Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

\# Bruns & Herzog → Cohen-Macaulay rings
- 1st Part

\# Affine algebra

X

Derived Category

Gelfand & Manin

E. Riehl's : Cat theory in context

Dwyer & Spalinski

$R \to$ ring (possibly non comm.)

R-complexes :

$$\cdots \to M_{i+1} \xrightarrow{\partial} M_i \xrightarrow{\partial} M_{i-1} \to \cdots \qquad \partial^2 = 0$$

$$\mathrm{im}(\partial_{i+1}) \subset \ker(\partial_i)$$

$$H_i(M) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

$$H(M) = (H_i(M))_{i\in\mathbb{Z}}$$

$\mathcal{C}(R)$ = category of R-complexes (morphisms as usual)

If $f : M \to N$, we get an induced map $H(f) : H(M) \to H(N)$.

Def^n f is a quasi isomorphism (or weak equivalence) if $H(f)$ is bijective. (Automatically iso.)

$W :=$ collection of weak equivalences in $\mathcal{C}(R)$

$D(R) := \mathcal{C}(R)[W^{-1}]$. (or $W^{-1}\mathcal{C}(R)$)

- Key property : W has the 2-out-of-6 property :

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,

- Key property: W has the 2-out-of-6 property:

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,

if gf and $hg \in W$, then f, g, h, hgf are in W.

Ex: $\Rightarrow$ 2-out-of-3 property

If f, g, fg are defined and 2 are in W, then so is the third.

Concretely: $\mathcal{C}(R) \rightsquigarrow K(R) \rightsquigarrow D(R)$.

$\uparrow$ homotopy category

$M, N \to$ R-complexes

$\mathrm{Hom}_R(M, N)$:= Hom-complex of abelian groups (when R is comm. this is an R-complex)

$\mathrm{Hom}_R(M,N)_n$:= Maps of degree n from M to N (no compatibility!)

$$\begin{tikzcd} \cdots \arrow[r] & M_i \arrow[r] \arrow[d] & M_{i+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & N_{i+n} \arrow[r] & N_{i+n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$$

$$\partial : \mathrm{Hom}_R(M, N)_{n+1} \longrightarrow \mathrm{Hom}_R(M, N)$$

$$\partial(f) = \partial^N f - (-1)^{n+1} f \partial^M.$$

Check: $\partial^2 = 0$.

Observe: $Z_0(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_{\mathcal{C}}(M,N)$.

Def. $f, g \in \mathrm{Hom}_{\mathcal{C}}(M,N)$ are homotopic if

$f - g \in B_0(\mathrm{Hom}_R(M,N))$, i.e.,

$f - g = \partial h$ for some $h \in \mathrm{Hom}_R(M,N)$.

$K(R) := \mathcal{C}(R)/_{\text{homotopy relation}}$.

Object = R-complexes

$\mathrm{Hom}_K(M,N) = H_0(\mathrm{Hom}_R(M,N))$.

- $f \sim g$ in $\mathcal{C} \Rightarrow f = g$ in $K(R)$.

$\Rightarrow H(f) = H(g)$

Def^n M an R-complex.

ΣM (or $M[1]$) is the R-complex

$$(\Sigma M)_i = M_{i-1}$$

with $\partial^{\Sigma M} = -\partial^M$.

$\mathrm{Proj}\, R :=$ Projective R-modules

$$\begin{tikzcd} K(R) \arrow[dr, "q"] & \\ \cup\!\mid & \\ K(\mathrm{Proj}\, R) \arrow[r, "q"'] & D(R) \arrow[l, dashed, bend right, "p"'] \end{tikzcd}$$

q = localisation

$\exists p : D(R) \to K(\mathrm{Proj}\ R)$, a full and faithful embedding

"projective resolutions"

left adjoint to q.

$$\mathrm{Hom}_K(pM, N) = \mathrm{Hom}_D(M, qN)$$

$f : M \to N$ morphism

$\mathrm{cone}(f) := N \oplus \Sigma M$ with differential

$$\begin{matrix} N_{i+1} \\ \oplus \\ M_i \end{matrix} \to \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \qquad \partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$0 \to N \hookrightarrow \mathrm{cone}(f) \to \Sigma M \to 0$.

f is w.e. $\Leftrightarrow H(\mathrm{cone}(f)) = 0$.

Image of p?

K-projectives.

P an R-complex is K-projective if given any solid diagram

$$\begin{tikzcd} & M \arrow[d, "\pi", "?\ \text{w.e.}"'] \\ P \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & N \end{tikzcd}, \quad \exists \text{ lift } \tilde{\alpha}.$$

FACT: $p: D(R) \xrightarrow{\sim} K\text{-}\mathrm{Proj}(R) \subseteq K(\mathrm{Proj}\ R)$.

morphisms here are up to homotopy

• $\mathrm{Hom}_K(R, M) = Z_0(M)$

$$\begin{tikzcd} 0 \arrow[d] & \\ R \arrow[r] \arrow[d] & M_0 \arrow[d, "\partial"] \\ 0 \arrow[r] & M_{-1} \end{tikzcd}$$

Using this,

check: R is K-projective.

Use:
surjective + w.e.
$\Downarrow$
$Z(M) \to Z(N)$ onto

• $(P_\lambda)_\lambda$ family of K-projectives

Then, $\bigoplus P_\lambda$ is also K-projective.

Conversely, closed under direct summands.

. K-projectives are closed under suspensions.

$\cdots \xrightarrow{\partial} P_{i+1} \xrightarrow{\partial} P_i \xrightarrow{\partial} P_{i-1} \xrightarrow{\partial} \cdots$

is K-proj, if P_i projective $\forall i$.

Ex: $0 \to P' \xrightarrow{f} P \xrightarrow{g} P'' \to 0$ exact seq. of complexes.

Then, if P' and P'' are K-projective, so is P.

If P', P, P'' are complexes of projectives, then any two being K-projective $\Rightarrow$ third is K-proj.

of ... any two being K-projective $\Rightarrow$ third is K-proj.

<u>Corollary</u>. Any bounded complex of projectives is K-projective.

<u>Proof</u>. $P_\cdot$: $0 \to P_b \to \cdots \to P_a \to 0$, each P_i proj.

Induce on $b-a$.

$b-a=0$ done earlier.

$$0 \to P_{\leq b-1} \to P \to \Sigma^b P_b \to 0. \qquad \blacksquare$$

$$P_{\leq b-1} = 0 \to P_{b-1} \to \cdots \to P_a \to 0$$

Next: Any complex of projectives with $P_i = 0$ $\forall i \ll 0$ is K-projective.

$$P_\cdot : \quad \cdots \to P_{a+1} \to P_a \to 0$$

$$P = \varinjlim_{n \geq a} P_{\leq n}, \quad \text{each } P_{\leq n} \text{ is projective since bounded.}$$

$$0 \to \bigoplus_n P_{\leq n} \xrightarrow{1-\delta} \bigoplus_n P_{\leq n} \to P \to 0.$$

K-proj

projectives

Use 2-out-of-3.

Do directly...

- $$\begin{array}{c} M \\ \pi \big\downarrow \text{w.e.} \\ N \end{array}$$

P is K-projective

$\Updownarrow$

$$\operatorname{Hom}_K(P, M) \cong \operatorname{Hom}_K(P, N).$$

Lecture 2 (11-01-2023)

11 January 2023 13:24

R ring.

$D(R) \simeq kProj(R)$.

Recall: $P \in \mathcal{C}(Proj\ R)$ is K-projective if

$$\begin{tikzcd} & X \arrow[d, "\simeq"] \\ P \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Example $P \in \mathcal{C}(Proj\ R)$ with $P_i = 0$ for all $i << 0$.

$$\cdots \to P_{a-1} \to P_a \to 0 \to 0 \to \cdots$$

Sketch. Construct lifting one step at a time.

Suppose $\tilde{\alpha} : P_{\leq n} \to X$ is a lifting.

Want $\tilde{\alpha} : P_{\leq n+1} \to X$ compatibly.

$$\begin{tikzcd} & & X \arrow[d, "\varepsilon"', "\simeq"] \\ \cdots \arrow[r] & P_{n+1} \arrow[r] & (P_n \to P_{n-1} \to \cdots \to P_a \to 0) \arrow[u, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

Let $s \in P_{n+1}$

We must have

- $\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$
- $\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$

Check that the above can be solved.

This uses three things:
- ε surjective $\Rightarrow \varepsilon$ surjective on boundaries
- $H(\varepsilon)$ iso $\Rightarrow \varepsilon$ surjective on cycles
- $\ker(\varepsilon)$ is acyclic.

$\Rightarrow$ Every module has a K-projective resolution.

$$\begin{tikzcd}[column sep=small] F: \cdots & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[dr, "\varepsilon"] & \\ & & \ker \partial_1 \arrow[ur] & & \ker \varepsilon \arrow[ur] & & M \end{tikzcd}$$

(Can avoid choices by taking generating set to be $M, \ker\varepsilon, \ker \partial_1, \ldots$)

$$F \xrightarrow{\varepsilon} M.$$

Defn: A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism

$$\varepsilon : P_\cdot \to M \quad \text{s.t.}$$

① ε is a quasi iso,

② $P_\cdot$ is K-projective.

(Not insisting surjective.)

② $P_\cdot$ is K-projective.

This makes it functorial!

Thm $\forall\, M_\cdot \in \mathcal{C}(R)$, $\exists$ surjective K-projective resolution:

$$\underset{\text{K-proj.}}{P_\cdot} \xrightarrow{\sim} M_\cdot$$

Def^n. An R-complex F is semi-free if F admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \cdots \subseteq \bigcup_{n\geq 0} F(n) = F$$

s.t. ① $F(n) \subseteq F$ is a subcomplex,

② $F(n+1)/F(n)$ graded free module with $\partial = 0$,

i.e. $\partial(F(n+1)) \subseteq F(n)$

Fact/Ex. semifree $\Rightarrow$ K-proj

Example. $\cdots \to F_{a+1} \to F_a \to 0$

$$F(n) = F_{\leq n} = 0 \to F_n \to \cdots \to F_a \to 0$$

$$\frac{F(n+1)}{F(n)} = \cdots \to 0 \to F(n+1) \to 0 \to \cdots$$

$$= \Sigma^{n+1} F_{n+1}.$$

Thm. Each $M_\cdot \in \mathcal{C}(R)$ has a surjective semi-free resolution

$$\underset{\text{K-projective}}{F_\cdot} \xrightarrow{\sim} M_\cdot$$

Corollary. Every K-projective is a retract of a semifree.

Proof. P is K-projective:

$$\begin{tikzcd} & F \arrow[d, "\wr"] \\ P \arrow[ur, dashed] \arrow[r, equal, "id"'] & P \end{tikzcd}$$

$\square$

→ Sketch (Baby ver of Quillen's "small object argument")

Construct $0 = F(0) \hookrightarrow F(1) \overset{i(1)}{\hookrightarrow} F(2) \overset{i(2)}{\hookrightarrow} \cdots \hookrightarrow F(n) \overset{i(n)}{\hookrightarrow} \cdots \hookrightarrow \bigcup F(n) = F$

Construct

$$\begin{tikzcd}
0 = F(0) \arrow[r, hook] \arrow[dr, "\varepsilon(0)"'] & F(1) \arrow[r, hook, "i(1)"] \arrow[d, "\varepsilon(1)"] & F(2) \arrow[r, hook, "i(2)"] \arrow[dl, "\varepsilon(2)"] & \cdots \arrow[r, hook] & F(n) \arrow[r, hook, "i(n)"] \arrow[dlll, bend left, "\varepsilon(n)"] & \cdots \arrow[r, hook] & \bigcup_{n\geq 0} F(n) = F \\
& M & & & & &
\end{tikzcd}$$

s.t. ① $F(n+1)/F(n)$ is graded free with $\partial = 0$,

② $\varepsilon(1)$ is surjective on homology.

(In turn, each $\varepsilon(n)$ is surjective on homology.)

③ $\ker(H(\varepsilon(n))) \subseteq H(F(n))$ maps to 0 under $H(i(n))$.

This does the job. ["Something 0 in colim is 0 at finite stage."]

Why is ε surjective?

Remark. $\varepsilon : X \to Y$ s.t.

$Z(\varepsilon)$ surjective + $H(\varepsilon)$ bijective.

Then, ε is surjective.

