

$$\int (\overset{\circ}{5} \overset{\circ}{5}) dx$$

MA 5106

Introduction to Fourier Analysis

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Preliminaries

- Rectangle in \mathbb{R}^d : $R = [a_1, b_1] \times \dots \times [a_d, b_d]$. } closed
- Cube in \mathbb{R}^d : $Q = [a_1, b_1] \times \dots \times [a_d, b_d]$
where $b_1 - a_1 = \dots = b_d - a_d$.
- Volume of R : $|R| = \prod_{i=1}^d (b_i - a_i)$
- Exterior measure of $E \subseteq \mathbb{R}^d$: (Exterior measure)
 $m_*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \text{ are cubes} \right\}$

Observations:

- (1) Any singleton has exterior measure 0.
- (2) Exterior measure of any (closed/open) rectangles is equal to its volume.
- (3) $m_*(\mathbb{R}^d) = \infty$.
- (4) $m_*(\text{Cantor set}) = 0$.

Properties:

$$(1) E \subseteq F \Rightarrow m_*(E) \leq m_*(F)$$

$$(2) m_* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} m_*(E_j) \quad \left(\begin{array}{l} \text{equality needn't} \\ \text{hold even if disjoint} \end{array} \right)$$

Measurable set

(Measurable set, Lebesgue measurable set)

Def' A set $E \subseteq \mathbb{R}^d$ is called (Lebesgue) measurable if for every $\epsilon > 0$, \exists an open set O with $O \supseteq E$ s.t. $m_*(O \setminus E) = 0$.

If $E \subseteq \mathbb{R}^d$ is measurable, then (Lebesgue) measure of E is denoted by $m(E)$ and defined as

$$m(E) = m_*(E).$$

(Lebesgue measure)

Examples of measurable sets

- (1) Any open set is measurable.
- (2) E s.t. $m_*(E) = 0 \Rightarrow E$ is measurable
- (3) Countable union of measurable sets are measurable.
- (4) Complement of a meas. set is meas.
- (5) Any closed set. Any countable intersection of meas. sets.

Thm. (1) Let E_1, E_2, \dots be disjoint measurable sets.

Then,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

(2) $m(E+h) = m(E)$ \forall measurable $E \subseteq \mathbb{R}^d$, $\forall h \in \mathbb{R}^d$

$$(E+h := \{y+h \mid y \in E\})$$

" $E+h$ is also measurable" is implicit. Similar for next ones.

$$(3) m(cE) = c^d m(E), \quad c > 0$$

$$(cE := \{cy \mid y \in E\})$$

$$(4) m(-E) = m(E).$$

$$(-E := \{-y \mid y \in E\})$$

Defn. $f: \mathbb{R}^d \rightarrow [-\infty, \infty]$ is said to be measurable if for any $a > 0$,

$$f^{-1}([- \infty, a]) \subseteq \mathbb{R}^d$$

is measurable.

(Measurable function)

Examples

(1) Any continuous function is measurable.

(2) If f is measurable and $p: \mathbb{R} \rightarrow \mathbb{R}$ is measurable, then $p \circ f$ is measurable.

(3) If $\{f_n\}_n$ is a sequence of measurable functions, then the functions
 $\sup f_n, \inf f_n, \limsup f_n, \liminf f_n$.

are all measurable.

(4) Limit of a sequence of measurable functions is measurable.
(Pointwise)

(5) If f, g are measurable, then so are $f+g, f \cdot g$.

Ex. Characteristic function. Let $E \subseteq \mathbb{R}^d$.

Ex. Characteristic function. Let $E \subseteq \mathbb{R}^d$.

Define

$$\chi_E(x) := \begin{cases} 1 & ; \text{ if } x \in E \\ 0 & ; \text{ if } x \notin E \end{cases}$$

Then, χ_E is a measurable $f \Leftrightarrow E$ is measurable.

Note $f^{-1}([-a, a]) = \begin{cases} E^c : & 0 < a \leq 1 \\ \mathbb{R}^d : & 1 < a \end{cases}$

Thus, χ_E is a meas. $f \Leftrightarrow E^c$ is meas $\Leftrightarrow E$ is.

Defn. A function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be (Simple function)
simple if

$$f = \sum_{k=1}^n a_k \chi_{E_k}. \quad (a_k \in \mathbb{R} \text{ constants})$$

($m(E_k) < \infty$)

Thm. Let f be a non-negative measurable function on \mathbb{R}^d .
Then, \exists an increasing seq. of non-neg simple functions $\{\varphi_k\}_k$
s.t.

$$\lim_{k \rightarrow \infty} \varphi_k = f \quad \text{pointwise.}$$

$$(\varphi_k(n) \leq \varphi_{k+1}(m) + \epsilon)$$

Integration

(Integration)

(1) Let f be a simple function.

$$f = \sum_{k=1}^n a_k \chi_{E_k}, \quad (E_k \text{ measurable} \& m(E_k) < \infty)$$

$$\int_{\mathbb{R}^d} f := \sum_{k=1}^n a_k m(E_k).$$

(Has to be checked that this is independent of $((a_k), (E_k), N)$.)

Example. $\int_{\mathbb{R}} \chi_{[0,1]} = 1.$

(2) Let f be a bounded measurable function with

$$m(\text{supp } f) < \infty \text{ where}$$

$$\text{supp } f = \{x : f(x) \neq 0\}. \rightarrow \text{will be measurable since } f \text{ is}$$

Then, $\exists \{\varphi_n\}_n$ of simple functions s.t. $\varphi_n \leq M$ and

$$\varphi_n \rightarrow f \text{ a.e.}$$

(i.e., the set of points x for which $\varphi_n(x) \rightarrow f(x)$ is of measure zero.)

and $\text{supp } \varphi_n \subseteq \text{supp } f$.

Then, $\int_{\mathbb{R}^d} f := \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_n.$

\uparrow
this defined by (1)

(Again, independent of this $\{\varphi_n\}_n$.)

(3) Assume $f \geq 0$.

$$\int_{\mathbb{R}^d} f := \sup \left\{ \int_{\mathbb{R}^d} g : 0 \leq g \leq f, g \text{ is bounded, measurable} \right\}$$

with $m(\text{supp } g) < \infty$

$$\int_E f := \int_{\mathbb{R}^d} f \cdot \chi_E \quad (E \subseteq \mathbb{R}^d \text{ is measurable})$$

this is defined earlier
note $f \cdot \chi_E$ is measurable and ≥ 0 .

Defn.: $f \geq 0$ is integrable if $\int_{\mathbb{R}^d} f < \infty$. (Integrable)

Now, if $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is any function, we can write

$$f(x) = f^+(x) - f^-(x)$$

$$f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}.$$

Note that $f^+, f^- \geq 0$.

Defn.: f is integrable if $\int_{\mathbb{R}^d} |f| < \infty$ and

$$\int_{\mathbb{R}^d} f := \int_{\mathbb{R}^d} f^+ - \int_{\mathbb{R}^d} f^-$$

Can be extended to $f: \mathbb{R}^d \rightarrow \mathbb{C}$ component-wise

Example.: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1 & ; x \in \mathbb{Q} \cap [0, 1] \\ 0 & ; x \notin \mathbb{Q} \cap [0, 1] \end{cases}$$

Then, f is not Riemann integrable on $[0, 1]$.
However,

$$\int_{[0,1]} f = \int_{\mathbb{Q} \cap [0,1]} f + \int_{[0,1] \setminus \mathbb{Q}} f$$

$$= 0$$

Thm. Let f be Riemann integrable on $[a, b]$. Then, f is measurable and both the integrals (Riemann & Lebesgue) coincide.

Lecture 2 (08-01-2021)

08 January 2021 09:24

Recap.

$$f \geq 0$$

1. $f = \sum a_i \chi_{E_i}$, then $\int f := \sum a_i m(E_i)$

2. $m(\text{supp } f) < \infty$, then $\exists \{\varphi_n\}$ simple s.t. $\varphi_n \rightarrow f$ a.e.
(f bounded)

$$\int f := \lim_{n \rightarrow \infty} \int \varphi_n$$

3. $\int f \, dx := \sup \left\{ \int g \, dx : \begin{array}{l} 0 \leq g \leq f \\ g \text{ bounded} \end{array} \right\}$

PROPERTIES.

1. $\int (af + bg) = a \int f + b \int g \quad \forall a, b \in \mathbb{C}$

2. $E \cap F = \emptyset$ and E, F measurable, then

$$\int_f = \int_E f + \int_F f$$

3. $|\int f| \leq \int |f|$

4. $f \geq 0$ and $\int_{\mathbb{R}^d} f = 0 \Rightarrow f = 0$ a.e.

If $f = 0$ a.e., then $\int f = 0$.

... . . .

5. $\int_{\mathbb{R}^d} |f| < \infty \Rightarrow |f| < \infty \text{ a.e.}$

Suppose $f_n \rightarrow f$ pointwise.

$$\lim_{n \rightarrow \infty} \int f_n = \int f$$

(We know above is true if uniform conv. & f_n Riemann integ.)

Thm. (Monotone Convergence Theorem) (Monotone Convergence Theorem)

Let $\{f_n\}_n$ be a sequence of non-negative measurable functions, converging pointwise to f and $f_n \leq f_{n+1}$.

Then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.$$

Thm. (Dominated Convergence Theorem) (Dominated Convergence Theorem) (DCT)

Let $\{f_n\}_n$ be a sequence of measurable functions such that

$$f_n \rightarrow f \quad \text{a.e.}$$

Assume further that \exists an integrable function g s.t.

$$|f_n(x)| \leq g(x).$$

Then,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.$$

$$\text{Thm.} \cdot \int_{\mathbb{R}^d} f(x-h) dx = \int_{\mathbb{R}^d} f(x) dx$$

$$\cdot \int_{\mathbb{R}^d} f(-x) dx = \int_{\mathbb{R}^d} f(x) dx$$

$$\cdot \int_{\mathbb{R}^d} f(cx) dx = \frac{1}{c^d} \int_{\mathbb{R}^d} f(x) dx ; c > 0$$

Thm. (Fubini's Theorem) (Fubini's Theorem)

(a) Let f be a non-negative measurable function on

$$\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^{d_1+d_2}$$

Then,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx$$

// //

$$\int_{\mathbb{R}^{d_1+d_2}} f(x, y) d(x, y)$$

(b) Let f be integrable on $\mathbb{R}^{d_1+d_2}$ (i.e., $\int_{\mathbb{R}^{d_1+d_2}} |f| < \infty$).

$$\text{Then, } \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx$$

// //

$$\int_{\mathbb{R}^{d_1+d_2}} f(x, y) d(x, y)$$

To use (b), we need to check if $\int_{\mathbb{R}^{d_1+d_2}} |f|^p < \infty$. However,

since $|f| \geq 0$, we can compute the above integral using (a).

Def. $L^p(\Omega) = \left\{ f: \Omega \rightarrow \mathbb{C} \mid f \text{ is meas. and } \int_{\mathbb{R}^d} |f|^p < \infty \right\}$.

$1 \leq p < \infty$

(L^p spaces, L_p spaces)

- Normed linear space: $(X, \| \cdot \|)$
(NLS)
 $X \rightarrow$ vector space over \mathbb{R} or \mathbb{C}
and $\| \cdot \|: X \rightarrow [0, \infty)$ s.t.

$$\begin{aligned} \text{(i)} \quad & \|x\| = 0 \iff x = 0 \\ \text{(ii)} \quad & \|\alpha x\| = |\alpha| \|x\|, \quad \alpha \in \mathbb{F} \\ \text{(iii)} \quad & \|x + y\| \leq \|x\| + \|y\| \end{aligned}$$

Any NLS is a metric space with $d(x, y) = \|x - y\|$.

- $L^p(\mathbb{R}^d)$ is a vector space, easy to see.
(linear space)

Moreover, defining $\|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \right)^{\frac{1}{p}}$

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

$$\|\alpha f\|_p = |\alpha| \|f\|_p$$

$$\|f\|_p = 0 \Rightarrow \int_{\mathbb{R}^d} |f|^p = 0 \Rightarrow |f|^p = 0 \text{ a.e.}$$

$$\cdot \|f\|_p = 0 \Rightarrow \int_{\mathbb{R}^d} |f|^p = 0 \rightarrow |f|^p = 0 \text{ a.e.}$$

↓

$$f = 0 \text{ a.e.}$$

not necessarily 0

In fact, $L^p(\mathbb{R})$ is actually classes of functions where $f \sim g \Leftrightarrow f = g \text{ a.e.}$

Then, $L^p(\mathbb{R}^d)$ is an NLS.

$$\cdot L^{\infty}(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \rightarrow \mathbb{C} \text{ meas. } f^h \text{ which are bounded a.e.} \right\}$$

$$\|f\|_{\infty} := \text{ess sup } |f|$$

$$\therefore |f(x)| \leq \|f\|_{\infty} \text{ a.e.}$$

- An NLS $(X, \|\cdot\|)$ is called a Banach space if X is complete as a metric space.
- $L^p(\mathbb{R}^d)$ is a Banach space for $1 \leq p \leq \infty$.

Thm. (Hölder's Theorem) (Hölder's Theorem, Holder's Theorem)

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

$$(p \geq 1, \quad p = \infty \Rightarrow \frac{1}{p} = 0)$$

Result. (Using Hahn-Banach Theorem)

$$\|f\|_p = \sup \left\{ \left| \int_{\mathbb{R}^d} fg \right| : \|g\|_q < 1 \right\}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Convolution

(Convolution)

Defn. Let f, g be integrable functions on \mathbb{R}^d ($f, g \in L^1(\mathbb{R}^d)$).

Then, convolution of f and g is defined as

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x-y) dy.$$

Q. Does RHS exist? Yes, for almost every $x \in \mathbb{R}^d$.

Proof. Note

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)| |g(x-y)| dy \right) dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)| |g(x-y)| dx \right) dy \\ &= \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |g(x-y)| dx \right) dy \\ &= \int_{\mathbb{R}^d} |f(y)| \underbrace{\left(\int_{\mathbb{R}^d} |g(z)| dz \right)}_{\text{constant}} dy \\ &= \left[\int_{\mathbb{R}^d} |f| \right] \left[\int_{\mathbb{R}^d} |g| \right] < \infty \quad \text{since } f, g \in L^1 \end{aligned}$$

$$\Rightarrow x \mapsto \int_{\mathbb{R}^d} |f(z)| |g(x-z)| dz \quad \text{is finite a.e. } \blacksquare$$

Thus, $(f * g)(x)$ exists for almost every x .

Also, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, by the above.

Thm. Let $p \in [1, \infty)$. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, then

$$f * g \in L^p(\mathbb{R}^d)$$

and

$$\|f * g\| \leq \|f\|_p \|g\|_1.$$

$$\begin{aligned} \cdot (f * g)(x) &= \int_{\mathbb{R}^d} f(y) g(x-y) dy \\ &= \int_{\mathbb{R}^d} f(x-z) g(z) dz \\ &= (g * f)(x) \end{aligned}$$

• Convolution can be defined on any measurable group (G, \cdot) .

$f, g \in L^1(G)$, then

$$(f * g)(x) = \int_G f(y) g(xy^{-1}) dy$$

Can define convolution on $\mathcal{T} = \mathbb{T} \cong [0, 2\pi] / \sim$.

$$\cdot f * (g * h) = (f * g) * h$$

$$\cdot (f + g) * h = f * h + g * h$$

$$\text{Now, } \text{supp } f = \overline{\{x : f(x) \neq 0\}}.$$

Thm. Let $C_c(\mathbb{R}^d)$ be the set of ^{continuous} func $f: \mathbb{R}^d \rightarrow \mathbb{C}$ with compact support. ($C_c(\mathbb{R}^d)$)

Obs. $C_c(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. (well, technically L^p is equiv. classes but note that if cont. f^n are equal a.e., then they are equal.)

Proof. $f \in C_c(\mathbb{R}^d)$

$$\Rightarrow \int_{\mathbb{R}^d} \|f\|^p = \int_{\text{supp } f} \|f\|^p \leq \|f\|_\infty \int_{\text{supp } f} 1 = \|f\|_\infty m(\text{supp } f) < \infty.$$

Thm. 1. $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. Here, $1 \leq p \leq \infty$.

2. $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. Here, $1 \leq p \leq \infty$.

↳ In f. differentiable

Def. (Approximate identity in $L'(\mathbb{R}^d)$) (Approximate identity)

A sequence $\{k_n\}_n$ in $L'(\mathbb{R}^d)$ is called approximate identity for $L'(\mathbb{R}^d)$ if

$$(1) \quad k_n \geq 0 \quad \forall n \in \mathbb{N}$$

$$(2) \quad \int_{\mathbb{R}^d} k_n = 1 \quad \forall n \in \mathbb{N}$$

$$(3) \quad \text{For any } \delta > 0,$$

$$\lim_{n \rightarrow \infty} \int_{\|x\| > \delta} k_n = 0.$$

Thm. Let $\{k_n\}_n$ be an approximate identity for $L'(\mathbb{R}^d)$. Let $f \in L'(\mathbb{R}^d)$.

Then,

$$f * k_n \rightarrow f \text{ in } L' \text{ as } n \rightarrow \infty.$$

That is,

$$\lim_{n \rightarrow \infty} \|f * k_n - f\| = 0.$$

Remark.

($L'(\mathbb{R}^d)$, $*$) does not have an identity.

That is, $\nexists g \in L'(\mathbb{R}^d) \forall f \in L'(\mathbb{R}^d) (f * g = f)$

We prove the theorem in the next class. Before that, we have the following lemma.

Lemma) Let $f \in L'(\mathbb{R}^d)$. Then, the map $y \mapsto T_y f$ is a continuous function $\mathbb{R}^d \rightarrow L'(\mathbb{R}^d)$, where

$$T_y f(x) := f(x - y).$$

That is, for any $\epsilon > 0$, $\exists \delta > 0$ s.t. $\|y_1 - y_2\|_2 \leq \delta \Rightarrow \|T_{y_1} f - T_{y_2} f\|_1 < \epsilon$.

