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§0. Notations and Preliminaries

§§0.0. Notations

$\mathbb{C}$ will denote the field of complex numbers and V, W vector spaces over $\mathbb{C}$. (We will stick to finite dimensional vector spaces.)

1. If V is a vector space and W a subspace, then we write $W \leq V$.
2. If $X \subset V$, then $\mathbb{C}X = \operatorname{span} X$. (cf. Section 0.1.3 for the definition of $\mathbb{C}X$ when X is an arbitrary set.)
3. R^* denotes the group of units of a ring R.
4. $M_{m\times n}(\mathbb{C})$ is the vector space $m \times n$ matrices with entries in $\mathbb{C}$.
5. $M_n(\mathbb{C}) = M_{n\times n}(\mathbb{C})$.
6. $\operatorname{Hom}_{\mathbb{C}}(V, W)$ is the vector space of linear maps from V to W.
7. $\operatorname{End}(V) = \operatorname{Hom}_{\mathbb{C}}(V, V)$ is the *ring of endomorphisms*. This is isomorphic to $M_{\dim V}(V)$.
8. $\operatorname{GL}(V) = \{A \in \operatorname{End}(V) \mid A \text{ is invertible}\} = \operatorname{End}(V)^*$ is the *general linear group* of V.
9. $\operatorname{GL}_n(\mathbb{C}) = M_n(\mathbb{C})^*$. This is isomorphic to $\operatorname{GL}(\mathbb{C}^n)$.
10. We have the usual sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, S_n, \mathbb{Z}/n\mathbb{Z}$. For us, $0 \notin \mathbb{N}$.
11. D_n will denote the dihedral group with $2n$ elements.
12. Given $\mathbb{C}_n$, we denote by e_i, the i-th standard basis vector.
13. $\omega_n := \exp\left(\frac{2\pi\iota}{n}\right)$.

§§0.1. Linear Algebra Preliminaries

0.1.1 Inner product spaces

Definition 0.1. Let V be a vector space and $T \in \operatorname{End}(V)$. If $W \leq V$ is such that $Tw \in W$ for all $w \in W$, then W is said to be T-invariant.

Proposition 0.2. Let $W \leq V$ be vector spaces and $T \in \operatorname{GL}(V)$. Then, W is T-invariant iff $T(W) = W$.

Proof. $\Longleftarrow$ is trivial. We prove the other direction.

By hypothesis, we know that $T(W) \leq W$. However, since W is finite-dimensional and T

an isomorphism, we see that

$$\dim(T(W)) = \dim(W)$$

and hence, $T(W) = W$. (If a subspace of a finite dimensional vector space has the same dimension, then the subspace must be the whole space.) $\square$

Proposition 0.3. Let $W \leq V$ be vector spaces and $T \in \mathrm{GL}(V)$ be such that W is T-invariant. Then, W is also T^{-1}-invariant.

Proof. Using Proposition 0.2, we know that $T(W) = W$. Since T is a bijection, this immediately yields that $W = T^{-1}(W)$, proving the desired result. $\square$

Proposition 0.4. Let $W \leq V$ be vector spaces and $T, S \in \mathrm{GL}(V)$ be such that W is T-invariant and S-invariant. Then, W is also $S \circ T$-invariant.

Proof. Let $w \in W$. Then, $Tw \in W$ since W is T-invariant. In turn, $S(Tw) \in W$ since W is S-invariant. Thus, $(S \circ T)(w) \in W$ for all $w \in W$, as desired. $\square$

Definition 0.5. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $T \in \mathrm{End}(V)$. The adjoint of T is the unique linear operator T^* such that the following equality holds for all $v, w \in V$:

$$\langle Tv, w \rangle = \langle v, T^* w \rangle.$$

Proposition 0.6. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $T \in \mathrm{GL}(V)$. Suppose that $W \leq V$ is T-invariant. Then, $W^\perp$ is T^*-invariant.

Proof. Let $v \in W^\perp$ and $w \in W$ be arbitrary. It suffices to show that $\langle T^* v, w \rangle = 0$. However, this is immediate since

$$0 = \langle v, Tw \rangle = \langle T^* v, w \rangle.$$

The first equality is true since $Tw \in W$ by T-invariance of W and $v \in W^\perp$, by hypothesis. $\square$

Definition 0.7. Let V be an inner product space and $U \in \mathrm{GL}(V)$. U is said to be unitary if

$$\langle Uv, Uw \rangle = \langle v, w \rangle$$

for all $v, w \in V$. The subset $U(V) \subset \mathrm{GL}(V)$ of all unitary operators forms a subgroup.

In other words, one sees that
$$\langle v, U^*Uw\rangle = \langle v, w\rangle$$
for all $v, w \in V$. In other words, U^*U is the adjoint of the identity map. However, since identity is its own adjoint, we see that U^*U is the identity map. In other words, $U^* = U^{-1}$.

Definition 0.8. A matrix $U \in \mathrm{GL}_n(\mathbb{C})$ is said to be unitary if $UU^* = I$. A set of all such matrices is denoted by $U_n(\mathbb{C})$ and forms a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

As usual, U^* denotes the conjugate transpose of U. One can show that the matrix U is unitary (Definition 0.8) iff the corresponding linear operator is unitary (Definition 0.7), with respect to the standard inner product on $\mathbb{C}^n$.

Corollary 0.9. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and $T \in U(V)$. Suppose that W is T-invariant. Then, $W^\perp$ is also T-invariant.

Proof. By Proposition 0.6, we see that $W^\perp$ is T^* invariant and hence, T^{-1}-invariant.
(Note that $T^{-1} = T^*$ since T is unitary.)
By Proposition 0.3, we then see that $W^\perp$ is T-invariant.
(We are using that $(T^{-1})^{-1} = T$.) □

0.1.2 Minimal polynomials and diagonalisation

Definition 0.10. Let $T \in \mathrm{End}(V)$. The minimal polynomial of T is the unique monic polynomial $m(X) \in \mathbb{C}[X]$ such that $m(T)$ is the zero operator.

Definition 0.11. Let $T \in \mathrm{End}(V)$. T is said to be diagonalisable if there exists a basis B of V consisting of eigenvectors of T.

For the remainder, T will denote an element of $\mathrm{End}(V)$ and $m(X)$ its minimal polynomial.

Proposition 0.12. Let $p(X) \in \mathbb{C}[X]$ be any polynomial such that $p(T) = 0$. Then $p(\lambda) = 0$ for any eigenvalue $\lambda \in \mathbb{C}$ of T. In particular, all eigenvalues of T (in $\mathbb{C}$) are roots of the minimal polynomial.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T. Let $v \neq 0$ be an eigenvector corresponding to λ. Then, note that
$$T^k v = \lambda^k v$$
for all $k \geq 0$. In particular, if
$$p(X) = a_0 + a_1X + \cdots + a_rX^r,$$

then we have

$$\begin{aligned} 0 = p(T)v &= (a_0 + a_1 T + \cdots + a_r T^r)v \\ &= a_0 v + a_1 T v + \cdots + a_r T^r v \\ &= a_0 v + a_1 \lambda v + \cdots + a_r \lambda^r v \\ &= p(\lambda) v. \end{aligned}$$

Thus, $p(\lambda)v = 0$. But since $v \neq 0$, we get that $p(\lambda) = 0$, as desired. □

Remark 0.13. Of course, the eigenvalues of T are precisely the roots of the characteristic polynomial of T. Thus, the above proposition tells us that the minimal polynomial and characteristic polynomial have precisely the same roots. (One way implication is in the above, the other is obvious since the minimal polynomial must divide the characteristic polynomial.)

Proposition 0.14. If T is diagonalisable, then $m(X)$ has distinct roots.

Proof. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ be the distinct eigenvalues of T. Let

$$p(X) = (X - \lambda_1) \cdots (X - \lambda_r).$$

Then, by the previous proposition, we know that $p(X) \mid m(X)$. Since both are monic, it suffices to show that $m(X) \mid p(X)$ to conclude that $m(X) = p(X)$. And to do that, it suffices to show that $p(T)$ is the zero operator. And to do *that*, it suffices to show that $p(T)$ annihilates some basis of V. To this end, let B be an eigenbasis of V with respect to T (which exists since T is diagonalisable). Then, any $v \in B$ is annihilated by some $T - \lambda_i$. Since all the $T - \lambda_j$ commute, we see that $p(T)v = 0$ and we are done. □

Proposition 0.15. Suppose that $m(X)$ has distinct roots. Then, T is diagonalisable.

Proof. By hypothesis, $m(X) = (X - \lambda_1) \cdots (X - \lambda_r)$ for some distinct $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$. Since $m(X)$ divides the characteristic polynomials, it follows that each λ_i is an eigenvalue. We wish to show that

$$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}.$$

Note that since we know that eigenspaces intersect trivially, it suffices to show that

$$V = E_{\lambda_1} + \cdots + E_{\lambda_r}.$$

Now, consider the polynomials

$$f_i(X) = \frac{m(X)}{X - \lambda_i} = \prod_{j \neq i} (X - \lambda_j)$$

for $i = 1, \ldots, r$. Put
$$g_i(X) = \frac{f_i(X)}{f_i(\lambda_i)}.$$
(Note that each $f_i(\lambda_i)$ is non-zero since the roots are distinct.)
Note that $g_i(\lambda_j) = \delta_{i,j}$.

Now, note that
$$1 = \sum_{i=1}^{r} g_i(X).$$
(Both sides are polynomials of degree at most $r-1$ which agree on the r points $\lambda_1, \ldots, \lambda_r$.)
Thus, $g_i(T)$ is the identity operator.

Thus, given any $v \in V$, we have
$$v = \sum_{i=1}^{r} g_i(T)v. \tag{Σ}$$
However, note now that
$$\begin{aligned}(T-\lambda_i)g_i(T)v &= (T-\lambda_i)\frac{f_i(T)}{f_i(\lambda_i)}v\\ &= \frac{p(T)}{f_i(\lambda_i)}v\\ &= 0.\end{aligned}$$
Thus, $g_i(T)v \in E_{\lambda_i}$ for each i and (Σ) shows that $V = \bigoplus E_{\lambda_i}$. □

The above two propositions are summarised in the following theorem.

Theorem 0.16. Let V be a vector space over $\mathbb{C}$. Let $T \in \operatorname{End}(V)$ and $m(X) \in \mathbb{C}[X]$ be the minimal polynomial of T. Then, T is diagonalisable if and only if $m(X)$ has distinct roots.

0.1.3 Linearisation

Definition 0.17 (Linearisation)**.** Given a non-empty finite set X, we define a $\mathbb{C}$-vector space $\mathbb{C}X$ whose elements are formal linear combinations
$$\sum_{x \in X} c_x x$$
where $c_x \in \mathbb{C}$.

The addition is given by adding the corresponding scalar coefficients and scalar multiplication is defined similarly.

X is identified as a subset of $\mathbb{C}X$ be identifying x with $1x$. Under this, X is a basis for $\mathbb{C}X$.

This is an inner product space with the product defined as

$$\left\langle \sum_{x \in X} a_x x, \sum_{x \in X} b_x x \right\rangle = \sum_{x \in x} a_x \overline{b_x}.$$

Note very carefully that we have assumed that X is finite. This avoids the complication of having to make sure that the sums are finite.

The above construction has the following property.

Proposition 0.18. Given non-empty finite sets X, Y and a function $f : X \to Y$, there exists a unique linear transformation

$$\mathbb{C}f : \mathbb{C}X \to \mathbb{C}Y$$

such that $\mathbb{C}f|_X = f$.

Those familiar with category theory can actually verify that the above defines a *functor* from the category of sets (and functions) to that of $\mathbb{C}$-vector spaces (and $\mathbb{C}$-linear function).

We can also note the following construction.

Proposition 0.19. Let G be a group which acts on a set X. Then, extending the action in the natural way gives an action on $\mathbb{C}X$. In other words, we get a homomorphism $\varphi : G \to S_{\mathbb{C}X}$. Moreover, we have the property that not only is $\varphi(g)$ is a bijection for each $g \in G$ but also an isomorphism.

Proof. The "natural way" of extension is to define

$$\cdot : G \times \mathbb{C}X \to \mathbb{C}X$$

as

$$g \cdot \left(\sum_{x \in X} c_x x \right) := \sum_{x \in X} c_x (g \cdot x).$$

The $\cdot$ on the right is the original action.
(The right hand side makes sense because $g \cdot x \in X$.)

With the above explicit formula, it is clear that the group action axioms are satisfied. We now show that the last part. It suffices to show that $\varphi(g)$ is linear.

In other words, we need to show that $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$ for all $g \in G$ and

$v_1, v_2 \in V$. This is simple, for we note that

$$\begin{aligned} g \cdot \left(\sum_{x \in X} c_x x + \sum_{x \in X} d_x x \right) &= g \cdot \left(\sum_{x \in X} (c_x + d_x) x \right) \\ &= \sum_{x \in X} (c_x + d_x)(g \cdot x) \\ &= \sum_{x \in X} c_x g \cdot x + \sum_{x \in X} d_x g \cdot x \\ &= g \cdot \left(\sum_{x \in X} c_x x \right) + g \cdot \left(\sum_{x \in X} d_x x \right). \end{aligned}$$

□

In other words, what we have above is actually a *representation*, the central topic of study in this report.

§§0.2. Group Theory Preliminaries

0.2.1 Group of complex homomorphisms

Definition 0.20. Given a group G, let $\widehat{G}$ denote the set of all group homomorphisms from G to $\mathbb{C}^*$. This is a group under point-wise operations.

Proposition 0.21. Let $G,\ G_1$, and G_2 be (not necessarily abelian) groups. If $G = G_1 \times G_2$, then $\widehat{G} \cong \widehat{G_1} \times \widehat{G_2}$.

Proof. Given $\varphi \in \widehat{G}$, we define $\varphi_1 : \widehat{G_1} \to \mathbb{C}$ by

$$\varphi_1(g_1) := \varphi(g_1, 1)$$

and similarly, $\varphi_2 : \widehat{G_1} \to \mathbb{C}$ by

$$\varphi_1(g_2) := \varphi(1, g_2).$$

It is easy to see that each φ_i is a homomorphism. That is, $\varphi_i \in \widehat{G}_i$ for $i = 1, 2$.

Now, we define $\Phi : \widehat{G} \to \widehat{G_1} \times \widehat{G_2}$ as follows:

$$\Phi(\varphi) = (\varphi_1, \varphi_2).$$

It is easy to verify that Φ is a homomorphism using the fact that

$$(\varphi\varphi')_i = \varphi_i \varphi_i'$$

for every $\varphi, \varphi' \in \widehat{G}$ and $i = 1, 2$.

Moreover, if $\Phi(\varphi) = (1,1)$, then $\varphi_1(g_1) = 1 = \varphi_2(g_2)$ for all $(g_1, g_2) \in G$. Thus, $\varphi \equiv 1$, showing that Φ is injective.

To show surjectivity, let $(\rho, \rho') \in \widehat{G_1} \times \widehat{G_2}$ be arbitrary. Then, define $\varphi : G \to \mathbb{C}^*$ by

$$\varphi(g_1, g_2) = \rho(g_1)\rho'(g_2).$$

Note that

$$\begin{aligned}
\varphi(g_1g_1', g_2g_2') &= \rho(g_1g_1')\rho'(g_2g_2') \\
&= \rho(g_1)\rho(g_2')\rho'(g_2)\rho'(g_2') \\
&= \rho(g_1)\rho'(g_2)\rho(g_2')\rho'(g_2') \\
&= \varphi(g_1, g_2)\varphi(g_1', g_2')
\end{aligned}$$

and hence, $\varphi \in \widehat{G}$. It is now easy to see that

$$\Phi(\varphi) = (\rho, \rho'),$$

proving surjectivity. □

Proposition 0.22. If $G = \mathbb{Z}/n\mathbb{Z}$, then $G \cong \widehat{G}$.

Proof. Note that we have the n distinct homomorphisms $\varphi^{(0)}, \ldots, \varphi^{(n-1)}$ given by

$$\varphi^{(k)}([m]) = \omega_n^{km}.$$

(It can be verified easily that this is indeed a well-defined map and a homomorphism by using the fact that $\omega_n^n = 1$ and $\omega^a\omega^b = \omega^{a+b}$.)

Moreover, these are the only homomorphisms since any homomorphism is uniquely determined once we map $[1]$ to an element, and that element is forced to be an n-th root of unity.

This shows that $\left|\widehat{G}\right| = n$. To note that it is cyclic, we simply observe that

$$\left(\varphi^{(1)}\right)^k = \varphi^{(k)}.$$

□

Corollary 0.23. Let G be a finite abelian group, then $G \cong \widehat{G}$.

Proof. Using the structure theorem of finite abelian groups, we know that

$$G \cong G_1 \times \cdots \times G_n$$

for some finite cyclic groups $G_1, \ldots, G_n$. From the previous two propositions, the result follows. □

0.2.2 Sign of a permutation

In the following, e_i denotes the i-th standard basis vector of $\mathbb{C}^n$.

Definition 0.24 (Matrix of a permutation)**.** We define $M : S_n \to M_n(\mathbb{C})$ as follows: Given $\sigma \in S_n$, we define $M(\sigma)$ to be the matrix representing the linear transformation determined by $e_i \mapsto e_{\sigma(i)}$.

We immediately note that M actually maps into $M_n(\mathbb{Z})$ since the i-th column of $M(\sigma)$ is simply $e_{\sigma(i)}$, i.e., all 0s with a 1 in the i-th place.

Proposition 0.25 (M is multiplicative)**.** Given $\sigma, \tau \in S_n$, we have

$$M(\sigma\tau) = M(\sigma)M(\tau).$$

Proof. It suffices to those that the matrices on either side of the equation act the same way on each e_i. To this end, note that

$$\begin{aligned} M(\sigma\tau)e_i &= e_{(\sigma\tau)(i)} \\ &= e_{\sigma(\tau(i))} \\ &= M(\sigma)e_{\tau(i)} \\ &= M(\sigma)M(\tau)e_i. \end{aligned}$$ □

We now state corollaries of the above proposition.

Corollary 0.26. Given any $\sigma \in S_n$, the matrix $M(\sigma)$ has determinant ± 1. In particular, each such matrix is invertible.

Proof. Note $M(\mathrm{id}) = I$ and thus,

$$M(\sigma)M(\sigma^{-1}) = I,$$

by the above proposition.

Since $M(\sigma)$ and $M(\sigma^{-1})$ have integer entries, their determinants are also integers. Taking det on both sides above yields the result. □

Definition 0.27 (Sign of a permutation)**.** Define the function $\operatorname{sign} : S_n \to \{-1, 1\}$ as the following composition

$$S_n \xrightarrow{M} M_n(\mathbb{C}) \xrightarrow{\det} \{-1, 1\}.$$

By the above corollary, the above composition is well-defined.

Corollary 0.28. The sign function is a homomorphism from S_n to $\{-1, 1\} = \mathbb{Z}^\times$.

Proof. Follows from the above proposition and the fact that det is multiplicative. □

Proposition 0.29 (Sign in terms of transpositions)**.** Let $\sigma \in S_n$ and suppose that we can write

$$\sigma = \tau_1 \cdots \tau_n$$

for transpositions $\tau_i \in S_n$.

Then, $\operatorname{sign} \sigma$ is 1 iff n is even.

Proof. Let τ be a transposition, say (ij). Then, $M(\tau)$ is the elementary row matrix that swaps the rows i and j. Thus,

$$\operatorname{sign}(\tau) = \det M(\tau) = -1.$$

By the earlier proposition, it follows that

$$M(\sigma) = M(\tau_1) \cdots M(\tau_n)$$

and hence,

$$\operatorname{sign}(\sigma) = (-1)^n,$$

which immediately proves the result. □

Corollary 0.30. Given any decompositions of a permutation into transpositions, the parity of the number of transpositions is fixed.

