Analytically, the line is :

$$t(x, y, z) + (1-t)(0, 0, 1).$$

We need $tz + 1 - t = 0$ or $t = \frac{1}{1-z}$.

$$\therefore \quad x = \frac{x}{1-z} \quad \text{and} \quad y = \frac{y}{1-z}. \qquad (\text{Note: } z \neq 1.)$$

Finally, $N \longmapsto \infty$.

(Eg.: Under the above map, $(0, 0, -1) \longmapsto (0, 0)$ or $0 + 0i$.)

To sum it up: Define $\theta : S^2 \longrightarrow \hat{\mathbb{C}}$ by

$$\theta(x, y, z) = \begin{cases} \frac{x}{1-z} + \frac{iy}{1-z} & ; \quad z \neq 1, \\ \infty & ; \quad \text{else}. \end{cases}$$

Check: θ is a bijection.

To see that it is onto, let $z = x + iy \in \mathbb{C}$ be arbit.

Check that

$$P(x, y, z) := \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right)$$

maps to z. (As usual, $|z| = x^2 + y^2$.)

Q. What happens to P above as $|z| \longrightarrow \infty$?

Evidently $P \longrightarrow N(0, 0, 1)$.

→ Using the above, we can define a topology on $\hat{\mathbb{C}}$.

In fact, we now define a metric on $\hat{\mathbb{C}}$ as follows:

For $w, z \in \hat{\mathbb{C}}$, define the distance between w and z to be the length of the straight line segment joining $\theta^{-1}(w)$ and $\theta^{-1}(z)$, i.e.,

$$d(w, z) := \|\theta^{-1}(w) - \theta^{-1}(z)\|_3.$$

Lecture 2 (06-01-2022)

06 January 2022 14:01

Integration:

Integration

Let Ω be a domain in $\mathbb{C}$, and $\gamma : [a,b] \longrightarrow \Omega$ is piecewise - C^1. For any $f \in C^0(\Omega)$, $(f : \Omega \to \mathbb{C})$

$$\int_\gamma f := \int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\, \gamma'(t)\, dt.$$

Index of a point wrt. a path:

Fix $\gamma : [a,b] \longrightarrow \mathbb{C}$ is piecewise C^1. Assume γ is closed, i.e., $\gamma(a) = \gamma(b)$. Let $\Omega := \mathbb{C} \setminus \operatorname{im}(\gamma)$.

Then, Ω has possibly many connected components, out of which exactly one is unbounded.

Let $z_0 \in \Omega$. We define

$$\operatorname{Ind}_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z_0}\, d\xi$$

well-defined since $z \notin \operatorname{im}(\gamma)$.

$$= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt.$$

Properties:

(1) $\operatorname{Ind}_\gamma$ is an integer-valued function on Ω.

(2) Thus, $\operatorname{Ind}_\gamma$ is constant on the connected components of Ω.

(3) $\operatorname{Ind}_\gamma \equiv 0$ on the unbounded component.

Prop^n. (Cauchy's Theorem)

Cauchy's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $f: \Omega \to \mathbb{C}$ be continuous.
TFAE:

(i) $\int_\gamma f = 0$ for every closed γ in Ω.

(ii) $\exists F \in \mathcal{O}(\Omega)$ such that $F' \equiv f$ on Ω.

Consequently, $f \in \mathcal{O}(\Omega)$ (since once differentiable $\Rightarrow$ always differentiable).

Example. Let γ be ... in $\mathbb{C}$.
If $a \notin \text{im}(\gamma)$, then evaluate

$$I_n := \int_\gamma (z-a)^n \, dz \quad \text{for} \quad n \in \mathbb{Z}.$$

If $n \neq -1$, we have an antiderivative for the integrand on $\mathbb{C} \setminus \{a\}$. $\therefore I_n = 0$.

If $n = -1$, then we simply have $I_n = 2\pi i \, \text{Ind}_\gamma(a)$.

Def^n. Path homotopy

Path homotopy

Given $\gamma_0, \gamma_1 : [0,1] \to \Omega$ two closed paths in Ω based at x_0.
A path homotopy between γ_0 and γ_1 is a function

$$H: [0,1] \times [0,1] \to \Omega$$

s.t. ① H is continuous,

② $H(s,0) = \gamma_0(s) \quad \forall s \in [0,1]$,

③ $H(s,1) = \gamma_1(s) \quad \forall s \in [0,1]$

④ $H(0,t) = x_0 = H(1,t) \quad \forall t \in [0,1]$.

Recall: $\gamma_0 \sim \gamma_1$, (equiv. rel'n) path-homotopic, null-homotopic ($\gamma \sim 0$).

EXAMPLES. (1) $\Omega = \mathbb{C}$.

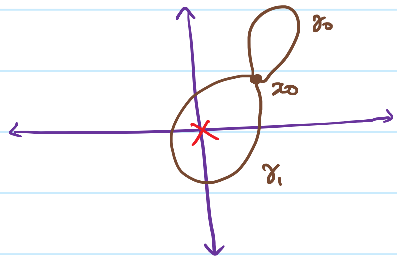
Any two loops are homotopic.

Indeed, $H(s,t) = (1-t)\gamma_0(s) + t\,\gamma_1(s)$ does the job.

② $\Omega = \mathbb{C}\setminus\{0\}$.

$x_0 = 1+i$.

The drawn loops are not homotopic.



Theorem. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let γ_0, γ_1 be loops based at the same point with $\gamma_0 \sim \gamma_1$. Then,

$$\int_{\gamma_0} f = \int_{\gamma_1} f \qquad \text{for all } f \in \mathcal{O}(\Omega).$$

EXAMPLE. The paths $\gamma_1, \gamma_2 : [0,1] \to \Omega = \mathbb{C}\setminus\{0\}$ defined as

$$\gamma_1(t) := \frac{1}{\gamma_0(t)} := e^{-2\pi i t}$$

cannot be path homotopic (in Ω) since $f = (z \mapsto \frac{1}{z}) \in \mathcal{O}(\Omega)$ and

$$\int_{\gamma_0} f = 2\pi i \neq -2\pi i = \int_{\gamma_1} f.$$

Corollary. Let Ω be a domain and γ be a loop in Ω with $\gamma \sim 0$. Then,

$$\int_{\gamma} f = 0 \quad \forall f \in \mathcal{O}(\Omega).$$

Def^n. An open set $\Omega \subseteq \mathbb{C}$ is said to be simply-connected if Ω is connected and $\gamma \sim 0$ for every loop γ in Ω.

Simply-connected, simply connected

(NON-)EXAMPLES:

- $\mathbb{C}$, $D(0,1)$, convex sets, star-shaped domains, $\mathbb{C}\setminus[0,\infty)$ ↳ s-c
- $\mathbb{C}\setminus\{0\}$, $D(0,1)\setminus\{0\}$, $D(0,a)\setminus D(0,b)$ for $a>b>0$

↳ not s-c.

