Lecture 8 (01-09)

01 September 2020 10:29 AM

If $R \longrightarrow R/I \times R/J$ is onto, then $(0,1)$ & $(1,0)$ must have preimage.

Notation: Given a ring R, we denote the set of maximal ideals in R by $\text{Max}(R)$, and if R is commutative, then the set of prime ideals is denoted $\text{Spec}(R)$.

called the prime spectrum of R.

[We shall consider R to be comm. when talking about prime ideals.]

[Natural q. after defining a set $\to$ is it non-empty?]

[Is $\max(R) \neq \phi$? Well, no, if $R = 0$.]

Okay, assume $R \neq 0$.

Claim: $\max(R) \neq \phi$.

Proof. We prove this by using Zorn's Lemma.

Let Λ be the set of all proper ideals in R.

① $\Lambda \neq \phi$ since $\{0\} \in \Lambda$.

② Λ is a poset by $\subseteq$.

③ Let $\{I_j\}_{j \in \gamma} \subseteq \Lambda$ be a chain (totally ordered).

We claim that $\{I_j\}_{j \in \gamma}$ has an upper bound in Λ,

i.e., $\exists\, I \in \Lambda$ s.t. $\forall j \in \gamma$, $I_j \subset I$.

Indeed, define $I := \bigcup\limits_{j \in \gamma} I_j$.

[Of course, we clearly have that $I_j \subset I \ \forall j$.]

Claim. $I \in \Lambda$. That is, I is an ideal (in R) which is proper.

Proof. Let $a, b \in I$.

$a \in I_{j_1}$ & $b \in I_{j_2}$ for some $j_1, j_2 \in \gamma$.

Since $\{I_j\}_{j \in \gamma}$ was a chain, either $I_{j_1} \subset I_{j_2}$ or $I_{j_2} \supset I_{j_1}$.

WLOG

Thus, $a \in I_{j_2}$ as well. Then, $a + b \in I_{j_2} \subset I$.

Similarly, given $r \in R$, we have $ar, ra \in I_{j_i} \subset I$.

Thus, I is actually an ideal.

($I \neq \emptyset$ is obvious.)

Lastly, to see that I is proper, note that $1 \notin I_j \ \forall j$ since each I_j was proper.

Thus, $1 \notin I$. $\therefore$ I is proper. ☺

Now, by ①, ② and ③, we see that Λ satisfies the hypothesis of Zorn's Lemma. Thus, Λ has a maximal element $\mathfrak{m}$.

Claim. $\mathfrak{m}$ is a maximal ideal in R. (That is, $\mathfrak{m} \in \mathrm{Max}(R)$.)

Proof. Let $I \subset R$ be an ideal such that $\mathfrak{m} \subsetneq I$.

If $I \subsetneq R$, then $I \in \Lambda$ which contradicts maximality of $\mathfrak{m}$.

Thus, $I = R$, proving that $\mathfrak{m}$ is maximal. ☺

Corollaries: ($R \neq 0$)

① Every proper ideal is contained in a maximal ideal.

② Let $a \in R$. Then,

a is a unit $\iff \nexists\, m \in \mathrm{Max}(R)$ s.t. $a \in \mathfrak{m}$.

} Commutative ring. Otherwise "left max." or "right max."

Ex. $\mathrm{Max}(\mathbb{Z}) = \{p\mathbb{Z} : p \text{ is prime}\}$

$\mathrm{Spec}(\mathbb{Z}) = \{p\mathbb{Z} : p \text{ is prime or } p = 0\}$.

In general, $\mathrm{Max}(R) \subset \mathrm{Spec}(R)$.

Max ideals are prime

That is, if $\mathfrak{m}$ is a maximal ideal in R, then $\mathfrak{m}$ is prime.

Recall: $\mathfrak{p}$ is prime if (if $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$).

Working rule: $ab \in \mathfrak{p}$, $a \notin \mathfrak{p} \Rightarrow b \in \mathfrak{p}$.

... $b \in R$ be such that

Proof. Let $\mathfrak{m}$ be a maximal ideal and $a, b \in R$ be such that $ab \in \mathfrak{m}$ and $a \notin \mathfrak{m}$.