Remark 0.31. The above way seems to have avoided all difficulties of showing that sign is well-defined by avoiding the definition in terms of transpositions. In fact, we get that as a corollary!
It seems that the work has gone in the fact that det is multiplicative. Note that we are actually using this result from linear algebra (over fields, that is) and not necessarily that from ring theory.

0.2.3 Conjugacy classes of S_n

Proposition 0.32. Let $\sigma, \tau \in S_n$. Suppose that a disjoint cycle decomposition of σ is given as

$$(a_1 \ldots a_{m_1})(a_{m_1+1} \ldots a_{m_2}) \ldots (a_{m_{k-1}+1} \ldots a_{m_k}),$$

where $\{a_1, \ldots, a_{m_k}\} = \{1, \ldots, n\}$ and $m_k = n$. Then, the cycle decomposition of $\tau\sigma\tau^{-1}$ is given by

$$(\tau(a_1) \ldots \tau(a_{m_1}))(\tau(a_{m_1+1}) \ldots \tau(a_{m_2})) \ldots (\tau(a_{m_{k-1}+1}) \ldots \tau(a_{m_k})).$$

Proof. Let

$$\rho := (\tau(a_1) \ldots \tau(a_{m_1}))(\tau(a_{m_1+1}) \ldots \tau(a_{m_2})) \ldots (\tau(a_{m_{k-1}+1}) \ldots \tau(a_{m_k})).$$

We wish to show that $\tau\sigma\tau^{-1} = \rho$. It suffices to show that $\tau\sigma = \rho\tau$. To this end, let $i \in [n]$. Then, $i = a_j$ for some j.

If j is of the form m_r, then (with $m_0 := 0$)

$$\rho(\tau(i)) = \rho(\tau(a_{m_r})) = \tau(a_{m_{r-1}+1}) = \tau(\sigma(a_{m_r})) = \tau(\sigma(i)).$$

Otherwise, we have

$$\rho(\tau(i)) = \rho(\tau(a_j)) = \tau(a_{j+1}) = \tau(\sigma(a_j)) = \tau(\sigma(i)),$$

completing the proof. □

Corollary 0.33. Any two conjugates have the same cycle type.

Proof. Immediate. □

Corollary 0.34. If two permutations have the same cycle type, then they are conjugates.

Proof. Let σ and σ' have the same cycle type. Then, we have write

$$\begin{aligned} \sigma &= (a_1 \ldots a_{m_1})(a_{m_1+1} \ldots a_{m_2}) \ldots (a_{m_{k-1}+1} \ldots a_{m_k}) \\ \sigma' &= (b_1 \ldots b_{m_1})(b_{m_1+1} \ldots b_{m_2}) \ldots (b_{m_{k-1}+1} \ldots b_{m_k}) \end{aligned}$$

Then, define $\tau : [n] \to [n]$ by

$$\tau(a_i) = b_i.$$

This defines a bijection since both $(a_1, \ldots, a_{m_k})$ and $(b_1, \ldots, b_{m_k})$ are permutations of $[n]$. By the earlier proposition, τ conjugates σ to σ'. □

The above two corollaries put together gives us:

Theorem 0.35 (Description of conjugacy classes)**.** The conjugacy classes of S_n consist precisely of permutations of the same cycle type.

Remark 0.36. We have assumed that every permutation does have a (unique, up to ordering) disjoint cycle decomposition.

0.2.4 Group actions

Definition 0.37. An action of a group G on a (finite) set X is a homomorphism $\sigma : G \to S_X$. We often write σ_g for $\sigma(g)$. The cardinality of X is called the degree of the action.

For $g \in G$ and $x \in X$, we often denote $\sigma_g(x)$ by $g \cdot x$.

Remark 0.38. We shall implicitly assume that $|X| \neq 2$ from hereon, even though the definition doesn't explicitly demand that. Note that S_X would be the trivial group if $|X| = 0, 1$ and there isn't much to discuss about that.

Remark 0.39. In the more suggestive notation $g \cdot x$ for $\sigma_g(x)$, we get the following identities for all $g_1, g_2 \in G$ and $x \in X$:

1. $1 \cdot x = x$,
2. $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

Both follow from the fact that σ is a homomorphism and hence, $\sigma_1 = \mathrm{id}_X$ and $\sigma_{g_1 g_2} = \sigma_{g_1} \circ \sigma_{g_2}$.

At this point, it would be educational to recall Remark 1.4.

Definition 0.40. Let $\sigma : G \to S_X$ be a group action. Then orbit of $x \in X$ under G is the set

$$G \cdot x = \{\sigma_g(x) \mid g \in G\} = \{g \cdot x \mid g \in G\}.$$

Proposition 0.41. The orbits form a partition of X.

Proof. We define the relation $\sim$ on X by $x_1 \sim x_2$ iff there exists $g \in G$ such that $\sigma_g(x_1) =$

x_2.

From the definition, it is clear that $G \cdot x$ is simply the collection of all those $y \in X$ such that $x \sim y$. Thus, to prove the proposition, it suffices to prove that $\sim$ is an equivalence relation.

1. (Reflexive) Note that $1 \cdot x = x$ for all $x \in X$.
2. (Symmetric) Note that $g \cdot x = y \implies x = g^{-1} \cdot y$ for all $x, y \in X$ and $g \in G$.
3. (Transitive) Let $x, y, z \in X$ and $g_1, g_2 \in G$ be such that

$$g_1 \cdot x = y \quad \text{and} \quad g_2 \cdot y = z.$$

Then,

$$(g_2 g_1) \cdot x = z.$$

□

Definition 0.42. A group action $\sigma : G \to S_X$ is said to be transitive if there is a unique orbit.

Remark 0.43. By the earlier proposition, it is clear that the above definition is equivalent to saying that given any $x, y \in X$, there exists $g \in G$ such that $\sigma_g(x) = y$.

Example 0.44 (Regular action)**.** Let G be a group and consider $X = G$. Then, G acts on X by left multiplication. That is, $\lambda : G \to S_X$ defined by

$$\lambda_g(x) = gx$$

for all $g, x \in G$ is a group action.

This is a transitive action as can be easily verified.

Remark 0.45. Recall Definition 2.54 from earlier. The "Regular" there and in the previous example are indeed related. This will be made more precise in the next subsection.

Example 0.46 (Coset action)**.** Let G be a group and H a (not necessarily normal) subgroup. Let G/H be the set of all *left* cosets of H. Then, G acts on G/H by left multiplication. That is, $\sigma : G \to S_{G/H}$ defined by

$$\sigma_g(C) = gC$$

for all $g \in G$ and $c \in G/H$ is a group action.

Let us check this. Note that for $g_1, g_2 \in C$, we have

$$\begin{aligned}\sigma_{g_1g_2}(C) &= (g_1g_2)C \\ &= \{g_1g_2c \mid c \in C\} \\ &= g_1\{g_2c \mid c \in C\} \\ &= g_1(g_2C) \\ &= \sigma_{g_1}(\sigma_{g_2}(C)),\end{aligned}$$

proving that

$$\sigma_{g_1g_2} = \sigma_{g_1} \cdot \sigma_{g_2}.$$

Using the fact that $\sigma_1 = \mathrm{id}_{G/H}$, we see that each σ_g is a bijection and that σ is a homomorphism.

Moreover, this a transitive. Indeed, given any two cosets $x_1H, x_2H \in G/H$, we see that $g = x_2x_1^{-1}$ satisfies

$$\sigma_g(x_1H) = x_2H.$$

Definition 0.47. An action $\sigma : G \to S_X$ on X is 2-transitive if given any two pairs of <u>distinct</u> elements $(x, y) \in X^2$ and $(x', y') \in X^2$, there exists $g \in G$ such that

$$\sigma_g(x) = x' \quad \text{and} \quad \sigma_g(y) = y'.$$

Note that the “distinct” above means that $x \neq y$ and $x' \neq y'$.

Proposition 0.48. A 2-transitive action is transitive.

Proof. Let $x, y \in X$ be arbitrary. We wish to show that there exists $g \in G$ such that $\sigma_g(x) = y$. Note that since $\sigma_1(x) = x$, we may assume $x \neq y$.

Put $(x', y') = (y, x)$. By 2-transitivity, there exists $g \in G$ such that

$$\sigma_g(x) = x' = y,$$

as desired. □

Example 0.49 (Action of D_4)**.** The converse of the above is not true. Consider the action of D_4 on the four vertices of a square. It is easy to see that his action is transitive.

Label the vertices $1, \ldots, 4$. Any $g \in D_4$ takes opposite vertices to opposite vertices. Thus, considering the pairs

$$(x, y) = (1, 3) \quad \text{and} \quad (x', y') = (2, 3)$$

shows that the action is not 2-transitive.

Example 0.50 (Action of symmetric groups)**.** As before, there's a natural action of S_n on $X := \{1, \ldots, n\}$.

To be more explicit, we define $\tau \cdot i = \tau(i)$ for all $\tau \in S_n$ and $i \in X$.

For $n \geq 2$, this action is 2-transitive. Indeed, let $i \neq j$ and $i' \neq j'$ be elements in X. Define

$$Y_1 := X \setminus \{i, j\} \quad \text{and} \quad Y_2 := X \setminus \{i', j'\}.$$

Since $|Y_1| = |Y_2|$, there exists a bijection $\alpha : Y_1 \to Y_2$. Define $\tau \in S_n$ by

$$\tau(k) = \begin{cases} i' & k = i, \\ j' & k = j, \\ \alpha(k) & \text{otherwise.} \end{cases}$$

The above is an element of S_n precisely because $i \neq j$ and $i' \neq j'$. Noting that $\tau \cdot i = i'$ and $\tau \cdot j = j'$ establishes that the action is 2-transitive.

In terms of cycles, we can see that τ is simply $(ii')(jj')$, assuming that all four are distinct. One can take different cases considering $i = i'$ and so on to explicitly get a cycle representation in each case.

Definition 0.51. Let $\sigma : G \to S_X$ be a group action. Define $\sigma^2 : G \to S_{X \times X}$ by

$$\sigma^2_g(x_1, x_2) = (\sigma_g(x_1), \sigma_g(x_2)).$$

This is a group action of G on $S \times S$. An orbit of σ^2 is called an orbital of σ. The number of orbitals is called the rank of σ.

Remark 0.52. Let $\Delta = \{(x, x) \mid x \in X\}$. Note that

$$\sigma^2_g(x, x) = (\sigma_g(x), \sigma_g(x)) \in \Delta.$$

That is Δ is *closed* under the action of σ^2. Moreover, Δ is an orbital iff σ is transitive.

Remark 0.53. Note that σ being 2-transitive is precisely the same as saying that

$$X^2 \setminus \Delta = \{(x, y) \in X \times X \mid x \neq y\}$$

is an orbital.

Proposition 0.54. Let $\sigma : G \to S_X$ be a group action (with $|X| \geq 2$). Then, σ is 2-transitive if and only if σ is transitive with $\operatorname{rank}(\sigma) = 2$.

Proof. Assume that σ is 2-transitive. By Proposition 0.48, it follows that σ is transitive. By the earlier remarks, we see that Δ and $X^2 \setminus \Delta$ are (distinct) orbitals. Since their union is X^2, it follows that $\operatorname{rank}(\sigma) = 2$.

Conversely, suppose that σ is transitive and $\operatorname{rank}(\sigma) = 2$. Since Δ is an orbital and orbitals partition $X \times X$ (Proposition 0.41), it follows that $X^2 \setminus \Delta$ is the other orbital. As before, this is precisely saying that σ is 2-transitive. $\square$

In the above, we $|X| \geq 2$ was implicitly used in asserting that $X^2 \setminus \Delta$ is nonempty.

Remark 0.55. The proof also shows that the rank is at least 2, whenever $|X| > 1$, regardless of σ being transitive.

Example 0.56 (Rank of D_4)**.** As noted earlier, the action of D_4 on $\{1, \ldots, 4\}$ is not 2-transitive. Let us now compute the rank. Since the action is transitive, we know that Δ is an orbital.

We now see how to partition $X^2 \setminus \Delta$ into orbitals. Note that if $(i, j) \in X^2 \setminus \Delta$, then $i \neq j$. There are precisely two distinct possibilities: Either i and j are adjacent, or i and j are opposite.

It is easy to see that if (i, j) and (i', j') are both adjacent (resp. opposite) pairs of vertices, then they are in the same orbit. Moreover, as commented earlier, no opposite pair is in the orbit of any adjacent pair.

Thus, there are precisely three orbitals:

$$\begin{aligned}
\Delta &= \{(x, y) \in X \times X \mid x = y\}, \\
\mathcal{O}_{\text{opp}} &= \{(x, y) \in X \times X \mid x - y \equiv 2 \bmod 4\}, \\
\mathcal{O}_{\text{adj}} &= \{(x, y) \in X \times X \mid x - y \equiv 1 \bmod 2\}.
\end{aligned}$$

Example 0.57 (Rank of S_n)**.** As noted in Example 0.50, the (natural) action of S_n is 2-transitive if $n \geq 2$. Thus, it has rank 2.

Definition 0.58. Let $\sigma : G \to S_X$ be a group action. For $g \in G$, we define

$$\mathrm{Fix}(g) = \{x \in X \mid \sigma_g(x) = x\}$$

to be the set of fixed points of g. Let $\mathrm{Fix}^2(g)$ denote the set of fixed points of g for the action σ^2.

Note that $\mathrm{Fix}^2(g)$ could also possibly denote the Cartesian product of the set $\mathrm{Fix}(g)$ with itself. The following proposition states that this is unambiguous since the two are equal.

Proposition 0.59. Let $\sigma : G \to S_X$ be a group action. Then,

$$\mathrm{Fix}^2(g) = \mathrm{Fix}(g) \times \mathrm{Fix}(g).$$

In particular, $\left|\mathrm{Fix}^2(g)\right| = |\mathrm{Fix}(g)|^2$.

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}
(x, y) \in \mathrm{Fix}^2(g) &\iff \sigma_g^2(x, y) = (x, y) \\
&\iff (\sigma_g(x), \sigma_g(y)) = (x, y) \\
&\iff \sigma_g(x) = x \quad \text{and} \quad \sigma_y(y) = y \\
&\iff x \in \mathrm{Fix}(g) \quad \text{and} \quad y \in \mathrm{Fix}(g) \\
&\iff (x, y) \in \mathrm{Fix}(g) \times \mathrm{Fix}(g).
\end{aligned}$$

$\square$

§1. Group representations

§§1.1. Definition and Examples

Definition 1.1. A representation of a group G is a homomorphism $\varphi : G \to \mathrm{GL}(V)$ for some (finite-dimensional) vector space V. The dimension of V is called the *degree* of φ. We write φ_g for $\varphi(g)$ and $\varphi_g(v)$ or simply $\varphi_g v$, for the action of $\varphi_g \in \mathrm{GL}(V)$ on $v \in V$.

Remark 1.2. We shall implicitly assume that $V \neq 0$ from hereon, even though the definition doesn't explicitly demand that. Note that $\mathrm{GL}(V)$ would be the trivial group if $V = 0$ and there isn't much to discuss about that.

Remark 1.3. Since representation are homomorphisms, if a group G is generated by X, then a representation φ of G is determined by its values on X. Of course, one must keep in mind that not every assignment of values to X will actually determine a homomorphism.

Remark 1.4. Recall the group S_X which is the group of all bijections from X to itself. If we consider a vector space V, we see that $\mathrm{GL}(V)$ is a subgroup of S_V. Recall from basic algebra, the concept of group actions. One may define it (G acting on a set X) as a certain map satisfying some properties but one sees that it was simply equivalent to a group homomorphism $\varphi : G \to S_X$.

In this sense, we see that representations are special group actions where we don't just want φ_g to be *bijections* but also to be *linear*.

Example 1.5. Recall from Section 0.1.3, the concept of Linearisation. Given a set X, we can construct the $\mathbb{C}$-vector space $\mathbb{C}X$ with X as a basis.

Now given a group G which acts on X, we saw in Proposition 0.19 that the action can actually be extended to an action on $\mathbb{C}X$. Moreover, it has the property that each element acts linearly, i.e., we get a representation.

Note that if V is a $\mathbb{C}$ vector space of dimension 1, then $V \cong \mathbb{C}$ and $\mathrm{GL}(V) \cong \mathbb{C}^*$. For psychological reasons, we may sometimes use z instead of φ for a degree one representation, to remind us that φ_g is simply multiplication by a complex number.

Example 1.6 (Trivial representation)**.** The trivial representation of a group G is the homomorphism $z : G \to \mathbb{C}^*$ given by $z_g = 1$ for all $g \in G$.

This is a degree one representation.

Example 1.7 (Degree one representations of $\mathbb{Z}/n\mathbb{Z}$)**.** Given $n \in \mathbb{N}$, note that a homomorphism $z : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ is determined by mapping $[1]$ to an element $\zeta \in \mathbb{C}^*$ such that $\zeta^n = 1$. Thus, we must map $[1]$ to an n-th root of unity. One such example of a representation is

$$z([m]) := \omega_n^m.$$

This is again a degree one representation. Another such is

$$z([m]) := \omega_n^{-m}$$

In fact, for each $k = 0, \ldots, n-1$, we get a different degree one representation given by

$$z^{(k)}([m]) = \omega_n^{mk}.$$

We will eventually show that the above are the "only" representations (in some sense) of a finite cyclic group.

On the other hand, for $\mathbb{Z}$, we note that giving a homomorphism $z : \mathbb{Z} \to \mathbb{C}^*$ is the same as giving an element of $\mathbb{C}^*$. Thus, we have uncountably many distinct representations of $\mathbb{Z}$. Moreover, if z_1 is not some n-root of unity, then z will be a injection of $\mathbb{Z}$ into $\mathbb{C}^*$.

Soon, we shall define a concept of "equivalence" of representations. We shall then show that all the representations mentioned above are actually inequivalent.

Remark 1.8. Note that if G is a finite group and z a degree one representation, then $z : G \to \mathbb{C}^*$ is actually very restrictive. Note that we must have $z_g^{|G|} = 1$ and thus, φ actually maps into[1] the unit circle. In fact, it further maps into the subgroup which is the $|G|$-th roots of unity.

Example 1.9 (Degree one representations of S_n)**.** Note that we already have two obvious degree one representations of S_n. The first is the trivial one and the second is the sign homomorphism mapping each permutation to its sign. (Recall the definition Sign of a permutation and the fact it is a homomorphism, Corollary 0.28.)

We now show that these are all. Firstly, note that all transpositions are conjugates. (Recall Description of conjugacy classes.) Hence, if the kernel of a representation contains one transposition, it must contain all. (Since kernels are normal.)

Also note that transpositions generate S_n. Thus, any homomorphism is completely determined by its values on the transpositions.

Now, noting that any transposition has order 2, we see that it can only mapped to ± 1. If a single transposition is mapped to 1, then all are; this gives us the trivial representation. Thus, if the representation is non-trivial, then each transposition is mapped to -1. However, then it must agree with sign.

Example 1.10 (Degree one representations of non-abelian groups)**.** Let G be a group and let $z : G \to \mathbb{C}^*$ be a degree one representation. Noting that $\mathbb{C}^*$ is abelian, we see that the $\ker z$ must contain the (normal) commutator subgroup $[G, G]$. Thus, it must factor through the quotient as follows:

$$\begin{tikzcd} G \arrow[r, "z"] \arrow[d, two heads] & \mathbb{C}^* \\ G/[G,G] \arrow[ur, dashed, "\tilde{z}"'] & \end{tikzcd}$$

In other words, it then suffices to study the degree one representations of the abelian group $G/[G, G]$.

