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Differential Topology

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Spring 2022

Last Updated: May 18, 2022

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§0. Preface

I am making this while I study *Differential Topology* by Victor Guillemin and Alan Pollack. These notes will likely not be helpful to anyone who is looking to learn this material from scratch. I am just going to be noting down the theorems and definitions from the book, not bothering with basic notations/definitions from topology. I also skip proofs.

§1. Manifolds and Smooth Maps

§§1.1. Definitions

Notation. "$U \subseteq_{op} X$" stands for "U is a nonempty open subset of X".
Given a function $f : X \to \mathbb{R}^m$, we can write $f = (f_1, \ldots, f_m)$ for functions $f_i : X \to \mathbb{R}$ $(i = 1, \ldots, m)$. These f_i will be referred to as component functions (of f).

Definition 1.1.1. A function f from $U \subseteq_{op} \mathbb{R}^n$ into $\mathbb{R}^m$ is called smooth if each component function f_i has partial derivatives of all orders.

More generally, if $X \subseteq \mathbb{R}^n$, then a map $f : X \to \mathbb{R}^m$ is called smooth if for each point $x \in X$, there exists an open set $U_x \subseteq_{op} \mathbb{R}^n$ containing x and a smooth function $F : U \to \mathbb{R}^m$ such that $F = f$ on $U_x \cap X$.

Definition 1.1.2. A map $f : X \to Y$ between subsets of Euclidean spaces is called a diffeomorphism if f is smooth and bijective with f^{-1} also smooth.

X and Y are said to be homeomorphic if such a map exists.

Example 1.1.3. Show that if $f : X \to Y$ is smooth, then f is continuous.
In particular, diffeomorphic spaces are homeomorphic.

Definition 1.1.4. Let $X \subseteq \mathbb{R}^N$. X is said to be a k-dimensional manifold if each $x \in X$ possesses a neighbourhood $V \subseteq_{op} X$ which is diffeomorphic to an open subset $U \subseteq_{op} \mathbb{R}^k$. We define the dimension of X as $\dim(X) = k$.

A diffeomorphism $\phi : U \to V$ is called a parametrisation of the neighbourhood V. The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a coordinate system on V.
Writing $\phi^{-1} = (x_1, \ldots, x_k)$, the component functions $x_1, \ldots, x_k$ are called coordinate functions.

Note that $\dim X = k$ is well-defined. Indeed, if $U \subseteq_{op} \mathbb{R}^n$ and $U' \subseteq_{op} \mathbb{R}^m$ are nonempty and homeomorphic, then $n = m$.

Example 1.1.5. The circle $S^1 \subseteq \mathbb{R}^2$ is 1-dimensional manifold.

Example 1.1.6. If $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ are manifolds, then so is $X \times Y$ with

$$\boxed{\dim(X \times Y) = \dim(X) + \dim(Y).}$$

Indeed, let $k := \dim(X)$, $l := \dim(Y)$, and let $(x, y) \in X \times Y$ be arbitrary. Let $U \subseteq_{op} \mathbb{R}^k$ (resp. $W \subseteq_{op} \mathbb{R}^l$) be open and $\phi : U \to X$ (resp. $\psi : W \to Y$) be a parametrisation around x (resp. y).

Define $\phi \times \psi : U \times W \to X \times Y$ by

$$(\phi \times \psi)(u, w) := (\phi(u), \psi(w)).$$

Note that $(U \times W) \subseteq_{op} \mathbb{R}^{k+l}$ and $f := \phi \times \psi$ is smooth (the component functions of f are the component functions of ϕ followed by those of ψ). We only need to verify that this is indeed a local parametrisation.

Note that ϕ and ψ are diffeomorphisms onto their images (and the images are open in X and Y respectively). Thus, $V := \phi(U) \times \psi(W)$ is an open neighbourhood of (x, y) in $X \times Y$. Moreover, $g : V \to U \times W$ by $(x', y') \mapsto (\phi^{-1}(x'), \psi^{-1}(y'))$ is the inverse of f.
The only check that needs to be done is that g is smooth. We leave this to the reader. (Use the smoothness of ϕ^{-1} and ψ^{-1} defined in the more general sense.)

Definition 1.1.7. If X and Y are both manifolds in $\mathbb{R}^N$ and $Z \subseteq X$, then Z is a submanifold of X.

Example 1.1.8. S^1 is a submanifold of $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < 2\}$.
Note that the dimensions are different.

Remark 1.1.9. We have defined manifolds only as subsets of Euclidean spaces.

Remark 1.1.10. Note that any open ball in $\mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k$ (check). Thus, the domains of local parametrisations may be assumed to be $\mathbb{R}^k$.

§§1.2. Derivatives and Tangents

Definition 1.2.1. Let $U \subseteq_{op} \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ be smooth, and $x \in U$. The derivative of f at x is the function

$$df_x : \mathbb{R}^n \to \mathbb{R}^m$$

defined by

$$df_x(v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

Note that df_x is defined on all of $\mathbb{R}^n$ even if $U \neq \mathbb{R}^n$.

Remark 1.2.2. df_x is a linear map. In particular, we may represent df_x as a matrix using the standard bases. If $f = (f_1, \ldots, f_m)$, then we have

$$df_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

Example 1.2.3. If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $L_x = L$ for all $x \in \mathbb{R}^n$.
In particular, if $i : U \hookrightarrow \mathbb{R}^n$ is the inclusion map, then $i_x = \mathrm{id}_{\mathbb{R}^n}$ for all $x \in U$.