Corollary. Let Ω be a s-c domain in $\mathbb{C}$ and let γ be a loop in Ω. Then,

$$\int_\gamma f = 0 \qquad \forall\, f \in \mathcal{O}(\Omega).$$

Cor. Let Ω be a s-c domain in $\mathbb{C}$. Let $f \in \mathcal{O}(\Omega)$. Then, $\exists F \in \mathcal{O}(\Omega)$ s.t. $F' = f$ on Ω.

Cor. Let Ω be a s-c domain in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$ be st. $f(z) \neq 0 \;\; \forall\, z \in \Omega$. Then, $\exists g \in \mathcal{O}(\Omega)$ s.t.

Analytic branch of logarithm

$$f = \exp \circ g.$$

(g is an analytic branch of logarithm of f.)

Proof. Since $f \neq 0$, $\quad \frac{f'}{f} \in \mathcal{O}(\Omega)$.

$\because \Omega$ is s-c, $\exists h \in \mathcal{O}(\Omega)$ s.t. $h' = \frac{f'}{f}$.

Let $\tilde{g} := \exp \circ h$

Then, $\tilde{g} \neq 0$. $\therefore \frac{f}{\tilde{g}} \in \mathcal{O}(\Omega)$.

$$\tilde{g}^2 \left(\frac{f}{\tilde{g}}\right)' = f'\tilde{g} - f\tilde{g}'$$

$$= f'\tilde{g} - f\,(\exp \circ h)\cdot h'$$

$$= f'\tilde{g} - f\tilde{g}h' = \tilde{g}\cdot(f' - fh') = 0.$$

$$\therefore \frac{f}{\tilde{g}} \equiv c \neq 0.$$

$$\Rightarrow f = c\cdot\tilde{g} = c\cdot \exp \circ h = \exp \circ (h + \underline{c'})$$

Lecture 3 (10-01-2022)

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Maximum Principle

① Let $\Omega \subseteq \mathbb{C}$ be a domain, and $f \in \mathcal{O}(\Omega)$.
Let $a \in \Omega$ such that $\exists r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$.
Then,

$$|f(a)| \leq \max_{0 \leq \theta \leq 2\pi} |f(a + re^{i\theta})|.$$

Moreover, equality holds iff f is constant.

② Let Ω be a bounded open set in $\mathbb{C}$. (Maximum Modulus Theorem)
Let $f \in C^0(\overline{\Omega}) \cap \mathcal{O}(\Omega)$. Then,

Maximum Modulus Theorem

$$|f(z)| \leq \max_{\partial\Omega} |f| \qquad \forall z \in \Omega.$$

In words, $|f|$ attains its maximum on the boundary.
Equivalently:

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

EXAMPLE: $\mathbb{H} := \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$.
Define $f(z) = \exp(-z^2)$ on $\overline{\mathbb{H}}$.
$f \in \mathcal{O}(\mathbb{H}) \cap C^0(\overline{\mathbb{H}})$.

Note that $|f(z)| \leq 1$ for $z \in \mathbb{R} = \partial\mathbb{H}$.
But

$$|f(iy)| = e^{y^2} \text{ grows rapidly on } i\mathbb{R}.$$

Thus, MMT need not hold if Ω is unbounded.
Now, we wish to formulate a similar theorem for unbounded.

- Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$.
For $a \in \overline{\Omega}$, define

$$\limsup_{\Omega \ni z \to a} f(z) := \lim_{r \to 0^+} \sup\{|f(z)| : z \in \Omega \cap D(a,r)\}.$$

this limit exists in $[0, \infty]$.

If a is the point at infinity, $D(a,r)$ is the neighbourhood of a in the metric d on the extended complex plane.

The extended boundary of Ω in $\mathbb{C} \cup \{\infty\}$ is denoted by $\partial_\infty \Omega$.
Note:

$$\partial_\infty \Omega = \begin{cases} \partial \Omega & ; \ \Omega \text{ is bounded}, \\ \partial\Omega \cup \{\infty\} & ; \ \text{else}. \end{cases}$$

③ **MMT**: Let Ω be a domain in $\mathbb{C}$, $f \in \mathcal{O}(\Omega)$.
(Not necessarily bounded!)

Suppose that $\limsup_{z \to a} |f(z)| \le M$ for all $a \in \partial_\infty \Omega$.

Then, $|f| \le M$ on Ω.

Proof. Let $\delta > 0$, and define $S := \{z \in \Omega : |f(z)| > M + \delta\}$.
We will show that $S = \emptyset$ and be done.
Note that $|f|$ is continuous and thus, S is open.
We claim that S is bounded in $\mathbb{C}$.
If Ω is bounded, then done.
Suppose Ω is unbounded. Then, $\infty \in \partial_\infty \Omega$.
But $\limsup_{\Omega \ni z \to \infty} |f(z)| \le M$.

$\therefore \exists R > 0$ s.t. $|f(z)| < M + \delta$ for all $|z| \geq R$, $z \in \Omega$.
(Check: $D(\infty, r)$ is the complement of a compact set.)

$\therefore S \subseteq D(0, R)$.

Applying MMT (2) to $f|_S$, we see that

$$|f(z)| \leq \max_{\partial S} |f| \qquad \forall z \in S.$$

But by def^n of S it follows that $|f| = M+\delta$ on ∂S.

$$\therefore \quad |f(z)| \leq M+\delta \qquad \forall z \in S.$$

But by def^n of S, we have $|f(z)| > M+\delta$ for $z \in S$.
$\therefore S = \phi$. ∎

Remark. In our previous example, we have $\limsup_{\mathbb{H} \ni z \to \infty} f(z) = \infty$.

Thus, this MMT did not apply!

Generalisations of MMT to unbounded domains.

Phragmén-Lindelöf Theorems.

Phragmén-Lindelöf, Phragmen-Lindelof

\# Liouville's Theorem: Bounded + entire $\Rightarrow$ constant
Also, recall the following exercise (using Cauchy's estimate, for example):

If $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)| \leq 1 + |z|^{1/3}$, then f is constant.

↳ "Generalisation" of Liouville.
Similarly, we generalise MMT.

(Phragmén - Lindelöf)

Theorem A. Let $\Omega \subseteq \mathbb{C}$ be simply-connected, and $f \in \mathcal{O}(\Omega)$. Fix $M > 0$.
Let $\partial_\infty \Omega = \mathrm{I} \cup \mathrm{II}$ be such that

(1) $\limsup_{\Omega \ni z \to a} |f(z)| \leq M$ for all $a \in \mathrm{I}$, and

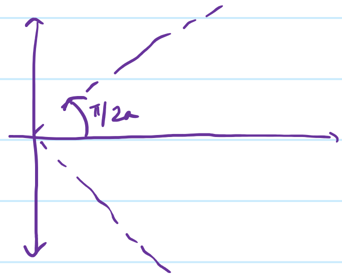
(2) $\exists\, \phi \in \mathcal{O}(\Omega)$, nonvanishing and bounded on Ω such that

$$\limsup_{\Omega \ni z \to a} |f(z)\,(\phi(z))^{\eta}| \le M$$

for all $a \in \mathbb{I}$ and for all $\eta > 0$.

