

$$\int (\circlearrowleft \circlearrowright) dx$$

MA 408

Measure Theory

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Spring 2020-21

Lecture 1

06 January 2021 22:06

Idea behind measure

Simplified case: Subsets of \mathbb{R}

Given $E \subseteq \mathbb{R}$, want to assign "length" or "content" to E .

Ideally, want a map

$$\mu: \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$$

s.t.

$$(1) \quad \mu(\emptyset) = 0$$

(2) For any $E \subseteq \mathbb{R}$ and $x \in E$,

$$\mu(E) = \mu(x + E).$$

$$(x + E := \{x + y : y \in E\})$$

↑ translation by x

(3) Given a countable collection $\{E_i\}_{i=1}^{\infty}$ of disjoint subsets of \mathbb{R} , we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

(So far, $\mu \equiv 0$ will satisfy above properties!)

$$(4) \quad \mu([0, 1]) = 1. \quad (\text{"Normalisation"})$$

Any such μ would be a "candidate" for our content.

However, no such μ exists!

Consider the following sets:

- (1) Define \sim on \mathbb{R} by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
 Clearly, \sim is an equiv. relation.

Let $E \subseteq [0, 1]$ be a set containing exactly one element from each equivalence class in \mathbb{R}/\sim .

(Existence is given by Axiom of Choice. Note that)
 distinct equiv. classes are disjoint. and a small argument
 that lets you conclude $E \subset [0, 1]$.

Q. What could $\mu(E)$ be?

Note that $\{E+r\}_{r \in \mathbb{Q} \cap [0, 1]}$ is a collection of pairwise disjoint sets.

Sketch. If $x \in (E+r_1) \cap (E+r_2)$, then $x = r_1 + e_1 = r_2 + e_2$ for some $e_1, e_2 \in E$

$$\begin{aligned} & \Rightarrow e_1 - e_2 = r_2 - r_1 \in \mathbb{Q} \\ & \Rightarrow e_1 \sim e_2 \Rightarrow e_1 = e_2 \end{aligned}$$

Moreover, $[0, 1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [0, 1]} (E+r) \subseteq [0, 2] = [0, 1] \cup [1, 2]$

An easy consequence of (1) - (3) is that $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$.

Proof. $\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$. \square

$$\Rightarrow \mu([0, 1]) \leq \mu\left(\bigcup_{i=1}^{\infty} (E+r_i)\right) \leq \mu([0, 1]) + \mu([1, 2])$$

enumerate $\mathbb{Q} \cap [0, 1]$ as $\{r_1, \dots\}$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E+r_i) \leq 2$$

$[1, 2] = [0, 1] + 1$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E + r_i) \leq 2 \quad [1, 2] = [0, 1] + 1$$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E) \leq 2$$

If $\mu(E) = 0$ $\rightarrow \leftarrow$

If $\mu(E) = r > 0$ $\rightarrow \leftarrow$

$$\sum_{i=1}^{\infty} \mu(E) = \infty \leq 2$$

Possible way to salvage : Replace (3) to have "finite union" instead of "countable".

Turns out that that's still not enough.

(2) BANACH - TARSKI THEOREM (1924) : (Using AC)

For any open sets $U, V \subseteq \mathbb{R}^n$ where $n \geq 3$, there exists $k \in \mathbb{N}$ and set $U_1, \dots, U_k, V_1, \dots, V_k$ s.t.

$$(1) \quad U_i \cap U_j = \emptyset, \quad V_i \cap V_j = \emptyset, \quad 1 \leq i \neq j \leq k.$$

$$(2) \quad U = \bigcup_{i=1}^k U_i, \quad V = \bigcup_{i=1}^k V_i.$$

$$(3) \quad U_i \cong V_i, \text{ i.e.,}$$

U_i is obtained from V_i by a sequence of rotations, reflections, and translations.

In other words, by isometries.

Thus, the analogue of (2) implies $\mu(U_i) = \mu(V_i) \ \forall i$.

$$\Rightarrow \mu(U) = \mu(V).$$

\rightsquigarrow Absurd conclusions.

As it turns out, the problem is NOT in the infinite union but rather the demand that μ is defined on all of $\mathcal{P}(\mathbb{R})$!

Thus, we restrict our attention to a smaller collection of subsets of \mathbb{R} . (Not too small!)

σ -ALGEBRAS

Let X be an arbitrary set.

