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MA 408

Measure Theory

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Lecture 1

06 January 2021 22:06

Idea behind measure

Simplified case: Subsets of $\mathbb{R}$

Given $E \subseteq \mathbb{R}$, want to assign "length" or "content" to E.

Ideally, want a map

$$\mu : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$$

s.t.

(1) $\mu(\emptyset) = 0$

(2) For any $E \subset \mathbb{R}$ and $x \in E$,

$$\mu(E) = \mu(x + E).$$

$(x + E := \{x + y : y \in E\})$ translation by x

(3) Given a countable collection $\{E_i\}_{i=1}^{\infty}$ of disjoint subsets of $\mathbb{R}$, we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

(So far, $\mu \equiv 0$ will satisfy above properties!)

(4) $\mu([0, 1]) = 1.$ ("Normalisation")

Any such μ would be a 'candidate' for our content.

However, no such μ exists!

Consider the following sets:

(1) Define $\sim$ on $\mathbb{R}$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
Clearly, $\sim$ is an equiv. relation.

Let $E \subseteq [0,1]$ be a set containing exactly one element from each equivalence class in $\mathbb{R}/\sim$.

(Existence is given by Axiom of Choice. Note that distinct equiv. classes are disjoint.)

and a small argument that lets you conclude $E \subset [0,1]$.

Q. What could $\mu(E)$ be?

Note that $\{E+r\}_{r \in \mathbb{Q} \cap [0,1]}$ is a collection of pairwise disjoint sets.

Sketch. If $x \in (E+r_1) \cap (E+r_2)$, $(r_2 \geq r_1)$ then $x = r_1 + e_1 = r_2 + e_2$ for some $e_1, e_2 \in E$

$$\Rightarrow e_1 - e_2 = r_2 - r_1 \in \mathbb{Q}$$

$$\Rightarrow e_1 \sim e_2 \Rightarrow e_1 = e_2$$

Moreover, $$[0,1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [0,1]} (E+r) \subseteq [0,2] = [0,1] \cup [1,2]$$

(An easy consequence of (1) – (3) is that $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$.

Proof. $\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$. $\square$)

$$\Rightarrow \mu([0,1]) \leq \mu\left(\bigcup_{i=1}^{\infty} (E+r_i)\right) \leq \mu([0,1]) + \mu([1,2])$$

enumerate $\mathbb{Q}\cap[0,1]$ as $\{r_1,\dots\}$

$$\Rightarrow \quad 1 \le \sum_{i=1}^{\infty} \mu(E+r_i) \le 2$$

$[1,2] = [0,1]+1$

$$\Rightarrow \quad 1 \le \sum_{i=1}^{\infty} \mu(E) \le 2$$

If $\mu(E)=0$: $1 \le 0$ $\rightarrow\leftarrow$

If $\mu(E) = r > 0$: $\sum_{i=1}^{\infty} \mu(E) = \infty \le 2$ $\rightarrow\leftarrow$

Possible way to salvage: Replace (3) to have "finite union" instead of "countable".

Turns out that that's still not enough.

(2) BANACH-TARSKI THEOREM ~~PARADOX~~ (1924): (Using AC)

For any open sets $U, V \subseteq \mathbb{R}^n$ where $n \ge 3$, there exists $k \in \mathbb{N}$ and set $U_1, \dots, U_k$, $V_1, \dots, V_k$ s.t.

(1) $U_i \cap U_j = \emptyset$, $V_i \cap V_j = \emptyset$, $1 \le i \ne j \le k$.

(2) $U = \bigsqcup_{i=1}^{k} U_i$, $V = \bigsqcup_{i=1}^{k} V_i$.

(3) $U_i \simeq V_i$, i.e., U_i is obtained from V_i by a sequence of rotations, reflections, and translations.

In other words, by isometries.

Thus, the analogue of (2) implies $\mu(U_i) = \mu(V_i)$ $\forall i$.

$\Rightarrow \quad \mu(U) = \mu(V).$

$\rightsquigarrow$ Absurd conclusions.

As it turns out, the problem is NOT in the infinite union but rather the demand that μ is defined on all of $\mathcal{P}(\mathbb{R})$!

Thus, we restrict our attention to a smaller collection of subsets of $\mathbb{R}$. (Not too small!)

σ - ALGEBRAS

Let X be an arbitary set.

Def^n. (1) An algebra ("field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

① $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$

② $A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$ for any $n \in \mathbb{N}$.

