

$$\int (\cap \cup) dx$$

MA 408

Measure Theory

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Spring 2020-21

Last Updated: January 11, 2021

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Lecture 1.

Theorem 1.1 (Non existence of ideal measure). There is no map $\mu : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. $\mu(E) = \mu(x + E)$ for all $x \in \mathbb{R}$ and $E \in \mathcal{P}(\mathbb{R})$,
where $x + E := \{x + y \mid y \in E\}$,
3. for any disjoint countable collection $\{E_i\}_i^\infty$ of subsets of \mathbb{R} , we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i),$$

4. $\mu([0, 1]) = 1$.

Note that the last is a “normalisation” property. Otherwise $\mu \equiv 0$ or $\mu(X) = \begin{cases} 0 & X = \emptyset, \\ \infty & \text{otherwise} \end{cases}$ would also satisfy and give us “useless” functions.

Replacing “countable union” with “finite union” also won’t do the trick in general due to the Banach-Tarski “paradox” (theorem).

Both the above required a use of the Axiom of Choice.

Definition 1.2 (Algebra). Let X be a non-empty set.

An **algebra** (“**field**”) on X is a non-empty collection $\mathcal{F} \subset \mathcal{P}(X)$ satisfying

1. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
2. $A_1, \dots, A_n \in \mathcal{F} \implies \bigcup_{i=1}^n A_i \in \mathcal{F}$.

Definition 1.3 (σ -algebra). Let X be a non-empty set.

A **σ -algebra** (“ **σ -field**”) on X is a non-empty collection $\mathcal{F} \subset \mathcal{P}(X)$ satisfying

1. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
2. $A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Example 1.4 (Countable-cocountable σ -algebra). Let $X \neq \emptyset$. Then,

$$\mathcal{F} = \{E \subset X \mid E \text{ or } E^c \text{ is countable}\}$$

is a σ -algebra on X .

Definition 1.5 (σ -algebra generated by a set). Let $\mathcal{E} \subset \mathcal{P}(X)$. Then,

$$\mathcal{M}(\mathcal{E}) := \bigcap_{\substack{\mathcal{E} \subset \mathcal{B} \\ \mathcal{B} \text{ is a } \sigma\text{-algebra}}} \mathcal{B}$$

is a σ -algebra. Moreover, it is the smallest σ -algebra containing \mathcal{E} .

This is called the σ -algebra generated by \mathcal{E} .

Definition 1.6 (Borel σ -algebra). Let (X, \mathcal{T}) be a topological space. The σ -algebra generated by \mathcal{T} is called the Borel σ -algebra on X , denoted $\mathcal{B}(X)$.

In other words, $\mathcal{B}(X)$ is the σ -algebra generated by the open sets of X .

Proposition 1.7. All of the following are contained in $\mathcal{B}(\mathbb{R})$:

1. All closed sets.
2. All open sets.
3. All F_σ and G_δ sets.

Recall that an F_σ set is a set which can be written as countable union of closed sets. Similarly, G_δ as countable intersection of open sets.

Proposition 1.8. $\mathcal{B}(\mathbb{R})$ is generated by any of the following collections.

1. $\{(a, b) \mid a < b\}$ or $\{[a, b) \mid a < b\}$,
2. $\{(a, b) \mid a < b\}$ or $\{[a, b) \mid a < b\}$,
3. $\{(a, \infty) \mid a \in \mathbb{R}\}$ or $\{(-\infty, b) \mid b \in \mathbb{R}\}$,
4. $\{[a, \infty) \mid a \in \mathbb{R}\}$ or $\{(-\infty, b) \mid b \in \mathbb{R}\}$.

Definition 1.9 (Product of σ -algebras). Let $\{(X_i, \mathcal{M}_i)\}_{i=1}^n$ be a finite collection of sets and σ -algebras.

Put $X := \prod_{i=1}^n X_i$ and let $\pi_i : X \rightarrow X_i$ denote the projection onto the i -th coordinate.

Let

$$\mathcal{B} = \{\pi_i^{-1}(E) \mid E \in \mathcal{M}_i, i = 1, \dots, n\}.$$

Then, $\mathcal{M} = \mathcal{M}(\mathcal{B})$ is the product σ -algebra induced by $\{\mathcal{M}_i\}_{i=1}^n$ which we (misleadingly)

denote by $\prod_{i=1}^n \mathcal{M}_i$.

With the above, we get two (possibly different) σ -algebrae on \mathbb{R}^n . One is the Borel σ -algebra on it, by virtue of it being a topological space, i.e., $\mathcal{B}(\mathbb{R}^n)$ and the other is the product of σ -algebra, i.e., $\prod_{i=1}^n \mathcal{B}(\mathbb{R})$. As it turns out, both are equal.

Theorem 1.10. $\mathcal{B}(\mathbb{R}^n) = \prod_{i=1}^n \mathcal{B}(\mathbb{R})$.

Remark 1.11. In general, the above can be generalised to a product of separable metric spaces. (Note that the product of metric spaces in the product topology is metrisable.)

Definition 1.12 (Measure). Suppose X is a non-empty set and \mathcal{M} a σ -algebra on X . A **measure** on X is a map

$$\mu : X \rightarrow [0, \infty]$$

satisfying

1. $\mu(\emptyset) = 0$,
2. if $\{E_i\}_1^\infty \subset \mathcal{M}$ are pairwise disjoint, then

$$\mu \left(\bigsqcup_{i=1}^\infty E_i \right) = \sum_{i=1}^\infty \mu(E_i).$$

(X, \mathcal{M}, μ) is called a **measure space**.

