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Differential Topology

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§0. Preface

I am making this while I study *Differential Topology* by Victor Guillemin and Alan Pollack. These notes will likely not be helpful to anyone who is looking to learn this material from scratch. I am just going to be noting down the theorems and definitions from the book, assuming the definitions and notations I want to assume. I also skip proofs.

Notations

1. "$U \subseteq_{op} X$" stands for "U is a nonempty open subset of X".
2. Given a function $f : X \to \mathbb{R}^m$, we can write $f = (f_1, \dots, f_m)$ for functions $f_i : X \to \mathbb{R}$ ($i = 1, \dots, m$). These f_i will be referred to as component functions (of f).
3. $S^k \subseteq \mathbb{R}^{k+1}$ is the unit sphere.
4. $B_r(p)$ will denote the open ball of radius r around the point p. The ambient metric space will be clear from context.
5. Given a function $T : V \to W$, $\mathrm{im}(T)$ denotes the image of T.

§1. Manifolds and Smooth Maps

§§1.1. Definitions

Definition 1.1.1. A function f from $U \subseteq_{op} \mathbb{R}^n$ into $\mathbb{R}^m$ is called smooth if each component function f_i has partial derivatives of all orders.

More generally, if $X \subseteq \mathbb{R}^n$, then a map $f : X \to \mathbb{R}^m$ is called smooth if for each point $x \in X$, there exists an open set $U \subseteq_{op} \mathbb{R}^n$ containing x and a smooth function $F : U \to \mathbb{R}^m$ such that $F = f$ on $U \cap X$.

Definition 1.1.2. A map $f : X \to Y$ between subsets of Euclidean spaces is called a diffeomorphism if f is smooth and bijective with f^{-1} also smooth.

X and Y are said to be diffeomorphic if such a map exists.

Exercise 1.1.3. Show that if $f : X \to Y$ is smooth, then f is continuous.
In particular, diffeomorphic spaces are homeomorphic.

Definition 1.1.4. Let $X \subseteq \mathbb{R}^N$. X is said to be a k-dimensional manifold if each $x \in X$ possesses a neighbourhood $V \subseteq_{op} X$ which is diffeomorphic to an open subset $U \subseteq_{op} \mathbb{R}^k$. We define the dimension of X as $\dim(X) = k$.

A diffeomorphism $\phi : U \to V$ is called a parametrisation of the neighbourhood V. The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a coordinate system on V.
Writing $\phi^{-1} = (x_1, \ldots, x_k)$, the component functions $x_1, \ldots, x_k$ are called coordinate functions.

Note that $\dim X = k$ is well-defined. Indeed, if $U \subseteq_{op} \mathbb{R}^n$ and $U' \subseteq_{op} \mathbb{R}^m$ are homeomorphic, then $n = m$.

Example 1.1.5. Any nonempty open subset of $\mathbb{R}^N$ is an N-dimensional manifold.

Example 1.1.6. The circle $S^1 \subseteq \mathbb{R}^2$ is a 1-dimensional manifold.

The open disc $B_1(0) = \{x \in \mathbb{R}^2 : \|x\| < 1\}$ is a 2-dimensional manifold but the closed disc $\overline{B_1(0)}$ is not.

Similarly, $(0, 1)$ is a 1-manifold but $[0, 1]$ is not. This is due to the boundary points.

Later, we shall see the concept of *manifold with boundary*.

Example 1.1.7. If $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ are manifolds, then so is $X \times Y$ with

$$\boxed{\dim(X \times Y) = \dim(X) + \dim(Y).}$$

Indeed, let $k := \dim(X)$, $l := \dim(Y)$, and let $(x, y) \in X \times Y$ be arbitrary. Let $U \subseteq_{op} \mathbb{R}^k$ (resp. $W \subseteq_{op} \mathbb{R}^l$) be open and $\phi : U \to X$ (resp. $\psi : W \to Y$) be a parametrisation around x (resp. y).

Define $\phi \times \psi : U \times W \to X \times Y$ by

$$(\phi \times \psi)(u, w) := (\phi(u), \psi(w)).$$

Note that $(U \times W) \subseteq_{op} \mathbb{R}^{k+l}$ and $f := \phi \times \psi$ is smooth (the component functions of f are the component functions of ϕ followed by those of ψ). We only need to verify that this is indeed a local parametrisation.