Proof. Let $g \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned} \|T_{y_1} g - T_{y_2} g\|_1 &= \int_{\mathbb{R}^d} |T_{y_1} g(x) - T_{y_2} g(x)| dx \\ &= \int_{\mathbb{R}^d} |g(x - y_1) - g(x - y_2)| dx \\ &= \int_{\mathbb{R}^d} |g(x + y_2 - y_1) - g(x)| dx \\ &\quad \text{let } k = \sup_{\text{compact}} g \end{aligned}$$

$\therefore g$ is continuous, can choose $\delta > 0$ s.t.

$$\|y_1 - y_2\|_2 < \delta \Rightarrow |g(x + y_2 - y_1) - g(x)| < \epsilon / \sup_{\text{compact}} g$$

$$\|y_1 - y_2\| < \delta \Rightarrow |g(x+y_2-y_1) - g(x)| < \frac{\epsilon}{m(\kappa_0(\kappa+y_2-y_1))}$$

$$< \epsilon \quad \text{if } \|y_1 - y_2\| < \delta.$$

Now, use the fact that $C_c(\mathbb{R}^d)$ is dense in $L'(\mathbb{R}^d)$.

Note the following to conclude:

$$\|T_{y_1}f - T_{y_2}f\|_1 \leq \|T_{y_1}f - T_{y_1}g\|_1 + \|T_{y_1}g - T_{y_2}g\|_1 + \|T_{y_2}g - T_{y_2}f\|_1.$$

$$= \|T_{y_1}(f-g)\|_1 + \|T_{y_1}g - T_{y_2}g\|_1 + \|T_{y_2}g - f\|_1$$

$$= \|f-g\|_1 + \|T_{y_1}g - T_{y_2}g\|_1 + \|f-g\|_1$$

can be made $< \epsilon$.

Lecture 3 (13-01-2021)

13 January 2021 09:20

Recall:

A sequence $\{k_n\}_n$ in $L'(\mathbb{R}^d)$ is called approximate identity for $L'(\mathbb{R}^d)$ if

$$(1) \quad k_n \geq 0 \quad \forall n \in \mathbb{N}$$

$$(2) \quad \int_{\mathbb{R}^d} k_n = 1 \quad \forall n \in \mathbb{N}$$

(3) For any $\delta > 0$,

$$\lim_{n \rightarrow \infty} \int_{\|x\| > \delta} k_n = 0.$$

Thm. Let $\{k_n\}_n$ be an approximate identity for $L'(\mathbb{R}^d)$. Let $f \in L'(\mathbb{R}^d)$.

Then,

$$f * k_n \rightarrow f \text{ in } L' \text{ as } n \rightarrow \infty.$$

That is,

$$\lim_{n \rightarrow \infty} \|f * k_n - f\| = 0.$$

To prove that, we had seen the following lemma.

Lemma. Let $f \in L'(\mathbb{R}^d)$. Then, the map $y \mapsto T_y f$ is a continuous function $\mathbb{R}^d \rightarrow L'(\mathbb{R}^d)$, where

$$T_y f(x) := f(x - y).$$

Remark. In fact, if $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, then

$y \mapsto T_y f$ is continuous as a $f: \mathbb{R}^d \rightarrow L^p(\mathbb{R}^d)$.

Proof (of Thm). $f \in L^1(\mathbb{R}^d)$ and $\{k_n\}_n$ is approximate identity in L^1 .

$$\|f * k_n - f\|_1 = \int_{\mathbb{R}^d} |(f * k_n)(x) - f(x)| dx$$

$$= \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} f(x-y) k_n(y) dy - \underbrace{\int_{\mathbb{R}^d} f(x) k_n(y) dy}_{\left(\because \int_{\mathbb{R}^d} k_n = 1 \right)} \right\} dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y) - f(x)| k_n(y) dy dx \quad (k_n \geq 0)$$

we'll interchanging,
show it's finite (Fubini)

$$= \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} |T_y f(x) - f(x)| dx \right\} k_n(y) dy$$

$$= \int_{\mathbb{R}^d} \|T_y f - f\|_1 k_n(y) dy$$

$\therefore y \mapsto T_y f$ is continuous, for every $\epsilon > 0$, $\exists \delta > 0$
 s.t. $\|T_y f - f\|_1 < \epsilon$ if $\|y\| < \delta$.

$\|T_y f\| = \|f\|$

$$= \int_{\|y\| < \delta} \|\bar{T}_y f - f\|_1 k_n(y) dy + \int_{\|y\| \geq \delta} \|\bar{T}_y f - f\|_1 k_n(y) dy$$

By defⁿ of
approx. id.

$$\leq \frac{\epsilon}{2} \int_{\|y\| < \delta} k_n(y) dy + 2\|f\| \int_{\|y\| \geq \delta} k_n(y) dy$$

$$\left(\int_{\mathbb{R}^d} k_n = 1 \right)$$

$$\begin{aligned}
 & \text{for our approx. id.} \\
 & \int_{\mathbb{R}^d} k_n \rightarrow 0 \quad \leftarrow \quad \int_{\mathbb{R}^d} k_n = 1 \\
 & \leq \frac{\varepsilon}{2} \cdot 1 + 2\|f\| \frac{\varepsilon}{4\|f\|}, \quad \forall n \geq N \\
 & = \varepsilon
 \end{aligned}$$

$$\therefore f * k_n \rightarrow f \text{ in } L^1.$$

Remark. We shall see later that $(L^1(\mathbb{R}^d), *)$ does not have an identity but it has (many) approximate identities.

Construction of an Approximate Identity

Let $\varphi \geq 0$ be an integrable function.

(That is, $\int_{\mathbb{R}^d} \varphi < \infty$. That is $\varphi \in L^1(\mathbb{R}^d)$)

Suppose $\int_{\mathbb{R}^d} \varphi = 1$.

For each $n \in \mathbb{N}$, let $\varphi_n(t) := n^d \varphi(nt)$, $t \in \mathbb{R}^d$.

(nt is the usual scalar multiplication in the n -space \mathbb{R}^d)

Then, $\{\varphi_n\}_n$ is an approximate identity in $L^1(\mathbb{R}^d)$.

Check. (i) $\varphi_n \geq 0 \quad \forall n \in \mathbb{N}$ is obvious.

$$(ii) \int_{\mathbb{R}^d} \varphi_n = n^d \int_{\mathbb{R}^d} \varphi(nt) dt$$

$$\int_{\mathbb{R}^d} \varphi(y) dy = n^d \int_{\mathbb{R}^d} \varphi(y) \frac{dy}{n^d} = \int_{\mathbb{R}^d} \varphi(y) dy = 1.$$

iii) Fix $\delta > 0$.

$$\begin{aligned} \int_{|t| \geq \delta} \varphi_n &= n^d \int_{|t| \geq \delta} \varphi(nt) dt \\ &= \int_{|y| \geq n\delta} \varphi(y) dy \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Example. For $d=1$, $\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ works.

Obsr. $f \in L^1$, $g \in L^p \Rightarrow f * g \in L^p$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Q. Let $g \in L^p$ and $\{k_n\}_n$ an approx. id. in L^1 . ($1 \leq p < \infty$)
Is it true that

$$\|g * k_n - g\|_p \rightarrow 0 ?$$

(That is, $\lim_{n \rightarrow \infty} g * k_n = g$?)

Yes! will show later.

Thm. (Minkowski's integral inequality)

Given two measure spaces (X, μ) and (Y, ν) with σ -finite

Given two measure spaces (X, μ) and (Y, ν) with σ -finite measures:

$$\left(\int_X \left(\int_Y |f(x, y)|^p d\nu(y) \right)^{1/p} d\mu(x) \right)^p \leq \int_Y \left(\int_X |f(x, y)|^p d\mu(x) \right)^{1/p} d\nu(y)$$

Thm. Let $\{k_n\}_n$ be an approximate identity in $L^1(\mathbb{R}^d)$. Then, $f \in L^p(\mathbb{R}^d)$. Then,

$$\|f * k_n - f\|_p \rightarrow 0$$

Proof.

$$\begin{aligned} \|f * k_n - f\|_p &= \left(\int_{\mathbb{R}^d} |(f * k_n)(x) - f(x)|^p dx \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [f(x-y) - f(x)] k_n(y) dy \right|^p dx \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [T_y f(x) - f(x)] k_n(y) dy \right|^p dx \right)^{1/p} \\ &\stackrel{\text{Markovki}}{\leq} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |T_y f(x) - f(x)|^p \cdot |k_n(y)|^p dx \right)^{1/p} dy \\ &= \int_{\mathbb{R}^d} k_n(y) \left\{ \int_{\mathbb{R}^d} |T_y f(x) - f(x)|^p dx \right\}^{1/p} dy \\ &= \int_{\mathbb{R}^d} \|T_y f - f\|_p \cdot k_n(y) dy \end{aligned}$$

Now we are in the same position as earlier. \square

($y \mapsto T_y f$ is continuous on $\mathbb{R}^d \rightarrow L^p(\mathbb{R}^d)$ since $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, for $1 \leq p \leq \infty$.)

Thm. Let f be a continuous function with compact support. (Then, $f \in L^\infty$.)

Let $\{k_n\}_n$ be an approximate identity in ℓ^1 .

Then,

$$\|f * k_n - f\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

(Need to prove that for $f \in C_c(\mathbb{R}^d)$, $y \mapsto T_y f$ is $c\epsilon$ on $\mathbb{R}^d \rightarrow L^\infty(\mathbb{R}^d)$)

. Let $T := S' = \mathbb{R}/\mathbb{Z}$. $x \mapsto e^{2\pi i x}$.

We will do analysis on T . (Torus)

Then, we identify f with a function on \mathbb{R} which is periodic with period 1.

. $L^p(T) = \{f: T \rightarrow \mathbb{C} \text{ measurable} \mid \int_0^1 |f|^p < \infty\}$.

$$\|f\|_{L^p(T)} := \left(\int_0^1 |f|^p \right)^{\frac{1}{p}}.$$

for emphasis. Sometimes we will simply write $\|f\|_p$.

. $L^p(T) \subset L^q(T)$ if $q \leq p$
(note the reversal)

$$(L^1(T) \supseteq L^2(T) \supseteq \dots)$$

↳ true for any finite measure space.
Not true in \mathbb{R} .

Proof. We show for 1 and 2 using Cauchy Schwarz.

$$\begin{aligned} \int_0^1 |f(t)| dt &= \int_0^1 |f(t)| \cdot 1 dt \\ &\leq \left(\int_0^1 |f(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^1 1 dt \right)^{\frac{1}{2}} \\ &= \|f\|_2 < \infty \end{aligned}$$

Thus, $f \in L^1$ and $\|f\|_1 \leq \|f\|_2$.

In general, we use Hölder.

$$\begin{aligned} \int_0^1 |fg|^q dt &= \int_0^1 (|f(t)|^\alpha)^{\frac{q}{\alpha}} \cdot (|g(t)|^\beta)^{\frac{q}{\beta}} dt \\ &\leq \left(\int_0^1 ((|f(t)|^\alpha)^{\frac{q}{\alpha}})^{\frac{p}{p}} dt \right)^{\frac{q}{p}} \cdot 1 \\ &= \left(\int_0^1 |f|^p dt \right)^{\frac{q}{p}} \\ \Rightarrow \left(\int_0^1 |f|^q dt \right)^{\frac{1}{q}} &\leq \left(\int_0^1 |f|^p dt \right)^{\frac{1}{p}} \\ \therefore f \in L^q & \end{aligned}$$

□

$$L^1(\mathbb{T}) \supseteq L^p(\mathbb{T}) \quad \forall p \geq 1$$

Thus, we only do Fourier Analysis for L^1 , which takes care of all.

—

Lecture 4 (15-01-2021)

15 January 2021 09:30

Let f be a function on an interval ($\in \mathbb{R}$).

Q. When can f be expressed in terms of \sin and \cos ?

That is,

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \sin(2\pi k x) + b_k \cos(2\pi k x)$$

for some complex sequences $\{a_k\}_{k=-\infty}^{\infty}$ and $\{b_k\}_{k=-\infty}^{\infty}$.

Obs.: the RHS is periodic with period 1. Thus, the LHS must also have period 1. (not necessarily fundamental)

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \quad \text{for some complex sequence } \{c_k\}_{k=-\infty}^{\infty}$$

Assume convergence unif. & abs.

Then,

$$c_k = \int_0^1 f(x) e^{-2\pi i k x} dx.$$

$$L'(\mathbb{T}) = \{ f: \mathbb{T} \rightarrow \mathbb{C} \text{ measurable} : \int |f| < \infty \}.$$

Def. Let $f \in L'(\mathbb{T})$. Then the Fourier coefficient of f is defined by

$$\hat{f}(k) := \int_0^1 f(x) e^{-2\pi i k x} dx \quad \text{for } k \in \mathbb{Z}.$$

Also, the series $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x}$ is called the Fourier series of f .

Q. When and in which sense does the Fourier series of f converge to f ?

Thm. There exists a continuous function f such that its Fourier series diverges at a point.

Thm. (Dini's theorem)

Let $f \in L^1(\mathbb{T})$. If, for some x_0 , $\exists \delta > 0$ s.t.

$$\int_{|t|<\delta} \left| \frac{f(x_0+t) - f(x_0)}{t} \right| dt < \infty.$$

Then,

$$f(x_0) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x_0} = \lim_{N \rightarrow \infty} \sum_{k=-N}^N \hat{f}(k) e^{2\pi i k x_0}$$



Note that it is only at that point.

Lemma

(Riemann Lebesgue Lemma)

Let $f \in L^1(\mathbb{T})$. Then,

$$\lim_{|k| \rightarrow \infty} \hat{f}(k) = 0.$$

Proof. Let g be a continuous function on \mathbb{T} .

Then,

$$\hat{g}(k) = \int g(x) e^{-2\pi i k x} dx \quad \text{--- (1)}$$

$$y = x + \frac{1}{2k} \quad \left(\begin{array}{l} \\ \end{array} \right) \quad = - \int_0^{1/2k} g(x) e^{-2\pi i k (x + \frac{1}{2k})} dx$$

$$\left(\begin{array}{l} \text{periodic} \\ \text{continuous} \\ \text{function.} \end{array} \right) \quad = - \int_{1/2k}^{1/2k} g\left(x - \frac{1}{2k}\right) e^{-2\pi i k x} dx$$

$$= - \int g\left(x - \frac{1}{2k}\right) e^{-2\pi i k x} dx \quad \text{--- (2)}$$

Using (1) and (2), we get

$$\hat{g}(k) = \int g(x) e^{-2\pi i k x} dx.$$

$$\hat{g}(k) = \frac{1}{2} \int_0^{\pi} \left(g(x) - g\left(x - \frac{1}{2k}\right) \right) e^{-2\pi i k x} dx$$

$$\Rightarrow |\hat{g}(k)| \leq \frac{1}{2} \int_0^{\pi} |g(x) - g\left(x - \frac{1}{2k}\right)| dx$$

$$\Rightarrow |\hat{g}(k)| \leq \frac{1}{2} \max_{x \in [0, \pi]} |g(x) - g\left(x - \frac{1}{2k}\right)|$$

But g is uniformly continuous. Thus, by choosing $|k|$ large, the RHS can be arbitrarily small.

Thus, the lemma is true for continuous functions.

Now, we use the fact that $\mathcal{C}(\mathbb{T})$ is dense in $L^1(\mathbb{T})$.

$\mathcal{C}(\mathbb{T})$ is compact, not
(need to write \mathcal{C}_c)

Take $f \in L^1(\mathbb{T})$. Let $\epsilon > 0$ be given.

Find $g \in \mathcal{C}(\mathbb{T})$ s.t.

$$\|g - f\|_1 < \epsilon/2$$

(g exists, by density)

Also, fix N s.t.

$$\|\hat{g}(k)\| < \epsilon/2 \quad \forall |k| \geq N$$

(Note $|\hat{f}(k)| = \left| \int_0^{\pi} f(x) e^{2\pi i k x} dx \right| \leq \int_0^{\pi} |f(x)| dx = \|f\|_1$)

Thus, for $|k| \geq N$, we have

$$\begin{aligned} |\hat{f}(k)| &\leq |\hat{f}(k) - \hat{g}(k)| + |\hat{g}(k)| \\ &\leq \widehat{|f-g|}(k) + |\hat{g}(k)| \\ &\leq \|f-g\|_1 + |\hat{g}(k)| \\ &< \epsilon. \end{aligned}$$

□

We shall use the above to prove Dini's Theorem.

(Dirichlet kernel)

Defn. The N -th partial sum of the Fourier series of f

is defined as

$$\begin{aligned}
 S_N f(x) &:= \sum_{k=-N}^N \hat{f}(k) e^{2\pi i k x} \\
 &= \sum_{k=-N}^N \left\{ \int_0^1 f(t) e^{-2\pi i k t} dt \right\} e^{2\pi i k x} \\
 &= \int_0^1 f(t) \cdot \sum_{k=-N}^N e^{2\pi i k (x-t)} dt
 \end{aligned}$$

Let $D_N(t) := \sum_{k=-N}^N e^{2\pi i k t}$ be the Dirichlet kernel.

Then,

$$\begin{aligned}
 S_N f(x) &= \int_0^1 f(t) D_N(x-t) dt \\
 &= (f * D_N)(x).
 \end{aligned}$$

Some properties

$$\begin{aligned}
 D_N(t) &= \sum_{k=-N}^N e^{2\pi i k t} \\
 &= e^{-2\pi i N t} + e^{-2\pi i (N-1)t} + \dots + e^{2\pi i N t} \\
 &= e^{-2\pi i N t} \left(\frac{1 - (e^{2\pi i t})^{2N+1}}{1 - e^{2\pi i t}} \right) \quad (\text{assuming } e^{2\pi i t} \neq 1) \\
 &= \frac{e^{-2\pi i N t} - e^{(4N+2)\pi i t - (2N)\pi i t}}{1 - e^{2\pi i t}} \\
 &= \frac{e^{-2\pi i N t} - e^{2\pi i t(N+1)}}{1 - e^{2\pi i t}} \cdot \frac{\frac{e^{-\pi i t}}{e^{-\pi i t}}}{\frac{e^{-\pi i t}}{e^{-\pi i t}}} \\
 &= \frac{e^{-\pi i (2N+1)t} - e^{\pi i (2N+1)t}}{e^{-\pi i t} - e^{\pi i t}} \\
 &= \frac{\sin((2N+1)\pi t)}{\sin \pi t}
 \end{aligned}$$

(note that if $e^{2\pi i t} = 1$, then the sum is $2N+1$, work as limit.)