Theorem 1.2.4 (Chain rule)**.** Let $U \subseteq_{op} \mathbb{R}^n$, $V \subseteq_{op} \mathbb{R}^m$. Suppose $f : U \to V$ and $g : V \to \mathbb{R}^l$ are smooth. For all $x \in U$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Definition 1.2.5. Let $X \subseteq \mathbb{R}^N$, $x \in X$, $U \subseteq_{op} \mathbb{R}^k$, and $\phi : U \to X$ be a local parametrisation around x. For convenience, assume that $0 \in U$ and $\phi(0) = x$.

The tangent space of X at x to be the image of the map $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^N$. This is denoted by $T_x(X)$.
A tangent vector to X at x is a point $v \in T_x(X) \subseteq \mathbb{R}^N$.

Note that in the above, we are making use of the fact that X is a subset of $\mathbb{R}^N$. The issue to clarify above is whether the above subspace $T_x(X)$ depends on ϕ or not. Suppose that $\psi : V \to X$ is another local parametrisation with $\psi(0) = x$. Note that $\phi(U)$ and $\psi(V)$ are both (relatively) open neighbourhoods of x. By passing to a subset, we may assume $\phi(U) = \psi(V)$. Thus, $h = \psi^{-1} \circ \phi : U \to V$ is a diffeomorphism. Using the chain rule on the relation $\phi = \psi \circ h$ gives

$$d\phi_0 = d\psi_0 \circ dh_0.$$

Thus, $\mathrm{im}(d\phi_0) \subseteq \mathrm{im}(d\psi_0)$. By symmetry, the converse is true too, as desired.

Theorem 1.2.6. With above notations,

$$\boxed{\dim(T_x(X)) = \dim(X),}$$

where the dimension on the left is the dimension as a vector space over $\mathbb{R}$.

In particular, $d\phi_0 : \mathbb{R}^k \to T_x(X)$ is an isomorphism.

Definition 1.2.7. Let $f : X \to Y$ be a smooth map of arbitrary manifolds. Let $x \in X$ and $y := f(x)$. The derivative of f at x is a linear map

$$df_x : T_x(X) \to T_y(Y)$$

defined as follows: Fix parametrisations $\phi : U \to X$ and $\psi : V \to Y$ around x and y. ($U \subseteq_{op} \mathbb{R}^k$ and $V \subseteq_{op} \mathbb{R}^l$.) Assume $\phi(0) = x$ and $\psi(0) = y$.

After passing to a subset of U, we have the following commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "h=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \end{tikzcd}.$$

We define df_x to be the unique map making the following square commute:

$$\begin{tikzcd} T_x(X) \arrow[r, "df_x"] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^l \arrow[u, "d\psi_0"'] \end{tikzcd}.$$

Note that $d\phi_0$ is an isomorphism and thus, df_x is uniquely determined as

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}.$$

Example 1.2.8. Check that the above does not depend on choice of ϕ or ψ.

Theorem 1.2.9 (Chain rule)**.** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x,$$

for all $x \in X$.

§§1.3. The Inverse Function Theorem and Immersions

Definition 1.3.1. Let $f : X \to Y$ be a smooth map of manifolds, and $x \in X$. f is called a local diffeomorphism at x if f maps a neighbourhood of x diffeomorphically onto a neighbourhood of $y := f(x)$.

f is called a local diffeomorphism if it is a local diffeomorphism at x for every $x \in X$.

Theorem 1.3.2 (Inverse Function Theorem)**.** Suppose that $f : X \to Y$ is a smooth map of manifolds, and let $x \in X$.
f is a local diffeomorphism at x iff df_x is an isomorphism.

Remark 1.3.3. If df_x is an isomorphism, one can choose local coordinates around x and y so that f appears to be the identity $f(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$ on some neighbourhood of x.

More precisely: there exists $U \subseteq_{op} \mathbb{R}^k$ and local parametrisations $\phi : U \to X$, $\psi : U \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\text{identity}"'] & U \arrow[u, "\psi"'] \end{tikzcd}.$$

Definition 1.3.4. Two maps $f : X \to Y$ and $f' : X' \to Y'$ are said to be equivalent (or same up to diffeomorphism) if there exist diffeomorphisms α and β making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\alpha"] \arrow[r, "f'"'] & U \arrow[u, "\beta"'] \end{tikzcd}.$$

Definition 1.3.5. $f : X \to Y$ smooth map of manifolds, $x \in X$.

f is said to be an immersion at x if $df_x : T_x(X) \to T_x(Y)$ is injective.
f is said to be an immersion if f is an immersion at x for all $x \in X$.

The canonical immersion is the standard inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^l$ for $k \leqslant l$ given by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$.

Theorem 1.3.6 (Local immersion theorem)**.** Suppose that $f : X \to Y$ is an immersion at $x \in X$, and $y = f(x)$. Then, there exist local coordinates around x and y such that

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

In other words, f is locally equivalent to the canonical immersion around x.

Corollary 1.3.7. If f is an immersion at x, then it is an immersion in a neighbourhood of x.

Remark 1.3.8. If $\dim(X) = \dim(Y)$, then local immersions and local diffeomorphisms are the same.

Remark 1.3.9. If $f : X \to Y$ is a smooth map, it is not necessary that $f(X)$ is a manifold. This is not true even if f is assumed to an immersion and injective.

One can construct a smooth map $f : \mathbb{R} \to \mathbb{R}^2$ which is an injective immersion but the image of f is the figure eight.

Definition 1.3.10. $f : X \to Y$ is called proper if the preimage of every compact set in Y is a compact subset of X. An immersion which is injective and proper is called an embedding.

Theorem 1.3.11. An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.