Then, $|f| \le M$ on Ω.

EXAMPLE Fix $a \ge 1/2$. Let $\Omega = \{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{2a} \}$.



Let $f \in \mathcal{O}(\Omega)$ be s.t.

(a) $\overline{\lim}_{z \to \xi \in \partial\Omega} |f(z)| \le M$, and

(b) $|f(z)| \le A \exp(|z|^b)$ for $|z| \gg 1$,

where A and b are positive constants such that $b < a$.

using above theorem

Then, $|f(z)| \le M \quad \forall\, z \in \Omega$.

Clearly Ω is s-c.

Now, we find $\phi \in \mathcal{O}(\Omega)$ as in P-L.

Consider $\phi(z) = \exp(-z^c)$, where $c > 0$ is chosen later.

Note that this is holo on Ω.

Also, $\phi(z) \neq 0 \quad \forall\, z \in \Omega$

$$|\phi(z)| = |\exp(-z^c)| \qquad z = re^{i\theta},\ |\theta| < \pi/2a.$$

$$= |\exp(-r^c e^{i\theta c})|$$

$$= \exp(-r^c \cos(c\theta)) \le 1.$$

if $c < a$, then $\cos(c\theta) \ge 0$.

Thus, ϕ is bdd.

Take $I = \partial\Omega$ and $II = \{\infty\}$.

Now, fix $\eta > 0$ and for $z = re^{i\theta} \in \Omega$.

For large z, we have

$$|f(z)\,\phi(z)^{\eta}| \leq A\exp(|z|^b)\,|\exp(-z^c)|^{\eta}$$
$$= A\exp(r^b - \eta r^c \cos c\theta) \quad \delta := \inf_{0 \leq \theta < \pi/2a} \cos(c\theta).$$
$$\leq A\exp(r^b - \eta r^c \delta).$$

The above goes to 0 if $c > b$.

Thus, we can choose any $c \in (b, a)$.

We are now done. ∎

Proof of P-L. Since Ω is s-c and ϕ nonvanishing, ϕ has an analytic log, and ϕ^{η} makes sense as a holo. function.

$$\phi^{\eta} = \exp(\eta \log \phi)$$

Let $K > 0$ be s.t. $|\phi| \leq K$ on Ω.

Consider $g_{\eta}(z) := g(z) := \dfrac{f(z)\,\phi(z)^{\eta}}{K^{\eta}}$. $\quad g \in \mathcal{O}(\Omega)$.

Note: $|g(z)| \leq |f(z)|$.

Thus,

$$\limsup_{z \to \xi \in I} |g(z)| \leq M.$$

On II:

$$\limsup_{z \to \xi \in II} \left| \frac{f(z)\,\phi(z)^{\eta}}{K^{\eta}} \right| \leq \frac{M}{K^{\eta}}.$$

Now, MMT (3) from earlier applied to g, we get that

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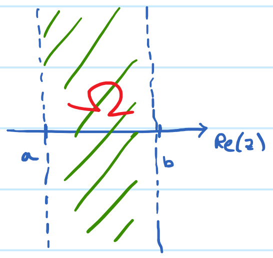
(Phragmén - Lindelöf)

Theorem B. Fix reals $a < b$, and $B > 0$.
Let $\Omega = \{z \in \mathbb{C} : a < Re(z) < b\}$, and $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.
Assume that:

$$|f| \leq B \quad \text{on} \quad \Omega,$$
$$|f| \leq 1 \quad \text{on} \quad \partial\Omega.$$



Then,

$$|f| \leq 1 \quad \text{on} \quad \Omega.$$

Remark: Note that the above is a type of MMT.

Idea: Introduce a typical multiplicative factor g_ε with $\lim_{\varepsilon \to 0} g_\varepsilon = 1$, such that $|fg_\varepsilon| < M$ on the boundary of a BOUNDED subdomain Ω_ε of Ω. Then, apply usual MMT on Ω_ε.
Moreover, pick the family $\{\Omega_\varepsilon\}_{\varepsilon > 0}$ nicely enough to cover all of Ω. Then take $\varepsilon \to 0$.

Proof. For each $\varepsilon > 0$, define $g_\varepsilon : \bar{\Omega} \to \mathbb{C}$ by

$$g_\varepsilon(z) := \frac{1}{1 + \varepsilon(z-a)}.$$

$\hookrightarrow$ denominator is 0 iff $z = a - \frac{1}{\varepsilon} \notin \bar{\Omega}$.

For $z \in \bar{\Omega}$, we have:

$$|f(z)\, g_\varepsilon(z)| \leq \frac{1}{|1 + \varepsilon(z-a)|} \leq \frac{1}{|Re(1 + \varepsilon(z-a))|}$$
$$= \frac{1}{|1 + \varepsilon(Re(z) - a)|}$$

$$|1 + \varepsilon(\mathrm{Re}(z) - a)|$$
$$\leq 1.$$

For $z = x + iy \in \overline{\Omega}$, we have:

(Note: $|f| \leq B$ on $\overline{\Omega}$ by continuity.)

$$|f(z)\, g_\varepsilon(z)| \leq \frac{B}{|1+\varepsilon(z-a)|} \leq \frac{B}{|\mathrm{Im}(1+\varepsilon(z-a))|}$$
$$= \frac{B}{|\varepsilon y|}. \qquad (*)$$

Now, define $\Omega_\varepsilon := \{ z \in \overline{\Omega} : -B/\varepsilon < y < B/\varepsilon \}$.

We have proven above that $|f g_\varepsilon| \leq 1$ on $\partial \Omega_\varepsilon$.
By the usual MMT, we have $|f g_\varepsilon| \leq 1$ on Ω_ε.

But also, by $(*)$, we see that $|f g_\varepsilon(z)| \leq 1$
if $y > \frac{B}{\varepsilon}$ as well!

Thus, for all z and for all $\varepsilon > 0$, we have the inequality. Fix z and let $\varepsilon \to 0^+$ to conclude. $\blacksquare$

Theorem C. Fix reals $a < b$, and $B > 0$.
Let $\Omega = \{ z \in \mathbb{C} : a < \mathrm{Re}(z) < b \}$, and $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega}) \setminus \{0\}$.
Assume that $|f| < B$. Define $M : [a,b] \to [0, \infty)$ by

$$M(x) := \sup_{y \in \mathbb{R}} |f(x+iy)|.$$

Then $\log \circ M$ is a convex function on (a, b).

Remarks: (i) For $a \leq x < v < y \leq b$:

$$M(v)^{(y-x)} \leq M(x)^{(y-v)} \cdot M(y)^{(v-x)}.$$

$$M(v)^{(y-x)} \leq M(x)^{(y-v)} \cdot M(y)^{(v-x)}.$$

(ii) Since $\log \circ M$ is convex on (a, b), we get

$$M(x) \leq \max\{M(a), M(b)\} \quad \forall x \in [a, b].$$

In particular, if $M(a) = M(b)$, then we get Theorem B.

