

# Morphisms of Schemes: Chevalley's Theorem

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Moreover, the “obvious diagrams” must commute.

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The collection  $\{D(f) : f \in A\}$  forms a basis for the above topology.

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Let  $k$  be a field. We denote  $\operatorname{Spec} k[x]$  by  $\mathbb{A}_k^1$ .

Since  $k[x]$  is a PID, the prime ideals are  $\langle 0 \rangle$  and the maximal ideals.

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To conclude, the only closed singleton subset of  $\mathbb{A}_k^1$  is  $\{\langle 0 \rangle\}$ .

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This is called the **structure sheaf** on  $\text{Spec } A$ .

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In fact, (it follows that) the affine opens form a basis for  $X$ .

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The above is a [morphism of affine schemes](#). That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

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# Some definitions

Definition 14 (Compact morphism)

Definition 15 (Finite type morphism)

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Let  $K$  be generated by  $x_1, \dots, x_n$ , as a  $k$ -algebra.

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