Def.(1) An algebra ("field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

$$\textcircled{1} A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$$

$$\textcircled{2} A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{F} \quad \text{for any } n \in \mathbb{N}.$$

(2) A σ -algebra (" σ -field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

$$\textcircled{1} A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$$

$$\textcircled{2} A_1, \dots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

Note that complements and unions give no intersections. Also, $\emptyset, X \in \mathcal{F}$.

EXAMPLES

$$\textcircled{1} \mathcal{F} = \mathcal{P}(X) \leftarrow \text{both}$$

2 (Countable-cocountable σ -algebra)

$$\mathcal{F} = \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$$

Proof: Clearly closed under complement.

Let $A_1, \dots \in \mathcal{F}$.

If all A_i are countable, then $\bigcup A_i$ is.

Suppose A_i not countable. Then, A_i^c is.

But

$$A_i \subset \bigcup A_i \Rightarrow (\bigcup A_i)^c \subset A_i^c$$

$$\Rightarrow (\bigcup A_i)^c \text{ is countable. } \square$$

③ Given any $\mathcal{F} \subseteq P(X)$, we can talk about σ -algebra generated by \mathcal{F} denoted $M(\mathcal{F})$ defined by

$$M(\mathcal{F}) = \bigcap B$$

$$\begin{matrix} f \subseteq B \\ B \text{ is a } \sigma\text{-alg} \end{matrix}$$

(Note that the intersection is non-empty because of $P(X)$.
Easy to see that intersection of σ -algebras is again a σ -alg.)

By construction, $M(\mathcal{F})$ is the smallest σ -alg containing \mathcal{F} .

BOREL σ -ALGEBRA.

Defn: Let (X, τ) be a topological space.

The σ -algebra generated by τ is called the Borel σ -algebra on X , denoted $\mathcal{B}(X)$.

(Abuse of notation that we don't mention τ .)

In other words, it is generated by the open sets of X .

Borel σ -algebra on \mathbb{R} : Smallest σ -alg on \mathbb{R} containing all the open sets.

Consequences:

- (1) All open sets are in $\mathcal{B}(\mathbb{R})$.
- (2) All closed sets are in $\mathcal{B}(\mathbb{R})$.
- (3) All F_σ, G_δ sets are in $\mathcal{B}(\mathbb{R})$.

$$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i \quad (F_i \text{ closed}) ; \quad G_\delta \equiv \bigcap_{i=1}^{\infty} G_i \quad (G_i \text{ open})$$

Prop.: Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$.

Then, \mathcal{B} is also generated by any of the following:

- (i) $\{(a, b) : a < b\}$ or $\{[a, b] : a < b\}$
- (ii) $\{[a, b) : a < b\}$ or $\{(a, b] : a < b\}$
- (iii) $\{(a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a) : a \in \mathbb{R}\}$
- (iv) $\{[a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a] : a \in \mathbb{R}\}$

Proof: Easy. \square

Borel σ -algebra on \mathbb{R}^n :

Suppose $\{X_i\}_{i=1}^n$ are metric spaces.

Let $X = \prod_{i=1}^n X_i$ with the product metric.
↓ defn

If f_i is the metric on X_i , then
 f on $\prod X_i$ is defined as

$$f(x, y) = \max_{1 \leq i \leq n} f_i(x_i, y_i) \quad \begin{cases} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{cases}$$

Def.: Suppose (X_i, M_i) are σ -algebras. One can define a σ -algebra on $X := \prod X_i$ as follows:

Consider the projection maps $\pi_i : X \rightarrow X_i$.
 Let

$$\mathcal{F} = \{ \pi_i^{-1}(E) : E \in M_i, i=1, \dots, n \}$$

$$= \{E \times x_2 \times \dots \times x_n : E \in M_1\} \\ \cup \{x_1 \times E \times \dots \times x_n : E \in M_2\} \\ \cup \dots \cup \{x_1 \times \dots \times x_{n-1} \times E : E \in M_n\}.$$

$M := M(\mathcal{F}) \subseteq \mathcal{P}(X)$ is the product σ -algebra induced by $\{M_i\}_{i=1}^n$.

We often write the above as $M = \overline{\prod}_{i=1}^n M_i$.

Caution: The above $\overline{\prod}_{i=1}^n$ is NOT the set-theoretic cartesian product.

Now, we get two (possibly different) σ -algebras on \mathbb{R}^n .