Note that ϕ and ψ are diffeomorphisms onto their images (and the images are open in X and Y respectively). Thus, $V := \phi(U) \times \psi(W)$ is an open neighbourhood of (x, y) in $X \times Y$. Moreover, $g : V \to U \times W$ by $(x', y') \mapsto (\phi^{-1}(x'), \psi^{-1}(y'))$ is the inverse of f.
The only check that needs to be done is that g is smooth. We leave this to the reader. (Use the smoothness of ϕ^{-1} and ψ^{-1} defined in the more general sense.)

Definition 1.1.8. If X and Y are both manifolds in $\mathbb{R}^N$ and $Z \subseteq X$, then Z is a submanifold of X.

The codimension of Z (in X) is defined by $\operatorname{codim}_X(Z) = \dim(X) - \dim(Z)$.

If the ambient manifold X is clear, we will simply write $\operatorname{codim}(Z)$.

Example 1.1.9. S^1 is a submanifold of $B_2(0) \subseteq \mathbb{R}^2$ of codimension 1.

Remark 1.1.10. We have defined manifolds only as subsets of Euclidean spaces.

Remark 1.1.11. Note that any open ball in $\mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k$ (check). Thus, the domains of local parametrisations may be assumed to be $\mathbb{R}^k$.

§§1.2. Derivatives and Tangents

Definition 1.2.1. Let $U \subseteq_{op} \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ be smooth, and $x \in U$. The derivative of f at x is the function

$$df_x : \mathbb{R}^n \to \mathbb{R}^m$$

defined by

$$df_x(v) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

Note that df_x is defined on all of $\mathbb{R}^n$ even if $U \neq \mathbb{R}^n$.

Remark 1.2.2. df_x is a linear map. In particular, we may represent df_x as a matrix using the standard bases. If $f = (f_1, \ldots, f_m)$, then we have

$$df_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

Example 1.2.3. If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $dL_x = L$ for all $x \in \mathbb{R}^n$.
In particular, if $i : U \hookrightarrow \mathbb{R}^n$ is the inclusion map, then $di_x = \mathrm{id}_{\mathbb{R}^n}$ for all $x \in U$.

Theorem 1.2.4 (Chain rule)**.** Let $U \subseteq_{op} \mathbb{R}^n$, $V \subseteq_{op} \mathbb{R}^m$. Suppose $f : U \to V$ and $g : V \to \mathbb{R}^l$ are smooth. For all $x \in U$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Definition 1.2.5. Let $X \subseteq \mathbb{R}^N$, $x \in X$, $U \subseteq_{op} \mathbb{R}^k$, and $\phi : U \to X$ be a local parametrisation around x. For convenience, assume that $0 \in U$ and $\phi(0) = x$.

The tangent space of X at x to be the image of the map $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^N$. This is denoted by $T_x(X)$.
A tangent vector to X at x is a point $v \in T_x(X) \subseteq \mathbb{R}^N$.

Note that in the above, we are making use of the fact that X is a subset of $\mathbb{R}^N$. Also note that $T_x(X)$ is a very concrete subspace of $\mathbb{R}^N$ – it is not "just defined up to isomorphism". No matter what ϕ we choose, we always get the same explicit subspace of $\mathbb{R}^N$. The translate $x + T_x(X)$ will pass through x and will be "tangent" (in a geometric manner) to X. Similarly, a tangent vector $v \in T_x(X)$ is to be imagined as a segment from x to $x + v$.

The issue to clarify above is whether the above (concrete!) subspace $T_x(X)$ depends on ϕ or not. Suppose that $\psi : V \to X$ is another local parametrisation with $\psi(0) = x$. Note that $\phi(U)$ and $\psi(V)$ are both (relatively) open neighbourhoods of x. By passing to a subset, we may assume $\phi(U) = \psi(V)$. Thus, $h = \psi^{-1} \circ \phi : U \to V$ is a diffeomorphism. Using the chain rule on the relation $\phi = \psi \circ h$ gives

$$d\phi_0 = d\psi_0 \circ dh_0.$$

Thus, $\mathrm{im}(d\phi_0) \subseteq \mathrm{im}(d\psi_0)$. By symmetry, the converse is true too, as desired.

