

$$\int (\circlearrowleft \circlearrowright) dx$$

MA 408

Measure Theory

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Spring 2020-21

Lecture 1

06 January 2021 22:06

Idea behind measure

Simplified case: Subsets of \mathbb{R}

Given $E \subseteq \mathbb{R}$, want to assign "length" or "content" to E .

Ideally, want a map

$$\mu: \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$$

s.t.

$$(1) \quad \mu(\emptyset) = 0$$

(2) For any $E \subseteq \mathbb{R}$ and $x \in E$,

$$\mu(E) = \mu(x + E).$$

$$(x + E := \{x + y : y \in E\})$$

↑ translation by x

(3) Given a countable collection $\{E_i\}_{i=1}^{\infty}$ of disjoint subsets of \mathbb{R} , we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

(So far, $\mu \equiv 0$ will satisfy above properties!)

$$(4) \quad \mu([0, 1]) = 1. \quad (\text{"Normalisation"})$$

Any such μ would be a "candidate" for our content.

However, no such μ exists!

Consider the following sets:

- (1) Define \sim on \mathbb{R} by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
 Clearly, \sim is an equiv. relation.

Let $E \subseteq [0, 1]$ be a set containing exactly one element from each equivalence class in \mathbb{R}/\sim .

(Existence is given by Axiom of Choice. Note that)
 distinct equiv. classes are disjoint. and a small argument
 that lets you conclude $E \subset [0, 1]$.

Q. What could $\mu(E)$ be?

Note that $\{E+r\}_{r \in \mathbb{Q} \cap [0, 1]}$ is a collection of pairwise disjoint sets.

Sketch. If $x \in (E+r_1) \cap (E+r_2)$, then $x = r_1 + e_1 = r_2 + e_2$ for some $e_1, e_2 \in E$

$$\begin{aligned} & \Rightarrow e_1 - e_2 = r_2 - r_1 \in \mathbb{Q} \\ & \Rightarrow e_1 \sim e_2 \Rightarrow e_1 = e_2 \end{aligned}$$

Moreover, $[0, 1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [0, 1]} (E+r) \subseteq [0, 2] = [0, 1] \cup [1, 2]$

An easy consequence of (1) - (3) is that $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$.

Proof. $\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$. \square

$$\Rightarrow \mu([0, 1]) \leq \mu\left(\bigcup_{i=1}^{\infty} (E+r_i)\right) \leq \mu([0, 1]) + \mu([1, 2])$$

enumerate $\mathbb{Q} \cap [0, 1]$ as $\{r_1, \dots\}$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E+r_i) \leq 2$$

$[1, 2] = [0, 1] + 1$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E + r_i) \leq 2 \quad [1, 2] = [0, 1] + 1$$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E) \leq 2$$

If $\mu(E) = 0$ $\rightarrow \leftarrow$

If $\mu(E) = r > 0$ $\rightarrow \leftarrow$

$$\sum_{i=1}^{\infty} \mu(E) = \infty \leq 2 \rightarrow \leftarrow$$

Possible way to salvage : Replace (3) to have "finite union" instead of "countable".

Turns out that that's still not enough.

(2) BANACH - TARSKI THEOREM (1924) : (Using AC)

For any open sets $U, V \subseteq \mathbb{R}^n$ where $n \geq 3$,
there exists $k \in \mathbb{N}$ and set $U_1, \dots, U_k, V_1, \dots, V_k$
s.t.

$$(1) \quad U_i \cap U_j = \emptyset, \quad V_i \cap V_j = \emptyset, \quad 1 \leq i \neq j \leq k.$$

$$(2) \quad U = \bigcup_{i=1}^k U_i, \quad V = \bigcup_{i=1}^k V_i.$$

$$(3) \quad U_i \cong V_i, \text{ i.e.,}$$

U_i is obtained from V_i by a sequence of rotations,
reflections, and translations.

In other words, by isometries.

Thus, the analogue of (2) implies $\mu(U_i) = \mu(V_i) \forall i$.

$$\Rightarrow \mu(U) = \mu(V).$$

\rightsquigarrow Absurd conclusions.

As it turns out, the problem is NOT in the infinite union but rather the demand that μ is defined on all of $\mathcal{P}(\mathbb{R})$!

Thus, we restrict our attention to a smaller collection of subsets of \mathbb{R} . (Not too small!)

σ -ALGEBRAS

Let X be an arbitrary set.

Def.(1) An algebra ("field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

(algebra, field)

$$\textcircled{1} A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$$

$$\textcircled{2} A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{F} \quad \text{for any } n \in \mathbb{N}.$$

(2) A σ -algebra (" σ -field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

(\sigma-algebra, \sigma-field)

$$\textcircled{1} A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$$

$$\textcircled{2} A_1, \dots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

Note that complements and unions give no intersections. Also, $\emptyset, X \in \mathcal{F}$.

EXAMPLES

$$\textcircled{1} \mathcal{F} = \mathcal{P}(X) \leftarrow \text{both}$$

$$\textcircled{2} (\text{Countable-cocountable } \sigma\text{-algebra})$$

(Countable-cocountable \sigma-algebra)

$$\mathcal{F} = \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$$

Proof. Clearly closed under complement.

Let $A_1, \dots \in \mathcal{F}$.

If all A_i are countable, then $\bigcup A_i$ is.

Suppose A_i not countable. Then, A_i^c is.

But

$$A_i \subset \bigcup A_i \Rightarrow (\bigcup A_i)^c \subset A_i^c$$

$$\Rightarrow (\bigcup A_i)^c \text{ is countable. } \square$$

③ Given any $\mathcal{F} \subseteq P(X)$, we can talk about
 σ -algebra generated by \mathcal{F} denoted $M(\mathcal{F})$
defined by (\sigma-algebra generated)

$$M(\mathcal{F}) = \bigcap B$$

$$\begin{matrix} f \subseteq B \\ B \text{ is a } \sigma\text{-alg} \end{matrix}$$

Note that the intersection is non-empty because of $P(X)$.
Easy to see that intersection of σ -algebras is again a σ -alg.

by construction, $M(\mathcal{F})$ is the smallest σ -alg containing \mathcal{F} .

BOREL σ -ALGEBRA.

Defn: Let (X, τ) be a topological space.

The σ -algebra generated by τ is called the Borel σ -algebra on X , denoted $\mathcal{B}(X)$.

(Borel \sigma-algebra)

(Abuse of notation that we don't mention τ .)

In other words, it is generated by the open sets of X .

Borel σ -algebra on \mathbb{R} : Smallest σ -alg on \mathbb{R} containing all the open sets.

Consequences:

- (1) All open sets are in $\mathcal{B}(\mathbb{R})$.
- (2) All closed sets are in $\mathcal{B}(\mathbb{R})$.
- (3) All F_σ, G_δ sets are in $\mathcal{B}(\mathbb{R})$.

$$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i \quad (F_i \text{ closed}) ; \quad G_\delta \equiv \bigcap_{i=1}^{\infty} G_i \quad (G_i \text{ open})$$

Prop.: Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$.

Then, \mathcal{B} is also generated by any of the following:

- (i) $\{(a, b) : a < b\}$ or $\{[a, b] : a < b\}$
- (ii) $\{[a, b) : a < b\}$ or $\{(a, b] : a < b\}$
- (iii) $\{(a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a) : a \in \mathbb{R}\}$
- (iv) $\{[a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a] : a \in \mathbb{R}\}$

Proof: Easy. \square

Borel σ -algebra on \mathbb{R}^n :

Suppose $\{X_i\}_{i=1}^n$ are metric spaces.