- ① Borel σ -alg. on $(\mathbb{R}^n, \mathcal{J})$
- ② Product of Borel σ -alg. of $\mathcal{B}(\mathbb{R})$.

Prop: $\mathcal{B}(\mathbb{R}^n) = \overline{\prod}_{i=1}^n \mathcal{B}(\mathbb{R})$. That is, both the σ -alg above are same.

Proof: We will prove this by a sequence of observations.

- ① Suppose $\{(x_i, M_i)\}_{i=1}^n$ are σ -algebras and $f_i \subseteq M_i$ are such that $M_i = M(f_i)$. ($i=1, \dots, n$)

Then, if $X = \overline{\prod}_{i=1}^n X_i$ and $M = \overline{\prod}_{i=1}^n M_i$, then

M is generated by $\{\prod_{i=1}^n f_i^{-1}(E_i) : E_i \in f_i, i=1, \dots, n\}$.

- ② M is generated by $\{E_1 \times \dots \times E_n : E_i \in f_i\}$ if we further assume $x_i \in f_i$.

Assuming ① and ② for now, we now note the following.

Clearly, one has $\prod_{i=1}^n \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^n)$.

[Proof: using ②, $\prod_{i=1}^n \mathcal{B}(\mathbb{R})$ is gen. by sets of the form $U_1 \times \dots \times U_n$, each $U_i \subseteq \mathbb{R}$ open]

Each such set is open in the metric space \mathbb{R}^n .
Thus, it is in $\mathcal{B}(\mathbb{R}^n)$.

We show $\mathcal{B}(\mathbb{R}^n) \subseteq \prod \mathcal{B}(\mathbb{R})$.

(*) { It suffices to show that every set of the form

$$U_1 \times \dots \times U_n \quad \text{where } U_i \subseteq \mathbb{R} \text{ are open}$$

are in the product $\prod \mathcal{B}(\mathbb{R})$.

{ Why? Every open set in \mathbb{R}^n is a countable union of sets of aforementioned form. In turn, the open sets generate $\mathcal{B}(\mathbb{R}^n)$.

Proving (*) is easy because

$$U_1 \times \dots \times U_n = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \dots \cap \pi_n^{-1}(U_n).$$

these are in $\prod \mathcal{B}(\mathbb{R})$, by def

Proof of ①

Want to show that $\tilde{\mathcal{F}} = \{\pi_i^{-1}(E) : E \in \mathcal{F}_i, 1 \leq i \leq n\}$ gen. $\prod M_i$.

Clearly $M(\tilde{\mathcal{F}}) \subseteq M$. ($\tilde{\mathcal{F}} \subseteq M$ and M is σ -alg)

It now suffices to show that every generator of M is in $M(\tilde{\mathcal{F}})$. (standard)

Note $M = \langle \pi_i^{-1}(E) : E \in M_i, 1 \leq i \leq n \rangle$
 $\tilde{M} := \langle \pi_i^{-1}(E) : E \in \mathcal{F}_i, 1 \leq i \leq n \rangle = M|\tilde{\mathcal{F}}$

Let $\tilde{M}_i := \{E \in M_i : \pi_i^{-1}(E) \in \tilde{M}\} \subseteq P(X_i)$.

We shall show that $\tilde{M}_i = M_i$.

We know, by defⁿ that $\mathcal{F}_i \subseteq \tilde{M}_i$. $(E \in \mathcal{F}_i \stackrel{e.M_i}{\Rightarrow} \pi_i^{-1}(E) \in \tilde{M} \cap M_i)$
 \Downarrow
 $E \in M_i$.

Moreover, $M(\mathcal{F}_i) = M_i$. Thus, it suffices to show that \tilde{M}_i is a σ-alg.

To that end, let $A \in \tilde{M}_i$. Then, $\pi_i^{-1}(A) \in \tilde{M}$.

Then, $\pi_i^{-1}(A)^c \in \tilde{M}$. But $\pi_i^{-1}(A^c) = \pi_i^{-1}(A)^c \in M$.
 $\Rightarrow \pi_i^{-1}(A^c) \in M_i$

Similarly, noting that $\pi_i^{-1}\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigcup_{j=1}^{\infty} \pi_i^{-1}(A_j)$ yields the result.

Proof of ②

Let $f' := \{E_1 \times \dots \times E_n : E_i \in \mathcal{F}_i\}$.