Theorem 1.2.6. With above notations,

$$\boxed{\dim(T_x(X)) = \dim(X),}$$

where the dimension on the left is the dimension as a vector space over $\mathbb{R}$.

In particular, $d\phi_0 : \mathbb{R}^k \to T_x(X)$ is an isomorphism.

We now define derivative for an arbitrary smooth map $f : X \to Y$.

Definition 1.2.7. Let $f : X \to Y$ be a smooth map of arbitrary manifolds. Let $x \in X$ and $y := f(x)$. The derivative of f at x is a linear map

$$df_x : T_x(X) \to T_y(Y)$$

defined as follows: Fix parametrisations $\phi : U \to X$ and $\psi : V \to Y$ around x and y. ($U \subseteq_{\text{op}} \mathbb{R}^k$ and $V \subseteq_{\text{op}} \mathbb{R}^l$.) Assume $\phi(0) = x$ and $\psi(0) = y$.

After passing to a subset of U (so that $f(\phi(U)) \subseteq \psi(V)$), we have the following commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "h=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \end{tikzcd}.$$

We define df_x to be the unique map making the following square commute:

$$\begin{tikzcd} T_x(X) \arrow[r, "df_x"] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^l \arrow[u, "d\psi_0"'] \end{tikzcd}.$$

Note that $d\phi_0$ is an isomorphism and thus, df_x is uniquely determined as

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}.$$

Exercise 1.2.8. Check that the above does not depend on choice of ϕ or ψ.

One way of the doing the above exercise is to consider this alternate definition of df_x. Let $X \subseteq \mathbb{R}^N$, $Y \subseteq \mathbb{R}^M$, f, x, y, $U \subseteq_{op} \mathbb{R}^k$, $V \subseteq_{op} \mathbb{R}^l$, $\phi : U \to X$, $\psi : V \to Y$ have the usual meanings.
By definition of f being smooth, there exists an open subset $U' \subseteq_{op} \mathbb{R}^N$ and $F : U' \to \mathbb{R}^M$ smooth such that $F = f$ on $U' \cap X$. Now, by shrinking U and U' if necessary, we get a diagram as follows:

$$\begin{tikzcd}
U' \arrow[r, "F"] & \mathbb{R}^M \\
\phi(U) \arrow[u, hook] \arrow[r, "f"] & \psi(V) \arrow[u, hook] \\
U \arrow[u, "\phi"] \arrow[r, "h=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"']
\end{tikzcd}.$$

The upper and lower squares commute and thus, the big outer rectangle commutes. In turn, the usual chain on open subsets of Euclidean spaces tells us that the following big rectangle commutes:

$$\begin{tikzcd}
\mathbb{R}^N \arrow[r, "dF_x"] & \mathbb{R}^M \\
T_x(X) \arrow[u, hook] & T_y(Y) \arrow[u, hook] \\
\mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^l \arrow[u, "d\psi_0"']
\end{tikzcd}.$$

A simple diagram chase shows that the image of dF_X restricted to $T_x(X)$ lands within $T_y(Y)$. Thus, we get an induced map

$$\begin{tikzcd}
\mathbb{R}^N \arrow[r, "dF_x"] & \mathbb{R}^M \\
T_x(X) \arrow[u, hook] \arrow[r, dashed, "dF_X|_{T_x(X)}"] & T_y(Y) \arrow[u, hook] \\
\mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^l \arrow[u, "d\psi_0"']
\end{tikzcd}$$

which makes the lower square commute. But we had already checked that there is a unique such map. Thus, we have

$$dF_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}.$$

Note that the left side is independent of parametrisations and the right side is independent of the extension F. In turn, both sides are independent of both and hence, we have our well-defined candidate for df_x.

The above definition also makes it easy to solve the following exercise.

Exercise 1.2.9. For a submanifold X of Y, let $i : X \to Y$ be the inclusion map. Check that di_x is the inclusion map of $T_x(X)$ into $T_x(Y)$.