(iii) By continuity, we have $|f| \leq B$ on $\partial\Omega$. Thus, $\sup_{\partial\Omega} |f| < \infty$.

By the above, we get

$$|f| \leq \sup_{\partial\Omega} |f| \quad \text{on } \Omega.$$

Proof Suffices to show that

$$M(v)^{b-a} \leq M(a)^{b-v} \cdot M(b)^{v-a}$$

for $v \in (a, b)$.

What if $M(a)$ or $M(b) = 0$?
Take care of this separately.

Consider the entire function g defined as

$$g(z) := M(a)^{\frac{b-z}{b-a}} \cdot M(b)^{\frac{z-a}{b-a}}.$$

$(\lambda^z := \exp(z \log \lambda))$

Also note that g is nonvanishing.

- $$|g(z)| = M(a)^{\frac{b-x}{b-a}} \cdot M(b)^{\frac{x-a}{b-a}}.$$

The above is continuous as a function of x and is non-vanishing. Thus, $\exists C > 0$ s.t. $\frac{1}{\cdot} \leq C$ on $\bar{\Omega}$.

$|g(z)|$

Now, consider $\frac{f}{g} \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\Omega)$.

$$\left|\frac{f(a+iy)}{g(a+iy)}\right| = \left|\frac{f(a+iy)}{M(a)}\right| \leq 1 \qquad \forall y \in \mathbb{R}.$$

$||^{ly}$ $$\left|\frac{f}{g}\right| \leq 1 \quad \text{on} \quad \partial\Omega.$$

Moreover, $\left|\frac{f}{g}\right| \leq CB$ on Ω

Thus, by Theorem B, we have $\left|\frac{f}{g}\right| \leq 1$ on Ω or

$|f| \leq |g|$ on Ω.

Expanding out, we get

$$|f(x+iy)|^{b-a} \leq M(a)^{b-x} \; M(b)^{x-a}$$

$\forall x \in (a,b)$
$\forall y \in \mathbb{R}$.

Take sup over $y \in \mathbb{R}$ and we are done. ∎

Consequences of MMT.

Schwarz Lemma: Let $f \in \mathcal{O}(D(0,1))$ such that $f(0) = 0$ and $|f| \leq 1$.

Then,

(a) $|f'(0)| \leq 1$ and

(b) $|f(z)| \leq |z| \quad \forall z \in D(0,1)$.

Moreover if equality holds either in (a) or for some $z \neq 0$ in (b), then $\exists \lambda \in S^1$ s.t. $f(z) = \lambda z$.

Sketch: Define $g : D(0,1) \to \mathbb{C}$ by $g(z) := \frac{f(z)}{z}$ holomorphically.
Fix $r \in (0,1)$. On $|z| = r$, we have

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- $\mathbb{D} := D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

- $\mathrm{Aut}(\mathbb{D}) := \{ f : \mathbb{D} \to \mathbb{D} \mid f \text{ is bijective, } f, f^{-1} \in \mathcal{O}(\mathbb{D}) \}$.

  ↳ group under composition

Aut(D)

Automorphisms of $\mathbb{D}$ fixing the origin:

Automorphisms of the disc

Theorem. If $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = 0$, then f is a rotation, i.e., $\exists \lambda \in \partial\mathbb{D}$ s.t. $f(z) = \lambda z \quad \forall z \in \mathbb{D}$.

Proof. Let $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0) = 0$.
By Schwarz, $|f'(0)| \leq 1$ and $|f(z)| \leq |z| \quad \forall z$.
Moreover, $f^{-1} \in \mathrm{Aut}(\mathbb{D})$ also fixes the origin.
By Schwarz again, $|f^{-1\prime}(0)| \leq 1$ and $|f^{-1}(z)| \leq |z| \quad \forall z$.
Thus, it follows that $|f(z)| = |z| \quad \forall z$ and hence, $\exists \lambda \in S^1$ s.t. $f = (z \mapsto \lambda z)$. ∎

Möbius transforms:

Möbius, Mobius

Let $\alpha \in \mathbb{D}$, and consider $z \xmapsto{\varphi_\alpha} \dfrac{z - \alpha}{1 - \bar{\alpha} z}$, $z \in \mathbb{D}$.

- Note that φ_α makes sense on $\mathbb{C} \setminus \{1/\bar{\alpha}\} \supseteq \mathbb{D}$.

Moreover, φ_α is holomorphic on $\mathbb{D}$, i.e., $\varphi_\alpha \in \mathcal{O}(\mathbb{D})$.

- $\varphi_\alpha(\alpha) = 0$.

- $\varphi_\alpha(\mathbb{D}) = ?$

  Check : $|\varphi_\alpha(e^{it})| \leq 1$ for $t \in \mathbb{R}$.

  Thus, by MMT $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$.

  Also, $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ has inverse as $\varphi_{-\alpha}$.

  Thus, $\varphi_\alpha \in \mathrm{Aut}(\mathbb{D}) \quad \forall \alpha \in \mathbb{D}$.

Theorem. $\mathrm{Aut}(\mathbb{D}) = \{ \lambda \varphi_\alpha : \lambda \in \partial\mathbb{D}, \ \alpha \in \mathbb{D} \}$.

Proof. ($\supseteq$) is clear.

($\subseteq$) Let $f \in \mathrm{Aut}(\mathbb{D})$.

Put $\alpha := f^{-1}(0)$.

Then, $(f \circ \varphi_{-\alpha}) \in \mathrm{Aut}(\mathbb{D})$ and $(f \circ \varphi_{-\alpha})(0) = 0$.

Thus, $f \circ \varphi_{-\alpha}$ is r_λ (rotation by λ) for some $\lambda \in \partial\mathbb{D}$.

$\Rightarrow f = r_\lambda \circ \varphi_\alpha$. ∎

Let $\alpha, \beta \in \mathbb{D}$. Let $f : \mathbb{D} \longrightarrow \overline{\mathbb{D}}$ be holomorphic and $f(\alpha) = \beta$. Among all such f, what is the maximum possible value of $|f'(\alpha)|$?

$$\begin{tikzcd} \alpha \quad \mathbb{D} \arrow[r, "f"] & \overline{\mathbb{D}} \quad \beta \arrow[d, "\psi_\beta"] \\ 0 \quad \mathbb{D} \arrow[u, "\psi_{-\alpha}"] \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"] & \overline{\mathbb{D}} \quad 0 \end{tikzcd}$$

$g := \psi_\beta \circ f \circ \psi_{-\alpha}$.

$g(0) = 0$.

By Schwarz, we have

$|g'(0)| \leq 1$.