Example 1.11 (Determining conjugacy using representations)**.** Note that since φ is a homomorphism, the (images of the) relations that hold in G must also hold in $\mathrm{GL}(V)$. In particular, some relations which are easier to solve in $\mathrm{GL}(V)$ may help in solving those in G.

To give a specific example, consider the problem of having $x, y \in G$ and wanting to find $g \in G$ such that $gxg^{-1} = y$. A priori, there may not even be a way of deducing whether such a g exists. However, considering a representation φ, we can try to solve

$$\varphi_g \varphi_x \varphi_g^{-1} = \varphi_y.$$

Since similarity of matrices is better solvable (at least, in theory for up to degree 4 representations, we can find all eigenvalues of φ_x, φ_y) by Jordan form, we can hope to get a better answer. If φ_x and φ_y are *not* similar (even consideration of trace or determinant could possibly tell us that), we know for a fact that such a g cannot exist.

Now, if we see that they are similar, we can find a matrix M such that $M\varphi_x M^{-1} = \varphi_y$. Then, elements in the set $\varphi^{-1}(M)$ are good candidates for g. (Of course, it could be possible that that set is empty or that none of them actually work.)

We now make the following observation:
Let $\varphi : G \to \mathrm{GL}(V)$ be a representation of degree n.
Now, suppose that we have two bases B, B' of V. Corresponding to these, we get two

isomorphisms

$$T : V \to \mathbb{C}^n \quad \text{and} \quad S : V \to \mathbb{C}^n$$

mapping the basis elements to the standard basis vectors of $\mathbb{C}^n$. Now, we have two representations

$$\psi : \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{C}^n) \quad \text{and} \quad \psi' : \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{C}^n)$$

obtained by setting

$$\psi_g := T\varphi_g T^{-1} \quad \text{and} \quad \psi'_g := S\varphi_g S^{-1}.$$

Note that ψ and ψ' are related to each other by

$$\psi'_g = (ST^{-1})\psi_g(ST^{-1})^{-1}.$$

This resembles a "change of basis" and we would wish for φ, ψ, ψ' to be considered as the "same" representation. To this end, we have the following definition.

Definition 1.12. Two representations $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are said to be equivalent if there exists an isomorphism (called an equivalence) $T : V \to W$ such that

$$\psi_g = T\varphi_g T^{-1}$$

for all $g \in G$. In such a case, we write $\varphi \sim \psi$. Note that the above definition is saying that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"'] & W \end{tikzcd}$$

for all $g \in G$.

The above equivalence can easily checked to be an actual "equivalence relation."

Remark 1.13. Note that in the above, we have not assumed $V = W$. However, V and W must be *isomorphic*. In particular, φ and ψ have the same degree.

Example 1.14 (Equivalent degree two representations of $\mathbb{Z}/n\mathbb{Z}$). Define $\varphi, \psi : \mathbb{Z}/n\mathbb{Z} \to GL_2(\mathbb{C})$ by

$$\varphi_{[m]} = \begin{bmatrix} \cos\left(\dfrac{2\pi m}{n}\right) & -\sin\left(\dfrac{2\pi m}{n}\right) \\ \sin\left(\dfrac{2\pi m}{n}\right) & \cos\left(\dfrac{2\pi m}{n}\right) \end{bmatrix} \quad \text{and} \quad \psi_{[m]} = \begin{bmatrix} \omega_n^m & 0 \\ 0 & \omega_n^{-m} \end{bmatrix}.$$

Then, we have the automorphism of $\mathrm{GL}_2(\mathbb{C})$ induced by

$$A = \begin{bmatrix} \iota & -\iota \\ 1 & 1 \end{bmatrix}.$$

We have

$$A^{-1} = \frac{1}{2\iota}\begin{bmatrix} 1 & \iota \\ -1 & \iota \end{bmatrix}$$

and a direct computation shows that

$$A^{-1}\varphi_{[m]}A = \psi_{[m]}$$

for all $[m] \in \mathbb{Z}/n\mathbb{Z}$. Thus, we have $\varphi \sim \psi$.

Proposition 1.15. Let G be a group and $z, z' : G \to \mathbb{C}^*$ be degree one representations. Then, $z \sim z'$ iff $z = z'$.

Proof. Clearly, equality implies equivalence. We show the converse.

Let $T : \mathbb{C} \to \mathbb{C}$ be an isomorphism such that $z'_g = Tz_gT^{-1}$ for all $g \in G$. Then, for any $v \in \mathbb{C}$, we have

$$\begin{aligned} z'_g(v) &= T\left(z_gT^{-1}(v)\right) \\ &= z_gT\left(T^{-1}(v)\right) && \textit{T is linear} \\ &= z_gv. \end{aligned}$$

Thus, $z_g = z'_g$ for all $g \in G$ and hence, $z = z'$. □

Corollary 1.16. All the distinct representations in Example 1.7 are actually inequivalent.

Corollary 1.17. There are exactly n distinct inequivalent degree one representations of $\mathbb{Z}/n\mathbb{Z}$.

Proof. We had constructed n distinct (and hence, inequivalent) representations in Example 1.7.

To see that these are all, note that φ is completely determined once we define $\varphi_{[1]}$. Moreover, $\varphi_{[1]}^n$ must be 1, since φ is a homomorphism. That is, $\varphi_{[1]}$ must be an n-th root of unity. Thus, we have at most n homomorphisms. □

In fact, the above can be generalised to all finite abelian groups.

Corollary 1.18. Let G be a finite abelian group. There are exactly $|G|$ inequivalent degree one representations of G.

Proof. By Corollary 0.23, there are exactly $|G|$ homomorphisms from G to $\mathbb{C}^*$. By Proposition 1.15, these are all inequivalent as well and we are done. □

Example 1.19 (Degree one representations of non-abelian groups revisited)**.** As noted in Example 1.10, if we wish to study degree one representations of G, it suffices to study those of the *abelian* group $G' = G/[G, G]$. Now, if $G/[G, G]$ is finite (which is certainly the case if G is finite), then the above corollary tells us that there are exactly $|G'|$ many such representations. In fact, Corollary 0.23 actually tells us the description of the representations as well. In fact, there's something more that we know which we isolate as a corollary.

Corollary 1.20. Let G be a finite group. Then, the number of (distinct, inequivalent) degree one representations of G divides $|G|$.

Proof. As noted in Example 1.10, any degree one representation factors through $G/[G, G]$ and all degree one representations are obtained in precisely this way.

Thus, the number of degree one representations of G is that of $G/[G, G]$. Since $G/[G, G]$ is abelian, there are exactly $|G/[G, G]|$ many such. This number divides $|G|$, by elementary group theory. □

Example 1.21 (Standard representation of S_n)**.** We define $\varphi : S_n \to \mathrm{GL}_n(\mathbb{C})$ as follows: Given $\sigma \in S_n$, we define φ_σ to be the matrix representing the linear transformation determined by $e_i \mapsto e_{\sigma(i)}$.

One checks easily that this is indeed a homomorphism. One can verify that the matrix φ_σ is explicitly given by permuting the columns of the identity matrix according to σ. To be more explicit, the i-th columns of φ_σ will be the $\sigma(i)$-th column of the identity matrix. This is because we wish to map e_i to $e_{\sigma(i)}$.

As an example, for $n = 3$, we have

$$\varphi_{(123)} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

For the above example, note that

$$\begin{aligned}\varphi_\sigma(e_1 + \cdots + e_n) &= \varphi_\sigma(e_1) + \cdots + \varphi_\sigma(e_n)\\ &= e_{\sigma(1)} + \cdots + e_{\sigma(n)} \\ &= e_1 + \cdots + e_n \end{aligned} \quad \textit{since } \sigma \textit{ is a permutation}$$

Thus, the subspace $\mathbb{C}(e_1 + \cdots + e_n)$ is φ_σ-invariant for all $\sigma \in S_n$. This motivates the following definition.

Definition 1.22. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \leq V$ is said to be G-invariant if, for all $g \in G$ and $w \in W$, we have $\varphi_g(w) \in W$.

Remark 1.23. Note that the invariance depends on G as well the representation being considered. To emphasise on the representation at times, we may add "with respect to φ."

Proposition 1.24. If $W \leq V$ is a G-invariant subspace with respect to $\varphi : G \to \mathrm{GL}(V)$, then $\varphi|_W : G \to \mathrm{GL}(W)$ by setting $(\varphi|_W)_g(w) = \varphi_g(w)$ for $w \in W$ is a representation.

Proof. We first show that the $\varphi|_W$ so defined actually maps into $\mathrm{GL}(W)$. By the hypothesis that W is G-invariant, we get that

$$\varphi_g(w) \in W$$

for all $w \in W$ and thus, φ_g restricts to a function from W to W, for all $g \in G$.
The fact that this is linear follows from the fact that φ was a representation to begin with. Moreover, it is invertible since $\varphi_{g^{-1}}$ also restricts from W to W and thus, we have that $\varphi_g \in \mathrm{GL}(W)$ for all $g \in G$. This shows that $\varphi|_W$ is actually a function from G to $\mathrm{GL}(W)$.

The fact that it is a homomorphism follows from the fact that φ was one to begin with. □

Definition 1.25. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. If $W \leq V$ is a G-invariant subspace, then $\varphi|_W : G \to \mathrm{GL}(W)$ is again a representation and we call $\varphi|_W$ is a subrepresentation of φ.

Going back to Example 1.14, it is easy to note that $\mathbb{C}e_1$ and $\mathbb{C}e_2$ are $\mathbb{Z}/n\mathbb{Z}$-invariant subspaces with respect to ψ (and not φ!). Moreover, we have that $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. This motivates the following.

Definition 1.26. Let $\varphi^{(1)} : G \to \mathrm{GL}(V_1)$ and $\varphi^{(2)} : G \to \mathrm{GL}(V_2)$ be representations. Then, their (external) direct sum is the representation

$$\varphi^{(1)} \oplus \varphi^{(2)} : G \to \mathrm{GL}(V_1 \oplus V_2)$$

given by

$$\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g (v_1, v_2) = \left(\varphi_g^{(1)}(v_1), \varphi_g^{(2)}(v_2)\right)$$

for all $g \in G$ and for all $(v_1, v_2) \in V_1 \oplus V_2$.

Note that in the above, we are using the tuple notation for representing the (external) direct sum of the vector spaces V_1 and V_2. This sum can be visualised naturally in terms of matrices.
If $\varphi^{(1)} : G \to \mathrm{GL}_n(\mathbb{C})$ and $\varphi^{(2)} : G \to \mathrm{GL}_m(\mathbb{C})$ are representations, then each $\varphi_g^{(i)}$ is a matrix. Then, the matrix $\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g \in \mathrm{GL}_{n+m}(\mathbb{C})$ is given as the block matrix

$$\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g = \begin{bmatrix} \varphi_g^{(1)} & \\ & \varphi_g^{(2)} \end{bmatrix}.$$

(The empty places are 0 matrices of appropriate sizes.)

Example 1.27. The two representations from Example 1.7 have their (external) direct sum as ψ from Example 1.14.

A slightly less direct (but simple) calculation shows that

$$\mathbb{C}\begin{bmatrix} \iota \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbb{C}\begin{bmatrix} -\iota \\ 1 \end{bmatrix}$$

are also $\mathbb{Z}/n\mathbb{Z}$-invariant subspaces with respect to φ. (Note that are just the columns of A in the example, multiplied by a scalar.)

Thus, φ can also be written as a sum of subrepresentations. This should not be surprising as one would expect that equivalent representations behave similarly in this aspect. This will be made more precise and proven at the end of this section.

Example 1.28. The representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ given by $\rho_g = I_n$ for all $g \in G$ is equivalent to the direct sum of n copies of the Trivial representation. Note that if $n > 1$, then it is *not* equivalent to the trivial representation since the degrees are different.

Example 1.29. Let $\rho : S_3 \to \mathrm{GL}_2(\mathbb{C})$ be specified on the generators by

$$\rho_{(12)} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \rho_{(123)} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

(It must be checked that this defined a representation. We do this at the end.)

Let $\psi : S_3 \to \mathbb{C}^* \cong \mathrm{GL}_1(\mathbb{C})$ be the trivial representation, i.e., $\psi_g = 1$. Then, we have the representation $\rho \oplus \psi$ which is specified on the generators by

$$(\rho \oplus \psi)_{(12)} = \begin{bmatrix} -1 & 1 & \\ 0 & 1 & \\ & & 1 \end{bmatrix}, \quad (\rho \oplus \psi)_{(123)} = \begin{bmatrix} -1 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix}.$$

We shall see later that $\rho \oplus \psi$ is equivalent to the standard representation as considered in Example 1.21.

To check that ρ actually gives us a representation (group homomorphism), we must verify that the matrices satisfy the relations that the generators satisfy. That is,

$$\rho_{(12)}^2 = I_2, \ \rho_{(123)}^3 = I_2, \ \rho_{(12)}\rho_{(123)} = \rho_{(123)}^2\rho_{(12)}.$$

(We are using the fact from group theory that the above relations completely determine S_3.)

One can compute and see that the above relations do hold.

Proposition 1.30. If $V_1, V_2 \leq V$ are G-invariant subspaces with respect to φ and $V = V_1 \oplus V_2$, then φ is equivalent to the (external) direct sum $\varphi|_{V_1} \oplus \varphi|_{V_2}$.

Proof. Let $T : V \to V_1 \oplus V_2$ be the natural map $v_1 + v_2 \mapsto (v_1, v_2)$. (Here we are considering the external direct sum of the vector spaces.)
This map is well-defined and an isomorphism because V is the (internal) direct sum of V_1 and V_2.

Now, put $\psi = \varphi|_{V_1} \oplus \varphi|_{V_2}$. Then, for any $g \in G$, we have

$$\begin{aligned}
\psi_g(v_1, v_2) &= \left((\varphi|_{V_1})_g (v_1), (\varphi|_{V_2})_g (v_2) \right) \\
&= (\varphi_g(v_1), \varphi_g(v_2)) && \text{\textit{since } } V_1, V_2 \text{ \textit{are} } G\text{\textit{-invariant}} \\
&= T(\varphi_g(v_1) + \varphi_g(v_2)) && \varphi_g \text{ \textit{is a linear map}} \\
&= T(\varphi_g(v_1 + v_2)) \\
&= T(\varphi_g(T^{-1}(v_1, v_2))),
\end{aligned}$$

showing that

$$\psi_g = T\varphi_g T^{-1},$$

as desired. □

The above can also be visualised in terms of matrices. Let B_i be a basis for V_i. Then, $B := B_1 \cup B_2$ is a basis for V (since V is the internal direct sum of the V_i). Since V_i is G-invariant, we see that $\varphi_g(B_i) \subset \mathbb{C}B_i$. Thus, the matrix representation with respect to B is as follows:

$$[\varphi_g]_B = \begin{bmatrix} \left[\varphi^{(1)}\right]_{B_1} & \\ & \left[\varphi^{(2)}\right]_{B_2} \end{bmatrix}.$$

One who has studied algebra would be familiar with the idea of breaking down structures into simpler "irreducibles" (similar to the prime factorisation of an integer). To such a reader, the following definition should not come as a surprise.

Definition 1.31. A non-zero representation $\varphi : G \to \mathrm{GL}(V)$ of a group G is said to be irreducible if the only G-invariant subspaces of V are 0 and V.

In the above, 0 refers to the 0 *subspace*, i.e., $\{0\}$.

Example 1.32. Any degree one representation is irreducible since there is no non-trivial proper subspace of a dimension 1 vector space.

Example 1.33. If $G = \{1\}$, the trivial group, then the only irreducible representation is a degree one representation. (In other words, the converse of the previous example holds too.)

Indeed, the only representation $\varphi : G \to \mathrm{GL}(V)$ is $\varphi_1 = I$ and thus, every subspace of V is a G-invariant subspace.

Remark 1.34. Note that in the above case, we actually have that the representation is actually a direct sum of subrepresentations. However, irreducibility does not demand that. The reader can see this happening in Example 1.62.

However, we will soon show that the above is actually true when the group is finite.

Example 1.35 (Irreducible representations of dihedral type groups)**.** Let G be a finite group with generators a and b. (By hypothesis, a and b have finite order.) Suppose further that every element of G can be written as $a^i b^j$ for some non-negative integers i and j. (Note that given any $g \in G$, g^{-1} can be written as $a^i b^j$ and hence, $g = b^{-j}a^{-i}$. Using that a and b have finite orders, we can actually write every element of G as $b^{j'}a^{i'}$ for non-negative integers as well.)

By the parenthetical remark, we can assume without loss of generality that $|a| \leq |b|$. ($|g|$ denotes the order of $g \in G$.)

Let $n := |a|$. We show that any irreducible representation of G has order at most n.

To this end, let $\varphi : G \to \mathrm{GL}(V)$ be an irreducible representation and let v be an eigenvector of φ_b. Consider the following subspace W of V given by

$$W := \langle v, \varphi_a v, \ldots, \varphi_{a^{n-1}} v \rangle.$$

Clearly, $0 < \dim W \leq n$. We show that W is G-invariant. Then, since φ is irreducible, it would follow that $V = W$, proving our claim.

By hypothesis, an arbitrary element of G can be written as $a^i b^j$. Pick an arbitrary element of the spanning set given above for W. It is of the form $\varphi_{a^k} v$ for some $0 \leq k \leq n-1$. It suffices to show that

$$\varphi_{a^i b^j}\left(\varphi_{a^k} v\right) \in W.$$

Note that, by hypothesis, $a^i b^j a^k = a^p b^q$ for some non-negative integers p and q. Since $n = |a|$, we may assume that $p < n$. Since φ is a homomorphism, we get that

$$\varphi_{a^i b^j}\left(\varphi_{a^k} v\right) = \varphi_{a^p}\left(\varphi_{b^q} v\right).$$

Since v is an eigenvector of φ_b, we see that $\varphi_{b^q} v$ is some linear multiple of v, say λv. Then, the right side of the above equation becomes

$$\varphi_{a^p}\left(\varphi_{b^q} v\right) = \varphi_{a^p}\left(\lambda v\right) = \lambda \varphi_{a^p}(v) \in W,$$

as desired.

Example 1.36 (Irreducible representations of dihedral groups)**.** Consider the dihedral group D_n with r denoting a rotation and s a reflection. Then, the hypothesis of the previous example applies with $a = s$ and hence, $n = 2$. This tells us that every irreducible representation of D_n has degree at most two.

We now see when are degree 2 representations irreducible.

Proposition 1.37. If $\varphi : G \to \mathrm{GL}(V)$ is a degree 2 representation, then φ is irreducible if and only if there is no common eigenvector v to all φ_g with $g \in G$.

Proof. One direction is easy. Suppose that $v \in V$ is such that v is an eigenvector of φ_g for all $g \in G$. In that case, $\mathbb{C}v$ is a one dimensional G-invariant subspace of V and hence, is proper and non-trivial. (Recall that eigenvectors are non-zero, by definition.)

Now, suppose the converse. Let W be a proper non-trivial G-invariant subspace of V. Then, $W = \mathbb{C}v$ for some $0 \neq v \in V$. Then, given any $g \in G$, we have that

$$\varphi_g v \in W$$

and hence, $\varphi_g v = \lambda_g v$ for some $\lambda_g \in \mathbb{C}$. This shows that v is an eigenvector for all φ_g. (Since it was non-zero to begin with.) □

Remark 1.38. For finite groups, the above proposition can also be generalised to degree three representations, using an almost identical proof. The only extra ingredient required is that if a representation of a finite group is reducible, then we can actually write $V = W \oplus W'$ for non-zero G-invariant subspaces.

For infinite groups, the above proposition does not generalise to degree three representation, as seen in Example 1.41.

The above does not generalise to degree four representations, even in the case of finite groups. This is seen in Example 1.63.

Example 1.39. The representation $\rho : S_3 \to \mathrm{GL}_2(\mathbb{C})$ in Example 1.29 is irreducible.