Solution. Let $X \subseteq Y \subseteq \mathbb{R}^N$. Consider the extension $I : \mathbb{R}^N \to \mathbb{R}^N$ which is the identity map. Then, $di_x = dI_x|_{T_x(X)}$. As noted earlier, $dI_x = I$ and we are done. □

I got the above from https://math.stackexchange.com/a/861132/427810.

Note that in particular, the above exercise implies that if $Z \subseteq X$ is a submanifold and $z \in Z$, then $T_z(Z) \subseteq T_z(X)$. (Once again, we emphasise that it makes sense to talk about these inclusions since the tangent spaces are concrete subspaces.)

Theorem 1.2.10 (Chain rule)**.** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x,$$

for all $x \in X$.

§§1.3. The Inverse Function Theorem and Immersions

Definition 1.3.1. Let $f : X \to Y$ be a smooth map of manifolds, and $x \in X$. f is called a local diffeomorphism at x if f maps a neighbourhood of x diffeomorphically onto a neighbourhood of $y := f(x)$.

f is called a local diffeomorphism if it is a local diffeomorphism at x for every $x \in X$.

Theorem 1.3.2 (Inverse Function Theorem)**.** Suppose that $f : X \to Y$ is a smooth map of manifolds, and let $x \in X$.
f is a local diffeomorphism at x iff df_x is an isomorphism.

Remark 1.3.3. If df_x is an isomorphism, one can choose local coordinates around x and y so that f appears to be the identity $f(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$ on some neighbourhood of x.

More precisely: there exists $U \subseteq_{op} \mathbb{R}^k$ and local parametrisations $\phi : U \to X$, $\psi : U \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\text{identity}"'] & U \arrow[u, "\psi"'] \end{tikzcd}.$$

Note that the same U is being used to parametrise.

Definition 1.3.4. Two maps $f : X \to Y$ and $f' : X' \to Y'$ are said to be equivalent (or same up to diffeomorphism) if there exist diffeomorphisms α and β making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "\alpha"] \arrow[r, "f'"'] & Y' \arrow[u, "\beta"'] \end{tikzcd}.$$

Definition 1.3.5. $f : X \to Y$ smooth map of manifolds, $x \in X$, $y := f(x)$.

f is said to be an immersion at x if $df_x : T_x(X) \to T_y(Y)$ is injective.
f is said to be an immersion if f is an immersion at x for all $x \in X$.

The canonical immersion is the standard inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^l$ for $k \leqslant l$ given by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$.

Check that the canonical immersion is an immersion.

Theorem 1.3.6 (Local immersion theorem)**.** Suppose that $f : X \to Y$ is an immersion at $x \in X$, and $y = f(x)$. Then, there exist local coordinates around x and y such that

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

In other words, f is locally equivalent to the canonical immersion around x.

Corollary 1.3.7. If f is an immersion at x, then it is an immersion in a neighbourhood of x.

Remark 1.3.8. If $\dim(X) = \dim(Y)$, then local immersions and local diffeomorphisms are the same.

Remark 1.3.9. If $f : X \to Y$ is a smooth map, it is not necessary that $f(X)$ is a manifold. This is not true even if f is assumed to an immersion and injective.

One can construct a smooth map $f : \mathbb{R} \to \mathbb{R}^2$ which is an injective immersion but the image of f is the figure eight.

Definition 1.3.10. $f : X \to Y$ is called proper if the preimage of every compact set in Y is a compact subset of X. An immersion which is injective and proper is called an embedding.

Theorem 1.3.11. An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.

Remark 1.3.12. If X is compact, then every $f : X \to Y$ is proper. In this case, embeddings are same as injective immersions.

To see why f is proper, note that if $K \subseteq Y$ is compact, then K is closed and hence, $f^{-1}(K)$ is closed in X by continuity. Any closed subset of a compact space is compact and we are done.

§§1.4. Submersions

We now study the case where $\dim(X) \geqslant \dim(Y)$. If $f : X \to Y$ carries x to y, then we can demand surjectivity of $df_x : T_x(X) \to T_y(Y)$. Identical to the definitions in Definition 1.3.5, we have the following.

