Extending Conformal Mappings Onto the Unit Disc

Aryaman Maithani

IIT Bombay

16th March 2022

Notations and Conventions

1. $\mathbb{D}$ will denote the open unit disc. $S^1 := \partial\mathbb{D}$ is the unit circle.
2. $\mathcal{O}^\infty$ denotes the set of bounded holomorphic functions on $\mathbb{D}$.
3. Ω will always denote a nonempty, open, bounded, and simply-connected subset of $\mathbb{C}$.
4. Recall that a conformal mapping of Ω onto $\mathbb{D}$ is simply a biholomorphism $\Omega \to \mathbb{D}$.
5. A curve shall mean a continuous function with domain $[0, 1]$. Typically, γ will be a curve such that $\gamma([0, 1)) \subseteq \Omega$ and $\gamma(1) \in \partial\Omega$. Similarly, Γ will be a curve such that $\Gamma([0, 1)) \subseteq \mathbb{D}$ and $\Gamma(1) \in \partial\mathbb{D}$.

Introduction

Let Ω be a bounded simply-connected domain in $\mathbb{C}$. By the Riemann Mapping Theorem, we know that there exists a biholomorphism $f : \Omega \to \mathbb{D}$. The following is a natural question.

Question

Can f be continuously extended up to $\overline{\Omega}$?

The obvious way to extend f is via sequences. In fact, if an extension exists, this *is* how it must be obtained. In particular, this extension is unique and we must have $f(\overline{\Omega}) \subseteq \overline{\mathbb{D}}$. This must force $f(\overline{\Omega}) = \overline{\mathbb{D}}$. (Why?)

Thus, it also makes sense to ask whether *the* extension is a homeomorphism onto $\overline{\mathbb{D}}$.

Spoilers

What we shall see is the following: By imposing a *simple* topological restriction on Ω, one gets that *any* biholomorphism $\Omega \to \mathbb{D}$ can be extended to a continuous *injective* map $\overline{\Omega} \to \overline{\mathbb{D}}$. Moreover, this will be a *homeomorphism*.

Remark 1

The last line is not difficult to see. Indeed, once we have continuously extended f to $\widetilde{f} : \overline{\Omega} \to \overline{\mathbb{D}}$, we have

$$\mathbb{D} \subseteq \widetilde{f}(\overline{\Omega}) \subseteq \overline{\mathbb{D}}.$$

As $\widetilde{f}(\overline{\Omega})$ is compact, we have $\widetilde{f}(\overline{\Omega}) = \overline{\mathbb{D}}$.

Furthermore, if $\widetilde{f}$ is an injection, then compactness again tells us that $\widetilde{f}$ is a homeomorphism (as $\widetilde{f}$ is a bijection).

Simple Boundary Points

Definition 2

A boundary point β of Ω is called a simple boundary point if β has the following property:
For every sequence $(\alpha_n)_n$ in Ω such that $\alpha_n \to \beta$ as $n \to \infty$, there exists a curve γ and a strictly increasing sequence $(t_n)_n$ in $(0, 1)$ such that

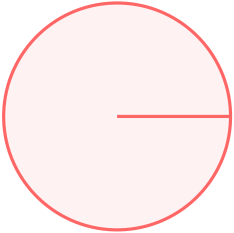
$$t_n \to 1,\ \gamma(t_n) = \alpha_n\ (n = 1, 2, \ldots,),\ \gamma([0, 1)) \in \Omega.$$

$\gamma(1) = \beta$ follows by continuity.

In words: there is a curve in Ω which passes through α_n and ends at β.

(Non-)Examples

1. Every boundary point of $\mathbb{D}$ is a simple boundary point.
2. Let $\Omega := \mathbb{D} \setminus [0, 1)$.



The boundary points of Ω lying on the real axis are not simple. Note that Ω is indeed bounded and simply-connected and thus, biholomorphic to $\mathbb{D}$. However, $\partial\Omega$ is clearly not homeomorphic to $\partial\mathbb{D}$ and thus, no biholomorphism can be extended to a homeomorphism $\overline{\Omega} \to \overline{\mathbb{D}}$.

Some constraints

Theorem 3 (Helper Theorem)

Let Ω be a bounded simply-connected domain, and let f be a conformal mapping of Ω onto $\mathbb{D}$.

1. If β is a simple boundary point of Ω, then f has a continuous extension to $\Omega \cup \{\beta\}$. If f is so extended, then $|f(\beta)| = 1$.
2. If β_1 and β_2 are distinct simple boundary points of Ω and *if* f is continuously extended to $\Omega \cup \{\beta_1, \beta_2\}$, then $f(\beta_1) \neq f(\beta_2)$.

We give the proof after proving the main theorem assuming the above.

As remarked, the extension in 1 is unique and would have to be attained as follows: given a sequence $(\alpha_n)_n$ in Ω converging to β, we have $f(\beta) := \lim f(\alpha_n)$. Once we show that this limit (exists and) is independent of the sequence (α_n), we would have shown continuity.

