Fourier Inversion for L^1 Functions

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Overview

Heat kernel

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1. $(f * h_t)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{-\pi t\|\xi\|^2}e^{2\pi\iota x\cdot\xi}\,\mathrm{d}\xi$ for <u>all</u> $x \in \mathbb{R}^n$.

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2. Using DCT, we let $t \to 0$ in the above to conclude that

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for those $x \in \mathbb{R}^n$ for which $(\star)$ holds.

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We will actually prove the result for a broader class of approximate identities.

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Note that the above $\mathrm{Leb}(f)$ is actually a superset of the $\mathrm{Leb}(f)$ we defined it in class. So, we shall prove a stronger result.

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for all $x \in \mathrm{Leb}(f)$.

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It is now clear that proving the Main Theorem will show that $(\star)$ holds for $x \in \mathrm{Leb}(f)$.

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$$\int_{B(x,r)} 1 = V_n r^n$$

for $x \in \mathbb{R}^n$ and $r > 0$.

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Proof of the Main Theorem

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Note that for all $t > 0$, we have $\int_{\mathbb{R}^n} \varphi_t = \int_{\mathbb{R}^n} \varphi = \int_{\mathbb{R}^n} |\varphi| = 1$.

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Using this, we first show that $I_2(t) \xrightarrow{t\to 0} 0$.

Taking I_2 down

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l_2 down, l_1 to go

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With these notations, we do some more calculations.

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This completes the proof.

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Theorem (General Theorem)

Suppose $\varphi \in L^1(\mathbb{R}^n)$.

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Reference: *Introduction to Fourier Analysis on Euclidean Spaces* by *Stein and Weiss*