Using chain rule, we have:

$$g'(0) = \psi_\beta'\big(f(\psi_{-\alpha}(0))\big) \cdot f'(\psi_{-\alpha}(0)) \cdot \psi_{-\alpha}'(0)$$

$$= \psi_\beta'(f(\alpha)) \cdot f'(\alpha) \cdot \psi_{-\alpha}'(0)$$

$$= \psi_\beta'(\beta) \cdot f'(\alpha) \cdot \psi_{-\alpha}'(0)$$

$\psi_\beta(z) := \dfrac{z-\beta}{1-\bar\beta z}$

$\Rightarrow \psi_\beta'(z) = \dfrac{1-\bar\beta z - (z-\beta)(-\bar\beta)}{(1-\bar\beta z)^2}$.

$$= \frac{1-\bar\beta\beta}{(1-\bar\beta\beta)^2} \cdot f'(\alpha) \cdot \frac{1-(\alpha\bar\alpha)}{1^2}$$

$$= \frac{1-|\alpha|^2}{1-|\beta|^2} \cdot f'(\alpha)$$

$$\therefore \ |f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}.$$

Note that equality is possible. For example, $f = \varphi_{-\beta} \circ \varphi_\alpha$.

In fact, it happens iff $\exists \lambda \in S^1$ s.t.

$$f = \varphi_{-\beta} \circ r_{\lambda} \circ \varphi_{\alpha}.$$

Ex. Calculate $\mathrm{Aut}(\mathbb{D}\setminus\{0\})$.

Towards the Riemann-Mapping Theorem

$$\mathcal{O}(\Omega) \subseteq \mathcal{C}^0(\Omega; \mathbb{C}).$$

want to make this a metric space.

Let us consider $\Omega = \mathbb{D}$.
There is a sequence $\{K_n\}$ of compact sets in $\mathbb{C}$ s.t.:

(1) $\mathbb{D} = \bigcup_{n=1}^{\infty} K_n^\circ$,

(2) $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$,

(3) for each compact $K \subset \mathbb{D}$, $\exists n \in \mathbb{N}$ s.t. $K \subseteq K_n$.

One can take $K_n := \overline{D(0, 1-\frac{1}{n})}$, for example.

Claim. One can do the above for any open $\Omega \subseteq \mathbb{C}$.
Given any open $\Omega \subseteq \mathbb{C}$, $\exists$ a sequence $\{K_n\}_n$ of compact subset of $\mathbb{C}$ s.t.

(Compact exhaustion)

(1) $\Omega = \bigcup_{n=1}^{\infty} K_n^\circ$,

(2) $K_n \subseteq K_{n+1}^\circ$ $\forall n \in \mathbb{N}$,

(3) for any compact $K \subseteq \Omega$, $\exists n \in \mathbb{N}$ s.t. $K \subseteq K_n$.

Proof. For each $n \in \mathbb{N}$, let

$$K_n := \overline{D(0,n)} \cap \{ z \in \Omega : \operatorname{dist}(z, \mathbb{C}\setminus\Omega) \geq 1/n \}.$$

Check that K_n satisfies (1) − (3).

Using the above, we define a metric on $\mathcal{C}^0(\Omega; \mathbb{C})$.

Fix some $\{K_n\}_n$ as given by compact exhaustion.
Let $f, g \in \mathcal{C}^0(\Omega; \mathbb{C})$.
Define

$$\rho_n(f, g) := \sup_{z \in K_n} |f(z) - g(z)|.$$

Finally, define

$$\rho(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f,g)}{1 + \rho_n(f,g)}.$$

Ex. ① $(\mathcal{C}^0(\Omega; \mathbb{C}), \rho)$ is a metric space.

② A sequence $\{f_k\}_{k \geq 1}$ converges to f in $(\mathcal{C}^0(\Omega; \mathbb{C}), \rho)$ iff $f_k \longrightarrow f$ uniformly on compact subsets of Ω.
What are open sets in $(\mathcal{C}^0(\Omega; \mathbb{C}), \rho)$?

This ex. shows that the topology does not depend on $\{K_n\}_{n \geq 1}$.

Lecture 6 (20-01-2022)

20 January 2022 14:19

$\mathcal{O}(\Omega) \subseteq \mathcal{C}^0(\Omega; \underset{\text{metric space}}{\mathbb{C}})$.

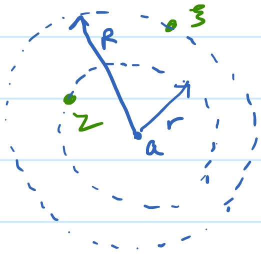
↳ subspace topology

Prop^n. $\mathcal{O}(\Omega)$ is closed in $\mathcal{C}^0(\Omega;\mathbb{C})$.
That is, if $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$ and $f_n \longrightarrow f$ in $\mathcal{C}^0(\Omega;\mathbb{C})$, then $f \in \mathcal{O}(\Omega)$.
Moreover, $f_n^{(k)} \longrightarrow f^{(k)}$ in $\mathcal{O}(\Omega)$ for all $k \geq 1$.

Proof. To show $f \in \mathcal{O}(\Omega)$, we may assume Ω is a disc and use Morera's theorem and that $\int_T f_n = 0$ for every triangle $T \subseteq \Omega$.

We now show that $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of Ω. Suffices to prove it for $k=1$ and use induction.

$$(f_n' - f')(z) = \frac{1}{2\pi i} \int_{|\xi - a| = R} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2}\, d\xi$$



for all $z \in \overline{D(a, r)}$.

$[\, D(a,r) \subsetneq D(a,R) \subseteq \Omega. \,]$

$$\Rightarrow |f_n'(z) - f'(z)| \leq \frac{1}{2\pi} \int_{|\xi - a| = R} \frac{|f_n(\xi) - f(\xi)|}{|\xi - z|^2}\, d\xi$$

$$\leq \frac{1}{(R-r)^2} \Big(\sup_{\partial D(a,R)} |f_n - f| \Big)$$

$\downarrow$

0 as $n \to \infty$.

$$\Rightarrow |f_n'(z) - f'(z)| \longrightarrow 0 \text{ uniformly for } z \in \overline{D(a,r)}.$$

Thus, $f_n' \to f'$ uniformly on closed discs.
Now, given any arbitrary $K \subseteq \Omega$, we can cover it by finitely many closed discs contained in Ω. ∎

Normal Families

Normal family

Defn. Let $\Omega \subseteq \mathbb{C}$ be a domain, and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$.
$\mathcal{F}$ is said to be normal if for every sequence $(f_n)_n \in \mathcal{F}^{\mathbb{N}}$, it is possible to extract a subsequence $(f_{n_k})_k$ such that either

(a) $(f_{n_k})_k$ converges uniformly on compact subsets of Ω, or

(b) given any pair of compact sets $K \subset \Omega$, $L \subset \mathbb{C}$,
$\exists\, k_0 = k_0(K, L) \in \mathbb{N}$ s.t.

$$f_{n_k}(K) \cap L = \emptyset \quad \forall\, k \geq k_0.$$

($f_{n_k} \to \infty$ uniformly on compact subset of Ω.)

