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MA 839

Advanced Commutative Algebra

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Lecture 0 (07-01-2021)

07 January 2021 08:08

A Quick Intro.

Setup: A ring is commutative with 1.

Let M be an R-module.

Observation: ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$),
we get an R-linear map $R \twoheadrightarrow M$ which is onto.
$a \mapsto ax$

Then, $M \cong R/I$ where I is the kernel.
In this case, $I = \text{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R.

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$.
$= \{ax + by \mid (a, b) \in R^{\oplus 2}\}$

Then, we get an onto R-linear map $\underbrace{Re_1 \oplus Re_2}_{\{e_1, e_2\} \text{ is a basis}} \xrightarrow{\varphi} M$
$e_1 \mapsto x$
$e_2 \mapsto y$ } extend this

this lets us extend the map

In particular, $M \cong R^2/\ker\varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle\ \rangle} \oplus \frac{R}{\langle\ \rangle} ?$$

This has a positive answer: ① R is a field
② R is a PID

CAUTION: We won't include fields as PID.
That is, when we say "PID", we exclude fields!!

③ Suppose M is a finitely generated (f.g.) R-module.
(That is, suppose $M = \langle x_1, \ldots, x_n \rangle$.)

way to get

Then, M is a quotient of $R^{\oplus n}$.

way to get this

Then, M is a quotient of $R^{\oplus n}$.

Define $R^{\oplus n} \overset{\varphi}{\twoheadrightarrow} M$ by $e_i \longmapsto x_i$.

$$M \cong R/\ker\varphi.$$

④ In general, consider a free module with "M as basis", call it $F(M)$. Then $F(M)$ maps onto M.

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$, then $F(A)$ maps onto M.

Thus, M can be written as a quotient of a free-module.

To Summarise: If M is an R-module, then M can be written as a quotient of a free R-module. Moreover, if M is f.g., then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R-module mapping onto M with kernel K. That is, $\varphi : F \twoheadrightarrow M$ is onto R-linear and $\ker\varphi = K$.

Now, $\exists$ a free R-module G and an onto map $\psi : G \to K$

We capture this in the following diagram

← compose ψ with $K \hookrightarrow F$

$$\begin{tikzcd}
G \arrow[rr, dashed, "\psi"] \arrow[dr, "\psi"'] & & F \arrow[r, "\varphi"] & M \arrow[r] & 0 \\
& K \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

means φ is onto

means the map $K \to F$ is 1-1

Note that $\operatorname{im}\psi = K = \ker\varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \longrightarrow 0$.

① φ is onto and $\ker\varphi = \operatorname{im}\psi$.
② G and F are free R-modules.

Note that we can repeat the above process with K instead of F.

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \varphi$, $\varphi_1 := \psi$.

$$\begin{tikzcd}
 & K_1 \arrow[dr] & & & & \\
F_2 \arrow[ur, "\varphi_2"] \arrow[rr, "\varphi_2"'] & & F_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\varphi_1"'] & & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
 & & & K_0 \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 & &
\end{tikzcd}$$

Thus, we get free modules $\{F_n, \varphi_n : F_n \to F_{n-1}\}$ such that $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$ written as

$$\cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

F_i s are free, φ_0 is onto & $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to \underbrace{F_1 \xrightarrow{\varphi_1} F_0 \to 0}$$ as an

R free resolution of M.

im $\varphi_1 = K$, this is not exact here. φ_1 not onto (nec.)

Q: ① If M is f.g.:
Can we get F_i s so that $\operatorname{rank}(F_i) < \infty$ $\forall i$.

② If yes, are $\operatorname{rank}(F_i)$ s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_i s s.t. the maps are "described nicely"?