(2) A σ-algebra ("σ-field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

① $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$

② $A_1, \dots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Note that complements and unions give us intersections. Also, $\phi, X \in \mathcal{F}$.

EXAMPLES

① $\mathcal{F} = \mathcal{P}(X) \leftarrow$ both

② (Countable - cocountable σ-algebra)

$\mathcal{F} = \{ E \subseteq X : E \text{ or } E^c \text{ is countable} \}$

Proof. Clearly closed under complement.
Let $A_1, \ldots \in \mathcal{F}$.
If all A_i are countable, then $\bigcup A_i$ is.
Suppose A_1 not countable. Then, A_1^c is.
But

$$A_1 \subset \bigcup A_i \Rightarrow \left(\bigcup A_i\right)^c \subset A_1^c$$

$$\Rightarrow \left(\bigcup A_i\right)^c \text{ is countable. } \square$$

③ Given any $\mathcal{F} \subseteq \mathcal{P}(X)$, we can talk about **σ-algebra generated by $\mathcal{F}$** denoted $\mathcal{M}(\mathcal{F})$ defined by

$$\mathcal{M}(\mathcal{F}) = \bigcap_{\substack{\mathcal{F} \subseteq B \\ B \text{ is a } \sigma\text{-alg}}} B$$

(Note that the intersection is non-empty because of $\mathcal{P}(X)$. Easy to see that intersection of σ-algebrae is again a σ-alg.)

By construction, $\mathcal{M}(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$.

Borel σ-Algebra.

Def^n: Let $(X, \mathcal{T})$ be a topological space.
The σ-algebra generated by $\mathcal{T}$ is called the **Borel σ-algebra** on X, denoted $\mathcal{B}(X)$.

(Abuse of notation that we don't mention $\mathcal{T}$.)

In other words, it is generated by the open sets of X.

Borel σ-algebra on $\mathbb{R}$: Smallest σ-alg on $\mathbb{R}$ containing all the open sets.

Consequences:

1. All open sets are in $\mathcal{B}(\mathbb{R})$.
2. All closed sets are in $\mathcal{B}(\mathbb{R})$.
3. All F_σ, G_δ sets are in $\mathcal{B}(\mathbb{R})$.

$$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i \ (F_i \text{ closed}) \ ; \quad G_\delta \equiv \bigcap_{i=1}^{\infty} G_i \ (G_i \text{ open})$$

Prop^n: Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$.
Then, $\mathcal{B}$ is also generated by any of the following:

(i) $\{(a, b) : a < b\}$ or $\{[a, b] : a < b\}$

(ii) $\{[a, b) : a < b\}$ or $\{(a, b] : a < b\}$

(iii) $\{(a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a) : a \in \mathbb{R}\}$

(iv) $\{[a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a] : a \in \mathbb{R}\}$

Proof Easy. $\square$

Borel σ-algebra on $\mathbb{R}^n$:

Suppose $\{X_i\}_{i=1}^{n}$ are metric spaces.

Let $X = \prod_{i=1}^{n} X_i$ with the product metric.

def^n

If ρ_i is the metric on X_i, then ρ on $\prod X_i$ is defined as

$$\rho(x, y) = \max \rho_i(x_i, y_i) \qquad \left| x = (x_1, \dots, x_n) \right|$$

$$\rho(x, y) = \max_{1 \le i \le n} \rho_i(x_i, y_i) \qquad \left(\begin{array}{l} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{array} \right)$$

Def^n. Suppose $(X_i, \mathcal{M}_i)$ are σ-algebrae. One can define a σ-algebra on $X := \prod X_i$ as follows:

Consider the projection maps $\pi_i : X \to X_i$.
Let
$$\mathcal{F} = \{ \pi_i^{-1}(E) : E \in \mathcal{M}_i, \ i = 1, \dots, n \}$$
$$= \{ E \times X_2 \times \cdots \times X_n : E \in \mathcal{M}_1 \}$$
$$\cup \{ X_1 \times E \times \cdots \times X_n : E \in \mathcal{M}_2 \}$$
$$\cup \cdots \cup \{ X_1 \times \cdots \times X_{n-1} \times E : E \in \mathcal{M}_n \}.$$

$\mathcal{M} := \mathcal{M}(\mathcal{F}) \subseteq \mathcal{P}(X)$ is the product σ-algebra induced by $\{\mathcal{M}_i\}_{i=1}^n$.

We often write the above as $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i$.

Caution. The above $\prod_{i=1}^{n}$ is NOT the set-theoretic cartesian product.