Definition 1.4.1. $f : X \to Y$ smooth map of manifolds, $x \in X$, $y := f(x)$.

f is said to be a submersion at x if $df_x : T_x(X) \to T_y(Y)$ is surjective.
f is said to be a submersion if f is an submersion at x for all $x \in X$.

The canonical submersion is the standard inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^l$ for $k \geqslant l$ given by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_l)$.

Theorem 1.4.2 (Local submersion theorem)**.** Suppose that $f : X \to Y$ is a submersion at $x \in X$, and $y = f(x)$. Then, there exist local coordinates around x and y such that

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_l).$$

In other words, f is locally equivalent to the canonical submersion around x.

Corollary 1.4.3. If f is a submersion at x, then it is a submersion in a neighbourhood of x.

Definition 1.4.4. For a smooth map $f : X \to Y$, a point $y \in Y$ is called a regular value for f if $df_x : T_x(X) \to T_y(Y)$ is surjective for every $x \in f^{-1}(y)$. Else, y is called a critical value.

Points not in the image of f are also regular values. In fact, if $\dim(X) < \dim(Y)$, then the regular values are precisely the points in $Y \setminus f(X)$.

Theorem 1.4.5 (Preimage Theorem)**.** If y is a regular value for $f : X \to Y$, then $f^{-1}(y)$ is a submanifold of X, with

$$\boxed{\dim(f^{-1}(y)) = \dim(X) - \dim(Y).}$$

Exercise 1.4.6. Use the above to show that S^{k-1} is a manifold of dimension $k-1$. (Use the map $f : \mathbb{R}^k \to \mathbb{R}$ defined by $x \mapsto \|x\|^2$ and check that 1 is a regular value.)

Exercise 1.4.7. Note that $M(n)$ – the space of all $n \times n$ matrices – can be identified with $\mathbb{R}^{n^2}$ in a natural way and is thus a manifold of dimension n^2. Check that that subset $S(n)$ of symmetric matrices is a submanifold diffeomorphic to $\mathbb{R}^k$ where $k = n(n+1)/2$.

Check that we have a map $f : M(n) \to S(n)$ given by $A \mapsto AA^\top$ which is smooth. Show that the identity matrix $I \in S(n)$ is a regular value. Conclude that $O(n)$ – the subspace of orthogonal matrices – is a submanifold of $M(n)$ with dimension $n(n-1)/2$.

Definition 1.4.8. A group that is a manifold such that the basic operations are smooth is called a Lie group.

By "basic operations", we mean the maps $(a, b) \mapsto ab$ and $a \mapsto a^{-1}$.

Example 1.4.9. $O(n)$ is a Lie group.

Definition 1.4.10. Let $g_1, \ldots, g_l : X \to \mathbb{R}$ be smooth functions, and $x \in X$.
$g_1, \ldots, g_l$ are said to be independent at x if $d(g_1)_x, \ldots, d(g_l)_x$ are linearly independent functionals on $T_x(X)$, i.e., linearly independent as elements of the dual $T_x(X)^*$.

Proposition 1.4.11. Let $g_1, \ldots, g_l$ be as in the above definition. Define the function

$$g : X \to \mathbb{R}^l$$

by $g := (g_1, \ldots, g_l)$.

Then, dg_x is a surjection iff $g_1, \ldots, g_l$ are independent at x.

Note that in the above notation, $g^{-1}(0)$ is the set of common zeroes of $g_1, \ldots, g_l$. This gives us the following.

Theorem 1.4.12. If the smooth, real-valued functions $g_1, \ldots, g_l$ on X are independent at each common zero, then the set Z of common zeroes is a submanifold of X with $\operatorname{codim}(Z) = l$.

There are two partial converses to the above.

Proposition 1.4.13. If y is a regular value of a smooth map $f : X \to Y$, then the preimage submanifold $f^{-1}(y)$ can be cut out by independent functions.

Proposition 1.4.14. Every submanifold of X is *locally* cut out by independent functions.

More precisely: let Z be a submanifold of codimension l, and $z \in Z$. Then, there exist l independent functions $g_1, \ldots, g_l$ defined on some neighbourhood $W \subseteq_{\text{op}} X$ of z such that $Z \cap W$ is the common vanishing set of the g_i.

