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MA 406

General Topology

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Lecture 1 (07-01-2021)

07 January 2021 15:04

Def^n. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties: (Topology)

(1) ϕ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(Open set)

Any $U \in \mathcal{T}$ is called an open set of X w.r.t. $\mathcal{T}$.
The pair $(X, \mathcal{T})$ or just the set X is called a topological space. (abuse of notat^n.)

Can reconcile the above with open sets in $\mathbb{R}$, or in general, any metric space X. That can be seen as a motivation for the definition.

Examples

(1) $X = \{a, b, c\}$

$\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ (Can be seen (fairly easily) that this is a topology)

$\mathcal{T}_2 = \{\phi, X\}$ trivial (pun intended, cf. next example)

(2) If X is any set, the collection of all subsets of X is a topology on X, it is called the discrete topology. ($\mathcal{T} = \mathcal{P}(X)$, that is) (Discrete topology)

The collection $\{\phi, X\}$ is also a topology on X called the indiscrete topology or trivial topology. (Indiscrete topology
Trivial topology)

(3) Let X be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\emptyset\}.$$

(Finite complement topology)

Then, $\mathcal{T}_f$ is a topology on X, called the finite complement topology on X.

- $\emptyset \in \mathcal{T}_f$ is clear. $X \in \mathcal{T}_f$ since $|X \setminus X| = 0 < \infty$.
- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be sets in $\mathcal{T}_f$. WLOG, $U_\alpha \neq \emptyset$ $\forall \alpha$.

Note $X \setminus \left(\bigcup_\alpha U_\alpha \right) = X \cap \left(\bigcup_\alpha U_\alpha \right)^c$

$= \bigcap_\alpha (U_\alpha^c)$

Note that each U_α^c is finite. ($U_\alpha \neq \emptyset$)
Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to $\bigcup_{i=1}^{n} (U_i^c)$ and conclude as earlier.

(Here, if some U_i were $\emptyset$, then so would be the intersection.)

(If X is finite, the $\mathcal{T}_f = \mathcal{P}(X)$. Thus, we get discrete.)

(4) Let X be a set.
Let $\mathcal{T}_c$ be the collection of subsets such that $X \setminus U$ is either countable or all of X.
Called the co-countable topology.
(Generalising the previous)

(Cocountable topology
Co-countable topology)

Def^n Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X.
If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and that $\mathcal{T}$ is coarser than $\mathcal{T}'$!
If $\mathcal{T}' \supsetneq \mathcal{T}$, then the above is strictly finer and strictly ...

coarser, respectively. (Finer, coarser, strictly finer, strictly coarser)

(The above gives us a way to compare two topologies)

EXAMPLE We have the usual topology on $\mathbb{R}$. ← strictly coarser than this
We also have the discrete topology on $\mathbb{R}$.

Def^n. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that (Basis)

(1) for each $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Note that in the above, $\mathcal{B}$ is just some collection of subsets of X satisfying (1) & (2). No topology is mentioned so far.

EXAMPLES

(1) $X = \mathbb{R}^2$, $\mathcal{B}$ is the collection of all discs w/o boundary.
(2) ——— " ——— " - rectangles —"—
(3) Any X. The singletons form a basis.

We now get a topology out of a basis:

Def^n. If $\mathcal{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows: (Topology generated)

A subset U of X is said to be open if for every $x \in U$, there exists $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

$$x \in B \subset U.$$

(By "open" in above, we mean element of ***T***. Same thing for what we see in the proof below.)

EXAMPLES (1) & (2) $\to$ gives standard topology on $\mathbb{R}^2$

(3) $\to$ gives discrete topology on X.

We still have to show that it is topology.

Proof.

- $\phi \in \mathcal{T}$ vacuously

  $X \in \mathcal{T}$ since given any $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.

  $B \subset X$ is by definition.

- Let $\{U_\alpha\}_{\alpha \in \Lambda \neq \phi}$ be open. Let $U := \bigcup_\alpha U_\alpha$.

  Fix $\alpha_0 \in \Lambda$.

  Let $x \in U$ be arbitrary. Then, $x \in U_{\alpha_0} \leftarrow$ open

  $\therefore \exists B \in \mathcal{B}$ s.t. $x \in B \subset U_{\alpha_0} \subset U$.

  $\therefore U \in \mathcal{T}$.

- Let U_1 and U_2 be open. Put $U := U_1 \cap U_2$.

  Let $x \in U$.

  Then $x \in U_1$ and $x \in U_2$

  $\Downarrow$ $\Downarrow$

  $\exists B_1 \in \mathcal{B}$ $\exists B_2 \in \mathcal{B}$

  s.t. $x \in B_1 \subset U_1$ s.t. $x \in B_2 \subset U_2$

  $$\therefore \underbrace{x \in B_1 \cap B_2}_{\Downarrow} \subset U_1 \cap U_2$$

  $$\exists B_3 \in \mathcal{B} \text{ s.t. } x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U.$$

  $\Rightarrow U \in \mathcal{T}$.