Note that $\mu(\bigsqcup E_i)$ makes sense because \mathcal{M} is a σ -algebra and hence $\bigsqcup E_i \in \mathcal{M}$.

Proposition 1.13. Suppose (X, \mathcal{M}, μ) is a measure space. All sets mentioned below are in \mathcal{M} . Then,

1. $E \subset F \implies \mu(E) \leq \mu(F)$,
2. $\mu(\bigcup_1^\infty E_i) \leq \sum_1^\infty \mu(E_i)$,
3. If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \dots$), then

$$\mu \left(\bigcup_{i=1}^\infty E_i \right) = \lim_{n \rightarrow \infty} \mu(E_n).$$

Definition 1.14 (Null set). A **null set** in a measure space (X, \mathcal{M}, μ) is a set N such that $N \subset F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.

Note that N need not necessarily be in \mathcal{M} . Of course, F in the above is also a null set.

Definition 1.15 (Completion). Given a measure space (X, \mathcal{M}, μ) , the **completion** of \mathcal{M} , denote $\overline{\mathcal{M}}$ is the collection of all subsets of the form $E \cup N$ where $E \in \mathcal{M}$ and N is a null set.

Clearly, $\mathcal{M} \subset \overline{\mathcal{M}}$ since \emptyset is a null set.

Proposition 1.16 (Extension to completion). Let (X, \mathcal{M}, μ) be a measure space.

1. $\overline{\mathcal{M}}$ is a σ -algebra.
2. There is a unique measure

$$\overline{\mu} : \overline{\mathcal{M}} \rightarrow [0, \infty]$$

such that $\overline{\mu}|_{\mathcal{M}} = \mu$.

Lecture 2.

Definition 2.1 (Outer measure). An **outer measure** on X is a map

$$\mu^* : \mathcal{P}(X) \rightarrow [0, 1]$$

satisfying

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Note that we don't demand equality even if disjoint.

Proposition 2.2 (A construction of an outer measure). Suppose $\mathcal{F} \subset \mathcal{P}(X)$ and $\rho : \mathcal{F} \rightarrow [0, \infty]$ is a map such that

1. $\emptyset, X \in \mathcal{F}$,
2. $\rho(\emptyset) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \mid E_i \in \mathcal{F}, E \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

Then, μ^* is an outer measure.

Note that the above had just the bare minimum requirement for both ρ and \mathcal{F} and still gave us that μ^* is an outer measure.

Definition 2.3 (μ^* -measurable). Given an outer measure μ^* on a set X , a set $A \subset X$ is said to be **μ^* -measurable** if for all $E \subset X$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition 2.4 (Complete measure). A measure μ on (X, \mathcal{M}) is said to be **complete** if \mathcal{M} contains all null sets of (X, \mathcal{M}, μ) .

Theorem 2.5 (Carathéodory). Let μ^* be an outer measure on X . Let

$$\mathcal{M} := \{E \subset X \mid E \text{ is } \mu^*\text{-measurable}\}.$$

Then,

1. \mathcal{M} is a σ -algebra.
2. $\mu^*|_{\mathcal{M}}$ is a complete measure.

Definition 2.6 (Pre-measure). Suppose \mathcal{F} is an algebra on X . A map

$$\mu_0 : \mathcal{F} \rightarrow [0, \infty]$$

is called a **pre-measure** if

1. $\mu_0(\emptyset) = 0$,
2. if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$ are pairwise disjoint such that $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{F}$, then

$$\mu_0\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

Note that by putting all but finitely many $A_i = \emptyset$, the above equality holds for finite unions as well. (The finite union *will* be in \mathcal{F} since it's an algebra.)

Proposition 2.7. Suppose μ_0 is a pre-measure on an algebra \mathcal{F} . Then, if μ_* is the outer measure as defined in Proposition 2.2 (with $\rho = \mu_0$), then

1. $\mu^*|_{\mathcal{F}} = \mu_0$,
2. every set in \mathcal{F} is μ^* -measurable.

Theorem 2.8. Suppose $\mathcal{F} \subset \mathcal{P}(X)$ is an algebra and let \mathcal{M} be the σ -algebra generated by \mathcal{F} .

Let μ_0 be a pre-measure defined on \mathcal{F} and let μ^* be the outer measure as before. Then

1. $\mu^*|_{\mathcal{M}}$ is a measure on (X, \mathcal{M}) . Put $\mu = \mu^*|_{\mathcal{M}}$ for the next part.
2. If ν is any measure extending μ_0 , then

$$\nu(E) = \mu(E)$$

whenever $\mu(E) < \infty$.

Definition 2.9. A **half-interval** is a subset of \mathbb{R} of one of the following forms:

1. $(a, b]$ for $-\infty \leq a < b < \infty$,
2. (a, ∞) for $-\infty \leq a < \infty$,
3. \emptyset .

Proposition 2.10. The collection of all finite unions of half-intervals is an algebra on \mathbb{R} .

Proposition 2.11. Let \mathcal{F} be the algebra consisting of finite unions of half-intervals. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing and right continuous function. Define

$$\mu_0 \left(\bigsqcup_{i=1}^n (a_j, b_j] \right) := \sum_{i=1}^n [F(b_j) - F(a_j)],$$

and let $\mu_0(\emptyset) = 0$.

Then, μ_0 is a well-defined pre-measure on \mathcal{F} .