Let $X = \prod_{i=1}^n X_i$ with the product metric.
↓ defn

If f_i is the metric on X_i , then
 f on $\prod X_i$ is defined as

$$f(x, y) = \max_{1 \leq i \leq n} f_i(x_i, y_i) \quad \begin{cases} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{cases}$$

Def.: Suppose (X_i, M_i) are σ -algebras. One can define a σ -algebra on $X := \prod X_i$ as follows:

(Product of \sigma-algebras)

Consider the projection maps $\pi_i : X \rightarrow X_i$.
 Let

$$\mathcal{F} = \{ \pi_i^{-1}(E) : E \in M_i, i = 1, \dots, n \}$$

$$= \{E \times x_2 \times \dots \times x_n : E \in M_1\} \\ \cup \{x_1 \times E \times \dots \times x_n : E \in M_2\} \\ \cup \dots \cup \{x_1 \times \dots \times x_{n-1} \times E : E \in M_n\}.$$

$M := M(\mathcal{F}) \subseteq \mathcal{P}(X)$ is the product σ -algebra induced by $\{M_i\}_{i=1}^n$.

We often write the above as $M = \overline{\prod}_{i=1}^n M_i$.

Caution: The above $\overline{\prod}_{i=1}^n$ is NOT the set-theoretic cartesian product.

Now, we get two (possibly different) σ -algebras on \mathbb{R}^n .

- ① Borel σ -alg. on $(\mathbb{R}^n, \mathcal{J})$
- ② Product of Borel σ -alg. of $\mathcal{B}(\mathbb{R})$.

Prop: $\mathcal{B}(\mathbb{R}^n) = \overline{\prod}_{i=1}^n \mathcal{B}(\mathbb{R})$. That is, both the σ -alg above are same.

Proof: We will prove this by a sequence of observations.

- ① Suppose $\{(x_i, M_i)\}_{i=1}^n$ are σ -algebras and $f_i \subseteq M_i$ are such that $M_i = M(f_i)$. ($i=1, \dots, n$)

Then, if $X = \overline{\prod}_{i=1}^n X_i$ and $M = \overline{\prod}_{i=1}^n M_i$, then

M is generated by $\{\prod_{i=1}^n (E_i) : E_i \in f_i, i=1, \dots, n\}$.

- ② M is generated by $\{E_1 \times \dots \times E_n : E_i \in f_i\}$ if we further assume $x_i \in f_i$.

Assuming ① and ② for now, we now note the following.

Clearly, one has $\prod_{i=1}^n \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^n)$.

[Proof: using ②, $\prod_{i=1}^n \mathcal{B}(\mathbb{R})$ is gen. by sets of the form $U_1 \times \dots \times U_n$, each $U_i \subseteq \mathbb{R}$ open]

Each such set is open in the metric space \mathbb{R}^n .
Thus, it is in $\mathcal{B}(\mathbb{R}^n)$.

We show $\mathcal{B}(\mathbb{R}^n) \subseteq \prod \mathcal{B}(\mathbb{R})$.

(*) { It suffices to show that every set of the form

$$U_1 \times \dots \times U_n \quad \text{where } U_i \subseteq \mathbb{R} \text{ are open}$$

are in the product $\prod \mathcal{B}(\mathbb{R})$.

{ Why? Every open set in \mathbb{R}^n is a countable union of sets of aforementioned form. In turn, the open sets generate $\mathcal{B}(\mathbb{R}^n)$.

Proving (*) is easy because

$$U_1 \times \dots \times U_n = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \dots \cap \pi_n^{-1}(U_n).$$

these are in $\prod \mathcal{B}(\mathbb{R})$, by def

Proof of ①

Want to show that $\tilde{\mathcal{F}} = \{\pi_i^{-1}(E) : E \in \mathcal{F}_i, 1 \leq i \leq n\}$ gen. $\prod M_i$.

Clearly $M(\tilde{\mathcal{F}}) \subseteq M$. ($\tilde{\mathcal{F}} \subseteq M$ and M is σ -alg)

It now suffices to show that every generator of M is in $M(\tilde{\mathcal{F}})$. (standard)

Note $M = \langle \pi_i^{-1}(E) : E \in M_i, 1 \leq i \leq n \rangle$
 $\tilde{M} := \langle \pi_i^{-1}(E) : E \in \mathcal{F}_i, 1 \leq i \leq n \rangle = M|\tilde{\mathcal{F}}$

Let $\tilde{M}_i := \{E \in M_i : \pi_i^{-1}(E) \in \tilde{M}\} \subseteq P(X_i)$.

We shall show that $\tilde{M}_i = M_i$.

We know, by defⁿ that $\mathcal{F}_i \subseteq \tilde{M}_i$. ($E \in \mathcal{F}_i \stackrel{e \in M_i}{\Rightarrow} \pi_i^{-1}(E) \in \tilde{M} \cap M_i$)

Moreover, $M(\mathcal{F}_i) = M_i$. Thus, it suffices to show that \tilde{M}_i is a σ-alg.

To that end, let $A \in \tilde{M}_i$. Then, $\pi_i^{-1}(A) \in \tilde{M}$.

Then, $\pi_i^{-1}(A)^c \in \tilde{M}$. But $\pi_i^{-1}(A^c) = \pi_i^{-1}(A)^c \in M$.

$$\Rightarrow \pi_i^{-1}(A^c) \in M_i$$

Similarly, noting that $\pi_i^{-1}\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigcup_{j=1}^{\infty} \pi_i^{-1}(A_j)$ yields the result.

Proof of ②

Let $f' := \{E_1 \times \dots \times E_n : E_i \in \mathcal{F}_i\}$.

Note that $E_1 \times \dots \times E_n = \pi_1^{-1}(E_1) \cap \dots \cap \pi_n^{-1}(E_n) \in M$.

Thus, $M(f') \subseteq M$.

Or else, $\pi_i^{-1}(E_i) = E_1 \times x_2 \times \dots \times x_n \in \mathcal{F}'$

or in general,

$$\pi_i^{-1}(E_i) \subset \prod_{j=1}^n A_j \in \mathcal{F}' \quad \text{where} \quad A_j = \begin{cases} E_i & ; j=i \\ X_j & ; j \neq i \end{cases}$$

$$\Rightarrow M \subseteq M(f')$$

REMARKS.

① The argument above generalises for a separable

metric spaces.

- ② If $(X_i, M_i)_{i \in A}$, ^{and A is COUNTABLE} then again, $X = \prod X_i$, $M = \prod M_i$ generated by $\{\pi_i^{-1}(E) : E \in M_i, i \in A\}$ is also generated by sets of the form $\left(\prod_{i \in A} E_i\right)$, $E_i \in \mathcal{F}_i$.

MEASURE

(Measure)

Defn. Suppose (X, M) is a measure space, i.e., M is a σ -algebra on X . A measure on X is a map $\mu: M \rightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) if $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

$\hookrightarrow M$

By abuse, we shall interchangeably call (X, M) or (X, M, μ) a measure space.

EXAMPLES.

(1) $X = \{x_1, x_2, \dots\}$ is countable. Suppose $p_i \geq 0$ are reals s.t. $\sum_{i=1}^{\infty} p_i = 1$. Let $M = P(X)$ and define $\mu: M \rightarrow [0, 1]$ as

$$\mu(E) = \sum_{i: x_i \in E} p_i.$$

(2) (X, M) be s.t. M is the countable-co-countable σ -alg. s.t. X itself is uncountable

Define

$$\mu(E) := \begin{cases} 0 & ; E \text{ is countable} \\ 1 & . . . \end{cases}$$

L

, \mathcal{E} is uncountable

Prop. Suppose (X, \mathcal{M}, μ) is a measure space.

