) continuous in the norm topology.

Definition 1.5.2. Let X be a topological space and $f : X \to \mathbb{R}$ a function. f is lower semi-continuous if

$$f^{-1}[(-\infty, \alpha]] = \{x \in X : f(x) \leqslant \alpha\}$$

is closed in X, for every $\alpha \in \mathbb{R}$.

Definition 1.5.3. Let V and W be NLS, and let $T : V \to W$ be a linear mapping. T is said to be weakly continuous if T is continuous as a mapping from V to W, each space being endowed with its weak topology.

Definition 1.5.4. Let V be a Banach space. The weak* topology on V^* is the coarsest (smallest) topology such that the functionals $\{J_x : x \in V\}$ are all continuous.

If $(x_n)_n$ converges to x in the weak* topology, we write this as $x_n \overset{*}{\rightharpoonup} x$.

Note that V^* is a Banach space in its own right and has a weak topology $\mathcal{W}$ and norm topology $\mathcal{S}$. Let $\mathcal{W}^*$ denote the weak* topology on V^*. Then, we have

$$\mathcal{W}^* \subseteq \mathcal{W} \subseteq \mathcal{S}.$$

If V is reflexive, then $\mathcal{W}^* = \mathcal{W}$.

§2. Examples

§§2.1. Banach spaces

Definition 2.1.1. Let $p \in [1, \infty)$. For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we define

$$\|x\|_p := \left(\sum_{i=1}^{N} |x_i|^p \right)^{1/p}.$$

For $p = \infty$, define

$$\|x\|_\infty := \max_{1 \leqslant i \leqslant N} |x_i|.$$

$\| \cdot \|_p$ is a norm on $\mathbb{R}^N$. The pair $(\mathbb{R}^N, \| \cdot \|_p)$ will be denoted by ℓ_p^N.

Definition 2.1.2. $\mathbb{F}^{\mathbb{N}}$ denotes the vector space of all $\mathbb{F}$-sequences. An element of $\mathbb{F}^{\mathbb{N}}$ is written as $x = (x_n)_n$ or $(x_n)_{n \geqslant 1}$.

For $p \in [1, \infty)$, we define the space

$$\ell_p := \left\{ x \in \mathbb{F}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

For $x \in \ell_p$, we define

$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

For $p = \infty$, we define

$$\ell_\infty := \left\{ x \in \mathbb{F}^{\mathbb{N}} : \sup_i |x_i| < \infty \right\}.$$

For $x \in \ell_\infty$, we define

$$\|x\|_\infty := \sup_i |x_i|.$$

ℓ_p^N and ℓ_p are Banach spaces for all $p \in [1, \infty]$ and all $N \geqslant 1$.

Definition 2.1.3. Given a topological space X, $\mathcal{C}(X)$ denotes the vector space of all real-valued functions on X.

If X is compact, $\mathcal{C}(X)$ has a norm given by

$$\|f\|_\infty := \sup_{x \in X} |f(x)|,$$

for $f \in \mathcal{C}(X)$. This is also called the sup-norm.

Example 2.1.4. $\mathcal{C}(X)$ is a Banach space, whenever X is compact. Convergence in this metric is uniform convergence.

$\mathcal{C}^1[0,1]$ – the space of continuously differentiable functions – is a dense and proper subspace of $\mathcal{C}[0,1]$.

Example 2.1.5. $\mathcal{L}(V, W)$ is a Banach space, whenever W is so. V^* is always a Banach space (even if V is not). $\mathcal{L}(W)$ is a Banach algebra whenever W is a Banach space.

Example 2.1.6. Let $\mathcal{C}^1[0,1]$ be the subspace of $\mathcal{C}[0,1]$ consisting of continuously differentiable function (this is also endowed with the sup-norm). Then, the map

$$T : \mathcal{C}^1[0,1] \longrightarrow \mathcal{C}[0,1]$$

defined by

$$f \longmapsto f'$$

is linear but not continuous.

However, if we endow the domain with the new norm

$$\|f\|_1 := \|f\|_\infty + \|f'\|_\infty,$$

then the map is continuous.

Example 2.1.7. Consider the new norm on $\mathcal{C}[0,1]$ given by

$$\|f\|_1 := \int_0^1 |f(t)| \, dt.$$

Then, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent. (Consider the functions $f_n(x) = x^n$ and their limits.)