We show this by showing that no eigenvector of $\rho_{(12)}$ is also an eigenvector of $\rho_{(123)}$. (That is, they have no common eigenvectors.) Then, we are done, by the above proposition.

To this end, we first compute the eigenvalues of $\rho_{(12)}$ to be ± 1. Corresponding to these, we get the eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. (Note that since the eigenvalues are distinct, any other eigenvector must be a scalar multiple (as opposed to a linear combination) of either of these.)

A direct computation gives us that neither is an eigenvector of $\rho_{(123)}$. Indeed, we have

$$\rho_{(123)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \rho_{(123)} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Example 1.40 (An irreducible representation of D_4)**.** Consider the group D_4. Let r be rotation by $\pi/2$ and s be a reflection about a perpendicular bisector of a side. We know that

$$D_4 = \langle r, s \mid r^4 = s^2 = rsrs^{-1} = 1 \rangle.$$

Using the above, we see that the following is a representation:

$$\varphi(r) := \begin{bmatrix} \iota & \\ & -\iota \end{bmatrix} \quad \text{and} \quad \varphi(s) := \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

Clearly, the only eigenvectors of $\varphi(r)$ (up to scaling) are e_1 and e_2, neither of which is an eigenvector of $\varphi(s)$. Thus, φ is irreducible.

Example 1.41. We now show that Proposition 1.37 is not true for degree three representations when the group is infinite.
Let $G := F(a, b)$ be the free group on two generators a and b. (Recall that a homomorphism from G to any group is defined uniquely by specifying its values on a and b.)

Consider the representation $\varphi : G \to \mathrm{GL}_3(\mathbb{C})$ defined by

$$\varphi_a := \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} \quad \text{and} \quad \varphi_b := \begin{bmatrix} & 1 & 1 \\ 1 & & 1 \\ & & 1 \end{bmatrix}.$$

Note that φ_a and φ_b are indeed elements of $\mathrm{GL}_3(\mathbb{C})$ as can be checked by noting that they both have nonzero determinant. Thus, the above defines a representation.

Claim 1. φ_a and φ_b have no common eigenvector. In particular, there is no $v \in \mathbb{C}^3$ which is a common eigenvector for all $\{\varphi_g\}_{g \in G}$.

Proof. This is simple for the only eigenvectors of φ_a (up to scaling) are e_1, e_2, and e_3. Clearly, none of them is an eigenvector of φ_b. □

Claim 2. $W = \mathbb{C}\{e_1, e_2\}$ is a G-invariant subspace.

Proof. Clearly, W is φ_a and φ_b-invariant. By Proposition 0.3, it follows that it is also $\varphi_{a^{-1}} = (\varphi_a)^{-1}$ and $\varphi_{b^{-1}}$ invariant.

Since any element $g \in G$ is a product of positive powers of a, b, a^{-1}, b^{-1}, it follows that W is φ_g-invariant, by Proposition 0.4. □

Thus, we have an example of a degree three representation which is reducible but there's no common eigenvector.

Similar to irreducible representations, we define some terms which the reader should find natural.

Definition 1.42. Let G be a group. A representation $\varphi : G \to \mathrm{GL}(V)$ is said to completely reducible

Remark 1.43. In view of Proposition 1.30, φ is completely reducible is equivalent to saying that $\varphi \sim \varphi^{(1)} \oplus \cdots \oplus \varphi^{(n)}$ for some irreducible representations $\varphi^{(i)}$.

Remark 1.44. As funny as it may seem, an irreducible representation *is* completely reducible. We did not demand for the V_is to be proper subspaces of V.

The above is similar to a sort of prime factorisation or diagonalisation. Our eventual goal is to show that any representation of a finite group is completely reducible. Thus, one can then just study irreducible representations.

Definition 1.45. A non-zero representation φ is said to be decomposable if $V = V_1 \oplus V_2$ for some non-zero G-invariant subspaces $V_1, V_2 \leq V$. Otherwise, V is said to be indecomposable.

Note that the above is, a priori, stronger than saying that φ is irreducible. However, we shall see later that the two coincide for when G is finite.

We now wish to show irreducible, completely reducible, and decomposability are actually notions that depend on the equivalence class of the representation. To this end, we first prove the following lemma.

Lemma 1.46. Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be equivalent representations and let $T : V \to W$ be an isomorphism such that the desired diagram commutes. If $V_1 \leq V$ is G-invariant, then so is $W_1 := T(V_1) \leq W$.

Proof. Let $w \in W_1$ and let $g \in G$. Then, we have

$$\psi_g = T\varphi_g T^{-1}.$$

Note that $T^{-1}w \in V_1$ and thus, $\varphi_g T^{-1}w \in V_1$ since V_1 is T-invariant. In turn, we get that

$$\psi_g w = T\varphi_g T^{-1} \in T(V_1) = W_1,$$

as desired. □

For the following three propositions, let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be equivalent representations and let $T : V \to W$ be an isomorphism such that the desired diagram commutes.

Proposition 1.47. ψ is irreducible if φ is so.

Proof. Let $V_1 \leq V$ be a G-invariant subspace which is non-zero and proper. Then, $W_1 := T(V_1)$ is non-zero and proper since T is an isomorphism. By Lemma 1.46, this is also G-invariant and we are done. □

Proposition 1.48. ψ is decomposable if φ is so.

Proof. If $V = V_1 \oplus V_2$ for non-zero subspaces, then $W = T(V_1) \oplus T(V_2)$ (with $T(V_1) \neq 0 \neq T(V_2)$) since T is an isomorphism. If V_1, V_2 are G-invariant, then so are $T(V_1)$ and $T(V_2)$, by Lemma 1.46. □

Proposition 1.49. ψ is completely reducible if φ is so.

Proof. By a similar argument as earlier, we see that if

$$V = V_1 \oplus \cdots \oplus V_n,$$

then

$$W = W_1 \oplus \cdots \oplus W_n,$$

where $W_i := T(V_i)$ and each subspace on the right is G-invariant.

We now wish to show that if $\varphi|_{V_i}$ is irreducible, then $\psi|_{W_i}$ is too. This is simple because we note that the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} V_i \arrow[r, "\varphi_g|_{V_i}"] \arrow[d, "T|_{V_i}"'] & V_i \arrow[d, "T|_{V_i}"] \\ W_i \arrow[r, "\psi_g|_{W_i}"'] & W_i \end{tikzcd}$$

and thus, $\varphi|_{V_i} \sim \psi|_{W_i}$. (Note that $T|_{V_i}$ is indeed an isomorphism.) Thus, by Proposition 1.47, we are done □

Theorem 1.50 (Irreducible representations of finite cyclic groups)**.** Let G be a finite cyclic group. All irreducible representations of G are of degree one.

Proof. Without loss of generality, we may assume $G = \mathbb{Z}/n\mathbb{Z}$. Suppose that $\varphi : G \to \mathrm{GL}_m(\mathbb{C})$ is a representation with $m \geq 2$. We show that it is reducible.

Note that $\varphi_{[1]}^n = I$. Thus, the minimal polynomial of $\varphi_{[1]}$ is a factor of $X^n - 1$ and hence, has distinct roots. This shows that $\varphi_{[1]}$ is diagonalisable. (Theorem 0.16)

Let $T \in \mathrm{GL}_m(\mathbb{C})$ be such that

$$T\varphi_{[1]}T^{-1} = D$$

for some diagonal matrix D. Note that raising both sides to the power k yields

$$T\varphi_{[1]}^k T^{-1} = D^k$$

or

$$T\varphi_{[k]}T^{-1} = D^k.$$

In other words, the equivalent representation $\psi : G \to \mathrm{GL}_m(\mathbb{C})$ given by $\psi_{[k]} = T\varphi_{[k]}T^{-1}$ has the property that $\psi_{[k]}$ is diagonal for all $[k] \in G$.

This shows that ψ can be decomposed as m non-zero proper sub-representations, proving reducibility. As a consequence, φ is reducible. $\square$

In the above, we used the fact from Linear Algebra that if the minimal polynomial of a matrix has distinct roots, then it is diagonalisable. In the next section, we shall prove the above theorem again without the fact and in turn, get the above fact as a corollary. (Note that there is no circular reasoning.)

§§1.2. Maschke's Theorem and Complete Reducibility

We recall the following definitions from linear algebra.

Definition 1.51. Let V be an inner product space. A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be unitary if $\varphi_g \in U(V)$ for all $g \in G$.

In other words, we can view φ as a map $\varphi : G \to U(V)$. In yet other words, we have

$$\langle \varphi_g v, \varphi_g w \rangle = \langle v, w \rangle$$

for all $g \in G$ and all $v, w \in V$.

Definition 1.52 (Unit circle)**.** We define $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Identifying $\mathrm{GL}_1(\mathbb{C})$ with $\mathbb{C}^*$, we see that $U_1(\mathbb{C})$ is identified with S^1. Hence, a degree-one unitary representation is a homomorphism $\varphi : G \to S^1$.

Example 1.53. $\varphi : \mathbb{R} \to S^1$ given by $t \mapsto \exp(2\pi\iota t)$ is a degree one-unitary representation of the additive group $(\mathbb{R}, +)$ since $\varphi(s+t) = \varphi(s)\varphi(t)$.

As we had noted earlier, decomposability was a stronger statement than reducibility. Now, we show that the two coincide for unitary representations.

Proposition 1.54. Let $\varphi : G \to \mathrm{GL}(V)$ be a unitary representation. Then, φ is either irreducible or decomposable.

Proof. Suppose that φ is not irreducible. Then, there exists a non-zero proper subspace $W \leq V$ which is G-invariant. Then, we have $V = W \oplus W^\perp$ and $W^\perp$ is non-zero proper. Thus, it now suffices to show that $W^\perp$ is G-invariant.

Now, given any $g \in G$, we know that φ_g is unitary and W is φ_g-invariant. Thus, by Corollary 0.9, we see that $W^\perp$ is φ_g-invariant. Since this is true for all $g \in G$, we see that $W^\perp$ is G-invariant, as desired. $\square$

Now, we show that for finite groups, every representation is equivalent to a unitary representation and thus, conclude that decomposable and reducible are equivalent for finite groups. To make the final proof simpler, we first state two lemmata.

Lemma 1.55. Let G be a finite group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^n$. Define the new product $(\cdot, \cdot)$ on $\mathbb{C}^n$ as

$$(v, w) := \sum_{g \in G} \langle \rho_g v, \rho_g w \rangle.$$

Then, $(\cdot, \cdot)$ is an inner product.

Note that the finiteness of G tells us that the above sum is well-defined. (Of course, along with the fact that addition is commutative.)

Proof. Let $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w, w \in \mathbb{C}^n$ be arbitrary.

First, note

$$\begin{aligned}
(c_1v_1 + c_2v_2, w) &= \sum_{g\in G} \langle \rho_g(c_1v_1 + c_2v_2), \rho_g w\rangle \\
&= \sum_{g\in G} \langle c_1\rho_g v_1 + c_2\rho_g v_2, \rho_g w\rangle \\
&= \sum_{g\in G} \left[c_1\langle \rho_g v_1, \rho_g w\rangle + c_2\langle \rho_g v_2, \rho_g w\rangle\right] \\
&= c_1 \sum_{g\in G} \langle \rho_g v_1, \rho_g w\rangle + c_2 \sum_{g\in G} \langle \rho_g v_2, \rho_g w\rangle \\
&= c_1(v_1, w) + c_2(v_2, w).
\end{aligned}$$

Next,

$$\begin{aligned}
(w, v) &= \sum_{g\in G} \langle \rho_g w, \rho_g v\rangle \\
&= \sum_{g\in G} \overline{\langle \rho_g v, \rho_g w\rangle} \\
&= \overline{\sum_{g\in G} \langle \rho_g v, \rho_g w\rangle} \\
&= \overline{(v, w)}.
\end{aligned}$$

Lastly,

$$(v, v) = \sum_{g\in G} \langle \rho_g v, \rho_g v\rangle \geq 0$$

since each term is non-negative and hence,

$$(v, v) = 0 \implies \langle \rho_g v, \rho_g v\rangle = 0 \text{ for all } g \in G$$

and thus, $\rho_g v = 0$ for all $g \in G$ since $\langle \cdot, \cdot \rangle$ is an inner-product.
In particular, $v = \rho_1 v = 0$, as desired. ☐

Lemma 1.56. With the same notation as in Lemma 1.55, we have that ρ is unitary with respect to the inner product $(\cdot, \cdot)$.

Proof. Let $v, w \in V$ and $g \in G$. Then,

$$\begin{aligned}
(\rho_g v, \rho_g w) &= \sum_{g'\in G} \langle \rho_{g'}\rho_g v, \rho_{g'}\rho_g w\rangle \\
&= \sum_{g'\in G} \langle \rho_{g'g} v, \rho_{g'g} w\rangle.
\end{aligned}$$

Note that $g' \mapsto g'g$ is a bijection and thus, the above is simplified as

$$(\rho_g v, \rho_g w) = \sum_{h\in G} \langle \rho_h v, \rho_h w\rangle = (v, w),$$

as desired. ☐

Note that in the previous two lemmata, we worked in $\mathrm{GL}_n(\mathbb{C})$ and used the standard inner product on $\mathbb{C}^n$. However, this was just for the sake of concreteness. Instead of which, we could've worked with any inner product space $(V, \langle \cdot, \cdot \rangle)$.

Proposition 1.57. Every representation of a finite group G is equivalent to a unitary representation.

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation and let $n := \dim V$. Fix an isomorphism $T : V \to \mathbb{C}^n$ and put $\rho_g := T\varphi_g T^{-1}$ for all $g \in G$. This defines a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ which is equivalent to φ. We now show that ρ is unitary.

Let $(\cdot, \cdot)$ be the inner product as in Lemma 1.55. Then, by Lemma 1.56, we know that ρ is a unitary representation and we are done. □

We now state the corollary alluded all along.

Corollary 1.58. Let $\varphi : G \to \mathrm{GL}(V)$ be a non-zero representation of a finite group. Then, φ is either irreducible or decomposable.

Proof. By Proposition 1.57, $\varphi \sim \rho$ for some unitary representation ρ. By Proposition 1.54, ρ is either irreducible or decomposable. By Propositions 1.47 to 1.48, we see that the same is true for φ as well. □

Remark 1.59. For any group, we obviously have that decomposable $\implies$ reducible. The above says that the converse is true for finite groups.

What the above says that if we have a G-invariant subspace W, then we can actually decompose V as $W_1 \oplus W_2$ (with them having the usual properties). In fact, our proof of Proposition 1.54 actually shows that we can take $W_1 = W$ and W_2 is then the orthogonal subspace (after suitably finding an isomorphism which transports the inner product structure).

With the above remark in mind, we rewrite the previous corollary as follows.

Corollary 1.60. Let $\varphi : G \to \mathrm{GL}(V)$ be a non-zero representation of a finite group. Suppose that W is a non-zero proper G-invariant subspace of V. Then, we can write

$$V = W \oplus W'$$

for a G-invariant subspace W'. (It follows that W' is also non-zero and proper.)

With the above, we can strengthen Proposition 1.37 to degree 3 representations as well when G is finite.

Proposition 1.61. If $\varphi : G \to \mathrm{GL}(V)$ is a degree 3 representation, then φ is reducible if and only if there is a common eigenvector v to all φ_g with $g \in G$.

Proof. As before, $\Longleftarrow$ is obvious. (That is true for all groups and all non-zero degree representation, in fact.)

We show the other direction. Suppose that φ is reducible. Then, by Corollary 1.58, φ is decomposable and we can write

$$V = W \oplus W'$$

for non-zero G-invariant subspaces W and W'. By looking at dimensions, we see that one of W or W' is one-dimensional. Thus, mimicking the proof of Proposition 1.37 shows that there is a common eigenvector. $\square$

One can observe that the above proof is similar to how shows that if a three degree polynomial is reducible, then it has a root. However, we really did need the finiteness of G as the following example shows us.

Example 1.62. Let $\varphi : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ be the representation

$$\varphi(n) = \begin{bmatrix} 1 & n \\ & 1 \end{bmatrix}.$$

Then, φ is reducible since $\mathbb{C}e_1$ is a $\mathbb{Z}$-invariant subspace. However, one sees that there is no other common eigenvector to all $\varphi(n)$ and hence, there is no other $\mathbb{Z}$-invariant subspace. Thus, φ is not decomposable.

That is, φ is neither neither irreducible nor decomposable, showing that Corollary 1.58 is false for infinite groups. In turn, Proposition 1.57 is also false for infinite groups. (Note that Proposition 1.54 had no assumption of finiteness of group.)

Moreover, the previous cannot be strengthened to degree four representations (even for finite groups) as the next example shows us.

Example 1.63. Let $\varphi : D_4 \to \mathrm{GL}_2(\mathbb{C})$ be the representation as in Example 1.40. Put $\psi := \varphi \oplus \varphi$. Then, $\psi : G \to \mathrm{GL}_4(\mathbb{C})$ is a degree four representation and we have

$$\psi(r) = \begin{bmatrix} \iota & & & \\ & -\iota & & \\ & & \iota & \\ & & & -\iota \end{bmatrix} \quad \text{and} \quad \psi(s) = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

Clearly, the eigenvectors of $\psi(r)$ are the standard basis vectors (up to scaling) and none of them is an eigenvector of $\psi(s)$.

Thus, ψ is reducible even though there is no $v \in V$ which is a common eigenvector for all ψ_g.

The next result is again something we had prompted earlier. It is similar to the existence (but not uniqueness yet) to the decomposition of integers into their prime factors.

Theorem 1.64 (Maschke)**.** Every representation of a finite group is completely reducible.

Proof. We prove this by induction on the degree of the representation. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation.

If $\dim V = 1$, then φ is irreducible (and hence, completely reducible) and we are done.

Now, let $n \geq 1$ and assume the statement is degree of representation of degree $\leq n$. Let $\dim V = n + 1$. If φ is irreducible, then we are done. If not, then

$$V = U \oplus W$$

for non-zero G-invariant subspaces, by Corollary 1.58. Since U, W both have dimension strictly less than $\dim V$, the induction hypothesis applies and we can write

$$\begin{aligned} U &= U_1 \oplus \cdots \oplus U_n \\ W &= W_1 \oplus \cdots \oplus W_m \end{aligned}$$

for G-invariant subspaces such that $\varphi|_{U_i}$ and $\varphi|_{W_j}$ is irreducible for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In turn, we have

$$V = U_1 \oplus \cdots \oplus U_n \oplus W_1 \oplus \cdots \oplus W_m,$$

as desired. $\square$

§2. Character Theory and the Orthogonality Relations

§§2.1. Morphisms of Representations

Definition 2.1. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations. A morphism from φ to ρ is a linear map $T : V \to W$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\rho_g"'] & W \end{tikzcd}$$

for all $g \in G$.

The set of all morphisms from φ to ρ is denoted by $\mathrm{Hom}_G(\varphi, \rho)$.

Note that $\mathrm{Hom}_G(\varphi, \rho) \subset \mathrm{Hom}(V, W)$.

The above definition can be seen as follows: Recall from Remark 1.4 that a representation can be viewed as giving a group action. With this understanding, we may write gv for $\varphi_g v$ and gw for $\rho_g w$ (where $v \in V$ and $w \in W$). Now, under this notation, we see that a morphism from φ to ρ is simply a linear transformation $T : V \to W$ such that

$$Tgv = gTv$$

for all $g \in G$ and all $v \in V$.

Remark 2.2. If $T \in \mathrm{Hom}_G(\varphi, \rho)$ is an isomorphism, then T is actually an *equivalence* and $\varphi \sim \rho$.