By induction, any finite intersection is in $\mathcal{T}$. $\boxtimes$

$$n \geq 3 \left(\bigcap_{i=1}^{n} U_i = U_n \cap \left(\bigcap_{i=1}^{n-1} U_i \right). \right.$$

Lecture 2 (11-01-2021)

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Lemma 1 Let $\mathcal{B}$ be a basis and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then, $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Note that $\emptyset$ is empty union.

Proof. Given $\{U_\alpha\} \subset \mathcal{B}$, it is clear that $\bigcup U_\alpha \in \mathcal{T}$ since $\mathcal{T}$ is a topology and U_α are open. (By defn)

Conversely, let $U \in \mathcal{T}$. Given any $x \in U$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. (By defn of $\mathcal{T}$.)

Thus,
$$\bigcup_{x \in U} B_x = U. \quad \blacksquare$$

($\subseteq$) since $B_x \subset U$

($\supseteq$) Each $x \in U$ is in B_x.

(Note that if $U = \emptyset$, the last union is the empty union!)

[The above gives us a way of extracting a basis $\mathcal{B}$ if we are already given a topology $\mathcal{T}$. Namely, pick any subcollection $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{T}$ is precisely the collection of all unions of elements of $\mathcal{B}$.]

Lemma 2. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$. (recall this means $\mathcal{T} \subset \mathcal{T}'$)

(ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \in B$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and $B \in \mathcal{B}$ be arbitrary. Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$. Thus $B \in \mathcal{T}'$. (by (i))

Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.
Thus, $B \in \mathcal{T}'$. (by (i))
Since B is open in $\mathcal{T}'$, $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.
(Def^n of top. generated.)

(ii) $\Rightarrow$ (i) Suppose $U \in \mathcal{T}$. We show that $U \in \mathcal{T}'$.
Let $x \in U$. By def^n of $\mathcal{T}$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B \subset U$.
By (ii), $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B \subset U$.

Since x was arbit, we see that $U \in \mathcal{T}'$. (By def^n of $\mathcal{T}'$)
Thus, $\mathcal{T} \subset \mathcal{T}'$. ☒

<u>Lemma 3</u>. Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X s.t. for each open set $U \subset X$ and each $x \in U$, $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset U$.
Then $\mathcal{C}$ is a basis for the topology.

<u>Proof</u>. • Showing $\mathcal{C}$ is a basis.
(i) Given any $x \in X$, X is an open set containing x.
Thus, by hypothesis, $\exists\, C \in \mathcal{C}$ s.t. $x \in C$.

(ii) Let $C_1, C_2 \in \mathcal{C}$ s.t. $x \in C_1 \cap C_2$.
Note that C_1, C_2 are open and hence, $C_1 \cap C_2$ is open.
By hypothesis $\exists\, C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$.

Thus, $\mathcal{C}$ satisfies both properties of a topology.

• $\mathcal{C}$ generates the topology.
Let $\mathcal{T}$ denote the topology of X. Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$.

Let $U \in \mathcal{T}'$, then U is some union of elements of $\mathcal{C}$.

But elements of $\mathcal{C}$ are elements of $\mathcal{T}$ and thus, $U \in \mathcal{T}$.
($\mathcal{T}$ is topo.)

Thus, $\mathcal{T}' \subset \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. For each $x \in U$, $\exists C_x \in \mathcal{C}$ s.t.
$x \in C_x \subset U$.

As earlier,
$$U = \bigcup_{x \in U} C_x \in \mathcal{T}'$$

Thus, $\mathcal{T} \subset \mathcal{T}'$. ∎

Def^n. Let $\mathcal{B}$ be the collection of all bounded intervals.
That is,
$$\mathcal{B} = \{ (a, b) : -\infty < a < b < \infty \}.$$

$\mathcal{B}$ is a basis and the topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$. (Standard topology on \Bbb R)

If $\mathcal{B}'$ is the collection of all half open intervals of the form $[a, b)$, then $\mathcal{B}'$ is also a basis and the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$. (Lower limit topology on \Bbb R)

Lemma 4. The lower limit topology is strictly finer than the standard topology.

Proof. Let $\mathcal{T}$ denote the standard topology and $\mathcal{T}'$ the lower limit.

- $\mathcal{T} \subseteq \mathcal{T}'$. Let (a, b) be an arbit. basis element and let $x \in (a, b)$.
  Then, $[x, b)$ is a basis element for $\mathcal{T}'$ &
  $x \in [x, b) \subset (a, b)$.

Thus, $\mathcal{T} \subseteq \mathcal{T}'$, by Lemma 2.

- $\mathcal{T}' \neq \mathcal{T}$. Note that $[0, 1) \in \mathcal{T}'$.
  But given $0 \in [0, 1)$, there is no $(a,b) \ni 0$ s.t. $(a, b) \subset [0, 1)$. ∎

Def^n: A subbasis $\mathcal{S}$ for a topology is a collection of subsets of X whose union is X. (Subbasis, sub basis)

(Note that no topology given so far. Similar to what we saw for basis.)