Now, we get two (possibly different) σ-algebrae on $\mathbb{R}^n$.
① Borel σ-alg. on $(\mathbb{R}^n, \mathcal{T})$
② Product of Borel σ-alg of $\mathcal{B}(\mathbb{R})$.

Prop^n: $\mathcal{B}(\mathbb{R}^n) = \prod_{i=1}^{n} \mathcal{B}(\mathbb{R})$. That is, both the σ-alg above are same.

Proof. We will prove this by a sequence of observations.

① Suppose $\{(X_i, \mathcal{M}_i)\}_{i=1}^n$ are σ-algebrae and $\mathcal{F}_i \subseteq \mathcal{M}_i$ are such that $\mathcal{M}_i = \mathcal{M}(\mathcal{F}_i)$. $(i=1,\dots,n)$

Then, if $X = \prod_{i=1}^n X_i$ and $\mathcal{M} = \prod_{i=1}^n \mathcal{M}_i$, then

$\mathcal{M}$ is generated by $\{\pi_i^{-1}(E) : E \in \mathcal{F}_i,\ i=1,\dots,n\}$.

② $\mathcal{M}$ is generated by $\{E_1 \times \dots \times E_n : E_i \in \mathcal{F}_i\}$.

Assuming ① and ② for now, we now note the following.

Clearly, one has $\prod_{i=1}^n \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^n)$.

Proof: using ②, $\prod_{i=1}^n B(\mathbb{R})$ is gen. by sets of the form $U_1 \times \dots \times U_n$, each $U_i \subseteq \mathbb{R}$ open

Each such set is open in the metric space $\mathbb{R}^n$.
Thus, it is in $\mathcal{B}(\mathbb{R}^n)$.

We show $\mathcal{B}(\mathbb{R}^n) \subseteq \prod \mathcal{B}(\mathbb{R})$.

(*) It suffices to show that every set of the form

$U_1 \times \dots \times U_n$ where $U_i \subset \mathbb{R}$ are open

are in the product $\prod \mathcal{B}(\mathbb{R})$.

Why? Every open set in $\mathbb{R}^n$ is a countable union of sets of aforementioned form. In turn, the open sets generate $\mathcal{B}(\mathbb{R}^n)$.

Proving (*) is easy because

$$U_1 \times \dots \times U_n = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \dots \cap \pi_n^{-1}(U_n).$$

these are in $\prod \mathcal{B}(\mathbb{R})$, by def^n

Proof of ①

Want to show that $\tilde{\mathcal{F}} = \{\pi_i^{-1}(E) : E \in \mathcal{F}_i,\ 1 \le i \le n\}$ gen. $\underset{\mathcal{M}}{\overset{\shortparallel}{\prod \mathcal{M}_i}}$.

Clearly $\mathcal{M}(\tilde{\mathcal{F}}) \subseteq \mathcal{M}$. ($\tilde{\mathcal{F}} \subset \mathcal{M}$ and $\mathcal{M}$ is a σ-alg)

It now suffices to show that every (standard) generator of $\mathcal{M}$ is in $\mathcal{M}(\tilde{\mathcal{F}})$.

Note $\mathcal{M} = \langle \pi_i^{-1}(E) : E \in \mathcal{M}_i,\ 1 \le i \le n \rangle$

$\tilde{\mathcal{M}} := \langle \pi_i^{-1}(E) : E \in \mathcal{F}_i,\ 1 \le i \le n \rangle = \mathcal{M}(\tilde{\mathcal{F}})$

Let $\tilde{\mathcal{M}}_i := \{E \in \mathcal{M}_i : \pi_i^{-1}(E) \in \tilde{\mathcal{M}}\} \subseteq \mathcal{P}(X_i)$.

We shall show that $\tilde{\mathcal{M}}_i = \mathcal{M}_i$.

We know, by def^n that $\mathcal{F}_i \subseteq \tilde{\mathcal{M}}_i$. ($E \in \mathcal{F}_i \overset{\subseteq \mathcal{M}_i}{\Rightarrow} \pi_i^{-1}(E) \in \tilde{\mathcal{M}} \cap \mathcal{M}_i$ $\Downarrow$ $E \in \tilde{\mathcal{M}}_i$.)

Moreover, $\mathcal{M}(\mathcal{F}_i) = \mathcal{M}_i$. Thus, it suffices to show that $\tilde{\mathcal{M}}_i$ is a σ-alg.

Also, $\tilde{\mathcal{M}}_i \subseteq \mathcal{M}_i$.

