Morphisms of Schemes: Chevalley's Theorem

Aryaman Maithani Mentor: Prof. Arvind Nair

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- **②** Given f ∈ A, A_f will denote the localisation of A at the multiplicative set $\{1, f, f^2, ...\}$.



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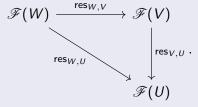
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Moreover, the "obvious diagrams" must commute.

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Proposition 8 (A basis for the Zariski topology)

The collection $\{D(f): f \in A\}$ forms a basis for the above topology.

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To conclude, the only closed singleton subset of \mathbb{A}^1_k is $\{\langle 0 \rangle\}$.

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This is called the structure sheaf on Spec A.

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In fact, (it follows that) the affine opens form a basis for X.

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The above is a morphism of affine schemes. That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

Morphisms of schemes

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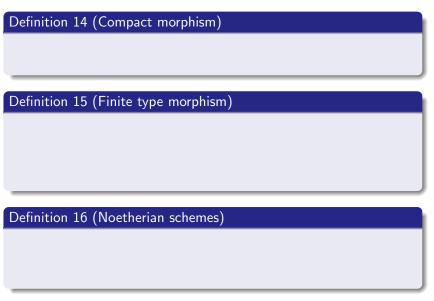
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More precisely, for each choice of affine open sets Spec $A \subset X$, Spec $B \subset Y$, such that $\pi(\operatorname{Spec} A) \subset \operatorname{Spec} B$, the restricted morphism is one of affine schemes.



Definition 14 (Compact morphism)

A morphism $\pi:(X,\mathscr{O}_X)\to (Y,\mathscr{O}_Y)$ of schemes is compact

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A morphism $\pi:(X,\mathcal{O}_X)\to (Y,\mathcal{O}_Y)$ of schemes is compact if the preimage of any compact open subset is compact.

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A scheme (X, \mathcal{O}_X) is said to be Noetherian if X can be covered by finitely many affine opens Spec A_i such that each A_i is a Noetherian ring.

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