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

We need to show that the topology defined above is actually a basis.

Let $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{S}$. We show $\mathcal{B}$ is a basis. (This suffices. Why? Lemma 1.)

(i) Let $x \in X$. Then, $\exists S \in \mathcal{S}$ s.t. $x \in S$. $\left(\because \bigcup_{S \in \mathcal{S}} S = X \right)$
But $S \in \mathcal{B}$.

(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$.
But note that $B_1 \cap B_2 \in \mathcal{B}$. (Why?)

Thus, both the conditions are satisfied.

Remark: The standard topology of $\mathbb{R}$ is also called the order topology on $\mathbb{R}$, because of the order relation of $\mathbb{R}$.

(We will see this in general, later.)

Def^n Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.

(Product topology)

(Open in the repective topologies, i.e.)

Note that $\mathcal{B}$ is a basis because:

(i) $X \times Y$ is itself a basis element

(ii) $U \times V,\ U' \times V' \in \mathcal{B} \Rightarrow (U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V') \in \mathcal{B}$

intersection of open sets

Note $\mathcal{B}$ itself won't be the topology. (In general)

Thm 5. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}_x$ on X, and $\mathcal{C}$ for $\mathcal{T}_y$ on Y, then the collection

$$\mathcal{D} = \{ B \times C : B \in \mathcal{B},\ C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We check that the hypotheses of Lemma 3 are satisfied.

Let $W \subset X \times Y$ be open and $(x, y) \in W$.
Then, by def^n of prod. top., $\exists\ U \in \mathcal{T}_x,\ V \in \mathcal{T}_y$ s.t.

$$(x, y) \in U \times V \subset W.$$

Since $\mathcal{B}$ is a basis for $\mathcal{T}_x$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

$\|^{ly}$ $\exists\, C \in \mathcal{C}$ s.t. $y \in C \subset V$.

$\Rightarrow (x, y) \in \underset{\substack{\uparrow \\ \mathcal{D}}}{B \times C} \subset U \times V \subset W$. $\blacksquare$

Lecture 3 (14-01-2021)

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By last lecture's discussion, we know that

$$\{ (a, b) \times (c, d) : a, b, c, d \in \mathbb{R} \}$$

is a basis for the product topology on $\mathbb{R}^2$.
This is called the standard topology on $\mathbb{R}^2$.

Defn. Given any two sets X and Y, we have the two projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given as

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y \quad \forall (x, y) \in X \times Y.$$

(Projections)

Note that $\pi_1^{-1}(U) = U \times Y$ for any $U \subseteq X$ and similarly $\pi_2^{-1}(V) = X \times V$ for any $V \subseteq Y$.

Thm 1. The collection

$$\mathcal{S} = \{ \pi_1^{-1}(U) \mid U \subseteq X \text{ open} \} \cup \{ \pi_2^{-1}(V) \mid V \subseteq Y \text{ open} \}. \quad \subseteq \mathcal{P}(X \times Y)$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}_p$ denote the product topology on $X \times Y$.
Let $\mathcal{T}_S$ — " — topology generated by $\mathcal{S}$.

Note that any element of $\mathcal{S}$ is of the form $U \times Y$ or $X \times V$.
$U \subseteq X$ open, $V \subseteq Y$ open

Thus, $\mathcal{S} \subseteq \mathcal{T}_p$ since both the above are actually basis elements. Since $\mathcal{T}_p$ is a topology, it is closed under arbitrary unions of finite intersections. Thus, $\mathcal{T}_S \subseteq \mathcal{T}_p$.

On the other hand, consider any arbitrary basis elt. of $\mathcal{T}_p$. It is of the form $U \times V$. $U \subseteq X$, $V \subseteq Y$ open.
Note now

$$U \times V = \underset{\underset{\mathcal{S}}{\cap}}{\pi_1^{-1}(U)} \cap \underset{\mathcal{S}}{\pi_2^{-1}(V)} \in \mathcal{T}_S$$

Thus, $U \times V \subset \mathcal{T}_S$. Since $\mathcal{T}_S$ is a topology, arbitrary union of basis elements of $\mathcal{T}_p$ is in $\mathcal{T}_S$.
Thus, $\mathcal{T}_p \subseteq \mathcal{T}_S$. ☒

Def^n. Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.
With this topology, $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$.

(Subspace topology)

(We will often just say "Y is a subspace of X" if it is clear.)

We now check that $\mathcal{T}_Y$ is actually a topology.

Check. (i) Since $\phi \in \mathcal{T}$, we get $\phi = \phi \cap Y \in \mathcal{T}_Y$.
$\quad$ $X \ni Y$, $Y = Y \cap X \in \mathcal{T}_Y$.