Then,

$$\textcircled{1} \quad E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$$

$$\textcircled{2} \quad \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i) \quad (\mu \text{ is "sub-additive"})$$

(sub-additive, subadditive, sub additive)

\textcircled{3} If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \dots$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i).$$

Proof. \textcircled{1} \& \textcircled{2} are trivial

\textcircled{3} Define $F_i = E_i \setminus E_{i-1}$ for $i \geq 2$.
 $F_1 = E_1$

Then, $\bigcup_{i=1}^n F_i = \bigcup_{i=1}^n E_i$. Also, $F_i \in \mathcal{M}$ for each i .
 $(n = \infty \text{ as well})$ Moreover, $F_i \cap F_j = \emptyset$ for $i \neq j$.

$$\begin{aligned} \text{Thus, } \mu\left(\bigcup E_i\right) &= \mu\left(\bigcup F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(F_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i) = \sum_{i=1}^{\infty} E_i. \end{aligned}$$

Def. \textcircled{1} A null set in a measure space (X, \mathcal{M}, μ) is a set E s.t. $E \subseteq F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.
 $(E \in \mathcal{M} \text{ NOT necessary.})$ (Null set)

\textcircled{2} Given a measure space (X, \mathcal{M}, μ) , the completion of \mathcal{M} , denoted $\bar{\mathcal{M}}$ is the collection of all sets of the form $F \cup N$ where $F \in \mathcal{M}$ and N is a null set. (Completion)

Prop. ① If (X, \mathcal{M}, μ) is a measure space, then $\bar{\mathcal{M}}$ is a σ -alg.

② Moreover, there exists a unique measure

$$\bar{\mu}: \bar{\mathcal{M}} \rightarrow [0, \infty] \text{ s.t.}$$

$$\bar{\mu}|_{\mathcal{M}} = \mu.$$

(That is, there is a unique extension of μ to a measure $\bar{\mu}$ on $\bar{\mathcal{M}}$.)

Lecture 2

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Prop. ① If (X, \mathcal{M}, μ) is a measure space, then $\bar{\mathcal{M}}$ is a σ -alg.
 ② Moreover, there exists a unique measure

$$\bar{\mu}: \bar{\mathcal{M}} \rightarrow [0, \infty] \quad \text{s.t.}$$

$$\bar{\mu}|_{\mathcal{M}} = \mu.$$

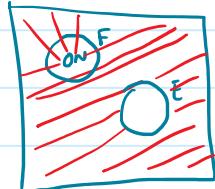
Proof ① To show that $\bar{\mathcal{M}}$ is a σ -algebra, we need to show:

- (i) $A \in \bar{\mathcal{M}} \Rightarrow A^c \in \bar{\mathcal{M}}$
- (ii) $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bar{\mathcal{M}}$.

(i) $A = E \cup N$, $N \subseteq F$, $\mu(F) = 0$, $E, F \in \mathcal{M}$
 $A^c = E^c \cap N^c$

Now, $E \cup F \in \mathcal{M}$ and hence, $E^c \cap F^c$.

Note that $E^c \cap N^c = (\underbrace{E^c \cap F^c}_{\in \mathcal{M}}) \cup (\underbrace{F \setminus N}_{\subseteq F \text{ and } \mu(F) = 0}) \in \bar{\mathcal{M}}$.



Now, if $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}}$, we can write

$$A_i = E_i \cup N_i, \quad N_i \subseteq F_i, \quad \mu(F_i) = 0, \quad E_i, F_i \in \mathcal{M}.$$

$$\bigcup A_i = \left(\bigcup \underbrace{E_i}_{=: E} \right) \cup \left(\bigcup \underbrace{N_i}_{=: N} \right)$$

Note $E \in \mathcal{M}$. Also, put $F = \bigcup F_i$. Then, $F \in \mathcal{M}$.
 Moreover, $N \subseteq F$ & $\mu(F) = \sum \mu(F_i) = 0$.

$\therefore \bigcup A_i = E \cup N$, in the desired form.

(iv) Define $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ as

$$\bar{\mu}(E \cup N) := \mu(E).$$

To show: $\bar{\mu}$ is well-defined.

Suppose $E_1 \cup N_1 = E_2 \cup N_2$.

$$\rightarrow E_1 \subset E_2 \cup N_2 \subseteq E_2 \cup F_2$$

$$\text{by } \Rightarrow \mu(E_1) \leq \mu(E_2) + \mu(F_2) = \mu(E_2).$$

$$\mu(E_2) \leq \mu(E_1). \quad \therefore \mu(E_1) = \mu(E_2).$$

Thus, $\bar{\mu}$ is well-defined

Note that this was also
the only way to define $\bar{\mu}$.
Thus, uniqueness will follow.

To show: $\bar{\mu}$ is a measure.

$$\textcircled{1} \quad \bar{\mu}(\emptyset) = 0 \text{ is trivial since } \bar{\mu}(\emptyset) = \bar{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0.$$

\textcircled{2} Suppose $\{E_i \cup N_i\}_{i=1}^{\infty}$ are pairwise disjoint.

$$\begin{aligned} \bar{\mu}(U(E_i \cup N_i)) &= \bar{\mu}((\cup E_i) \cup (\cup N_i)) = \mu(\cup E_i) = \sum \mu(E_i) \\ &= \sum \bar{\mu}(E_i \cup N_i) \end{aligned}$$

some logic
earlier

OUTER MEASURE

Defn: An outer measure on X is a map (Outer measure)

$\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$ satisfying

$$(i) \mu^*(\emptyset) = 0,$$

$$(ii) A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B),$$

$$(iii) \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Motivation for μ^* comes from the intuitive idea that knowing areas enclosed by rectangles, we "approximate" areas bounded by arbitrary sets by covering these by countable union of rectangles.

Propn. Suppose $f \subseteq \mathcal{P}(X)$ and $g: f \rightarrow [0, \infty]$ s.t.

- (i) $\emptyset, X \in f$
- (ii) $g(\emptyset) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} g(E_i) : E_i \in f, E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

Then, μ^* is an outer measure.

Proof. We need to show μ^* is well-defined \leftarrow This follows because and it satisfies $X \in f$.

$$(i) \mu^*(\emptyset) = 0 \quad \leftarrow \text{trivial since } \mu^*(\emptyset) \geq 0 \text{ since inf over non-negative reals}$$

$$\text{Also, } \emptyset \subseteq \emptyset \text{ & } g(\emptyset) = 0$$

$$(ii) A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$$

Any cover of B is also a cover of A .
 \therefore We are taking inf over a larger set & hence, it will be smaller.
 $\Rightarrow \mu^*(A) \leq \mu^*(B)$

$$(iii) \text{ Suppose } \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{P}(X). \text{ Fix } \varepsilon > 0.$$

for each E_i , let $\{A_j^{(i)}\}_{j=1}^{\infty} \subseteq f$ be a cover of E_i :

$$\dots * (i-1) > \sum_{j=1}^{\infty} g(A_j^{(i)}) < \dots$$

with $\mu^*(E_i) \geq \sum_{j=1}^{\infty} \rho(A_j^{(i)}) - \frac{\epsilon}{2^i} + \epsilon$

Then, $\bigcup_i \bigcup_j A_j^{(i)}$ covers $\bigcup E_i$.

$$\begin{aligned} \text{Thus, } \mu^*\left(\bigcup E_i\right) &\leq \sum_{i,j} A_j^{(i)} \leq \left(\sum_{i=1}^{\infty} \mu^*(E_i) + \frac{\epsilon}{2^i} \right) \\ &= \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon \\ \Rightarrow \mu^*\left(\bigcup E_i\right) &\leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbit., this is completes the proof. \square

Defn. Given an outer measure μ^* , we say that a set $A \subseteq X$ is μ^* -measurable if for all $E \in P(X)$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c). \quad (\text{(\mu^*-measurable})$$

Defn. A measure μ on (X, M) is complete if for all $F \in M$ with $\mu(F) = \phi$, we have $P(F) \subseteq M$.