Indeed, we have s.e.s.es:

$$\begin{tikzcd}
0 \arrow[r] & B(X) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & H(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B(Y) \arrow[r] & Z(Y) \arrow[r] & H(Y) \arrow[r] & 0
\end{tikzcd}$$

Snake lemma $\Downarrow$ $B(X) \to B(Y)$ is epi $\Downarrow$ $X \to Y$ is surj

$$\begin{tikzcd}
0 \arrow[r] & Z(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \Sigma B(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] & Y \arrow[r] & \Sigma B(Y) \arrow[r] & 0
\end{tikzcd}$$

Construction of $F(n), \varepsilon(n)$:

$\varepsilon(1) : F(1) \to M$ free cover of cycles.

0 diffs

($\varepsilon(1)$ morphism since mapping on cycles)

Say we have constructed $\varepsilon(n) : F(n) \longrightarrow M$.

Choose cycles $(z_\lambda)_\lambda \subseteq F(n)$ that map to a generating set of $\ker(H(\varepsilon(n)))$

Pick w_λ s.t. $\partial(w_\lambda) = \varepsilon(n)(z_\lambda)$.

Set $F(n+1) = F(n) \oplus R e_\lambda$ $\qquad \deg(e_\lambda) = \deg(z_\lambda) + 1$.

with $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_\lambda) = z_\lambda.$$

Define $\varepsilon(n+1) : F(n+1) \longrightarrow M$

$$\varepsilon(n+1)|_{F(n)} = \varepsilon(n),$$

$$\varepsilon(n+1)(e_\lambda) = \omega_\lambda.$$

$\boxtimes$

Remarks. ① As before the above construction can be made functorial by avoiding choices (consider all choices!).

② Depending on what we wish to do with the resolution, there are other constructions.

Given a module, we have the graded homology module $H(M) = \langle H_i(M) \rangle_{i \in \mathbb{Z}}$.

(Recall: for us, a graded module is a collection of modules.)

If $P_\cdot \xrightarrow{\sim} H(M)$ is a projective res^n, one can "perturb" the differentials of $P_\cdot$ to construct a K-projective resolution of M.

(Adam's resolution, Cartan/Eilenberg resolution)

Exercise. $P_\cdot$ K-proj $\Rightarrow$ P_i projective $\forall i$.

Converse of above NOT true.

Example (Dold's): Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the complex

$$P_\cdot : \quad \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} \cdots$$

One way of seeing that the above is not K-projective is to do the following exercise and note that $P_\cdot$ is acyclic but not contractible.

is to do the following exercise and ...

$P_\cdot$ is acyclic but not contractible.

Exercise. If P is K-proj and $H(P)=0$, then P is contractible, i.e., $\mathrm{id}_P \sim 0$.

(or: P is the mapping cone of some id.)

We saw we have an inclusion
$$\mathrm{KProj}(R) \hookrightarrow K(\mathrm{Proj}\ R).$$

FACT. Let R be comm. Noetherian.
The above inclusion is an equality iff R is regular.

Examples of reg. rings: $\mathbb{Z}$, $k[x_1,\dots,x_n]$.

Derived functors

Let $M_\cdot$ be an R-complex.

FACT. If $P_\cdot \xrightarrow{\simeq} M_\cdot$ and $Q_\cdot \xrightarrow{\simeq} M_\cdot$ are K-projective resolutions, then $P_\cdot \cong Q_\cdot$ in $K(\mathrm{Proj}\ R)$.

Given any $N_\cdot \in \mathcal{C}(R)$, set
$$\mathrm{RHom}_R(M_\cdot, N_\cdot) := \mathrm{Hom}_R(P_\cdot, N_\cdot),$$
where $P_\cdot$ is a K-proj. res^n of $M_\cdot$

The object on the right is defined in the homotopy category of abelian groups, i.e.,

$\mathrm{RHom}_R(-, N_\cdot)$ is a functor

$\mathcal{C}(R) \to K(\mathbb{Z})$.

$R\mathrm{Hom}_R(-, N_\cdot)$ is a functor

$$\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$$

(If R is comm, then $\longrightarrow K(R)$.)

Define $\mathrm{Ext}^i_R(M_\cdot, N_\cdot) := H^i(R\mathrm{Hom}_R(M_\cdot, N_\cdot))$

$$= H_{-i}(\mathrm{Hom}_R(P_\cdot, N_\cdot))$$

$$\mathrm{Ext}^0_R(M, N) = H_0(\mathrm{Hom}_R(P_\cdot, N_\cdot))$$

$$= \text{morphisms } P_\cdot \to N_\cdot \text{ up to homotopy}$$

$$\mathrm{Ext}^i_R(M, N) = \text{—}\|\text{——} \; P_\cdot \to \Sigma^i N_\cdot \text{ —}\|\text{—}$$

If $Q_\cdot \xrightarrow{\sim} N$ is a K-proj resol^n, then

$$\mathrm{Hom}_R(P_\cdot, Q_\cdot) \cong \mathrm{Hom}_R(P_\cdot, N_\cdot).$$

quasi-iso

Tensors Let $M_\cdot$ be a chain complex of <u>right</u> R-modules.

Let $N_\cdot \in \mathcal{C}(R)$.

$M_\cdot \otimes N_\cdot$ is a complex of $\mathbb{Z}$-modules defined by

$$(M_\cdot \otimes N_\cdot)_i = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-1)^{|m|} m \otimes \partial n$$

FACT. If $X_\cdot \xrightarrow{\sim} Y_\cdot$ is a quasi iso, then $P_\cdot \otimes_R X_\cdot \xrightarrow{\sim} P_\cdot \otimes Y_\cdot$ for any K-proj $P_\cdot$.

Defn. $M_\cdot \otimes_R^{L} N_\cdot := P_\cdot \otimes_R N_\cdot$ where $P_\cdot \xrightarrow{\sim} M_\cdot$ is a K-proj. resolⁿ.

$$\mathrm{Tor}_i^R(M_\cdot, N_\cdot) = H_i(P_\cdot \otimes_R N_\cdot).$$

$X \xrightarrow{\sim} Y$ quasi iso $\Rightarrow \mathrm{Tor}_i^R(M_\cdot, X_\cdot) = \mathrm{Tor}_i^R(M_\cdot, Y_\cdot)$.

Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \to$ comm. Noetherian ring

$M \to$ R-module

$r \in R$ is a zerodivisor on M if $r \cdot m = 0$ for some $m \neq 0$.

nzd = "not a zero divisor"

$$Z_R(M) = \{ r \in R : r \text{ is a z.d. on } M \}$$

$$= \bigcup_{p \in \mathrm{Ass}(M)} p.$$

(M need not be finite. Union need not be)

Fix R, M. Let $\underline{x} = x_1, \dots, x_n$ be a sequence in R.

$\underline{x}$ is weakly M-regular or a weakly regular sequence on M

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \dots, x_i)M} \quad \text{for } 0 \leq i \leq n-1.$$

$\underline{x}$ is M-regular (or ...) if further $M/(\underline{x})M \neq 0$.

Ex. $R = k[x_1, \dots, x_n]$.
$\underline{x} := x_1, \dots, x_n$ is a regular sequence on R.

Koszul Complexes. Given $r \in R$,

$$K(r; R) = 0 \to \underset{\deg 1}{R} \xrightarrow{r} \underset{\deg 0}{R} \to 0.$$

$H_1(K(r;R)) = 0 \iff r$ is nzd on R.

Given $\underline{x} = x_1, \dots, x_n$, we define

$$K(\underline{x}; R) = \bigotimes_{i=1}^{n} K(x_i; R).$$

$$K(\underline{x}; R) = 0 \to R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \dots \to R^{\binom{n}{2}} \to R^n \xrightarrow{\cdot (x_1 \cdots x_n)} R \to 0$$

$$K(\underline{x};R) = 0 \to R \xrightarrow{\pm\begin{bmatrix}x_1\\-x_2\\\vdots\end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{\cdot(x_1\cdots x_n)} R \to 0$$

$$\begin{bmatrix}r_1\\\vdots\\r_n\end{bmatrix} \mapsto \sum r_i x_i$$

Now, given $M \in \mathcal{C}(R)$,

$$K(\underline{x};M) := K(\underline{x},R)\otimes M.$$

$\hookrightarrow$ Koszul complex on $\underline{x}$ with coefficients in M.

$$H_i(\underline{x};M) := H_i(K(\underline{x};M)).$$ $\to$ Koszul homology

---

If M is simply an R-module (viewed in degree 0), then

$K(\underline{x};M)$:

$$0 \to M \to M^n \to \cdots \to M^n \to M \to 0$$

"same" differentials

$$H_0(\underline{x};M) = M/\underline{x}M,$$

$$H_n(\underline{x};M) = \{m \in M : x_i m = 0 \ \forall i\}$$
$$= (0 :_M (\underline{x})).$$

---

① $K(\underline{x};M) = K(x_1;R)\otimes_R K(x_2;R)\otimes_R \cdots \otimes K(x_n;R)\otimes M$

$$= K(x_1;R)\otimes K(x_{\geq 2},M)$$

$$= K(x_1;K(x_{\geq 2},M))$$

② $X,Y\in\mathcal{C}(R) \rightsquigarrow X\otimes_R Y \xrightarrow{\sim} Y\otimes_R X$ as R-complexes.

$$x\otimes y \longmapsto (-1)^{|x||y|}\, y\otimes x$$

$\therefore\ K(\underset{\sim}{x}; R) \cong K(\underset{\sim}{x}^{\sigma}; R)$ for any $\sigma \in S_n$.

$\Rightarrow K(\underset{\sim}{x}; M) \cong K(\underset{\sim}{x}^{\sigma}; M)$ — " —.

(Can apply this to Obs ①.)

2nd Perspective: "Koszul complexes are iterative mapping cones."

$f: X \to Y$ morphism of complexes

$$\mathrm{Cone}(f) = \left(Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ & -\partial^X \end{pmatrix}\right).$$

s.e.s. : $0 \to Y \to \mathrm{Cone}(f) \to \Sigma X \to 0.$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$

$\begin{pmatrix} y \\ x \end{pmatrix} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow[H_i(f)]{} H_i(Y) \to H_i(\mathrm{Cone}(f)) \to \underset{\displaystyle H_{i-1}(X)}{\overset{\rotatebox{90}{$\cong$}}{H_i(\Sigma X)}} \to \cdots$$

connecting map

Consider: $x \in R$

$f: R \xrightarrow{x} R$

$1 \mapsto x$

$$\mathrm{Cone}(R \xrightarrow{x} R) = \left(\underset{\deg 0}{R} \oplus \underset{\deg 1}{R}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = K(x; R).$$

Ditto: If $x \in R$, and $M \in \mathscr{C}(R)$ → complex, not necessarily in deg 0

$$\mathrm{Cone}(M \xrightarrow{x} M) = K(x; M)$$

$\underline{x} = x_1, x_2, \ldots, x_n$

$$K(\underline{x}; M) = K(x_1 ; K(x_{\geq 2}; M))$$

$$= \mathrm{Cone}\left(K(x_{\geq 2}; M) \xrightarrow{x_1} K(x_{\geq 2}; M)\right)$$

on homology

iterate

$$H_i(x_{\geq 2}; M) \xrightarrow{\pm x_1} H_i(x_{\geq 2}; M) \to H_i(\underline{x}; M) \to H_{i-1}(x_{\geq 2}; M) \xrightarrow{\pm x_1} \cdots$$

s.e.s.

$$0 \to \frac{H_i(x_{\geq 2}; M)}{x_1 H_i(x_{\geq 2}; M)} \to H_i(\underline{x}; M) \to (0 :_{H_{i-1}(x_{\geq 2}; M)} x_1) \to 0$$

$M \to$ R-module

$$K(\underline{x}; M) \rightsquigarrow H_0(x; M) = \frac{M}{\underline{x}M}$$

So, $K(\underline{x}, M) \to \dfrac{M}{\underline{x}M}$ is a w.e. quasi iso

$\Leftrightarrow H_i(\underline{x}; M) = 0 \quad \forall i > 0.$

Def^n. $\underline{x}$ is Koszi-regular on M (or...) if

$$H_i(\underline{x}; M) = 0 \ \forall\, i \geq 1.$$

Lemma. When $\underline{x}$ is weakly M-reg, ($M \to$ R-mod)

(weakly-reg $\Rightarrow$ Koszi-reg)

$$K(\underline{x}, M) \twoheadrightarrow M/(\underline{x})M \text{ is a w.eq.}$$

Proof. $n=1$: $\quad 0 \to M \xrightarrow{x} M \to 0.$

$H_1(x; M) = 0 \Leftrightarrow x$ nzd on M

$n \geq 2$: $\quad K(\underline{x}; M) = K(x_n ; K(x_{<n}; M)).$

By induction,

$$K(x_{<n}; M) \xrightarrow{\sim} \frac{M}{(x_{<n})M}.$$

Now, $0 \to R \xrightarrow{x_n} R \to 0$ is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x}; M) = K(x_n; R) \otimes_R K(x_{<n}; M)$$

$$\simeq K(x_n; R) \otimes_R \frac{M}{(x_{<n})M}$$

now note that x_n is a nzd on $M/(x_{<n})M$ and we are done. ☒

Instead of semifree, can use l.e.s. of homology and induction.