Proposition 1.4.15. Let Z be the preimage of a regular value $y \in Y$ under the smooth map $f : X \to Y$. Then the kernel of the derivative $df_x : T_x(X) \to T_y(Y)$ at any point $x \in Z$ is precisely the tangent space $T_x(Z)$.

§§1.5. Transversality

So far we discussed the problem of when $f^{-1}(y)$ is a manifold, given a smooth map $f : X \to Y$. Now, we wish to study the more general case of when $f^{-1}(Z)$ is a manifold, where $Z \subseteq Y$ is a submanifold. Note that being a manifold is a local problem. More precisely: $f^{-1}(Z)$ is a manifold iff every $x \in f^{-1}(Z)$ has a neighbourhood $U \subseteq_{op} X$ such that $f^{-1}(Z) \cap U$ is a manifold. This observation along with some calculations leads to the following definition and theorem.

Definition 1.5.1. Let $f : X \to Y$ be a smooth map, and $Z \subseteq Y$ a manifold. The map f is said to be transversal to the submanifold Z, abbreviated $f \pitchfork Z$, if the following equation holds true at each point $x \in f^{-1}(Z)$:

$$\boxed{\mathrm{im}(df_x) + T_y(Z) = T_y(Y),}$$

where $y := f(x)$.

Theorem 1.5.2. If the smooth map $f : X \to Y$ is transversal to a submanifold $Z \subseteq Y$, then $f^{-1}(Z)$ is a submanifold of X. Moreover,

$$\mathrm{codim}(f^{-1}(Z)) = \mathrm{codim}(Z).$$

Note that when Z is a single point, then $T_y(Z)$ is the zero subspace and the transversality condition reduces to $\mathrm{im}(df_x) = T_y(Y)$, i.e., df_x is surjective. Thus, the case of y being a regular value was a special case.

Definition 1.5.3. Let Y be a manifold, and let X and Z be submanifolds of Y. We say that X and Z are transversal, abbreviated $X \pitchfork Z$ if

$$\boxed{T_x(X) + T_x(Z) = T_x(X \cap Z),}$$

for every $x \in X \cap Z$.

The above is simply saying that $i \pitchfork Z$, where i is the inclusion $i : X \hookrightarrow Y$. By symmetry, it is also the same as $j \pitchfork X$ for $j : Z \hookrightarrow Y$. The earlier theorem then specialises to the following.

Theorem 1.5.4. The intersection of two transversal submanifold manifolds of Y is again a submanifold. Moreover,

$$\mathrm{codim}(X \cap Z) = \mathrm{codim}(X) + \mathrm{codim}(Z).$$

In terms of dimensions, the above says

$$\dim(X \cap Z) = \dim(X) + \dim(Z) - \dim(Y).$$

In particular, note that if $\dim(X) + \dim(Z) < \dim(Y)$, then X and Z can only intersect transversally by not intersecting at all.

Example 1.5.5. The two coordinate axes intersect transversally in $\mathbb{R}^2$ but not when considered submanifolds of $\mathbb{R}^3$.

Exercise 1.5.6. Suppose $A : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and V is a vector space of $\mathbb{R}^n$. Check that $A \pitchfork V \Leftrightarrow A(\mathbb{R}^k) + V = \mathbb{R}^n$.

In particular, if W is another subspace of $\mathbb{R}^n$, then $V \pitchfork W \Leftrightarrow V + W = \mathbb{R}^n$.

§§1.6. Homotopy and Stability

Let I denote the unit interval $[0, 1] \subseteq \mathbb{R}$.

Definition 1.6.1. Let $f_0, f_1 : X \to Y$ be smooth maps. We say that f_0 and f_1 are homotopic, abbreviated $f_0 \sim f_1$, if there exists a smooth map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$.
F is called a homotopy between f_0 and f_1.