(That is, all null sets are in M .) (Complete measure)

Thm. (CARATHÉODORY) (Carathéodory, Caratheodory)

Let μ^* be an outer measure on X .

Let $M := \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$.

Then,

(i) M is a σ -algebra.

(ii) μ^* restricted to M is a complete measure.

Proof. (i) M is closed under (\setminus) since the defn of μ^* -meas. is symmetric under (\setminus) .

Now, if $A, B \in M$ and $E \subset X$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A^c \cup B^c)) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)\end{aligned}$$

Of course, we also have $\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \leq \mu^*(E)$.

$$\therefore \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Thus, M is closed under finite unions. To show that it is a σ -alg, it suffices to show closure under disjoint unions.

(why? Given $\{E_i\}_{i=1}^{\infty} \subseteq M$, consider $F_i = E_i$ and $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right)$ for $n \geq 2$.)
Note that $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and that each $F_i \in M$.

Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint collection in M . Put $B_n = \bigcup_{i=1}^n A_i$
and $B = \bigcup_{i=1}^{\infty} A_i$.

Then, for any $E \subset X$,

$$\begin{aligned}B_n \cap A_n &= \bigcup_{i=1}^n (A_n \cap A_i) = A_n \\ B_n \cap A_n^c &= \bigcup_{i=1}^n (A_n^c \cap A_i) = \bigcup_{i=1}^{n-1} A_i \\ &= B_{n-1},\end{aligned}$$

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) \quad \because \underbrace{\mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2})}_{\dots} \\ &= \dots\end{aligned}$$

$$\Rightarrow \mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \left[\sum_{i=1}^n \mu^*(E \cap A_i) \right] + \mu^*(E \cap B_n^c)$$

$$(B_n \subset B \Rightarrow B^c \subset B_n^c)$$

Take $n \rightarrow \infty$

$$\geq \left[\sum_{i=1}^{\infty} \mu^*(E \cap A_i) \right] + \mu^*(E \cap B^c)$$

take $n \rightarrow \infty$

$$\Rightarrow \mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)$$

$$\begin{aligned}
 (\star) \quad & \geq \mu^*\left(\bigcup(E \cap A_i)\right) + \mu^*(E \cap B^c) \\
 & = \mu^*\left(E \cap \left(\bigcup A_i\right)\right) + \mu^*(E \cap B^c) \\
 & = \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).
 \end{aligned}$$

Thus, we have equality throughout giving

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c) \text{ & hence, } B \in M.$$

Moreover, taking $E = B$; the (\star) equation gives

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Thus, μ^* is countably additive on M .

Thus, μ is a measure on M .

To show completeness: Let $F \in M$ be s.t. $\mu(F) = 0$ and $A \subseteq F$.

Then, $\mu^*(A) = 0$. Also, for any $E \subseteq X$, we have

$$\begin{aligned}
 \mu^*(E) & \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(A) + \mu(E \cap A^c) \\
 & = \mu^*(E \cap A^c) \leq \mu^*(E).
 \end{aligned}$$

Thus, $A \in M$.

B

Def.: Suppose \mathcal{F} is an algebra on X . (Pre-measure)
A map

$$\mu_0: \mathcal{F} \rightarrow [0, \infty]$$

is called a pre-measure if

- (i) $\mu_0(\emptyset) = 0$,
- (ii) If $\{\mathcal{A}_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint s.t. $\bigcup_{i=1}^{\infty} \mathcal{A}_i \in \mathcal{F}$,

then

$$\mu_0\left(\bigcup_{i=1}^{\infty} \mathcal{A}_i\right) = \sum_{i=1}^{\infty} \mu_0(\mathcal{A}_i).$$

(Note that the above also gives the thing for finite unions.
guaranteed to be in \mathcal{F} .)

Propn: Suppose μ_0 is a premeasure on an algebra \mathcal{F} .
Then, if μ^* is the outer measure as defined in the
earlier proposition, then

- (i) $\mu^*|_{\mathcal{F}} = \mu_0$
- (ii) Every set in \mathcal{F} is μ^* -measurable.

An immediate corollary:

Thm: Suppose $\mathcal{F} \subseteq P(X)$ is an algebra and suppose M is the σ -algebra generated by \mathcal{F} .
Let μ_0 be a premeasure defined on \mathcal{F} and let μ^* be the outer measure as before. Then,

- (i) $\mu^*|_M$ is a measure on (X, M) . Put $\mu = \mu^*|_M$.
- (ii) If ν is any measure extending μ_0 , then

$$\nu(E) = \mu(E)$$

whenever $\mu(E) < \infty$

Proof of Propn.

(i) For $E \in \mathcal{F}$, want to show, $\mu^*(E) = \mu_0(E)$.

Considering $E_1 = E$ and $E_i = \emptyset$ for $i \geq 2$ gives

$$\sum_{i=1}^{\infty} \mu_0(E_i) = \mu_0(E) \Rightarrow \mu^*(E) \leq \mu_0(E).$$

\downarrow inf over all covers

To show \geq : let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$ be a cover for E .

$$\text{Let } F_i := E \cap (E_i \setminus \bigcup_{j=1}^{i-1} E_j).$$

Clearly (i) $F_i \in \mathcal{F}$ by closure properties

(ii) $F_i \cap F_j = \emptyset$ if $i \neq j$.

$$(iii) \bigcup_{i=1}^{\infty} F_i = E.$$

If $\{F_i\}$ is a cover s.t. $\sum_{i=1}^{\infty} \mu_0(F_i) \leq \mu^*(E) + \varepsilon$
(such a cover exists)

$$\text{Note } \mu_0(E) = \mu_0\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu_0(F_i) \leq \mu^*(E) + \varepsilon$$

μ_0 is a premeasure

$$\Rightarrow \mu_0(E) \leq \mu^*(E) + \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \mu_0(E) \leq \mu^*(E).$$

This establishes (i).

(ii) To show: Every $A \in \mathcal{F}$ is μ^* -measurable.

Let $A \in \mathcal{X}$. Let $\{E_n\}$ be a cover for A s.t.

$$\sum_n \mu_0(E_n) \leq \mu^*(A) + \varepsilon$$

Then,

$$\begin{aligned} \mu^*(A) + \varepsilon &\geq \sum_n \mu_0(E_n) \xrightarrow{\substack{E_n, A, A^c \in \mathcal{F} \\ \text{and } \mu_0 \text{ is pre-meas.}}} \\ &= \sum [I_{E_n \cap A} + I_{E_n \cap A^c}] \end{aligned}$$

$$= \sum_n [\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c)]$$

$$= \left[\sum_n \underbrace{\mu_0(E_n \cap A)}_{\substack{\text{cover for} \\ E \cap A}} \right] + \left[\sum_n \underbrace{\mu_0(E_n \cap A^c)}_{E \cap A^c} \right]$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

$$\Rightarrow \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad (E \text{ was arbit.})$$

\leq is anyway true for any outer measure.

$$\Rightarrow \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Thus, f is μ^* -measurable.

Proof of Thm.

(i) Follows from Carathéodory and Propn. (Since F is μ^* -measurable and M is the smallest σ -alg containing F .)

(ii) First note that if ν extends μ_0 , then

for any $E \in M = M(F)$

if $E \subseteq \bigcup_n E_n$, $E_n \in F$, then

$$\nu(E) \leq \sum_n \nu(E_n) = \sum_n \mu_0(E_n).$$

So if the cover is s.t. $\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon = \mu(E) + \varepsilon$,

this gives $\nu(E) \leq \mu(E) + \varepsilon$ for arbit ε . That is,

$$v(E) \leq \mu(E).$$

Now, if $\mu(E) < \infty$, we show \geq .