Note that $E_1 \times \dots \times E_n = \pi_1^{-1}(E_1) \cap \dots \cap \pi_n^{-1}(E_n) \in M$.

Thus, $M(f') \subseteq M$.

Or else, $\pi_i^{-1}(E_i) = E_1 \times x_2 \times \dots \times x_n \in \mathcal{F}'$

or in general,

$$\pi_i^{-1}(E_i) \subset \prod_{j=1}^n A_j \in \mathcal{F}' \quad \text{where} \quad A_j = \begin{cases} E_i & ; j=i \\ X_j & ; j \neq i \end{cases}$$

$$\Rightarrow M \subseteq M(f')$$

REMARKS.

① The argument above generalises for a separable

metric spaces.

- ② If $(X_i, M_i)_{i \in A}$, and A is Countable, then again, $X = \prod X_i$, $M = \prod M_i$ generated by $\{\pi_i^{-1}(E) : E \in M_i, i \in A\}$ is also generated by sets of the form $\left(\prod_{i \in A} E_i\right)$, $E_i \in F_i$.

MEASURE

Defn. Suppose (X, M) is a measure space, i.e., M is a σ -algebra on X . A measure on X is a map $\mu: M \rightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) if $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

↓
in M

EXAMPLES.

(1) $X = \{x_1, x_2, \dots\}$ is countable. Suppose $p_i \geq 0$ are reals s.t. $\sum_{i=1}^{\infty} p_i = 1$. Let $M = P(X)$ and define $\mu: M \rightarrow [0, 1]$ as

$$\mu(E) = \sum_{i: x_i \in E} p_i.$$

(2) (X, M) be s.t. M is the countable-co-countable σ -alg. s.t. X itself is uncountable

Define

$$\mu(E) := \begin{cases} 0 & ; E \text{ is countable} \\ 1 & . . . \end{cases}$$

L

, \mathcal{E} is uncountable

Prop. Suppose (X, \mathcal{M}, μ) is a measure space.

Then,

$$\textcircled{1} \quad E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$$

$$\textcircled{2} \quad \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i) \quad (\mu \text{ is "sub-additive"})$$

\textcircled{3} If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \dots$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i).$$

Proof. \textcircled{1} \& \textcircled{2} are trivial

\textcircled{3} Define $F_i = E_i \setminus E_{i-1}$ for $i \geq 2$.
 $F_1 = E_1$

Then, $\bigcup_{i=1}^n F_i = \bigcup_{i=1}^n E_i$. Also, $F_i \in \mathcal{M}$ for each i .
 $(n = \infty \text{ as well})$ Moreover, $F_i \cap F_j = \emptyset$ for $i \neq j$.

$$\begin{aligned} \text{Thus, } \mu\left(\bigcup E_i\right) &= \mu\left(\bigcup F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(F_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i) = \sum_{i=1}^{\infty} E_i. \end{aligned}$$

Def. \textcircled{1} A null set in a measure space (X, \mathcal{M}, μ) is a set E s.t. $E \subseteq F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.
 $(E \in \mathcal{M} \text{ NOT necessary.})$

\textcircled{2} Given a measure space (X, \mathcal{M}, μ) , the completion of \mathcal{M} , denoted $\bar{\mathcal{M}}$ is the collection of all sets of the form $F \cup N$ where $F \in \mathcal{M}$ and N is a null set.

Prop. ① If (X, \mathcal{M}, μ) is a measure space, then $\bar{\mathcal{M}}$ is a σ -alg.

② Moreover, there exists a unique measure

$$\bar{\mu}: \bar{\mathcal{M}} \rightarrow [0, \infty] \text{ s.t.}$$

$$\bar{\mu}|_{\mathcal{M}} = \mu.$$

(That is, there is a unique extension of μ to a measure $\bar{\mu}$ on $\bar{\mathcal{M}}$.)

Lecture 2

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Prop. ① If (X, \mathcal{M}, μ) is a measure space, then $\bar{\mathcal{M}}$ is a σ -alg.
 ② Moreover, there exists a unique measure

$$\bar{\mu}: \bar{\mathcal{M}} \rightarrow [0, \infty] \quad \text{s.t.}$$

$$\bar{\mu}|_{\mathcal{M}} = \mu.$$

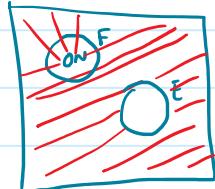
Proof ① To show that $\bar{\mathcal{M}}$ is a σ -algebra, we need to show:

- (i) $A \in \bar{\mathcal{M}} \Rightarrow A^c \in \bar{\mathcal{M}}$
- (ii) $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bar{\mathcal{M}}$.