(Semifree business $\Rightarrow$ ($M \simeq N$ quasi $\Downarrow$ $K(\underline{x}; M) \simeq K(\underline{x}; N)$ quasi))

Note: ① $\underline{x}$ Koszi-reg $\Rightarrow$ $\underline{x}^\sigma$ is Koszi-reg $\forall \sigma \in S_n$.

② Not true for weakly regular.

Theorem. Say $\underline{x} \subseteq J(R)$ and $M \overset{\neq 0}{}$ f.g. R-module.

TFAE:

1) $\underline{x}$ is M-regular. ($\equiv$ weakly M-reg, by NAK.)

2) $\underline{x}$ is Koszi M-regular, i.e., $H_i(\underline{x}; M) = 0$ $\forall i \geq 1$.

3) $H_1(\underline{x}; M) = 0$.

In particular, take R local and x_i nonunits.

Proof. ① $\Rightarrow$ ② $\Rightarrow$ ③ is clear.

Only need to prove ③ $\Rightarrow$ ①.

Already saw for $n=1$.

Induction:

$$K(\underline{x}; M) = K(x_n; K(x_{<n}; M)).$$

l.e.s.

$$0 \to \frac{H_i(x_{<n}; M)}{x_n H_i(x_{<n}; M)} \to H_i(\underline{x}; M) \to (0 :_{H_{i-1}(x_{<n}; M)} x_n) \to 0. \qquad (*)$$

Put $i=1$ to get $\dfrac{H_1(x_{<n}; M)}{x_n H_1(x_{<n}; M)} = 0$

$$\xRightarrow{\text{NAK}} H_1(x_{<n}; M) = 0.$$

(Note: Koszul homology modules are f.g. when M is f.g.)

$\xRightarrow{\text{induction}}$ $x_1, \ldots, x_{n-1}$ is M-reg. — (1)

Moreover, (*) now tells us

$$(0 :_{H_0(x_{<n}; M)} x_n) = 0.$$
$$\| $$
$$\ker\left(\frac{M}{x_{<n} M} \xrightarrow{x_n} \frac{M}{x_{<n} M}\right).$$

$\therefore$ x_n is nzd on $M/(x_{<n})M$. — (2)

(1) & (2) finish. ∎

Corollary. $\mathfrak{q} \subseteq J(R)$, M f.g., the property of $\underline{x}$ being regular is not dependent on the order of x_i.

(Permutation of regular is regular.)

$R = k[x, y, z]$

$x, y(1-x), z(1-x)$ reg

$y(1-x), z(1-x), x$ NOT

Lemma. If $\underline{x} = x_1, \ldots, x_n \subseteq R$, M an R-module.

TFAE:

① $\underline{x}$ is M-Koszul-regular.

② $\underline{x}^{a}$ is M-Koszul-regular for some $\underline{a} \geq \underline{1}$.

Proof. Suffices to prove:

$x_1, \boxed{x_2, \ldots, x_n}$ is M-KR

$\iff x_1^a, \boxed{x_2, \ldots, x_n}$ is M-KR for some $a \geq 1$.
(/all)

$x_1, x_2, \ldots, x_n$ KR

$$\Rightarrow K(x_1 ; K(x_{\geq 2} ; M)) \xrightarrow{\sim} K\left(x_1 ; \frac{M}{(x_{\geq 2})M}\right).$$

Replacing M by $M/(x_{\geq 2})M$ we are reduced to $n = 1$.

But

x is reg on M

$\iff x$ is nzd on M

$\iff x^a$ is nzd on M for some $a \geq 1$

$\iff x^a$ is reg on M ——"——.

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$ and M f.g. R-module.

Let $i \geq 0$ be s.t. $H_i(\underline{x} ; M) = 0$.

Then, $H_j(\underline{x} ; M) = 0 \quad \forall j \geq i$.

HW.

Lecture 4 (23-01-2023)

Monday, January 23, 2023 1:19 PM

$R \to$ commutative Noetherian

Given complexes, $M, N \in \mathcal{C}(R)$.

$$R\mathrm{Hom}_R(M, N) := \mathrm{Hom}_R(pM, N)$$

$pM \xrightarrow{\sim} M$ is a K-proj res^n

$$\mathrm{Ext}^*_R(M,N) := H^i(R\mathrm{Hom}_R(pM, N)).$$

Given $M, N, P \in \mathcal{C}(R)$, we have a morphism

$$\theta : R\mathrm{Hom}_R(M, N) \otimes^L_R P \longrightarrow R\mathrm{Hom}_R(M, N \otimes^L_R P).$$

Lemma. θ is a w.e. when P is perfect, i.e,

$$P \simeq (0 \to P_0 \to \cdots \to P_n \to 0),$$

each P_i f.g. R-module.

$$\mathrm{Hom}_R(pM, N) \otimes_R pP \longrightarrow \mathrm{Hom}_R(pM, N \otimes_R pP)$$

$$f \otimes x \longmapsto m \mapsto (-1)^{|m||x|} f(m) \otimes x.$$

in $\mathcal{C}(R)$

Check this is a morphism.

Proof. Replace P with $0 \to P_0 \to \cdots \to P_n \to 0$.

"Things true for $R \rightsquigarrow$ true for perfect."

Then it reduces to checking for one f.g. projective P.

But that can be reduced to f.g. free.

That reduces to R. Obvious. ∎

Rees' Lemma

Setup. $\underline{x} \subset R$ finite subset. ($\underline{x} = x_1, \dots, x_c$)

$N \rightsquigarrow R$-module s.t. $\underline{x} N = 0$.
$M \rightsquigarrow R$-module s.t. $\underline{x}$ is Koszul-regular on M.

i.e. $K(\underline{x}; M) \xrightarrow{\sim} M/\underline{x}M$

Lemma. Then, $\mathrm{RHom}_R(N, M/\underline{x}M) \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

In particular, $\mathrm{Ext}^*_R\left(N, \frac{M}{\underline{x}M}\right) \cong \mathrm{Ext}^*_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$.

$\Lambda^*(\Sigma R^c)$: $\underset{\text{deg } c}{0 \to R} \xrightarrow{0} R^c \xrightarrow{0} \cdots \xrightarrow{0} R^{\binom{c}{2}} \xrightarrow{0} R^c \to \underset{\text{deg } 0}{R} \to 0$

Proof. $\mathrm{RHom}_R(N, M/\underline{x}M) \simeq \mathrm{RHom}_R(N, K(\underline{x}; M))$

$\simeq \mathrm{RHom}_R(N, M \otimes^L_R K(\underline{x}; R))$

$\simeq \mathrm{RHom}_R(N, M) \otimes^L_R K(\underline{x}; R)$ perfect

Since $\underline{x} N = 0$, $\underline{x}\, \mathrm{Ext}^*_R(N, M) = 0$.

Alter: $\mathrm{RHom}_R(N, M) \simeq \mathrm{Hom}_R(N, I)$ where $M \xrightarrow{\simeq} I$ is an injective resol^n.

Now, $\underline{x} \cdot \mathrm{Hom}_R(N, I) = 0$.

$\therefore$

$\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{x}; R)$

$\simeq K(\underline{x}; \mathrm{Hom}_R(N, I))$

$\simeq K(\underset{\text{length } c}{\underline{0}}; \mathrm{Hom}_R(N, I))$

$\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{0}; R)$

$\cong \mathrm{Hom}_R(N, I) \otimes_R K(Q; R)$

$= \mathrm{Hom}_R(N, I) \otimes_R \Lambda^*(\Sigma R^c)$. ▨

Last statement means:

$$\mathrm{Ext}^n_R\left(N, \frac{M}{\underline{x}M}\right) \cong \left(\mathrm{Ext}_R(N, M) \otimes_R \Lambda(\Sigma R^c)\right)^n$$

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R (\Lambda(\Sigma R^c))^{n-i} \quad \} \text{think}$$

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R R^{\binom{c}{i-n}}.$$

$V \to \mathbb{Z}$-graded object

View V as having both an upper grading and lower grading via $V^i = V_{-i}$.

Notation:

$$\sup V^* = \sup\{i : V^i \neq 0\}$$

$$\inf V^* = \inf\{i : V^i \neq 0\}.$$

$$\sup V_* = \sup\{i : V_i \neq 0\} \quad \ldots$$

$$= -\inf V^*$$

Corollary. $\rho := \inf \mathrm{Ext}^*_R(N, M) = \inf \mathrm{Ext}^*_R(N, M/\underline{x}M) + c.$

$$\mathrm{Ext}^\rho_R(N, M) = \mathrm{Ext}^{\rho-c}_R(N, M/\underline{x}M)$$

Defn. Fix $I \subseteq R$ ideal. $M \in \mathcal{C}(R)$.

$$\mathrm{depth}_R(I, M) := \inf \mathrm{Ext}^*(R/I, M).$$

I-depth of M

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences of complexes on . .

Properties. ① $0 \to L \to M \to N \to 0$ exact sequence (of complexes or modules).

Then, $\operatorname{depth}_R(I, M) \geq \min\{\operatorname{depth}_R(I,L), \operatorname{depth}_R(I,N)\}$.

Similarly relations with other two using l.e.s.

$$\to \operatorname{Ext}^i_R(R/I, L) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, N)$$
$$\to \operatorname{Ext}^{i+1}_R(R/I, L) \to \cdots$$

Thm (depth).

② Let $\underline{x} = x_1, \ldots, x_c$ be a gen. set for the ideal I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M) \qquad \forall M \in \mathcal{C}(R).$$

Will prove when M is an R-module. (recall Koszul homology)

Koszul complexes revisited

Giving $x_1, \ldots, x_c$ in R

$\leftrightarrow$ Giving a map $f : F \to C$, where F is a free mod of rank c and chosen basis

$K(f) := (\Lambda^*(\Sigma F), \partial)$, where ∂ is

$$e_{i_1} \wedge \cdots \wedge e_{i_n} \longmapsto \sum_{j=1}^{n} (-1)^{j-1} f(e_{i_j})\, x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_n}.$$

Lemma. Fix $\underline{x} = x_1, \ldots, x_c \subseteq R$. For any $y \in (\underline{x})$, we have

$$K(\underbrace{\underline{x}, y}_{c+1 \text{ seq.}}; M) \cong K(\underline{x}, 0; M)$$
$$\cong K(\underline{x}; M) \otimes (0 \to \underset{\deg 0}{R} \to R \to 0).$$

Proof.

$$\begin{tikzcd} R^{c+1} \arrow[r, "{[x_1 \ \cdots \ x_c \ y]}"] & R \\ \arrow[u, "\cong"', "{\begin{bmatrix} 1 & & & -r_1 \\ & \ddots & & \vdots \\ & & 1 & -r_c \\ & & & 1 \end{bmatrix}}"] \arrow[r, phantom, "2"] & \arrow[u, equal] \end{tikzcd}$$

$y = \sum r_i x_i$

$$R^{c+1} \xrightarrow{[x_1 \cdots x_c \;\; 0]} R \qquad y = \sum r_i x_i$$

In particular,

$$\sup H_*(\underline{x}, y; M) = 1 + \sup H_*(\underline{x}; M)$$

Thus,

$$c+1 - \sup H_*(\underline{x}, y; M) = c - \sup H_*(\underline{x}; M).$$

Corollary. $c - \sup_* H(\underline{x}, y; M)$ is independent of a generating set of I. (If $\underline{x}$, $\underline{y}$ generate, look at $\underline{x}, \underline{y}$ & $\underline{y}, \underline{x}$.)

Proof of Thm (Depth). When M is a module

$$\begin{aligned} \operatorname{depth}(I, M) = 0 &\iff \operatorname{Hom}(R/I, M) \neq 0 \\ &\iff I \subseteq \operatorname{Zdv}_R(M) \\ &\iff H_c(\underline{x}; M) \neq 0 \\ &\iff \sup H_*(\underline{x}; M) = c. \end{aligned}$$

Actually, $H_c(\underline{x}; M) \cong \operatorname{Hom}(R/I, M)$

Can assume $\operatorname{depth}_R(I, M) \geq 1$, i.e., $\exists y \in I$ nzd on M

Then, in particular, y is Koszul-reg on M.