From the summation defⁿ, we get

$$\int_0^1 D_N(t) dt = 1$$

$$\therefore S_N f(x) = \int_{-y_2}^{y_2} f(t) D_N(x-t) dt,$$

(Note that D_n is also periodic with period 1)

$$D_N(t) = \frac{\sin((2N+1)\pi t)}{\sin \pi t}$$

$$\Rightarrow S_N f(x) = \int_{-y_2}^{y_2} f(t) D_N(x-t) dt$$

$$= \int_{x-y_2}^{x+y_2} f(x-t) D_N(t) dt$$

$$= \int_{-y_2}^{y_2} f(x-t) D_N(t) dt$$

$$= \int_{-1/2}^{y_2} f(x-t) \frac{\sin((2N+1)\pi t)}{\sin \pi t} dt$$

Proof of Dini's Theorem. Let $f \in L^1(\mathbb{T})$ and fix x s.t.
 $\exists \delta > 0$

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty.$$

We wish to show $S_N f(x) \xrightarrow{n \rightarrow \infty} f(x)$. (Note x is fixed.)

Note

$$S_N f(x) - f(x) = \int_{-y_2}^{y_2} f(x-t) D_N(t) dt - f(x)$$

$$= \int_{-y_2}^{y_2} f(x-t) D_N(t) dt - \int_{-1/2}^{y_2} f(x) D_N(t) dt$$

$-y_2$

$$= \int_{-y_2}^{y_2} [f(x-t) - f(x)] D_n(t) dt$$

Choose $\delta_1 < \delta$ s.t.

$$\sin \pi t \asymp \pi t \quad \text{if } |t| < \delta_1$$

That is, $\exists C_1, C_2 > 0$ s.t.

$$C_1 \pi t \leq \sin \pi t \leq C_2 \pi t \quad \forall |t| < \delta_1$$

$$\Rightarrow \sum_n [f(x) - f(x)] = \int_{|t| < \delta_1} [f(x-t) - f(x)] \frac{\sin(2N+1)\pi t}{\sin \pi t} dt \quad := I_1$$

$$+ \int_{\frac{1}{2} \geq |t| \geq \delta_1} [] \frac{\sin(\)}{\sin(\)} dt \quad := I_2$$

$$I_1 = \int_{-y_2}^{y_2} \frac{f(x-t) - f(x)}{\sin \pi t} \cdot \chi_{|t| < \delta_1}(t) \sin(2N+1)\pi t dt$$

want to show this is in L^1
Then Riemann-Lebesgue shows that $I_1 \rightarrow 0$.

$$\text{Let } g_1(t) := \frac{f(x-t) - f(x)}{\sin \pi t} \cdot \chi_{|t| < \delta_1}(t)$$

By hypothesis in the Thm,

$$\begin{aligned} \int_0^{y_2} |g_1| &\leq \int_{-y_2}^{y_2} \left| \frac{f(x-t) - f(x)}{\sin \pi t} \right| \chi_{|t| < \delta_1}(t) dt \\ &\leq C_1 \int_{-y_2}^{y_2} \left| \frac{f(x-t) - f(x)}{t} \right| \chi(t) dt \end{aligned}$$

$$\text{For } I_2 : \text{ define } g_2(t) = \frac{f(x-t) - f(x)}{\sin \pi t} \chi_{|t| \geq \delta_1}(t)$$

Again, we show $g_2 \in L^1$. Note $|\sin \pi t| \geq \sin \pi \delta_1$
for $|t| \geq \delta_1$.

Again, we show $g_2 \in L_1$. Note $|\sin \pi t| \geq \sin \pi \delta_1$
 for $\frac{1}{2} > |t| \geq \delta_1$.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |g_2(t)| dt \leq \frac{1}{|\sin \pi \delta_1|} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x-t) - f(x)| \chi_{|t| \geq \delta_1}(t) dt$$

$< \infty$ since f is in L^1

Thus, $g_2 \in L^1(\mathbb{T})$ and $I_2 \rightarrow 0$.

Thus, we have $S_n f(x) - f(x) = I_1 + I_2 \rightarrow 0$.

Hence, $\lim_{n \rightarrow \infty} S_n f(x) = f(x)$. B

(for the fixed x which satisfies Dini's condition.)

C. Suppose $f \in L^1(\mathbb{T})$ and f satisfies Lipschitz condition in a neighbourhood of some $x \in \mathbb{T}$.
 That is,

$$|f(x+t) - f(x)| \leq c \cdot |t|^a$$

for some $c, a > 0$ and $|t| < \delta$ for some $\delta > 0$.

Then, $\lim_{n \rightarrow \infty} S_n f(x) = f(x)$.

Proof. Just need to check Dini's condition. Let δ be as in the Lips. condition.

Then,

$$\int_{|t| < \delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt \leq c \int_{|t| < \delta} \frac{1}{|t|^{1-a}} dt < \infty$$

$(a > 0 \Rightarrow 1-a < 1)$

Now we will prove the first theorem. First we recall the following from Functional Analysis.

- Let X, Y be normed linear spaces.

Let $T: X \rightarrow Y$ be a linear map.

T is bounded (linear map) if $\exists c > 0$ s.t.

$$\|Tx\|_y \leq c \cdot \|x\|_x.$$

Since T is linear, we have

boundedness $\Leftrightarrow T$ is continuous

$$\|T\|_{op} = \inf \{c > 0 : \|Tx\|_y \leq c \|x\|_x \ \forall x\}$$

norm on all
(bounded) linear maps from X to Y

$$= \sup_{x \neq 0} \frac{\|Tx\|_y}{\|x\|_x}$$

(1) any linear $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is always bounded.

(2) derivative $D : C([0,1]) \rightarrow C([0,1])$
in sup norm $f \mapsto f'$

is a linear map which is not bounded.

Proof mainly uses Baire Category Theorem

(Uniform Boundedness principle)

X - Banach space

Y - normed linear space

Let $\{T_\alpha\}_{\alpha \in \Lambda}$ be a collection of linear maps

$$T_\alpha : X \rightarrow Y \text{ s.t.}$$

for each $x \in X$,

$$\sup \{\|T_\alpha x\|_y : \alpha \in \Lambda\} < \infty.$$

Then, $\sup_{\alpha \in \Lambda} \|T_\alpha\|_{op} < \infty$.

Lecture 5 (20-01-2021)

20 January 2021 09:26

Recall that we were asking the question:

When does the Fourier series of f converge to f ?

Dini's Theorem had given one condition for convergence at a point.

We now give a ^{continuous} example for which this does not happen.

To show the existence of such a f , we use the uniform boundedness principle.

Thm. There exists a continuous function f whose Fourier series diverges at a point.

Proof. $X = C(\mathbb{T})$ (space of continuous function on \mathbb{T})
with sup norm \leftarrow Banach space

$Y = \mathbb{R}$ \leftarrow with $|\cdot|$ norm
 \leftarrow NLS

Define, for each $n \in \mathbb{N}$,

$$T_n : X \rightarrow Y \quad \text{by} \quad T_n(f) = S_n f(x).$$

$$\begin{aligned} \text{Recall: } S_n f(x) &= \sum_{k=-n}^n \hat{f}(k) e^{2\pi i k x} \\ &= \int_{-y_2}^{y_2} f(t) D_n(x-t) dt \end{aligned} \quad \left(D_n(-x) = D_n(x) \right)$$
$$\therefore T_n(f) = \int_{-y_2}^{y_2} f(t) D_n(t) dt$$

We first prove that $\{T_n\}_{n \in \mathbb{N}}$ is a family of bounded operators.

$$\begin{aligned} \text{Note } |T_n(f)| &= \left| \int_{-Y_2}^{Y_2} f(t) D_n(t) dt \right| \\ &\leq \|f\|_\infty \int_{-Y_2}^{Y_2} |D_n(t)| dt \\ \Rightarrow \|T_n\|_{op} &\leq \int_{-Y_2}^{Y_2} |D_n(t)| dt \quad \leftarrow \text{finite} \end{aligned}$$

$$\text{Claim. (a)} \quad \|T_n\|_{op} = \int_{-Y_2}^{Y_2} |D_n(t)| dt$$

$$\text{(b)} \quad \lim_{n \rightarrow \infty} \int_{-Y_2}^{Y_2} |D_n(t)| dt = \infty$$

Supposing the Claim is true for now, we complete the proof.

$\{T_n\}_{n \in \mathbb{N}}$ - family of bdd. operators from X to Y .

$$\text{Also, } \|T_n\|_{op} = \int_{-Y_2}^{Y_2} |D_n(t)| dt \rightarrow \infty.$$

By VBP, $\exists f \in C(\overline{T})$ s.t.

$$\sup_{n \in \mathbb{N}} |T_n f| = \infty.$$

This means that the Fourier series of f at 0 diverges.

Proof of the Claim:

(a) Need to prove \geq .

Fix N . Define $g(t) = \begin{cases} 1 & \text{if } D_n(t) \geq 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}$

Step function with finitely many discontinuities

Then, $\exists \{f_j\}_j$ continuous s.t. $f_j \rightarrow g$ (pt-wise, \mathcal{L}_2)

$$\text{with } -1 \leq f_j \leq 1 \quad \forall j.$$

$$\text{Then, } T_n(f_j) = S_N f_j(0) = \int_{-Y_2}^{Y_2} f_j(t) D_n(t) dt$$

$$\lim_{j \rightarrow \infty} T_n(f_j) = \lim_{j \rightarrow \infty} \int_{-Y_2}^{Y_2} f_j(t) D_n(t) dt \quad \xrightarrow{\text{DCT}} |f_j| \leq 1$$

$$= \int_{-Y_2}^{Y_2} \lim_{j \rightarrow \infty} f_j(t) D_n(t) dt$$

$$= \int_{-Y_2}^{Y_2} g(t) D_n(t) dt \quad \xrightarrow{\text{defn of } g}$$

$$\lim_{j \rightarrow \infty} T_n(f_j) = \int_{-Y_2}^{Y_2} |D_n(t)| dt$$

$$\Rightarrow \|T_n\|_{op} = \sup_{f \neq 0} \frac{\|T_n(f)\|}{\|f\|}$$

$$\geq \sup_{f_j \neq 0} \frac{\|T_n(f_j)\|}{\|f_j\|} \geq \sup_{f_j \neq 0} \frac{\|T_n(f_j)\|}{\int_{-Y_2}^{Y_2} |D_n(t)| dt}$$

$$\text{Thus, } \|T_n\|_{op} \geq \int_{-Y_2}^{Y_2} |D_n(t)|, \text{ as desired.}$$

$$(b) \int_{-Y_2}^{Y_2} |D_n(t)| dt = \int_{-Y_2}^{Y_2} \left| \frac{\sin((2n+1)\pi t)}{\sin \pi t} \right| dt$$

$$= 2 \int_0^{Y_2} \left| \frac{\sin((2n+1)\pi t)}{\sin \pi t} \right| dt$$

$$\geq \frac{2}{\pi} \int_0^{\frac{2n+1}{2}} \frac{\sin((2n+1)\pi t)}{t} dt \quad (2n+1)t = s$$

$$\frac{2n+1}{2}$$

$$= \frac{2}{\pi} \cdot \frac{1}{(2n+1)} \int_0^{\frac{2n+1}{2}} \frac{|\sin \pi s|}{s} \cdot (2n+1) ds$$

$$= \frac{2}{\pi} \int_0^{n+\frac{1}{2}} \frac{|\sin \pi s|}{s} ds$$

$$\geq \frac{2}{\pi} \sum_{k=0}^{n-1} \int_k^{k+1} \frac{|\sin \pi s|}{s} ds$$

$$\geq \frac{2}{\pi} \sum_{k=0}^{n-1} \int_k^{k+1} \frac{|\sin \pi s|}{k+1} ds$$

$$= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \int_k^{k+1} |\sin \pi s| ds$$

$$= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \int_0^1 \sin \pi s ds$$

$$= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1}$$

$\rightarrow \infty$
harmonic series

Cesaro summability

Defn: $\sum_{n=1}^{\infty} a_n$ is Cesaro summable to A if

$$\left\{ \frac{s_1 + \dots + s_N}{N} \right\}_n \rightarrow A.$$

$$(s_n = \sum_{k=1}^n a_k)$$

Prop: If $\sum_{n=1}^{\infty} a_n$ converges, then it is Cesaro summable to the same sum.

~~to~~ to the same sum.

Proof. Omitted. \square

Prop. There exists a series which does not converge but it is Cesaro summable.

Proof. let $a_n = (-1)^{n+1}$. Then $\sum a_n$ does not converge.

$$\text{Note } S_n = \begin{cases} 1 & ; n \text{ odd} \\ 0 & ; n \text{ even} \end{cases}$$

$$\Rightarrow \frac{S_1 + \dots + S_N}{N} = \begin{cases} \frac{N/2}{N} = \frac{1}{2} & ; N \text{ even} \\ \frac{(N+1)/2}{N} = \frac{1}{2} + \frac{1}{N} & ; N \text{ odd} \end{cases}$$

$\downarrow n \rightarrow \infty$

θ

Thus, Cesaro summable is more relaxed. We now wish to do so with our Fourier Series.

Cesaro Summability of Fourier Series

$$S_N f(x) = \sum_{k=-N}^N \hat{f}(k) e^{2\pi i k x}.$$

$$\text{Define } \sigma_N f(x) := \frac{1}{N+1} \left(\sum_{k=0}^N S_k f(x) \right).$$

We now wish to see if $\sigma_N f(x)$ converges.

$$\sigma_N f(x) = \frac{1}{N+1} \left(\sum_{k=0}^N \int_0^x f(t) D_k(x-t) dt \right)$$

$$= \left(f(x) \cdot \left(- \sum_{k=0}^N D_k(x) \right) \right) dt$$

$$= \int_0^x f(t) \cdot \left(\frac{1}{N+1} \sum_{k=0}^N D_k(x-t) \right) dt$$

$$= \int_0^x f(t) F_N(x-t) dt = (f * F_N)(x).$$

where $F_N(t) = \frac{1}{N+1} \sum_{k=0}^N D_k(t)$ - Fejer kernel
(Fejer kernel)

$$\begin{aligned} &= \frac{1}{N+1} \sum_{k=0}^N \frac{\sin((2k+1)\pi t)}{\sin \pi t} \\ &= \frac{1}{\sin \pi t} \cdot \frac{1}{N+1} \cdot \sum_{k=0}^N \sin((2k+1)\pi t) \\ &= \frac{1}{(N+1) \sin \pi t} \cdot \frac{\sin^2((N+1)\pi t)}{\sin \pi t} \end{aligned}$$

$$= \frac{1}{N+1} \cdot \left(\frac{\sin((N+1)\pi t)}{\sin \pi t} \right)^2$$

(i) $F_N(t) \geq 0 \quad \forall t \quad \forall N$

(ii) $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t) dt = 1 \quad \forall N$ (from the \sum defn,
we have $\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N = 1$)

(iii) $F_N \quad \delta > 0, \quad \int_{\frac{1}{2} \geq |t| > \delta} F_N(t) dt \rightarrow 0 \quad \text{as } N \rightarrow \infty$

$$\int_{\delta \leq |t| \leq \frac{1}{2}} F_N(t) dt \leq \frac{1}{\sin^2 \pi \delta} \cdot \frac{1}{N+1} \int_{\delta \leq |t| \leq \frac{1}{2}} (\sin((N+1)\pi t))^2 dt$$

$$\leq \left(\frac{2(\frac{1}{2} - \delta)}{\sin^2 \pi \delta} \right) \cdot \frac{1}{N+1} \rightarrow 0.$$

Cor. $\{F_N\}_N$ is an approximate identity in L'

Tm. (1) If $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$, then

$$\|f - f * F_N\|_p \rightarrow 0 \text{ as } N \rightarrow \infty.$$

(That is, $\|\sigma_N f - f\|_p \rightarrow 0$ as $N \rightarrow \infty$.)

(2) Let f be a continuous function. Then,

$$(f - f * F_N) \rightarrow 0 \text{ uniformly.}$$

That is, $\sigma_N f \rightarrow f$ uniformly.

($\|\cdot\|_\infty$ is sup norm.)

This proves that the Fourier series of f is Cesaro summable to f , if f is continuous.

Ex. $F_N(t) = \frac{1}{N+1} \sum_{k=0}^N D_k(t)$ $\left(\begin{array}{l} \text{recall} \\ D_k(t) = \sum_{m=k}^N e^{-2\pi i m t} \end{array} \right)$

$$= \sum_{j=-N}^N \left(1 - \frac{|j|}{N+1} \right) e^{2\pi i j t}$$

$$\therefore f * F_N(t) = \sum_{j=-N}^N \left(1 - \frac{|j|}{N+1} \right) (\underbrace{f * e^{2\pi i j t}}_?)$$

$$\begin{aligned} (f * e^{2\pi i j t})(x) &= \int f(t) e^{2\pi i j(x-t)} dt \\ &= e^{2\pi i j x} \hat{f}(j) \end{aligned}$$

$$\therefore f * F_N(x) = \sum_{j=-N}^N \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j) e^{2\pi j x}$$

Defn: A trigonometric polynomial of degree n is of the form

$$\sum_{j=-N}^N a_k e^{2\pi j x}.$$

For. (1) The trigonometric polynomials are dense in $L^p(\mathbb{T})$ for $1 \leq p < \infty$.

(2) The trigonometric functions are dense in $C(\mathbb{T})$ with sup norm.

For. Suppose $f \in L^1(\mathbb{T})$ and $\hat{f}(k) = 0 \quad \forall k \in \mathbb{Z}$.

Then, $f = 0$ in L^1 .

(If f is continuous, then $f = 0$.)

$$\left(\begin{aligned} \hat{f}(k) = 0 \quad \forall k &\Rightarrow f * F_N = 0 \\ &\Rightarrow \|f - f * F_N\|_1 = \|\hat{f}\|_1 \rightarrow 0 \\ &\Rightarrow \|f\|_1 = 0. \end{aligned} \right)$$

Defn: Hilbert space

X with \langle , \rangle s.t. $(X, \|\cdot\|)$ is complete (Banach).
wrt \langle , \rangle

ℓ_p for $p = 2$ only is Hilbert space. } can't define
wrt \langle , \rangle otherwise

\mathbb{C}^n , \mathbb{R}^n are Hilbert spaces.

Defn.: Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

A sequence $\{e_n\}$ in H is orthogonal if $\langle e_i, e_j \rangle = 0 \quad \forall i \neq j$.

• orthonormal if orthogonal
and $\langle e_i, e_i \rangle = 1 \quad \forall i$.

We say $\{e_n\}$ is complete orthonormal (or orthonormal basis or o.n.b.) if

- (i) $\{e_n\}$ are orthonormal
- (ii) $\langle f, e_n \rangle = 0 \quad \forall n \Rightarrow f = 0$

Tm.: let $(H, \langle \cdot, \cdot \rangle)$ be an Hilbert space and $\{e_n\}_n$ be an o.n.b. Then, for $f \in H$,

$$(i) \quad f = \sum_n \langle f, e_n \rangle e_n,$$

$$(ii) \quad \|f\|^2 = \sum_n |\langle f, e_n \rangle|^2.$$

Consider $(L^2(\mathbb{T}), \langle \cdot, \cdot \rangle)$ where

$$\langle f, g \rangle := \int_0^1 f(t) \overline{g(t)} dt$$

$$\langle f, f \rangle = \|f\|_{L^2(\mathbb{T})}^2$$

$\therefore L^2(\mathbb{T})$ is a Hilbert space.