To that end, let $A \in \tilde{\mathcal{M}}_i$. Then, $\pi_i^{-1}(A) \in \tilde{\mathcal{M}}$.

Then, $\pi_i^{-1}(A)^c \in \tilde{\mathcal{M}}$. But $\pi_i^{-1}(A^c) = \pi_i^{-1}(A)^c \in \mathcal{M}$.

$\Rightarrow \pi_i^{-1}(A^c) \in \mathcal{M}_i$

Similarly noting that $\pi_i^{-1}\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigcup_{j=1}^{\infty} \pi_i^{-1}(A_j)$ yields the result.

Proof of ②.

Now, put $\tilde{\mathcal{F}} := \{E_1 \times \cdots \times E_n : E_i \in \mathcal{F}_i\}$ and
$\tilde{\mathcal{M}} := \mathcal{M}(\tilde{\mathcal{F}})$.

Since $E_1 \times \cdots \times E_n = \bigcap_{i=1}^{n} \pi_i^{-1}(E_i)$, we see that $\tilde{\mathcal{F}} \subseteq \mathcal{M}$.

Thus, $\tilde{\mathcal{M}} \subseteq \mathcal{M}$.

REMARKS.

① The argument above generalises for a separable metric spaces.

② If $(X_i, \mathcal{M}_i)_{i \in A}$, and A is COUNTABLE then again, $X = \prod X_i$, $\mathcal{M} = \prod \mathcal{M}_i$ generated by $\{\pi_i^{-1}(E) : E \in \mathcal{M}_i, i \in A\}$ is also generated by sets of the form

$$\left(\prod_{i \in A} E_i\right), \quad E_i \in \mathcal{F}_i.$$

MEASURE

Def^n. Suppose $(X, \mathcal{M})$ is a measure space, i.e., $\mathcal{M}$ is a σ-algebra on X. A measure on X is a map $\mu : \mathcal{M} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) if $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, then

$$\mu\Big(\underbrace{\bigcup_{i=1}^{\infty} E_i}_{\text{in } \mathcal{M}}\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

EXAMPLES.

(1) $X = \{x_1, x_2, \dots\}$ is countable. Suppose $p_i \geq 0$ are reals s.t. $\sum_{i=1}^{\infty} p_i = 1$. Let $\mathcal{M} = \mathcal{P}(X)$ and define $\mu : \mathcal{M} \to [0,1]$ as

$$\mu(E) = \sum_{i:\, x_i \in E} p_i.$$

(2) $(X, \mathcal{M})$ be s.t. $\mathcal{M}$ is the countable-cocountable σ-alg. s.t. X itself is uncountable

Define

$$\mu(E) := \begin{cases} 0 & ; E \text{ is countable} \\ 1 & ; E \text{ is uncountable} \end{cases}$$

Prop^n. Suppose $(X, \mathcal{M}, \mu)$ is a measure space.

Then,

① $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$

② $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$ (μ is "sub-additive")

③ If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \cdots$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i).$$

Proof. ① & ② are trivial

③ Define $F_i = E_i \setminus E_{i-1}$ for $i \geq 2$.

$$F_1 = F_1$$

Then, $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. Also, $F_i \in \mathcal{M}$ for each i.
($n = \infty$ as well) Moreover, $F_i \cap F_j = \emptyset$ for $i \neq j$.

Thus, $\mu\left(\bigcup E_i\right) = \mu\left(\bigcup F_i\right) = \sum_{i=1}^{\infty} \mu(F_i)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{\infty} E_i.$$

Def^n. ① A null set in a measure space $(X, \mathcal{M}, \mu)$ is a set E s.t. $E \subseteq F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.
($E \in \mathcal{M}$ NOT necessary.)

② Given a measure space $(X, \mathcal{M}, \mu)$, the completion of $\mathcal{M}$, denoted $\overline{\mathcal{M}}$ is the collection of all sets of the form $F \cup N$ where $F \in \mathcal{M}$ and N is a null set.

Prop^n. ① If $(X, \mathcal{M}, \mu)$ is a measure space, then $\overline{\mathcal{M}}$ is a σ-alg.
② Moreover, there exists a unique measure

$$\bar{\mu} : \overline{\mathcal{M}} \to [0, \infty] \quad \text{s.t.}$$

$$\bar{\mu}\big|_{\mathcal{M}} = \mu.$$

(That is, there is a unique extension of μ to a measure $\bar{\mu}$ on $\overline{\mathcal{M}}$.)