$a \notin \mathfrak{m} \Rightarrow \mathfrak{m} \subsetneq \mathfrak{m} + \langle a \rangle \Rightarrow \mathfrak{m} + \langle a \rangle = R$

$\Rightarrow \exists m \in \mathfrak{m}, r \in R$ s.t. $m + ra = 1$

$$\Rightarrow \underbrace{\underbrace{mb}_{\in \mathfrak{m}} + \underbrace{rab}_{\in \mathfrak{m}}}_{\in \mathfrak{m}} = b$$

$\therefore b \in \mathfrak{m}$.

Remark. Corollary ② is not necessarily true if R not comm. Take $R = M_n(\mathbb{Q})$. $(n \geq 2)$

Proof of Cor.

① Let $I \subsetneq R$ be an ideal. Then, R/I is a ring which is not the zero ring.
Let $\mathfrak{m}$ be a max. ideal in R/I.
Then, $\pi_I^{-1}[\mathfrak{m}]$ is a max. ideal in R containing I.

② Let $a \in R$.

a is not a unit $\Leftrightarrow \langle a \rangle \neq R$

$\Updownarrow$ $\Downarrow$ we proved $\Uparrow$ obvious since max ideals are proper

$\langle a \rangle$ is conta. in max ideal

a is cont. in a max ideal $\Longleftrightarrow$

Lecture 9 (03-09)

03 September 2020 11:27 AM

Note: A prime ideal has to be proper.
(Commutative ring is also assumed.)

Also, 0 is not an integral domain.

Ex. Let $\mathfrak{p} \subset R$ be prime, $I, J \subset R$ be ideals in R.

(Prime ideal Exercise)

If $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.

Q. Let $\mathfrak{m} \in Max(R)$, $a \in \mathfrak{m}$. What can you say about $1+a$?

Comm.

- $1+a \notin \mathfrak{m}$. (Otherwise $1 \in \mathfrak{m}$ and $\mathfrak{m} = R$. $\rightarrow\leftarrow$)
- $1+a \in U(R)$? No. Take $R = \mathbb{Z}$, $\mathfrak{m} = 2\mathbb{Z}$, $a = 1$.

Recall: $u \in U(R)$ iff $u \notin \mathfrak{m}$ for any $\mathfrak{m} \in max(R)$.

Q. What conditions can you put on $a \in R$ so that $1+a$ is a unit?

$$\bigcup_{\mathfrak{m} \in Max(R)} \mathfrak{m} = R \setminus U(R)$$

What if we take $J = \bigcap_{\mathfrak{m} \in Max(R)} \mathfrak{m}$ and $a \in J$.

Is $1+a$ a unit? Yes. If $1+a \notin U(R)$, then $1+a \in \mathfrak{m} \in M(R)$, then $1 \in \mathfrak{m}$ $(\because a \in \bigcap \mathfrak{m})$.

$\rightarrow\leftarrow$

Def. Let R be a commutative ring. The Jacobson radical $J(R)$ of R is defined as

Jacobson radical

$$J(R) = \bigcap_{\mathfrak{m} \in Max(R)} \mathfrak{m}.$$

Prop. $\mathcal{N}(R) \subset J(R)$. That is, if a is nilpotent, then $a \in \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof: If $a \in N(R)$, then $a^k = 0$ for so—-

$\Rightarrow a^k \in \mathfrak{m} \quad \forall \mathfrak{m}$ max

max. ideals are prime

$\Rightarrow a \in \mathfrak{m} \quad \forall \mathfrak{m}$

$\Rightarrow a \in J(R)$.

In fact, $\underbrace{\mathcal{N}(R) \subset \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} \subset J(R)}$.

↳ any equality?

Thm. In fact, $J(R)$ is a radical ideal, by (almost) the same argument.

Q. If $1+a$ is a unit, does $a \in J(R)$?
No. Take, $R = \mathbb{Z}$, $1+a = -1$.

Note: $a \in J(R) \Rightarrow \forall r \in R (1 + ra \in U(R))$

Does this converse hold now?
Yes!

Prop. $a \in J(R) \Leftrightarrow \forall r \in R \ (1 + ra \in U(R))$

Proof: ($\Rightarrow$) Let $a \in J(R)$ and $r \in R$ be arbitrary.
$ra \in J(R)$ since $J(R)$ is an ideal.
Thus, $ra \in \mathfrak{m}$ for every max. ideal $\mathfrak{m}$.
$\Rightarrow 1 + ra \notin \mathfrak{m}$ for any max ideal $\mathfrak{m}$
$\Rightarrow 1+ra$ is a unit.