Let $\{E_i\}_{i=1}^n$ be a cover for $E \in \mathcal{M}$ and let $A := \bigcup_{i=1}^n E_i$.

$$\begin{aligned} \text{Note that } v(A) &= \lim_{n \rightarrow \infty} \left(v\left(\bigcup_{i=1}^n E_i\right) \right) = \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{i=1}^n E_i\right) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n E_i\right) \\ &= \mu(A) \end{aligned}$$

Since $\mu(E) < \infty$ (by assumption), we can pick a cover $\{E_i\}_{i=1}^n$ s.t.

$$\mu(A) \underset{\substack{\text{H(A)} \\ + \text{H(A \setminus E)}}}{<} \mu(E) + \epsilon \text{ and thus, } \mu(A \setminus E) < \epsilon.$$

$$\begin{aligned} \text{Thus, } \mu(E) &\leq \mu(A) = v(A) = v(E) + v(A \setminus E) \leq v(E) + \mu(A \setminus E) \\ &< v(E) + \epsilon \end{aligned}$$

This gives $\mu(E) \leq v(E)$, as desired.

TOWARDS "GOOD" BOREL MEASURES

The idea is to extend/define a measure μ on $\mathcal{B}(\mathbb{R})$ from the notion of length of bounded intervals. Whatever done so far leads to that.

(Half-interval, half interval)

Defn. A half-interval is a subset of \mathbb{R} of the form:

- (i) $(a, b]$ for $-\infty \leq a < b < \infty$, or
- (ii) (a, ∞) for $-\infty \leq a < \infty$, or
- (iii) \emptyset .

Can be checked that the collection of fin unions of half-intervals is an algebra on \mathbb{R}

Propⁿ: Let F be the algebra consisting of finite unions of half-intervals.

Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing and right continuous function.

$$\left(\lim_{\delta \downarrow 0} F(x + \delta) = F(x) \quad \forall x \in \mathbb{R} \right)$$

Define

$$\mu_0 \left(\bigcup_{j=1}^n (a_j, b_j] \right) := \sum_{j=1}^n (F(b_j) - F(a_j)),$$

$$\text{and } \mu_0(\emptyset) = 0.$$

Then, μ_0 is a premeasure on F .

Remarks:

(1) Note that F above actually generates $\mathcal{B}(\mathbb{R})$, as seen in Lec 1.

(2) If we take $F(x) = x$, then $F(b) - F(a) = b - a$
= length of $[a, b]$.

So, the above extends the notion of measure arising from lengths of intervals onto the Borel σ -field.

(3) Why right-continuity?

Suppose μ is a finite Borel measure. Let

$$F(x) := \mu((-\infty, x]).$$

$$\begin{aligned} \text{Then, if } x_n \downarrow x, \text{ then } \lim_{n \rightarrow \infty} \mu((-\infty, x_n]) &= \mu \left(\bigcap_{n=1}^{\infty} (-\infty, x_n] \right) \\ &\stackrel{\text{def}}{=} \mu((-\infty, x]) = F(x) \end{aligned}$$

Thus, this F above is right-continuous.

Note that closure on right

Proof of the Propⁿ

First, we need to check that μ_0 is well-defined.

Let $\{(a_j, b_j]\}$ ($j=1, \dots, n$) be pairwise disjoint
and let $\bigcup_{j=1}^n (a_j, b_j] = (a, b]$.

Then, by re-arranging indices, if necessary, it follows that

$a = a_1 < b_1 = a_2 < b_2 = \dots = a_n < b_n = b$ and in
this case

$$\mu_0((a, b)) = F(b) - F(a)$$

$$\mu_0\left(\bigcup_{j=1}^n (a_j, b_j]\right) = \sum F(b_j) - F(a_j) = F(b) - F(a)$$

Thus, μ_0 is well-defined in this case.

More generally, if $\{I_i\}_{i=1}^n$ and $\{J_j\}_{j=1}^m$ are s.t.

$$\bigcup_{i=1}^n I_i = \bigcup_{j=1}^m J_m, \text{ then}$$

$$\sum_i \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_j \mu_0(J_j)$$

since intersection of half-intervals is again
a half-interval, for which the defⁿ is consistent.

This shows that μ_0 is well-defined on \mathcal{F} .

It now remains to show that μ_0 is indeed a premeasure
on \mathcal{F} .

Lecture 3

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To show μ_0 is a pre-measure, we need to prove:

If $A_i \in \mathcal{F}$ are pairwise disjoint s.t. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, then

$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

Note the μ_0 being finitely additive is clear.

Suppose $\{I_j\}_{j=1}^{\infty}$ is a collection of p-wise disjoint half intervals and

$$\bigcup_j I_j = [a, b] = I, \text{ say.}$$

Case i. $-\infty < a < b < \infty$.

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} I_j\right) &= \mu_0\left(\bigcup_{j=1}^n I_j\right) + \mu_0\left(I \setminus \bigcup_{j=1}^n I_j\right) \\ &\stackrel{\mathcal{F}}{\geq} \mu_0\left(\bigcup_{j=1}^n I_j\right) \\ &= \sum_{j=1}^n \mu_0(I_j) \end{aligned}$$

and this holds $\forall n \in \mathbb{N}$. Thus, $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$.

For the other side, let $\varepsilon > 0$ be arbitrary.

We know (by defⁿ of μ_0) that $\mu_0(I) = f(b) - f(a)$.
 (Write $I_j = (a_j, b_j]$.)

By right continuity, let $\delta > 0$ be s.t. $f(a + \delta) - f(a) < \delta$.
 Similarly, let $\delta_j > 0$ be s.t.

$$F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j} \quad \forall j.$$

Note that $\{(a_j, b_j + \delta_j)\}_{j=1}^\infty$ covers $[a+\delta, b]$.

Since $[a+\delta, b]$ is compact, there is a finite subcover.

By removing those intervals that are contained in larger intervals, we assume wlog that

- (i) $(a_1, b_1 + \delta_1), \dots, (a_N, b_N + \delta_N)$ cover $[a+\delta, b]$.
- (ii) for each j , $b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1})$ for $j=1, \dots, N-1$.

$$\mu_o(I) = F(b) - F(a) = F(b) - F(a+\delta) + \underbrace{F(a+\delta) - F(a)}_{<\varepsilon, \text{ by choice}}$$

$$\begin{aligned} F(b) - F(a+\delta) &\leq F(b_N + \delta_N) - F(a_1) \quad (\text{since } F \uparrow) \\ &= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j+1}) - F(a_j)] \\ &\leq F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] \\ &\leq F(b_N) - F(a_N) + \underbrace{\sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)]}_{\mu_o(I_N)} + \varepsilon \end{aligned}$$

$$\text{So, } \mu_o(I) \leq \sum_{j=1}^N \mu_o(I_j) + 2\varepsilon \leq \sum_{j=1}^\infty \mu_o(I_j) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\mu_o(I) \leq \sum_{j=1}^\infty \mu_o(I_j)$.

$$\text{Thus, } \mu_o(I) = \sum_{j=1}^\infty \mu_o(I_j).$$

Case ii. $a = -\infty$, i.e., $I = (-\infty, b]$

In this case, $(a_j, b_j + \delta_j)$ cover $[-M, b]$ for M large. By the previous part, we have

By the previous part, we have

$$F(b) - F(-M) \leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\epsilon.$$

As $M \rightarrow \infty$, the LHS $\rightarrow \mu_0(I)$.

(Note that the other inequality did not need $a > -\infty$.)