Tm

$\{t \mapsto e^{2\pi i nt}\}_{n \in \mathbb{Z}}$ is an o.n.b for $L^2(\mathbb{T})$.

Proof: Put $e_n(t) := e^{2\pi i n t}$.

Try to see that $\langle e_n, e_m \rangle = \begin{cases} 0 & ; n \neq m \\ 1 & ; n = m \end{cases}$

Now, suppose $\langle f, e_n \rangle = 0 \quad \forall n$ for $f \in L^2(\mathbb{T})$.

To show: $f = 0$

$$\int f(t) e^{-2\pi i n t} dt = 0 \quad \forall n$$

$$\Rightarrow \hat{f}(n) = 0 \quad \forall n$$

$$\Rightarrow f = 0 \quad \text{in } L^2.$$

□

Lecture 6 (22-01-2021)

22 January 2021 09:33

Recall:

$$f_N(t) = \frac{1}{N+1} \sum_{k=0}^N D_k(t)$$

(1) Trig. polynomials are dense in $L^p(\mathbb{T})$, $1 \leq p < \infty$
and also in $L^1(\mathbb{T})$.

(2) If $f \in L^1(\mathbb{T})$ and $\hat{f}(k) = 0 \quad \forall k \in \mathbb{Z}$,
then $f = 0$ in L^1 .

$$L^2(\mathbb{T}) : \langle f, g \rangle := \int_0^1 f(t) \overline{g(t)} dt$$

(inner product)

$$\|f\|_2^2 = \langle f, f \rangle.$$

$$\text{Let } e_n(t) := e^{2\pi i n t}, \quad t \in [0, 1]$$

Then, $e_n \in L^2(\mathbb{T})$.

$$\begin{aligned} 1. \quad \langle e_n, e_m \rangle &= \int_0^1 e_n \overline{e_m} \\ &= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases} \end{aligned}$$

$\therefore \{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$.

2. Claim. $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T})$.

That is, $\langle f, e_n \rangle = 0 \quad \forall n \in \mathbb{Z} \Rightarrow f = 0 \in L^2$.

Note that $f \in L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$ and

$$0 = \langle f, e_n \rangle = \int f \bar{e}_n = \hat{f}(n) \quad \forall n \in \mathbb{Z}$$

$$\Rightarrow f = 0 \text{ in } L^2$$

Thus, $\{e_n\}_{n \in \mathbb{Z}}$ is an o.n.b. for $L^2(\mathbb{T})$

Then (by the Hilbert space theorem):

$$(1) \quad f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n, \text{ i.e.,}$$

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n, \text{ i.e.,}$$

$$f(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n t}.$$

Equality in $L^2(\mathbb{T})$

$$\left(\text{That is, } \left\| f - \sum_{n=-N}^N \hat{f}(n) e_n \right\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty \right)$$

$$(2) \quad \|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$

↳ Parseval's Theorem

This means that the map

$$\mathcal{F} : L^2(\mathbb{T}) \rightarrow l_2(\mathbb{Z})$$

$$\mathcal{F}(f) = \{\hat{f}(n)\}_{n \in \mathbb{Z}}$$

is an isometry.

In particular, it will be one-one.

Is it onto? (Yes. Needs an argument.)

SUMMARY SO FAR:

- (Q) Does the Fourier series of f converge to f ?
1. We showed that if f satisfies Dini's theorem, then $S_n f \rightarrow f$ at that point.
 2. \exists a continuous function f whose Fourier series does not converge to f at 0.
 3. We looked at Cesaro summation. Then,

$$\sigma_n f \rightarrow f \text{ in } L^p(\mathbb{T}), \quad 1 \leq p < \infty$$
 and $\sigma_n f \rightarrow f \text{ uniformly if } f \in \ell(\mathbb{T}).$
 4. (Q) Does $\|S_n f - f\|_p \rightarrow 0$ as $N \rightarrow \infty$?

$$1 \leq p < \infty$$

L^p convergence of Fourier series)

That is,

$$S_n f \rightarrow f \text{ in } L^p \text{ as } n \rightarrow \infty?$$

(Ans) Not true if $p = 1$. True if $1 < p < \infty$.
Going to prove this

Remark. Note that for $p = 2$, we have shown it above as $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$.
 (The Hilbert Space theory.)

Fix $p \in [1, \infty]$

Lemma 1. $S_n f \rightarrow f \text{ in } L^p(\mathbb{T}) \quad \forall f \in L^p$
 $\Leftrightarrow \exists C_p > 0$ (independent of N) s.t. $\forall f \in L^p$

$$\|S_n f\|_p \leq C_p \|f\|_p \quad \forall N$$

$$(\text{i.e., } \|S_n\|_p \leq C_p)$$

Proof. (\Rightarrow) Suppose $S_n f \rightarrow f$ in L^p , $1 < p < \infty$.

That is, $\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}$ s.t.

$$\|S_N f - f\|_p < \varepsilon \quad \forall N \geq N_0$$

Hence, $\|S_N f\|_p = \|S_N f - f + f\|_p < \varepsilon + \|f\|_p$
 $\forall N \geq N_0$

\therefore For a fixed $f \in L^p$:

$$\sup_{n \in \mathbb{N}} \|S_n f\|_p < \infty$$

\therefore By VBP, $\sup_n \|S_n\|_{op}$ is finite.

$$c_p = \sup_n \|S_n\|_{op} \text{ works.}$$

(\Leftarrow) Suppose $\exists c_p > 0$ s.t.

$$\|S_N f\|_p \leq c_p \|f\|_p \quad \forall N.$$

We will prove $S_n f \rightarrow f$ in L^p .

Let g be a trig poly of degree M , i.e.,

$$g(t) = \sum_{k=-M}^M a_n(k) e^{2\pi i k t}$$

$$\begin{aligned} S_N g(t) &= \sum_{l=-N}^N \hat{g}(l) e^{2\pi i l t} \\ &= \sum_{l=-N}^M a_n(l) e^{2\pi i l t} \quad \text{if } N \geq M \\ &= g(t) \end{aligned} \quad \left| \begin{array}{l} \hat{g}(l) = \int_0^1 g(t) e^{-2\pi i l t} dt \\ = \sum_{k=-M}^M \int_0^1 a_n(k) e^{2\pi i (k-l)t} dt \\ = \begin{cases} a_n(l) & M \geq |l| \\ 0 & M < |l| \end{cases} \end{array} \right.$$

$$\therefore S_N g = g \quad \text{if} \quad N \geq \deg g.$$

Now we use the fact that trigonometric polynomials
are dense in L^p . Given $f \in L^p$. That is, for $\epsilon > 0$,

$\exists g \rightarrow \text{trig polyomial s.t.}$

$$\|f - g\|_p < \epsilon.$$

Let $N \geq \deg g$. Then,

$$\begin{aligned} \|S_N f - f\|_p &= \|S_N f - S_N g + S_N g - g + g - f\|_p \\ &\stackrel{=} {\|S_N f - S_N g\|_p + \|g - f\|_p} \\ &\leq \|S_N(f - g)\|_p + \|f - g\|_p \\ &\leq C_p \|f - g\|_p + \|f - g\|_p \\ &< (1 + C_p) \epsilon. \end{aligned}$$

Thus, $S_N f \rightarrow f$ in L^p .

- For $p = 1$, we will see that $\|S_N\|_{L^1 \rightarrow L^1} = \int_{-y_2}^{y_2} |D_N|$.

We had seen that RHS $\rightarrow \infty$ as $N \rightarrow \infty$.

This will show $S_N f \rightarrow f$ in L^1 is not true
for all f , by Lemma 1.

(That is, $\exists f \in L^1$ s.t. $S_N f \not\rightarrow f$ in L^1 .)

- Harmonic function

$f: \Omega \rightarrow \mathbb{R}$, f continuous and $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ exist.

Let $\Delta f := \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$.

$$\text{Let } \Delta f := \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f.$$

If $\Delta f = 0$, then f is called harmonic.

- Dirichlet problem

$$D := \{z \in \mathbb{C} : |z| < 1\}.$$

Then, $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and

$$\partial D = \{z \in \mathbb{C} : |z| = 1\}.$$

Given $f: \partial D \rightarrow \mathbb{R}$ continuous function,

consider $\begin{cases} \Delta u = 0 & \text{on } D \text{ and } u \text{ harmonic} \\ u = f & \text{on } \partial D. \end{cases}$

Then, does u exist? What is u ?

We have

$$(P_r f(\theta)) := u(r e^{2\pi i \theta}) = \int_0^{2\pi} p_r(\theta - t) f(e^{2\pi i t}) dt$$

(Poisson integral)

$$(0 \leq r \leq 1, 0 \leq \theta \leq 2\pi) = (P_r * f)(\theta)$$

- where P_r is called Poisson kernel

$$p_r(\theta) := \sum_{n=-\infty}^{\infty} |r|^n e^{2\pi i n \theta}.$$

$$= \operatorname{Re} \left(\frac{1+z}{1-z} \right), \quad \text{where } z = r e^{2\pi i \theta}$$

$$= \frac{1-r^2}{1-r \cos(2\pi \theta) + r^2} = \frac{1-r^2}{|1-r e^{2\pi i \theta}|^2}$$

- Properties of Poisson Kernel

$$(a) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta) d\theta = 1$$

$$(b) p_r(\theta) > 0 \quad \forall \theta \quad \forall 0 \leq r < 1$$

$$(c) p_r(\theta) = p_r(-\theta)$$

$$(d) p_r(\theta) < p_r(\delta) \quad \text{if } 0 < \delta < |\theta| \leq \pi.$$

$$(e) \text{ For each } \delta > 0, \lim_{r \rightarrow 1^-} p_r(\theta) = 0$$

uniformly in θ for $\delta \leq |\theta| \leq \pi$.

$$(f) \lim_{r \rightarrow 1^-} \int_{\delta \leq |\theta| \leq \pi} p_r(\theta) d\theta = 0.$$

• Hence $\{p_r\}_r$ is an approximate identity in $L^1(\mathbb{T})$
and

$$\|f * p_r - f\|_p \rightarrow 0 \quad \text{as } r \rightarrow 1.$$

(Not a sequence, a continuous family.)

Poisson integral

for $f \in C(\mathbb{T})$

$$(P_r f)(t) = (p_r * f)(t)$$

$$= \left(\sum_{n \in \mathbb{Z}} r^{|n|} e^{2\pi i n t} \right) * f(t)$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \left(e^{2\pi i n t} * f \right)(t)$$

$$P_r f(t) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{2\pi i n t}$$

defined on \mathbb{T} ,
will extend to \mathbb{D} by t with $e^{2\pi i t}$

Conjugate Poisson integral

$$f \in C(\mathbb{T}).$$

$$\begin{aligned} Q_n f(t) &:= (-i) \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) \hat{f}(n) r^{|n|} e^{2\pi i n t} \\ &= (g_r * f)(t) \quad \operatorname{sgn}(n) = \begin{cases} -1 & n < 0 \\ 0 & n = 0 \\ 1 & n > 0 \end{cases} \end{aligned}$$

where

$$g_r(t) = -i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) r^{|n|} e^{2\pi i n t}$$

(Conjugate Poisson kernel)

Note

$$\int_0^1 g_r(t) dt = 0$$

$$\cdot P f(t) = \sum_{k=0}^{\infty} \hat{f}(k) e^{2\pi i k t} \quad \text{for } f \in C(\mathbb{T}).$$

Extend P to L^2 functions

Can do so since $\|P f\|_2^2 \leq \sum_{k=0}^{\infty} |\hat{f}(k)|^2$

$$\leq \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 = \|f\|_2^2$$

Aim: $S_N f \rightarrow f$ in L^p for all f ($1 < p < \infty$)
 Have proven:

Lemma 1. $S_N f \rightarrow f \quad \forall f \in L^p$
 $\Leftrightarrow \|S_N\|_{L^p} \leq C_p \quad \forall N$

will show: Lemma 2. S_N is uniformly bounded on L^p
 $\Leftrightarrow P: L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$ is a

bounded linear operator

Observe:

$$\begin{aligned} P_r f(t) + i Q_r f(t) &= \sum_{k=1}^{\infty} 2 \hat{f}(k) r^k e^{2\pi i k t} + \hat{f}(0) \\ &= 2 P f(r e^{2\pi i t}) - \hat{f}(0) \\ &\quad (\text{extend } Pf \text{ to ID.}) \end{aligned}$$

Lemma 3. $P: L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$ is bounded linear

$\Leftrightarrow P_r, Q_r$ are uniformly bounded on $L^p(\mathbb{T})$
 $\forall 0 < r < 1$.

$$\left(\text{i.e., } \|P_r f\|_p \leq C \|f\|_p \quad \forall 0 < r < 1, \quad \|Q_r f\|_p \leq C' \|f\|_p \quad \begin{matrix} C, C' \\ \text{independent of } r \end{matrix} \right)$$

Lemma 4. $\|P_r f\|_p \leq \|f\|_p \quad \forall 0 < r < 1, \quad 1 \leq p < \infty$

Lemma 5. $\|Q_r f\|_p \leq C \|f\|_p \quad \forall 0 < r < 1, \quad 1 \leq p < \infty$

Lecture 7 (27-01-2021)

27 January 2021 09:29

Recall

Lem 1. $S_N f \rightarrow f$ if f in L_p $\Leftrightarrow \exists C_p > 0$ s.t. $\|S_N f\|_p \leq C_p \|f\|_p$

Lem 2. S_N is uniformly bounded on $L^p(\mathbb{T})$
 $\Leftrightarrow P: L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$ extends to a bounded linear operator

$$e_n(t) := e^{2\pi i n t} ; S_N f(t) = \sum_{k=-N}^N \widehat{f}(k) e^{2\pi i k t}$$

$$\begin{aligned} \widehat{f}(k-N) &= \int_0^1 f(t) e^{-2\pi i (k-N)t} dt \\ &= \sum_{k=0}^{2N} \widehat{f}(k-N) e^{2\pi i (k-N)t} \\ &= \int_0^1 (f(t) e^{2\pi i Nt}) e^{-2\pi i kt} dt = e_{-N}(t) \sum_{k=0}^{2N} \widehat{e_N f}(k) e^{2\pi i k t} \\ &= \widehat{e_N f}(k) \end{aligned}$$

$$\Rightarrow e_n(t) S_N f(t) = \sum_{k=0}^{2N} \widehat{f} \cdot e_n(k) e_k(t)$$

$$= P_{2N}(f \cdot e_{-N})(t)$$

where $P_N f(t) = \sum_{k=0}^N \widehat{f}(k) e_k(t) = \sum_{k=0}^N \widehat{f} \cdot e_N(k) e_k(t)$

$$\therefore e_n \cdot S_N f = P_{2N}(f \cdot e_{-N})$$

$\therefore S_N$ is uniformly bounded $L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$
 $\Leftrightarrow P_{2N}$ is linear

Proof (\Rightarrow) Suppose S_N is uniformly bounded on $L^p(\mathbb{T})$.
~~of lemma~~ $\therefore P_{2N}$ is $\xrightarrow{\text{u}}$ on $L^p(\mathbb{T})$

$$\therefore \|P_{2N}f\|_p \leq c \|f\|_p \quad \forall f \in L^p$$

c indep. of N

Let f be a trigonometric polynomial of deg. m , i.e.,

$$f = \sum_{k=-m}^m c_k e_k$$

$$\hat{f}(p) = 0 \quad \text{if } |p| > m$$

$$\begin{aligned} \Rightarrow P_{2N}(f \cdot e_{-2N})(t) &= \sum_{k=0}^{2N} \hat{f}(k) e_k(t) \\ &= \sum_{k=0}^{\infty} \hat{f}(k) e_k(t) \quad \text{if } N > \deg f \\ &= Pf \end{aligned}$$

$$\therefore P_{2N}(f \cdot e_{-2N}) = Pf \quad \forall \text{ trig poly } f \text{ with } \deg f < N$$

$$\begin{aligned} \Rightarrow \|Pf\|_p &= \|P_{2N}(f \cdot e_{-2N})\|_p \\ &\leq c \cdot \|f \cdot e_{-2N}\|_p \\ &= c \cdot \|f\|_p \quad \leftarrow \text{indep of } N \end{aligned}$$

Thus, $\|Pf\|_p \leq c \cdot \|f\|_p$ \forall trig polynomials f
 Use density of trig poly to conclude the above
 is true for all f .

(\Leftarrow) Suppose $\|Pf\|_p \leq c \|f\|_p \quad \forall f \in L^p$

Let f be a trig poly. Then

$$P_{2N}(f \cdot e_{-2N}) = Pf \quad \text{if } N > \deg f$$

$$\text{or } P_{2N}(f) = P(f \cdot e_{2N}) \text{ if } N > \deg f.$$

Fix an f with $\deg f < n$

Then,

$$\begin{aligned} \|P_{2N}(f)\|_p &= \|P(f \cdot e_{2N})\| \leq \|P\|_{op} \|f \cdot e_{2N}\|_p \\ &= \|P\|_{op} \|f\|_p \end{aligned}$$

Use the denseness of trig poly in L^p to get

$$\sup_n \|P_{2N}(f)\| \leq C_f < \infty$$

so in S_N .

↗

Extension of Pf : $\underset{t \in [0, 1]}{Pf(t)} = \sum_{k=0}^{\infty} \hat{f}(k) e^{2\pi i k t} ; f \in \ell(\pi)$

$$\text{Let } z = r e^{2\pi i t}, \quad 0 \leq r < 1, \quad t \in [0, 1]$$

$$\begin{aligned} Pf(z) &= Pf(re^{2\pi i t}) := \sum_{k=0}^{\infty} \hat{f}(k) r^k e^{2\pi i k t} \\ &= \sum_{k=0}^{\infty} \hat{f}(k) z^k. \end{aligned}$$

Thus, Pf is holomorphic on D . Moreover, the convergence is uniform on compact subsets.

Recall: $P_r f(t) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e_k(t)$

$$Q_r f(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) r^{|k|} \hat{f}(k) e_k(t)$$

$$P_r f(t) + i Q_r f(t) = 2 P f(r e^{2\pi i t}) - \hat{f}(0). \quad \text{--- (1)}$$

Lem 3. If P_r and Q_r are uniformly bounded on L^p $\forall 0 \leq r < 1$, then $P: L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$ is bounded.