($\Leftarrow$) Fix $a \in R$.

Assume that $1+ra$ is a unit for every $r \in R$.

Assumption: Suppose that $\exists \mathfrak{m} \in Max(R)$ s.t. $a \notin \mathfrak{m}$.

Then, $\mathfrak{m} + \langle a \rangle = R$.

$\Rightarrow m - ra = 1$ for some $r \in R$, $m \in \mathfrak{m}$.

$\Rightarrow m = 1+ra \in \mathfrak{m}$ is a unit $\rightarrow\leftarrow$

Thus, our assumption was incorrect. In other words,

$a \in \mathfrak{m}$ for all $\mathfrak{m} \in Max(R)$.

Thus, $a \in J(R)$. :)

Q. Prove or disprove: $J(R) = 0$ for any $0 \neq R$ comm.

Sol^n. Disproof. We construct a counterexample.

$R = \mathbb{Z}/4\mathbb{Z}$.

Ideals of $R = \{\{0\}, \{0,2\}, R\}$

↳ maximal!

Thus, $J(R) = \{0, 2\} \neq \{0\}$. :(

(Prime ideal Exercise solution)

Let R be a comm. ring and $I, J \subset R$ be ideals. Let $\mathfrak{p} \in Spec(R)$ s.t.

$IJ \subset \mathfrak{p}$ and $I \not\subset \mathfrak{p}$.

We show that $J \subset \mathfrak{p}$.

Proof. Let $j \in J$ be arbit.

Since $I \not\subset \mathfrak{p}$, $\exists\, i \in I$ s.t. $i \notin \mathfrak{p}$. $ij \in IJ \subset \mathfrak{p}$.

$\Rightarrow ij \in \mathfrak{p}$ and $i \notin \mathfrak{p}$.

$\therefore j \in \mathfrak{p}$ since $\mathfrak{p}$ is prime.

$\Rightarrow J \subset \mathfrak{p}$ (j was arbit) :)

Lecture 10 (07-09)

07 September 2020 09:22 AM

From this point on, unless otherwise mentioned, we shall assume rings to be commutative

Q. Consider the natural map $\varphi : R \longrightarrow R/I \times R/J$. Is this onto?

A. Well, if φ is onto, then $(\bar{1}, \bar{0})$ must have a preimage.

$\therefore \varphi(a) = (\bar{1}, \bar{0})$ for some $a \in R$.

$\Rightarrow a \equiv 1 \mod I$ & $a \equiv 0 \mod J$

$\Rightarrow 1 - a \in I$ and $a \in J$

$\Rightarrow 1 = (1-a) + a \in I + J$.

Leads to the following def^n.

Def^n. Let $I, J \subsetneq R$ be ideals. We say (I, J) is co-maximal if
$$I + J = R.$$

Co-maximal, comaximal ideals

Thus, if $\varphi : R \to R/I \times R/J$ is onto, then (I, J) is co-max.

[Assuming they are proper]

Q. Is the converse true? That is, if (I, J) is co-max, then is
$$\varphi : R \longrightarrow R/I \times R/J \quad \text{surjective?}$$

A. Yes! Note that $\exists\, i \in I, j \in J$ s.t. $i + j = 1$. $(\because I + J = R)$

Now, let $(\bar{a}, \bar{b}) \in R/I \times R/J$ be arbitrary.

Fix some pre-im. $a \in R$, $b \in R$.

Consider $r = bi + aj \in R$.

Then, $\varphi(r) = (bi + aj + I,\ bi + aj + J)$

$= (aj + I,\ bi + J) = (a - ai + I,\ b - bj + J)$

$= (a + I,\ b + J)$

$= (\bar{a}, \bar{b})$. ☺

Q. Is φ one-one? **Ans.** Note that $\ker \varphi = I \cap J$.
Thus, $1\text{-}1 \iff I \cap J = 0$.

Thus, we see that for proper ideals I, J in R

$$\begin{tikzcd} R/I\cap J \arrow[r, hook, "\tilde{\varphi}"] & R/I \times R/J \\ R \arrow[u, "\pi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

which is an isomorphism if the pair (I, J) is co-max.