By Rees, ($\because y(R/I) = 0$)

$$\inf \operatorname{Ext}^*_R(R/I, M/yM) = \inf \operatorname{Ext}^*_R(R/I, M) - 1.$$

$$\therefore \operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R\left(I, \frac{M}{yM}\right)$$

{induction now applies

$$= 1 + \left(c - \sup H_*\left(\underline{x}; \frac{M}{yM}\right)\right)$$

$H_*(\underline{x}; M/yM)$

$$\begin{aligned}
&H_*(\underline{x};\ M/yM)\\
&\cong H_*(\underline{x};\ K(y;M))\\
&\cong H_*(K(\underline{x},y;M))\\
&\cong H_*(K(\underline{x},0;M))
\end{aligned}$$

$$= c + 1 - \sup H_*(\underline{x},0;\ M)$$

$$= c - \sup H_*(\underline{x};M).$$

Ex. Show $\mathrm{depth}_R(I, M)$ is the length of longest ___ M-regular in I.

Lecture 5 (25-01-2023)

Wednesday, January 25, 2023 1:19 PM

$R \to$ comm. noetherian ring

$I \subsetneq R$ ideal

$M \to$ R-complex

$$\operatorname{depth}(I, M) = \inf \operatorname{Ext}^*_R(R/I, M)$$

$I = (x_1, \ldots, x_c)$

$$= c - \sup H_*(\underline{x}; M)$$

- When M is a module, $\operatorname{depth}(I, M) =$ length of any maximal M-Koszul-regular sequence in I

(Maybe $IM \neq M$ needed?)

If M f.g. and $I \subseteq \operatorname{Jac}(R)$

$=$ length of any maximal M-reg. seq in I

<u>Observation.</u> $\underline{x} = x_1, \ldots, x_c$

$\underline{y} = y_1, \ldots, y_d$

$$\sup H_*(\underline{x}, \underline{y}; M) \leq \sup H_*(\underline{x}; M) + d.$$

(Can use l.e.s. to see this.)

$$K(\underline{x}, \underline{y}; M) \cong K(\underline{x}; K(\underline{y}; M))$$

This implies

$$I \subseteq J \Rightarrow \operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(J, M).$$

Also, $\sqrt{I} = \sqrt{J} \Rightarrow \operatorname{depth}_R(I, M) = \operatorname{depth}_R(J, M)$.

<u>Today.</u> $(R, \mathfrak{m}, k)$ is a local ring:

- R is commutative Noetherian,
- $\mathfrak{m}$ is the unique maximal ideal of R,

- $k = R/\mathfrak{m}$.

In this case, $\operatorname{depth}(M) = \mathfrak{m}$-depth of M

$$= \inf \operatorname{Ext}^*_R(k, M).$$

It suffices to compute the above using $\underline{x} = x_1, \dots, x_c$

s.t. $\sqrt{(\underline{x})} = \mathfrak{m}$.

Thus we can take c minimal as $c = \dim(R)$.

(Then, $\underline{x}$ is a system of parameters.)

$\therefore$ Can compute using $\dim(R)$ elements.

×

Ausland-Buchsbaum Equality

- $F \rightarrow$ an R-complex

F has finite flat dimension if

$$F \simeq (0 \to F_b \to \cdots \to F_a \to 0)$$

with F_i flat.

We write $\operatorname{flat\,dim}_R F < \infty$.

Examples
- Flat modules.
- Perfect complexes.
- Koszul complexes.

If $\operatorname{flat\,dim}_R F < \infty$, then $\operatorname{Tor}_i(-, F) = 0 \quad \forall |i| \gg 0$ on $\operatorname{Mod} R$.

$$\mathrm{Tor}_i^R(M, F) = H_i\big(M \otimes_R (0 \to F_b \to \cdots \to F_a \to 0)\big)$$
$$= 0 \qquad \text{for } i \notin [a, b].$$

In fact, the above characterises flat $\dim_R F < \infty$.

Theorem (AB equality) $(R, \mathfrak{m}, k)$ local.

$F \to$ finite flat dimension. Then,

$$\operatorname{depth}_R(M \otimes^L_R F) = \operatorname{depth}_R M - \underbrace{\sup H_*(k \otimes^L_R F)}_{\mathrm{Tor}_*^R(k, F)}$$

for **ANY** R-complex M.

Specialise: ① $M = R$.

$$\operatorname{depth}_R(F) = \operatorname{depth}_R(R) - \sup H_*(k \otimes^L_R F).$$

Now, let N be a f.g. R-module.

Such an N has a minimal free resolution.

$$\begin{tikzcd}
\cdots \arrow[r] & R^{b_2} \arrow[rr, "\partial_2"] \arrow[dr, two heads] & & R^{b_1} \arrow[rr, "\partial_1"] \arrow[dr, two heads, "\text{min}"] & & R^{b_0} \arrow[r, two heads, "\varepsilon"] & N \\
& & \ker \partial_1 \arrow[ur] & & \ker \varepsilon \arrow[ur] & &
\end{tikzcd}$$

minimally

$\ker \partial_1 \subseteq \mathfrak{m} R^{b_1}$ $\quad$ $\ker \varepsilon \subseteq \mathfrak{m} R^{b_0}$, by minimality

This gives a complex

$$\begin{array}{l} G : (\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to 0) \\ \qquad\qquad\qquad\qquad\qquad\quad \downarrow \simeq \\ \qquad\qquad\qquad\qquad\qquad\quad N \end{array}$$

$\partial G \subseteq \mathfrak{m} G$. $\quad$ G turns out to be unique up to isomorphism of complexes.

up to isomorphism of complexes.

"The" minimal free resolution of N.

$$\mathrm{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes G)_i.$$

H_i = everything since $\partial \otimes k = 0$

$$\therefore \quad \mathrm{Tor}_i^R(k, N) = 0 \iff G_i = 0.$$

$\therefore$ flat $\dim_R N < \infty \iff$ N has a finite free resolution, i.e., N is perfect.

$$\sup \mathrm{Tor}_*^R(k, N) = \text{length of } G$$

$$=: \mathrm{proj}\dim_R N.$$

② If N is a f.g. R-module with $\mathrm{proj}\dim_R N < \infty$, then

$$\mathrm{proj}\dim_R(N) + \mathrm{depth}_R(N) = \mathrm{depth}_R(R).$$

(Classical AB Equality.)

<u>Corollary</u>. $\mathrm{proj}\dim_R N < \infty \Rightarrow \mathrm{depth}(N) \leq \mathrm{depth}(R)$.

Furthermore, equality iff N is projective.

False without assumption of proj dim $< \infty$

Note we had:

$$\mathrm{depth}_R(F) = \mathrm{depth}_R(R) - \sup H_*(k \otimes_R^L F)$$

$$\mathrm{depth}_R(M \otimes_R^L F) = \mathrm{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

Subtract:

$$\operatorname{depth}_R(F) - \operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}(R) - \operatorname{depth}(M).$$

Note: Some terms above may be ∞.

when $H_i(\underline{x}; M) = 0 \ \forall i$, then

$\sup H_*(\underline{x}; M) = -\infty$.

Proof of AB Equality

$$\operatorname{RHom}_R(k, M \otimes_R^L F) \xleftarrow{\simeq} \operatorname{RHom}_R(k, M) \otimes_R^L F$$

quasi iso because k is a f.g. module

$\therefore \exists\, G \xrightarrow{\simeq} k$ where each G_i is finite free

AND flat dim $F < \infty$.

More generally:

$$\operatorname{Hom}_R(N, M) \otimes R \xrightarrow{\simeq} \operatorname{Hom}_R(N, M \otimes_R F)$$

N f.g. R-mod, F flat

Key observation:

$\operatorname{RHom}_R(k, M) \simeq$ complexes of k-vector spaces

(take injective resolution of M)

$$\operatorname{RHom}_R(k, M) \otimes_R^L F \simeq \operatorname{RHom}_R(k, M) \otimes_k^L (k \otimes_R^L F)$$

$$\therefore \operatorname{Ext}_R^*(k, M \otimes_R^L F) = \operatorname{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^L F). \qquad \blacksquare$$

Observation: Let M be an R-complex.

Suppose $s := \sup H_*(M)$ is finite.

Then, $\operatorname{depth}_R M \geq -s$

Then, $\operatorname{depth}_R M \geq -s$

with equality iff $\operatorname{depth}_R H_s(M)$.

Note. For an R-module M, $\operatorname{depth} M = 0 \Leftrightarrow \inf \operatorname{Ext}^{i}(k, M) = 0$

$\Leftrightarrow \operatorname{Hom}(k, M) \neq 0$

$\Leftrightarrow k \hookrightarrow M$

$\Leftrightarrow \mathfrak{m} \in \operatorname{Ass}_R M$.

One proof. M as above.

$$\operatorname{Ext}_R^{-s}(N, M) = \operatorname{Hom}_R(N, H_s(M))$$

N any R-module.

- key. $M \simeq M'$ with $M_i' = 0 \;\forall\, i > s$.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{s+1} \arrow[r, "\partial"] \arrow[d, "0"] & M_s \arrow[r] \arrow[d] & M_{s-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[d, equal] & = M \arrow[d, "\wr"] \\
\cdots \arrow[r] & 0 \arrow[r] & \dfrac{M_s}{\partial(M_{s+1})} \arrow[r] & M_{s-1} \arrow[r] & \cdots & = M'
\end{tikzcd}$$

$\therefore$ Can assume $M_i = 0$ for $i \geq s+1$.

In particular,

$$0 \to \Sigma^s H_s(M) \hookrightarrow M \to M'' \to 0.$$

iso on homology in degrees $\leq s-1$

$H_i(M'') = 0 \quad i \geq s$.

Let $\underline{x} = x_1, \dots, x_n$ gen set for $\mathfrak{m}$.

Then,

Then,

$$H_{i+1}(\underline{x}; M'') \to H_i(\underline{x}, \Sigma^* H_s(M)) \to H_i(\underline{x}; M) \to H_i(\underline{x}; M'') \quad \text{--- } (*)$$

$H_j(M'') = 0 \ \forall j \geq s.$

So, $M'' \simeq M'''$ with $M_j''' = 0$ for $j \geq s$.

$K(\underline{x}; M''')_j = 0$ for $j \geq s+n+1$.

Thus, $H_j(\underline{x}; M''') = 0 \quad \forall j \geq s+n+1.$

$\Rightarrow H_i(\underline{x}; \Sigma^s H_s(M)) = 0 \quad \forall j \geq s+n+1.$

Put $i \geq n+s$ in $(*)$:

$H_j(\underline{x}; M) = 0 \ \forall j \geq n+s+1$

$\Rightarrow \sup H_*(\underline{x}; M) \leq n+s$

$\Rightarrow -s \leq n - \sup H_*(\underline{x}; M) = \text{depth } M.$

Moreover,

$$H_{n+s}(\underline{x}; M) \cong H_{n+s}(\underline{x}; \Sigma^s H_s(M))$$
$$\cong H_n(\underline{x}, H_s(M))$$

The above is nonzero iff depth $H_s(M) = 0$. ☒

---

① flat $\dim_R F < \infty$. Then $\forall M$

$$\text{depth}_R(M \otimes^L_R F) = \text{depth}_R(M) - \sup H_*(k \otimes^L_R F).$$

② $s = \sup H_*(M)$ is finite.

Then, $\operatorname{depth}_R(M) \geq -s$.

Equality $\Leftrightarrow$ $\operatorname{depth}(H_s(M)) = 0$.

<u>Application.</u> Say

$$F = \quad 0 \to F_d \to \cdots \to F_0 \to 0$$

minimal, $\partial F \subseteq \mathfrak{m} F$

is a finite free complex s.t.

$$0 < \operatorname{length}_R H_*(F) < \infty.$$

(All $H_i(F)$ are finite length and at least one is nonzero.)

Then, for any M,

$$\operatorname{depth}_R M = d - \sup H_*(F \otimes_R M).$$

Thus, such an F is depth sensitive.

When $\mathfrak{m}M \neq M$, one can check some $H_*(F \otimes_R M) \neq 0$.

Then, one gets

$$d \geq \operatorname{depth}_R(M).$$

<u>Note</u>: Over any local ring, $\exists M$ s.t. $\mathfrak{m}M \neq M$ and $\operatorname{depth}_R M = \dim R$.