It is easily checked that $\sim$ is an equivalence relation on smooth maps from X to Y, and the equivalence class to which a mapping belongs is its homotopy class. We also get a family of smooth maps $(f_t)_{t\in[0,1]}$ from X to Y given by

$$f_t(x) := F(x, t).$$

We sometimes also say "$f_t : X \to Y$" is a homotopy to denote a family $(f_t)_{t\in I}$ of smooth maps from X to Y such that the map $F : X \times I \to Y$ defined by $(x, t) \mapsto f_t(x)$ is a homotopy between f_0 and f_1.

Definition 1.6.2. A property of smooth maps is said to be stable provided that whenever $f_0 : X \to Y$ possesses the property and $f_t : X \to Y$ is a homotopy of f_0, then, for some $\varepsilon > 0$, each f_t with $t < \varepsilon$ also possesses the property.

The collection of maps that posses a particular property may be referred to as a stable class of maps.

Example 1.6.3. Consider curves in the planes, i.e., smooth maps from $\mathbb{R}^1$ to $\mathbb{R}^2$. The property that a curve pass through the origin is not stable. Nor is the property of intersecting the x-axis stable. (Think about the mapping $x \mapsto (x, x^2)$ and the homotopy $(x,t) \mapsto (x, x^2 + t)$ in both cases.)

However, transversal intersection with the x-axis *is* a stable property.

Exercise 1.6.4. Verify the last line above.

Theorem 1.6.5 (Stability theorem)**.** Let X be a compact manifold, and Y an arbitrary manifold. The following classes of smooth maps are stable classes:

1. local diffeomorphisms,
2. immersions,
3. submersions,
4. maps transversal to a fixed submanifold $Z \subseteq Y$,
5. embeddings,
6. diffeomorphisms.

§2. Sard's Theorem and Morse Functions

Recall when a subset of a Euclidean space is said to have measure zero.

Definition 2.0.1. Let Y be a manifold and $C \subseteq Y$. C is said to have measure zero if for every local parametrisation ψ of Y, the preimage $\psi^{-1}(C)$ has measure zero in Euclidean space.

It can be checked that the following is reducible to the following: there exists a collection of local parametrisations $\{\psi_\alpha\}_\alpha$ such that C is covered by the images of ψ_α and $\psi_\alpha^{-1}(C)$ has measure zero for all α.

Theorem 2.0.2 (Sard's Theorem)**.** Let $f : X \to Y$ be any smooth map of manifolds. Almost every point in Y is a regular value, i.e., the set of critical values has measure zero.

Corollary 2.0.3. The regular values of any smooth map $f : X \to Y$ are dense in Y. In fact, if $f_i : X \to Y$ are any countable number of smooth maps, then the points of f_i that are simultaneously regular values for all of the f_i are dense.

Definition 2.0.4. Let $f : X \to Y$ be a smooth map, and $x \in X$.
If df_x is surjective, then we say that f is regular at x or that x is a regular point of f. Else, we say that x is a critical point of x.

Remark 2.0.5. Let $f : X \to Y$ be smooth, and $y \in Y$.
y is a regular value $\Leftrightarrow$ *every* point in $f^{-1}(y)$ is a regular point.
y is a critical value $\Leftrightarrow$ *some* point in $f^{-1}(y)$ is a critical point.

Definition 2.0.6. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a smooth function. The Hessian matrix of f is the $k \times k$ matrix

$$H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}.$$

If H is nonsingular at a critical point x, then x is said to be a nondegenerate critical point of f.

Proposition 2.0.7. Nondegenerate critical points are isolated from other critical points of f.

Sketch. Define $g : \mathbb{R}^k \to \mathbb{R}^k$ by $g = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}\right)$.
Then dg_x is nonsingular and thus, $g \neq 0$ on a punctured neighbourhood of x. □

Theorem 2.0.8 (Morse Lemma)**.** Suppose $a \in \mathbb{R}^k$ is a nondegenerate critical point of $f : \mathbb{R}^k \to \mathbb{R}$, and

$$(h_{ij}) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right)$$

is the Hessian of f at a. Then there exists a local coordinate system $(x_1, \ldots, x_k)$ around a such that

$$f = f(a) + \sum h_{ij} x_i x_j$$

near a.

We now extend the concept of nondegeneracy of critical points to arbitrary manifolds.