① Observation: If (I, J) is comaximal, then $IJ = I \cap J$.
Proof. ($\subseteq$) Always.
($\supseteq$) Let $a \in I \cap J$. $\quad i + j = 1$

$$a = \underset{\substack{\uparrow \\ JI}}{ai} + \underset{\substack{\uparrow \\ IJ}}{aj}$$

$IJ =$

☺

② Examples: Let $\mathfrak{m} \in \mathrm{Max}(R)$ and $I \not\subseteq \mathfrak{m}$ be a proper ideal.
Then, $(I, \mathfrak{m})$ is comaximal.

↳ However, this will not work if every proper ideal is contained in $\mathfrak{m}$.

↓

This means that $\mathrm{Max}(R) = \{\mathfrak{m}\}$.

↳ Such a ring is called **local.**

↳ Notation: $(R, \mathfrak{m})$.

③ A local ring does not contain a pair of comaximal ideals.

If I, J are prop. ideals, then $I, J \subseteq \mathfrak{m}$ & thus, $I + J \subseteq \mathfrak{m} \subsetneq R$.

Conversely, a non-local ring always contains a comaximal pair.

Choose two distinct maximal ideals!

$\mathfrak{m}_1, \mathfrak{m}_2 \subsetneq \mathfrak{m}_1 + \mathfrak{m}_2 \quad \therefore \mathfrak{m}_1 + \mathfrak{m}_2 = R$.

Q. Let $m, n \in \mathbb{Z}$. When is $m\mathbb{Z}$ maximal, prime or radical?
When is $(m\mathbb{Z}, n\mathbb{Z})$ comaximal?

Do the same for $\mathbb{k}[x]$.

Q. Let $I_1, \dots, I_n \subsetneq R$. What can we say about

$$\varphi : R \longrightarrow R/I_1 \times \cdots \times R/I_n \ ?$$

often called the "diagonal map"

(Note that $\ker \varphi = \bigcap_{i=1}^{n} I_i$)

Notation: $\bar{e_j} = (\bar{0}, \dots, \bar{0}, \bar{1}, \bar{0}, \dots, \bar{0})$

j^{th} pos.

If φ is onto, $\exists a_j \in R$ s.t. $\varphi(a_j) = \bar{e_j}$.

$\Rightarrow 1 - a_j \in I_j$ & $a_j \in I_k$ for $k \neq j$.

$\Rightarrow I_j$ & $\bigcap_{\substack{k=1 \\ k\neq j}}^{n} I_k$ are comaximal

Q. Suppose I_1 and $\bigcap_{j=2}^{n} I_j$ are comaximal.

Is (I_1, I_j) comaximal for all $j \neq 1$?

Lecture 11 (08-09)

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Recall the following q.

Q. Suppose I_1 and $\bigcap_{j=2}^{n} I_j$ are coximal. Is (I_1, I_j) comax $\forall j \geq 2$?

Ans. Yes. Let $2 \leq j \leq n$. Then

$$R = I_1 + \bigcap_{j=2}^{n} I_j \subseteq I + I_j \subseteq R.$$

$\Rightarrow I + I_j = R$ showing (I, I_j) is comax.
(we had assumed $I_j \subsetneq R$.)

★ In fact, if (I, J) is co-max, then so is (I, K) for all proper ideals $K \supset J$.

Proof. Same as above.

Thus, if $\varphi : R \longrightarrow R/I_1 \times \cdots \times R/I_n$ is onto,

then for all $j \neq k$, (I_j, I_k) is comax.

In other words: $I_1, \ldots, I_n$ are pairwise comax.

Q: Is converse true? That is, if $I_1, \ldots, I_n \subsetneq R$ are comax, is φ onto?

A. Recall we had seen in the last class that if (I, J) is comax, then the induced map $\tilde{\varphi} : R/(I \cap J) \to R/I \times R/J$ is an iso.

We can now prove the result by induction.
Suppose the result is true for $m < n$. (Induc. hyp.)
Base: $n = 2$ done.
Let $I_1, \ldots, I_n$ be pairwise comaximal.

Claim: I_1 and $\bigcap_{j=2}^{n} I_j$ are comaximal.