M need not be f.g.

In particular, $d \geq \dim(R)$. $\longrightarrow$ New Intersection Theorem

Hochster ('70)
André (2016)
Bhatt (2021)

Proof. Assume $s = \sup H_*(F \otimes_R M)$.

Take any prime $\mathfrak{p} \neq \mathfrak{m}$.

$$H_i(F \otimes_R M)_{\mathfrak{p}} \cong H_i(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$$

$$H_i(F_{\mathfrak{p}}) = 0 \quad (\because \text{length } H_i(F) < \infty)$$

i.e. $F_{\mathfrak{p}} \simeq 0$ in $D(R_{\mathfrak{p}})$.

$\therefore H_*(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$

Thus, $H_i(F \otimes_R M)$ is $\mathfrak{m}$-power torsion.

(I.e., each $a \in H_i(F \otimes_R M)$ is killed by some $\mathfrak{m}^n$.)

$\therefore$ depth $H_s(F \otimes_R M) = 0$.

Thus, by previous result,

$$\text{depth}_R(F \otimes_R M) = -s$$

$$\| \text{AB}$$

$$\text{depth}(M) - \sup H(k \otimes_R^L F)$$

$$\Rightarrow \text{depth}(M) = \underbrace{\sup H_*(k \otimes_R^L F)}_{d} - s. \qquad \blacksquare$$

Lecture 6 (30-01-2023)

Monday, January 30, 2023 1:26 PM

Recap. $I \subseteq R$ Comm. Noe
M an R-complex.

$$\operatorname{depth}_R(I, M) = \inf \operatorname{Ext}_R^*(R/I, M).$$

- Choose any fin gen set $\underline{x} = x_1, \dots, x_c$ of I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M).$$

(Focus on the case $H_*(M)$ bounded.)

$(R, \mathfrak{m}_R, k) \to$ local

$$\operatorname{depth}_R M := \operatorname{depth}_R(\mathfrak{m}_R, M).$$

$$\operatorname{depth}_R(M) \geq -\sup H_*(M). \quad \text{—— (1)}$$

Equality $\iff \mathfrak{m} \in \operatorname{Ass}(H_s(M))$

$s = \sup H_*(M).$

Exercise 1.

Suppose $M = 0 \to M_b \to \cdots \to M_a \to 0$

$$\operatorname{depth}_R M \geq \inf\{\operatorname{depth}(M_i) - i : a \leq i \leq b\}.$$

$$\operatorname{depth}_R M \geq \inf\{\operatorname{depth} H_i(M) - i : \inf H_*(M) \leq i \leq \sup H_*M\}. \quad \text{—— (2)}$$

(Note (2) $\Rightarrow$ (1).)

Setup. $H_*(M)$ bounded. $\underline{x} = x_1, \dots, x_c$. ($R$ not necessarily local.)

$$\sup H_*(M) \underset{(i)}{\leq} \sup H_*(\underline{x}; M) \underset{(ii)}{\leq} \sup H_*(M) + c$$

Lemma 2. (a) Inequality (ii) always holds.
Equality iff $\operatorname{depth}_R(\underline{x}; H_s(M)) = 0$ $\quad s = \sup H_*(M)$

(b) (i) holds if $\underline{x} \subseteq J(R)$, each $H_i(M)$ is f.g.

(b) (i) holds if $\underline{x} \subseteq J(R)$, each $H_i(M)$ is f.g.

Equality holds iff $\underline{x}$ is $H_s(M)$-regular.

Proof. (a) $H(\underline{x}; M) = H(x_1 ; K(x_2, \dots, x_c ; M))$.

Reduce to $c = 1$. $x := x_1$.

In this case, we have

$$H_{i+1}(M) \xrightarrow{x} H_{i+1}(M) \to H_{i+1}(x;M) \to H_i(M) \xrightarrow{x} H_i(M)$$

For $i \geq s$, we get $H_{i+1}(x;M) = 0$.

$\therefore$ (ii) follows.

Moreover, $H_{s+1}(x;M) \neq 0 \iff x$ is a zd on $H_s(M)$.

(b) $H_i(M) \neq 0 \overset{NAK}{\Longrightarrow} H_i(x;M) \neq 0$

$\overset{?}{=} \dfrac{H_i(M)}{x H_i(M)}$

$\therefore \sup H_*(x;M) \geq \sup H_*(M)$.

Moreover,

$$0 \to H_{s+1}(x;M) \to H_s(M) \xrightarrow{x} H_s(M).$$

Corollary 3. $\operatorname{depth}_R(\underline{x}; M) \geq -\sup H_*(M)$.

Equality $\iff \operatorname{depth}(\underline{x}, H_s(M)) = 0$

$\overset{?}{\iff} (\underline{x}) \subseteq \mathfrak{p} \in \operatorname{Ass} H_s(M)$.

Prop^n 4. $(R, \mathfrak{m}, k)$ local.

Let M be any bounded complex.

Then, for any $I \subset R$,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

In particular, if M is a f.g. module

$$\operatorname{depth}_R(M) \leq \inf\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M\}$$

$$\leq \dim_R(M). \rightsquigarrow \sup\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M\}$$

Proof. Say $I = (y_1, \dots, y_c)$ and $d := \dim(R/I)$.

... s.t. they form a SoP in R/I

Proof. Say $I = (y_1, \dots, y_c)$ and $d := \dim(R/I)$.
Let $x_1, \dots, x_d \in R$ s.t. they form a SoP in R/I.

Then, $\sqrt{(\underline{y}, \underline{x})} = \sqrt{\mathfrak{m}_R}$.

Apply Lemma (2)(b) to $K(\underline{y}; M)$ to get

$$\sup H_*(\underline{y}; M) \leq \sup H_*(\underline{x}; K(\underline{y}; M)) = \sup H_*(\underline{x}, \underline{y}; M).$$

$$\therefore\ d + \underbrace{c - \sup H_*(\underline{y}; M)} \geq \underbrace{d + c - \sup H_*(\underline{x}, \underline{y}; M)}$$

$$\Rightarrow d + \operatorname{depth}(I, M) \geq \operatorname{depth} M. \qquad \blacksquare$$

Local case:

AB Equality. F an R-complex with $\operatorname{flatdim}_R F < \infty$.
Then, for any R-complex M,

$$\operatorname{depth}(M \otimes^L_R F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes^L_R F)$$

$$\operatorname{depth}_R M \geq -\sup H_*(M) =s$$

with equality iff $\operatorname{depth}(H_s(M)) = 0$.

Prop^n. $R \to$ comm. Noe.
$I \subseteq R$ ideal.

$$\operatorname{depth}(I, M) = \inf \{ \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in V(I) \}.$$

Proof. $I \subseteq \mathfrak{p} \Rightarrow \operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(\mathfrak{p}, M) \leq \operatorname{depth}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$

check using Koszul

This gives $\leq$.

We now construct $\mathfrak{p}$ achieving $\operatorname{depth}_R(I, M)$.

Let $I = (x_1, \dots, x_c)$. Let $s := \sup H_*(\underline{x}; M)$.

We now construct $\mathfrak{p}$ achieving $\operatorname{depth}_R(I, M)$.

Let $I = (x_1, \dots, x_c)$. Let $s := \sup H_*(\underline{x}; M)$.

Pick $\mathfrak{p} \in \operatorname{Ass} H_s(\underline{x}; M)$. (Maybe even minimal.)

Then, $\operatorname{depth}_{R_\mathfrak{p}} H_s(\underline{x}; M)_\mathfrak{p} = 0$.

Consider $K(\underline{x}; M)_\mathfrak{p} = K(\underline{x}; M_\mathfrak{p})$.

$\sup H_*(K(\underline{x}; M_\mathfrak{p})) = \sup H_*(\underline{x}; M_\mathfrak{p}) = s$

$$\operatorname{depth}_{R_\mathfrak{p}} K(\underline{x}; M_\mathfrak{p}) = -s$$

$$\operatorname{depth}_{R_\mathfrak{p}} \left(K(\underline{x}; R_\mathfrak{p}) \otimes M_\mathfrak{p} \right)$$

$\| $ AB

$$\operatorname{depth}(M_\mathfrak{p}) - \sup H_*\left(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} K(\underline{x}; R_\mathfrak{p})\right)$$

$$\geqslant \operatorname{depth}(M_\mathfrak{p}) - c$$

$$\Rightarrow \quad -s \geqslant \operatorname{depth}(M_\mathfrak{p}) - c$$

$$\Leftrightarrow \quad \underbrace{c - s}_{\operatorname{depth}_R(I, M)} \geqslant \operatorname{depth}(M_\mathfrak{p})$$

$\blacksquare$

Remark. The proof says
$$\operatorname{depth}_R(I, M) = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})$$
$$\forall\, \mathfrak{p} \in \operatorname{Ann} H_s(\underline{x}; M)$$

Thm. $(R, \mathfrak{m}_R) \longrightarrow (S, \mathfrak{m}_S)$ local map.
(i.e., $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$.)

Let M be an R-complex, N an S-module s.t. N is flat as an R-module.

Then,
$$\operatorname{depth}_S(N \otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N).$$

Corollary When $(R, m_R) \to (S, m_S)$ is flat, then $\left(\begin{smallmatrix} M=R \\ N=S \end{smallmatrix} \text{ gives}\right)$

$$\operatorname{depth}_S(S) = \operatorname{depth}_R(R) + \operatorname{depth}_{S/m_RS}(S/m_RS).$$

$\underbrace{}_{\text{"fiber"}}$

$$R \to S \to \frac{S}{m_R S}$$

$$\begin{tikzcd}
\operatorname{Spec}(R/m_RS) \arrow[r, hook] \arrow[d] & \operatorname{Spec} S \arrow[d] \\
\{m_R\} \arrow[r, hook] & \operatorname{Spec}(R)
\end{tikzcd}$$

Under the same hypothesis, we have

$$\dim(S) = \dim(R) + \dim(S/m_RS).$$

Proof. $\underline{x} = x_1, \dots, x_c \in R$ s.t. $m_R = (\underline{x})$.

Pick $\underline{y} = y_1, \dots, y_d \in S$ s.t. $\underline{y}\left(\frac{S}{m_RS}\right)$ is the maximal ideal of S/m_RS (i.e., m_S/m_RS).

Then, $(\underline{x}S, \underline{y}) = m_S$.

$$K(\underline{x}S, \underline{y}; N \otimes_R M) \cong K(\underline{y}; N) \otimes_R K(\underline{x}; M)$$

$\hookrightarrow$ assoc. of $\otimes$

N flat over $R \Rightarrow K(\underline{y}, N)$ has fin flat $\dim_R$

$$\operatorname{depth}_R K(\underline{x}, \underline{y}; N \otimes_R M) = \operatorname{depth}_R K(\underline{x}; M) - \sup H_*(k \otimes K(\underline{y}, N))$$

$(k = R/m_R)$

Note. $(\underline{x}, \underline{y}) \cdot H_*(\underline{x}, \underline{y}; N \otimes_R M) = 0$

$m_S \; H_*(\underline{x}, \underline{y}; N \otimes_R M) = 0$

$\Rightarrow m_R \; H_*(\underline{x}, \underline{y}; N \otimes_R M) = 0$

Similarly $m_R H_*(\underline{x}; M) = 0$.

$$-\sup H_*(\underline{x}, \underline{y}; N\otimes_R M) = -\sup H_*(\underline{x}, M) - \sup H_*\left(\underline{y}; \frac{N}{M_R N}\right)$$

Add $c+d$

$\hookrightarrow$ depths $(N \otimes_R M)$

<u>Exercise</u> R local. M f.g. R-module

$$\operatorname{depth} R - \dim_R M \leq \operatorname{grade}_R M \leq \operatorname{codim}_R M$$

$$\leq \dim R - \dim M \leq \operatorname{pdim}_R M.$$

use intersection thm

$\operatorname{depth}_R(\operatorname{ann}_R M, R)$

$\operatorname{height}(\operatorname{ann} M)$
$\parallel$
$\inf\{\dim R_p : p \supseteq \operatorname{ann} M\}$

<u>Recall.</u> New Intersection Theorem (P. Roberts '86)

$$0 \to F_d \to \cdots \to F_0 \to 0 \quad \text{finite free}$$

$$0 < \operatorname{length} H_*(F) < \infty.$$

Then, $d \geq \dim$ dim (R). ⊛

"Simple consequence" of AB + Existence of big CM modules

<u>Corollary.</u> (Intersection theorem)

R local.