(i) $A = E \cup N$, $N \subseteq F$, $\mu(F) = 0$, $E, F \in \mathcal{M}$
 $A^c = E^c \cap N^c$

Now, $E \cup F \in \mathcal{M}$ and hence, $E^c \cap F^c$.

Note that $E^c \cap N^c = (\underbrace{E^c \cap F^c}_{\in \mathcal{M}}) \cup (\underbrace{F \setminus N}_{\subseteq F \text{ and } \mu(F) = 0}) \in \bar{\mathcal{M}}$.



Now, if $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}}$, we can write

$$A_i = E_i \cup N_i, \quad N_i \subseteq F_i, \quad \mu(F_i) = 0, \quad E_i, F_i \in \mathcal{M}.$$

$$\bigcup A_i = \left(\bigcup \underbrace{E_i}_{=: E} \right) \cup \left(\bigcup \underbrace{N_i}_{=: N} \right)$$

Note $E \in \mathcal{M}$. Also, put $F = \bigcup F_i$. Then, $F \in \mathcal{M}$.
 Moreover, $N \subseteq F$ & $\mu(F) = \sum \mu(F_i) = 0$.

$\therefore \bigcup A_i = E \cup N$, in the desired form.

(iv) Define $\bar{\mu}: M \rightarrow [0, \infty]$ as

$$\bar{\mu}(E \cup N) := \mu(E).$$

To show: $\bar{\mu}$ is well-defined.

Suppose $E_1 \cup N_1 = E_2 \cup N_2$.

$$\rightarrow E_1 \subset E_2 \cup N_2 \subseteq E_2 \cup F_2$$

$$\text{by } \Rightarrow \mu(E_1) \leq \mu(E_2) + \mu(F_2) = \mu(E_2).$$

$$\mu(E_2) \leq \mu(E_1). \quad \therefore \mu(E_1) = \mu(E_2).$$

Thus, $\bar{\mu}$ is well-defined

Note that this was also
the only way to define $\bar{\mu}$.
Thus, uniqueness will follow.

To show: $\bar{\mu}$ is a measure.

$$\textcircled{1} \quad \bar{\mu}(\emptyset) = 0 \text{ is trivial since } \bar{\mu}(\emptyset) = \bar{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0.$$

\textcircled{2} Suppose $\{E_i \cup N_i\}_{i=1}^{\infty}$ are pairwise disjoint.

$$\begin{aligned} \bar{\mu}(U(E_i \cup N_i)) &= \bar{\mu}((\cup E_i) \cup (\cup N_i)) = \mu(\cup E_i) = \sum \mu(E_i) \\ &= \sum \bar{\mu}(E_i \cup N_i) \end{aligned}$$

some logic
earlier

OUTER MEASURE

Defn: An outer measure on X is a map

$$\mu^*: P(X) \rightarrow [0, \infty] \text{ satisfying}$$

$$(i) \mu^*(\emptyset) = 0,$$

$$(ii) A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B),$$

$$(iii) \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Motivation for μ^* comes from the intuitive idea that knowing areas enclosed by rectangles, we "approximate" areas bounded by arbitrary sets by covering these by countable union of rectangles.

Propn. Suppose $f \subseteq \mathcal{P}(X)$ and $g: f \rightarrow [0, \infty]$ s.t.

- (i) $\emptyset, X \in f$
- (ii) $g(\emptyset) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} g(E_i) : E_i \in f, E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

Then, μ^* is an outer measure.

Proof. We need to show μ^* is well-defined \leftarrow This follows because and it satisfies $X \in f$.

$$(i) \mu^*(\emptyset) = 0 \quad \leftarrow \text{trivial since } \mu^*(\emptyset) \geq 0 \text{ since inf over non-negative reals}$$

$$\text{Also, } \emptyset \subseteq \emptyset \text{ & } g(\emptyset) = 0$$

$$(ii) A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$$

Any cover of B is also a cover of A .
 \therefore We are taking inf over a larger set & hence, it will be smaller.
 $\Rightarrow \mu^*(A) \leq \mu^*(B)$

$$(iii) \text{ Suppose } \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{P}(X). \text{ Fix } \varepsilon > 0.$$

for each E_i , let $\{A_j^{(i)}\}_{j=1}^{\infty} \subseteq f$ be a cover of E_i :

$$\dots * (i-1) > \sum_{j=1}^{\infty} g(A_j^{(i)}) < \dots$$

with $\mu^*(E_i) \geq \sum_{j=1}^{\infty} \rho(A_j^{(i)}) - \frac{\epsilon}{2^i} + \epsilon$