Q. If two modules have "isomorphic" free resolutions, are they isomorphic?

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_3"] & F_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2", "\wr"'] & F_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1", "\wr"'] & F_0 \arrow[r] \arrow[d, "\psi_0", "\wr"'] & 0 \\
\cdots \arrow[r, "\varphi_3'"] & F_2' \arrow[r, "\varphi_2'"] & F_1' \arrow[r, "\varphi_1'"] & F_0' \arrow[r] & 0
\end{tikzcd}$$

$M \simeq F_0 / \operatorname{im} \varphi_1$

$M' \simeq F_0' / \operatorname{im} \varphi_1'$

$$\varphi_1' \psi_1 = \psi_0 \varphi_1 \qquad (*)$$

Claim. $\gamma_0(\mathrm{im}\ \varphi_1) = \mathrm{im}\ \varphi_1'$

$(\subseteq)$ clear by (*)

$(\supseteq)$ clear again since $\varphi_1' = \gamma_0 \varphi_1 \gamma_1^{-1}$

Thus, $M_0 \simeq F_0 / \mathrm{im}\varphi \simeq \dfrac{\psi_0(F_0)}{\psi_0(\mathrm{im}\,\varphi)} = \dfrac{F_0'}{\mathrm{im}\varphi_1'} \simeq M_0'$

Lecture 1 (11-01-2021)

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Free modules:

As usual : R is a (commutative) ring (with 1).
M is an R-module.

Def^n. ① Let $A \subset M$. A is said to be a generating set of M (as an R-module) if
$\forall x \in M$, $\exists x_1, \ldots, x_n \in A$ and $(a_1, \ldots, a_n) \in R^n$ s.t.

$$x = a_1 x_1 + \cdots + a_n x_n .$$

(Note that A need not be finite.)

Notation : $M = \langle A \rangle$

If $A = \{x_1, \ldots, x_n\}$ is finite, then $M = \langle x_1, \ldots, x_n \rangle$ and M is said to be finitely generated.

②ⓐ Let $x_1, \ldots, x_n \in M$. We say $\{x_1, \ldots, x_n\}$ is linearly independent (over R) if for $(a_1, \ldots, a_n) \in R^n$,

$$a_1 x_1 + \cdots + a_n x_n = 0 \Rightarrow (a_1, \ldots, a_n) = 0 \text{ in } R^n .$$

ⓑ A subset $A \subset M$ is linearly independent (over R) if every finite subset of A is linearly independent (over R).

③ A subset $A \subset M$ is a basis of M (over R) if $M = \langle A \rangle$ (over R) and A is linearly independent (over R).

④ M is free (over R) if M has a basis (over R).

REMARKS ① Not every R-module has a basis.
② A minimal generating set need not be lin. indep.
③ A maximal lin indep. set need not be a gen. set.

Q. If every R-module has a basis, is R a field?
(Yes. Take a non-field ring R and any non-trivial ideal $I \subsetneq R$.
Then, R/I has no lin. indep. set over R.)

Q. If an R-module M has a basis, does every basis have the same cardinality?
Ans. Yes. This is called the Invariant Basis Number (IBN) property of R.
Remark. This is not true if R is non-commutative. (That is, we can find a counterexample of a non-commutative ring.)
If R is a division ring, then again we have IBN.

Defn If M has a finite basis, say B, then we define
$$\text{rank}(M) := |B|.$$
If M is free with an infinite basis, $\text{rank}(M) := \infty$.
} well-defined, by IBN

(When we do say "rank", we will usually mean "finite rank.")

EXAMPLES. ① $R^{\oplus n}$ is a free R-module of rank n
$M_{m\times n}(R)$ of rank mn
$R[x]$ of rank ∞

② Let A be a non-empty set and
$$\mathcal{F}_0(A, R) = \{f : A \to R \mid f(a) = 0 \text{ for all but fin. many } a \in A\}.$$
Then, $\mathcal{F}_0(A, R)$ is an R-module under pointwise operations.
In fact, $\mathcal{F}_0(A, R)$ is a free R-module with basis $\{\chi_a\}_{a\in A}$,
where

$$\chi_a(b) = \begin{cases} 0 & ; b \neq a \\ 1 & ; b = a \end{cases}$$

To see where the above set is generating, given any $f \in \mathcal{F}_0(A, R)$, we can write

$$f = \sum_{a \in A} f(a)\, \chi_a .$$

↑ the sum is actually finite since $f(a)=0$ for all but finitely many.
(it is to be understood that 0s are ignored.)