EXAMPLES. (i) $\Omega_1 = D(0,1)$.
$\mathcal{F}_1 = \{ z \underset{z \in \Omega}{\longmapsto} z^n : n \in \mathbb{N} \}$. → normal because of (a)

(ii) $\Omega_2 = \{ z \in \mathbb{C} : |z| > 1 \}$.
$\mathcal{F}_2 = \{ z \mapsto z^n : n \in \mathbb{N} \}$. → normal because of (b)

(iii) $\Omega_3 = \{ z \in \mathbb{C} : \frac{1}{2} < |z| < 2 \}$.
$\mathcal{F}_3 = \{ z \mapsto z^n : n \in \mathbb{N} \}$.

The above is not normal. Consider the sequence $f_n = (z \longmapsto z^n) \in \mathcal{F}_3$.
Consider $K = D(1, \varepsilon)$ for some small $\varepsilon > 0$ s.t. $K \subset \Omega$.

Consider $K \cap \Omega_1$ and $K \cap \Omega_2$ to see $\mathcal{F}_3$ is NOT NORMAL.

(iv) Let $\Omega \subseteq \mathbb{C}$ be a domain.
$\mathcal{F} = \{ z \mapsto z^n : n \in \mathbb{N} \}$ is NOT NORMAL
if $\partial D(0,1) \subseteq \Omega$.

REMARKS. (i) If (a) is true and $f_{n_k} \to f$, the $f \in \mathcal{O}(\Omega)$.
(ii) However, f above need not be in $\mathcal{F}$.

Theorem (Montel's Theorem) Montel's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be locally uniformly bounded on Ω, i.e., for all compact $K \subseteq \Omega$, $\exists M = M(K) > 0$ such that

$$|f(z)| \le M \qquad \forall f \in \mathcal{F}, \ \forall z \in K.$$

Then, $\mathcal{F}$ is a normal family.
In fact, $\mathcal{F}$ is normal and satisfying (a) of the def^n.

EXAMPLE. Let $\Omega \subseteq \mathbb{C}$ be a domain.
Then, given any subset $\mathcal{F} \subseteq \{ f \in \mathcal{O}(\Omega) : f(\Omega) \subseteq D(0,1) \}$, Montel's theorem asserts that $\mathcal{F}$ is normal!

Recall:

Theorem (Arzelà - Ascoli Theorem)

Let $\mathcal{F} \subset \mathcal{C}^0(\Omega; \mathbb{C})$.
Every sequence in $\mathcal{F}$ admits a convergent subsequence (in $(\mathcal{C}(\Omega;\mathbb{C}), \rho)$) iff:
(i) $\mathcal{F}$ is pointwise bounded, i.e., $\exists M : \Omega \to [0, \infty)$ s.t.

$$|f(z)| \le M(z) \qquad \forall z \in \Omega, \quad \text{and}$$

(ii) $\mathcal{F}$ is equicontinuous at each point of Ω.

Proof of Montel's Theorem: Let $\mathcal{F}$ be as given.
It suffices to show that $\mathcal{F}$ is

Lecture 7 (24-01-2022)

24 January 2022 14:02

EXAMPLE. Montel's Theorem fails on $\mathbb{R}$.

Indeed, consider the family $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$, where $f : \mathbb{R} \to \mathbb{R}$ is defined as $f_n(x) := \sin(nx)$.

Clearly, $\mathcal{F}$ is locally uniformly bounded as $|f_n(x)| \le 1$ $\forall x \in \mathbb{R}$ $\forall n \in \mathbb{N}$.

However, given any $\delta > 0$, pick n s.t. $x = \frac{\pi}{2n} < \delta$.

Then, $|f_n(x) - f_n(0)| = \left|\sin\left(\frac{\pi}{2}\right)\right| = 1$.

Thus, no δ exists for $\varepsilon = 1$.

Thus, $\mathcal{F}$ is not equicontinuous.

Hurwitz's theorem

Theorem (Hurwitz's Theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that $\exists\, a \in \Omega$, $r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$ such that f has no zeroes on $\partial D(a,r)$.
Then, $\exists\, N \in \mathbb{N}$ such that f and f_n have the same number of zeroes (counting multiplicities) in $D(a,r)$ for all $n \ge N$.

Remark. Note that if f is not identically zero, one can find $a \in \Omega$, $r > 0$ as stated. In fact, for any $a \in \Omega$, we can find an $r > 0$ since zeroes are isolated!

Proof. Since $f \ne 0$ on $\partial D(a,r)$, $\min_{\partial D(a,r)} |f| =: \delta > 0$.

Since $f_n \to f$ uniformly on compact subsets of Ω, it follows that $\exists\, N \in \mathbb{N}$ s.t.

$$|f_n(z) - f(z)| < \delta/2 \qquad \forall z \in \partial D(a,r)\ \forall n \ge N.$$

2 $\forall n \geq N$.

Thus, $|f_n(z) - f(z)| < |f(z)|$ $\forall z \in \partial D(a,r)$ and $n \gg 0$.

Now, by Rouché's theorem, we are done. ∎

Corollary 1. Let Ω be a domain in $\mathbb{C}$, $f_n \in \mathcal{O}(\Omega)$ $\forall n$, $f_n \longrightarrow f$ in $\mathcal{O}(\Omega)$.
Suppose that each f_n is non-vanishing on Ω.
Then, either $f \equiv 0$ or f is also nonvanishing.

Corollary 2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$, $f_n \longrightarrow f$ in $\mathcal{O}(\Omega)$.
Suppose that each f_n is injective on Ω, then f is injective on Ω.

Riemann Mapping Theorem

Theorem (RMT). Let $\Omega \subsetneq \mathbb{C}$ be simply-connected.
Then, Ω is biholomorphic to $D(0,1)$.

Remark ① $\mathbb{C}$ cannot be biholo. to $D(0,1)$, by Liouville.
② If Ω is biholo. to $D(0,1)$, then Ω is homeomorphic to $D(0,1)$ and thus, simply-connected.

Question Is this Riemann map unique?
$f: \Omega \to D(0,1)$

No. These will precisely "differ" by $\mathrm{Aut}(\mathbb{D})$.

Proof of RMT Let $\Omega \subsetneq \mathbb{C}$ be as specified.
Fix $p \in \Omega$.
Let

$$\mathcal{F} = \{ f \in \mathcal{O}(\Omega) : f(p) = 0,\ f \text{ is injective},\ f(\Omega) \subseteq D(0,1) \}.$$

If we can find $f_0 \in \mathcal{F}$ such that $f(\Omega) = D(0,1)$, then we are done since f_0^{-1} is also holomorphic.

Steps: (I) $\mathcal{F} \neq \emptyset$.

(II) $\sup_{f \in \mathcal{F}} |f'(p)| = |f_0'(p)|$ for some $f_0 \in \mathcal{F}$.

(III) f_0 (as above) is onto.

Motivation: Suppose we a compact exhaustion $(K_n)_{n \in \mathbb{N}}$ of Ω with $p \in K_n$ $\forall n$.

By choosing f as in (II), we get a function which "starts out fastest" at p. Then, $\bigcup_{n=1}^{\infty} f_0(K_n) = D(0,1)$ is likely.

(I) To show: $\mathcal{F} \neq \emptyset$.

