Fourier Inversion for L^1 Functions

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Overview

Recap

Heat kernel

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1. $(f * h_t)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t \|\xi\|^2} e^{2\pi \iota x \cdot \xi} \, \mathrm{d}\xi$ for all $x \in \mathbb{R}^n$.

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2. Using DCT, we let $t \to 0$ in the above via $\{t_n\}$ to conclude that
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$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi \iota x \cdot \xi} \, \mathrm{d}\xi$$
for those $x \in \mathbb{R}^n$ for which $(\star)$ holds.

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We will actually prove the result for a broader class of approximate identities.

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for all $x \in \mathrm{Leb}(f)$.

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It is now clear that proving the Main Theorem will show that ($\star$) holds for $x \in \mathrm{Leb}(f)$.

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$$\int_{B(x,r)} 1 = V_n r^n$$

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$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = \omega\left(S^{n-1}\right) \int_0^\infty r^{n-1} g(r)\,\mathrm{d}r.$$

Proof of the Main Theorem

Fix a point $x \in \mathrm{Leb}(f)$ and let $\epsilon > 0$ be arbitrary.

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Still proving the Main Theorem

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The A Game

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Using this, we first show that $I_2(t)\xrightarrow{t\to 0}0$.

Taking I_2 down

$I_2(t)$

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$$I_2(t) = \left| \int_{\|u\| \geq \delta} [f(x-u) - f(x)] \varphi_t(u) \, \mathrm{d}u \right|$$

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$$I_2(t) \leq \int_{\|u\| \geq \delta} |f(x-u)| \varphi_t(u) \, \mathrm{d}u + |f(x)| \int_{\|u\| \geq \delta} \varphi_t(u) \, \mathrm{d}u.$$

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as $t \to 0$. Thus, $I_2(t) \xrightarrow{t\to 0} 0$.

I_2 down, I_1 to go

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With these notations, we do some more calculations.

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Integrate by parts

We have

$$\begin{aligned} I_1(t) &\leq \int_{\|u\|<\delta} |f(x-u) - f(x)| t^{-n} \varphi(u/t) \, \mathrm{d}u \\ &= \int_0^\delta r^{n-1} g(r) t^{-n} \psi_0(r/t) \, \mathrm{d}r \\ &= G(r) t^{-n} \psi_0(r/t) \Big|_0^\delta - \int_0^\delta G(r) \, \mathrm{d}(t^{-n} \psi_0(r/t)) \end{aligned}$$

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Note that $\mathrm{d}\psi_0 \leq 0$.

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The Green Integral

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We showed that I_1 is bounded independent of t.

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This completes the proof.

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Reference: *Introduction to Fourier Analysis on Euclidean Spaces* by *Stein and Weiss*