Case iii. If $b = \infty$, we use a similar argument with
 $F(M) - F(a) \leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\epsilon.$ \square

To summarise, we have:

Thm. Suppose $G, F : \mathbb{R} \rightarrow \mathbb{R}$ is \uparrow and right continuous. Then there is a unique Borel measure $\mu = \mu_F$ s.t.
 $\mu(a, b] = F(b) - F(a).$

Furthermore

(i) $\mu_F = \mu_G \Leftrightarrow F - G = \text{constant}$.

(ii) Conversely, if μ is a Borel measure s.t.
 $\mu(a, b] < \infty$ whenever $|a|, |b| < \infty$ and we define

$$F(x) = \begin{cases} \mu(0, x] & ; x > 0 \\ 0 & ; x = 0 \\ -\mu(-x, 0] & ; x < 0 \end{cases}$$

then F is \uparrow , right cts and $\mu = \mu_F$.

Prof. The "immediate corollary" theorem from last lecture shows that extension of pre measure is unique on sets with finite measures. Actually true in more generality. First a def.

Def. A measure μ on (X, \mathcal{M}) is called σ -finite if there exist sets $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ s.t. $\bigcup_{j=1}^{\infty} E_j = X$.

there exist sets $\{E_j\}_{j=1}^{\infty} \subset M$ s.t.
 $\mu(E_j) < \infty \forall j$ and $X = \bigcup_{j=1}^{\infty} E_j$

(\sigma-finite)

The last theorem (extension of outer measure measure from a pre-measure on F to a measure on $M(F)$) has an additional statement:

Thm. If ν is a measure extending μ_0 and suppose μ (coming from outer measure) is σ -finite, then $\nu = \mu$.

Proof. let $\{A_i\}_{i=1}^{\infty} \rightarrow$ p-wise disj. s.t. $\bigcup_{i=1}^{\infty} A_i = X$ & $\mu(A_i) < \infty$
 (can always disjointify the E_i in defⁿ of σ -finite.)

For any $E \in M$

$$\begin{aligned}\mu(E) &= \mu\left(\bigcup_{i=1}^{\infty}(E \cap A_i)\right) = \sum \mu(E \cap A_i) \xrightarrow[\text{by earlier thm \&}]{\nu(E \cap A_i) < \infty} \\ &= \sum \nu(E \cap A_i) \\ &= \nu\left(\bigcup_{i=1}^{\infty} E \cap A_i\right) \\ &= \nu(E).\end{aligned}$$

□

Remark. The measure μ_F is called the Lebesgue-Stieltjes measure associated with F .

(Lebesgue-Stieltjes measure)

Prop. Let F be \uparrow and right-continuous. Let μ be the corresponding measure, $M = M_F = M_{\mu_F}$ (μ complete).

For any $E \in M$,

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}.$$

Proof. Let us denote

$$\inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} =: \nu(E).$$

Want to show $\mu(E) = \nu(E)$ $\forall E \in \mathcal{M}$.

First, note that $(a_j, b_j) = \bigcup_{i=1}^{\infty} I_j^{(i)}$ for some

half-intervals $I_j^{(i)}$ and in particular

$$E \subset \bigcup_{i,j} I_j^{(i)} \text{ and hence, } \mu(E) \leq \sum_{i,j} \mu(I_j^{(i)}) \\ = \sum_i \mu(a_i, b_i)$$

$$\Rightarrow \mu(E) \leq \nu(E).$$

For the other inequality, let $\varepsilon > 0$ be arbit.

Let $\{(a_j, b_j]\}_{j=1}^{\infty}$ be s.t.

$$\sum_j \mu(a_j, b_j] \leq \mu(E) + \varepsilon. \quad (*)$$

As before, let $\delta_j > 0$ be s.t.

$$F(b_j + \delta_j) - F(b_j) < \varepsilon/2^j$$

$$\begin{aligned} \sum_j \mu(a_j, b_j + \delta_j) &\leq \sum_j \mu(a_j, b_j] + \sum_j \mu(b_j, b_j + \delta_j] \\ &= \sum_j \mu(a_j, b_j] + \sum_j [F(b_j + \delta_j) - F(b_j)] \\ &< \sum_j \mu(a_j, b_j] + \varepsilon \quad \text{use } (*) \\ &\leq \mu(E) + 2\varepsilon \end{aligned}$$

$$\Rightarrow \nu(E) \leq \mu(E) + 2\varepsilon \quad \forall \varepsilon > 0.$$

$$\Rightarrow \gamma(E) \leq \mu(E)$$

$$\Rightarrow \gamma(E) = \mu(E).$$

Thm.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is \uparrow and right continuous. Let $\mu = \mu_f$.

Then,

$$\begin{aligned}\mu(E) &= \inf \left\{ \mu(U) : E \subset U, U \stackrel{\text{open}}{\subset} \mathbb{R} \right\} \\ &= \sup \left\{ \mu(K) : K \subset E, K \stackrel{\text{compact}}{\subset} \mathbb{R} \right\}\end{aligned}$$

Proof.

(i) Suppose $U \supseteq E$, then $\mu(U) \geq \mu(E)$ and hence

$$\inf \{ \dots \} \geq \mu(E).$$

Now, let $\epsilon > 0$ be arbitrary.

By the previous proposition,

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(a_i, b_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}.$$

Then, $\exists \{(a_i, b_i)\}_{i=1}^{\infty}$ s.t.

$$\sum_{i=1}^{\infty} \mu(a_i, b_i) < \mu(E) + \epsilon.$$

Let $U = \bigcup_{i=1}^{\infty} (a_i, b_i)$. Then, $\mu(U) < \sum_{i=1}^{\infty} \mu(a_i, b_i)$.

Then, $\mu(U) < \mu(E) + \epsilon$. But U is open.

Thus, $\mu(E) = \inf \{ \mu(U) : E \subseteq U \text{ open} \}$.

(ii) First assume that E is bounded.

- If E is closed, then E is compact and the result is trivially true.

Consider the closure $\bar{E} \leftarrow$ this is closed and bounded.

By the previous part, $\exists U$ open s.t.

$$U \supseteq \bar{E} \setminus E \quad \text{and} \quad \mu(U) < \mu(\bar{E} \setminus E) + \varepsilon.$$

↳ measurable since
 $\bar{E} \in \mathcal{B}_R$ and $E \in \mathcal{M}$

Let $K = \bar{E} \setminus U$. Note $K \subseteq E$ and

$$\begin{aligned}\mu(K) &= \mu(E) - \mu(E \cap U) = \mu(E) - (\mu(U) - \mu(U \setminus E)) \\ &= \mu(E) - \mu(U) + \mu(U \setminus E) \\ &\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E) > \mu(E) - \varepsilon\end{aligned}$$

$$\Rightarrow \mu(K) > \mu(E) - \varepsilon.$$

$$\Rightarrow \mu(E) = \sup \{\mu(K) : E^c \supset K \rightarrow \text{compact}\}.$$

Now, if E is unbounded, define

$$E_j = E \cap [j, j+1] \quad \text{for } j \in \mathbb{Z}.$$

By the above argument, for any $\varepsilon > 0$, \exists compact $k_j \subset E_j$
with

$$\mu(k_j) \geq \mu(E_j) - \frac{\varepsilon}{2^j}.$$

let $H_n = \bigcup_{j=-n}^n k_j$. Then, H_n is compact, $H_n \subset E$ and

$$\mu(H_n) \geq \mu\left(\bigcup_{j=-n}^n E_j\right) - \varepsilon.$$

Since $\mu(E) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{j=-n}^n E_j\right)$, the result follows.

Ex Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is \uparrow and right continuous. Let $\mu = \mu_f$.
TFAE:

(i) E is measurable,

(ii) $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$,

(iii) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

(iii) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

$$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i \quad (F_i \text{ closed}) ; \quad G_\delta \equiv \bigcap_{i=1}^{\infty} G_i \quad (G_i \text{ open})$$

Proof.