(ii, iii) Let $\{\mathcal{U}_i\}_{i \in I} \subseteq \mathcal{T}_Y$. Then, we have $\{\mathcal{U}_i'\}_{i \in I} \subseteq \mathcal{T}$ s.t.

$$\mathcal{U}_i' \cap Y = \mathcal{U}_i \qquad \forall i \in I.$$

Then, $$\bigcup_{i \in I} \mathcal{U}_i = \bigcup_{i \in I} (\mathcal{U}_i' \cap Y) = \underset{\underset{\mathcal{T}}{\cap}}{\left(\bigcup_{i \in I} \mathcal{U}_i' \right)} \cap Y$$

and similarly, $$\bigcap_{i \in I} \mathcal{U}_i = \underset{\underset{}{\cap}}{\left(\bigcap_{i \in I} \mathcal{U}_i' \right)} \cap Y.$$ ☒

and similarly, $\bigsqcup_{i\in I} \mathcal{U}_i = \left(\bigsqcup_{i\in I} \mathcal{U}_i\right) \cap Y$. ☒

$\mathcal{J}$ if $|I|<\infty$

Lemma 2. If $\mathcal{B}$ is a basis for $(X, \mathcal{J})$, then the collection

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for $(Y, \mathcal{J}_Y)$, i.e., the subspace topology on Y.

Proof We use Lemma 3 from Lecture 2.
Using that, it suffices to show that for any $\mathcal{U} \in \mathcal{J}_Y$ and any $x \in \mathcal{U}$, $\exists B \in \mathcal{B}_Y$ s.t. $x \in B \subset \mathcal{U}$.

To this end, let x, $\mathcal{U}$ be as given. Then,
$\mathcal{U} = \mathcal{U}' \cap Y$ for some $\mathcal{U}' \in \mathcal{J}$.
Clearly, $x \in \mathcal{U}'$.
Then, $\exists B' \in \mathcal{B}$ s.t. $x \in B' \subset \mathcal{U}'$. ($\mathcal{B}$ is a basis for $\mathcal{J}$.)

Then, $B = B' \cap Y \in \mathcal{B}_Y$ and
$x \in B \subset \mathcal{U}$. ☒

Lemma 3. Let Y be a subspace of X. If $\mathcal{U}$ is open in Y and Y is open in X, then $\mathcal{U}$ is open in X.

Proof. $\mathcal{U} = \mathcal{U}' \cap Y$ for some $\mathcal{U}' \subset X$ open.
Since $\mathcal{U}'$ and Y are open in X, so is $\mathcal{U} = \mathcal{U}' \cap Y$. ☒
(finite intersection of open sets.)

EXAMPLES. (i) If $Y = [0,1] \subset X = \mathbb{R}$ in subspace topology, it has a basis given as
$$\{(a, b) \cap Y : a < b \in \mathbb{R}\}.$$

More explicitly, here we have

$$(a, b) \cap Y = \begin{cases} (a, b) & a \in Y \ni b \\ [0, b) & a \notin Y \ni b \\ (a, 1] & a \in Y \not\ni b \\ \phi \text{ or } Y & a \notin Y \not\ni b \end{cases}$$

(2) Consider $Y = [0, 1) \cup \{2\} \subseteq \mathbb{R}$.

Note that

$$\{2\} = (1.5, 2.5) \cap Y.$$

Thus, $\{2\}$ is open in Y. (Was not open in $\mathbb{R}$!)

Similarly, $[0, 1)$ is open in Y but not $\mathbb{R}$.

Thm 4. If A is a subspace of X and B of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Note that the above tells us that the following two ways of topolog-ising $A \times B$ are the same:

- consider A and B as spaces by themselves and give $A \times B$ the product topology
- consider the topological space $X \times Y$ in product topology. Note that $A \times B$ is a <u>subset</u> of $X \times Y$ and hence, can be given the subspace topology.

$$\text{subspace} \nearrow \searrow \text{product} \;\overset{?}{=}\; \text{product} \searrow \nearrow \text{subspace}$$

Proof. Note the following:

typical basis elt of $X \times Y$

$$\{ (U \times V) \cap (A \times B) : U \subseteq X,\ V \subseteq Y \text{ open} \}$$

basis for subspace topology on $A \times B$ by Lemma 2

$= \{ (U \cap A) \times (V \cap B) : U \subseteq X,\ V \subseteq Y \text{ open}\}$ basis for prod top. on $A \times B$

$U \cap A$, $V \cap B$: a general open set in the subspace topology of $A \subseteq X$ or $B \subseteq Y$

Thus, both the topologies have a common basis.

Def^n. A subset of a topological space is said to be *closed* if its complement is open.

(Closed set)

Examples. (1) $[a, b] \subseteq \mathbb{R}$ is closed because
$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open.

(2) $[0, \infty) \times [0, \infty) \subseteq \mathbb{R}^2$ is closed because
$\mathbb{R}^2 \setminus [0,\infty)^2 = \big((-\infty, 0) \times \mathbb{R}\big) \cup \big(\mathbb{R} \times (-\infty, 0)\big)$ is open.

(3) In the discrete topology, every set is open and hence, every set is closed.

(4) Consider $Y = [-1, 0] \cup (2, 3) \subseteq \mathbb{R}$.
Both $[-1,0]$ and $(2,3)$ are open in Y.