Remark 2.3. $T \in \mathrm{Hom}(V, V)$ is an element of $\mathrm{Hom}_G(\varphi, \varphi)$ if and only if $T \circ \varphi_g = \varphi_g \circ T$ for all $g \in G$. In other words, T commutes with every element of $\varphi(G)$. In particular, the identity map is always an element of $\mathrm{Hom}_G(\varphi, \varphi)$.

Proposition 2.4. Let $T : V \to W$ be in $\mathrm{Hom}_G(\varphi, \rho)$. Then $\ker T$ and $\operatorname{im} T$ are G-invariant subspaces of V and W, respectively.

Proof. $\ker T$: Let $v \in \ker T$ and $g \in G$ be arbitrary. Then,

$$T(\varphi_g v) = \rho_g(Tv) = \rho_g(0) = 0$$

and hence, $\varphi_g v \in \ker T$, as desired.

$\operatorname{im} T$: Let $w \in \operatorname{im} T$ and $g \in G$ be arbitrary. Then, $w = Tv$ for some $v \in V$. Then,

$$\rho_g w = \rho_g(Tv) = T(\varphi_g v)$$

showing that $\rho_g w \in \operatorname{im} T$, as desired. $\square$

As we had earlier observed, $\operatorname{Hom}_G(\varphi, \rho) \subset \operatorname{Hom}(V, W)$. In fact, more is true as the following proposition shows.

Proposition 2.5. Let G be a group and $\varphi : G \to \mathrm{GL}(V),\ \rho : G \to \mathrm{GL}(W)$ be representations. Then, $\operatorname{Hom}_G(\varphi, \rho)$ is a subspace of the vector space $\operatorname{Hom}(V, W)$.

Proof. Clearly, the zero operator $0 : V \to W$ is an element of $\operatorname{Hom}_G(\varphi, \rho)$.

Now, suppose that $S, T \in \operatorname{Hom}_G(\varphi, \rho)$ and $\alpha \in \mathbb{C}$ are arbitrary. Let $g \in G$ and $v \in V$ be arbitrary. Then,

$$\begin{aligned} (S + \alpha T)(\varphi_g v) &= S(\varphi_g v) + \alpha T(\varphi_g v) && \textit{$S, T \in \operatorname{Hom}_G(\varphi, \rho)$} \\ &= \rho_g S v + \alpha \rho_g T v && \textit{ρ_g is linear} \\ &= \rho_g(Sv + \alpha T v). \end{aligned}$$

Thus, $S + \alpha T \in \operatorname{Hom}_G(\varphi, \rho)$. $\square$

Proposition 2.6. Let $\varphi : G \to \mathrm{GL}(V),\ \varphi' : G \to \mathrm{GL}(V'),\ \rho : G \to \mathrm{GL}(W),$ and $\rho' : G \to \mathrm{GL}(W')$ be representations.

If $\varphi \sim \varphi'$ and $\rho \sim \rho'$, then $\dim \operatorname{Hom}_G(\varphi, \rho) = \dim \operatorname{Hom}_G(\varphi', \rho')$.

Proof. Let $T : V \to V'$ and $T' : W \to W'$ be isomorphisms showing the equivalences $\varphi \sim \varphi'$ and $\rho \sim \rho'$, respectively. (That is, they make the desired rectangles commute.)

Then, define the obvious map $\Phi : \operatorname{Hom}_G(\varphi, \rho) \to \operatorname{Hom}_G(\varphi', \rho')$ by

$$\Phi(S) = T' \circ S \circ T^{-1}.$$

That is, we wish to make the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "S"'] & V' \arrow[d, "\Phi(S)"] \\ W \arrow[r, "T'"'] & W' \end{tikzcd}$$

First, we verify that Φ actually maps into $\mathrm{Hom}_G(\varphi', \rho')$. This is simple. Let $g \in G$, $S \in \mathrm{Hom}_G(\varphi, \rho)$ and $v' \in V'$ be arbitrary. We then note

$$\begin{aligned}
\Phi(S)(\varphi'_g v') &= (T' \circ S \circ T^{-1})(\varphi'_g v') \\
&= T'S(T^{-1}(\varphi'_g v')) \\
&\qquad\qquad\qquad \textit{T and hence, } T^{-1} \textit{ is an equivalence} \\
&= T'S(\varphi_g T^{-1} v') \\
&\qquad\qquad\qquad S \in \mathrm{Hom}_G(\varphi, \rho) \\
&= T'(\rho_g S T^{-1} v') \\
&\qquad\qquad\qquad T' \textit{ is an equivalence} \\
&= \rho'_g(T'ST^{-1}v') \\
&= \rho'_g(\Phi(S)v'),
\end{aligned}$$

as desired.

It is easy to see that Φ is linear. Indeed, this follows simply because T is linear. Lastly, to see that it is a bijection, note that we have a two-sided inverse for Φ defined in the similar manner. □

Lemma 2.7 (Schur's lemma)**.** Let φ, ρ be irreducible representations of G, and $T \in \mathrm{Hom}_G(\varphi, \rho)$. Then either T is invertible or $T = 0$. Consequently:

1. If $\varphi \not\sim \rho$, then $\mathrm{Hom}_G(\varphi, \rho) = 0$;
2. If $\varphi = \rho$, then $T = \lambda I$ with $\lambda \in \mathbb{C}$. In other words, T is simply multiplication with a scalar. (Here is where we use that the base field is $\mathbb{C}$.)

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be irreducible representations.

If $T = 0$, then we are done. Thus, assume that $T \neq 0$. In this case, $\ker T \neq V$. On the other hand, by Proposition 2.4, we know that $\ker T$ is G-invariant. Hence, irreducibility of φ forces that $\ker T = 0$. In other words, T is injective.

Similarly, we know that $\operatorname{im} T$ is G-invariant and hence, $\operatorname{im} T = 0$ or $\operatorname{im} T = W$. As $T \neq 0$, the former is not possible. Thus, we see that $\operatorname{im} T = W$, i.e., T is onto.

Thus, we conclude that T is invertible. We now prove the consequences.

1. This is immediate for if $\varphi \not\sim \rho$, then T cannot be invertible for otherwise it would be an equivalence. Thus, the only possible morphism is the zero map.
2. Let λ be an eigenvalue of T (which exists because the base field is the algebraically closed $\mathbb{C}$).
   Now, recall that the identity map I is an element of $\mathrm{Hom}_G(\varphi, \varphi)$. (Remark 2.3)
   By Proposition 2.5, we then see that $T - \lambda I \in \mathrm{Hom}_G(\varphi, \varphi)$. Now, by definition of an eigenvalue, $T - \lambda I$ cannot be invertible. Thus, $T - \lambda I = 0$ which establishes the consequence.

Thus, we are done. □

Corollary 2.8. If φ and ρ are equivalent irreducible representations, then $\dim \mathrm{Hom}_G(\varphi, \rho) = 1$.

Proof. By Proposition 2.6, it suffices to show that $\dim \mathrm{Hom}_G(\varphi, \varphi) = 1$. By the previous part, we see that $\{I\}$ is a basis for $\mathrm{Hom}_G(\varphi, \varphi)$. □

We now generalise the result of Theorem 1.50 (in fact, this also gives an alternate proof of Theorem 1.50).

Theorem 2.9 (Irreducible representations of abelian groups)**.** Let G be an abelian group. Then any irreducible representation of G has degree 1.

Proof. The idea is simple. We first show that every φ_h is a morphism from φ to itself. Using that, we construct a dimension one invariant subspace of V forcing V to be one dimensional.

To this end, fix $h \in H$. Put $T := \varphi_h$ and let $g \in G$ be arbitrary. Then, we have

$$T\varphi_g = \varphi_h \varphi_g = \varphi_{hg} = \varphi_{gh} = \varphi_g \varphi_h = \varphi_g T$$

proving that $\varphi_h \in \mathrm{Hom}_G(\varphi, \varphi)$. Consequently, Lemma 2.7 (which is applicable since φ is irreducible) tells us that $\varphi_h = \lambda_h I$ for some $\lambda_h \in \mathbb{C}$.

Now, fix a non-zero vector $v \in V$. Then, $\varphi_h v = \lambda_h v \in \mathbb{C}v$. This shows that $\mathbb{C}v$ is φ_h invariant. Note that h was arbitrary and v did not dependent on h. Thus, $\mathbb{C}v$ is a G-invariant subspace and irreducibility forces $V = \mathbb{C}v$. □

Remark 2.10. By Corollary 1.18, we already have a description of the degree one representations of the finite abelian groups.

Corollary 2.11. Let G be a finite abelian group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a representation. Then, there exists an invertible matrix T such that $T^{-1}\varphi_g T$ is invertible all $g \in G$.

Note that the matrix T is independent of g.

Proof. Since G is finite, φ is completely reducible, by Theorem 1.64. Thus, we can write

$$\varphi \sim \varphi^{(1)} \oplus \cdots \oplus \varphi^{(m)}$$

where each $\varphi^{(i)}$ is irreducible. By the previous corollary, it follows that that each $\varphi^{(i)}$ is of degree 1 and hence, we also get $m = n$.

If $T : \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism giving the above equivalence, then we see that

$$T^{-1}\varphi_g T = \operatorname{diag}\left(\varphi_g^{(1)}, \ldots, \varphi_g^{(n)}\right),$$

as desired. □

Corollary 2.12. Let $A \in \mathrm{GL}_m(\mathbb{C})$ be a matrix of finite order. Then, A is diagonalisable.

Proof. Let $n > 0$ be the order of A. Then, we get a representation $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_m(\mathbb{C})$ given by $\varphi([k]) = A^k$. Then, by Corollary 2.11, $\varphi([1]) = A$ is diagonalisable. (In fact, the collection $I, \ldots, A^{n-1}$ is *simultaneously* diagonalisable.) □

§§2.2. The Orthogonality Relations

From now on, the group G will be assumed to be finite unless otherwise mentioned.

Definition 2.13. Let G be a group and let $L(G)$ denote the set of all functions from G to $\mathbb{C}$. That is,

$$L(G) := \mathbb{C}^G = \{f \mid f : G \to \mathbb{C}\}.$$

Then, $L(G)$ is a vector space over $\mathbb{C}$ in the natural way. It is also an inner product space with inner product defined as

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

$L(G)$ is called the group algebra of the group G.

The last sum makes sense without any convergence issues because G is finite.

Note that given a representation $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$, we get n^2 functions $\varphi_{ij} : G \to \mathbb{C}$, corresponding to the n^2 entries. We now wish to study properties of $\varphi_{ij} \in L(G)$ when φ is irreducible and unitary.

Our eventual goal will be to prove Theorem 2.18.

Proposition 2.14. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations and suppose that $T : V \to W$ is a linear transformation. Then,

1. $T^\# = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g \in \mathrm{Hom}_G(\varphi, \rho)$.

2. If $T \in \mathrm{Hom}_G(\varphi, \rho)$, then $T^\# = T$.
3. The map $P : \mathrm{Hom}_{\mathbb{C}}(V, W) \to \mathrm{Hom}_G(\varphi, \rho)$ defined by $T \mapsto T^\#$ is an onto linear map.

Proof. The proof of (1) is by direct computation. Let $h \in G$ be arbitrary. Note that

$$T^\# \varphi_h = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_{gh} = \frac{1}{|G|} \sum_{g' \in G} \rho_{hg'^{-1}} T \varphi_{g'} = \rho_h T^\#.$$

The middle inequality follows by the (bijective) change of variable $gh = g'$. The above then establishes (1).

Now, if $T \in \mathrm{Hom}_G(\varphi, \rho)$, then we get

$$T^\# = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} \rho_g T = \frac{1}{|G|} |G| T = T,$$

which proves (2).

Note that the above also proves that $T \mapsto T^\#$ is onto. Thus, to prove (3), we only need to prove linearity of P. This again follows by direct computation. Let $c \in \mathbb{C}$ and $T_1, T_2 \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ be arbitrary.

$$\begin{aligned} P(cT_1 + T_2) &= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} (cT_1 + T_2) \varphi_g \\ &= c \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} (T_1) \varphi_g + \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} (T_2) \varphi_g \\ &= cP(T_1) + P(T_2), \end{aligned}$$

as desired. □

Proposition 2.15. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be irreducible representations of G and let $T : V \to W$ be a linear map. Then:

1. If $\varphi \not\sim \rho$, then $T^\# = 0$;
2. If $\varphi = \rho$, then $T^\# = \frac{\mathrm{trace}\, T}{\deg \varphi} I$.

Proof. (1) is simple for $T^\# \in \mathrm{Hom}_G(\varphi, \rho) = 0$, by Schur's lemma. Now, if $\varphi = \rho$, then $T^\# = \lambda I$ for some $\lambda \in \mathbb{C}$, again by Schur's lemma. We now wish to determine λ.

Note that trace $T^\# = \mathrm{trace}(\lambda I) = \lambda \dim V = \lambda \deg \varphi$. That is,

$$T^\# = \lambda I = \frac{\mathrm{trace}\, T^\#}{\deg \varphi} I. \qquad (*)$$

We may also calculate trace $T^{\#}$ separately, using the definition of $T^{\#}$ and the fact that $\operatorname{trace}(ABC) = \operatorname{trace}(CAB)$. This gives us that

$$\begin{aligned}\operatorname{trace}(T^{\#}) &= \frac{1}{|G|}\sum_{g\in G}\operatorname{trace}(\varphi_{g^{-1}}T\varphi_g)\\ &= \frac{1}{|G|}\sum_{g\in G}\operatorname{trace}(\varphi_g\varphi_{g^{-1}}T)\\ &= \frac{1}{|G|}\sum_{g\in G}\operatorname{trace}(T)\\ &= \operatorname{trace}(T).\end{aligned}$$

Putting the above back in ($*$), we get

$$T^{\#} = \frac{\operatorname{trace} T}{\deg\varphi} I. \qquad \square$$

If we consider $V = \mathbb{C}^n$ and $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ (and similarly for $W = \mathbb{C}^m$), then Proposition 2.14 tells us that we can consider P as a linear from $\mathrm{GL}(V, W) = M_{m\times n}(\mathbb{C})$ to itself. It is now natural to ask what is the matrix representation of P with respect to the standard basis $\{E_{ij}\}$. (Recall that E_{ij} is the $m \times n$ matrix with 1 in the (i, j)-th entry and 0 everywhere else.)

Lemma 2.16. Let $A \in M_{r\times m}(\mathbb{C})$, $E_{ki} \in M_{m\times n}(\mathbb{C})$, and $B \in M_{n\times s}(\mathbb{C})$. Then, we have

$$(AE_{ki}B)_{lj} = a_{lk}b_{ij},$$

where $A = (a_{ij})$ and $B = (b_{ij})$.

Proof. By definition, we have

$$(AE_{ki}B)_{lj} = \sum_{x,y} a_{lx}(E_{ki})_{xy}b_{yj}.$$

The only (possibly) non-zero term appearing in the summation is when $(x, y) = (k, i)$ which proves the result since $(E_{ki})_{ki} = 1$. $\square$

Lemma 2.17. Let $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ be unitary representations of G. Let $A = E_{ki} \in M_{m\times n}(\mathbb{C})$. Then, $A^{\#}_{lj} = \langle \varphi_{ij}, \rho_{kl}\rangle$.

Note that we had remarked earlier that given a function $\varphi : G \to U_n(\mathbb{C})$, we actually get n^2 $\mathbb{C}$-valued functions. The inner product appearing in the above lemma is the one defined in Definition 2.13.

Proof. Let $g \in G$. Then $\rho_g \in U_n(\mathbb{C})$. Note that we have

$$\rho_{g^{-1}} = (\rho_g)^{-1} = \rho_g^*$$

because ρ_g is unitary.

Thus, we see that

$$\rho_{lk}(g^{-1}) = \overline{\rho_{kl}(g)}.$$

With the above, we note that

$$\begin{aligned} A^\#_{lj} &= \frac{1}{|G|} \sum_{g \in G} (\rho_{g^{-1}} E_{ki} \varphi_g)_{lj} \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_{lk}(g^{-1}) \varphi_{ij}(g) && \text{Lemma 2.16} \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\rho_{kl}(g)} \varphi_{ij}(g) \\ &= \langle \varphi_{ij}, \rho_{kl} \rangle, && \text{Definition 2.13} \end{aligned}$$

as desired. ∎

We now prove the desired theorem.

Theorem 2.18 (Schur's orthogonality relations)**.** Let G be a finite group. Suppose that $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ are inequivalent irreducible unitary representations. Then:

1. $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$,
2. $\langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} 1/n & \text{if } (i,j) = (k,l), \\ 0 & \text{otherwise.} \end{cases}$

In particular, $\{\varphi_{ij} \mid 1 \le i,j \le n\} \cup \{\rho_{kl} \mid 1 \le k,l \le m\}$ is a linearly independent set.

The last part follows since the theorem tells us that the above set of functions form an orthogonal set of non-zero vectors.

Proof. Letting $A = E_{ki} \in M_{m \times n}(\mathbb{C})$, we see that $A^\# = 0$ by Item 1 of Proposition 2.15. On the other hand, $\langle \varphi_{ij}, \rho_{kl} \rangle = (A^\#)_{lj}$, by Lemma 2.17. This proves (1).

Now, we put $\rho = \varphi$. We apply the same proposition and lemma again. Letting $A = E_{ki} \in M_n(\mathbb{C})$, we see that

$$A^\# = \frac{\operatorname{trace} A}{n} I$$

by Item 2 of Proposition 2.15. By Lemma 2.17, we see that

$$\langle \varphi_{ij}, \varphi_{kl} \rangle = (A^{\#})_{lj} = \frac{\operatorname{trace} A}{n} I_{lj}.$$

Now if $i \neq k$, then trace $A = 0$. On the other hand, if $l \neq j$, then $I_{lj} = 0$. Now, if $(i, j) = (k, l)$, then trace $A = 1$ and $I_{lj} = 1$. These three cases put together prove (2). □

Corollary 2.19. Let φ be an irreducible unitary representation of G of degree n. Then, the following set of n^2 functions

$$\{\sqrt{n}\varphi_{ij} \mid 1 \leq i, j \leq n\}$$

forms an orthonormal set.

Proof. By the previous theorem, we already know that any two distinct elements of the set are orthogonal. The multiplication by $\sqrt{n}$ simply makes all the functions have unit norm. □

Proposition 2.20. Let G be a finite group. Then, the following hold.

1. There are only finitely many equivalence classes of irreducible representations of G.
2. Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of unitary representatives of the equivalence classes of irreducible representations of G. Set $d_i := \deg \varphi^{(i)}$. Then, the set of functions
$$\left\{\sqrt{d_k}\varphi_{ij}^{(k)} \mid 1 \leq k \leq s,\ 1 \leq i, j \leq d_k\right\}$$
forms an orthonormal set in $L(G)$.
3. In particular, $s \leq d_1^2 + \cdots + d_s^2 \leq |G|$.

Proof. All of these follow from Corollary 2.19.

1. Note that given any set of equivalence classes of (not necessarily irreducible) representations, each class contains a unitary representation, by Proposition 1.57. Now, since $\dim L(G) = |G|$, no linearly independent set of vectors from $L(G)$ can contain more than $|G|$ many elements. Since orthonormal sets are linearly independent, Corollary 2.19 shows that there can only be finitely many classes of irreducible representations.
2. This part again follows mainly from Corollary 2.19. The orthogonality of two functions of representations of different degrees follows from Schur's orthogonality relations since the representations $\varphi^{(i)}$ and $\varphi^{(j)}$ are inequivalent if $d_i \neq d_j$.
3. $s \leq d_1^2 + \cdots + d_s^2$ is clear since each d_i^2 is at least 1. On the other hand, the orthonormal set given has $d_1^2 + \cdots + d_s^2$ elements in a vector space of dimension $|G|$, proving the second inequality. □

Remark 2.21. We shall later see that we actually have the equality

$$|G| = d_1^2 + \cdots + d_s^2.$$

§§2.3. Some Examples

Example 2.22 (Degree one representations of D_n)**.** Recall that D_n has the following presentation

$$D_n = \langle r, s \mid r^n = s^2 = rsrs = 1 \rangle.$$

In other words, to define a representation $z : G \to \mathbb{C}^*$, we only need to specify z_r and z_s which satisfy the above relations. (In the sense that this gives all the representations and that every representation is obtained this way.)