Lecture 2

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Prop^n. ① If $(X, \mathcal{M}, \mu)$ is a measure space, then $\bar{\mathcal{M}}$ is a σ-alg.
② Moreover, there exists a unique measure

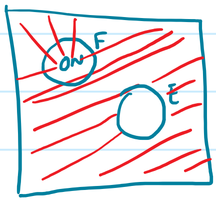
$$\bar{\mu} : \bar{\mathcal{M}} \longrightarrow [0, \infty] \quad \text{s.t.}$$

$$\bar{\mu}\big|_{\mathcal{M}} = \mu .$$

Proof: ① To show that $\bar{\mathcal{M}}$ is a σ-algebra, we need to show:

(i) $A \in \bar{\mathcal{M}} \Rightarrow A^c \in \bar{\mathcal{M}}$

(ii) $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bar{\mathcal{M}}$.



(i) $A = E \cup N, \quad N \subset F, \quad \mu(F) = 0, \quad E, F \in \mathcal{M}$

$A^c = E^c \cap N^c$

Now, $E \cup F \in \mathcal{M}$ and hence, $E^c \cap F^c$.

Note that $E^c \cap N^c = \underbrace{(E^c \cap F^c)}_{\in \mathcal{M}} \cup \underbrace{(F \setminus N)}_{\subseteq F \text{ and } \mu(F) = 0} \in \bar{\mathcal{M}}$.

Now, if $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}}$, we can write

$$A_i = E_i \cup N_i, \qquad N_i \subseteq F_i, \quad \mu(F_i) = 0, \quad E_i, F_i \in \mathcal{M}.$$

$$\bigcup A_i = \underbrace{\left(\bigcup E_i\right)}_{=: E} \cup \underbrace{\left(\bigcup N_i\right)}_{=: N}$$

Note $E \in \mathcal{M}$. Also, put $F = \bigcup F_i$. Then, $F \in \mathcal{M}$.
Moreover, $N \subseteq F$ & $\mu(F) = \sum \mu(F_i) = 0$.

$\therefore \bigcup A_i = E \cup N$, in the desired form.

(ii) Define $\bar{\mu} : \mathcal{M} \to [0, \infty]$ as

$$\bar{\mu}(E \cup N) := \mu(E).$$

• To show: $\bar{\mu}$ is well-defined.

Suppose $E_1 \cup N_1 = E_2 \cup N_2$.

$\Rightarrow E_1 \subset E_2 \cup N_2 \subseteq E_2 \cup F_2$

$\Rightarrow \mu(E_1) \leq \mu(E_2) + \mu(F_2) = \mu(E_2)$.

$\|^{ly}$ $\mu(E_2) \leq \mu(E_1)$. $\therefore \mu(E_1) = \mu(E_2)$.

Thus, $\bar{\mu}$ is well-defined (Note that this was also the only way to define $\bar{\mu}$. Thus, uniqueness will follow.)

• To show: $\bar{\mu}$ is a measure.

① $\bar{\mu}(\phi) = 0$ is trivial since $\bar{\mu}(\phi) = \bar{\mu}(\phi \cup \phi) = \mu(\phi) = 0$.

② Suppose $\{E_i \cup N_i\}_{i=1}^{\infty}$ are pairwise disjoint.

$$\bar{\mu}(\cup(E_i \cup N_i)) = \bar{\mu}((\cup E_i) \cup (\cup N_i)) = \mu(\cup E_i) = \sum \mu(E_i)$$
$$= \sum \bar{\mu}(E_i \cup N_i) \blacksquare$$

same logic earlier

OUTER MEASURE

Def. An outer measure on X is a map

$$\mu^* : \mathcal{P}(X) \to [0, \infty] \text{ satisfying}$$

(i) $\mu^*(\phi) = 0$,

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$,

(iii) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Motivation for μ^* comes from the intuitive idea that knowing areas enclosed by rectangles, we "approximate" areas bounded by arbitrary sets by covering these by <u>countable union</u> of rectangles.

Prop^n. Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{F} \to [0, \infty]$ s.t.

(i) $\phi, X \in \mathcal{F}$

(ii) $\rho(\phi) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \ : \ E_i \in \mathcal{F}, \ E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

Then, μ^* is an outer measure.

Proof. We need to show μ^* is well-defined $\leftarrow$ This follows because $X \in \mathcal{F}$.
and it satisfies

(i) $\mu^*(\phi) = 0 \quad \longleftarrow$ trivial since $\mu^*(\phi) \geq 0$ since inf over non-ve reals

Also, $\phi \subseteq \phi$ & $\rho(\phi) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

Any cover of B is also a cover of A.
$\therefore$ We are taking inf over a larger set & hence, it will be smaller.
$\Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) Suppose $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{P}(X)$. Fix $\varepsilon > 0$.