Q. What if we take $\mathcal{F}(A, R)$? (All functions.)

Universal Property of free modules:

Def^n: Given a non-empty set A, a free R-module on A is a pair $(F(A), e)$ where (i) $F(A)$ is an R-module,
(ii) $e : A \to F(A)$ is a (set) function

satisfying:

Given an R-module M and a function $f : A \to M$, there exists a unique R-linear $\tilde{f} : F(A) \to M$ making the following diagram commute.

$$\begin{tikzcd} & F(A) \arrow[d, dashed, "\tilde{f}"] \\ A \arrow[ur, "e"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=4, "\circlearrowright"] & M \end{tikzcd}$$

(That is, $\tilde{f} e = f$.)

REMARKS. ① Given $A \neq \emptyset$, a free R-module on A exists, and is unique up to isomorphism.
Moreover, $e : A \to F(A)$ is one-one and $F(A)$ is free with basis $\{e_a\}_{a \in A}$, where $e_a := e(a)$.

② If M is a free R-module, then $M \cong F(B)$, where B is a(ny) basis of M.

Thus, an R-module M is free iff $M \cong F(A)$ for some A.

What the universal property is really saying is that: given a free R-module M with basis A, every R-linear

$$M \to N \quad R\text{-module}$$

is completely determined by its action on A.

(The above is in the sense that given any assignment of values on A, we do get an R-linear map.)

EXAMPLE: Given an R-module M, such that $M = \langle A \rangle$, we can write M as a quotient of $F(A)$.
(What we did last lec.)

Lecture 2 (12-01-2021)

12 January 2021 08:35

Weyl Algebra

Ex. $\mathbb{k}$ is a field, $\mathbb{k}[x_1, \dots, x_d]$

$\partial_1, \dots, \partial_d \to$ partial diff op.

$A_d(k) = k[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$ D-modules

non-comm. How would you define product?

Tensor Product

Tensor product (of two modules) essentially converts the study of bilinear maps to linear maps.

Defn. Given R-modules M and N, the tensor product of M and N (over R) is a pair (T, θ), where T is an R-module, $\theta : M \times N \to T$ is R-bilinear satisfying:

Given (K, φ) where K is an R module, $\varphi : M \times N \to K$ is R-bilinear, there exists a unique R-linear map $\tilde{\varphi} : T \to K$ making the following diagram commute

$$\begin{tikzcd} & T \arrow[d, dashed, "\tilde{\varphi}"] \\ M \times N \arrow[ur, "\theta"] \arrow[r, "\varphi"'] & K \end{tikzcd}$$

, i.e., $\tilde{\varphi}\,\theta = \varphi$.

(We are using "with", but can use "and" once we prove $M \otimes_R N \cong N \otimes_R M$.)

Thm. A tensor of M with N exists and is unique, up to isomorphism.

Uniqueness follows by universal property.

Notation: $M \otimes_R N$

Construction:

Want

$$\begin{tikzcd} M\times N \arrow[rr, dashed, "\theta"] \arrow[dr, "\varphi"'] & & T \arrow[dl, dashed, "\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

$\theta(x_1+x_2, y) = \theta(x_1, y) + \theta_2(x, y)$

Step 1: Let $F = F(M\times N)$, the free module on the set $M\times N$.
We get a map $e : M\times N \longrightarrow F(M, N)$
$$(x, y) \longmapsto e_{(x,y)}$$

$\{e_{(x,y)} : x \in M,\ y \in M\}$ is a basis for F.