Then, $\bigcup_i \bigcup_j A_j^{(i)}$ covers $\bigcup E_i$.

$$\begin{aligned} \text{Thus, } \mu^*\left(\bigcup E_i\right) &\leq \sum_{i,j} A_j^{(i)} \leq \left(\sum_{i=1}^{\infty} \mu^*(E_i) + \frac{\epsilon}{2^i} \right) \\ &= \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon \end{aligned}$$

$$\Rightarrow \mu^*\left(\bigcup E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon$$

Since $\epsilon > 0$ is arbit, this is completes the proof. \square

Defn. Given an outer measure μ^* , we say that a set $A \subseteq X$ is μ^* -measurable if for all $E \in P(X)$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Defn. A measure μ on (X, M) is complete if for all $F \in M$ with $\mu(F) = \phi$, we have $P(F) \subseteq M$.

(That is, all null sets are in M .)

Thm. (CARATHÉODORY)

Let μ^* be an outer measure on X .

Let $M := \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$.

Then,

(i) M is a σ -algebra.

(ii) μ^* restricted to M is a complete measure.

Proof. (i) M is closed under (\setminus) since the defn of μ^* -meas. is symmetric under (\setminus) .

Now, if $A, B \in M$ and $E \subset X$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A^c \cap B^c)) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)\end{aligned}$$

Of course, we also have $\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \leq \mu^*(E)$.

$$\therefore \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Thus, M is closed under finite unions. To show that it is a σ -alg, it suffices to show closure under disjoint unions.

(why? Given $\{E_i\}_{i=1}^{\infty} \subseteq M$, consider $F_i = E_i$ and $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right)$ for $n \geq 2$.)
Note that $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and that each $F_i \in M$.

Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint collection in M . Put $B_n = \bigcup_{i=1}^n A_i$
and $B = \bigcup_{i=1}^{\infty} A_i$.

Then, for any $E \subset X$,

$$\begin{aligned}B_n \cap A_n &= \bigcup_{i=1}^n (A_n \cap A_i) = A_n \\ B_n \cap A_n^c &= \bigcup_{i=1}^n (A_n^c \cap A_i) = \bigcup_{i=1}^{n-1} A_i \\ &= B_{n-1},\end{aligned}$$

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) \quad \because \underbrace{\mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2})}_{\dots} \\ &= \dots\end{aligned}$$

$$\Rightarrow \mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \left[\sum_{i=1}^n \mu^*(E \cap A_i) \right] + \mu^*(E \cap B_n^c)$$

$$(B_n \subset B \Rightarrow B^c \subset B_n^c)$$

Take $n \rightarrow \infty$

$$\geq \left[\sum_{i=1}^{\infty} \mu^*(E \cap A_i) \right] + \mu^*(E \cap B^c)$$

take $n \rightarrow \infty$

$$\Rightarrow \mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)$$

$$\begin{aligned}
 (\star) \quad & \geq \mu^*\left(\bigcup(E \cap A_i)\right) + \mu^*(E \cap B^c) \\
 & = \mu^*\left(E \cap \left(\bigcup A_i\right)\right) + \mu^*(E \cap B^c) \\
 & = \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).
 \end{aligned}$$

Thus, we have equality throughout giving

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c) \text{ & hence, } B \in M.$$

Moreover, taking $E = B$; the (\star) equation gives

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Thus, μ^* is countably additive on M .

Thus, μ is a measure on M .

To show completeness: Let $F \in M$ be s.t. $\mu(F) = 0$ and $A \subseteq F$.

Then, $\mu^*(A) = 0$. Also, for any $E \subseteq X$, we have

$$\begin{aligned}
 \mu^*(E) & \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(A) + \mu(E \cap A^c) \\
 & = \mu^*(E \cap A^c) \leq \mu^*(E).
 \end{aligned}$$

Thus, $A \in M$.

B

Def.: Suppose \mathcal{F} is an algebra on X .