For each E_i, let $\{A_j^{(i)}\}_{j=1}^{\infty} \subseteq \mathcal{F}$ be a cover of E_i

$\therefore$ ... $\mu^*(E_i) \geq \sum^{\infty} \rho(A^{(i)})$ ...

with $\mu^*(E_i) \geq \sum_{j=1}^{\infty} \rho(A_j^{(i)}) - \frac{\varepsilon}{2^i} \quad \forall i$

Then, $\bigcup_i \bigcup_j A_j^{(i)}$ covers $\bigcup E_i$.

Thus, $\mu^*(\bigcup E_i) \leq \sum_{i,j} A_j^{(i)} \leq \left(\sum_{i=1}^{\infty} \mu^*(E_i) + \frac{\varepsilon}{2^i} \right)$

$$= \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$$

$$\Rightarrow \mu^*\left(\bigcup E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$$

Since $\varepsilon > 0$ is arbt, this is completes the proof. ☒

Def^n. Given an outer measure μ^*, we say that a set $A \subseteq X$ is μ^*-measurable if for all $E \in \mathcal{P}(X)$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Def^n. A measure μ on $(X, \mathcal{M})$ is complete if for all $F \in \mathcal{M}$ with $\mu(F) = \phi$, we have $\mathcal{P}(F) \subseteq \mathcal{M}$.

(That is, all null sets are in $\mathcal{M}$.)

Thm. (CARATHÉODORY)

Let μ^* be an outer measure on X.

Let $\mathcal{M} := \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$.

Then,

(i) $\mathcal{M}$ is a σ-algebra.

(ii) μ^* restricted to $\mathcal{M}$ is a complete measure.

Proof. (i) $\mathcal{M}$ is closed under $(\ )^c$ since the def^n of μ^*-meas. is symmetric under $(\ )^c$.

Now, if $A, B \in \mathcal{M}$ and $E \subset X$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)\end{aligned}$$

Of course, we also have $\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \leq \mu^*(E)$.

$$\therefore \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Thus, $\mathcal{M}$ is closed under finite unions. To show that it is a σ-alg, it suffices to show closure under disjoint unions.

(Why? Given $\{E_i\}_{i=1}^\infty \subseteq \mathcal{M}$, consider $F_1 = E_1$ and $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$ for $n \geq 2$.
Note that $\bigcup_{i=1}^\infty F_i = \bigcup_{i=1}^\infty E_i$ and that each $F_n \in \mathcal{M}$.)

Let $\{A_i\}_{i=1}^\infty$ be a disjoint collection in $\mathcal{M}$. Put $B_n = \bigcup_{i=1}^n A_i \in \mathcal{M}$ and $B = \bigcup_{i=1}^\infty A_i$.

$$\left(\begin{aligned} B_n \cap A_n &= \bigcup_{i=1}^n (A_n \cap A_i) = A_n \\ B_n \cap A_n^c &= \bigcup_{i=1}^n (A_n^c \cap A_i) = \bigcup_{i=1}^{n-1} A_i = B_{n-1}\end{aligned}\right)$$

Then, for any $E \subset X$,

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i)\end{aligned}$$

$= \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2})$

$$\Rightarrow \mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \left[\sum_{i=1}^n \mu^*(E \cap A_i)\right] + \mu^*(E \cap B_n^c)$$

$(B_n \subseteq B \Rightarrow B^c \subseteq B_n^c)$

$$\geq \left[\sum_{i=1}^n \mu^*(E \cap A_i)\right] + \mu^*(E \cap B^c)$$

take $n \to \infty$

(take $n \to \infty$)

$$\Rightarrow \mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)$$

(★)
$$\geq \mu^*\left(\bigcup (E \cap A_i)\right) + \mu^*(E \cap B^c)$$

$$= \mu^*\left(E \cap \left(\bigcup A_i\right)\right) + \mu^*(E \cap B^c)$$

$$= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).$$

Thus, we have equality throughout giving

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$$ & hence, $B \in \mathcal{M}$.

Moreover, taking $E = B$; the (★) equation gives

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Thus, μ^* is countably additive on $\mathcal{M}$.

Thus, μ is a measure on $\mathcal{M}$.