Proof.

$$\begin{aligned} \|P_r f\| &\leq c \cdot \|f\|_p \quad \forall r \in [0, 1] \\ \|Q_r f\| &\leq c' \cdot \|f\|_p \quad \forall r \in [0, 1] \end{aligned}$$

Fix an r . Then, $t \mapsto r e^{2\pi i t}$ is a function

$$\Rightarrow \|P f(r e^{2\pi i \cdot})\|_p \leq c \cdot \|f\|_p \quad \forall r$$

by (1)

Take $r \rightarrow 1^-$. (Justify exchange using DCT.)

$$\Rightarrow \|P f\|_p \leq C \|f\|_p$$

□

Lem 4. $\|P_r f\| \leq \|f\|_p \quad \forall r \quad \forall 1 \leq p < \infty$

Proof. We know $P_r f = p_r * f$ and that $\{p_r\}$ is an approximate identity.

$$\therefore \|P_r f\|_p = \|f * p_r\|_p \leq \|p_r\|_1 \|f_p\| = \|f_p\| \quad \left(\|fg\|_p \leq \|f\|_1 \|g\|_p \right)$$

$$\Rightarrow \|P_r f\|_p \leq \|f_p\|$$

Lemma 5. $\|Q_r f\|_p \leq c_p \|f\|_p \quad \forall r \quad \forall 1 < p < \infty$

Proof. $Q_r f = f * q_r$ where $q_r \rightarrow$ conjugate Poisson kernel
and

Proof. $Q_r f = f * q_r$ where $q_r \rightarrow$ conjugate Poisson Kernel
and

$$q_r(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) r^{|k|} e_n(t)$$

$$= \operatorname{Im} \left(\frac{1}{1 - re^{2\pi i t}} \right)$$

$$= \frac{2r \sin 2\pi t}{1 + r^2 - 2r \cos 2\pi t}$$

$$\int_0^1 q_r(t) dt = 0$$

• Let $f \geq 0$ and $1 < p < 2$:

$$F(z) = f(re^{2\pi i t}) := P_r f(t) + i Q_r f(t) \quad \leftarrow \text{holomorphic}$$

$$= f * (P_r + i Q_r) \quad \text{on } \mathbb{D}$$

$$\therefore \operatorname{Re} F(z) = (f * P_r)(t) > 0$$

Then, let $G(z) := f(z)^p = e^{p \log f(z)}$

Note that

well defined
since f is
non-vanishing
and \mathbb{D} is simply
connected

$$G(0) = (f(0))^p$$

$$= \hat{f}(0)$$

Let $\gamma < \pi/2$ and $\frac{\pi}{2} < pr < \pi$.
(Possible since $1 < p < 2$)

$$\text{let } A_\gamma = \{t \in [0, 1] : \arg f(re^{2\pi i t}) < \gamma\}$$

$$\text{and } B_\gamma = [0, 1] \setminus A_\gamma$$

We prove:

$$\int_0^1 |f(re^{2\pi i t})|^p dt \leq c \cdot \|f\|_p^p$$

The well ... $\int_0^1 |f(re^{2\pi i t})|^p dt \leq r \cdot \|f\|_h^p$

$$\text{This would give } \int_0^r |Q_m f(re^{2\pi it})|^p dt \leq c \|f\|_p^p$$

$$\int_0^r |Q_r f(t)|^p dt$$

- $f(re^{2\pi it}) = |f(re^{2\pi it})| \cdot e^{i \arg f(re^{2\pi it})}$

$$\therefore \operatorname{Re} f(re^{2\pi it}) = |f(re^{2\pi it})| \cos(\arg(f(re^{2\pi it})))$$

for $t \in A_\gamma$, $\geq |f(re^{2\pi it})| \cos \gamma$

Hence, $|f(re^{2\pi it})| \leq \frac{1}{\cos \gamma} \operatorname{Re} f(re^{2\pi it})$ for $t \in A_\gamma$.

$$\begin{aligned} \therefore \int_{A_\gamma} |f(re^{2\pi it})|^p dt &\leq (\cos \gamma)^{-p} \int_{A_\gamma} \operatorname{Re} f(re^{2\pi it})^p dt \\ &= (\cos \gamma)^{-p} \int_{A_\gamma} (\operatorname{Re} f(t))^p dt \quad \left. \begin{array}{l} \operatorname{Re} f > 0 \\ \text{since } f \neq 0 \text{ and } A_\gamma \subset [0, 1] \end{array} \right) \\ &\leq (\cos \gamma)^{-p} \int_0^1 (\operatorname{Re} f(t))^p dt \\ &\leq (\cos \gamma)^{-p} \|f\|_p^p \quad \leftarrow \text{independent of } r \end{aligned}$$

- $\operatorname{Re} G(re^{2\pi it})$ — is a harmonic function and hence, it satisfies the mean value property, i.e.,

$$\int_0^1 \operatorname{Re} G(re^{2\pi it}) dt = \operatorname{Re} G(0) = \hat{f}(0)$$

- $\operatorname{Re} G(re^{2\pi it}) = |f(re^{2\pi it})|^p \cos(p \cdot \arg f(re^{2\pi it}))$

$$\begin{aligned} \int_{A_\gamma} |\operatorname{Re} G(re^{2\pi i t})| dt &= \int_{A_\gamma} |f(re^{2\pi i t})|^p |\cos(\operatorname{p.arg}(f(re^{2\pi i t})))| dt \\ &\leq C \cdot \int_{A_\gamma} |f(re^{2\pi i t})|^p dt \end{aligned}$$

$\leftarrow \because \frac{\pi}{2} < \operatorname{p.arg} f(z) < \pi$

$$\begin{aligned} \left| \int_{B_\gamma} \operatorname{Re} G(re^{2\pi i t}) dt \right| &\leq \hat{f}(0) + \int_{A_\gamma} |\operatorname{Re} G(re^{2\pi i t})| dt \\ &\leq C \cdot \int_{A_\gamma} |f(re^{2\pi i t})|^p dt \end{aligned}$$

$$|G(re^{2\pi i t})| = |\operatorname{Re} G(re^{2\pi i t})| \cdot [\cos(\operatorname{p.arg} f(z))]^{-1}$$

$$\begin{aligned} \text{If } t \in B_\gamma, \quad \operatorname{arg} f(z) &\geq r \\ \therefore \operatorname{p.arg} f(z) &> pr > \pi/2 \end{aligned}$$

$$|\operatorname{Re} f(z)| \Rightarrow |\operatorname{arg} f(z)| < \pi/2$$

$$\therefore \frac{\pi}{2} < \operatorname{p.arg} f(z) < \frac{\pi}{2} + \pi < \pi$$

$$\begin{aligned} \therefore |G(re^{2\pi i t})| &= |\operatorname{Re} G(z)| \cdot |\cos(\operatorname{p.arg} f(z))|^{-1} \\ &\leq |\cos pr|^{-1} \cdot (-\operatorname{Re} G(z)) \end{aligned}$$

$$\begin{aligned} \int_{B_\gamma} |G(re^{2\pi i t})| dt &\leq |\cos pr|^{-1} \left| \int_{B_\gamma} \operatorname{Re} G(z) \right| \end{aligned}$$

$$\leq C \cdot |\cos pr|^{-1} \int_{A_\gamma} |f(z)|^p$$

$$\leq C \cdot \|f\|_p^p$$

$$\therefore \int_0^1 |f(re^{2\pi i t})|^p dt \leq C \|f\|_p^p \quad \forall r$$

$$\text{Thus, } \int_0^1 |Q_r f|^p dt \leq c \|f\|_p^p \quad \forall r$$

Can extend this for all f by

$$f = f^+ - f^- + i(f_i^+ - f_i^-)$$

• $p = 2$ works by Hilbert space theory

• $p > 2$ works by duality.

Lecture 8 (29-01-2021)

29 January 2021 09:26

$$\text{For } p > 2 : Q_r f(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) \hat{f}(k) e^{2\pi i k t}$$

$$\langle Q_r^* f, g \rangle := \langle f, Q_r g \rangle \quad (\text{in } L^2)$$

adjoint ↗

$$\text{Then, } Q_r^* = -Q_r.$$

$$\frac{1}{p} + \frac{1}{p'} = 1 \\ p > 2 \Rightarrow p' < 2$$

$$\|Q_p f\| = \sup_{\|g\|_{p'} \leq 1} \left| \int Q_r f \cdot \bar{g} \right| \quad (g \in L^{p'})$$

$$= \sup_{\|g\|_{p'} \leq 1} \left| \int f \cdot \overline{Q_r^* g} \right| \quad \xrightarrow{Q_r^* = -Q_r} 0$$

$$= \sup_{\|g\|_{p'} \leq 1} \left| \int f \cdot \overline{Q_r g} \right|$$

$$\leq \sup_{\|g\|_{p'} \leq 1} \|f\|_p \|Q_r g\|_{p'} \quad \xrightarrow{\text{since } p' < 2, \text{ we can't part}}$$

$$\leq \sup_{\|g\|_{p'} \leq 1} \|f\|_p c \cdot \|g\|_{p'} \quad \xrightarrow{\sup = 1}$$

$$\|Q_p f\| \leq c \cdot \|f\|_p$$

Then, we are done for all $p \in (1, 2) \cup (2, \infty)$. \square
 For $p = 2$, we already knew convergence of partial.

For $p = 1$?

We show $\|S_n\|_{L^1}$ is not uniformly bounded.
 $(S_n : L^1 \rightarrow L^1)$

$$S_N f(t) = \sum_{k=-N}^N \hat{f}(k) e^{2\pi i k t}$$

$$= (f * D_n)(t)$$

$$\therefore \|S_n f\|_1 = \|f * D_n\|_1 \leq \|f\|_1 \cdot \|D_n\|_1$$

$$\therefore \|S_n\|_{op} \leq \|D_n\|_1$$

$$\begin{aligned}\|S_n\|_{op} &= \sup_{\|f\|_1 \neq 0} \frac{\|S_n f\|_1}{\|f\|_1} \\ &\geq \sup_M \frac{\|S_n F_M\|_1}{\|F_M\|_1} \\ &= \|D_n\|_1\end{aligned}$$

(fns are Fejér kernel)
 $(S_n F_M = F_M * D_N \rightarrow D_N \text{ in } L^1)$

$$\therefore \|S_n\|_{op} = \|D_n\|_1.$$

For $\|D_n\|_1$, we have already shown divergence.

Thus, we are done.

(For $p > 1$, the argument does not hold since $\|D_n\|_p \not\rightarrow 0$.)

Conclusion

For $p \in (1, \infty)$: $S_n f \rightarrow f$ in L^p $\forall f \in L^p$

For $p = 1$: $\exists f \in L^1$ s.t. $S_n f \not\rightarrow f$ in L^1

For $p = [1, \infty)$: $\sigma_n f \rightarrow f$ in L^p $\forall f \in L^p$

For $p = \infty$: $\sigma_n f \rightarrow f$ in L^∞ if f continuous

Isoperimetric problem

Given a string of length $L > 0$, find the maximum area enclosed by the string.
 What is the position?

$$\underline{\text{Ans.}} \quad \text{Position: circle, area} = \pi \left(\frac{L}{2\pi} \right)^2 = \frac{L^2}{4\pi}$$

Defn. A parameterized curve is a function $\gamma: [a, b] \rightarrow \mathbb{R}^2$

which is continuous.

γ is closed if $\gamma(a) = \gamma(b)$.

γ is simple if $\gamma|_{[a, b]}$ is injective.

We will take simple closed γ which is C^1 . We assume $\gamma'(t) \neq 0 \quad \forall t \in [a, b]$.

The length $L(\gamma)$ of a curve γ is defined as

$$\begin{aligned} L(\gamma) &:= \int_a^b |\gamma'(t)| dt \\ \gamma &= (\gamma_1, \gamma_2) \\ &= \int_a^b \left[(\gamma_1'(t))^2 + (\gamma_2'(t))^2 \right]^{\frac{1}{2}} dt \end{aligned}$$

The area $A(\gamma)$ enclosed by γ :

$$\begin{aligned} A(\gamma) &:= \frac{1}{2} \left| \int \gamma_1 dy - \gamma_2 dx \right| \\ &= \frac{1}{2} \left| \int_a^b \gamma_1 \gamma_2' - \gamma_1' \gamma_2 \right| \end{aligned}$$

- The length and area above is independent of parametrisation.
- We assume the orientation is such that it moves at a constant speed, i.e., $|\gamma'|$ is constant.
(Can show such a parameterisation exists.)
- We will assume $\gamma: [0, 1] \rightarrow \mathbb{R}^2$.

Write $\gamma(t) = (x(t), y(t))$ and the speed as u , i.e,

$$(x(t))^2 + (y(t))^2 = u^2 \quad \forall t \in [0, 1].$$

Let the length of curve $\ell(\gamma)$ be L .

We have $\gamma(0) = \gamma(1)$. Thus, we can extend it periodically.

Note that $L = \ell(\gamma)$
 $= \int |y'(t)| dt = \int u dt = u$.

Thus,

$$|x'(t)|^2 + |y'(t)|^2 = L^2. \quad \text{--- (1)}$$

$t \mapsto x(t)$ is periodic, $x(0) = x(1)$

$t \mapsto y(t)$ is periodic, $y(0) = y(1)$

x, y are both C^1

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t} \quad \forall t \in [0, 1] \quad \left(\begin{array}{l} x \in C^1 \text{ and thus,} \\ \text{it satisfies Lipschitz} \\ \text{condition for Dini's} \end{array} \right)$$

$$\text{Also, } y(t) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n t} \quad \forall t \in [0, 1]$$

The convergence is uniform and one can show

$$x'(t) = \sum_{n=-\infty}^{\infty} (2\pi i) a_n n e^{2\pi i n t}$$

$$y'(t) = \sum_{n=-\infty}^{\infty} (2\pi i) b_n n e^{2\pi i n t}$$

$$\int_0^1 |x'(t)|^2 dt = \sum_{n=-\infty}^{\infty} 4\pi^2 |a_n|^2 n^2 \quad (\text{by Parseval})$$

$$\text{(ii)} \quad \int_0^1 |y'(t)|^2 dt = \sum_{n=-\infty}^{\infty} 4\pi^2 |b_n|^2 n^2$$

$$\therefore \int L^2 dt = L^2 = 4\pi^2 \sum_{n=-\infty}^{\infty} n^2 (|a_n|^2 + |b_n|^2) \quad [\text{by (1)}]$$

$$\text{or} \quad \sum_{n=-\infty}^{\infty} (|a_n|^2 + |b_n|^2) = \frac{L^2}{4\pi^2}. \quad \text{--- (2)}$$

$$\begin{aligned} \therefore A &= \frac{1}{2} \left| \int_0^L x(t) y'(t) dt - \int_0^L y(t) x'(t) dt \right| \\ &= \frac{1}{2} \left| \sum_{n=-\infty}^{\infty} a_n \overline{2\pi i n b_n} - \sum_{n=-\infty}^{\infty} b_n \overline{2\pi i n a_n} \right| \quad \text{Parseval} \\ &= \pi \left| \sum_{n=-\infty}^{\infty} n (a_n \bar{b}_n - \bar{a}_n b_n) \right| \end{aligned}$$

Now,

$$L^2 - 4\pi A = 4\pi^2 \left[\sum_{n=-\infty}^{\infty} n^2 (|a_n|^2 + |b_n|^2) - \left| \sum_{n=-\infty}^{\infty} n (a_n \bar{b}_n - \bar{a}_n b_n) \right| \right]$$

$$\begin{aligned} \|a_n \bar{b}_n - \bar{a}_n b_n\| &\leq 2|a_n||b_n|\|h\| \leq 2n^2 |a_n||b_n| \\ \therefore L^2 - 4\pi A &\geq 4\pi^2 \left[\sum_{n=-\infty}^{\infty} n^2 (|a_n|^2 + |b_n|^2) - 2|a_n||b_n|\right] \\ &= 4\pi^2 \left[\sum_{n=-\infty}^{\infty} n^2 [|a_n| - |b_n|]^2 \right] \\ \therefore L^2 - 4\pi A &\geq 0 \quad \text{or} \quad A < \frac{L^2}{4\pi} \end{aligned}$$

Moreover, the equality holds as can be seen by

$$\gamma: [0, 1] \rightarrow \mathbb{R}^2 \text{ given by } t \mapsto \left(\frac{\cos 2\pi t}{2\pi}, \frac{\sin 2\pi t}{2\pi} \right).$$

When else?

Note that if $|n| > 2$, then $n \geq 2$, $|n| < n^2$.

$$\text{Thus, } L^2 = 4\pi A \Rightarrow a_n = b_n = 0.$$

$$\begin{aligned} \text{Thus, } x(t) &= a_{-1} e^{-2\pi i t} + a_0 + a_1 e^{2\pi i t} \\ y(t) &= b_{-1} e^{-2\pi i t} + b_0 + b_1 e^{2\pi i t} \end{aligned}$$

$$y(t) = b_{-1} e^{-2\pi i t} + b_0 + b_1 e^{2\pi i t}$$

Note $x(t) = \overline{z(t)}$ and $y(t) = \overline{y(t)}$ if $t \in [0, 1]$.

$$\therefore \bar{a}_1 = a_1 \quad \text{and} \quad \bar{b}_{-1} = b_1$$

$$\begin{aligned} \text{Also, } L^2 &= 4\pi^2 \sum_{n=-1}^1 n^2 (|a_n|^2 + |b_n|^2) \\ &= 4\pi^2 2(|a_1|^2 + |b_1|^2) = 8\pi^2 (|a_1|^2 + |b_1|^2) \end{aligned}$$

$$\Rightarrow |a_1|^2 + |b_1|^2 = \frac{L^2}{8\pi^2} \quad \text{--- (3)}$$

$$\text{Also, } L^2 - 4\pi A = 0 \quad \text{and thus}$$

$$\begin{aligned} |a_1|^2 + |b_1|^2 - |\bar{a}_1 \bar{b}_1 - b_1 \bar{a}_1| &= 0 \quad \text{--- (4)} \\ \Rightarrow 0 &\geq (|a_1| - |b_1|)^2 \end{aligned}$$

Thus, $|a_1| = |b_1|$ and hence, (3) tells us that

$$|a_1|^2 = \frac{L^2}{16\pi^2} \quad \text{or} \quad |a_1| = |b_1| = \frac{L}{4\pi}$$

$$\therefore a_1 = \frac{L}{4\pi} e^{i\theta}, \quad b_1 = \frac{L}{4\pi} e^{i\varphi}$$

for some $\theta, \varphi \in [0, 2\pi]$.