$[-1,0] = (-2, 1) \cap Y$

Since they are complements of each other (in Y), we have that both the sets are closed as well, in Y.

Thm 5. Let X be a topological space. Then,
(i) ϕ and X are closed,
(ii) arbitary intersection of closed sets is closed,
(iii) finite union of closed sets is closed.

Proof. $X \setminus \phi = X$, $X \setminus X = \phi$.

Proof. $X \setminus \phi = X, \quad X \setminus X = \phi.$

$$X \setminus \left(\bigcap_{i \in I} C_i \right) = \bigcup_{i \in I} (X \setminus C_i).$$

$$X \setminus \left(\bigcup_{i=1}^{n} C_i \right) = \bigcap_{i \in I} (X \setminus C_i).$$

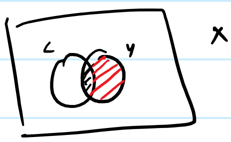
Conclude. ∎

Remark. The above is also a way to define a topology.

Thm 6. Let Y be a subspace of X and $A \subseteq Y$. Then, A is closed in Y iff A equals the intersection of a closed set (in X) with Y.

Proof. ($\Leftarrow$) Suppose $A = C \cap Y$ for some closed set $C \subseteq X$. Then,

$$Y \setminus A = \underbrace{\underset{\text{open in } X}{(X \setminus C)} \cap Y}_{\text{open in } Y}$$



$\therefore$ A is closed in Y.

($\Rightarrow$) Suppose A is closed in Y.
Then, $Y \setminus A$ is open in Y. Thus, $\exists\, \mathcal{U} \subseteq X$ open s.t.

$$Y \setminus A = \mathcal{U} \cap Y$$

Then,

$$\begin{aligned} \underset{A}{\overset{\shortparallel}{Y \setminus (Y \setminus A)}} &= Y \setminus (\mathcal{U} \cap Y) \\ &= Y \cap (\mathcal{U} \cap Y)^c \\ &= Y \cap (\mathcal{U}^c \cup Y^c) \\ &= (Y \cap \mathcal{U}^c) \cap (Y \cup Y^c) \end{aligned}$$

(The $(\;)^c$ is complement in X.)

$$\Rightarrow \quad A = Y \cap \mathcal{U}^c$$

Since $\mathcal{U}^c \subseteq X$ is closed, we are done.

Remark: A set can be both open and closed. For example, $\emptyset$ and X.

A less trivial example: Take $X = [0, 1] \cup [2, 3]$.

Then, $A = [0, 1] \subset X$ is both open & closed.

Lecture 4 (18-01-2021)

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Def^n. Given a topological space X and $A \subset X$, we define:

(Interior) The interior of A as the union of all open sets contained in A.

Notation: int A or $\mathring{A}$. $(\phi \subset A)$

(Closure) The closure of A as the intersection of all closed sets containing A.

Notation: $cl(A)$ or $\bar{A}$. $(A \subset X)$

Remark. $\mathring{A}$ is an open set and $\bar{A}$ is a closed set. Further,

$$\mathring{A} \subset A \subset \bar{A}.$$

A is open iff $A = \mathring{A}$.

A is closed iff $A = \bar{A}$.

Def^n. Let $x \in X$. A neighbourhood of x is any set A such that there is an open set $U \subset X$ with

$$x \in U \subseteq A.$$

(Neighbourhood)

(That is, a neighbourhood is any set containing an open set containing the point. This is different from the def^n in Munkres!)

Thm 1. Let A be a subset of a topological space X.
Then, $x \in \bar{A}$ iff every neighbourhood U of x intersects A.

Proof. $(\Leftarrow)$ $x \notin \bar{A} \Rightarrow U = X \setminus \bar{A}$ is a nbd of x not intersecting A.

$(\Rightarrow)$ Suppose $\exists$ nbd C of x s.t. $C \cap \bar{A} = \phi$.

Let U be open s.t. $x \in U \subseteq C$. (Def^n of nbd.)

Then, $X \setminus U$ is a closed set s.t. $A \subset X \setminus U$.

$\Rightarrow \bar{A} \subset X \setminus U$ (Why? $\because \bar{A}$ is the inter. of all closed sets cont. A.)

$\Rightarrow \bar{A} \cap U = \phi$

$\Rightarrow \quad \bar{A} \cap U = \phi$

$\Rightarrow \quad x \notin \bar{A}.$ $\quad \blacksquare$

Examples. ① $X = \mathbb{R}$ and $A = (0,1]$. Then, $\bar{A} = [0,1]$.

However, if $X = (0,1] = A$, then $\bar{A} = A$.

② $X = \mathbb{R}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, $\bar{B} = B \cup \{0\}$.

③ $C = \{0\} \cup (1, 2)$. Then, $\bar{C} = \{0\} \cup [1, 2]$

④ $\bar{\mathbb{Q}} = \mathbb{R}$

⑤ $\bar{\mathbb{N}} = \mathbb{N}$

⑥ $\bar{\mathbb{R}_+} = \mathbb{R}_+ \cup \{0\} = [0, \infty)$.