(a) If Ω is bounded, then $z \longmapsto \frac{z-p}{M}$ works for an appropriate $M > 0$.

(b) As $\Omega \subsetneq \mathbb{C}$, pick $Q \in \mathbb{C} \setminus \Omega$.

Let $\phi(z) := z - Q$ is nonvanishing on Ω.

As Ω is simply-connected, $\exists$ a holomorphic square root of ϕ.

$\exists h \in \mathcal{O}(\Omega)$ s.t. $(h(z))^2 = \phi(z)$ $\forall z \in \Omega$.

Note that since ϕ is injective, we get

$h(z_1) \neq h(z_2)$ and $h(z_1) \neq -h(z_2)$

for $z_1 \neq z_2 \in \Omega$.

In particular, h is nonconstant on Ω.

Thus, h is an open map.

Let $b \in h(\Omega)$. Then, $D(b, r) \subseteq h(\Omega)$ for some $r > 0$.

Then, $D(-b, r) \cap h(\Omega) = \emptyset$ by earlier observation.

For $z \in \Omega$, define $f(z) := \frac{r}{}$.

Then, $\mathbb{H} \subsetneq \mathbb{C}$ is a simply-connected domain.
We have an explicit biholomorphism $f: \mathbb{H} \to D(0,1)$
given by $z \mapsto \dfrac{z-i}{z+i}$.

Next up: Weierstrass Factorisation Theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{C}$. Then, f is either identically zero on Ω or $Z(f) := \{z \in \Omega : f(z) = 0\}$ is discrete in Ω.

Q. Let $A \subseteq \mathbb{C}$ be discrete. Can we find $f \in \mathcal{O}(\Omega)$ such that $Z(f) = A$?

Note: A must be countable. If finite, consider polynomials.
Now, assume $(a_n)_{n\in\mathbb{N}}$ is an enumeration of A.

Naive guess: $f(z) = (z-a_1)(z-a_2)\cdots(z-a_n)\cdots$.
How to make sense of f?

Another attempt: Construct $f_1, f_2, \ldots \in \mathcal{O}(\mathbb{C})$ s.t. $Z(f_n) = \{a_n\}$
and put $f = \prod_n f_n$.
↳ need to make sense of infinite products.

Infinite Products.

Infinite products

Def^n. Suppose that $(u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Define the sequence $(p_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ by

$$p_n := (1+u_1)\cdots(1+u_n).$$

If $\lim_{n\to\infty} p_n =: p$ exists (in $\mathbb{C}$), then we write

∞

Lecture 9 (31-01-2022)

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Theorem Let X be a metric space. Let $u_n : X \to \mathbb{C}$ be a sequence of functions such that $\sum_{n=1}^{\infty} |u_n|$ converges uniformly to a bounded function. (Say, bounded by $M > 0$.)

Then, (1) $\prod_{n=1}^{\infty} (1+u_n)$ converges uniformly on X.

$$\text{Define} \quad f(x) := \prod_{n=1}^{\infty} (1+u_n(x)) \quad \text{for } x \in X.$$

(2) For $x_0 \in X$: $\quad f(x_0) = 0 \iff u_M(x_0) = -1$ for some $M \in \mathbb{N}$.

(3) For every permutation $\sigma \in S_{\mathbb{N}}$, the infinite product

(Rearrangement)
$$\prod_{k=1}^{\infty} \left(1 + u_{\sigma(k)}(x)\right) \quad \text{converges to } f(x),$$
for all $x \in X$.

Proof. (1) Let $P_N(x) := \prod_{n=1}^{N} (1+u_n(x))$, $\quad x \in X$.

We will show that $(P_N)_{N=1}^{\infty}$ is uniformly Cauchy on X.

For $M > N$, note

$$\begin{aligned}
|P_M(x) - P_N(x)| &= \left| P_N(x) \cdot \prod_{n=N+1}^{M} (1+u_n(x)) - P_N(x) \right| \\
&= |P_N(x)| \cdot \left| \prod_{n=N+1}^{M} (1+u_n(x)) - 1 \right| \\
&\leq |P_N(x)| \left[\prod_{n=N+1}^{M} (1+|u_n(x)|) - 1 \right] \\
&\leq |P_N(x)| \left[\exp\left(\sum_{n=N+1}^{M} |u_n(x)| \right) - 1 \right]
\end{aligned}$$

last lec's last lemma

—"—

↳ this term is uniformly Cauchy since $\sum |u_n|$ converges uniformly

Cauchy uniformly since $\sum |u_n|$ converges

$$\leq \exp(M) \cdot \left(\text{small}\right).$$ ✓

(2) Let f denote the limit. Let $x \in K$ be s.t. $P_N(x) \neq 0 \ \forall x$.

From the above, given $\varepsilon = 1/4$, we can get N_0 s.t.

$$|P_M(x) - P_{N_0}(x)| < 2|P_{N_0}(x)|\varepsilon \quad \forall M > N_0.$$

Then, $|f(x)| \geq (1 - 2\varepsilon) P_{N_0}(x)$.

In particular, $f(x) \neq 0$.

Thus, $f(x) = 0 \Rightarrow P_N(x) = 0$ for some N ⟩ finite product

$\Rightarrow 1 + u_n(x) = 0$ for some n

$\Rightarrow u_n(x) = -1$ for some n. ✓

(3) Exercise.

Theorem. Let Ω be a domain in $\mathbb{C}$. Let $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$ be such that no f_n is identically zero.

Suppose that $\sum_{n=1}^{\infty} |1 - f_n|$ converges uniformly on compact subsets of Ω.

(1) Then, $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of Ω.

Consequently $f := \prod_{n=1}^{\infty} f_n$ is holomorphic.

(2) Let $a \in \Omega$. If $f(a) = 0$, then $f_n(a) = 0$ for some n.

Moreover, this is true for only finitely many n.

Lastly,

$$\operatorname{ord}_f(a) = \sum_{n=1}^{\infty} \operatorname{ord}_{f_n}(a).$$

multiplicity

this is only nonzero for finitely many.

Lecture 10 (03-02-2022)

03 February 2022 14:00

If we can find $g_k \in \mathcal{O}(\Omega)$ for $k \in \mathbb{N}$ s.t.

(i) g_k has no zeroes on Ω, and

(ii) $\sum_{k=1}^{\infty} |1 - (z - z_k) g_k(z)|$ converges uniformly on compact ...,

$$\text{then } z \mapsto \prod_{k=1}^{\infty} (z - z_k) g_k(z) \in \mathcal{O}(\Omega)$$

and the zeroes are precisely $\{z_k\}_{k=1}^{\infty}$.

Guess: $g_k = \exp(h_k)$ for some $h_k \in \mathcal{O}(\Omega)$.
($\because$ we want $g_k \neq 0$.)

Elementary Factors :

Weierstrass elementary factors

Defn. $E_0(z) = 1 - z$ for $z \in \mathbb{C}$.

For $p \in \mathbb{N}$, define

$$E_p(z) := (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right).$$

These functions are called (Weierstrass) Elementary factors.