Assume claim for now.

Then, $\varphi: R \longrightarrow R/I_1 \times R/\bigcap_{j=2}^{n} I_j$ is onto (by $n=2$)

By induction, $R/\bigcap_{j=2}^{n} I_j \simeq R/I_2 \times \cdots \times R/I_n$. (and the iso thm.)

Moreover, this iso was induced by φ.

$$(a + \bigcap I_j) \longmapsto (a + I_2, \ldots, a + I_n).$$

Using this, we get that

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[rr, bend right, "\text{is onto}"'] & R/I_1 \times R/(\bigcap I_j) \arrow[r, two heads] & R/I_1 \times \cdots \times R/I_n \end{tikzcd}$$

Now, we prove the claim.

Claim: I_1 and $\bigcap_{j=2}^{n} I_j$ are comaximal.

Proof. $\overset{\in I_2}{a_2}, \ldots, \overset{\in I_n}{a_n}$. $\overset{\in I_1}{b_2}, \ldots, \overset{\in I_1}{b_n}$

$a_2 + b_2 = 1$, $a_3 + b_3 = 1$, ..., $a_n + b_n = 1$

$$1 = (a_2 + b_2) \cdots (a_n + b_n)$$
$$= \underbrace{a_2 a \cdots a_n}_{\bigcap I_j} + \underbrace{b_2(\ ) + b_3(\ ) + \cdots + b_n(\ )}_{I_1}$$

$\Rightarrow (I_1, \bigcap_{j=2}^{n} I_j)$ is comaximal. ☺

Thus, we have proved the Chinese Remainder Theorem.

Thm. (Chinese Remainder Theorem)
Let R be a non-zero commutative ring.
Let $I_1, \ldots, I_n \subsetneq R$ be pairwise comaximal ideals.
Then,

$$R/ \quad \simeq R/ \times \cdots \times R/I .$$

Then,

$$\frac{R}{I_1 \cap \cdots \cap I_n} \simeq R/I_1 \times \cdots \times R/I_n .$$

Note that we also proved:

① The natural map $R \longrightarrow \prod R/I_j$ is onto.

② $I_1 \cap \cdots \cap I_n = I_1 \cdots I_n$.

Ex. Write a text book proof of CRT. → Assignment, due before class on Thursday. The statement

Prime Ideals.

How do you find prime ideals?
How do you find prime but not maximal?

1 Q: Is $\mathcal{N}(R) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$? ($R \neq 0$ comm.)

We had observed ($\subseteq$).
What about ($\supseteq$)?

Let $A = \bigcap \mathfrak{p}$ and $B = \mathcal{N}(R)$.

Claim. $A \subset B$.

Proof We show $B^c \subset A^c$.
Let $a \in R \setminus \mathcal{N}(R)$. We show that $a \notin \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$.

Idea in general: Take some collection of proper ideals
Show it has a max.
Show it is prime.

Consider the collection

$$\Delta = \{ I \subsetneq R \mid I \text{ is an ideal, } a \notin I \}.$$

$\Delta \neq \emptyset$ since $\mathcal{N}(R) \in \Delta$. Δ is a poset by $\subseteq$.

Given a chain $\{I_i\}_{i \in I}$, take $I = \bigcup I_i$.

$I \in \Lambda$, clearly. By Zorn, $\exists$ maximal $\mathfrak{p} \in \Lambda$.

Claim. $\mathfrak{p}$ is prime.

Let $b, c \in R$ s.t. $b \notin \mathfrak{p}$ and $c \notin \mathfrak{p}$. (want to show $bc \notin \mathfrak{p}$.)

By maximality of $\mathfrak{p}$ in Λ, we get

$$\mathfrak{p} + \langle b \rangle \notin \Lambda \not\ni \mathfrak{p} + \langle c \rangle.$$

Thus, $a \in \mathfrak{p} + \langle b \rangle$ and $a \in \mathfrak{p} + \langle c \rangle$.

$\Rightarrow$ $a = p_1 + r_1 b = p_2 + r_2 c$, $\quad p_1, p_2 \in \mathfrak{p}$, $r_1, r_2 \in R$

$$a^2 = p + r_1 r_2 bc$$

$$bc \in \mathfrak{p} \Leftrightarrow a^2 \in \mathfrak{p}$$

Now what?