A map

$$\mu_0: \mathcal{F} \rightarrow [0, \infty]$$

is called a pre-measure if

- (i) $\mu_0(\emptyset) = 0$,
- (ii) If $\{\mathcal{A}_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint s.t. $\bigcup_{i=1}^{\infty} \mathcal{A}_i \in \mathcal{F}$,

then

$$\mu_0\left(\bigcup_{i=1}^{\infty} \mathcal{A}_i\right) = \sum_{i=1}^{\infty} \mu_0(\mathcal{A}_i).$$

(Note that the above also gives the thing for finite unions.
guaranteed to be in \mathcal{F} .)

Propn: Suppose μ_0 is a premeasure on an algebra \mathcal{F} .
Then, if μ^* is the outer measure as defined in the
earlier proposition, then

- (i) $\mu^*|_{\mathcal{F}} = \mu_0$
- (ii) Every set in \mathcal{F} is μ^* -measurable.

An immediate corollary:

Thm: Suppose $\mathcal{F} \subseteq P(X)$ is an algebra and suppose M is the σ -algebra generated by \mathcal{F} .
Let μ_0 be a premeasure defined on \mathcal{F} and let μ^* be the outer measure as before. Then,

- (i) $\mu^*|_M$ is a measure on (X, M) . Put $\mu = \mu^*|_M$.
- (ii) If ν is any measure extending μ_0 , then

$$\nu(E) = \mu(E)$$

whenever $\mu(E) < \infty$

Proof of Propn.

(i) For $E \in \mathcal{F}$, want to show, $\mu^*(E) = \mu_0(E)$.

Considering $E_1 = E$ and $E_i = \emptyset$ for $i \geq 2$ gives

$$\sum_{i=1}^{\infty} \mu_0(E_i) = \mu_0(E) \Rightarrow \mu^*(E) \leq \mu_0(E).$$

\downarrow inf over all covers

To show \geq : let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$ be a cover for E .

$$\text{Let } F_i := E \cap (E_i \setminus \bigcup_{j=1}^{i-1} E_j).$$

Clearly (i) $F_i \in \mathcal{F}$ by closure properties

(ii) $F_i \cap F_j = \emptyset$ if $i \neq j$.

$$(iii) \bigcup_{i=1}^{\infty} F_i = E.$$

If $\{F_i\}$ is a cover s.t. $\sum_{i=1}^{\infty} \mu_0(F_i) \leq \mu^*(E) + \varepsilon$
(such a cover exists)

$$\text{Note } \mu_0(E) = \mu_0\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu_0(F_i) \leq \mu^*(E) + \varepsilon$$

μ_0 is a premeasure

$$\Rightarrow \mu_0(E) \leq \mu^*(E) + \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \mu_0(E) \leq \mu^*(E).$$

This establishes (i).

(ii) To show: Every $A \in \mathcal{F}$ is μ^* -measurable.

Let $A \in \mathcal{X}$. Let $\{E_n\}$ be a cover for A s.t.

$$\sum_n \mu_0(E_n) \leq \mu^*(A) + \varepsilon$$

Then,

$$\begin{aligned} \mu^*(A) + \varepsilon &\geq \sum_n \mu_0(E_n) \xrightarrow[\text{and } \mu_0 \text{ is pre-meas.}]{} \\ &= \sum [I_{E_n \cap A} + I_{E_n \cap A^c}] \end{aligned}$$

$$= \sum_n [\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c)]$$

$$= \left[\sum_n \underbrace{\mu_0(E_n \cap A)}_{\substack{\text{cover for} \\ E \cap A}} \right] + \left[\sum_n \underbrace{\mu_0(E_n \cap A^c)}_{E \cap A^c} \right]$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

$$\Rightarrow \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad (E \text{ was arbit.})$$

\leq is anyway true for any outer measure.

$$\Rightarrow \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Thus, f is μ^* -measurable.

Proof of Thm.

(i) Follows from Carathéodory and Propn. (Since F is μ^* -measurable and M is the smallest σ -alg containing F .)

(ii) First note that if ν extends μ_0 , then

for any $E \in M = M(F)$

if $E \subseteq \bigcup_n E_n$, $E_n \in F$, then

$$\nu(E) \leq \sum_n \nu(E_n) = \sum_n \mu_0(E_n).$$

So if the cover is s.t. $\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon = \mu(E) + \varepsilon$,

this gives $\nu(E) \leq \mu(E) + \varepsilon$ for arbit ε . That is,

$$v(E) \leq \mu(E).$$

Now, if $\mu(E) < \infty$, we show \geq .