Put the above in (4) to get

$$\begin{aligned} \frac{L^2}{16\pi^2} &= |a_1 \bar{b}_1 - b_1 \bar{a}_1| = \frac{L^2}{16\pi^2} |e^{i(\theta-\varphi)} - e^{i(\varphi-\theta)}| \\ \Rightarrow |e^{i(\theta-\varphi)} - e^{i(\varphi-\theta)}| &= 2 \\ \Rightarrow |2 \sin(\theta-\varphi)| &= 2 \\ \Rightarrow |\sin(\theta-\varphi)| &= 1 \quad \text{or} \quad \theta - \varphi = (2k+1)\frac{\pi}{2} \end{aligned}$$

$$\therefore a_1 = \frac{L}{4\pi} e^{i(\varphi + (2k+1)\frac{\pi}{2})} \quad \text{and} \quad b_1 = \frac{L}{4\pi} e^{i\varphi}$$

$$\therefore a_1 = \pm i \frac{L}{4\pi} e^{i\varphi} \quad \text{and} \quad b_1 = \frac{L}{4\pi} e^{i\varphi}$$

$$\begin{aligned} \text{Thus, } x(t) &= a_0 + a_1 e^{2\pi i t} + \bar{a}_1 e^{-2\pi i t} \\ &= a_0 \pm i \left(\frac{L}{4\pi} e^{i(\varphi + 2\pi t)} - \frac{L}{4\pi} e^{i(\varphi - 2\pi t)} \right) \\ &= a_0 \pm i \frac{L}{4\pi} (2i \sin(\varphi + 2\pi t)) \\ &= a_0 \mp \frac{L}{2\pi} \sin(\varphi + 2\pi t) \end{aligned}$$

$$\text{Similarly, } y(t) = b_0 + \frac{L}{2\pi} \cos(\varphi + 2\pi t)$$

Thus, we only get a circle, at the end,
of radius $\frac{L}{2\pi}$. //

Lecture 9 (05-02-2021)

05 February 2021 09:31

Fourier Transform

$$L'(\mathbb{T}) \supseteq L^p(\mathbb{T}) \quad \text{for } p \geq 1.$$

No relation as above if we replace \mathbb{T} with \mathbb{R}^n .

- We will work in $L^p(\mathbb{R}^n)$, $p \geq 1$.
- Fourier transform:

$$L'(\mathbb{R}^n) = \left\{ f: \mathbb{R}^n \rightarrow \mathbb{C} \text{ measurable} \mid \int_{\mathbb{R}^n} |f| < \infty \right\}$$

$$\|f\|_1 = \|f\|_{L'(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |f| dx.$$

$$\text{Similarly, } \|f\|_p = \|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p \right)^{\frac{1}{p}}.$$

- $C_c(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Defn. For $f \in L'(\mathbb{R}^n)$, the Fourier transform \hat{f} of f is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

for all $\xi \in \mathbb{R}^n$.

Note $\langle x, \xi \rangle = x_1 \xi_1 + \dots + x_n \xi_n$ where

$$x = (x_1, \dots, x_n) \quad \text{and} \quad \xi = (\xi_1, \dots, \xi_n).$$

Since $f \in L'$, the above integral does exist since the

integrand is absolutely integrable.

Thus, for $f \in L^1(\mathbb{R}^n)$, we have $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$.

Q. Is \hat{f} continuous?

Note. (i) $\widehat{(\alpha f + \beta g)}(\xi) = \alpha \widehat{f}(\xi) + \beta \widehat{g}(\xi)$ $\forall f, g \in L^1(\mathbb{R}^n)$, $\forall \alpha, \beta \in \mathbb{C}, \forall \xi \in \mathbb{R}^n$

(ii) $|\widehat{f}(\xi)| \leq \left| \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx \right| \leq \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_1$.

$$\therefore \|\widehat{f}\|_\infty \leq \|f\|_1$$

$\Rightarrow \widehat{f}$ is a bounded function

(iii) $\lim_{|\xi| \rightarrow \infty} |\widehat{f}(\xi)| = 0$ for $f \in L^1(\mathbb{R}^n)$.

(Riemann-Lebesgue Lemma)

Proof. $n=1$:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} dx \quad - (1)$$

$$= - \int_{\mathbb{R}} f(x) e^{-2\pi i (x + \frac{1}{2\xi}) \xi} dx$$

$$= - \int_{\mathbb{R}} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i x \cdot \xi} dx \quad - (2)$$

By (1) and (2) :

$$\widehat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}} \left(f(x) - f\left(x - \frac{1}{2\xi}\right) \right) e^{-2\pi i x \cdot \xi} dx$$

$$\Rightarrow |\widehat{f}(\xi)| \leq \frac{1}{2} \int_{\mathbb{R}} \left| f(x) - f\left(x - \frac{1}{2\xi}\right) \right| dx$$

(Want to take $\lim_{|\xi| \rightarrow \infty}$ inside. Not sure if possible.)

Let $f \in C_c(\mathbb{R})$. let $\text{supp } f \subseteq [-M, M]$.

$$\begin{cases} \Rightarrow f(x) = 0 \text{ if } x > M, \\ \Rightarrow f\left(x - \frac{1}{2\xi}\right) = 0 \text{ if } x > M + \frac{1}{2\xi} \end{cases}$$

$$\text{Then, } |\hat{f}(\xi)| \leq \frac{1}{2} \int_{\mathbb{R}} |f(x) - f(x - \frac{1}{2\xi})| dx$$

$$\leq \frac{1}{2} \int_{-M - \frac{1}{2\xi}}^{M + \frac{1}{2\xi}} |f(x) - f(x - \frac{1}{2\xi})| dx$$

$$\text{if } |\xi| > M+1 \leq \frac{1}{2} \int_{-M-1}^{M+1} |f(x) - f(x - \frac{1}{2\xi})| dx$$

$$\therefore \lim_{|\xi| \rightarrow \infty} |\hat{f}(\xi)| \leq \frac{1}{2} \lim_{|\xi| \rightarrow \infty} \int_{-M-1}^{M+1} |f(x) - f(x - \frac{1}{2\xi})| dx$$

$$= \frac{1}{2} \int_{-M-1}^{M+1} \lim_{|\xi| \rightarrow \infty} |f(x) - f(x - \frac{1}{2\xi})| dx$$

DCT
compact domain

) f is continuous

$$= 0$$

$$\therefore \text{if } f \in C_c(\mathbb{R}), \text{ then } \lim_{|\xi| \rightarrow \infty} |\hat{f}(\xi)| = 0.$$

Let $f \in L^1(\mathbb{R})$, for any $\varepsilon > 0$, $\exists g \in C_c(\mathbb{R})$ s.t.

$$\|g - f\|_1 < \varepsilon/2. \quad (C_c(\mathbb{R}) \text{ is dense in } L^1(\mathbb{R})).$$

$$\therefore |\hat{f}(\xi)| \leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)|$$

$$\begin{aligned} &\leq |\widehat{f-g}(\xi)| + |\widehat{g}(\xi)| \\ &\leq \|f - g\|_1 + |\widehat{g}(\xi)| \\ |\widehat{f}(\xi)| &\leq \varepsilon/2 + |\widehat{g}(\xi)| \end{aligned}$$

Choose $M > 0$ s.t. $|\xi| > M \Rightarrow |\widehat{g}(\xi)| < \varepsilon/2$.

Then,

$$|\widehat{f}(\xi)| < \varepsilon \quad \forall \xi \text{ s.t. } |\xi| > M. \quad \square$$

Ex. Try for \mathbb{R}^n ?

$$(iv) \quad f, g \in L^1(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n)$$

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-t) g(t) dt$$

Then, $\widehat{f * g}$ makes sense and we have

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi) \quad \text{for } \xi \in \mathbb{R}^n. \quad (\text{ex.})$$

$$(v) \quad f \in L^1(\mathbb{R}^n). \quad \text{Translation of } f \text{ by } h \in \mathbb{R}^n:$$

$$(T_h f)(x) := f(x+h)$$

$$f \in L^1(\mathbb{R}^n) \Rightarrow (T_h f) \in L^1(\mathbb{R}^n)$$

$$\text{In fact, } \|f\|_1 = \|T_h f\|_1.$$

$$\widehat{T_h f}(\xi) = e^{2\pi i \xi \cdot h} \widehat{f}(\xi) \quad (\xi \cdot h := \langle \xi, h \rangle)$$

(Do a change of variable.)

$$(vi) \quad \text{Fix } h \in \mathbb{R}^n.$$

$$\text{Given } f \in L^1(\mathbb{R}^n), \text{ define } g(x) := f(x) e^{2\pi i h \cdot x}.$$

$$\text{Then, } \widehat{g}(\xi) = \widehat{f}(\xi - h)$$

$$= \mathcal{T}_{-h}(\hat{f})(\xi) \quad (\text{Not } \widehat{\mathcal{T}_h f}(\xi) !)$$

(vii) $n=1$. Suppose $f \in C_c^\infty$ - function.

Then,

$$\hat{f}'(\xi) = (2\pi i \xi) \hat{f}(\xi).$$

$$\begin{aligned} \text{Proof. } \hat{f}'(\xi) &= \int_{\mathbb{R}} f'(x) e^{-2\pi i x \xi} dx \\ &= e^{-2\pi i x \xi} f(x) \Big|_{-\infty}^{\infty} + 2\pi i \xi \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx \\ &\stackrel{f \in C_c(\mathbb{R})}{=} 0 + 2\pi i \xi \hat{f}(\xi) \end{aligned}$$

Similarly, $\left(\frac{\partial \hat{f}}{\partial x_j} \right)(\xi) = (2\pi i \xi_j) \hat{f}(\xi).$

(viii) Let $f \in C_c^\infty(\mathbb{R}^n)$.

Then, $(-\widehat{2\pi i x f})(\xi) = (\hat{f})'(\xi).$

$$(\hat{f})'(\xi) = \lim_{h \rightarrow 0} \frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h}$$

$$= \lim_{h \rightarrow 0} \int_{-\infty}^{\infty} f(x) \left[e^{\frac{-2\pi i x(\xi+h)}{h}} - e^{\frac{-2\pi i x \xi}{h}} \right] dx$$

$$= \int_{-\infty}^{\infty} f(x) \lim_{h \rightarrow 0} \left[e^{\frac{-2\pi i x(\xi+h)}{h}} - e^{\frac{-2\pi i x \xi}{h}} \right] dx$$

$$= \int_{-\infty}^{\infty} f(x) (-2\pi i x) e^{-2\pi i \xi x} dx = (-2\pi i x \hat{f}(x))(\xi)$$

Justification of taking \lim inside: we DCT

$$\text{Take } g_h(x) = \left(e^{\frac{-2\pi i x(\xi+h)}{h}} - e^{\frac{-2\pi i x \xi}{h}} \right) f(x)$$

$\xrightarrow{-2\pi i h} \text{ has finite limit}$

Then, $|g_n(x)| = |f(x)| \left| e^{-\frac{2\pi i n}{n}} - 1 \right|$ has finite limit $\leq M \cdot |f(x)| \in L'$.

Let $f(x) := e^{-\pi t|x|^2}$ for $x \in \mathbb{R}^n$. ($t > 0$ fixed)

Is $f \in L'(\mathbb{R}^n)$? Any $x \in \mathbb{R}^n \setminus \{0\}$ can be written (uniquely) as

$$x = r\omega, \quad 0 < r < \infty \text{ and } \omega \in S^{n-1}.$$

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \left(\int_{S^{n-1}} f(r\omega) d\omega \right) r^{n-1} dr$$

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\pi t|x|^2} dx &= \iint_0^\infty \int_{S^{n-1}} e^{-\pi t r^2} d\omega \ r^{n-1} dr \\ &= \left(\int_{S^{n-1}} 1 \cdot d\omega \right) \underbrace{\int_0^\infty e^{-\pi t r^2} r^{n-1} dr}_{< \infty} \\ &\quad \underbrace{\left(r^{n-1} e^{-\frac{\pi t r^2}{2}} \right)}_{< C \text{ after } r > M} \underbrace{e^{-\frac{\pi t r^2}{2}}}_{< \infty} \end{aligned}$$

$n=1, \quad f(x) = e^{-\pi t x^2}$

$$f(\xi) = \int_{\mathbb{R}} e^{-\pi t x^2} \cdot e^{-2\pi i n \xi} dx$$

$$= \int_{\mathbb{R}} e^{-\pi t (x^2 + \frac{2i\xi}{t} \cdot x)} dx$$

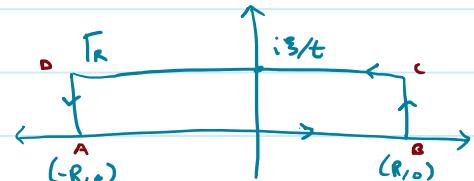
$$= \int_{\mathbb{R}} \exp \left(-\pi t \left[\left(x + \frac{i\xi}{t} \right)^2 - \left(\frac{i\xi}{t} \right)^2 \right] \right) dx$$

$$= \int_{\mathbb{R}} e^{-\pi t (x + \frac{i\xi}{t})^2} dx$$

$$= \exp\left(-\frac{\pi \xi^2}{t}\right) \int_{-\infty}^{\infty} e^{-\pi t \left(x + \frac{i\xi}{t}\right)^2} dx$$

Put $g(z) = e^{-\pi t z^2}$.
↑ entire

$$\therefore \int_{\mathbb{R}} g = 0.$$



(assuming $\xi > 0$)

$$\therefore \int_A^B g + \int_B^C g + \int_C^D g + \int_D^A g = 0$$

$$\Rightarrow \int_{-R}^R e^{-\pi t x^2} dx + \int_0^{\xi/t} e^{-\pi t (R+iy)^2} dy + \int_R^{-R} e^{-\pi t \left(x + \frac{i\xi}{t}\right)^2} dx \\ \xrightarrow{R \rightarrow \infty} 0 + \int_{\xi/t}^0 e^{-\pi t (-R+iy)^2} dy = 0$$

$$\text{Thus, } \int_{\mathbb{R}} e^{-\pi t x^2} dx = \int_{\mathbb{R}} e^{-\pi t \left(x + \frac{i\xi}{t}\right)^2} dx$$

$$\therefore \hat{f}(\xi) = \exp\left(-\frac{\pi \xi^2}{t}\right) \int_{-\infty}^{\infty} e^{-\pi t x^2} dx \\ = \exp\left(-\frac{\pi \xi^2}{t}\right) \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{\sqrt{\pi t}} dy \quad x = \frac{1}{\sqrt{\pi t}} y \\ = \exp\left(-\frac{\pi \xi^2}{t}\right) \frac{\sqrt{\pi}}{\sqrt{\pi t}} \\ = \frac{1}{\sqrt{t}} \exp\left(-\frac{\pi \xi^2}{t}\right)$$

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$$\text{For } \mathbb{R}^n: \quad f(x) = e^{-\pi t \|x\|^2} \quad t > 0 \text{ fixed}$$

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-\pi t(x_1^2 + \dots + x_n^2)} e^{-2\pi i(x_1 \xi_1 + \dots + x_n \xi_n)} dx_1 \dots dx_n \\ &= \left(\int_{\mathbb{R}} e^{-\pi t x_1^2 - 2\pi i x_1 \xi_1} dx_1 \right) \dots \left(\int_{\mathbb{R}} e^{-\pi t x_n^2 - 2\pi i x_n \xi_n} dx_n \right) \\ &= \frac{1}{(\sqrt{t})^n} e^{-\frac{\pi}{t} |\xi|^2}\end{aligned}$$

$C_c^\infty(\mathbb{R}^n)$: collection of C^∞ compactly supported functions

$$L^p(\mathbb{R}^n) \supseteq C_c^\infty(\mathbb{R}^n) \text{ dense}$$

Schwartz class functions

$$\underline{n=1}: S(\mathbb{R}) = \left\{ f: \mathbb{R} \rightarrow \mathbb{C} : f \in C^\infty \text{ and } \begin{array}{l} \sup_{x \in \mathbb{R}} (1+|x|)^m |f^{(n)}(x)| < \infty \\ \forall m, n \in \mathbb{N} \cup \{0\} \end{array} \right\}$$

$$\text{Clearly } C_c^\infty(\mathbb{R}) \subseteq S(\mathbb{R}).$$

$$\text{Moreover, } S(\mathbb{R}) \subset L^p(\mathbb{R}) \quad \forall p \geq 1$$

Proof.
Take $n = 0$ and $m > 2p$.

Then,

$$\sup_{x \in \mathbb{R}} (1+|x|)^m |f(x)| = M < \infty$$

$$\therefore |f(x)|^p \leq \frac{M^p}{(1+|x|)^{mp}}$$

$$\Rightarrow \int |f|^p < \infty.$$

3)

$$\therefore C_c^\infty(\mathbb{R}) \subseteq S(\mathbb{R}) \subseteq L^p(\mathbb{R})$$

$C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, so is $S(\mathbb{R})$.

$$s_{m,n}(f) = \sup_{x \in \mathbb{R}} (1+|x|)^m |f^{(n)}(x)|$$

is a seminorm on $S(\mathbb{R})$

(Topology on $S(\mathbb{R})$ is generated by the seminorms $s_{m,n}$. These seminorms are countable and hence, $S(\mathbb{R})$ is metrisable space.)

Convergence on $S(\mathbb{R})$:

Let $(f_j)_j$ be a sequence in $S(\mathbb{R})$ and $f \in S(\mathbb{R})$.

$$f_j \rightarrow f \text{ in } S(\mathbb{R}) \Leftrightarrow s_{m,n}(f_j - f) \rightarrow 0 \text{ as } j \rightarrow 0$$

$\forall m, n \in \mathbb{N} \cup \{\infty\}$

$(x \mapsto e^{-|x|}) \notin S(\mathbb{R})$ since not in C^∞)

General n : If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N}_0)^n$,
then $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and
 $D^\alpha f(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x).$

$$S(\mathbb{R}^n) := \left\{ f: \mathbb{R}^n \rightarrow \mathbb{C} : f \in C^\infty \text{ and } \sup_{x \in \mathbb{R}^n} |x^\alpha| |D^\beta f(x)| < \infty \right\},$$

$\forall \alpha, \beta \in (\mathbb{N}_0)^n$

As before, $C_c^\infty(\mathbb{R}^n) \subseteq S(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for $p \geq 1$.

As before, $C_0(\mathbb{R}^n) \subseteq S(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for $p \geq 1$.