Note that since $\mathbb{C}^*$ is commutative, for the last relation, we only need

$$z_r^2 z_s^2 = 1.$$

However, the second relation already tells us that $z_s^2 = 1$. Thus, we now have the equivalent job of finding $z_r, z_s \in \mathbb{C}^*$ satisfying

$$z_r^n = 1,\ z_r^2 = 1,\ z_s^2 = 1.$$

Note that, in the above, we have separated the relations into those for z_r and z_s separately. Thus, we have precisely two choices for z_s (namely, ± 1) for every choice for z_r.

We now turn to the case of determining z_r. There are two cases.

Case 1. n is even. In this case, the relation $z_r^n = 1$ is implied by $z_r^2 = 1$. Thus, we get precisely two choices for $z_r : \pm 1$.

Case 2. n is odd. Then, since $\gcd(n, 2) = 1$, one can conclude that $z_r^1 = 1$ and thus, we have only one choice.

Thus, we get the number of degree one representations of D_n as:

1. 4, if n is even,
2. 2, if n is odd.

Note that all of these are inequivalent since distinct degree one representations are inequivalent. (Proposition 1.15.)

Example 2.23 (An irreducible representation of D_n)**.** Consider the regular n-polygon as a subset of $\mathbb{C}$ with vertices as the n-th roots of unity. We can think of its set of symmetries as D_n. This gives us an embedding as follows

$$\varphi : D_n \to \mathrm{GL}_2(\mathbb{C})$$

defined as

$$\varphi_r := \begin{bmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{bmatrix} \quad \text{and} \quad \varphi_s := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\theta_n = \frac{2\pi}{n}$.

(Alternately, one can verify that $\varphi_r^n = \varphi_s^2 = (\varphi_r\varphi_s)^2 = 1$.)

Now, to see that it is irreducible, we note that the eigenvectors of φ_s (up to scaling) are e_1 and e_2. Thus, φ_r and φ_s have no common eigenvectors (note that $\sin\theta_n \neq 0$) and hence, φ is irreducible.

Example 2.24 (All irreducible representations of D_3 and D_4)**.** Note that by Example 2.22, we already know that there are 2 (inequivalent irreducible) degree one representations of D_3 and 4 of D_4.

By Example 2.23, we also have 1 irreducible degree two representation of both.

On the other hand, note that

$$1^1 + 1^1 + 2^2 = 6 = |D_3|,$$
$$1^1 + 1^1 + 1^1 + 1^1 + 2^2 = 8 = |D_4|.$$

Thus, by Proposition 2.20, we see that we have actually found all irreducible representations of D_3 and D_4! Note that Example 2.23 and Example 1.40 are two distinct degree two representations of D_4. The above analysis however tells us that the two are equivalent, even without us explicitly constructing any equivalence.

§§2.4. Characters and Class Functions

In this subsection, we will prove the uniqueness of decompositions. (That is, the uniqueness of the decomposition given in Maschke.)

We start by introducing the character of a representation. Recall that given a endomorphism of a (finite dimensional) vector space, we can talk about its trace. This is defined as the trace of any matrix representation obtained after fixing an ordered basis. It is easy to see that this is basis invariant.

Definition 2.25. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. The character $\chi_\varphi : G \to \mathbb{C}$ of φ is defined by $\chi_\varphi(g) = \operatorname{trace}\varphi_g$. The character of an irreducible representation is called an irreducible character.

As remarked earlier, the computation of character is independent of the basis we choose. For this reason, we may assume without loss of generality that we are talking about matrix representations. (In the cases where the general case is as simple, we need not do so.)

If $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ is a representation given by $\varphi_g = (\varphi_{ij}(g))$, then

$$\chi_\varphi(g) = \sum_{i=1}^{n} \varphi_{ii}(g).$$

Remark 2.26. If $z : G \to \mathbb{C}^* \hookrightarrow \mathbb{C}$ is a degree one representation, then $\chi_z = z$. From now on, we shall treat degree one representations and their characters as the same.

Proposition 2.27. If $\varphi : G \to \mathrm{GL}(V)$ is a representation, then $\chi_\varphi(1) = \deg \varphi$.

Proof. $\chi_\varphi(1) = \operatorname{trace} \varphi_1 = \operatorname{trace} \mathrm{id}_V = \dim V = \deg \varphi$. □

Proposition 2.28. If φ and ρ are equivalent representations, then $\chi_\varphi = \chi_\rho$.

Proof. As remarked earlier, we may assume the representations in the form

$$\varphi, \rho : G \to \mathrm{GL}_n(\mathbb{C}).$$

Since the representations are equivalent, there exists an invertible matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\varphi_g = T\rho_g T^{-1}$$

for all $g \in G$. Now fix $g \in G$ and using $\operatorname{trace}(ABC) = \operatorname{trace}(CAB)$, we see that

$$\begin{aligned}
\chi_\varphi(g) &= \operatorname{trace} \varphi_g \\
&= \operatorname{trace}(T\rho_g T^{-1}) \\
&= \operatorname{trace}(T^{-1}T\rho_g) \\
&= \operatorname{trace} \rho_g \\
&= \chi_\rho(g),
\end{aligned}$$

as desired. □

The same proof as above also tells us that the function $\chi_\varphi : G \to \mathbb{C}$ is constant on the conjugacy classes of G. More precisely:

Proposition 2.29. Let φ be a representation of G. Then, for all $g, h \in G$, we have that $\chi_\varphi(g) = \chi_\varphi(hgh^{-1})$.

Proof. Let $g, h \in G$ and note

$$\begin{aligned}
\chi_\varphi(g) &= \operatorname{trace} \varphi_g \\
&= \operatorname{trace}(\varphi_{h^{-1}}\varphi_h\varphi_g) \\
&= \operatorname{trace}(\varphi_h\varphi_g\varphi_{h^{-1}}) \\
&= \operatorname{trace} \varphi_{hgh^{-1}} \\
&= \chi_\varphi(hgh^{-1}).
\end{aligned}$$

□

Functions with the above property have a special name.

Definition 2.30. A function $f : G \to \mathbb{C}$ is called a class function if $f(g) = f(hgh^{-1})$ for all $g, h \in G$. The space of all class functions is denoted $Z(L(G))$.

Thus, we have shown that characters are class functions. Given a conjugacy class $C \subset G$ and a class function $f : G \to \mathbb{C}$, $f(C) \in \mathbb{C}$ will denote the (constant) value taken by elements of C.

Proposition 2.31. $Z(L(G))$ is a subspace of the vector space $L(G)$.

Proof. Let $c \in \mathbb{C}$, $f_1, f_2 \in L(G)$, and $h, g \in G$ be arbitrary. Then,

$$\begin{aligned}
(cf_1 + f_2)(hgh^{-1}) &= cf_1(hgh^{-1}) + f_2(hgh^{-1}) \\
&= cf_1(g) + f_2(g) \\
&= (cf_1 + f_2)(g),
\end{aligned}$$

showing that $Z(L(G))$ is closed under linear combinations. Also, note that the zero map is an element of $Z(L(G))$ proving that $Z(L(G)) \leq L(G)$. □

Definition 2.32. Given a group G, the set of conjugacy classes of G is denoted $\mathrm{Cl}(G)$. For $C \in \mathrm{Cl}(G)$, we define $\delta_C : G \to \mathbb{C}$ as

$$\delta_C(g) = \begin{cases} 1 & g \in C, \\ 0 & g \notin C. \end{cases}$$

In other words, δ_C is just the indicator function of $C \subset G$.

Proposition 2.33. The set $B = \{\delta_C \mid C \in \mathrm{Cl}(G)\}$ is a basis for $Z(L(G))$. In particular, $\dim Z(L(G)) = |\mathrm{Cl}(G)|$.

Proof. It is clear $\delta_C \in Z(L(G))$ for each $C \in \mathrm{Cl}(G)$. (Note that conjugacy classes partition G and thus, distinct conjugacy classes have empty intersection.)

Spanning. Let $f \in Z(L(G))$. One verifies

$$f = \sum_{C \in \mathrm{Cl}(G)} f(C)\delta_C$$

by computing each side at an arbitrary $g \in G$. This proves that $\operatorname{span} B = Z(L(G))$.

Linear independence. Note that

$$\langle \delta_C, \delta_{C'} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_C(g)\overline{\delta_{C'}(g)} = \begin{cases} 0 & C \neq C', \\ \frac{|C|}{|G|} & C = C'. \end{cases}$$

Thus, B is a set of orthogonal non-zero vectors and hence, is linearly independent.

Lastly, note that $|B| = |\mathrm{Cl}(G)|$ since $C \neq C' \implies \delta_C \neq \delta_{C'}$. Thus, $\dim Z(L(G)) = |B| = |\mathrm{Cl}(G)|$. □

Theorem 2.34 (First orthogonality relations)**.** Let φ, ρ be <u>irreducible</u> representations of G. Then

$$\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} 1 & \varphi \sim \rho, \\ 0 & \varphi \not\sim \rho. \end{cases}$$

Thus, the irreducible characters of G form an orthonormal set of class functions. In particular, they are linearly independent.

Note that technically, we should have said "distinct irreducible characters" in the last line but Proposition 2.28 tells us that equivalent representations have equal characters.

Proof. By Proposition 1.57 and Proposition 2.28, we may assume that $\varphi : G \to U_n(\mathbb{C})$ and

$\rho : G \to U_m(\mathbb{C})$. Now, note that

$$\begin{aligned}\langle \chi_\varphi, \chi_\rho \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \overline{\chi_\rho(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \left[\left(\sum_{i=1}^{n} \varphi_{ii}(g) \right) \left(\sum_{j=1}^{m} \overline{\rho_{jj}(g)} \right) \right] \\ &= \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \left[\frac{1}{|G|} \sum_{g \in G} \varphi_{ii}(g) \overline{\rho_{jj}(g)} \right] \\ &= \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \langle \varphi_{ii}, \rho_{jj} \rangle.\end{aligned}$$

Now, if $\varphi \nsim \rho$, then all the terms in the summation are 0, by Schur's orthogonality relations. Now, if $\rho \sim \varphi$, then we may assume $\rho = \varphi$, by Proposition 2.28. (Since we are making a statement about the characters only.)

In this case, Schur's orthogonality relations tell us that the only non-zero terms in the summation are when $i = j$, in which case

$$\langle \varphi_{ii}, \rho_{jj} \rangle = \langle \varphi_{ii}, \varphi_{ii} \rangle = \frac{1}{n}$$

and so,

$$\langle \chi_\varphi, \chi_\rho \rangle = n \cdot \frac{1}{n} = 1.$$

□

Corollary 2.35. Given two irreducible inequivalent representations φ and ρ of G, we have $\chi_\varphi \neq \chi_\rho$.

Proof. Note that $\langle \chi_\varphi, \chi_\rho \rangle = 0$. If $\chi_\varphi = \chi_\rho$, this would force $\chi_\varphi = 0$. However, this is not possible since $\langle \chi_\varphi, \chi_\varphi \rangle = 1 \neq 0$. □

Note that Proposition 2.28 already told us that equivalent characters have the same character. We have now proven the converse for irreducible representations. Thus, we have the following.

Theorem 2.36. Two irreducible representations are equivalent if and only they have the same character.

Corollary 2.37. There are at most $|\mathrm{Cl}(G)|$ equivalence classes of irreducible representations of G.

Proof. We have already shown that distinct equivalence classes will have distinct characters. Moreover, we have shown that picking a character from each set gives us a orthonormal (and hence, linearly independent) subset of $Z(L(G))$ and in turn, there can be at most $\dim Z(L(G)) = |\mathrm{Cl}(G)|$ many such. $\square$

We now introduce some notation for ease of writing.

Definition 2.38. If V is a vector space, φ a representation, and $m \in \mathbb{N}$, then

$$mV := \underbrace{V \oplus \cdots \oplus V}_{m} \quad \text{and} \quad m\varphi := \underbrace{\varphi \oplus \cdots \oplus \varphi}_{m}.$$

If $m = 0$, then we define $0V$ to be the zero vector space and 0φ to be the degree zero representation.

Remark 2.39. Note that we had said that we won't consider degree zero representations and we shall continue to do so. The only reason for considering $m = 0$ above is so that when we write an expression as

$$\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)},$$

then we allow that possibility for some m_i to be 0. In that case, we simply ignore $\varphi^{(i)}$. It will never be the case that each m_i is 0.

Our immediate goal now is to prove the uniqueness of decomposition. More precisely, if we are given a complete set of irreducible representatives $\varphi^{(1)}, \ldots, \varphi^{(s)}$ and have

$$\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)},$$

we want to show that each m_i is uniquely determined. We shall see that this information can be extracted from just the character of ρ.

Lemma 2.40. Let $\varphi = \rho \oplus \psi$. Then $\chi_\varphi = \chi_\rho + \chi_\psi$.

Proof. We may assume that $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ and $\psi : G \to \mathrm{GL}_m(\mathbb{C})$. Then, we have the block matrix form for $\varphi : G \to \mathrm{GL}_{n+m}(\mathbb{C})$ with

$$\varphi_g = \begin{bmatrix} \rho_g & \\ & \psi_g \end{bmatrix}$$

for all $g \in G$. From the above, it follows that

$$\operatorname{trace} \varphi_g = \operatorname{trace} \rho_g + \operatorname{trace} \psi_g$$

for all $g \in G$, as desired. $\square$

Theorem 2.41. Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of representatives of equivalence classes of irreducible representations of G. Suppose ρ is a representation such that

$$\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}.$$

Then, $m_i = \langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle$.

Proof. Note that by definition, $\varphi^{(i)} \not\sim \varphi^{(j)}$ if $i \neq j$. Thus, by Theorem 2.34, it follows that

$$\langle \chi_{\varphi^{(i)}}, \chi_{\varphi^{(j)}} \rangle = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases} \tag{$*$}$$

From the previous lemma, it follows that

$$\chi_\rho = m_1\chi_{\varphi^{(1)}} + \cdots + m_s\chi_{\varphi^{(s)}}.$$

Taking the inner product with $\chi_{\varphi^{(i)}}$ and using ($*$) prove the result. □

Corollary 2.42. The composition of ρ into irreducible characters is unique.

This is immediate for the "unique" just means that m_i is uniquely determined. This actually tells us that we can make sense of something as the "multiplicity" of an irreducible representation. This leads to Definition 2.51.

Corollary 2.43. ρ is determined, up to equivalence by its character.

Proof. Let $f := \chi_\rho$. We show that we can construct a representation equivalent to ρ just in terms of f.

To this end, define $n_i := \langle f, \chi_{\varphi^{(i)}} \rangle$ and set

$$\varphi := n_1\varphi^{(1)} \oplus \cdots \oplus n_s\varphi^{(s)}.$$

We claim that $\varphi \sim \rho$. To see this, note that by Maschke, ρ is completely reducible and there exists a decomposition of ρ as

$$\rho \sim \rho^{(1)} \oplus \cdots \oplus \rho^{(s')}.$$

By construction, $\varphi^{(1)}, \ldots, \varphi^{(s)}$ are the only irreducible representations, up to equivalence. Thus, each $\rho^{(j)}$ is equivalent to some $\varphi^{(i)}$. By clubbing the representations in the same equivalence class together, we get

$$\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}.$$

However, $m_i = n_i$ for each i, by the previous theorem and hence, $\rho \sim \varphi$. □

Corollary 2.44. A representation ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.

Proof. As before, write $\rho \sim m_1 \varphi^{(1)} \oplus \cdots \oplus m_s \varphi^{(s)}$ and note that

$$\langle \chi_\rho, \chi_\rho \rangle = m_1^2 + \cdots + m_s^2.$$

Thus, $\langle \chi_\rho, \chi_\rho \rangle = 1$ iff there exists j such that $m_j = 1$ and $m_i = 0$ for all $i \neq j$ iff $\rho \sim \varphi^{(j)}$ for some j iff ρ is irreducible. □

Remark 2.45. The above calculation also shows us that $\langle \chi_\rho, \chi_\rho \rangle$ is always a positive integer.

Corollary 2.46. Let $z : G \to \mathbb{C}^*$ be a degree one representation and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ an irreducible representation. Then, $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ defined as

$$\varphi_g = z_g \rho_g$$

is an irreducible representation.

Moreover, if there exists $g_0 \in G$ such that $z_{g_0} \neq 1$ and $\chi_\varphi(g_0) \neq 0$, then $\rho \not\sim \varphi$.

Proof. First we show that φ is indeed a representation. This simple for $z_{g_2} \rho_{g_1} = \rho_{g_1} z_{g_2}$ for any $g_1, g_2 \in G$ which gives

$$\varphi_{g_1 g_2} = \varphi_{g_1} \varphi_{g_2},$$

as desired.

Moreover, we also note that

$$\operatorname{trace} \varphi_g = \operatorname{trace}(z_g \rho_g) = z_g \operatorname{trace} \rho_g$$

or

$$\chi_\varphi(g) = z_g \chi_\rho(g) \tag{$\star$}$$

and hence,

$$|\chi_\varphi(g)|^2 = |z_g|^2 |\chi_\rho(g)|^2.$$

Recall that since G is finite, $z_g^{|G|} = 1$ and hence, $|z_g| = 1$, which gives us

$$|\chi_\varphi(g)|^2 = |\chi_\rho(g)|^2. \tag{$*$}$$

To see that it is irreducible, we use Corollary 2.44 (twice!) as follows

$$\begin{aligned}
\langle \chi_\varphi, \chi_\varphi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \overline{\chi_\varphi(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} |\chi_\varphi(g)|^2 \\
&= \frac{1}{|G|} \sum_{g \in G} |\chi_\rho(g)|^2 && (*) \\
&= \langle \chi_\rho, \chi_\rho \rangle \\
&= 1. && \textit{Corollary } 2.44
\end{aligned}$$

Thus, ρ is irreducible, by Corollary 2.44.

We now prove the last statement. For this, we will use Proposition 2.28.
Let g_0 be as in the theorem; then, by ($\star$), we see that

$$\chi_\rho(g_0) = z_{g_0}^{-1} \chi_\varphi(g_0) \neq \chi_\varphi(g_0).$$

Thus, by Proposition 2.28, we have $\rho \not\sim \varphi$. $\square$

Remark 2.47. Note that the last part of the theorem is really just asking us to look at the characters of ρ and φ and conclude inequivalence.

Also, note that ($\star$) tells us that the character of φ is obtained by multiplying χ_z and χ_ρ. (Recall that character of a degree one representation is the representation itself.)

Example 2.48. Let us use the above corollary to show that the representation ρ of S_3 in Example 1.29 is irreducible. (We had already done this earlier in Example 1.39.)

Recalling Description of conjugacy classes, we see that there are exactly three conjugacy classes in S_3, namely, $[1],\ [(12)],\ [(123)]$. These have cardinalities $1,\ 3,\ 2$, respectively.

Note that $\chi_\rho(1) = 2$, $\chi_\rho\left((12)\right) = 0$, and $\chi_\rho\left((123)\right) = -1$.

Since characters are class functions, we see that

$$\begin{aligned}
\langle \chi_\rho, \chi_\rho \rangle &= \frac{1}{6} \sum_{\sigma \in S_3} \chi_p(\sigma) \overline{\chi_p(\sigma)} \\
&= \frac{1}{6}(1 \cdot 2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) \\
&= \frac{1}{6}(6) = 1.
\end{aligned}$$

Example 2.49 (Character table of S_3)**.** The previous example gives us an irreducible degree two representation of S_3. Example 1.9 had given us two degree one (inequivalent and irreducible) representations. Since the number of conjugacy classes of S_3 is 3, these are all. (Of course, using that $S_3 \cong D_3$, we knew this already.)