Definition 2.0.9. Suppose that $f : X \to \mathbb{R}$ has a critical point at x and that ϕ is a local parametrisation carrying the origin to x. (Then, 0 is a critical point for $f \circ \phi$: use chain rule.)
We declare x to be nondegenerate for f if 0 is nondegenerate for $f \circ \phi$.

One can check that the above does not depend on ϕ.

Definition 2.0.10. A function $X \to \mathbb{R}$ whose critical points are all nondegenerate is called a Morse function.

Notation: If f is a function on $X \subseteq \mathbb{R}^N$ and $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, we define a new function $f_a : X \to \mathbb{R}$ by

$$f_a(x_1, \ldots, x_N) = f + a_1 x_1 + \cdots + a_N x_N.$$

Theorem 2.0.11. Let $f : X \to \mathbb{R}$ be an arbitrary smooth function. For almost every $a \in \mathbb{R}^N$, the function f_a is a Morse function on X.

§§2.1. Embedding Manifolds in Euclidean Space

We had defined a k-dimensional manifold as some subset of $\mathbb{R}^N$. We now "show" that we may assume $N = 2k+1$. That is, if X is a k-dimensional manifold and $n \geqslant 2k+1$, then there exists an embedding $X \to \mathbb{R}^n$. (Note that $\mathbb{R}^n$ naturally embeds inside $\mathbb{R}^m$ for $m \geqslant n$.)

Definition 2.1.1. Let $X \subseteq \mathbb{R}^N$ be a manifold. The tangent bundle of X, abbreviated $T(X)$, is defined as

$$T(X) := \{(x, v) \in X \times \mathbb{R}^N : v \in T_x(X)\} \subseteq X \times \mathbb{R}^{2N}.$$

$T(X)$ contains a copy of X: $X_0 = \{(x, 0) : x \in X\}$. Note that $T(X)$ is again a concrete subset of $\mathbb{R}^{2N}$.

Definition 2.1.2. Any smooth map $f : X \to Y$ induces a global derivative map $df : T(X) \to T(Y)$ defined by

$$df(x, v) = (f(x), df_x(v)).$$

Note that if $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$, then df is a map from a subset of $\mathbb{R}^{2N}$ into a subset of $\mathbb{R}^{2M}$, and it makes sense to talk about smoothness of df.

Proposition 2.1.3. df is a smooth map. Moreover, $d(g \circ f) = dg \circ df$. Consequently, if f is a diffeomorphism, then so is df.

In particular, the tangent bundle $T(X)$ is determined up to diffeomorphism, and does not depend on the ambient $\mathbb{R}^N$.

Proposition 2.1.4. The tangent bundle of a manifold is another manifold, with twice the dimension.

In symbols: If X is a manifold, then so is $T(X)$ and $\dim(T(X)) = 2\dim(X)$.

Theorem 2.1.5. Every k-dimensional manifold admits a one-to-one immersion in $\mathbb{R}^{2k+1}$.

Note that one-to-one immersions are weaker than embeddings (Remark 1.3.9) but the two coincide for compact spaces (Remark 1.3.12). To go from compact to arbitrary manifolds, we use partitions of unity.

Theorem 2.1.6. Let X be an arbitrary subset of $\mathbb{R}^N$. For any covering of X by (relatively) open subsets $(U_\alpha)_\alpha$, there exists a sequence $(\theta_i)_{i \geqslant 1}$ of smooth functions on X, called a partition of unity subordinate to $(U_\alpha)_\alpha$, with the following properties:

1. $0 \leqslant \theta_i(x) \leqslant 1$ for each $x \in X$ and all $i \geqslant 1$.
2. Each $x \in X$ has a neighbourhood on which all but finitely many θ_i are zero.

3. Each function θ_i is zero outside some closed set contained in some U_α.
4. For each $x \in X$, $\sum \theta_i(x) = 1$.

Note that the last sum is finite for all x, in view of the second point.

Corollary 2.1.7. On any manifold X, there exists a proper (smooth) map $\rho : X \to \mathbb{R}$.

Theorem 2.1.8 (Whitney Theorem)**.** Every k-dimensional manifold embeds in $\mathbb{R}^{2k+1}$.