Well, we didn't use the full power of $a \notin N(R)$.

To be continued...

Lecture 12 (10-09)

10 September 2020 11:14 AM

Changing the prev. proof.

Consider the collection

$$\Lambda = \{ I \subsetneq R \mid I \text{ is an ideal, } a^n \notin I \text{ for any } n \in \mathbb{Z} \}$$

$\Lambda \neq \emptyset$ since $\mathcal{N}(R), \langle 0 \rangle \in \Lambda$. Λ is a poset by $\subseteq$.

Given a chain $\{I_i\}_{i \in \mathcal{I}}$, take $I = \bigcup I_i$.

still goes through

$I \in \Lambda$, clearly. By Zorn, $\exists$ maximal $\mathfrak{p} \in \Lambda$.

<u>Claim</u>. $\mathfrak{p}$ is prime.

Let $b, c \in R$ s.t. $b \notin \mathfrak{p}$ and $c \notin \mathfrak{p}$. (want to show $bc \notin \mathfrak{p}$.)

By maximality of $\mathfrak{p}$ in Λ, we get

$$\mathfrak{p} + \langle b \rangle \notin \Lambda \not\ni \mathfrak{p} + \langle c \rangle.$$

Thus, $a^n \in \mathfrak{p} + \langle b \rangle$ and $a^m \in \mathfrak{p} + \langle c \rangle$. for some $n, m \in \mathbb{N}$

$\Rightarrow$ $a^n = p_1 + r_1 b$; $a^m = p_2 + r_2 c$, $\quad p_1, p_2 \in \mathfrak{p}$, $r_1, r_2 \in R$

$$a^{n+m} = p + r_1 r_2 bc$$

$$bc \in \mathfrak{p} \Rightarrow a^{n+m} \in \mathfrak{p}$$

not possible by def^n of $\mathfrak{p}$

Thus, $bc \notin \mathfrak{p}$.

Hence, $\mathfrak{p}$ is a prime and $a \notin \mathfrak{p}$. ∎

Thus, we have proven.

Thm. Let R be a non-zero commutative ring. Then,

$$\mathcal{N}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}.$$

Cor. Let $I \subsetneq R$ be a proper ideal. Then,

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ I \subset \mathfrak{p}}} \mathfrak{p}.$$

Proof. ① Either go mod I.
② Re-write earlier theorem with new Λ.

Notation: For an ideal $I \subset R$,
$$V(I) = \{ \mathfrak{p} \in \mathrm{Spec}(R) : I \subset \mathfrak{p} \}.$$

Prime Avoidance

Set Theoretic Q. Let $A, A_1, \ldots, A_n$ be sets. If $A \subset \bigcup_{i=1}^{n} A_i$, is it necessary that $A \subset A_i$ for some i?

Nope!

Take $(n=2)$ $A = \{0, 1\}$, $A_1 = \{0\}$, $A_2 = \{1\}$.

Is the above true if each A and A_i is an ideal in some (comm.) ring R?

Still nope!

Ex. Find a counterexample.

However, the statement is true for prime ideals.

Thm. (Prime Avoidance) Let $I \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$ for $\mathfrak{p}_i \in \operatorname{Spec}(R)$.

Then, $I \subset \mathfrak{p}_j$ for some j.

Proof. Note that $n=2$ is true in general. (Even if $\mathfrak{p}_1, \mathfrak{p}_2 \notin \operatorname{Spec}(R)$.)

We prove $n \geq 3$ by induction.

~~$n=3$: Suppose $I \not\subset \mathfrak{p}_j$; $j=1,2,3$.~~

~~We show $I \not\subset \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_3$.~~

~~$\Rightarrow \exists a_j \in I$ but $a_j \notin \mathfrak{p}_j$.~~

~~Verify that $a = a_1 a_2 + a_3 a_1 + a_2 a_3 \in I$ but not in union.~~

This actually WON'T work. (Or at least, we don't see why it should.)

The point is that we did not use the full info.

That is, $I \not\subset \mathfrak{p}_1 \cup \mathfrak{p}_2$, etc. (induction)

Counter example for non-prime ideals.