Let G be the submodule of F generated by

- $e_{(x_1+x_2, y)} - e_{(x_1, y)} - e_{(x_2, y)}$
- $e_{(x, y_1+y_2)} - e_{(x, y_1)} - e_{(x, y_2)}$
- $e_{(ax, y)} - a\, e_{(x,y)}$
- $e_{(x, ay)} - a\, e_{(x,y)}$

$\forall\, x, x_1, x_2 \in M,\ \forall\, y, y_1, y_2 \in M,\ \forall\, a \in R$

Step 2: Define $T = F/G$. Let $\pi : F \twoheadrightarrow T$ be the natural map.
Set $\pi(e_{(x,y)}) =: x \otimes y$.

Note that
$$\left.\begin{aligned} (x_1+x_2)\otimes y &= x_1\otimes y + x_2\otimes y \\ x\otimes(y_1+y_2) &= x\otimes y_1 + x\otimes y_2 \\ (ax)\otimes y &= a(x\otimes y) = x\otimes(ay) \end{aligned}\right\}\ \begin{aligned} &\forall\, x, \ldots \in M \\ &\forall\, y, \ldots \in N \\ &\forall\, a \in R \end{aligned}$$

Consider $\theta = \pi e : M\times N \longrightarrow T$
$$(x, y) \longmapsto x\otimes y$$

$$\begin{tikzcd} & F(M\times N) \arrow[dr, "\pi"] & \\ M\times N \arrow[ur, "e"] \arrow[rr, dashed, "\theta"] & & T \end{tikzcd}$$

$e(x,y)$ $\quad (x,y) \quad x\otimes y$

Step 3. Now, suppose we are given a bilinear
$\varphi : M\times N \longrightarrow K$. ($K$ is some R-module.)

$$\begin{tikzcd}
M \times N \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists! \tilde{\varphi}"] \\
 & K &
\end{tikzcd}$$

Q1: Is there even a func. $\tilde{\varphi}$?

Q2: Is it R-linear?

Q3: Is it unique?

Note also

$$\begin{tikzcd}
 & F \arrow[dd, dashed] & \\
M \times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists! \tilde{\varphi}"] \\
 & K &
\end{tikzcd}$$

Note we have a set map $M \times N \xrightarrow{\varphi} K$ which induces an R linear map $\bar{\varphi} : F \to K$. (UMP of free modules)

$$\begin{tikzcd}
 & F \arrow[dd, dashed, "\bar{\varphi}"] & \\
M \times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists! \tilde{\varphi}"] \\
 & K &
\end{tikzcd}$$

We now want to show that $\bar{\varphi}$ factors through T. It would suffice to show that $G \subseteq \ker \bar{\varphi}$.

Using bilinearity of φ, it follows that all our (four types of) generators of G are in $\ker \bar{\varphi}$.

Thus, $\bar{\varphi}$ factors through quotient. That is, $\exists!$ R-linear $\tilde{\varphi} : T \to K$ s.t.

$$\begin{tikzcd}
F \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi"'] & K \\
T \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

commutes. That is, $\bar{\varphi} = \tilde{\varphi}\pi$.

$$\begin{tikzcd}
 & F \arrow[dd, dashed, "\bar{\varphi}"] \arrow[dr, dashed, "\pi"] & \\
M \times N \arrow[ur, dashed, "e"] \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\tilde{\varphi}"] \\
 & &
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[dr, "\varphi"'] & {} \arrow[d, dashed] & {} \arrow[dl, dashed, "\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

Can now verify $\tilde{\varphi}\,\theta = \varphi$. (Use commutation of diff. triangles.)
Can also verify that $\tilde{\varphi}$ is unique R-linear such.

Basic Properties:

(1) ["Identity"] $R \otimes_R M \simeq M$

(2) [Commutativity] $M \otimes_R N \simeq N \otimes_R M$

(3) [Associativity] $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$

(4) [Distributivity] $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$

Q. How does get an R-linear map $M \otimes_R N \to L$?
A. Give an R-bilinear map $M \times N \to L$.
Pretty much the only way. $x \otimes y$ would be 0 even if $x, y \neq 0$.
Thus, checking "well-defined"ness would become quite difficult.

Q. Let M be an R-module. $R \subset S$ subring.
Can you identify a natural S-module on $S \otimes_R M$?
(Base change)