§§2.2. L^p and ℓ_p spaces

c_0 is the subspace of ℓ_∞ consisting of sequences that converge to 0 (equipped with the $\|\cdot\|_\infty$ norm). This is a Banach space.

Theorem 2.2.1. We have the following isometric isomorphisms.

1. $(c_0)^* \cong \ell_1$.
2. $(\ell_p)^* \cong \ell_{p^*}$ for $p \in [1, \infty)$ and not for $p = \infty$.

In both cases, the identification of the right space with the left is made as follows: Given an element $y \in \ell_{p^*}$, we get an element $f_y \in \ell_p^*$ (or c_0^* if $p^* = 1$) defined as

$$f_y(x) := \sum_{i=1}^{\infty} x_i y_i.$$

Theorem 2.2.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. We have the following isometric isomorphisms.

1. $(L^p(\mu))^* \cong L^{p^*}(\mu)$ for $p \in (1, \infty)$.
2. If μ is σ-finite, then $(L^1(\mu))^* \cong L^\infty(\mu)$.

In particular, the above is true when X is an open subset of $\mathbb{R}^n$ with the Lebesgue measure.

Theorem 2.2.3 (Riesz)**.** Given $l \in (\mathcal{C}[0,1])^*$, there exists $\alpha \in BV([0,1])$ such that $\alpha(0) = 0$ and

$$l(f) = \int_{[0,1]} f \, d\alpha$$

for all $f \in \mathcal{C}[0,1]$.

The integral above is the Lebesgue-Stieltjes integral, which is just integrating against the *signed* Borel measure induced by α.

In the following, Ω will be an open subset of $\mathbb{R}^N$ with Lebesgue measure.

Theorem 2.2.4 (Dense subsets)**.** Fix $p \in [1, \infty)$. The following subsets of $L^p(\Omega)$ are dense:

1. the set of all simple functions which vanish outside a set of finite measure,
2. $\mathcal{C}_c(\Omega)$ – the space of all continuous functions with compact support,

3. $\mathcal{C}_c^\infty(\Omega)$ – the space of all infinitely differentiable functions with compact support.

Theorem 2.2.5. ℓ_p and $L^p(\Omega)$ are separable for $1 \leqslant p < \infty$ and not separable for $p = \infty$.

Theorem 2.2.6. ℓ_p and $L^p(\Omega)$ are reflexive for $1 < p < \infty$ and not reflexive for $p \in \{1, \infty\}$.

Theorem 2.2.7 (Young's Inequality)**.** Let $1 < p < \infty$. Let $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$. The map

$$x \mapsto \int_{\mathbb{R}^N} f(y)g(x-y)\,dy$$

is well-defined almost everywhere in $\mathbb{R}^N$. The function thus defined is denoted $f * g$ and is called the convolution of f and g. Further, $f * g \in L^p(\mathbb{R}^N)$ and we have

$$\|f * g\|_p \leqslant \|f\|_1 \|g\|_p.$$

§§2.3. Weak and Weak* topologies

Example 2.3.1 (Weak convergence does not imply convergence)**.** Consider $V = \ell_p$ for $1 < p < \infty$. Consider the sequence $(e_n)_n$ in ℓ_p. Since $\|e_n - e_m\|_2 = 2^{1/p}$ for all $n \neq m$, it is clear that not only does $(e_n)_n$ not converge, but neither does any subsequence of $(e_n)_n$.

However, we $e_n \rightharpoonup 0$. Indeed, given any $f \in \ell_p^*$, it is of the form f_y, for some $y \in \ell_{p^*}$ (as discussed earlier). Thus,

$$f_y(e_n) = y_n.$$

Since $\sum |y_n|^p$ converges, it follows that $y_n \to 0$ (in $\mathbb{F}$) and thus, $f(e_n) \to f(0)$, as desired.