Let $\{E_i\}_{i=1}^n$ be a cover for $E \in \mathcal{M}$ and let $A := \bigcup_{i=1}^n E_i$.

$$\begin{aligned} \text{Note that } v(A) &= \lim_{n \rightarrow \infty} \left(v\left(\bigcup_{i=1}^n E_i\right) \right) = \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{i=1}^n E_i\right) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n E_i\right) \\ &= \mu(A) \end{aligned}$$

Since $\mu(E) < \infty$ (by assumption), we can pick a cover $\{E_i\}_{i=1}^n$ s.t.

$$\mu(A) \underset{\mu(A \cap E) + \mu(A \setminus E)}{<} \mu(E) + \epsilon \text{ and thus, } \mu(A \setminus E) < \epsilon.$$

$$\begin{aligned} \text{Thus, } \mu(E) &\leq \mu(A) = v(A) = v(E) + v(A \setminus E) \leq v(E) + \mu(A \setminus E) \\ &< v(E) + \epsilon \end{aligned}$$

This gives $\mu(E) \leq v(E)$, as desired.

TOWARDS "GOOD" BOREL MEASURES

The idea is to extend/define a measure μ on $\mathcal{B}(\mathbb{R})$ from the notion of length of bounded intervals. Whatever done so far leads to that.

Defn. A half-interval is a subset of \mathbb{R} of the form:

- (i) $(a, b]$ for $-\infty \leq a < b < \infty$, or
- (ii) (a, ∞) for $-\infty \leq a < \infty$, or
- (iii) \emptyset .

Can be checked that the collection of fin unions of half-intervals is an algebra on \mathbb{R}

Propⁿ: Let F be the algebra consisting of finite unions of half-intervals.

Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing and right continuous function.

$$\left(\lim_{\delta \downarrow 0} F(x + \delta) = F(x) \quad \forall x \in \mathbb{R} \right)$$

Define

$$\mu_0 \left(\bigcup_{j=1}^n (a_j, b_j] \right) := \sum_{j=1}^n (F(b_j) - F(a_j)),$$

$$\text{and } \mu_0(\emptyset) = 0.$$

Then, μ_0 is a premeasure on F .

Remarks:

(1) Note that F above actually generates $\mathcal{B}(\mathbb{R})$, as seen in Lec 1.

(2) If we take $F(x) = x$, then $F(b) - F(a) = b - a$
= length of $[a, b]$.

So, the above extends the notion of measure arising from lengths of intervals onto the Borel σ -field.

(3) Why right-continuity?

Suppose μ is a finite Borel measure. Let

$$F(x) := \mu((-\infty, x]).$$

$$\begin{aligned} \text{Then, if } x_n \downarrow x, \text{ then } \lim_{n \rightarrow \infty} \mu((-\infty, x_n]) &= \mu \left(\bigcap_{n=1}^{\infty} (-\infty, x_n] \right) \\ &\stackrel{\text{def}}{=} \mu((-\infty, x]) = F(x) \end{aligned}$$

Thus, this F above is right-continuous.

Note that closure on right

Proof of the Propⁿ

First, we need to check that μ_0 is well-defined.

Let $\{(a_j, b_j]\}$ ($j=1, \dots, n$) be pairwise disjoint
and let $\bigcup_{j=1}^n (a_j, b_j] = (a, b]$.

Then, by re-arranging indices, if necessary, it follows that

$a = a_1 < b_1 = a_2 < b_2 = \dots = a_n < b_n = b$ and in
this case

$$\mu_0((a, b)) = F(b) - F(a)$$

$$\mu_0\left(\bigcup_{j=1}^n (a_j, b_j]\right) = \sum F(b_j) - F(a_j) = F(b) - F(a)$$

Thus, μ_0 is well-defined in this case.

More generally, if $\{I_i\}_{i=1}^n$ and $\{J_j\}_{j=1}^m$ are s.t.

$$\bigcup_{i=1}^n I_i = \bigcup_{j=1}^m J_m, \text{ then}$$

$$\sum_i \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_j \mu_0(J_j)$$

since intersection of half-intervals is again
a half-interval, for which the defⁿ is consistent.

This shows that μ_0 is well-defined on \mathcal{F} .

It now remains to show that μ_0 is indeed a premeasure
on \mathcal{F} .