Extending Conformal Mappings Onto the Unit Disc

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Notations and Conventions

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4. Recall that a conformal mapping of Ω onto $\mathbb{D}$ is simply a biholomorphism $\Omega \to \mathbb{D}$.
5. A curve shall mean a continuous function with domain $[0, 1]$. Typically, γ will be a curve such that $\gamma([0, 1)) \subseteq \Omega$ and $\gamma(1) \in \partial\Omega$. Similarly, Γ will be a curve such that $\Gamma([0, 1)) \subseteq \mathbb{D}$ and $\Gamma(1) \in \partial\mathbb{D}$.

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Thus, it also makes sense to ask whether *the* extension is a homeomorphism onto $\overline{\mathbb{D}}$.

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Furthermore, if $\widetilde{f}$ is an injection, then compactness again tells us that $\widetilde{f}$ is a homeomorphism (as $\widetilde{f}$ is a bijection).

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In words: there is a curve in Ω which passes through α_n and ends at β.

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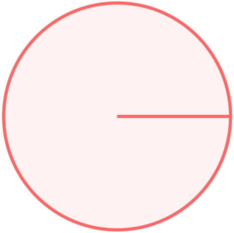

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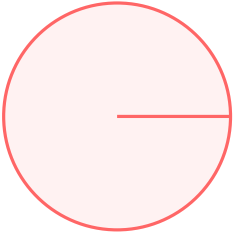

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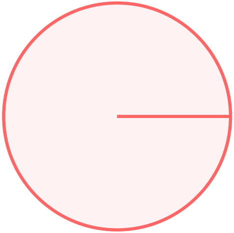

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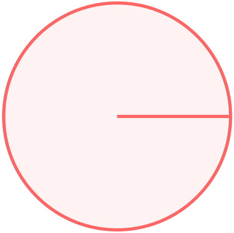

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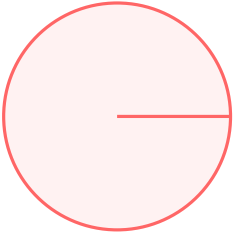

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To this end, let $(z_n)_n$ be an arbitrary sequence in $\overline{\Omega}$ that converges to z.

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