§3. Results

§§3.1. Banach spaces

Proposition 3.1.1. Let $p \in (1, \infty)$. If $a, b \geqslant 0$, then

$$a^{1/p} b^{1/p^*} \leqslant \frac{a}{p} + \frac{b}{p^*}.$$

Proposition 3.1.2 (Hölder's inequality)**.** Let $p \in [1, \infty)$. If $x \in \ell_p$ and $y \in \ell_{p^*}$, then

$$\sum_{i=1}^{\infty} |x_i y_i| \leqslant \|x\|_p \|y\|_{p^*}.$$

Proposition 3.1.3 (Minkwoski's Inequality)**.** Let $p \in [1, \infty)$. Let $x, y \in \ell_p$. Then

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p.$$

Proposition 3.1.4. Let V be an NLS, and W a *closed* subspace. Then,

$$\|x + W\|_{V/W} := \inf_{w \in W} \|x + w\|_V$$

defines a norm on V/W.

If V is Banach space, then so is V/W.

§§3.2. Continuous linear transformations

V and W will be assumed to be normed linear spaces throughout.

Proposition 3.2.1. Let $T : V \to W$ be linear. The following are equivalent:

1. T is continuous.
2. T is continuous at $\mathbf{0}$.
3. T is bounded.

Proposition 3.2.2. Let $T : V \to W$ be continuous, and $V \neq 0$. Then,

$$\begin{aligned} \|T\| &= \sup_{\|x\|_V = 1} \|Tx\|_W \\ &= \sup_{x \in V \setminus \{\mathbf{0}\}} \frac{\|Tx\|_W}{\|x\|_V} \\ &= \inf\{K > 0 : \|Tx\|_W \leqslant K\|x\|_V \text{ for all } x \in V\}. \end{aligned}$$

Corollary 3.2.3. $\|T(x)\| \leqslant \|T\|\|x\|$ for all $x \in V$.

Proposition 3.2.4. If W is complete, then so is $\mathcal{L}(V, W)$. In particular, V^* is always a Banach space, and $\mathcal{L}(W)$ is a Banach algebra whenever W is a Banach space.

Proposition 3.2.5. Any two norms on a finite-dimensional $\mathbb{F}$-vector space are equivalent. In particular, given any norm on $\mathbb{F}^N$, the topology induced is the Euclidean topology.

Corollary 3.2.6. Any finite dimensional NLS is complete. Any finite dimensional subspace of any NLS is closed. If $T : V \to W$ is linear and $\dim(V) < \infty$, then T is continuous.

Theorem 3.2.7 (Riesz' Lemma)**.** Let V be an NLS and $W \subsetneq V$ a *closed* and proper subspace. Then, for every $\varepsilon > 0$, there exists $u = u(\varepsilon) \in V$ such that

$$\|u\| = 1 \quad \text{and} \quad d(u, W) \geqslant 1 - \varepsilon.$$

Proposition 3.2.8. Let V be an NLS. V is finite-dimensional iff the closed unit ball is compact.

§§3.3. Hahn-Banach theorems

Theorem 3.3.1 (Hahn-Banach theorem)**.** Let V be a vector space over $\mathbb{R}$. Let $p : V \to \mathbb{R}$ be a sublinear functional. Let $W \subseteq V$ be a subspace and let $g : W \to \mathbb{R}$ be a linear map such that

$$g \leqslant p \quad \text{on } W.$$

Then, there exists a linear extension $f : V \to \mathbb{R}$ of g such that $f \leqslant p$.

Note that this was a statement purely about sublinear functionals, without any reference to norms or continuity.

Corollary 3.3.2 (Hahn-Banach Theorem)**.** Let V be an NLS over $\mathbb{F}$. Let $W \subseteq V$ be a subspace, and let $g \in W^*$. Then, there exists a continuous linear extension $f \in V^*$ of

g such that
$$\|f\|_{V^*} = \|g\|_{W^*}.$$

Corollary 3.3.3. Let V be an NLS and $x_0 \in V$ be nonzero. Then, there exists $f \in V^*$ such that $\|f\| = 1$ and $f(x_0) = x_0$.

In particular, if $x \neq y$ are elements of V, then there exists $f \in V^*$ such that $f(x) \neq f(y)$.

Corollary 3.3.4. Let V be an NLS, and $x \in V$. Then,
$$\|x\| = \sup_{f \in V^*, \|f\| \leqslant 1} |f(x)| = \max_{f \in V^*, \|f\| \leqslant 1} |f(x)|.$$

Note the duality with the *definition* of $\|f\|$. (The above is a result, and not a definition.) In particular, the above says that the supremum is always achieved, which need not be the case for the operator norm.