$M \neq 0$ f.g. R-module s.t. $\operatorname{pdim}_R M < \infty$.

Then, for any f.g. R-module N,

$$\dim_R(N) - \dim_R(M \otimes_R N) \leq \operatorname{pdim}_R(M).$$

Inspired by (Serre): R regular local (e.g. $k[[x_1, \dots, x_n]]$)

M, N any finite R-module.

(Serre) M, N any finite R-module.

$$\dim_R N - \dim_R (M \otimes_R N) \leq \dim R - \dim M.$$

Lecture 7 (01-02-2023)

Wednesday, February 1, 2023 1:24 PM

$R \to$ Commutative noetherian.

$I \subseteq R$ ideal

Invariance of domain

Let $R \to S$ finite map. (S is a finite R-module.)

Let M be an S-complex.

Then,

$$\operatorname{depth}_R(I, M) = \operatorname{depth}_S(IS, M).$$

[Only need S noetherian for this.]

Obvious using Koszul.

$I = (\underline{x})$

$$K(\underline{x}; M) = K(\underline{x}; R) \otimes_R R$$

$$= (K(\underline{x}; R) \otimes_R S) \otimes_S M$$

$$= K(\underline{x}; S) \otimes_S M.$$

$\dim_R M = \dim_S M.$ [Finiteness needed here.]

Say $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ is finite and local.

$$\begin{aligned}\operatorname{depth}_R(M) &= \operatorname{depth}(\mathfrak{m}_R, M)\\ &= \operatorname{depth}(\mathfrak{m}_R S, M)\\ &= \operatorname{depth}(\mathfrak{m}_S, M)\\ &= \operatorname{depth}_S(M).\end{aligned}$$

finiteness $\Rightarrow \sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$

Special case. R, M an R-module.

Take $S = R/\operatorname{ann}_R(M)$.

Recall. $(R, \mathfrak{m}_R)$ local, $M \overset{\neq 0}{a}$ f.g R-module.

$\operatorname{depth}(R) \quad \dim(M) \overset{①}{\leq} \operatorname{grade}(M) \overset{②}{\leq} \operatorname{codim}(M) \overset{③}{\leq} \dim R - \dim M$

$$\mathrm{depth}(R) - \dim(M) \overset{①}{\le} \mathrm{grade}(M) \overset{②}{\le} \mathrm{codim}(M) \overset{③}{\le} \dim R - \dim M$$
$$\overset{④}{\le} \mathrm{pdim}_R\, M.$$

$$\mathrm{depth}(M) \le \mathrm{depth}_R(I, M) + \dim(R/I).$$

Using invariance of domain, can tighten the above to

$$\mathrm{depth}_R M \le \mathrm{depth}_R(I, M) + \underbrace{\dim(M/IM)}_{= \dim\left(R/I + \mathrm{ann}\, M\right)}.$$

Examples. ① $R = \dfrac{k[[x,y]]}{(x^2, xy)}$

$\mathfrak{m}_R = (x, y).$

$\dim(R) = 1$: $\sqrt{(x^2, xy)} = (x)$

$\therefore \dim R = \dim \dfrac{k[[x,y]]}{(x)} = \dim k[[y]] = 1.$

$\mathrm{depth}(R) = 0$, i.e., $\mathrm{Hom}_R(k, R) \neq 0$

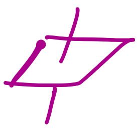
i.e, $(0 : \mathfrak{m}_R) \neq 0.$ socle

Taking $M = R$ shows
$\mathrm{depth}\, R - \dim M < \mathrm{grade}(M)$ is strict.

② Take same R.

$M = R/\mathfrak{m}$. Then, $\mathrm{grade}\, M < \mathrm{codim}\, M.$

③ $k[[x,y,z]]/(x) \cap (y,z) = R.$



Spec R $= V(x) \cup V(y,z)$

Pick $M = R/\mathfrak{p}$ where $\mathfrak{p}$ is a min'l prime
but $\dim(R/\mathfrak{p}) < \dim(R)$.

$\mathfrak{p} = (y, z).$

⑤ Say $\mathrm{pdim}_R\, M < \infty$.

$\dim R - \dim M \le \mathrm{pdim}_R M$

?, AB

$\dim R - \dim M \le \mathrm{pdim}_R M$

$\Updownarrow$ $\quad$ } AB

$\dim R - \dim M \le \operatorname{depth} R - \operatorname{depth} M$

$\Updownarrow$

$\dim R - \operatorname{depth} R \le \dim M - \operatorname{depth} M$ — (5)

Now, take $R = k[x,y]$ and $M = R/(x^2, xy)$.

Then, LHS of (5) is 0 but RHS is not.

Def^n: $(R, \mathfrak{m})$ local, $M \underset{}{\overset{\neq 0}{}}$ f.g. over R.

$$\mathrm{Cmd}_R(M) := \dim_R M - \operatorname{depth}_R M.$$

Cohen-Macaulay defect

If $\mathrm{Cmd}_R(M) = 0$, i.e., $\dim_R(M) = \operatorname{depth}_R(M)$, then

M is Cohen-Macaulay (CM).

M is maximal Cohen-Macaulay (MCM) if $\operatorname{depth}(M) = \dim(R)$.

0 is also considered CM and MCM.

{ $\underset{\infty}{\operatorname{depth} M} \ge \underset{-\infty}{\dim M}$ for $M = 0$.

One can define dim etc. for complexes as well.

$(R, \mathfrak{m}_R)$ local. $M \overset{\neq 0}{}$ f.g. module.

$$\operatorname{depth}_R(M) \le \inf\{\dim R/\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}_R M\} \le \dim_R M.$$

Corollary. M MCM $\Rightarrow$ $\mathrm{Ann}_R M = \mathrm{Min}_R M$

and $\dim(R/\mathfrak{p}) = \dim(M)$ $\forall \mathfrak{p} \in \mathrm{Min}(M)$.

Geometrically: $\mathrm{Supp}(M)$ has no "embedded components" and all components have the same dimension.

Prop^n. $(R, \mathfrak{m})$ local, M f.g. and CM.

① $\operatorname{depth}_R(I, M) = \dim M - \dim(M/IM)$ $\forall I \subseteq R$

② Given $\underline{x} = x_1, \dots, x_c \in \mathfrak{m}$, then

$\underline{x}$ is M-regular $\iff \dim(M) - \dim(M/\underline{x}M) = c.$

Note $(\Rightarrow)$ is always true.

$(\Leftarrow)$ Apply ① to $I = (\underline{x})$.

The hypothesis gives

$\operatorname{depth}_R(\underline{x}, M) = c$

$\iff H_i(\underline{x}; M) = 0 \ \forall i \geq 1$

$\iff \underline{x}$ is regular. ∎

③: $\underline{x}$ is part of an SoP for M ⇝ SoP for $R/\operatorname{ann}(M)$

$\iff \underline{x}$ is M-regular.

Thm. $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ local map.

M f.g. R-module, N f.g. S-module, flat over R.

$M \neq 0$, $N \neq 0$

Then,

$$\operatorname{Cmd}_S(N \otimes_R M) = \operatorname{Cmd}_R(M) + \operatorname{Cmd}_{(S/\mathfrak{m}_R S)}(N/\mathfrak{m}_R N).$$

Proof: We saw the above for depth instead of Cmd.

Same holds for dim. ∎

Corollary. Under same hypothesis,

$N \otimes_R M$ is CM$/S$ $\iff$ M CM$/R$ + $N/\mathfrak{m}_R N$ CM$/S/\mathfrak{m}_R S$

Special case: $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ flat local.

S CM $\iff$ R and $S/\mathfrak{m}_R S$ are CM.

Def^n A local ring R is CM if it is so as a module over itself.

Key Consequence: R CM: for any f.g. R-module M,

$$\operatorname{grade}_R(M) = \operatorname{codim}_R(M) = \dim(R) - \dim(M).$$

Exercise $(R,\mathfrak{m})$ M f.g.

M CM $\Rightarrow$ $M_{\mathfrak{p}}$ CM over $R_{\mathfrak{p}}$ $\forall\, \mathfrak{p} \in \operatorname{Supp} M$.

Def^n $R \to$ comm. noe.
$M \to$ f.g. R-module

Then, M is CM if $M_{\mathfrak{p}}$ CM$/R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}_R M$.

Equivalently, for all $\mathfrak{m} \in \operatorname{Max}(\operatorname{Supp}_R M)$.

R is CM if . . .

Examples: · $k[x_1, \dots, x_c]$ is CM. (How?)

· $k[[x_1, \dots, x_c]]$ is CM. $\because$ $\dim = c$
depth $= c$ since $x_1, \dots, x_c$.

· R CM $\Leftrightarrow$ $\Lambda^I R$ is CM for some ($\equiv$ all) I.
↳ completion wrt I

If R local,
$(R,\mathfrak{m}) \longrightarrow \Lambda^I R =: S$ is flat and local.

Check $S/\mathfrak{m}_R S = R/\mathfrak{m}_R \to$ field (CM)

$\therefore$ $\Lambda^I R$ is CM $\Leftrightarrow$ R is CM.

Thm: $(R,\mathfrak{m})$ CM local.
M f.g. R module s.t. $\operatorname{pdim}_R M < \infty$

Then,

M is CM $\Leftrightarrow$ $\operatorname{pdim}_R M = \operatorname{grade}_R M$.

Proof. $\operatorname{pdim}_R M = \operatorname{grade}_R M$

$\Updownarrow$

$\operatorname{depth} R - \operatorname{depth} M = \dim R - \dim M$

$\Updownarrow$

$\operatorname{depth} M = \dim M.$ $\blacksquare$

Lecture 8 (06-02-2023)

Monday, February 6, 2023 1:25 PM

Regular Rings

For now, $(R, \mathfrak{m}, k)$ is a comm. noetherian local ring.

Recall. $\text{depth}(R) \overset{①}{\leq} \dim(R) \overset{②}{\leq} \text{edim}(R)$.

$\text{edim}(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$

When equality holds in ①, then R is Cohen-Macaulay. (By defn.)

② holds by Krull's height theorem.

Notation: (embedding) codepth of R is

$$\text{codepth}(R) = \text{edim}(R) - \text{depth}(R).$$

Defn. R is regular if $\text{codepth}(R) = 0$.

Exercise. R is regular $\iff$ $\mathfrak{m}$ is generated by a regular sequence

$\iff \text{edim}(R) = \dim(R)$

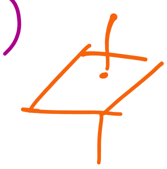
Example.

① $k[[x_1, \dots, x_n]]$ with k a field is regular since $\mathfrak{m}$ is gen by a reg. seq.

② $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ —— " ——.

③ $\mathbb{Z}_{(p)}$ similar.

④ Regular $\Rightarrow$ CM. (Non CM $\Rightarrow$ Non reg.)



$\dfrac{k[[x, y, z]]}{(xz, yz)}$ not reg.

⑤ $R = \dfrac{k[[x y]]}{(xy)}$. codepth $= 1 > 0$.

NOT regular.

Flat Maps

Let $\varphi : (R, m) \to (S, n)$ be a local flat extension.

Recall

S is CM $\iff$ R is CM and S/mS are CM.

Analog fails for regularity. ($\Leftarrow$) does hold.

Consider

$$\underset{R}{k[[x^2]]} \hookrightarrow \overset{S}{k[[x]]}.$$

(flat ext^n since S is free)

$\swarrow$ regular

$S/mS = k[[x]]/(x^2)$

↳ not regular codepth $= 1$

Also, if R is CM, then R_p is CM $\forall p$.

Question. ['30s, Krull, Zariski, ...]

Does R regular imply R_p regular $\forall p \in \operatorname{Spec} R$?

Homological characterisation of regularity. (Used to prove ↑)

Recall: for a f.g. R-module M, a minimal free resolution over (R, m, k) is a free resolution

$$F \xrightarrow{\sim} M$$

with $\partial(F) \subseteq \mathfrak{m} F$.

Recall/Prove. Minimal resolutions exist and are unique up to isomorphism of complexes.

The ith Betti number of M is

$$\beta_i^R(M) = \operatorname{rank}_R(F_i)$$
$$= \operatorname{rank}_k(\operatorname{Tor}_i^R(M, k))$$
$$= \operatorname{rank}_k(\operatorname{Ext}^i_R(M, k)).$$

Example. ① $R = k[[x_1, \ldots, x_n]]$. $f \in \mathfrak{m} \setminus \{0\}$.

