

Problem Set - pre-REU 2025

Problems marked with L are problems on linear algebra.

Problem set on Invariant Theory

1. Show that the following sets of matrices form a group under matrix multiplication. In each case, justify why the matrices are indeed invertible.
 - (a) The set of $n \times n$ matrices with determinant one.
 - (b) The set of $n \times n$ matrices M satisfying $MM^T = I_n$.
 - (c) The set of $2n \times 2n$ matrices M satisfying $M\Omega M^T = \Omega$, where Ω is the $2n \times 2n$ block matrix given as $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
2. Let R be a ring (such as the ring of polynomials). Let $f: R \rightarrow R$ be a function satisfying
 - $f(1) = 1$,
 - $f(x+y) = f(x) + f(y)$ for all $x, y \in R$,
 - $f(xy) = f(x)f(y)$ for all $x, y \in R$.

Let S be the set of fixed points of R , i.e., $S := \{r \in R : f(r) = r\}$. Show that S is closed under addition, multiplication, and contains 1.

3. From class, we know that any element of the orthogonal group looks like

$$M = \begin{bmatrix} \cos(\theta) & -\varepsilon \sin(\theta) \\ \sin(\theta) & \varepsilon \cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$ and some $\varepsilon \in \{1, -1\}$. Using this description, check that for the action of $G = O_2(\mathbb{R})$ on $R = \mathbb{R}[x, y]$, we have $x^2 + y^2 \in R^G$. (As before, the action is via $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto M \begin{bmatrix} x \\ y \end{bmatrix}$.)

4. Let $G = GL_2(\mathbb{R})$ act on $R = \mathbb{R}[X_{2 \times m}]$ in the usual way. Show that $R^G = \mathbb{R}$, i.e., the only invariant polynomials are the constants.

Hint: Think about what the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ does.

- L 5. Let A, B be $n \times n$ matrices such that $\exists P \in GL_n(\mathbb{R})$ with $PAP^{-1} = B$. Show that A and B have the same characteristic polynomial, i.e., show that $\det(A - \lambda I) = \det(B - \lambda I)$.
- 5.5. Recall that for the action of $G = GL_n(\mathbb{R})$ on $R = \mathbb{R}[X_{n \times n}]$ given by conjugation, R^G is generated by the coefficients of the characteristic polynomial. Interpret the previous problem in this context.

- ^L6. Let A be an $n \times m$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^n$. Show that the (i, j) th entry of $A^\top A$ is $\mathbf{a}_i \cdot \mathbf{a}_j$.
- 6.5. Recall that for the action of $G = O_n(\mathbb{R})$ on $R = \mathbb{R}[X_{n \times m}]$ given by left multiplication, R^G is generated by the entries of $X^\top X$. Using the previous problem, how does this description of R^G fit in with our geometrical definition of $O_n(\mathbb{R})$?
7. Consider the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $R = \mathbb{R}[X_{2 \times 2}]$ by left multiplication. For ease of notation, assume that the variables are denoted and arranged as

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}.$$

Show that $x_1 y_2 - x_2 y_1 \in R^G$. More generally, show that if SL_2 is acting on the polynomial with variables

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{bmatrix},$$

then $x_i y_j - x_j y_i \in R^G$ for all $1 \leq i < j \leq m$.

Hint for the last part: If A and B are matrices of compatible sizes, think about how the columns of AB look. In particular, the i -th column of AB is the product of A and the i -th column of B .

- 7.5. Generalise the previous to $\mathrm{SL}_n(\mathbb{R})$ acting on $\mathbb{R}[X_{n \times m}]$.