§§3.4. Baire's Theorem and Applications

Theorem 3.4.1 (Baire Category Theorem)**.** Let (X, d) be a complete metric space. Let $(U_n)_{n \geqslant 1}$ be a sequence of open dense sets. Then, $\bigcap_{n \geqslant 1} U_n$ is also dense (in particular, nonempty, if X is nonempty).

Theorem 3.4.2 (Banach-Steinhaus Theorem)**.** Let V be a Banach space and W an NLS. Let I be an arbitrary indexing set. Let $T_i \in \mathcal{L}(V, W)$ for each $i \in I$ be given. Then, one of the two is true:

1. $\sup_{i \in I} \|T_i\| < \infty$,
2. $\sup_{i \in I} \|T_i(x)\| = \infty$ for all x belonging to some G_δ set in V.

Corollary 3.4.3 (Uniform boundedness principle)**.** Let V be a Banach space, and W an NLS. Let $T_i \in \mathcal{L}(V, W)$ for all $i \in I$.
$$\sup_{i \in I} \|T_i(x)\| < \infty \text{ for all } x \in V \Rightarrow \sup_{i \in I} \|T_i\| < \infty.$$

Corollary 3.4.4. Let V be a Banach space, and W an NLS. Let $(T_n)_n$ be a sequence in $\mathcal{L}(V, W)$. Assume that $(T_n(x))_n$ converges in W, for all $x \in V$. Define

$$T(x) := \lim_{n\to\infty} T_n(x),$$

for $x \in V$.

Then $T \in \mathcal{L}(V, W)$ and $\|T\| \leqslant \liminf_n \|T_n\|$.

Corollary 3.4.5. Let V be a Banach space, and $B \subseteq V$ any subset. Assume that $f[B] \subseteq \mathbb{F}$ is bounded for all $f \in V^*$. Then, B is bounded in V.

Proposition 3.4.6. Let V and W be Banach spaces, and $T \in \mathcal{L}(V, W)$ be surjective. Then, there exists $c > 0$ such that

$$B_W(\mathbf{0}, c) \subseteq T[B_V(\mathbf{0}, 1)].$$

Theorem 3.4.7 (Open Mapping Theorem)**.** Let W and W be Banach spaces, and $T \in \mathcal{L}(V, W)$ be surjective. Then, T is an open map.

Corollary 3.4.8. Let W and V be Banach spaces, and $T \in \mathcal{L}(V, W)$ be bijective. Then, T is an isomorphism.

Theorem 3.4.9. Let W and V be Banach spaces, and $T : V \to W$ be linear.
The graph $G(T)$ is closed in $V \times W$ iff T is continuous.

§§3.5. Weak and Weak* Topologies

Proposition 3.5.1. The weak topology is Hausdorff.

Proposition 3.5.2. Let V be a Banach space and let W be a closed subspace of V. The weak topology on W is the topology induced on W by the weak topology on V.

Proposition 3.5.3. Let V be a Banach space, and $(x_n)_n$ a sequence in V.

1. $x_n \rightharpoonup x$ iff $f(x_n) \to f(x)$ for all $f \in V^*$.
2. $x_n \to x$ implies $x_n \rightharpoonup x$.
3. $x_n \rightharpoonup x$ implies $(\|x_n\|)_n$ is bounded and $\|x\| \leqslant \liminf_n \|x_n\|$.
4. $x_n \rightharpoonup x$ and $f_n \to f$ in V^* implies $f_n(x_n) \to f(x)$.

Remark 3.5.4. Contrary to sequences, note that "weakly open $\Rightarrow$ open".

Proposition 3.5.5. If V is finite dimensional, then norm and weak topologies coincide.

Proposition 3.5.6. Let V be a Banach space, and C be a convex and (norm) closed subset of V. Then C is also weakly closed.

Proposition 3.5.7. Let V be a Banach space. Consider the following subsets

$$\begin{aligned} D &:= \{x \in V : \|x\| < 1\} \quad \text{(open unit ball)}, \\ B &:= \{x \in V : \|x\| \leqslant 1\} \quad \text{(closed unit ball)}, \\ S &:= \{x \in V : \|x\| = 1\} \quad \text{(unit sphere)}. \end{aligned}$$

Suppose V is infinite-dimensional.