$M := R/(f)$.

Then the min'l free resol^n of M is

$$0 \to R \xrightarrow{\cdot f} R \to 0.$$

↪ Koszul

$$\beta_i(M) = \begin{cases} 1 & ; \quad i = 0, 1 \\ 0 & ; \quad \text{else} \end{cases}$$

The min'l resol^n of k is $K(\underline{x})$.

$$\beta_i(k) = \binom{n}{i} \quad \text{for } i \geq 0.$$

② $R = \dfrac{k[[x, y]]}{(xy)}$.

$M = R/(x)$ has min'l free resol^n:

$$\cdots \xrightarrow{\cdot y} \overset{3}{R} \xrightarrow{\cdot x} \overset{2}{R} \xrightarrow{\cdot y} \overset{1}{R} \xrightarrow{\cdot x} \overset{0}{R} \to 0$$

$$\therefore \beta_i(M) = 1 \quad \forall i.$$

Res^n for k: $\quad k = R/(x,y)$

$$\text{repeat} \cdots \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x \;\; y)} R \longrightarrow 0$$

$$\beta_i(M) = \begin{cases} 1 & ; \; i = 0 \\ 2 & ; \; i \geq 1 \end{cases}$$

FACT. (/Defn) $\quad 0 \neq M$ f.g.:

$$\operatorname{projdim}_R(M) = \sup\{i \geq 0 : \beta_i^R(M) \neq 0\}.$$

$\parallel$

length of min'l free res^n

$\parallel$

inf (length (free res^n))

Theorem. [Auslander-Buchsbaum-Serre, '50s] (ABS)

(R, m, k) local. TFAE:

① R is regular.

② $\operatorname{projdim}_R(M) < \infty$ for all f.g. M.

③ $\operatorname{projdim}_R(k) < \infty$.

Proof. ① $\Rightarrow$ ②

Let $\underline{x}$ ($x_1, \ldots, x_d$) be a min'l gen set for m. (Hence reg. seq.)

Then, $K(\underline{x})$ is a min'l free res^n for k.

$$\beta_i^R(M) = \operatorname{rank}_k(\operatorname{Tor}_i^R(M, k))$$

$$= \operatorname{rank}_k H_i(M \otimes_R K(\underline{x}))$$

$$= 0 \quad \text{for} \quad i > d.$$

$\therefore \operatorname{projdim}_R(M) < \infty$.

② $\Rightarrow$ ③. —

② ⇒ ③. ✓

③ ⇒ ①. Serre's proof of ABS Theorem relies on:

<u>Lemma</u>. $(R, \mathfrak{m}, k)$ local.

Then, $\beta_i^R(k) \geq \binom{\operatorname{edim}(R)}{i}$ for $i \geq 0$.

<u>Proof of Lemma</u>. Let $F \xrightarrow{\simeq} k$ be a min'l resol^n. $\left(\therefore \beta_i^R(k) = \operatorname{rk}_R F_i.\right)$

Let $\underline{x} = x_1, \ldots, x_e$ be a min gen set for $\mathfrak{m}$.

$$\begin{tikzcd} K := K(\underline{x}) \arrow[r] \arrow[dr, dashed] & k \\ & F \arrow[u, "\simeq"'] \end{tikzcd}$$

$\exists \varphi$ a map of complexes $(\because$ perfect$)$

<u>Claim</u>. φ_i is a split injection. $\forall i$ (Then it is clear.)

$\varphi_i : K_i \to F_i$ is split injective

$\underset{\text{NAK}}{\Leftrightarrow}$ $k \otimes_R K_i \to k \otimes_R F_i$ is an injection.

By induction, we show φ_i is split inj.

$i = 0$ ✓ $R \xrightarrow{\simeq} R$

$i > 0$: Let $a \in K_i$ with $\varphi_i(a) \in \mathfrak{m} F_i$. (we are using the NAK result.)

<u>WTS</u>: $a \in \mathfrak{m} K_i$.

$\therefore \partial\varphi(a) \in \mathfrak{m}^2 F_{i-1}$

$\quad \Vert$

$\varphi(\partial a)$

$$\begin{tikzcd} a \in K_i \arrow[r] \arrow[d] & K_{i-1} \arrow[d] \\ F_i \arrow[r] & F_{i-1} \end{tikzcd}$$

$F_i \supseteq \varphi(a) \in \mathfrak{m} F_i \quad \partial(F) \subseteq \mathfrak{m}F \quad \partial(\varphi(a)) \in \mathfrak{m}^2 F_{i-1}$

By induction,

$\partial^K a \in \mathfrak{m}^2 K_{i-1}$. $(*)$

Notice that by def^n of ∂^K and since x is a min gen set for $\mathfrak{m}$, Notice that [illegible] and

since $\underline{x}$ is a min gen set for $\mathfrak{m}$,

$(*) \Rightarrow a \in \mathfrak{m} K_i$. This does it. ∎

Back to: $\text{proj dim}_R (k) < \infty \Rightarrow R$ is regular.

From Serre's inequality:

$$\text{proj dim}_R\, k \geq \text{edim}\, R.$$

$\|$ ⇝ By AB Equality, since $\text{proj dim}\, k < \infty$.

$\text{depth}(R)$

But $\text{depth}(R) \leq \text{edim}(R)$ always true. ∎

The above now solves the localisation problem.

Corollary. R regular $\Rightarrow R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \text{Spec}\, R$.

Proof. R is reg.

$\Rightarrow \text{proj dim}_R\, R/\mathfrak{p} < \infty$

$\Rightarrow \text{proj dim}_{R_{\mathfrak{p}}} (k(\mathfrak{p})) < \infty \Rightarrow R_{\mathfrak{p}}$ is reg. ∎

Prop^n. $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{m}_S)$ local flat.

① R and $S/\mathfrak{m}S$ reg $\Rightarrow S$ is reg.

② S is regular $\Rightarrow R$ is regular.

Proof. ① $\bar{S} = S/\mathfrak{m}S$

$$\text{depth}\, S \underset{\text{flat}}{=} \text{depth}(R) + \text{depth}(\bar{S})$$

$$= \text{edim}(R) + \text{edim}(\bar{S})$$ regular hypothesis

There is always a right exact sequence

$$\frac{m_R}{m_R^2} \otimes_k \ell \to \frac{m_S}{m_S^2} \longrightarrow \frac{m_{\bar S}}{m_{\bar S}^2} \longrightarrow 0 \qquad \ell = S/m_S$$

$$\therefore \ \operatorname{edim} S \le \operatorname{edim} R + \operatorname{edim}(\bar S) = \operatorname{depth}(\bar S).$$

② Let $F \xrightarrow[R]{\sim} k$ min'l, since φ flat AND local,

$$F \otimes_R S \longrightarrow \bar S \quad \text{min'l.}$$

$$\beta_i^R(k) = \beta_i^S(\bar S) = 0 \quad \text{for} \quad i \gg 0.$$

$\therefore$ $\operatorname{projdim}_R k < \infty$.

$\Rightarrow R$ is reg. ∎

Lecture 9 (08-02-2023)

Wednesday, February 8, 2023 1:26 PM

Recall: R local is regular if $\operatorname{codepth}(R) = 0$,

i.e., $\mathfrak{m}$ generated by a regular sequence.

We proved Auslander-Buchsbaum Theorem:

$(R, \mathfrak{m}, k)$ TFAE:

① R is regular,

② $\operatorname{projdim}_R M < \infty$ for all f.g. M

③ $\operatorname{projdim}_R(k) < \infty$.

---

① $\Rightarrow$ ② $\Rightarrow$ ③ was ok.

③ $\Rightarrow$ ①: showed $\operatorname{projdim}_R(k) \geq \operatorname{edim}(R)$ and then AB Equality.

Sketch of second proof.

Theorem [Nagata]. Let $(R, \mathfrak{m}, k)$ be local (not necessarily regular) and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be nzd. Set $\bar{R} = R/x$.

Then, for any f.g. R/x-module M, we have

$$\operatorname{Tor}^R_*(M, k) \cong \operatorname{Tor}^{\bar{R}}_*(M, k) \otimes_k \underbrace{\Lambda(\Sigma k)}_{= \; 0 \to k \to k \to 0}.$$

In particular, $\beta^R_i(M) = \beta^{\bar{R}}_i(M) + \beta^{\bar{R}}_{i-1}(M)$.

In fact, one can show the min'l R-free res'n of M

has the form

$$\cdots \longrightarrow \begin{matrix} G_3 \\ \oplus \\ G_2 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_3 & x \\ \beta_3 & -\alpha_2 \end{pmatrix}} \begin{matrix} G_2 \\ \oplus \\ G_1 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_2 & x \\ \beta_2 & -\alpha_1 \end{pmatrix}} \begin{matrix} G_1 \\ \oplus \\ G_0 \end{matrix} \xrightarrow{(\alpha_1 \;\; x)} G_0$$

multiplication by x

$G_i \to$ free

and $\cdots \to \overline{G_3} \xrightarrow{\alpha_3} \overline{G_2} \xrightarrow{\alpha_2} \cdots$ is the min'l $\overline{R}$-resol^n of M.

(Stronger result but not proving this.)

<u>Example.</u> $R = k[[x,y]]/(xy)$ (non regular)

Minimal R-free resol^n of k:

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x \;\; y)} R \to 0$$

Consider $y - x^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$ (nzd on R).

By row/col operations, the min'l resol^n is iso to,

$$\cdots \xrightarrow{\begin{pmatrix} x & y-x^2 \\ & -y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & y-x^2 \\ & -x \end{pmatrix}} R^2 \xrightarrow{(x \quad y-x^2)} R \to 0$$

Now, going mod $y - x^2$, [using the result]

we get

$$\cdots \longrightarrow \overline{R} \xrightarrow{x} \overline{R} \xrightarrow{y} \overline{R} \xrightarrow{x} \overline{R} \to 0$$

$\overline{R} = k[[x,y]]$

$$\cdots \to \bar{R} \to \bar{R}$$

$$= \quad \cdots \to \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{x^2} \bar{R} \xrightarrow{x} \bar{R} \to 0$$

$$\bar{R} = \frac{k[[x,y]]}{(xy,\, y-x^2)} \cong \frac{k[[x]]}{(x^3)}$$

Proof of Nagata's Thm.

If $a \in R$ is nzd, and set $\bar{R} = R/a$.

$M \rightsquigarrow \bar{R}$-module.

There is a long exact sequence

$$\xrightarrow{\ \chi\ } \mathrm{Tor}^{\bar{R}}_{n-1}(M,k) \to \mathrm{Tor}^{R}_{n}(M,k) \to \mathrm{Tor}^{\bar{R}}_{n}(M,k) \xrightarrow{\ \chi\ } \mathrm{Tor}^{\bar{R}}_{n-2}(M,k) \to \mathrm{Tor}^{R}_{n-1}(M,k) \to \mathrm{Tor}^{\bar{R}}_{n-1}(M,k)$$

Specialising to our setting, we wish to show that each χ is 0.

One can compute χ in the following way:

$\bar{F} \xrightarrow{\sim} M$ min'l $\bar{R}$-res^n:

Lift this to a sequence of free R-modules:
(not saying complex!)

$$\cdots \to F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \to \cdots$$

∂^2 may not be 0 but

$\partial^2 = a \cdot \ldots$ where

∂^i may not be 0 ...

$\partial^2 = x\,\Theta$ where

$\Theta = \{\Theta_i : F_i \longrightarrow F_{i-2}\}$

is a chain map

and the following diagram commutes: (up to sign)

$$\begin{tikzcd}
F_i \otimes_R k \arrow[r, "\Theta_i \otimes_R k"] \arrow[d, phantom, "\cong"] & F_{i-2} \otimes_R k \arrow[dd, "\cong"] \\
\bar{F}_i \otimes_{\bar{R}} k \arrow[d, phantom, "="] & \\
\mathrm{Tor}_i^{\bar{R}}(M,k) \arrow[r, "x"'] & \mathrm{Tor}_{i-2}^{\bar{R}}(M,k)
\end{tikzcd} \qquad (*)$$

Now, since x is linear and $\partial^2 = x\,\Theta$,

we have that $\Theta(F) \subseteq \mathfrak{m} F$.

$\therefore$ the top map in $(*)$ is zero. $\therefore$ $x = 0$. ∎

Second proof of ③ ⇒ ① in ABT.

Given $\operatorname{projdim}_R(k) < \infty$. WTS: R is regular.

