Problem Set 5 - pre-REU 2025

Problem set on Invariant Theory
Problems marked with ^{L} are problems on linear algebra.

1. Show that the following sets of matrices form a group under matrix multiplication. In each case, justify why the matrices are indeed invertible.

   (a) The set of $n \times n$ matrices with determinant one.

   (b) The set of $n \times n$ matrices M satisfying $MM^\mathsf{T} = I_n$.

   (c) The set of $2n \times 2n$ matrices M satisfying $M\Omega M^\mathsf{T} = \Omega$, where Ω is the $2n \times 2n$ block matrix given as $\Omega = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.

2. Let R be a ring (such as the ring of polynomials). Let $f \colon R \to R$ be a function satisfying

   - $f(1) = 1$,
   - $f(x + y) = f(x) + f(y)$ for all $x, y \in R$,
   - $f(xy) = f(x)f(y)$ for all $x, y \in R$.

   Let S be the set of fixed points of R, i.e., $S \coloneqq \{r \in R : f(r) = r\}$. Show that S is closed under addition, multiplication, and contains 1.

3. From class, we know that any element of the orthogonal group looks like

   $$M = \begin{bmatrix} \cos(\theta) & -\varepsilon \sin(\theta) \\ \sin(\theta) & \varepsilon \cos(\theta) \end{bmatrix}$$

   for some $\theta \in \mathbb{R}$ and some $\varepsilon \in \{1, -1\}$. Using this description, check that for the action of $G = \mathrm{O}_2(\mathbb{R})$ on $R = \mathbb{R}[x, y]$, we have $x^2 + y^2 \in R^G$. (As before, the action is via $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto M \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.)

4. Let $G = \mathrm{GL}_2(\mathbb{R})$ act on $R = \mathbb{R}[X_{2\times m}]$ in the usual way. Show that $R^G = \mathbb{R}$, i.e., the only invariant polynomials are the constants.

   Hint: Think about what the matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ does.
   Start out with $m = 1$ or 2 to get an idea.

^{L}5. Let A, B be $n \times n$ matrices such that $\exists P \in \mathrm{GL}_n(\mathbb{R})$ with $PAP^{-1} = B$. Show that A and B have the same characteristic polynomial, i.e., show that $\det(A - \lambda I) = \det(B - \lambda I)$.

5.5. Recall that for the action of $G = \mathrm{GL}_n(\mathbb{R})$ on $R = \mathbb{R}[X_{n\times n}]$ given by conjugation, R^G is generated by the coefficients of the characteristic polynomial. Interpret the previous problem in this context.

^{L}6. Let A be an $n \times m$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. Show that the (i, j)th entry of $A^\mathsf{T} A$ is $\mathbf{a}_i \cdot \mathbf{a}_j$.

6.5. Recall that for the action of $G = \mathrm{O}_n(\mathbb{R})$ on $R = \mathbb{R}[X_{n\times m}]$ given by left multiplication, R^G is generated by the entries of $X^\mathsf{T} X$. Using the previous problem, how does this description of R^G fit in with our geometrical definition of $\mathrm{O}_n(\mathbb{R})$?

7. Consider the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $R = \mathbb{R}[X_{2\times 2}]$ by left multiplication. For ease of notation, assume that the variables are denoted and arranged as

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}.$$

Show that $x_1y_2 - x_2y_1 \in R^G$. More generally, show that if SL_2 is acting on the polynomial with variables

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & x_m \end{bmatrix},$$

then $x_iy_j - x_jy_i \in R^G$ for all $1 \le i < j \le m$.

Hint for the last part: If A and B are matrices of compatible sizes, think about how the columns of AB look. In particular, the i-th column of AB is the product of A and the i-th column of B.

7.5. Generalise the previous to $\mathrm{SL}_n(\mathbb{R})$ acting on $\mathbb{R}[X_{n\times m}]$.