Def^n. Let X be a top. space and $A \subset X$. (Limit point)

A point $x \in X$ is said to be a limit point of A if every neighbourhood of x intersects A in some point other than x.

Notation : A'

Examples

	Subset of $\mathbb{R}$	Set of limit points
①	$(1, 2]$	$[1, 2]$
②	$\{\frac{1}{n} : n \in \mathbb{N}\}$	$\{0\}$
③	$\{0\} \cup (1, 2)$	$[1, 2]$
④	$\mathbb{Q}$	$\mathbb{R}$
⑤	$\mathbb{N}$	ϕ
⑥	$\mathbb{R}_+$	$\bar{\mathbb{R}_+}$

Thm 2. $\bar{A} = A \cup A'$. (Proof at the end)

Corollary 3. A is closed iff $A' \subset A$.

Proof. A is closed $\Leftrightarrow A = \bar{A} \underset{\text{Thm 2.}}{\Leftrightarrow} A' \subset A$.

Def^n. (Order relation or Simple order)

A relation C on set A is called an order relation (or a simple order) if it has the following properties:

(1) (Comparability) For every $x, y \in A$, $x \neq y \Rightarrow x C y$ or $y C x$.

(2) (Non reflexity) $\nexists x \in A$ s.t. $x C x$

(3) (Transitivity) $x C y$ and $y C z \Rightarrow x C z$.

A set with a simple order is called an ordered set.

Example Usual '$<$' on $\mathbb{R}$ is a simple order.

Def^n. If X is a set and '$<$' a simple order relation. Then, we define "$x \leq y$" as "$x < y$ or $x = y$."

Let $A \subset X$. An element $a \in A$ is said to be the smallest element of A if

$$a \leq x \quad \forall x \in A.$$

Similarly, we define the largest element.

(We have used "the" since uniqueness is simple to check. Existence, however, is not guaranteed. $\mathbb{R}$ has no largest or smallest element. Neither does $(0, 1)$.)

Def^n. If $(X, <)$ is an ordered set, then for $a, b \in X$, we define the intervals

$$(a, b) := \{x \in X : a < x < b\},$$
$$(a, b] := \{x \in X : a < x \leq b\},$$
$$[a, b) := \{x \in X : a < x < b\},$$
$$[a, b] := \{x \in X : a < x < b\}.$$

(Intervals)

Def^n. (Order topology)

Let $(X, <)$ be an ordered set. Let $\mathcal{B}$ be the collection

Let $(X, <)$ be an ordered set. Let $\mathcal{B}$ be the collection of sets of the form:

(1) All (a, b) for $a, b \in X$,

(2) All $[a_0, b)$ for $b \in X$ where $a_0 \in X$ is the smallest element of X, if any.

(3) All $(a, b_0]$ for $a \in X$ where $b_0 \in X$ is the largest element of X, if any.

Then, $\mathcal{B}$ is a basis (check) and the topology generated is called the order topology on X.

Example. The standard topology on $\mathbb{R}$ is the order topology derived from the usual order on $\mathbb{R}$.

Def^n (Dictionary order)

Suppose that $(A, <_A)$ and $(B, <_B)$ are two ordered sets. We can define $<$ on $A \times B$ by

(we will denote elements of $A \times B$ by $a \times b$ instead of (a, b).)

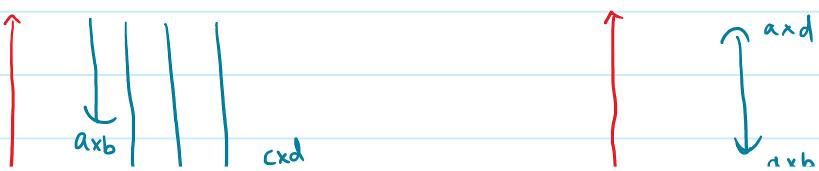
$$a_1 \times b_1 < a_2 \times b_2.$$

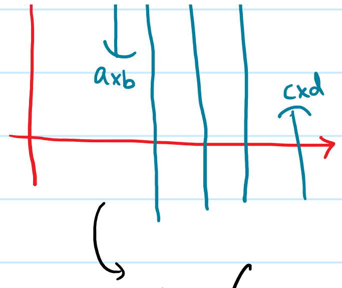
if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 < b_2$.

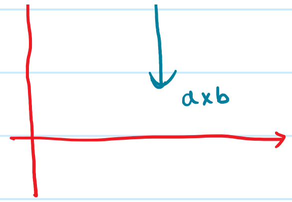
$<$ is a simple order on $A \times B$, called the dictionary order on $A \times B$.

Example $\mathbb{R} \times \mathbb{R}$ can be given an order topology in this dict. order. A basis will be

$\{ (a \times b, c \times d) \}$ where $a < c$ or $a = c$ & $b < d$.







$x \times y \in (a \times b, c \times d)$ iff

- $x = a$ and $b < y$ or
- $a < x < c$ and $y \in \mathbb{R}$
- $x = c$ and $y < d$

Remark If $Y = [0, 1) \cup \{2\}$, then $\{2\}$ is <u>NOT</u> open in the order topology.