We defined Fourier Transform for $L' \supseteq S$.

What if $f \in S(\mathbb{R}^n)$? What more can we say?

$\cdot f \in L' \Rightarrow \hat{f} \in C_0(\mathbb{R}^n) = \left\{ g \text{ continuous and } \lim_{|x| \rightarrow \infty} g(x) = 0 \right\}$

Prop. If $f \in S(\mathbb{R})$, then $\hat{f} \in S(\mathbb{R})$.

$$\begin{aligned} \text{Proof.} \quad & \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^m \left| \frac{d^n}{d\xi^n} \hat{f}(\xi) \right| \quad \left(\frac{d}{d\xi} \hat{f}(\xi) = C \cdot \widehat{\xi^n f(\xi)} \right) \\ &= C \cdot \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^m \left| \widehat{x^n f(\xi)} \right| \\ &\leq C \cdot \sup_{\xi \in \mathbb{R}} \widehat{(x^n f)^{(m)}}(\xi) \end{aligned}$$

If $(x^n f)^{(m)} \in L'$, then we are done, by Riemann-Lebesgue.

$$\int_{-\infty}^{\infty} |(x^n f)^{(m)}| dx \leq \sum_{m_1+m_2=m} \int_{-\infty}^{\infty} |(x^{n_1})^{(m_1)} f^{(m_2)}(x)| dx < \infty$$

$\curvearrowleft f \in S(\mathbb{R})$

Fourier Series: If $f \in C^1(\mathbb{T})$, then

$$f(x) = \sum \hat{f}(n) e^{2\pi i n x}$$

} inversion formula

- Fourier inversion for Schwartz class functions.

Step 1: $f, g \in S(\mathbb{R})$, then

$$\int f \hat{g} = \int \hat{f} \cdot g. \quad (\text{Note: } f, g \in S(\mathbb{R}) \Rightarrow fg \in S(\mathbb{R}))$$

Proof.

$$\begin{aligned} & \int_{\mathbb{R}} f(n) \hat{g}(n) dn \\ &= \int_{\mathbb{R}} f(n) \int_{\mathbb{R}} g(t) e^{-2\pi i t n} dt dn \quad \text{(*) Fubini} \\ &= \int_{\mathbb{R}} g(t) \int_{\mathbb{R}} f(n) e^{-2\pi i t n} dn dt \\ &= \int_{\mathbb{R}} g(t) \hat{f}(t) dt. \end{aligned}$$

(*) Fubini justification:

$$\begin{aligned} \text{Note} \quad & \iint_{\mathbb{R}^2} |f(n) g(t) e^{-2\pi i t n}| dt dn = \iint_{\mathbb{R}^2} |f(n)| |g(t)| dt dn \\ &= \int_{\mathbb{R}} |f(n)| dn \int_{\mathbb{R}} |g(t)| dt < \infty. \end{aligned}$$

\therefore Fubini is indeed applicable. □

$$\text{Step 2. } \int_{\mathbb{R}} f\left(\frac{x}{\lambda}\right) \hat{g}(x) dx = \int_{\mathbb{R}} \hat{f}(x) g\left(\frac{x}{\lambda}\right) dx \quad (\text{Ex.})$$

Take $\lambda \rightarrow \infty$ on both sides.

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}} f\left(\frac{x}{\lambda}\right) \hat{g}(x) dx = \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}} \hat{f}(x) g\left(\frac{x}{\lambda}\right) dx$$

$\int_{\mathbb{R}}, \quad \text{as } \lambda \rightarrow \infty \quad \int_{\mathbb{R}}, \quad \text{as } \lambda \rightarrow \infty$

use DCT

$$\Rightarrow \int_{\mathbb{R}} \lim_{\lambda \rightarrow \infty} f(\frac{x}{\lambda}) \hat{g}(x) dx = \int_{\mathbb{R}} \lim_{\lambda \rightarrow \infty} \hat{f}(\lambda) g\left(\frac{x}{\lambda}\right) dx$$

use DCT

$$\Rightarrow \int_{\mathbb{R}} f(0) \hat{g}(x) dx = \int_{\mathbb{R}} \hat{f}(0) g(x) dx$$

$$\Rightarrow f(0) \int_{\mathbb{R}} \hat{g}(x) dx = g(0) \int_{\mathbb{R}} \hat{f}(x) dx$$

$f, g \in S(\mathbb{R})$
and hence,
continuous

The above is true for all $f, g \in S(\mathbb{R})$.

$$\text{Take } g(x) = e^{-\pi x^2}. \quad \hat{g}(\xi) = e^{-\pi \xi^2}.$$

$$\therefore f(0) \underbrace{\int_{\mathbb{R}} e^{-\pi x^2} dx}_{=1} = \int_{\mathbb{R}} \hat{f}(x) dx$$

$$\therefore f(0) = \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad \text{--- (1)}$$

• Step 3. Fix $x \in \mathbb{R}$. Let $\tau_x f(y) := f(x+y)$.
 $f \in S(\mathbb{R}) \Rightarrow \tau_x f \in S(\mathbb{R})$

Use (1) for $\tau_x f$.

$$\tau_x f(0) = \int_{\mathbb{R}} \widehat{\tau_x f}(\xi) d\xi$$

$$\Rightarrow \int_{\mathbb{R}} f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \text{for all } f \in S(\mathbb{R}).$$

(Inversion formula for $S(\mathbb{R})$)

$$\begin{aligned} (\hat{f})(-x) &= f(x) \\ (\hat{\hat{f}})(x) &= f(x) \end{aligned} \quad \left. \right\} f \in S(\mathbb{R})$$

The above calculations go through for \mathbb{R}^n as well.

- Inversion formula for L' function:

Thm. Let $f \in L'(\mathbb{R})$ be such that $\hat{f} \in L'(\mathbb{R})$.

Then,

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \text{a.e. } x.$$

Proof. $h_t(x) = \underbrace{e^{-\pi t x^2}}_{\substack{\downarrow \\ \text{heat kernel}}} = \frac{1}{t^{1/2}} e^{-\frac{\pi}{t} x^2}$

Steps. 1. $(f * h_t)(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{-\pi t \xi^2} e^{2\pi i x \xi} d\xi$.

2. $(h_t)_{t>0}$ is an approximate identity in $L'(\mathbb{R})$.

3. $\|f * h_t - f\|_1 \rightarrow 0$ as $t \rightarrow 0$

4. Then, \exists a subsequence $(t_n)_n$ s.t.

$(t_n \rightarrow 0)$

$$f * h_{t_n}(x) \rightarrow f(x) \quad \text{a.e. } x.$$

} from measure theory

$$\left\| \int \hat{f}(\xi) e^{-\pi t_n |\xi|^2} e^{2\pi i x \xi} d\xi \right\| \rightarrow \int \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

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Thm. Fourier inversion formula for L^1

Let $f \in L^1(\mathbb{R}^n)$ be such that $\hat{f} \in L^1(\mathbb{R}^n)$. Then,

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} dx \quad a.e. x.$$

Fact: If $\{f_n\}_n$ is in L^p and $f \in L^p$ with $\|f_n - f\|_p \rightarrow 0$,

then \exists a subsequence $\{f_{n_k}\}_k$ s.t.

$$f_{n_k}(x) \rightarrow f(x) \quad a.e. x.$$

Proof (of Thm.): Define $h_t(x) := \frac{1}{t^{n/2}} e^{-\frac{\pi|x|^2}{t}}$; $t > 0$

heat kernel

$$\text{Note } \hat{h}_t(\xi) = e^{-\pi|\xi|^2}$$

$$\text{Step 1. } \text{Claim: } \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t |\xi|^2} e^{2\pi i x \cdot \xi} d\xi = (f * h_t)(x).$$

$$\text{Proof. } \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot \xi} dy \right) e^{-\pi t |\xi|^2} e^{2\pi i x \cdot \xi} d\xi \quad \text{(*) Fubini}$$

$$= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} e^{-\pi t |\xi|^2} e^{-2\pi i (y - x) \cdot \xi} d\xi dy$$

$$= \int_{\mathbb{R}^n} f(y) \underbrace{\left(\xi \mapsto e^{-\pi t |\xi|^2} \right)}_{\text{F}} (y - x) dy$$

$$= \int_{\mathbb{R}^n} f(y) \frac{1}{t^{n/2}} e^{-\frac{\pi}{t}|y-x|^2} dy$$

$$= \int_{\mathbb{R}^n} f(y) h_t(x-y) dy = (f * h_t)(x). \quad \square$$

Justification of Fubini:

Justification of Fubini:

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)| e^{-\pi t |\xi|^2} e^{-2\pi i (y-x) \cdot \xi} dy dx \\ &= \left(\int_{\mathbb{R}^n} f(y) dy \right) \left(\int_{\mathbb{R}^n} e^{-\pi t |\xi|^2} dy \right) < \infty \end{aligned}$$

$\hookrightarrow f \in L^1$ \hookrightarrow very nice function

Step 2. Claim: $\{h_t\}_{t>0}$ is an approximate identity in $L^1(\mathbb{R}^n)$.

Proof.

$$(1) h_t(x) \geq 0 \quad \forall x \quad \checkmark$$

$$\begin{aligned} (2) \int_{\mathbb{R}^n} h_t(x) dx &= \frac{1}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\pi t^{-1} |x|^2} dx \\ &= \frac{1}{t^{n/2}} \underbrace{e^{-\pi t^{-1} |x|^2}}_{(o)} (o) \\ &= \frac{1}{t^{n/2}} \cdot t^{n/2} \cdot e^{-\pi t^{-1} (0)^2} = 1. \end{aligned}$$

$$(3) \lim_{t \rightarrow 0} \int_{|x|>\delta} h_t(x) dx = 0.$$

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{|x|>\delta} h_t(x) dx &= \lim_{t \rightarrow 0} \frac{1}{t^{n/2}} \int_{|x|>\delta} e^{-\pi t^{-1} |x|^2} dx \\ &= \lim_{t \rightarrow 0} \frac{1}{t^{n/2}} \int_{|x|>\delta} \exp\left(-\frac{\pi}{2t} |x|^2\right) \exp\left(-\frac{\pi}{2t} |x|^2\right) dx \\ &\leq \lim_{t \rightarrow 0} \frac{1}{t^{n/2}} \exp\left(-\frac{\pi}{2t} \delta^2\right) \int_{|x|>\delta} \exp\left(-\frac{\pi}{2t} |x|^2\right) dx \\ &\quad \downarrow \text{for } t \leq 1 \\ &\leq \lim_{t \rightarrow 0} \frac{1}{t^{n/2}} \underbrace{\exp\left(-\frac{\pi}{2t} \delta^2\right)}_{\downarrow 0} \int_{|x|>\delta} \exp\left(-\frac{\pi}{2} |x|^2\right) dx \\ &\quad \text{finite constants, indep. of } t \end{aligned}$$

$$\therefore \lim_{t \rightarrow 0} \int_{|x|>\delta} h_t(x) dx = 0. \quad \square$$

$$\text{Hence, } \lim_{t \rightarrow 0} \|f * h_t - f\|_1 = 0.$$

Step 3. \exists a subsequence $\{t_k\}$ s.t. $f * h_{t_k}(x) \rightarrow f(x)$ a.e. x .

$$\text{By Step 1, } (f * h_{t_k})(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t_k |\xi|^2} e^{2\pi i x \cdot \xi} d\xi$$

Then,

$$\lim_{t_k \rightarrow 0} [f * h_{t_k}](x) = \lim_{t_k \rightarrow 0} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t_k |\xi|^2} e^{2\pi i x \cdot \xi} d\xi$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

$$\left[\text{DCT: Put } g_k(\xi) = \hat{f}(\xi) e^{-\pi t_k |\xi|^2} e^{-2\pi i x \cdot \xi} \right]$$

$$|g_k(\xi)| \leq |\hat{f}(\xi)| \text{ but } \hat{f} \in L^1, \text{ no given.}$$

Thus, can apply DCT.

Hence,

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = f(x) \quad \text{a.e. } x. \quad \blacksquare$$

Lor.

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f}(\xi) = 0$ a.e. ξ ,

then $f(x) = 0$ a.e. x .

• Heat kernel

Let $u: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$ nice function.
 $f \in L^1(\mathbb{R}^n)$ given.

$$\begin{aligned} \Delta u(x, t) - \frac{\partial}{\partial t} u(x, t) &= 0 \\ \ell u(x, 0) &= f(x) \end{aligned} \quad \left. \right\} \text{heat equation}$$

$$\left(\Delta := \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}, \text{ Laplacian in } \mathbb{R}^n \right)$$

Let us solve using Fourier transform:

$$\frac{\partial^2}{\partial x^2} u(x, t) - \frac{2}{\partial t} u(x, t) = 0 \quad (*) \quad ; x \in \mathbb{R}, t > 0$$

$$u(x, 0) = f(x)$$

In $(*)$, take $\hat{\quad}$ on both sides (in variable x) after fixing t .

$$\left(\widehat{\frac{\partial^2}{\partial x^2} u(x, t)} \right)(\xi) - \left(\widehat{\frac{2}{\partial t} u(x, t)} \right)(\xi) = 0$$

$$\Rightarrow (2\pi i \xi)^2 \hat{u}(\xi, t) - \frac{2}{\partial t} (\hat{u}(\xi, t)) = 0$$

$$\Rightarrow -4\pi^2 \xi^2 \hat{u}(\xi, t) - \frac{\partial}{\partial t} (\hat{u}(\xi, t)) = 0$$

$$\Rightarrow \frac{\partial}{\partial t} (\hat{u}(\xi, t)) = -4\pi \xi^2 \hat{u}(\xi, t)$$

$$\Rightarrow \hat{u}(\xi, t) = C e^{-4\pi^2 \xi^2 t}$$

Take $t \rightarrow 0$.

$$\begin{aligned} C &= \hat{u}(\xi, 0) \\ &= \int_{\mathbb{R}} u(x, 0) e^{-2\pi i x \cdot \xi} dx \\ &= \int f(x) e^{-2\pi i x \cdot \xi} dx \\ &= \hat{f}(\xi). \end{aligned}$$

$$\therefore \hat{u}(\xi, t) = \hat{f}(\xi) e^{-4\pi^2 \xi^2 t}$$

$$= \hat{f}(\xi) \left(\widehat{\frac{1}{(4\pi t)^2} e^{-\frac{1}{4t} |x|^2}} \right)(\xi)$$

call the inner part $h_t(x)$

$$= (\widehat{f * h_t})(\xi)$$

$$\therefore u(x, t) = (f * h_t)(x)$$

Let $f \in S(\mathbb{R}^n)$. $\therefore f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$.

$$\|f\|_2^2 = \int_{\mathbb{R}^n} f(x) \overline{f(x)} dx$$

$$= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} e^{2\pi i \xi \cdot x} d\xi dx$$

$$= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} e^{-2\pi i \xi \cdot x} d\xi dx \quad \xrightarrow[\text{(} \hat{f} \in S(\mathbb{R}^n) \subset L^1 \text{)}]{\text{Fubini}}$$

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{f}(\xi) d\xi = \|\hat{f}\|_2^2$$

Thus, for $f \in S(\mathbb{R}^n)$, $\|f\|_2 = \|\hat{f}\|_2$.

Aim: Extend the Fourier Transform to L^2 functions

Let $f \in L^2(\mathbb{R}^n)$. Then, $\exists \{f_k\}_k \subset S(\mathbb{R}^n)$ s.t.

$$\|f_k - f\|_2 \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

(S is dense in L^p for all $1 \leq p < \infty$.)

$f_k, f_m \in S$

Then, $\|f_k - f_m\| \rightarrow 0 \quad \text{as } m, k \rightarrow \infty$

$$\Rightarrow \|\widehat{f_k - f_m}\| \rightarrow 0 \quad \text{as } m, k \rightarrow \infty$$

$$\Rightarrow \|\hat{f}_k - \hat{f}_m\| \rightarrow 0 \quad \text{as } m, k \rightarrow \infty$$

$\Rightarrow \{\hat{f}_k\}_k$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.

But $L^2(\mathbb{R}^n)$ is complete. Thus, $\exists g \in L^2(\mathbb{R}^n)$ s.t.

$$\hat{f}_k \rightarrow g \text{ in } L^2.$$

By definition, we define

$$\hat{f} := g = \lim_{k \rightarrow \infty} \hat{f}_k.$$

↳ limit in L^2 , not pointwise!

Q Is the above well-defined?

Suppose $\exists \{f_k\}$ and $\{\hat{f}_k\} \subset S(\mathbb{R}^n)$ s.t.

$$\|f_k - f\|_2 \rightarrow 0 \quad \& \quad \|\hat{f}_k - \hat{f}\|_2 \rightarrow 0.$$

Then, both $\{\hat{f}_k\}$ and $\{\hat{F}_k\}$ are Cauchy, as earlier.

$$\text{let } g = \lim \hat{f}_k \text{ and } G = \lim \hat{F}_k \text{ (in } L^2).$$

$$\|G - g\|_2 = \lim_k \|\hat{F}_k - \hat{f}_k\|_2$$

$$= \lim_k \|f_k - f\|_2 = \|f - f\|_2 = 0.$$

$$\therefore G = g \text{ in } L^2. \quad \square$$

If $f \in S(\mathbb{R}^n)$, $\|f\|_2 = \|\hat{f}\|_2$.

Now, if $f \in L^2$, $\|f\| = \|\hat{f}\|_2$. (Plancherel Theorem)

$$\text{↳ } \|\hat{f}\| = \lim \|\hat{f}_k\| = \lim \|f_k\| = \|f\|.$$

$\mathcal{F}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$

$$f \mapsto \hat{f}$$

is an isometry and onto.
 ↳ Fourier inversion

In fact, $\mathcal{F}: S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ is also 1-1, onto, iso.

$\mathcal{F}: L'(\mathbb{R}^n) \rightarrow C_c(\mathbb{R}^n)$.

Is \mathcal{F} onto? No!

next class

- Paley Wiener Theorem: what is image of $C_c^\infty(\mathbb{R}^n)$ under \mathcal{F}

No characterisation of $\mathcal{F}(L^1(\mathbb{R}^n))$ so far!

Lecture 12 (17-02-2021)

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Paley-Wiener Theorem - We will characterise the image of $C_c^\infty(\mathbb{R})$ under the Fourier transform.