Take the ring $R = \dfrac{\mathbb{F}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$ and the ideal $I = \langle \bar{x}, \bar{y} \rangle \subset R$.

$$= \{0, 1, \bar{x}, \bar{y}, \bar{x}+1, \bar{y}+1, \bar{x}+\bar{y}, \bar{x}+\bar{y}+1\}$$

$I = \{0, \bar{x}, \bar{y}, \bar{x}+\bar{y}\}$ ← this is not principal. (Check manually.)

But $I \subset \langle \bar{x} \rangle \cup \langle \bar{y} \rangle \cup \langle \bar{x}+\bar{y} \rangle$ and not contained in any individual one. ☺

But $I \subset \langle \bar{x} \rangle \cup \langle \bar{y} \rangle \cup \langle \bar{x} + \bar{y} \rangle$ and not contained in any individual one. $\square$

Lecture 13 (14-09)

14 September 2020 09:35 AM

Thm. (Prime avoidance)

Let $I \subset R$ be an ideal and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec}(R)$.
If $I \subset \bigcup \mathfrak{p}_i$, then $I \subset \mathfrak{p}_i$ for some $1 \le i \le n$.

Proof. $n=1$. Nothing.

$n=2$. Suppose not. Take $a_1 \in I \setminus \mathfrak{p}_1$ & $a_2 \in I \setminus \mathfrak{p}_2$.

Then, $a_1 + a_2 \in I$.

But $a_1 + a_2 \notin \mathfrak{p}_1 \cup \mathfrak{p}_2$ $\rightarrow\leftarrow$

$n \geq 3$. By induction. Suppose not.

we know $I \not\subset \bigcup_{\substack{i=1 \\ i \neq k}}^{n} \mathfrak{p}_i$ for each k, by induc.

Choose $a_k \in I \setminus \bigcup_{\substack{i=1 \\ i \neq k}}^{n} \mathfrak{p}_i$.

Here is where we used primality

Then, $b_k = \prod_{\substack{j=1 \\ j \neq k}}^{n} a_j \notin \mathfrak{p}_k$ since $\mathfrak{p}_k$ is prime.

$\in \mathfrak{p}_j$ for all $j \neq k$.

Thus, $b_k \in \bigcup_{\substack{j=1 \\ j \neq k}}^{n} \mathfrak{p}_j \setminus \mathfrak{p}_k$, also $b_k \in I \setminus \mathfrak{p}_k$.

Let $b \in I$ be defined as

$b = b_1 + \dots + b_n$.

Then, $b \in I \setminus \bigcup_{j=1}^{n} \mathfrak{p}_j$. $\rightarrow\leftarrow$

Thm. (More general prime avoidance) In the above hypothesis, we can assume two are not necessarily prime.

Prof. We phrase the theorem as follows:

Let $n \geq 2$.

Let $I_1, \dots, I_n \subset R$ be ideals s.t. $I_n \in \operatorname{Spec}(R)$ for $n \geq 3$.

Let $I \subset R$ be an ideal such that

$$I \subset \bigcup_{i=1}^{n} I_i.$$

Then, $I \subset I_i$ for some $1 \leq i \leq n$.

Prof. For $n=2$, we know by earlier.

Assume true for $n-1$ for some $n \geq 3$.

Now, let $I_1, \dots, I_n$ be as in the theorem.

Assumption: Suppose $I \not\subset I_i$ for any $1 \leq i \leq n$ but $I \subset \bigcup_{i=1}^{n} I_i$.

By induction, $I \not\subset \bigcup_{\substack{i=1 \\ i \neq k}}^{n} I_i$ for any $1 \leq k \leq n$.

each such has at most 2 ideals which are possibly not prime

Thus, we may choose $a_k \in I \setminus \bigcup_{\substack{i=1 \\ i \neq k}}^{n} I_i$ for each $k = 1, \dots, n$.

Then, $a_k \in I_k$ for each $1 \leq k \leq n$.

Define $a = \underbrace{a_1 a_2 \cdots a_{n-1}}_{\in I_1, \dots, I_{n-1}} + \underbrace{a_n}_{\in I_n \setminus (I_1 \cup \dots \cup I_{n-1})}$.

This does not belong to I_n since I_n is prime, whereas each factor $\notin I_n$

Thus, $a \notin \bigcup_{i=1}^{n} I_i$, contradiction!