To show completeness: Let $F \in \mathcal{M}$ be s.t. $\mu^*(F) = 0$ and $A \subseteq F$.
Then, $\mu^*(A) = 0$. Also, for any $E \subset X$, we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(A) + \mu(E \cap A^c)$$
$$= \mu^*(E \cap A^c) \leq \mu^*(E).$$

Thus, $A \in \mathcal{M}$. ∎

Def^n. Suppose $\mathcal{F}$ is an algebra on X.
A map

$$\mu_0 : \mathcal{F} \to [0, \infty]$$

is called a pre-measure if

(i) $\mu_0(\emptyset) = 0$,
(ii) If $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint s.t. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$,

then
$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

(Note that the above also gives the thing for finite unions.
guaranteed to be in $\mathcal{F}$.)

Prop^n. Suppose μ_0 is a premeasure on an algebra $\mathcal{F}$.
Then, if μ^* is the outer measure as defined in the earlier proposition, then

(i) $\mu^*|_{\mathcal{F}} = \mu_0$
(ii) Every set in $\mathcal{F}$ is μ^*-measurable.

An immediate corollary:

Thm. Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$ is an algebra and suppose $\mathcal{M}$ is the σ-algebra generated by $\mathcal{F}$.
Let μ_0 be a premeasure defined on $\mathcal{F}$ and let μ^* be the outer measure as before. Then,

(i) $\mu^*|_{\mathcal{M}}$ is a measure on $(X, \mathcal{M})$. Put $\mu = \mu^*|_{\mathcal{M}}$.
(ii) If ν is any measure extending μ_0, then
$$\nu(E) = \mu(E)$$
whenever $\mu(E) < \infty$

Proof of Prop^n.

(i) For $E \in \mathcal{F}$, want to show, $\mu^*(E) = \mu_0(E)$.
Considering $E_1 = E$ and $E_i = \emptyset$ for $i \geq 2$ gives

$$\sum_{i=1}^{\infty} \mu_0(E_i) = \mu_0(E) \Rightarrow \mu^*(E) \leq \mu_0(E).$$

inf over all covers

To show $\geq$: let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ be a cover for E.

Let $F_i := E \cap \left(E_i \setminus \bigcup_{j=1}^{i-1} E_j \right)$.

Clearly (i) $F_i \in \mathcal{F}$ by closure properties
(ii) $F_i \cap F_j = \emptyset$ if $i \neq j$.
(iii) $\bigcup_{i=1}^{\infty} F_i = E$.

If $\{E_i\}$ is a cover s.t. $\sum_{i=1}^{\infty} \mu_0(E_i) \leq \mu^*(E) + \varepsilon$

(such a cover does exist)

Note $\mu_0(E) = \mu_0\left(\bigsqcup_{i=1}^{\infty} F_i \right) = \sum \mu_0(F_i) \leq \mu^*(E) + \varepsilon$

μ_0 is a premeasure

$\Rightarrow \mu_0(E) \leq \mu^*(E) + \varepsilon \quad \forall \varepsilon > 0$
$\Rightarrow \mu_0(E) \leq \mu^*(E)$.

This establishes (i).

(ii) To show: Every $A \in \mathcal{F}$ is μ^*-measurable.
Let $E \subset X$. Let $\{E_n\}$ be a cover for E s.t.

$$\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon$$

Then, $\mu^*(E) + \varepsilon \geq \sum_n \mu_0(E_n)$

$E_n, A, A^c \in \mathcal{F}$ and μ_0 is pre-meas.

$= \sum \left[\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c) \right]$

$$= \sum_n \left[\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c) \right]$$

$$= \left[\sum_n \underbrace{\mu_0(E_n \cap A)}_{\text{cover for } E \cap A} \right] + \left[\sum_n \underbrace{\mu_0(E_n \cap A^c)}_{E \cap A^c} \right]$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

$$\Rightarrow \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad (E \text{ was arbit.})$$

$\leq$ is anyway true for any outer measure.

$$\Rightarrow \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Thus, A is μ^*-measurable.

Proof of Thm.

(i) Follows from Carathéodory and Prop^n. (Since $\mathcal{F}$ is μ^*-measurable and $\mathcal{M}$ is the smallest σ-alg containing $\mathcal{F}$.)

(ii) First note that if ν extends μ_0, then

for any $E \in \mathcal{M} = \mathcal{M}(\mathcal{F})$

if $E \subseteq \bigcup_n E_n$, $E_n \in \mathcal{F}$, then

$$\nu(E) \leq \sum_n \nu(E_n) = \sum_n \mu_0(E_n).$$

So if the cover is s.t. $\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon = \mu(E) + \varepsilon$,

this gives $\nu(E) \leq \mu(E) + \varepsilon$ for arbit ε. That is,

$$\nu(E) \leq \mu(E).$$

Now, if $\mu(E) < \infty$, we show $\geq$.