(i) \Rightarrow (ii), (iii)

Suppose $\mu(E) < \infty$. Then, \exists open U_i and compact K_j s.t.

$$\mu(U_i) - \frac{1}{2^j} \leq \mu(E) \leq \mu(K_j) + \frac{1}{2^j}$$

$$\text{with } K_j \subseteq E \subseteq U_i.$$

$$\text{Let } V = \bigcap_i U_i \text{ and } H = \bigcup_j K_j. \text{ Then,}$$

$$H \subset E \subset V \text{ and } \mu(H) = \mu(E) = \mu(V) \text{ and}$$

so, $\mu(V \setminus E) = 0 = \mu(E \setminus H)$

If $\mu(E) = \infty$, put $E_j := E \cap (j, j+1]$ for $j \in \mathbb{Z}$.

Note that each E_j has finite measure. Then,

$$E_j = V_j \setminus N_j^{(1)} = H_j \cup N_j^{(2)}. \text{ Then,}$$

$$\begin{aligned} E &= \left(\bigcup H_i \right) \cup \left(\bigcup N_j^{(2)} \right) \\ &\quad \xleftarrow{\text{still } F_\sigma} \xrightarrow{\text{still null}} \\ &= \left(\bigcup V_i \right) \setminus \left(\bigcup N_j^{(1)} \right) \\ &\quad \xleftarrow{\text{still } G_\delta} \end{aligned}$$

(ii), (iii) \Rightarrow (i) is obvious because M_{μ_F} is complete and

contains F_σ, G_δ sets.

□

Defn

Consider $F : \mathbb{R} \rightarrow \mathbb{R}$ defined as $F(x) = x$.

Then, $m = \mu_F$ is called the Lebesgue measure and

we shall denote μ_F by L , the set of Lebesgue measurable sets.

(Lebesgue measure)

$$\mathcal{B}(\mathbb{R}) \subset L \subset \mathcal{P}(\mathbb{R})$$

\downarrow \downarrow
 not complete complete measure
 (why?)

Thm. For any $x \in \mathbb{R}$ and $E \in L$, define

$$x + E := \{x + y : y \in E\} \text{ and}$$

$$x \cdot E := \{xy : y \in E\}.$$

Then, $x + E, x \cdot E \in L$ and

$$m(x + E) = m(E), \text{ and}$$

$$m(x \cdot E) = |x| \cdot m(E).$$

Proof Consider the field F of disjoint union of half-intervals. Since F is invariant under translations and dilations, it follows that $\mathcal{B}(\mathbb{R})$ follows the same.

Let v_1 and v_2 be measures defined by

$$v_1(E) := m(x + E) \text{ and } v_2(E) := m(x \cdot E).$$

Note that v_1 and v_2 are pre-measures on F and

$$v_1(I) = m(I) \text{ and } v_2(I) = |x| \cdot m(I)$$

for intervals. Thus, by uniqueness of extension of pre-measures, it follows that

$$v_1(E) = m(x + E), \quad v_2(E) = |x| \cdot m(E) \quad \forall E \in \mathcal{B}_{\mathbb{R}}.$$

Thus, null sets remain null under v_1 and v_2 .

Thus, the general result follows. □

REMARKS.

(1) $L \neq P(\mathbb{R})$

(2) All countable sets have measure zero.

However, there are sets which are uncountable and have measure zero.

EXAMPLE. (CANTOR SET)

Let $C_0 = [0, 1]$. $C_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3})$.

$C_2 = C_1 \setminus ((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}))$,

Define $C = \bigcap_{n=0}^{\infty} C_n$

Facts :

(i) C is compact.

(ii) C consists of all those $x \in [0, 1]$ which have a ternary expansion have no 1s.

In particular, C is uncountable.

$$(iii) m(C_0) = 1 \quad \text{and} \quad m(C_n) = \frac{2}{3} m(C_{n-1}) \quad \forall n \geq 1$$

$$= \left(\frac{2}{3}\right)^n$$

$$\text{Thus, } m(C) = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0.$$

(iv) Since $C \subset L$ and $m(C) = 0$, every subset of C is in L .

$$\text{Thus, } |L| \geq 2^{|C|} = 2^{\omega} \rightarrow |\mathbb{R}|$$

$$\text{Of course, } L \subset P(\mathbb{R}) \Rightarrow |L| \leq 2^{\omega}$$

$$\text{Thus, } |L| = 2^{\omega}.$$

On the other hand, $|B(\mathbb{R})| = \omega$. (why?)

Lecture 4

19 January 2021 15:31

Integration on Measure Spaces

Drawbacks of Riemann integrable

① Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ 0 & ; x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

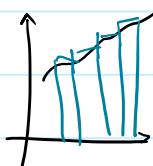
It "ought" to be integrable since it is "essentially" 0.

(It is 1 only on a set of measure 0.)

② The theory of Riemann integration does not admit suitable convergence theorems.

If $f_n \rightarrow f$ pointwise, then $\int f_n \rightarrow \int f$ is not necessary.
(Uniform continuity helps but that's more restrictive)

Recall: Riemann integration : We knew area of "rectangles".



Now we know more "areas" (via measure).

Defn: Suppose (X, \mathcal{M}) and (Y, \mathcal{N}) are measure spaces.
A function $f: X \rightarrow Y$ is called **measurable**
or **$(\mathcal{M}, \mathcal{N})$ -measurable**, if
 $f^{-1}(N) \in \mathcal{M}$ for all $N \in \mathcal{N}$.

Remark.

Composition of measurable functions is measurable.

That is, given (X, \mathcal{M}) , (Y, \mathcal{N}) , and (Z, \mathcal{O}) and

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$(\mathcal{M}, \mathcal{N})$ meas. $(\mathcal{N}, \mathcal{O})$ meas.

gof is $(\mathcal{M}, \mathcal{O})$ -measurable

Prop. 1. Suppose \mathcal{N} is generated by \mathcal{F} . Then,

$f: X \rightarrow Y$ is $(\mathcal{M}, \mathcal{N})$ measurable iff $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{N}$.

Proof. (\Rightarrow) Obvious.

(\Leftarrow) Define $\mathcal{N}' = \{E \in \mathcal{N} : f^{-1}(E) \in \mathcal{M}\}$.
By hypothesis, $\mathcal{F} \subset \mathcal{N}'$.

It is easy to see that \mathcal{N}' is a σ -alg.

Thus, $\mathcal{N} \subset \mathcal{N}'$. Thus, f is measurable. □

Cor. If X and Y are topological spaces, then any continuous function $f: X \rightarrow Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.

Cor. Suppose (X, \mathcal{M}) is a measure space and $f: X \rightarrow \mathbb{R}$ is a function. TFAE:

- (i) f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}))$ measurable,
- (ii) $f^{-1}((a, \infty)) \in \mathcal{M}$,
- (iii) $f^{-1}([a, \infty)) \in \mathcal{M}$,
- (iv) $f^{-1}(-\infty, a]) \in \mathcal{M}$.

Given a collection $\{(X_i, \mathcal{M}_i)\}_{i=1}^n$ of measure spaces and X is an arbitrary set.

Suppose $f_i: X \rightarrow X_i$ is a map for each i .

Then, consider the smallest σ -alg. M on X_i wrt which f_i are measurable. That is,

M is generated by $\{f_i^{-1}(E_i) \mid E_i \in \mathcal{M}_i, i=1, \dots, n\}$.

In particular, if $X = \prod X_i$ and $f_i = \pi_i$, then the above M is simply the product σ -alg as in the previous lectures.