Let χ_1 denote the character of the the trivial representation, χ_2 of the sign representation, and χ_3 of the representation from the previous example.

Each of these are class functions, that is, constant on the conjugacy classes. Thus, we can construct something called the "character table."

	$[1]$	$[(12)]$	$[(123)]$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	1	0	-1

Table 1: Character table of S_3

Example 2.50 (Revisiting a representation of S_3)**.** Let us again turn back to Example 1.29. We had remarked that we shall show that $\rho \oplus \psi$ is equivalent to the standard representation from Example 1.21.

To see this, now we simply compute the character of the standard representation φ.

Computing it at $1,\ (12),\ (123)$, we see that the table is as follows.

	$[1]$	$[(12)]$	$[(123)]$
χ_φ	3	1	0

From the above table, it is evident that

$$\chi_\varphi = \chi_1 + \chi_3,$$

where we have retained the notation from the previous example. In turn, this establishes the desired equivalence.

Definition 2.51. Let G be a finite group and $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of irreducible unitary representations of G, up to equivalence. Set $d_i := \deg \varphi^{(i)}$.

If $\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, then m_i is called the multiplicity of $\varphi^{(i)}$ in ρ. If $m_i > 0$, then we say that $\varphi^{(i)}$ is an irreducible constituent of ρ.

Remark 2.52. With the same notation, we have

$$\deg \rho = m_1 d_1 + \cdots + m_s d_s.$$

The result in the proof of Corollary 2.43 is important and so, we isolate it below.

Theorem 2.53. Let G be a finite group and ρ a representation. Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be as earlier. Define, $m_i := \langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle$. Then,

$$\rho \sim m_1 \varphi^{(1)} \oplus \cdots \oplus m_s \varphi^{(s)}.$$

Note the similarity with inner product spaces where the coefficients of a vector with respect to an orthonormal basis is given by the inner product. The similarity is not surprising since the theorems and corollaries above actually tell us how the above equivalence of representations translates to equality of characters in the inner product space $Z(L(G))$.

§§2.5. The Regular Representation

Recall from Section 0.1.3, the concept of Linearisation.

Definition 2.54. Let G be a finite group. The regular representation of G is the homomorphism $L : G \to \mathrm{GL}(\mathbb{C}G)$ defined by

$$L_g\left(\sum_{h \in G} c_h h\right) = \sum_{h \in G} c_h gh = \sum_{x \in G} c_{g^{-1}x} x$$

for all $g \in G$.

Remark 2.55. Note that since G is a basis for $\mathbb{C}G$, we have that $\deg L = |G|$.

Remark 2.56. Of course, one must now verify that L_g is actually an element of $\mathrm{GL}(\mathbb{C}G)$ and that L is a homomorphism.

The above can be seen permuting the coefficients of a given element of $\mathbb{C}G$. Its action on the (natural) basis vectors can be seen as follows:

$$L_g h = gh.$$

In other words, L_g acts on basis vectors by (left) multiplication by g and the (unique) map obtained by extending it linearly to all of $\mathbb{C}G$ gives us the map L_g. (cf. Proposition 0.18.) The L stands for "left."

Proposition 2.57. The regular representation is a unitary representation of G. In particular, it is indeed a representation.

Proof. The fact that L is a representation follows from Proposition 0.19.

To see that it is unitary, note that

$$\begin{aligned}\left\langle L_g \sum_{h\in G} c_h h, L_g \sum_{h\in G} k_h h\right\rangle &= \left\langle \sum_{x\in G} c_{g^{-1}x} x, \sum_{x\in G} k_{g^{-1}x} x\right\rangle \\ &= \sum_{x\in G} c_{g^{-1}x}\overline{k_{g^{-1}x}} \\ &= \sum_{y\in G} c_y \overline{k_y} \qquad x \mapsto gy \\ &= \left\langle \sum_{h\in G} c_h h, \sum_{h\in G} k_h h\right\rangle,\end{aligned}$$

as desired. □

Proposition 2.58. The character of the regular representation L is given as

$$\chi_L(g) = \begin{cases} |G| & g = 1, \\ 0 & g \neq 1. \end{cases}$$

Proof. For $g = 1$, note that $\chi_L(1) = \deg L$, by Proposition 2.27 and $\deg L = |G|$, by Remark 2.55.

We now compute the character for $g \neq 1$. Let $n := |G|$ and write

$$G = \{g_1, \ldots, g_n\}$$

in some order and fix it as above. Now, we look at the matrix representation $[L_g]$ of L_g with respect to this ordered basis G.

We contend that all the diagonal entries of $[L_g]$ are 0.
Indeed, for any $g_i \in G$, we have $gg_i = g_j \neq g_i$. (Since $g \neq 1$.)
Thus, the i-the entry in the i-th column will be 0. It follows at once that

$$\chi_L(g) = \operatorname{trace} L_g = \operatorname{trace}[L_g] = 0,$$

as desired. □

Remark 2.59. Note that from the above, we can conclude the following.

$$\begin{aligned}\langle \chi_L, \chi_L \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_L(g)} \\ &= \frac{1}{|G|} |G|^2 \\ &= |G|.\end{aligned}$$

In particular, if G is non-trivial, then L is *not* irreducible. In fact, the next proposition shows us exactly the description of L in terms of its decomposition.

We shall now fix the following notation: G is a finite group and $\left\{\varphi^{(1)}, \ldots, \varphi^{(s)}\right\}$ is a complete set of inequivalent irreducible **unitary** representatives of G. As usual, $d_i := \deg \varphi^{(i)}$. Moreover, $\chi_i := \chi_{\varphi^{(i)}}$.

Proposition 2.60. Let L denote the regular representation of G. Then,

$$L \sim d_1 \varphi^{(1)} \oplus \cdots \oplus d_s \varphi^{(s)}.$$

In particular, the equality $|G| = d_1^2 + \cdots + d_s^2$ holds.

Proof. By Theorem 2.53, it suffices to show that $\langle \chi_L, \chi_i \rangle = d_i$ holds. To that end, note that

$$\begin{aligned}\langle \chi_L, \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_i(g)} \\ &= \frac{1}{|G|} \chi_L(1) \overline{\chi_i(1)} \\ &= \frac{1}{|G|} |G| \deg \varphi^{(i)} \\ &= \deg \varphi^{(i)} = d_i,\end{aligned}$$

as desired.

Now, note that by the Proposition 2.60 and Lemma 2.40, we see that

$$\chi_L = d_1 \chi_1 + \cdots + d_s \chi_s.$$

Evaluating both sides at 1 finishes the proof. □

Corollary 2.61. The set $B = \left\{ \sqrt{d_k}\varphi_{ij}^{(k)} \mid 1 \le k \le s,\ 1 \le i, j \le d_k \right\}$ is an orthonormal basis of $L(G)$.

Proof. By Proposition 2.20, we already know that it is orthonormal and hence, linearly independent. On the other hand, note that

$$|B| = d_1^2 + \cdots + d_s^2 = |G| = \dim L(G).$$ □

Example 2.62 (Number of irreducible representations of D_n)**.** Note that by Example 2.22, we know the exact number of degree one representations of D_n. By Example 1.36, we know that all other irreducible representations must have degree two.

Now, let t_n denote the number of inequivalent irreducible degree two representations of D_n. We shall now calculate t_n, using Proposition 2.60.

Case 1. $n = 2k + 1$.
In this case, there are 2 inequivalent degree one representations. Thus, we see that

$$2 \cdot 1^2 + t_n \cdot 2^2 = |D_n| = 4k + 2$$

which gives us

$$t_n = k = \frac{n-1}{2}.$$

Case 2. $n = 2k$.
In this case, there are 4 inequivalent degree one representations. Thus, we see that

$$4 \cdot 1^2 + t_n \cdot 2^2 = |D_n| = 4k$$

which gives us

$$t_n = k - 1 = \frac{n}{2} - 1.$$

Thus, we get the total number of inequivalent irreducible representations as

$$\begin{aligned} &\frac{n+3}{2} && \text{if } n \text{ is odd,} \\ &\frac{n}{2} + 3 && \text{if } n \text{ is even.} \end{aligned}$$

Example 2.63 (Finishing off D_n)**.** With the above calculations, we now finish the study of irreducible representations of D_n. Fix $n \ge 3$.

Let us first set up the notation as follows: $\theta := \frac{2\pi}{n}$ and

$$A_k := \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

for $k \in \{0, \ldots, n-1\}$.

Also, let

$$A := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As the reader might have guessed, the above matrices do indeed satisfy the following relations:

$$A_k^n = A^2 = (A_k A)^2 = I_n$$

and hence $r \mapsto A_k$, $s \mapsto A$ defines a two dimensional representation φ_k of D_n.

Our goal is now to identify as many irreducible and pairwise inequivalent representations as possible. We shall end up showing that we get precisely t_n many. (t_n being as in Example 2.62.)

First, we note that the eigenvector of A (up to scaling) are e_1 and e_2. Thus, if $\sin k\theta \neq 0$, then φ_k is irreducible. (Proposition 1.37.) Thus, we need to ensure that $k\theta \neq 0, \pi$.

Second, we need to see when two irreducible representations above are actually inequivalent. The answer is actually quite simple, in view of Theorem 2.36. One notes that

$$\chi_{\varphi_k}(r) = \text{trace } \varphi_k(r) = 2\cos k\theta$$

and hence, $\varphi_k \not\sim \varphi_{k'}$ if $\cos k\theta \neq \cos k'\theta$. Noting that $k\theta, k'\theta \in [0, 2\pi)$, simple trigonometry tells us that

$$\cos k\theta = \cos k'\theta \iff k = k', \frac{2\pi}{\theta} - k \iff k = k', n-k.$$

Thus, if we looks at $k \in \{1, \ldots, n-1\}$ such that $k\theta < \pi$, we see that all the φ_k are pairwise inequivalent.

If n is even, then there are $\frac{n}{2} - 1$ such k and if n is odd, then there are $\frac{n-1}{2}$ many such. However, by Example 2.62, there are no more and we are done!

Theorem 2.64. The set $B = \{\chi_1, \ldots, \chi_s\}$ is an orthonormal basis for $Z(L(G))$.

Proof. We shall assume that $\varphi^{(i)} : G \to U_{d_i}(\mathbb{C})$ since we wish to use Proposition 2.15. Since our statement is about characters, which is unaffected by equivalence, our claim follows.

Note that we know that $B \subset Z(L(G))$ since characters are indeed class functions. Moreover, we know that B is an orthonormal set, by First orthogonality relations. Thus, only spanning needs to be shown.

To this end, let $f \in Z(L(G)) \leq L(G)$ be given. By the previous corollary, we see that

$$f = \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)},$$

for some $c_{ij}^{(k)} \in \mathbb{C}$, $1 \leq k \leq s$, $1 \leq i, j \leq d_k$. Let $x \in G$ be arbitrary. Note that

$$\begin{aligned}
f(x) &= \frac{1}{|G|} \sum_{g \in G} f(x) \\
&= \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg) && \textit{$f \in Z(L(G))$} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)}(g^{-1}xg) \\
&= \sum_{i,j,k} \frac{1}{|G|} \sum_{g \in G} c_{ij}^{(k)} \varphi_{ij}^{(k)}(g^{-1}xg) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{1}{|G|} \sum_{g \in G} \varphi_{ij}^{(k)}(g^{-1}xg) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{1}{|G|} \sum_{g \in G} \left[\varphi^{(k)}(g^{-1}xg)\right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{1}{|G|} \sum_{g \in G} \varphi^{(k)}(g^{-1}xg)\right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{1}{|G|} \sum_{g \in G} \varphi_{g^{-1}}^{(k)} \varphi_{x}^{(k)} \varphi_{g}^{(k)}\right]_{ij} && \textit{$\varphi^{(k)}$ is a representation} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[(\varphi_x^{(k)})^{\#}\right]_{ij} && \textit{\# with respect to (φ, φ)} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{\operatorname{trace} \varphi_x^{(k)}}{\deg \varphi^{(k)}} I_{ij} && \textit{Item 2 of Proposition 2.15} \\
&= \sum_{i,k} c_{ii}^{(k)} \frac{\operatorname{trace} \varphi_x^{(k)}}{\deg \varphi^{(k)}} && \textit{$I_{ij} = 0$ if $i \neq j$ and $I_{ii} = 1$} \\
&= \sum_{i,k} c_{ii}^{(k)} \frac{\chi_k(x)}{d_k}. && \textit{definition of d_k and χ}
\end{aligned}$$

This shows that

$$f = \sum_{1 \leq k \leq s} \left[\sum_{1 \leq i \leq d_k} \frac{c_{ii}^{(k)}}{d_k}\right] \chi_k. \qquad \square$$

Corollary 2.65. The number of equivalence classes of irreducible representations of G is number of conjugacy classes of G.

Proof. By the above theorem, we have $s = \dim Z(L(G))$. By Proposition 2.33, we have $\dim Z(L(G)) = |\mathrm{Cl}(G)|$, as desired. □

Example 2.66 (Number of conjugacy classes of D_n)**.** By Example 2.62, we know the number of inequivalent irreducible representations of D_n. By the previous corollary, this is also the number of conjugacy classes of D_n.

Corollary 2.67. Let G be a finite group. Then, G has $|G|$ equivalence classes of irreducible representations if and only if G is abelian.

Proof. $|G| = |\mathrm{Cl}(G)|$ holds if and only if G is abelian. □

Corollary 2.68. Let G be a finite group. Then, G is abelian if and only if all the irreducible representations of G have degree one.

Proof. The "only if" was proven in Theorem 2.9.

To prove the "if" part, note that if G is not abelian, then $s < |G|$. On the other hand

$$d_1^2 + \cdots + d_s^2 = |G|.$$

Thus, at least one d_i is at least 2. In other words, there is a non-degree-one irreducible representation of G. □

Definition 2.69. Let G be a finite group with irreducible $\chi_1, \ldots, \chi_s$ and conjugacy classes $C_1, \ldots, C_s$. The character table of G is the $s \times s$ matrix X with $\mathsf{X}_{ij} = \chi_i(C_j)$. In other words, the rows of X are indexed by the characters of G and columns by the conjugacy classes; the (ij)-th entry of X denotes the value of the i-th character on the j-th conjugacy class.

Note that the fact that the above table is square (that is, the number of irreducible characters equals the number of conjugacy classes) is due to Corollary 2.65. We had seen an example of the character table of S_3. (Recall Table 1.)

Example 2.70 (Character table of $\mathbb{Z}/n\mathbb{Z}$)**.** As noted earlier, the character of a degree one representation is simply the representation itself. Thus, we get the table as follows. To make the table look more natural, we shall consider $\mathbb{Z}/n\mathbb{Z}$ as the n-th roots of unity.

Recall the n representations $\varphi^{(0)}, \ldots, \varphi^{(n-1)}$ from Example 1.7. Letting $\chi_k := \chi_{\varphi^{(k)}}$, we get the following character table.

	$[1]$	$[\omega_n]$	$\cdots$	$[\omega_n^{n-1}]$
χ_0	1	1	$\cdots$	1
χ_1	1	ω_n	$\cdots$	ω_n^{n-1}
χ_2	1	ω_n^2	$\cdots$	$\omega_n^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_{n-1}	1	ω_n^{n-1}	$\cdots$	$\omega_n^{(n-1)^2}$

Table 2: Character table of $\mathbb{Z}/n\mathbb{Z}$

The astute reader might have noticed that the columns are orthogonal. To make things more concrete, let us consider $n = 4$, in which case the table becomes as follows.

	$[1]$	$[\iota]$	$[-1]$	$[-\iota]$
χ_0	1	1	1	1
χ_1	1	ι	-1	$-\iota$
χ_2	1	-1	1	-1
χ_3	1	$-\iota$	-1	ι

Note that this was the case in Table 1. In could do a computation for two general columns in Table 2 and conclude the same. Instead of doing that, we now prove that this is always the case.

To do that, we first note that if C and C' are conjugacy classes of G, then the inner product of their columns is given by

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)},$$

where g (resp. h) is any element of C (resp. C').

Retaining the notation as in Definition 2.69, we get the following theorem.

Theorem 2.71 (Second orthogonality relations)**.** Let C, C' be conjugacy classes of G and let $g \in C$ and $h \in C'$. Then

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |G|/|C| & C = C', \\ 0 & C \neq C'. \end{cases}$$

Consequently, the columns of the character table are orthogonal and the matrix X is invertible.

Proof. Note that since $\{\chi_i\}$ form an orthonormal basis for $Z(L(G))$ and $\delta_{C'} \in Z(L(G))$, we get that

$$\delta_{C'} = \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i.$$

Thus, (where g is as in the theorem) we get

$$\begin{aligned}
\delta_{C'}(g) &= \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i(g) \\
&= \sum_{i=1}^{s} \frac{1}{|G|} \sum_{x \in G} \delta_{C'}(x) \overline{\chi_i(x)} \chi_i(g) \\
&= \sum_{i=1}^{s} \frac{1}{|G|} \sum_{x \in C'} \delta_{C'}(x) \overline{\chi_i(x)} \chi_i(g) \\
&= \frac{1}{|G|} \sum_{i=1}^{s} \sum_{x \in C'} \overline{\chi_i(x)} \chi_i(g).
\end{aligned}$$

Noting that χ_i is a class function and that $h \in C'$, the above simplifies as following.

$$\begin{aligned}
\delta_{C'}(g) &= \frac{1}{|G|} \sum_{i=1}^{s} \sum_{x \in C'} \overline{\chi_i(h)} \chi_i(g) \\
&= \frac{1}{|G|} \sum_{i=1}^{s} |C'| \overline{\chi_i(h)} \chi_i(g) \\
&= \frac{|C'|}{|G|} \sum_{i=1}^{s} \chi_i(g) \overline{\chi_i(h)}.
\end{aligned}$$

Rearranging gives us

$$\sum_{i=1}^{s} \chi_i(g) \overline{\chi_i(h)} = \frac{|G|}{|C'|} \delta_{C'}(g).$$

Noting that $\delta_{C'}(g) \neq 0 \iff \delta_{C'}(g) = 1 \iff g \in C' \iff C = C'$ yields the result. $\square$

§§2.6. Representations of Abelian Groups

We now conclude this section with completing our discussion of finite abelian groups. By Theorem 2.9, we know that every degree one representation of G has degree one. Moreover, by Corollary 2.67, we know that there are $|G|$ many such. We now explicitly calculate all of these.

Note that the structure theorem of finite abelian groups tells us that every such group is a direct product of cyclic groups. Since we already know explicitly these representations (and their character tables) by Example 1.7, we would get a complete description for all abelian groups.

Proposition 2.72. Let G_1, G_2 be finite abelian groups with $m = |G_1|$ and $n = |G_2|$. Suppose that $\rho_1, \ldots, \rho_m$ and $\varphi_1, \ldots, \varphi_n$ are all the irreducible representations of G_1 and G_2, respectively. The functions $\alpha_{ij} : G_1 \times G_2 \to \mathbb{C}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ given by

$$\alpha_{ij}(g_1, g_2) = \rho_i(g_1)\varphi_j(g_2)$$

form a complete set of irreducible representations of $G_1 \times G_2$.

Proof. Note that it suffices to show that each α_{ij} is a homomorphism. Indeed, the fact that each α_{ij} irreducible follows from the fact that it is degree one. Moreover, the fact that $\{\alpha_{ij}\}_{1\leq i\leq m}^{1\leq j\leq n}$ forms a complete set will follow once we show that all the mn α_{ij}s are distinct.