B is closed in norm topology and convex. Hence, B is closed in weak topology.
B is the weak closure of S. In particular, S is never weakly closed (but it is norm closed).
Similarly, D is never weakly open.

In the proof of the above, one sees that given any $x_0 \in D$ and any weakly open neighbourhood U of x_0, there exists $y_0 \in V \setminus \{\mathbf{0}\}$ such that U contains the (affine) line $\{x_0 + ty_0 : t \in \mathbb{R}\}$.

Corollary 3.5.8. If V is an infinite-dimensional Banach space, the weak topology is strictly coarser than the norm topology.

Theorem 3.5.9 (Schur's Lemma)**.** In the space ℓ_1, a sequence is convergent in the weak topology iff it converges in the norm topology.

Observation 3.5.10. Note that the weak topology is still strictly coarser on ℓ_1 than the norm topology. The point is that topologies are not determined by sequences for general topological spaces. This also shows that the weak topology on ℓ_1 is not metrisable.

Proposition 3.5.11. If X is a topological space, $x_n \to x$ in X, and $f : X \to \mathbb{R}$ is lower semi-continuous, then $f(x) \leqslant \liminf_n f(x_n)$.

Corollary 3.5.12. Let V be a Banach space and $\varphi : V \to \mathbb{R}$ be convex and lower semi-continuous (with respect to norm topology). Then φ is also lower semi-continuous with respect to weak topology.

In particular, the map $x \mapsto \|x\|$, being continuous, is also lower semi-continuous with respect to the weak topology, and if $x_n \rightharpoonup x$ in V, we have $\|x\| \leqslant \liminf_n \|x_n\|$.

Theorem 3.5.13. Let V and W be NLS, and $T : V \to W$ be linear. The following are equivalent:

1. T is weakly continuous.
2. $f \circ T$ is weakly continuous for every $f \in W^*$.

Theorem 3.5.14. Let V and W be *Banach* spaces, and $T : V \to W$ be linear. T is continuous iff T is weakly continuous.

Proposition 3.5.15. Let V be a Banach space. The weak* topology on V^* is Hausdorff.

Proposition 3.5.16. Let V be a Banach space and let $(f_n)_n$ be a sequence in V^*.

1. $f_n \overset{*}{\rightharpoonup} f$ iff $f_n(x) \to f(x)$ for all $x \in V$.
2. $f_n \to f \Rightarrow f_n \rightharpoonup f \Rightarrow f_n \overset{*}{\rightharpoonup} f$.
3. $f_n \overset{*}{\rightharpoonup} f$ in V^* and $x_n \to x$ in V implies $f_n(x_n) \to f(x)$.

Proposition 3.5.17. Let V be a Banach space, and ϕ be a linear functional on V^* which is continuous with respect to the weak* topology. Then $\varphi = J_x$ for some $x \in V$.

In other words, $\{J_x : x \in V\}$ is the complete set of functions which are continuous with respect to the weak* topology.

Theorem 3.5.18 (Banach-Alaoglu Theorem)**.** Let V be a Banach space. Then, B^*, the closed unit ball in V^*, is weak* compact.

Theorem 3.5.19. Let V be a Banach space. Let B be the closed unit ball in V and B^{**} the closed unit ball in V^{**}. Let $J : V \to V^{**}$ be the canonical embedding.

B^{**} is the weak* closure of $J(B)$ in V^{**}.

Remark 3.5.20. Let V be a Banach space. Since J is an isometry, it follow that $J[B]$ is closed in V^{**}. Thus, either $J[B] = B^{**}$ (which happens iff V is reflexive) or $J[B]$ is a closed and proper subspace of B^{**} (and hence, not dense).

§§3.6. Reflexivity

Theorem 3.6.1. A Banach space is reflexive iff the closed unit ball is weakly compact.

Corollary 3.6.2. Let V be a reflexive Banach space. A subset of V is weakly compact iff it is bounded and weakly closed.

In particular, if $K \subseteq V$ is a closed, bounded and convex subset, then K is weakly compact.

Corollary 3.6.3. Let V and W be Banach spaces that are isometrically isomorphic. If V is reflexive, then so is W.

Corollary 3.6.4. A closed subspace of a reflexive space is reflexive.

Corollary 3.6.5. Let V be a Banach space. V is reflexive $\Leftrightarrow$ V^* is reflexive.