Note that any basis element containing is of the form
$B = (a, 2]$ with $a \in Y$.

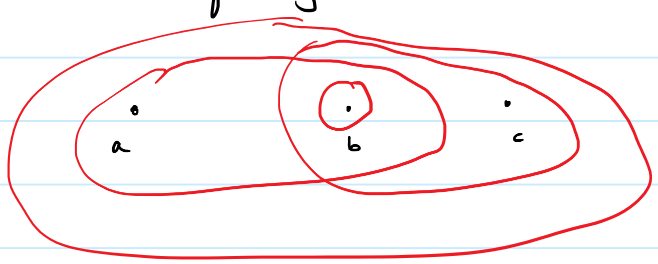
This means that $0 \leq a < 1$ and hence, $\frac{a+1}{2} \in B$.

Thus, it always contains a point distinct from 2.

This shows that subspace and order topologies do not "commute"

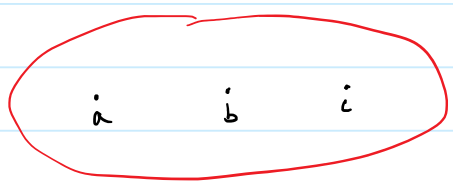
Remark. Singletons in $\mathbb{R}$ (or $\mathbb{R}^n$) are closed. This need not be true in general.

Consider the following topologies



$X = \{a, b, c\}$

$\mathcal{T} = \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\}$



$X = \{a, b, c\}, \quad \mathcal{T} = \{\phi, X\}$

$\{b\}$ is not closed in either of the above since $\{a, c\}$ is not open.

These spaces are not "nice". In fact, in the above spaces, a convergent sequence may have multiple limits. (Haven't defined this yet, though!)
We restrict ourselves to "nicer" spaces.

Def^n. A topological space X is called Hausdorff if for every distinct $x_1, x_2 \in X$, there exist neighbourhoods U_1, U_2 of x_1, x_2, respectively such that
$$U_1 \cap U_2 = \phi.$$

Thm 4. Every finite set in a Hausdorff space is closed.

Proof. It suffices to show the statement for singletons since finite unions of closed sets is closed.

Let $x_0 \in X$ be arbitrary. We show $\{x_0\}$ is closed.

Clearly, $\{x_0\} \subset \overline{\{x_0\}}$. Now, consider $y \in \{x_0\}^c$.

That is, $y \neq x_0$. By Hausdorffness, $\exists$ U_1, U_2 nbds s.t.
$$x \in U_1, \quad y \in U_2 \quad \text{and} \quad U_1 \cap U_2 = \phi.$$

Thus, $U_2 \cap \{x_0\} = \phi$. Thus, $y \notin \overline{\{x_0\}}$. ∎
(Thm 1)

Proof of Thm 2. $\bar{A} = A \cup A'$.

($\subseteq$) Let $x \in \bar{A}$. Suppose $x \notin A$. We show $x \in A'$.

Let U be an arbit. nbd of x.

By Thm 1, $U \cap A \neq \phi$.

By assumption, $x \notin U \cap A$.

Thus, $x \in A'$, by def^n of A'.

($\supseteq$) $A \subset \bar{A}$ is clear. $A' \subset \bar{A}$ is also clear by def^n of A' and Thm 1. ∎

Lecture 5 (21-01-2021)

21 January 2021 15:36

Thm 1. Let X be a Hausdorff space, $A \subset X$, and $x \in X$.
Then, $x \in \bar{A} \Leftrightarrow$ every nbd of x contains infinitely many points of A.

Proof. ($\Leftarrow$) Trivial since infinitely many points imply one point apart from x.

($\Rightarrow$) Let x be a limit point.
For the sake of contradiction, let N be a nbd of x s.t. $A \cap (U \setminus \{x\}) = \{x_1, \dots, x_n\}$ is finite.

$N \supseteq U \ni x$, open

Note $\{x_1, \dots, x_n\}$ is closed since X is Hausdorff.
Thus, $V := U \cap (X \setminus \{x_1, \dots, x_n\})$ is a nbd of x.
But $V \cap (A \setminus \{x\}) = \phi$. $\rightarrow\leftarrow$ ∎

Recall from tutorial:
(1) Order top. is Hausdorff.
(2) Product of Hausdorff spaces is Hausdorff
(3) Subspace of Hausdorff spaces is Hausdorff

Defn. Continuous functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
A function $f: X \to Y$ is said to be **continuous** if $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_Y$.

In other words, inverse image of open sets (in Y) is open (in X).

Remark. By our earlier discussions, it is easily to see that it suffices to check that inverse images of basis (or subbasis) elements are open.

Recall $$f^{-1}\left(\bigcup_{\alpha \in \Lambda} B_\alpha\right) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$$

$$f^{-1}\left(\bigcap_{\alpha\in\Lambda} B_\alpha\right) = \bigcap_{\alpha\in\Lambda} f^{-1}(B_\alpha)$$

Examples. (i) $f : \mathbb{R} \longrightarrow \mathbb{R}_\ell$, $f(x) := x$ is not continuous since the topology of $\mathbb{R}_\ell$ is strictly finer.