Homomorphism. Note that a degree one representation is simply a map into $\mathbb{C}^*$ and thus, commutativity gives us that

$$\begin{aligned}
\alpha_{ij}\left((g_1, g_2)(g_1', g_2')\right) &= \alpha_{ij}(g_1g_1', g_2g_2') \\
&= \rho_i(g_1g_1')\varphi_j(g_2g_2') \\
&= \rho_i(g_1)\rho_i(g_1')\varphi_j(g_2)\varphi_j(g_2') \\
&= \rho_i(g_1)\varphi_j(g_2)\rho_i(g_1')\varphi_j(g_2') \\
&= \alpha_{ij}(g_1, g_2)\alpha_{ij}(g_1', g_2').
\end{aligned}$$

Distinctness. Suppose that $\alpha_{ij} = \alpha_{kl}$. Then, note that

$$\rho_i(g_1) = \alpha_{ij}(g_1, 1) = \alpha_{kl}(g_1, 1) = \rho_k(g_1),$$

for all $g_1 \in G_1$. Thus, $i = k$. Similarly, analysing $\alpha_{ij}(1, g_2)$ for $g_2 \in G_2$ yields $j = l$, as desired. $\square$

Note that character of a degree one representation is the representation itself. The above proposition easily gives us the character table of the products now.

Example 2.73 (Character table of the Klein group)**.** Note that we have the following character table for $\mathbb{Z}/2\mathbb{Z}$.

	[0]	[1]
χ_1	1	1
χ_2	1	-1

Looking at the products, we get the following table for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

	$[(0,0)]$	$[(0,1)]$	$[(1,0)]$	$[(1,1)]$
χ_{11}	1	1	1	1
χ_{12}	1	-1	1	-1
χ_{21}	1	1	-1	-1
χ_{22}	1	-1	-1	1

§3. Permutation Representations

The reader is advised to recall Group actions. We shall continue with the notation established in that section.

Definition 3.1. Let $\sigma : G \to S_X$ be a group action. Define a representation $\sigma : G \to \mathrm{GL}(\mathbb{C}X)$ by setting

$$\widetilde{\sigma}_g\left(\sum_{x\in X} c_x x\right) = \sum_{x\in X} c_x \sigma_g(x).$$

$\widetilde{\sigma}$ is called the permutation representation associated to σ.

Remark 3.2. Note that $\widetilde{\sigma}$ is a representation by Proposition 0.19. Note that $\widetilde{\sigma}_g$ is the linear map defined by extending the map $x \mapsto \sigma_g(x)$. This can be done since X is a basis for $\mathbb{C}X$.

In more suggestive notation, the above representation can also be written as

$$\begin{aligned}\widetilde{\sigma}_g\left(\sum_{x\in X} c_x x\right) &= \sum_{x\in X} c_x \sigma_g(x)\\ &= \sum_{x\in X} c_x (g \cdot x)\\ &= \sum_{y\in X} c_{g^{-1}\cdot y}\, y.\end{aligned}$$

Remark 3.3. Recall Example 0.44 which was the action λ of G on G by left multiplication. Then, we have $\widetilde{\lambda} = L$, the Definition 2.54. Note that the degree of the action and of the representation coincide.

Remark 3.4. Recall the action σ of S_n on $\{1, \ldots, n\}$ as in Example 0.50. The corresponding $\widetilde{\sigma}$ is precisely the standard representation of S_n as in Example 1.21.

Proposition 3.5. Let $\sigma : G \to S_X$ be a group action. Then, the representation $\widetilde{\sigma} : G \to \mathrm{GL}(\mathbb{C}X)$ is unitary.

Proof. Let $g \in G$, $x, y \in X$ be arbitrary. Note that

$$\begin{aligned}
\left\langle \widetilde{\sigma}_g \sum_{x \in X} c_x x, \widetilde{\sigma}_g \sum_{x \in X} k_x x \right\rangle &= \left\langle \sum_{x \in X} c_{g^{-1} \cdot x} x, \sum_{x \in X} k_{g^{-1} \cdot x} x \right\rangle \\
&= \sum_{x \in X} c_{g^{-1} \cdot x} \overline{k_{g^{-1} \cdot x}} \qquad\qquad x \mapsto g \cdot y \\
&= \sum_{y \in X} c_y \overline{k_y} \\
&= \left\langle \sum_{x \in X} c_x, \sum_{x \in X} k_x x \right\rangle,
\end{aligned}$$

as desired. □

As before, we now wish to compute the character of such representations. As with the regular representation, we have a simple formula.

Proposition 3.6. Let $\sigma : G \to S_X$ be a group action. Then,

$$\chi_{\widetilde{\sigma}}(g) = |\mathrm{Fix}(g)|.$$

Proof. The proof is again almost identical to that of Proposition 2.58. Note that X acts as a basis for $\mathbb{C}X$. Fix an ordering $X = \{x_1, \ldots, x_n\}$. Let $g \in G$ be arbitrary. Note that the matrix $[\widetilde{\sigma}_g]$ with respect to this basis X will consists of columns with exactly with 1 and rest 0s.

More precisely, the i-th column will consist of all 0s and a 1 at the j-th position with j satisfies $x_j = g \cdot x_i$. In particular, $[\widetilde{\sigma}_g]_{ii} = 1$ iff $g \cdot x_i = x_i$ and 0 otherwise. The statement now follows at once. □

Corollary 3.7. Retaining the same notation, we have

$$\langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}} \rangle = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|^2 = \frac{|X|^2}{|G|} + \frac{1}{|G|} \sum_{1 \neq g \in G} |\mathrm{Fix}(g)|^2.$$

Corollary 3.8. Let $\sigma : G \to S_X$ be an action. If $|G| \nmid |X|^2$, then there exists $g \in G \setminus \{1\}$ and $x \in X$ such that $g \cdot x = x$.

Proof. Note that the statement is precisely saying that $|\mathrm{Fix}(g)| \neq 0$ for some $1 \neq g \in G$. Suppose not, that is, suppose that $|\mathrm{Fix}(g)| = 0$ for all $g \in G \setminus \{1\}$. Then, by the earlier corollary, we get that

$$\langle \chi_{\tilde{\sigma}}, \chi_{\tilde{\sigma}} \rangle = \frac{|X|^2}{|G|} \notin \mathbb{Z}.$$

However, this is a contradiction. (Remark 2.45.) $\square$

Definition 3.9. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. Then,

$$V^G := \{v \in V \mid \varphi_g(v) = v \text{ for all } g \in G\}$$

is a subspace of V, called the fixed subspace of G.

The check that V^G is a subspace is simple. We now show that it has some better properties.

Proposition 3.10. V^G is a G-invariant subspace.

Proof. Let $v \in V^G$ and $g \in G$. Then, $\varphi_g v = v$ by definition of V^G. Thus, $\varphi_g v \in V^G$. $\square$

Remark 3.11. The above proof also shows that the subrepresentation $\varphi|_{V^G}$ is the trivial one. By Example 1.28, we know that this can be written as a direct sum of $\dim V^G$ many trivial representations.

The next proposition shows that there are no more trivial representations in φ. To be more precise, given the decomposition

$$\varphi \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)},$$

the coefficient of the trivial representation is $\dim V^G$.

Proposition 3.12. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation and let χ_1 be the trivial representation of G. Then, $\langle \varphi, \chi_1 \rangle = \dim V^G$.

Proof. Since V^G is a G-invariant subspace, there exists a G-invariant subspace W such that

$$V = V^G \oplus W,$$

by Corollary 1.60. (The above is an *internal* direct product. In particular, $V^G \cap W = 0$.)

Let ψ and ρ denote the subrepresentations obtained by restricting φ to V^G and W, respectively. Then $\varphi \sim \psi \oplus \rho$, by Proposition 1.30.

Let $\varphi^{(1)}$ denote the trivial representation.

Claim. The multiplicity of $\varphi^{(1)}$ in ρ is 0.

Proof. Assume not. Let $W' \leq W$ be such that $\rho|_{W'} \sim \varphi^{(1)}$.
In particular, W' has dimension 1.
Choose a nonzero $w \in W' \leq W$. Then, $\rho_g(w) = w$ for all $g \in G$.[2] Thus, $w \in V^G$, a contradiction since $W \cap V^G = 0$. □

Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of inequivalent irreducible representations of G. Then, we know that

$$\psi \sim m_1\varphi^{(1)}$$

where $m_1 = \dim V^G$. (Remark 3.11.) The above claim shows that

$$\rho \sim 0\varphi^{(1)} \oplus m_2\varphi^{(2)} \oplus \cdots \oplus m_s\varphi^{(s)}.$$

Thus, we get

$$\varphi \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}.$$

Thus, we have $\langle \varphi, \chi_1 \rangle = m_1 = \dim V^G$.

□

Proposition 3.13. Let $\sigma : G \to S_X$ be a transitive group action. Define

$$v_0 := \sum_{x \in X} x \in \mathbb{C}X.$$

Then, $\mathbb{C}X^G = \mathbb{C}v_0$. In, particular, $\mathbb{C}X^G$ is one-dimensional.

Note that this is a special case of the immediate next proposition.

Proof. It is clear that $\mathbb{C}v_0 \leq \mathbb{C}X^G$ since every σ_g is simply a permutation of X. Thus, it suffices to show that v_0 spans $\mathbb{C}X^G$. The idea is simple. Consider $v \in \mathbb{C}X^G$. Then, it can represented in the standard basis as

$$v = \sum_{x \in X} c_x x.$$

We assert that c_x is independent of x. In other words, we show that $c_y = c_z$ for all $y, z \in X$.

Indeed, given $y, z \in X$, choose $g \in G$ such that $g \cdot y = z$. (We can do so since the action is transitive.)
Now, note that

$$\begin{aligned} v &= \widetilde{\sigma}_g(v) \\ \iff \sum_{x \in X} c_x x &= \sum_{x \in X} c_x g \cdot x. \end{aligned}$$

[2]We are using the fact that if a representation is equivalent to the trivial representation, then it acts as identity.

The coefficient of z is c_z on the left and c_y on the right and thus, $c_y = c_z$.

Thus, each $c_x = c$ for some $c \in \mathbb{C}$ and we get

$$v = \sum_{x \in X} c_x x = \sum_{x \in X} cx = c \sum_{x \in X} x = cv_0,$$

as desired. □

Proposition 3.14. Let $\sigma : G \to S_X$ be a group action. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be orbits of G on X. Define

$$v_i := \sum_{x \in \mathcal{O}_i} x \in \mathbb{C}X$$

for $i = 1, \ldots, m$. Then, $B = \{v_1, \ldots, v_m\}$ is a basis for $\mathbb{C}X^G$. In particular, $\dim \mathbb{C}X^G = m$, the number of orbits.

Proof. First, we show that B is indeed a subset of $\mathbb{C}X^G$. This is simple for if $1 \leq i \leq m$ and $g \in G$ are arbitrary, then

$$\begin{aligned} \widetilde{\sigma}_g v_i &= \widetilde{\sigma}_g \left(\sum_{x \in \mathcal{O}_i} x \right) \\ &= \sum_{x \in \mathcal{O}_i} \sigma_g(x) \\ &= \sum_{x \in \mathcal{O}_i} x \qquad \textit{$\sigma_g|_{\mathcal{O}_i}$ is a bijection} \\ &= v_i \end{aligned}$$

Second, we show that B is linearly independent. We do the usual by computing the inner product of elements of B. However, recall that the inner product on $\mathbb{C}X$ is essentially the "usual" dot product, just indexed by X. Since distinct orbits are disjoint, we get the following

$$\langle v_i, v_j \rangle = \begin{cases} |\mathcal{O}_i| & i = j, \\ 0 & i \neq j. \end{cases}$$

That is, B consists of non-zero orthogonal vectors and thus, is linearly independent.

Third, we show that B is spanning. Let $v \in \mathbb{C}X^G$ be an arbitrary vector given by

$$v = \sum_{x \in X} c_x x$$

for some scalars $c_x \in \mathbb{C}$. Note that G acts transitively on each orbit. Thus, by a similar argument as in the previous proof, we get that $c_z = c_y$ for all $y, z \in X$ if $z \in G \cdot y$.

Thus, for each $i = 1, \ldots, m$, there exists $c_i \in \mathbb{C}$ such that $c_x = c_i$ for all $x \in \mathcal{O}_i$. Hence, we may write v as

$$\begin{aligned} v &= \sum_{x \in X} c_x x \\ &= \sum_{i=1}^{m} \sum_{x \in \mathcal{O}_i} c_i x \\ &= \sum_{i=1}^{m} c_i \sum_{x \in \mathcal{O}_i} x \\ &= \sum_{i=1}^{m} c_i v_i \in \operatorname{span} B. \end{aligned}$$

□

Corollary 3.15. Suppose $\sigma : G \to S_X$ is a group action and $|X| > 1$. Then, $\widetilde{\sigma}$ is reducible.

Proof. Note that the degree of $\widetilde{\sigma}$ is $|X| > 1$. However, since X has at least one orbit, the previous proposition shows that the fixed subspace of G has dimension at least one. Thus, the trivial representation appears as a proper constituent in the decomposition of $\widetilde{\sigma}$. □

Corollary 3.16 (Burnside's lemma)**.** Let $\sigma : G \to S_X$ be a group action and let m be the number of orbits of G on X. Then,

$$m = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

That is, the number of orbits equals the average number of fixed points.

Proof. Let χ_1 denote the trivial character of G. Then, we note

$$\begin{aligned} &m \\ &= \dim \mathbb{C}X^G && \textit{Proposition 3.14} \\ &= \langle \chi_{\widetilde{\sigma}}, \chi_1 \rangle && \textit{Proposition 3.12} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\widetilde{\sigma}}(g) \overline{\chi_1(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\widetilde{\sigma}}(g) && \chi_1 \equiv 1 \\ &= \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|, && \textit{Proposition 3.6} \end{aligned}$$

finishing the proof. □

Corollary 3.17. Let $\sigma : G \to S_X$ be a group action. Then, the equalities

$$\operatorname{rank}(\sigma) = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|^2 = \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}} \rangle$$

hold.

Proof. The left equality follows by recalling that the definition of rank is the number of orbits of σ^2. Thus, applying Burnside's lemma to σ^2 yields the equality since $\left|\operatorname{Fix}^2(g)\right| = |\operatorname{Fix}(g)|^2$, by Proposition 0.59.

The right equality is simply Corollary 3.7. □

Definition 3.18. Let $\sigma : G \to S_X$ be a <u>transitive</u> action. Let $v_0 := \sum_{x \in X} x \in \mathbb{C}X$.

$\mathbb{C}v_0 = \mathbb{C}X^G$ is a G-invariant subspace. $V_0 := \mathbb{C}v_0^{\perp}$ is G-invariant. Let $\widetilde{\sigma}'$ denote the restriction of $\widetilde{\sigma}$ to V_0.

$\mathbb{C}v_0$ is called the trace of σ, V_0 the augmentation of σ, and $\widetilde{\sigma}'$ the augmentation representation associated to σ.

Remark 3.19. Let us justify the various statements in the definition.

$\mathbb{C}v_0 = \mathbb{C}X^G$ followed from Proposition 3.14.

Since $\widetilde{\sigma}$ is a unitary representation, $V_0 := \mathbb{C}v_0^{\perp}$ is G-invariant, by the proof of Proposition 1.54.

Theorem 3.20. Let $\sigma : G \to S_X$ be a transitive group action. Then, the augmentation representation $\widetilde{\sigma}'$ is irreducible if and only if G is 2-transitive.

Proof. Given that σ is transitive, we see that σ is 2-transitive if and only if $\operatorname{rank}(\sigma) = 2$, by Proposition 0.54. Also note that Lemma 2.40 gives us

$$\chi_{\widetilde{\sigma}'} = \chi_{\widetilde{\sigma}} - \chi_1.$$

Thus, we get

$$\begin{aligned}
\langle \chi_{\widetilde{\sigma}'}, \chi_{\widetilde{\sigma}'} \rangle &= \langle \chi_{\widetilde{\sigma}} - \chi_1, \chi_{\widetilde{\sigma}} - \chi_1 \rangle \\
&= \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}} \rangle - \langle \chi_{\widetilde{\sigma}}, \chi_1 \rangle - \langle \chi_1, \chi_{\widetilde{\sigma}} \rangle + \langle \chi_1, \chi_1 \rangle \\
&= \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}} \rangle - \langle \chi_{\widetilde{\sigma}}, \chi_1 \rangle - \langle \chi_1, \chi_{\widetilde{\sigma}} \rangle + 1 && \textit{Proposition 3.12} \\
&= \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}} \rangle - \dim \mathbb{C}X^G - \overline{\dim \mathbb{C}X^G} + 1 && \dim \mathbb{C}X^G = 1, \textit{ Proposition 3.14} \\
&= \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}} \rangle - 1 && \textit{Corollary 3.17} \\
&= \operatorname{rank}(\sigma) - 1.
\end{aligned}$$

By Corollary 2.44, $\widetilde{\sigma}'$ is irreducible iff $\langle \chi_{\widetilde{\sigma}'}, \chi_{\widetilde{\sigma}'} \rangle = 1$ iff $\operatorname{rank}(\sigma) - 1 = 1$ iff $\operatorname{rank}(\sigma) = 2$ iff σ is 2-transitive. □

Example 3.21 (Character table of S_4)**.** Note that we have five conjugacy classes in S_4. (Recall Description of conjugacy classes.) One set of representatives is

$$1, (12), (12)(34), (123), (1234).$$

We already know it has exactly two degree one representations. (Example 1.9.) Let χ_1 denote the character of the trivial representation and χ_2 of the sign representation.

Let ρ denote the standard representation of S_4. (Example 1.21.) Recall that this is the permutation permutation corresponding to the natural action of S_4 on $\{1, \ldots, 4\}$. (Remark 3.4.) Also, recall that this action is 2-transitive. (Example 0.50.) Thus, by Theorem 3.20, the augmentation representation is a degree three irreducible representation of S_4. Let us denote its character by χ_4. We know that $\chi_4 = \chi_\rho - \chi_1$.

Thus, there are two more left. By Proposition 2.60, we see the sums of squares of their degrees is 13. Thus, the degrees are two and three. (This is the reason we used χ_4 and not χ_3 for the earlier representation.)

Let χ_3 and χ_5 denote the characters of the unknown degree two and three representations, respectively. Thus, so far, we have the following table.

	[1]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2				
χ_4	3	1	-1	0	-1
χ_5	3				

(Note that $\chi_4 = \chi_\rho - \chi_1$ is easy to calculate because $\chi_\rho(\tau)$ is the number of elements fixed by τ. Thus, we just look at the number of elements fixed by τ and subtract 1 to get χ_4.)

From the above table, we note that

$$\chi_2((12)) \neq 1 \quad \text{and} \quad \chi_4((12)) \neq 0.$$

Thus, by Corollary 2.46, we see that multiplying the representations corresponding to χ_2 and χ_4 gives us a new inequivalent irreducible degree three representation. Thus, the character χ_5 is obtained by multiplying the corresponding characters to get the following.

	[1]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2				
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

The remaining entries of χ_2 are now easy to fill since the columns are orthogonal, by Theorem 2.71. Since we do know the first column completely, the other columns can be filled.

Computing the inner product for $g \neq 1$ with the first column, we get

$$1\chi_1(g) + 1\chi_2(g) + 2\chi_3(g) + 3\chi_4(g) + 3\chi_5(g) = 0$$

or

$$\chi_3(g) = -\frac{1}{2}(\chi_1(g) + \chi_2(g) + 3\chi_4(g) + 3\chi_5(g)).$$

(We have dropped the conjugate since everything is real.)

Thus, we fill the last row to obtain the table as follows.

	[1]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Table 3: Character table of S_4

List of Examples