Induct on $d := \operatorname{depth}(R)$

$d = 0$: AB Equality gives $\operatorname{projdim}_R(k) = 0$.

$\therefore$ $R = k$ is a field. R is reg.

$d > 0$: By prime avoidance, $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$ nzd.

$\therefore$ $\operatorname{codepth}(R) = \operatorname{codepth}(R/x)$.

By Nagata's thm,

$\operatorname{projdim}_{R/x} k = \operatorname{projdim}_R k - 1 < \infty$.

By Nagata's Thm,

$$\operatorname{projdim}_{R/x} k = \operatorname{projdim}_R k - 1 < \infty.$$

But R/x has smaller depth. ⊠

$(R, \mathfrak{m}, k)$ local

Prop^n: $x \in \mathfrak{m}$ nzd.

R is CM $\iff$ R/x is CM.

$(\mathrm{CMD}(R) = \mathrm{CMD}(R/x).)$

$(\mathrm{CMD} = \dim - \operatorname{depth}.)$

Proposition. $x \in \mathfrak{m}$ nzd.

① Suppose R is regular.

R/x is regular $\iff$ $x \notin \mathfrak{m}^2$.

② If R/x is regular, then R is regular.

(Hence, $x \notin \mathfrak{m}^2$.)

Proof.

$$\operatorname{codepth}(R/x) = \begin{cases} \operatorname{codepth}(R) & x \notin \mathfrak{m}^2, \\ \operatorname{codepth}(R)+1 & x \in \mathfrak{m}^2. \end{cases}$$

⊠

Global Setting

$R \to$ comm. noetherian, not necessarily local or of finite Krull dim.

Def^n. R is *regular* if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$

$(\iff \forall \mathfrak{m} \in \operatorname{mSpec}(R))$.

($\Leftrightarrow \forall \mathfrak{m} \in \mathrm{mSpec}(R)$).

(Since regularity localises for local rings, the above makes sense.)

Exercise. R is regular $\Leftrightarrow R[x]$ is regular

$\Leftrightarrow R[[x]]$ is regular

Example ① $k[x_1, \dots, x_n]$ is regular. (k a field.)

② $\mathbb{Z}[x_1, \dots, x_n]$ —"— . ($\mathbb{Z}$ is regular since every localisation is $\mathbb{Q}$ or a DVR.)

③ Nagata's example. (Infinite Krull dimension but regular)

Thm. [Bass–Murthy '60s].

$R \to$ comm noetherian

$M \to$ f.g.

$\mathrm{projdim}_R M < \infty \Leftrightarrow \mathrm{projdim}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

Corollary. $R \to$ comm noe.

R is regular $\Leftrightarrow \mathrm{projdim}_R M < \infty$ for all M f.g.

Proof. $(\Rightarrow)$ ✓

$(\Leftarrow)$ Let $F \xrightarrow{\sim} M$ be a free res^n of M, F_i f.g. for all $i \geq 0$.

For $n \geq 0$, define

For $n \geq 0$, define

$$D_n := \{ \mathfrak{p} \in \operatorname{Spec} R : \operatorname{proj\,dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leq n \}$$

$$= \{ \mathfrak{p} \in \operatorname{Spec} R : \operatorname{im}(\partial_n^F)_\mathfrak{p} \text{ is free over } R_\mathfrak{p} \}.$$

Since $\operatorname{im}(\partial_n^F)$ is f.g., the above set is open in $\operatorname{Spec}(R)$. "free locus is open"

$D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$

and $\operatorname{Spec} R = \bigcup_{n \geq 0} D_n$.

Since R is noetherian, $\operatorname{Spec} R$ is a noe. top space.

$\therefore \operatorname{Spec} R = D_n$ for some n.

$\Rightarrow \operatorname{im}(\partial_n^F)$ is locally free.

$\Rightarrow \operatorname{im}(\partial_n^F)$ is projective.

Now, $0 \to \operatorname{im}(\partial_n^F) \hookrightarrow F_{n-1} \to \cdots \to F_0 \to 0$ is a proj. res^n. $\boxtimes$

Lecture 10 (13-02-2023)

13 February 2023 13:26

$R \to$ comm noetherian

$$\varphi : \begin{array}{ccc} F_1 & \longrightarrow & F_0 \\ \cong & & \cong \\ R^s & \xrightarrow{(a_{ij})} & R^r \end{array}$$ free modules of finite rank

$I_c(\varphi) =$ ideal generated by $c \times c$ minors of (a_{ij}).

(well defined)

$I_0(R) = R \supseteq I_1(R) = (a_{11}, \ldots, a_{rs}) \supseteq \cdots \supseteq I_i(R) \supseteq I_{i+1}(R) = 0$

$i = \min(r, s)$

M f.g. R-module

$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0$ (F_0: rank $= r$)

finite free presentations

$G_1 \xrightarrow{\psi} G_0 \longrightarrow M \longrightarrow 0$ (G_0: rank $= s$)

Then, $I_{r-c}(\varphi) = I_{s-c}(\psi)$ for all c. (Exercise.)

Def^n. $\mathrm{Fitt}_c^R(M) := I_{r-c}(\varphi)$, c^{th} Fitting ideal of M.

We get $\mathrm{Fitt}_0(M) \subseteq \mathrm{Fitt}_1(M) \subseteq \cdots$ eventually R.

$\mathrm{Fitt}_0(M) \neq 0$ only if $r \leq s$

$\mathrm{Fitt}_c(M) = R$ for $c \geq r$.

Examples.

① $R = k$ a field, V some vector space (f.g.) of rank n.

$0 \to k^n$ is a presentation

$$\mathrm{Fitt}_c(V) = k \iff c \geq n.$$

② R a PID.

$$0 \to R^s \xrightarrow{\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \\ \hline & 0 & \end{pmatrix}} R^r \to M \to 0$$

(can assume $s \leq r$)

can put in normal form

$\mathrm{Fitt}_0(M) \neq 0 \iff r = s \iff M$ is torsion.

In this case, $\mathrm{Fitt}_0(M) = (d_1 \cdots d_r)$.

Properties. $R^s \xrightarrow{\varphi} R^r \to M \to 0.$

① If $R \to S$ is any map of rings.

$$\mathrm{Fitt}_c^S(S \otimes_R M) = S \cdot \mathrm{Fitt}_c^R(M).$$

② $(\mathrm{ann}_R(M))^r \subseteq \mathrm{Fitt}_0(M) \subseteq \mathrm{ann}_R(M).$

Proof. Pick an $(r \times r)$-minor 'a' in φ.

WTP: $a \cdot R^r \subseteq \mathrm{im}(\varphi)$

Can assume

$$\varphi = \underbrace{\left(\underbrace{\quad}_{r\times r} \Big| \quad \right)}_{r \times s} = A$$

$\det = a$

Take the $s \times r$ matrix $\begin{pmatrix} \text{signed co-factor matrix}_{r \times r} \\ \hline 0 \end{pmatrix}_{s \times r} = B$

Then, $AB = \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}.$

$\therefore\ a R^n \subseteq \operatorname{im} \varphi.$ Proves $\mathrm{Fitt}_0 \subseteq \mathrm{ann}.$

Next: $\operatorname{ann}(M)^r \subseteq \mathrm{Fitt}_0(M).$

Fix $a_1, \dots, a_r \in \operatorname{ann}(M).$

$$\begin{tikzcd} & R^r \arrow[dl, dashed] \arrow[d, "\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix}"] \\ R^s \arrow[r, "\varphi"'] & R^r \end{tikzcd}$$

Apply $\Lambda^r(\cdot)$ to the above to get

$$\begin{tikzcd} & R \arrow[dl] \arrow[d, "a_1 \cdots a_r"] \\ R^{\binom{s}{r}} \arrow[r, "\Lambda^r(\varphi)"'] & R \end{tikzcd}$$

(3) Fix $c \geq 0$ and $\mathfrak{p} \in \operatorname{Spec}(R).$
TFAE:

(i) $\mathrm{Fitt}_c(M) \not\subseteq \mathfrak{p}.$

(ii) $\operatorname{im}(\varphi)_{\mathfrak{p}}$ contains a free summand of $R_{\mathfrak{p}}^r$ of rank $\geq r-c.$

(iii) $\nu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq c.$

min'l # gens

Sketch. Can assume $(R, \mathfrak{p}, k)$ local. $\mathfrak{m} := \mathfrak{p}.$

$\mathrm{Fitt}_c(M) \not\subseteq \mathfrak{m} \Leftrightarrow \mathrm{Fitt}_c(R) = 0 \Leftrightarrow \mathrm{Fitt}_c(k \otimes_R M) \neq 0$

$\Updownarrow$ → first example

$\nu_R(M) \leq c \overset{\text{NAK}}{\Longleftrightarrow} \nu_k(k \otimes_R M) \leq c$

(4) Fix $c \geq 0$ and $\mathfrak{p} \in \operatorname{Spec} R.$ TFAE

(i) $\mathrm{Fitt}_{c-1}(M)_{\mathfrak{p}} = 0$ and $\mathrm{Fitt}_c(M)_{\mathfrak{p}} = R_{\mathfrak{p}}.$

(ii) $\operatorname{im}(\varphi)_{\mathfrak{p}}$ is a free summand of $R_{\mathfrak{p}}^r$ of rank $r-c.$

(iii) $M_p \cong (R_p)^c$.

Deduce from (3).

(5) $c \geq 0$. M is projective of rank c.

$\iff \mathrm{Fitt}_{c-1}(M) = 0$ and $\mathrm{Fitt}_c(M) = R$.

Hilbert-Birch Theorem

$R \to$ comm noetherian

Given $I \subseteq R$ with free resolution

$$0 \to R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \to 0.$$

Then, $\exists$ nzd $a \in R$ s.t. $I = a \cdot I_n(\varphi)$.

Moreover, if I is projective, then I is principal.

If $\mathrm{projdim}(I) = 1$, then $\mathrm{depth}(I_n(\varphi), R) \geq 2$.

Conversely, if $\varphi : R^n \to R^{n+1}$ is a matrix s.t. $\mathrm{depth}(I_n(\varphi), R) \geq 2$, then $0 \to R^n \to R^{n+1} \to I_n(\varphi) \to 0$

is a free resolution.

Regular Local Rings are UFDs.

Thm. $M \to$ projective module

If M has a finite free res^n, then it must have a free resol^n of length 1.

Say $0 \to F_s \to F_{s-1} \to \dots F_1 \to 0$.

$\downarrow \cong$
M

M projective $\Rightarrow \partial(F_i)$ proj $\forall i \geq 1 \ldots$ $\blacksquare$

So, Projective + FFR $\Rightarrow$ stably free.
↳ finite free resolution

Corollary. $I \subseteq R$ proj + FFR $\Rightarrow$ I principal.

Proof. $0 \to R^n \to R^{n+1} \to I \to 0$.
I proj $\Rightarrow I = (a)$. $\blacksquare$

Thm. R comm. noe. domain s.t. each f.g. R-modules has FFR. Then, R is a UFD.

Corollary. Regular local rings are UFDs.
↳ not true otherwise

Proof. Suppose R is local. ($\because$ Regular.)
- Induction on $\dim R$.
  - $\dim R \leq 1$ is clear.

Pick $\omega \in m_R \setminus m_R^2$. Then, R/ω is also RLR and hence a domain.

$\therefore \omega$ is prime.
Suffices to verify R_ω is a UFD.
Note $\dim R_\omega < \dim R$.

Pick $\mathfrak{p} \in \operatorname{Spec}(R_\omega)$ with ht $= 1$.

Suffices to show $\mathfrak{p}$ is principal.

Note. $\mathfrak{p}$ has a FFR. (For $\mathfrak{p}$ comes from R.)

It now suffices to prove it is projective.

(Earlier corollary.)

This can be tested locally. But localisations are RLRs of dim $<$ dim R.

$\therefore$ UFD.

$\therefore \mathfrak{p}$ is locally free and hence projective.

For general R, again fix $\mathfrak{p}$ of ht $=1$.

<u>WANT</u>: $\mathfrak{p}$ is principal.

Since $\mathfrak{p}$ has FFR, suffices to prove projective.

This is local. $\blacksquare$

R $\to$ PID, M torsion module.

$$0 \to R^r \xrightarrow{\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}} R^r \to M \to 0 \qquad d_i \neq 0$$

$$\mathrm{Fitt}_0(M) = \left(\prod d_i\right) \subseteq \mathrm{ann}_R(M).$$

$$\mathrm{length}_R(M) = \mathrm{length}\left(R/\mathrm{Fitt}_0(M)\right)$$