(ii) $g : \mathbb{R}_\ell \longrightarrow \mathbb{R}$, $g(x) := x$ is continuous.

Thm 2. Let X and Y be top. spaces and $f : X \to Y$.
TFAE

(i) f is continuous.
(ii) For every $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.
(iii) $f^{-1}(B)$ is closed for every closed $B \subset Y$.

Proof. (i) $\Rightarrow$ (ii)

Let $y \in f(\bar{A})$. Then, $y = f(x)$ for some $x \in \bar{A}$.
We show $x \in \overline{f(A)}$.
Let V be any open nbd. of y. (Want to show $V \cap f(A) \neq \phi$.)
Then, $f^{-1}(V)$ is an open nbd of x.
Then, $A \cap f^{-1}(V) \neq \phi$. Let $x' \in A \cap f^{-1}(V)$.
Then, $f(x') \in f(A) \cap f(f^{-1}(V))$
$\Rightarrow f(x') \in f(A) \cap V$ $(*)$
Thus, $f(A) \cap V \neq \phi$, as desired.
Since any nbd contains an open nbd, we are done.

(ii) $\Rightarrow$ (iii) Let $B \subset Y$ be closed.
Put $A = f^{-1}(B)$. To show: A is closed.
A is closed $\Leftrightarrow A = \bar{A} \Leftrightarrow \bar{A} \subset A$.
$x \in \bar{A} \Rightarrow f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$ $(*)$
$\Rightarrow x \in f^{-1}(B)$
$\Rightarrow x \in A$.

$(*)$ $f(f^{-1}(B)) \subset B$, in general. Equality if f onto.

(iii) $\Rightarrow$ (i) Obvious since $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$. ☒

Def^n. Let X and Y be topological spaces and $f: X \to Y$ be a bijection. f is said to be a homeomorphism if f and f^{-1} are both continuous.

X and Y are said to be homeomorphic if there exists a homeomorphism $f: X \to Y$.

(Homeomorphism, homeomorphic)

A homeomorphism can also be defined as a bijection $f: X \to Y$ s.t. $f(U)$ is open in Y iff U is open in X.

Thus, f is not only a bijection of X and Y but also of $\mathcal{T}_X$ and $\mathcal{T}_Y$.

Def^n. Let $f: X \to Y$ be an injective continuous function. Let $Z = f(X)$ be the image of X in the supspace topology. Then, the restriction

$$f': X \to Z \text{ is a bijection.}$$

If f' is a homeomorphism, then we say that $f: X \to Y$ is a topological imbedding or an imbedding of X in Y.

(Imbedding)

Example (i) $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) := 2x + 4$ is a homeomorphism.

(ii) $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined $f(x) := \tan x$ is a homeomorphism.

(iii) $g: \mathbb{R}_\ell \to \mathbb{R}$ defined $g(x) := x$ is bijective and continuous but not a homeomorphism.

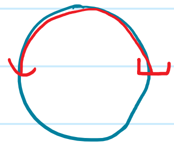
(iv) Let $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be in subspace topology of $\mathbb{R}^2$.
Let $f : [0,1) \longrightarrow S^1$ be defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t).$$

The f is bijective and continuous but f^{-1} is not continuous. To see the last part, consider

$$U = [0, 1/2) \subseteq [0,1).$$

U is open but $f(U) \to$ top arc of S^1



$\Downarrow$

not open in S^1

$\swarrow$

note that $1 \times 0 \in$ top arc but no basis elt around that point.

Thm 3. Let X, Y, and Z be topological spaces.
If $f : X \longrightarrow Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \longrightarrow Z$ is continuous.

Proof. Use $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. ∎

Def^n Box topology, Product Topology

Let J be an indexing set and $\{X_\alpha\}_{\alpha \in J}$ a collection of topological spaces.
Let us consider a basis for a topology on the Cartesian product

$$\prod_{\alpha \in J} X_\alpha,$$

the collection of all set of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where each U_α is open in X_α. The topology induced is called the box topology.

Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \longrightarrow X_\beta$ be the projection map

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta.$$

Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\}$ and let

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta.$$

Then $\mathcal{S}$ is a subbasis for a topology on $\prod_{\alpha \in J} X_\alpha$.
The topology generated is called the product topology.

Remark. ① A typical basis elt. for prod. topology is

$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \qquad [\beta_1, \dots, \beta_n \text{ p-wise distinct}]$$

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \begin{cases} U_{\beta_i} & ; \alpha = \beta_i \\ X_\alpha & ; \text{else} \end{cases}$$

② If J is finite, both box and product coincide.

③ In general, box topology is finer than product.
If $|J| = \infty$, then it can be strictly finer.
(If each $X_\alpha = \mathbb{R}$, then strictly finer.
If each $X_\alpha = \{0\}$, then not.
If each X_α is in indsicrete topology, then not.)