Below, we have $p \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$.

- Each E_p vanishes precisely at 1.

1 is a simple zero (order $= 1$) for each E_p.

- $E_p(0) = 1$

- For $|z| < 1$,

$$E_p(z) = (1 - z) \exp\left(\sum_{k=1}^{p} \frac{z^k}{k} \right)$$

Heuristic!

$$= (1 - z) \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \right) \exp\left(- \sum_{k=p+1}^{\infty} \frac{z^k}{k} \right)$$

$$\ldots \left(\sum_{k=1} \frac{\ \ }{k} \right) \ldots \left(\sum_{k=p+1} \frac{\ \ }{k} \right)$$

↳ branch of log of $z \mapsto \frac{1}{1-z}$ on $D(0,1)$

$$= (1-z) \cdot \frac{1}{1-z} \cdot \exp\left(- \sum_{k=p+1}^{\infty} \frac{z^k}{k} \right)$$

$$= \exp\left(- \sum_{k=p+1}^{\infty} \frac{z^k}{k} \right)$$

$$= 1 - \frac{z^{p+1}}{p+1} + \text{higher order}$$

HAND-WAVY: Thus, if p is large, we expect $E_p \sim 1$.

More precisely:

Lemma For every $p \geq 0$,

$$|1 - E_p(z)| \leq |z|^{p+1} \qquad \text{if} \quad |z| \leq 1.$$

Proof. Fix $p \geq 1$. ($p = 0$ is clear.)

Write $E_p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$.

(This expansion is valid on $\mathbb{C}$ since E_p is entire. $0 = E_p(1) = 1 + \sum_{n=1}^{\infty} a_n$.)

$$\Rightarrow \quad E_p'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

OTOH,

$$E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$$

$$\Rightarrow E_p'(z) = - \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$$

$$+ (1-z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right) \left(1 + z + \cdots + z^{p-1} \right)$$

$$= \exp\left(z + \cdots + \frac{z^p}{p} \right) \left[(-1) + (1-z) \left(\frac{1 - z^p}{1-z} \right) \right]$$

$$= \exp\left(\qquad \right) \left[(-1) + (1 - z^p) \right]$$

$$= - z^p \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right).$$

$\therefore E_p'$ has a zero of order p at the origin.
Thus, $a_1 = \cdots = a_p = 0$.

Thus, $E_p(z) = 1 + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots$.

Also, equating

$$-z^p \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) = \sum_{n=p+1}^{\infty} n a_n z^{n-1}$$

shows us that $a_n \in \mathbb{R} \ \forall n$ and $a_n \leq 0 \ \forall n \geq p+1$.

Coefficients here are +ve

For $|z| \leq 1$:

$$|E_p(z) - 1| = \left|\sum_{n=p+1}^{\infty} a_n z^n\right|$$

$$= |z|^{p+1} \left|\sum_{n=p+1}^{\infty} a_n z^{n-p-1}\right|$$

$$\leq |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n|$$

$\because a_n \leq 0 \ \forall n \geq p+1$

$$= -|z|^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$= -|z|^{p+1} \left(E_p(1) - 1\right)$$

$$= |z|^{p+1}.$$

☒

Remark. The function $z \mapsto E_p\left(\frac{z}{a}\right)$ has a simple zero at $z = a$ (and no other zeroes).

(Weierstrass Product Theorem)

Theorem Let $(a_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}}$ be such that $a_n \neq 0 \ \forall n \geq 1$ and $|a_n| \to \infty$ as $n \to \infty$.

(Note: the sequence need not consist of distinct points. However, $|a_n| \to \infty$ forces that no point appears inf. often.)

IF $(p_n)_n \in \mathbb{N}_0^{\mathbb{N}}$ is such that

$$\sum^{\infty} \left(\frac{r}{\cdot}\right)^{p_n + 1} < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n + 1} < \infty$$

for every $r > 0$, THEN:

(i) $\displaystyle\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ converges in $\mathcal{O}(\mathbb{C})$.

Write f for the above function.

(ii) $f \in \mathcal{O}(\mathbb{C})$ and $Z(f) = \{a_n : n \in \mathbb{N}\}$.

(iii) The multiplicity of any zero is precisely the number of times that it appears in the sequence.

Remarks: (1) Since $|a_n| \to \infty$ as $n \to \infty$, for every $r > 0$, $\exists N_0 = N_0(r) \in \mathbb{N}$ s.t.
$|a_n| > 2r$ for all $n \geq N_0$.

Thus,

$$\left(\frac{r}{|a_n|}\right) < \frac{1}{2} \qquad \forall\, n \geq N_0.$$

In turn,

$$\left(\frac{r}{|a_n|}\right)^{p_n+1} < \left(\frac{1}{2}\right)^{p_n+1} \qquad \forall\, n \geq N_0.$$

Thus, $p_n = n - 1$ ALWAYS works for any $(a_n)_n$ with $|a_n| \to \infty$.

(2) Suppose that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$.

Then, $p_n \equiv 0$ works!

In this case, $f(z) = E_0\left(\frac{z}{a_n}\right)$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \text{ works.}$$

Proof. Since zeroes are isolated, $|a_n| \to \infty$.

As discussed last line, $\exists (p_n)_n$ s.t. $\sum \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \quad \forall r > 0.$

(e.g.: $p_n = n-1$.)

Thus, $h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ is holo on $\mathbb{C}$ and has SAME zeroes as f (with mult).

Thus, f/h is entire and nonvanishing. $\therefore \exists g \in \mathcal{O}(\mathbb{C})$ s.t.

$$f/h = \exp \circ g.$$

$\blacksquare$

Theorem. Let $\Omega \subsetneq \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$ be an open set.

Suppose $A \subset \Omega$ has no limit points in Ω.

Let $m : A \to \mathbb{N}$ be any function.

Then, $\exists f \in \mathcal{O}(\Omega)$ such that $Z(f) = A$, and f has a zero of multiplicity $m(\alpha)$ for every $\alpha \in A$.

Proof. It suffices to prove the theorem in the special case where:

Ω is a deleted neighbourhood of ∞ and $\infty \notin \bar{A}$.

$\equiv \Omega = \mathbb{C} \setminus K$ for some compact $K \subseteq \mathbb{C}$.

Justification.

Let Ω_1 and A_1 be as in the hypothesis of the theorem.

Fix $\infty \neq a \in \Omega_1 \setminus A_1$. Define

$$T(z) = \frac{1}{z-a}.$$

T is a linear fractional transformation from $\hat{\mathbb{C}}$ onto itself.

T is a homeomorphism of Ω_1 onto $T(\Omega_1) =: \Omega$.

Define $A := T(A_1)$. Then, A is BOUNDED as well, has no limit points in Ω.

Now, Ω and A satisfy the requirements of the special case.

Now, if theorem holds for special case, we can translate it back.

Now, we prove the theorem for the special case.

If $A = \{a_1, \ldots, a_n\}$, take

$$f(z) := (z-a_1)^{m_1} \cdots (z-a_n)^{m_n}$$