- Let $n = 1$: Fix $R > 0$.

$$Pw_R(\mathbb{C}) = \left\{ h: \mathbb{C} \rightarrow \mathbb{C} \text{ entire: } \sup_{\lambda \in \mathbb{C}} (1+|\lambda|)^m |h(\lambda)| e^{-2\pi R \cdot |\operatorname{Im} \lambda|} < \infty \quad \forall m \in \mathbb{N} \cup \{0\} \right\}$$

$$\text{If } h \in Pw_R(\mathbb{C}), \quad |h(\lambda)| \leq \frac{C_m}{(1+|\lambda|)^m} e^{2\pi R |\operatorname{Im} \lambda|} \quad \forall \lambda \in \mathbb{C}$$

$$Pw(\mathbb{C}) := \bigcup_{R>0} Pw_R(\mathbb{C}).$$

↑ Paley - Wiener Space

note this means
that
 $h|_{\mathbb{R}} \in S(\mathbb{R})$

Th. $\mathcal{F}: C_c^\infty(\mathbb{R}) \rightarrow Pw(\mathbb{C})$ is an isomorphism.

Proof. Let $f \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp} f \subseteq [-R, R]$
 $(\operatorname{supp} h = \{\lambda: h(\lambda) \neq 0\})$

- First, we show that $\hat{f} \in Pw_R(\mathbb{C}) \subset Pw(\mathbb{C})$.

- We show \hat{f} is entire.

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-2\pi i \lambda x} dx$$

Note that $\hat{f}(\lambda)$ exists for $\lambda \in \mathbb{C}$ since f is compactly supported.

To show \hat{f} is entire, we use Morera's theorem.

Let Γ be a triangle in \mathbb{C} . Then,

$$\int_{\Gamma} \dots \int_{\Gamma} \dots e^{-2\pi i \lambda x} \dots$$

$$\begin{aligned}
 \int_{\Gamma} \hat{f}(\lambda) d\lambda &= \int_{\Gamma} \int_{\mathbb{R}} f(x) e^{-2\pi i \lambda x} dx d\lambda \\
 &= \int_{\mathbb{R}} f(x) \int_{\Gamma} e^{-2\pi i \lambda x} d\lambda dx \quad \left. \begin{array}{l} \lambda \mapsto e^{-2\pi i \lambda x} \\ \text{is holomorphic} \end{array} \right. \\
 &= \int_{\mathbb{R}} f(x) \cdot 0 dx = 0.
 \end{aligned}$$

$\therefore \hat{f}$ is entire.

$$\begin{aligned}
 |\hat{f}(\lambda)| / (1 + |\lambda|)^m &= |\hat{f}(\lambda)| \left(1 + c_1 |\lambda| + \dots + c_m |\lambda|^m \right) \\
 &\leq |\hat{f}(\lambda)| + c \cdot \left[|\lambda| |\hat{f}(\lambda)| + \dots + |\lambda|^m |\hat{f}(\lambda)| \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{Note } |\lambda^k| |\hat{f}(\lambda)| &= c \cdot |\widehat{f^{(k)}}(\lambda)| \\
 &= c \cdot \left| \int_{\mathbb{R}} f^{(k)}(x) e^{-2\pi i \lambda x} dx \right|
 \end{aligned}$$

$$\begin{aligned}
 |e^z| &= e^{\operatorname{Re} z} \\
 \text{supp } f &\subset [-R, R] \\
 \Rightarrow \text{supp } f^{(k)} &\subset [-R, R]
 \end{aligned}$$

$$\begin{aligned}
 &\leq c \cdot \int_{\mathbb{R}} |f^{(k)}(x)| e^{2\pi |x| |\operatorname{Im} \lambda|} dx \\
 &= c \cdot \int_{-R}^R |f^{(k)}(\lambda)| e^{2\pi |\lambda| |\operatorname{Im} \lambda|} d\lambda \\
 &\leq c \cdot \underbrace{\left(\int_{-R}^R |f^{(k)}(\lambda)| d\lambda \right)}_{C_k} e^{2\pi R |\operatorname{Im} \lambda|}
 \end{aligned}$$

$$\therefore |\lambda^k| \cdot |\hat{f}(\lambda)| \leq c_m e^{2\pi R |\operatorname{Im} \lambda|}$$

$$\Rightarrow \sup_{\lambda} |\lambda^k| |\hat{f}(\lambda)| e^{-2\pi R |\operatorname{Im} \lambda|} < \infty$$

$$\therefore \sup_{\lambda} (1 + |\lambda|)^m |\hat{f}(\lambda)| e^{-2\pi R |\operatorname{Im} \lambda|} < \infty$$

$$\therefore \sup_{\lambda} (1+|\lambda|)^m |\hat{f}(\lambda)| e^{-2\pi R |\operatorname{Im} \lambda|} < \infty$$

Thus, $\hat{f} \in \text{PWR}(\mathbb{C})$.

- Conversely, let $h \in \text{PWR}(\mathbb{C})$. We show that $\exists f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq [-R, R]$ s.t. $\hat{f} = h$.

Define $f(x) = \int_R h(\lambda) e^{2\pi i x \lambda} d\lambda \quad \text{for } x \in \mathbb{R}$

(The above integral exists since $|h(\lambda) e^{2\pi i x \lambda}| = |h(\lambda)| \leq \frac{C}{(1+|\lambda|)^2}$)

- Need to prove: $f \in C_c^\infty(\mathbb{R})$.

Easy to prove that f is smooth.

$$\begin{aligned}
 f'(x) &= \lim_{\xi \rightarrow 0} \frac{f(x+\xi) - f(x)}{\xi} \\
 &= \lim_{\xi \rightarrow 0} \int_R h(\lambda) \frac{e^{2\pi i (x+\xi)\lambda} - e^{2\pi i x \lambda}}{\xi} d\lambda \\
 &\stackrel{H_\xi(\lambda) = h(\lambda) \frac{e^{2\pi i (x+\xi)\lambda} - e^{2\pi i x \lambda}}{\xi}}{=} \int_R \lim_{\xi \rightarrow 0} h(\lambda) e^{2\pi i x \lambda} \cdot \frac{e^{2\pi i \xi \lambda} - 1}{\xi} d\lambda \\
 &= \int_R h(\lambda) 2\pi i \lambda e^{2\pi i x \lambda} d\lambda \\
 &= 2\pi i \int_R \lambda h(\lambda) e^{2\pi i x \lambda} d\lambda < \infty
 \end{aligned}$$

$\curvearrowleft h|_{\mathbb{R}} \in \text{SC}(\mathbb{R})$

Similarly, for $f^{(k)}$, we get $\int_R^k \lambda^k h(\lambda) e^{2\pi i x \lambda} d\lambda$.

- We now prove $\operatorname{supp} f \subseteq [-R, R]$.

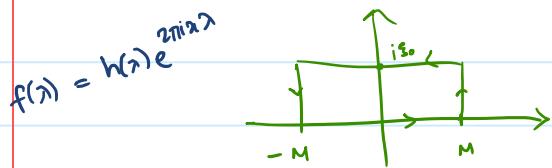
Fix $\xi_0 > 0$. we have

$$\int_{-\infty}^{\infty} e^{2\pi i x \lambda} d\lambda$$

$$f(x) = \int_{\mathbb{R}} h(\lambda) e^{2\pi i x \lambda} d\lambda$$

Claim

$$= \int_{\mathbb{R}} h(\lambda + i \xi_0) e^{2\pi i x (\lambda + i \xi_0)} d\lambda$$



Just need to show that integrals along vertical sides go to zero as $M \rightarrow \infty$. (Function has no poles.)

$$\text{Right arm: } R_M = \int_0^{\xi_0} h(M+iy) e^{2\pi i x(M+iy)} idy$$

$$\Rightarrow R_M \leq \int_0^{\xi_0} |h(M+iy)| e^{-2\pi xy} dy$$

$$\leq \int_0^{\xi_0} \frac{C_m}{(1+|M+iy|)^2} e^{2\pi Ry} e^{-2\pi xy} dy$$

$$\leq \frac{C_2}{(1+|M|)^2} \underbrace{\int_0^{\xi_0} e^{2\pi y(R-x)}}_{\text{fixed}} dy \xrightarrow{\text{as } M \rightarrow \infty} 0$$

Similarly, for left $\rightarrow 0$ as $M \rightarrow \infty$.

$$\text{Thus, } f(x) = \int_{\mathbb{R}} h(\lambda + i \xi_0) e^{2\pi i x (\lambda + i \xi_0)} d\lambda$$

for all $\xi_0 > 0$.

$$\begin{aligned} \therefore |f(x)| &\leq \int_{\mathbb{R}} |h(\lambda + i \xi_0)| e^{-2\pi \xi_0 \cdot x} d\lambda \\ &\leq C_3 \int_{\mathbb{R}} \frac{e^{2\pi R \xi_0}}{(1+|\lambda+i\xi_0|)^3} \cdot e^{-2\pi \xi_0 \cdot x} d\lambda \\ &= C_3 \cdot e^{2\pi \xi_0 (R-x)} \int_{-\infty}^x \frac{1}{(1+|\lambda+i\xi_0|)^3} d\lambda \end{aligned}$$

$$= C_3 \cdot e^{2\pi \xi_0(R-\alpha)} \int_{\mathbb{R}} \frac{1}{(1+|\lambda+i\xi_0|)^3} \cdot d\lambda \quad \text{--- (4)}$$

Now, if $\alpha > R$, then $R - \alpha < 0$.

However, (4) holds for all $\xi_0 > 0$. Taking $\xi_0 \rightarrow \infty$ gives $|f(\lambda)| \leq 0$ and thus, f vanishes on (R, ∞) .

Similarly, the above can be proven for $\xi_0 < 0$. This gives that f vanishes on $(-\infty, -R)$ as well.

Thus, $\text{supp } f \subset [-R, R]$.

$\therefore f \in C_c^\infty(\mathbb{R})$. By Fourier inversion, $\hat{f} = h$. \square

- Similar result for $n > 1$. One needs to know what entire for $h: \mathbb{C}^n \rightarrow \mathbb{C}$ means.

- Fourier transform of an L^p function, $1 < p < 2$.

If $f \in L^p$, $1 < p < 2$, then we can write

$$f = f_1 + f_2 \quad \text{for some } f_1 \in L^1 \text{ and} \\ f_2 \in L^2.$$

Proof. Let $A = \{\lambda : |f(\lambda)| > 1\}$ and $B = \mathbb{R}^n \setminus A$.

Put $f_1 = f \cdot \chi_A$ and $f_2 = f \cdot \chi_B$.

$$\int_{\mathbb{R}^n} |f_1| = \int_A |f_1| \leq \int_A |f|^p < \infty \quad \text{and thus, } f_1 \in L^1.$$

$$\int_{\mathbb{R}^n} |f_2|^2 = \int_B |f_2|^2 \leq \int_B |f|^p < \infty \quad \text{and thus, } f_2 \in L^2.$$

Now, we define $\hat{f} := \hat{f}_1 + \hat{f}_2$.

↑
defined pointwise

↑ normalize

Q. Is it well-defined?

Suppose $f = f_1 + f_2 = g_1 + g_2$ for $f, g_i \in L'$
 $\& f_1, g_1 \in L^2$.

$$\therefore f_1 - g_1 = g_2 - f_2 \in L' \cap L^2$$

and hence, the fourier definitions coincide and

$$\hat{f}_1 - \hat{g}_1 = \hat{g}_2 - \hat{f}_2$$

$$\Rightarrow \hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2.$$

Thm. $\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq c \|f\|_{L^p(\mathbb{R}^n)}$ $1 \leq p \leq 2$
 and $\frac{1}{p} + \frac{1}{p'} = 1$.

$Tf = \hat{f}$ is linear.

$T: L' \rightarrow L^\infty$

$$\|\hat{f}\|_\infty \leq \|f\|,$$

$T: L^2 \rightarrow L^2$

$$\|\hat{f}\|_2 = \|f\|_2$$

$$\begin{aligned} \|Tf\|_\infty &\leq 1 \cdot \|f\|, \\ \|Tf\|_2 &\leq 1 \cdot \|f\|_2 \end{aligned} \quad \left| \begin{array}{l} p_0 = 1, q_0 = \infty \\ p_1 = 2, q_1 = 2 \end{array} \right.$$

$$\frac{1}{p_0} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2} \quad \left\{ \frac{1}{p_0} + \frac{1}{q_0} = 1 \right.$$

$$\frac{1}{q_0} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \theta/2 \quad \left. \right\}$$

By Riesz - Thorin interpolation theorem,

$$\|Tf\|_{q_\theta} \leq \|f\|_{p_\theta}$$

As θ varies from 0 to 1, p_θ varies from 1 to 2.

This proves the theorem. □

Lecture 13 (19-02-2021)

19 February 2021 09:33

"Recall"

Riesz-Thorin

$$T: L^{p_0}(\mathbb{R}^n) + L^{q_0}(\mathbb{R}^n) \rightarrow L^{q_0}(\mathbb{R}^n) + L^{q_1}(\mathbb{R}^n)$$

sublinear

$$\begin{aligned} T \text{ is strong } & \leftarrow \|Tf\|_{q_0} \leq M_0 \|f\|_{p_0}, \\ (p_0, q_0) & \leftarrow \\ (p_1, q_1) & \leftarrow \|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}. \end{aligned}$$

Then,

$$\begin{array}{c} \text{strong} \\ (p_0, q_0) \end{array} \leftarrow \|Tf\|_{q_0} \leq M_0^{r_0} M_1^{r_1} \|f\|_{p_0} \text{ where}$$

$$\frac{1}{p} = \frac{1-r_0}{p_0} + \frac{r_0}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-r_0}{q_0} + \frac{r_1}{q_1}.$$

- Young's Inequality: $\|\hat{f}\|_{p'} \leq \|f\|_p$ for $1 \leq p \leq 2$
 $\frac{1}{p'} + \frac{1}{p} = 1$

Proof. Consider $Tf = \hat{f}$.

T is strong $(1, \infty)$ and $(2, 2)$.

Then, T is strong $(1, p')$. □

- Hausdorff Young Inequality

$$f \in L^p(\mathbb{R}^n), \quad g \in L^q(\mathbb{R}^n).$$

Then,

$$f * g \in L^r(\mathbb{R}^n), \quad \text{where} \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

More precisely,

$$\|f * g\|_r \leq C \|f\|_p \|g\|_q$$

Recall Minkowski's integral inequality:

$$\int_{\mathbb{R}^n} \int \left(\int_{\mathbb{R}^n} f(x, t) d\nu(x) \right)^p d\mu(t) \stackrel{?}{=} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(x, t)^p d\mu(t) \right]^{\frac{1}{p}} d\nu(x)$$

$$\left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x, t) d\nu(x) \right)^p d\mu(t) \right|^q \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(x, t)^p d\mu(t) \right]^q d\nu(x)$$

Proof (Haus-Young)

Step 1. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ and $1 \leq p \leq \infty$.

$$\begin{aligned} \|f * g\|_\infty &= \left(\int |f * g(x)|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int \left(\int f(x-y) g(y) dy \right)^p dx \right)^{\frac{1}{p}} \\ &\leq \left[\int \left(\int |f(x-y)| |g(y)| dy \right)^p dx \right]^{\frac{1}{p}} \quad \text{Minkowski} \\ &\leq \int \left[\int |f(x-y)|^p |g(y)|^p dy dx \right]^{\frac{1}{p}} dy \\ &= \int |g(y)| \left[\int |f(x-y)|^p dx \right]^{\frac{1}{p}} dy \\ &= \int |g(y)| \|f\|_p dy = \|g\|_1 \|f\|_p \\ \therefore \|f * g\|_p &\leq \|f\|_p \|g\|_1 \quad - \textcircled{1} \end{aligned}$$

Step 2. $\|f * g\|_\infty \leq \|f\|_p \|g\|_p \quad \frac{1}{p} + \frac{1}{p'} = 1$

$$\begin{aligned} |(f * g)(x)| &= \left| \int f(x-y) g(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)| |g(y)| dy \quad \text{Hölder's inequality} \end{aligned}$$

$$\begin{aligned}
 & \int_{\mathbb{R}^n} |f(x-y)g(y)| dy \\
 & \leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g(y)|^{p'} dy \right)^{\frac{1}{p'}} \\
 & = \|f\|_p \|g\|_{p'}
 \end{aligned}$$

Hölder's inequality

$$\begin{aligned}
 |\langle f * g(x) \rangle| & \leq \|f\|_p \|g\|_{p'} \quad \forall x \\
 \Rightarrow \|f * g\|_\infty & \leq \|f\|_p \|g\|_{p'} \quad \text{--- (2)}
 \end{aligned}$$

Step 3. So far, we have

$$\begin{aligned}
 \|f * g\|_p &= \|f\|_p \|g\|_1, \\
 \|f * g\|_\infty &\leq \|f\|_p \|g\|_{p'}.
 \end{aligned}$$

Now, for $f \in L^p$ and define

$$T_f(g) := f * g.$$

$$\begin{aligned}
 p=1, q_0=\infty \quad & \|T_f(g)\|_p \leq M_0 \cdot \|g\|_1, \quad M_0 = \|f\|_p, \\
 p=p', q_1=\infty \quad & \|T_f(g)\|_\infty \leq M_1 \cdot \|g\|_{p'}, \quad M_1 = \|f\|_p.
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{p_0} &= \frac{1-\theta}{p_1} + \frac{\theta}{p_1} = 1-\theta + \frac{\theta}{p'} = 1-\theta + \theta \left(1 - \frac{1}{p}\right) \\
 &= 1 - \frac{\theta}{p}
 \end{aligned}$$

$$\frac{1}{q_0} = \frac{1-\theta}{q_1} + \frac{\theta}{q_1} = \frac{1-\theta}{p}$$

By Riesz Thm,

$$\begin{aligned}
 \|T_f g\|_{q_0} &\leq \underbrace{M_0^{1-\theta} M_1^\theta}_{=\|f\|_p} \|g\|_{p_0}
 \end{aligned}$$

$$\Rightarrow \|T_f g\|_{q_0} \leq \|f\|_p \|g\|_{p_0}.$$

where $\frac{1}{p_0} = 1 - \frac{\theta}{p}$, $\frac{1}{q_0} = \frac{1-\theta}{p}$.

Note $1 + \frac{1}{q_0} = 1 + \frac{1-\theta}{p} = 1 + \frac{1}{p} - \frac{\theta}{p} = \frac{1}{p_0} + \frac{1}{p}$.

Take θ so that $\frac{1}{p_0} = \frac{1}{q}$. Then, $\frac{1}{q_0} = \frac{1}{r}$.

$$\therefore \|T_f g\|_r \leq \|f\|_p \|g\|_q$$

or $\|f * g\|_r \leq \|f\|_p \|g\|_q$, as desired. \square