Galois correspondence in Algebraic Topology

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Galois correspondence: Proof (contd.)

Existence. Let $(\tilde{X}, \widetilde{x_0})$ be the universal covering space. Let G be the group of covering transformations.
Recall the isomorphism $\Phi : \pi_1(X, x_0) \to G$. Let $H' = \Phi(H)$. H' acts evenly on $\tilde{X}$.

Consider $(E, e_0) := (\tilde{X}/H', H'\widetilde{x_0})$. Now, we get an induced covering map p as follows.

$$\begin{tikzcd}
(\tilde{X}, \widetilde{x_0}) \arrow[d, "\pi"'] \arrow[dr, "q"] & \\
(E, e_0) \arrow[r, "p"'] & (X, x_0)
\end{tikzcd}$$

We now show that $p_*\pi_1(E, e_0) = H$.

($\supset$) Let $[\sigma] \in H$. $\Phi([\sigma]) \in H'$.

Let $\tilde{\sigma}$ be a lift to $\tilde{X}$ with $\tilde{\sigma}(0) = \widetilde{x_0}$. Put $\widetilde{x_1} := \tilde{\sigma}(1)$.

Then, there is a homeomorphism $h \in H'$ such that $h(\widetilde{x_0}) = \widetilde{x_1}$. Thus, $H'\widetilde{x_0} = H'\widetilde{x_1}$ and hence, $\pi \circ \tilde{\sigma}$ is a loop. But we have $p \circ \pi \circ \tilde{\sigma} = q \circ \tilde{\sigma} = \sigma$.

Thus, $[\sigma] = p_*([\pi \circ \tilde{\sigma}]) \in p_*\pi_1(E, e_0)$.

Galois correspondence: Proof (contd.)

($\subset$) Consider a loop τ in E at e_0. We wish to show $[p \circ \tau] \in H$.

Let $\sigma = p \circ \tau$. This is a loop in X at x_0. Consider a lift $\tilde{\sigma}$ in $\tilde{X}$ with $\tilde{\sigma}(0) = \widetilde{x_0}$.

Now, note that $\tau = \pi \circ \tilde{\sigma}$ is a loop in E. Thus, $\pi(\widetilde{x_0}) = \pi(\widetilde{x_1})$. This tells us that $H'\widetilde{x_0} = H'\widetilde{x_1}$.

In other words, there exists $h \in H'$ such that $h(\widetilde{x_0}) = \widetilde{x_1}$. Thus, $\Phi([\sigma]) \in H'$ or $[\sigma] \in H$, as desired. $\square$

We now show that $p_*\pi_1(E, e_0) = H$.

Let $[\sigma]$ be an arbitrary element of $\pi_1(X, x_0)$.

Consider the lift $\tilde{\sigma}$ in $\tilde{X}$ with $\tilde{\sigma}(0) = \widetilde{x_0}$. Put $\widetilde{x_1} := \tilde{\sigma}(1)$.

Let $\tau = \pi \circ \tilde{\sigma}$. This is a path in E starting at e_0. Moreover, it is the unique lift of σ starting at e_0.
Thus, if τ is a loop, then $[\sigma] = p_*([\tau]) \in p_*\pi_1(E, e_0)$.
More importantly, if τ is *not* a loop, then $[\sigma] \notin p_*\pi_1(E, e_0)$.

Note that τ is a loop iff $\pi_1(\widetilde{x_0}) = \pi_1(\widetilde{x_1})$ iff there is a homeomorphism $h \in H'$ such that $h(\widetilde{x_0}) = \widetilde{x_1}$ iff $\Phi([\sigma]) \in H'$ iff $[\sigma] \in H$.

Thus, $[\sigma] \in H \iff [\sigma] \in p_*\pi_1(E, e_0)$.