Problem Set 5 - pre-REU 2025

Problem set on Invariant Theory

Problems marked with ^{L} are self-contained linear algebra problems. It may be useful to remember that $\det(AB) = \det(A)\det(B)$ for square matrices of the same size. In particular, if P is invertible, then $\det(P)\det(P^{-1}) = 1$.

1. Show that the following sets of matrices form a group under matrix multiplication. In each case, justify why the matrices are indeed invertible.
   (a) The set of $n \times n$ matrices with determinant one.
   (b) The set of $n \times n$ matrices M satisfying $MM^{\mathsf{T}} = I_n$.
   (c) The set of $2n \times 2n$ matrices M satisfying $M\Omega M^{\mathsf{T}} = \Omega$, where Ω is the $2n \times 2n$ block matrix given as $\Omega = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.

2. Let R be a polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$. Let $f\colon R \to R$ be a function satisfying
   - $f(1) = 1$,
   - $f(x + y) = f(x) + f(y)$ for all $x, y \in R$,
   - $f(xy) = f(x)f(y)$ for all $x, y \in R$.

   Let S be the set of fixed points of R, i.e., $S \coloneqq \{r \in R : f(r) = r\}$. Show that S is closed under addition, multiplication, and contains 1.

3. Let $\sigma \in \mathrm{GL}_2(\mathbb{R})$ be the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then, the group $\langle\sigma\rangle$ generated by σ consists of two elements: the identity I and σ. Show that $\mathbb{R}[x, y]^{\langle\sigma\rangle} = \mathbb{R}[x^2, xy, y^2]$.

3.5. Can you come up with an element $\sigma \in \mathrm{GL}_2(\mathbb{C})$ such that $\mathbb{C}[x, y]^{\langle\sigma\rangle} = \mathbb{C}[x^3, x^2y, xy^2, y^3]$? In what ways can you generalise this?

   Hint: There is a reason for stating this question over $\mathbb{C}$ instead of $\mathbb{R}$.

4. Let $\sigma, \tau \in \mathrm{GL}_2(\mathbb{R})$ be given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Show that $\mathbb{R}[x, y]^{\langle\sigma,\tau\rangle} = \mathbb{R}[x^2, y^2]$.

   You may the following fact: for an element $f \in \mathbb{R}[x, y]$ to be fixed by the group $\langle\sigma, \tau\rangle$, it is sufficient to be fixed by σ and τ alone.

5. From class, we know that any element of the orthogonal group looks like

$$M = \begin{bmatrix} \cos(\theta) & -\varepsilon\sin(\theta) \\ \sin(\theta) & \varepsilon\cos(\theta) \end{bmatrix}$$

   for some $\theta \in \mathbb{R}$ and some $\varepsilon \in \{1, -1\}$. Using this description, check that for the action of $G = \mathrm{O}_2(\mathbb{R})$ on $R = \mathbb{R}[x, y]$, we have $x^2 + y^2 \in R^G$. (As before, the action is via $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto M \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.)

^{L}6. Let A, B be $n \times n$ matrices such that there exists an invertible matrix $P \in \mathrm{GL}_n(\mathbb{R})$ with $PAP^{-1} = B$. Show that $\det(A) = \det(B)$. More generally, show that A and B have the same characteristic polynomial, i.e., show that $\det(A - \lambda I) = \det(B - \lambda I)$.

6.5. Recall that for the action of $G = \mathrm{GL}_n(\mathbb{R})$ on $R = \mathbb{R}[X_{n\times n}]$ given by conjugation, R^G is generated by the coefficients of the characteristic polynomial. Interpret the previous problem in this context.

^{L}7. Let A be an $n \times m$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. Show that the (i,j)th entry of $A^\top A$ is $\mathbf{a}_i \cdot \mathbf{a}_j$ for all $1 \leq i, j \leq m$.

7.5. Recall that for the action of $G = \mathrm{O}_n(\mathbb{R})$ on $R = \mathbb{R}[X_{n\times m}]$ given by left multiplication, R^G is generated by the entries of $X^\top X$. Using the previous problem, how does this description of R^G fit in with our geometrical definition of $\mathrm{O}_n(\mathbb{R})$?

8. Let $G = \mathrm{GL}_2(\mathbb{R})$ act on $R = \mathbb{R}[X_{2\times m}]$ in the usual way. Show that $R^G = \mathbb{R}$, i.e., the only invariant polynomials are the constants.

   Hint: Think about what the matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ does.
   Start out with $m = 1$ or 2 to get an idea.

9. Consider the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $R = \mathbb{R}[X_{2\times 2}]$ by left multiplication. For ease of notation, assume that the variables are denoted and arranged as

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}.$$

   Show that $x_1y_2 - x_2y_1 \in R^G$. More generally, show that if SL_2 is acting on the polynomial with variables

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & x_m \end{bmatrix},$$

   then $x_iy_j - x_jy_i \in R^G$ for all $1 \leq i < j \leq m$.

   Hint for the last part: If A and B are matrices of compatible sizes, think about how the columns of AB look. In particular, the i-th column of AB is the product of A and the i-th column of B.

9.5. Generalise the previous to $\mathrm{SL}_n(\mathbb{R})$ acting on $\mathbb{R}[X_{n\times m}]$.