The Main Extension Theorem

Theorem 4

If Ω is a bounded simply-connected domain and if every boundary point of Ω is simple, then every conformal mapping of Ω onto $\mathbb{D}$ extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{D}$.

Proof.

Let $f : \Omega \to \mathbb{D}$ be a biholomorphism. By the Helper Theorem and the remark following it, we see that we may extend f to $\overline{\Omega}$ using sequences. By 2, it follows that f so extended is one-one. We now check that it is continuous on $\overline{\Omega}$. As remarked earlier, this would finish the proof.

To this end, let $(z_n)_n$ be an arbitrary sequence in $\overline{\Omega}$ that converges to z. We can pick a corresponding sequence $(\alpha_n)_n$ in Ω such that $|\alpha_n - z_n| < 1/n$ and $|f(\alpha_n) - f(z_n)| < 1/n$. Thus, $\alpha_n \to z$ and hence, $f(\alpha_n) \to f(z)$. In turn, $f(z_n) \to f(z)$, as desired. $\square$

A Purely Topological Corollary

Recall that a Jordan curve is the image of an injective map $S^1 \to \mathbb{C}$.

Corollary 5

If every boundary point of a bounded simply-connected region Ω is simple, then the boundary of Ω is a Jordan curve, and $\overline{\Omega}$ is homeomorphic to $\overline{\mathbb{D}}$.

In fact, the converse is true too: If the boundary of Ω is a Jordan curve, then every boundary point of Ω is simple.

Blackboxes

Theorem 6 (Radial Limit Theorem)

To every $g \in \mathcal{O}^\infty$ corresponds a function $g^* \in L^\infty(S^1)$, defined almost everywhere by

$$g^*(e^{\iota\theta}) = \lim_{r\to 1} g(re^{\iota\theta}).$$

If $g^*(e^{\iota\theta}) = 0$ for almost all $e^{\iota\theta}$ on some arc $J \subseteq S^1$, then $g(z) = 0$ for every $z \in \mathbb{D}$.

Theorem 7 (Lindelöf's Theorem)

Suppose Γ is a curve such that $\Gamma([0,1)) \subseteq \mathbb{D}$ and $\Gamma(1) = 1$.
If $g \in \mathcal{O}^\infty$ and

$$\lim_{t\to 1^-} g(\Gamma(t)) = L,$$

then g has radial limit L at 1.

Proof of Helper Theorem

Now, we prove the earlier Helper Theorem assuming the earlier two results.

1. Suppose that f cannot be extended to β. Then, there exists a sequence $(\alpha_n)_n$ in Ω and points $w_1, w_2 \in \overline{\mathbb{D}}$ such that

$$\alpha_n \to \beta, f(\alpha_{2n}) \to w_1, f(\alpha_{2n+1}) \to w_2, w_1 \neq w_2.$$

Choose γ as given by β being a simple boundary point, and put $\Gamma := f \circ \gamma$. Let $g = f^{-1}$ and put $K_r := g(\overline{D}(0; r))$ for $0 < r < 1$. Then K_r is a compact subset of Ω. Since $\gamma(t) \to \beta$ as $t \to 1$, there exists $t^* < 1$ (depending on r) such that $\gamma(t) \notin K_r$ if $t^* < t < 1$. Thus, $|\Gamma(t)| \to 1$ as $t \to 1$. In particular, $|w_1| = |w_2| = 1$.

Let J be one of the open arcs of $S^1 \setminus \{w_1, w_2\}$ such that every radius of $\mathbb{D}$ which ends at a point of J intersects the range of Γ in a set which has a limit point on S^1. By the Radial Limit Theorem, g has radial limits a.e. on S^1 since $g \in \mathcal{O}^\infty$ (as Ω is bounded).

Continuing the Proof

We have that $g \circ \Gamma = \gamma$ and that g has radial limits a.e. on S^1. Now, for whichever $e^{it} \in J$ the radial limit does exist, we must have

$$\lim_{r \to 1} g(re^{it}) = \beta,$$

since $g(\Gamma(t)) = \gamma(t) \to \beta$ as $t \to 1$. Thus, by the Radial Limit Theorem again, applied to $g - \beta$, we see that $g \equiv \beta$ on $\mathbb{D}$, contradicting that g is an injection.

Thus, we have shown that $w_1 = w_2$ and $|w_1| = 1$.

Continuing the Proof

2 Now, we need to prove that an extension takes different values at different boundary points. Let β_1, β_2 be simple boundary points with $f(\beta_1) = f(\beta_2)$. We may assume $f(\beta_i) = 1$. Let γ_i be curves with $\gamma_i([0,1)) \subseteq \Omega$, and $\gamma_i(1) = \beta_i$, and let $\Gamma_i := f \circ \gamma_i$. Then, each Γ_i satisfies the condition of Lindelöf's Theorem with

$$\lim_{t \to 1} g(\Gamma_i(t)) = \lim_{t \to 1} \gamma_i(t) = \beta_i.$$

Thus, the radial limit of g at 1 is both β_1 and β_2 and hence, $\beta_1 = \beta_2$. $\square$