Let $\{E_n\}$ be a cover for $E \in \mathcal{M}$ and let $A := \bigcup_n E_n \in \mathcal{M}$.

Note that
$$\nu(A) = \lim_{n\to\infty}\left(\nu\left(\bigcup_{i=1}^{n} E_i\right)\right) = \lim_{n\to\infty} \mu_0\left(\bigcup_{i=1}^{n} E_i\right)$$
$$= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right)$$
$$= \mu(A)$$

Since $\mu(E) < \infty$ (by assumption), we can pick a cover $\{E_n\}$ s.t.
$$\underbrace{\mu(A)}_{\mu(A\cap E) + \mu(A\setminus E)} < \mu(E) + \varepsilon \text{ and thus, } \mu(A\setminus E) < \varepsilon.$$

Thus,
$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A\setminus E) \leq \nu(E) + \mu(A\setminus E)$$
$$< \nu(E) + \varepsilon$$

This gives $\mu(E) \leq \nu(E)$, as desired.

Towards "Good" Borel Measures

The idea is to extend/define a measure μ on $\mathcal{B}(\mathbb{R})$ from the notion of length of bounded intervals. Whatever done so far leads to that.

Def^n A half-interval is a subset of $\mathbb{R}$ of the form:
(i) $(a, b]$ for $-\infty \leq a < b < \infty$, or
(ii) (a, ∞) for $-\infty \leq a < \infty$, or
(iii) $\emptyset$.

Can be checked that the collection of fin unions of half-intervals is an algebra on $\mathbb{R}$

Prop^n. Let $\mathcal{F}$ be the algebra consisting of finite unions of half-intervals.
Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function.

$$\left(\lim_{\delta \downarrow 0} F(x+\delta) = F(x) \quad \forall\, x \in \mathbb{R} \right)$$

Define
$$\mu_0\left(\bigsqcup_{i=1}^{n} (a_j, b_j] \right) := \sum_{j=1}^{n} \left(F(b_j) - F(a_j) \right),$$

and $\mu_0(\phi) := 0$.
Then, μ_0 is a premeasure on $\mathcal{F}$.

Remarks
(1) Note that $\mathcal{F}$ above actually generates $\mathcal{B}(\mathbb{R})$, as seen in Lec 1.
(2) If we take $F(x) = x$, then $F(b) - F(a) = b - a$
= "length" of $(a, b]$.

So, the above extends the notion of measure arising from lengths of intervals onto the Borel σ-field.

(3) Why right-continuity?

Suppose μ is a finite Borel measure. Let
$$F(x) := \mu((-\infty, x]).$$
Then, if $x_n \downarrow x$, then
$$\underbrace{\lim_{n\to\infty} \mu((-\infty, x_n])}_{\lim_{n\to\infty} F(x_n)} = \mu\left(\bigcap_{n=1}^{\infty} (-\infty, x_n] \right) = \mu((-\infty, x]) = F(x)$$

Thus, this F above is right-continuous.
Note that closure on right

Proof of the Prop^n. First, we need to check that μ_0 is well-defined.

Let $\{(a_j, b_j]\}$ $(j=1, \dots, n)$ be pairwise disjoint and let

$$\dot{\bigcup_{j=1}^{n}} (a_j, b_j] = (a, b].$$

Then, by re-arranging indices, if necessary, it follows that

$a = a_1 < b_1 = a_2 < b_2 = \cdots = a_n < b_n = b$ and in this case

$$\mu_0((a, b]) = F(b) - F(a)$$
$$\parallel$$
$$\mu_0\left(\dot{\bigcup_{j=1}^{n}} (a_j, b_j]\right) = \sum F(b_j) - F(a_j) = F(b) - F(a)$$

Thus, μ_0 is well-defined in this case.

More generally, if $\{I_i\}_{i=1}^{n}$ and $\{J_j\}_{j=1}^{m}$ are s.t.

$$\dot{\bigcup_{i=1}^{n}} I_i = \dot{\bigcup_{j=1}^{m}} J_m, \quad \text{then}$$

$$\sum_i \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_j \mu_0(J_j)$$

since intersection of half-intervals is again a half-interval, for which the def^n is consistent.

This shows that μ_0 is well-defined on $\mathcal{F}$.

It now remains to show that μ_0 is indeed a premeasure on $\mathcal{F}$.