Defn. We define Borel sets in $\bar{\mathbb{R}}$ as

$$\mathcal{B}(\bar{\mathbb{R}}) = \{E \subset \bar{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}.$$

The above agrees with the Borel field of $\bar{\mathbb{R}}$ as a topological or metric space. ($p(x,y) = |\arctan x - \arctan y|$)

Moreover, it is generated by the rays $(a, \infty]$ or $[-\infty, a)$. ($a \in \mathbb{R}$)

$f: X \rightarrow \bar{\mathbb{R}}$ is measurable if it is $(M, \mathcal{B}(\bar{\mathbb{R}}))$ measurable.

Defn. $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be

- Borel measurable if it is $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ measurable,

Lebesgue measurable if it is $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$ measurable.

Likewise for $f: \mathbb{R} \rightarrow \mathbb{C}$.

Remark If we talk of Lebesgue measurability, we have to be more careful.

If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are Lebesgue measurable, it is not necessary that $f \circ g$ is Lebesgue measurable.

The problem is in terminology, by $f: \mathbb{R} \rightarrow \mathbb{R}$ being measurable,

we mean that it is $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$ measurable.

If $f^{-1}(E)$ is null, then $g^{-1}(f^{-1}(E)) \in \mathcal{L}$ is not guaranteed EVEN if g is continuous.

Prop 2: Suppose (X, \mathcal{M}) is a measure space and suppose $f, g: X \rightarrow \mathbb{R}$, $f_i: X \rightarrow \bar{\mathbb{R}}$ are measurable. ($i = 1, 2, \dots$)

Then,

- ① $f+g, fg$ are measurable (wrt Borel)
- ② g_1 defined as $g_1(x) := \sup_i f_i(x)$,
- g_2 $g_2(x) := \inf_i f_i(x)$
- g_3 $g_3(x) := \limsup_i f_i(x)$
- g_4 $g_4(x) := \liminf_i f_i(x)$

are all measurable.

Proof: Define $F: X \rightarrow \mathbb{R} \times \mathbb{R}$, $F(x) := (f(x), g(x))$.

Recall $\pm: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\cdot: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Then, $f \pm g = \frac{\pm}{cts} \circ F$.
measurable, by previous obs. of product
spaces
hence, measurable

Thus, $f \pm g$ and fg are measurable.

For g_1 : $g_1(x) = \sup_j f_j(x)$

$$\{x : g_1(x) > a\} = g_1^{-1}(a, \infty]$$

$$= \bigcup_{j=1}^{\infty} f_j^{-1}(a, \infty] \in \mathcal{M}.$$

By the previous prop, we are done since (a, ∞) generated $\mathcal{B}(\bar{\mathbb{R}})$.

$$g_2 : g_2^{-1}([-∞, a]) = \bigcup_{j=1} f_j^{-1}([-∞, a]).$$

More generally, $h_K(n) = \sup_{j>K} f_j(n)$ is measurable

Thus, $g_3 = \inf_K h_K$ is measurable. Similarly for g_4 .

Cor. 3. If $\lim_j f_j(n)$ exists $\forall n$, then $f = \lim_j f_j$ is measurable.

Proof. $f = g_3 = g_4$.

Cor. 4. $f, g \rightarrow$ measurable $\Rightarrow f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$ are measurable.

Proof. $f \wedge g = \inf \{f, g\}$; $f \vee g = \sup \{f, g\}$

Cor. 5. $f \rightarrow$ measurable $\Rightarrow |f|$ measurable

Proof. $|f| = f \vee -f$. Or: $|\cdot|$ is continuous.

Note: $f \wedge g = \frac{(f+g) - |f-g|}{2}$.

Defn. Let (X, \mathcal{M}) be a measurable space

Suppose $E_1, \dots, E_n \in \mathcal{M}$ are p-wise disjoint.

A simple function is a function of the form

$$f = \sum_{i=1}^n a_i \mathbb{1}_{E_i}, \quad a_i \in \mathbb{R}.$$

Prop. 6. Simple functions are (Borel) measurable.

Proof. Suffices to show $\mathbb{1}_E$ is measurable $\forall E \in \mathcal{M}$.

Note that, for $B \in \mathbb{B}_{\mathbb{R}}$,

$$\mathbb{1}_E(B) = \begin{cases} \emptyset & ; B \cap [0, 1] = \emptyset \\ E & ; B \cap [0, 1] \neq \emptyset \end{cases}$$

$$\mathbb{1}_E(B) = \begin{cases} \phi & ; B \cap \{0,1\} = \emptyset \\ E & ; B \cap \{0,1\} = \{1\} \\ E^c & ; B \cap \{0,1\} = \{0\} \\ X & ; B \cap \{0,1\} = \{0,1\} \end{cases}$$

Thus, $\mathbb{1}_E$ is measurable $\Leftrightarrow E \in \mathcal{M}$

□

Thm.1 Let (X, \mathcal{M}) be a measure space.

If $f: X \rightarrow [0, \infty]$ is a measurable function (Borel), there is a sequence $\{\phi_n\}$ of simple functions s.t.

$\phi_n \uparrow f$ pointwise.

Moreover $\phi_n \rightarrow f$ UNIFORMLY on any set where f is bounded.

Proof.

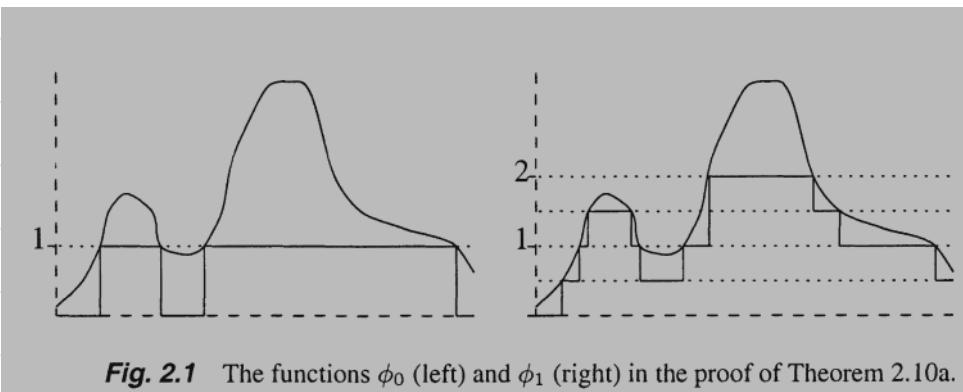


Fig. 2.1 The functions ϕ_0 (left) and ϕ_1 (right) in the proof of Theorem 2.10a.

Idea ↑

For $n \in \mathbb{N} \cup \{0\}$, and $0 \leq k \leq 2^n - 1$, define

$$E_n^k := f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \text{ and } F_n := f^{-1}((2^n, \infty]).$$

Note $E_n^k, F_n \in \mathcal{M}$ since f is measurable.

Moreover, all the sets appearing in the following eqⁿ are disjoint:

$$\phi_n = \sum_{k=0}^{2^n-1} k 2^{-n} \mathbb{1}_{E_n^k} + 2^n \mathbb{1}_{F_n}.$$

Then, $\phi_n \leq \phi_{n+1} \forall n$ and $0 \leq f - \phi_n \leq 2^{-n}$.
 Thus, both parts follow. □

Remark. When dealing with complete measures, one has to be careful as the following statement shows.

Prop. 2 The following hold iff μ is complete:

(a) If f is measurable and $f = g \mu$ a.e., then
 $(\mu \text{ a.e.} \equiv \mu \text{ almost everywhere} \equiv \{\{f(n) \neq g(n)\}\} \subset N \text{ with } \mu(N) = 0.)$
 $(\text{If } \mu \text{ is complete, then, } \mu\{f \neq g\} = 0.)$
 g is measurable.

(b) If $(f_n)_{n=1}^{\infty}$ are all measurable and suppose $f_n \rightarrow f$ μ a.e., then f is measurable.

Proof. (a) Suppose μ is complete, f is meas. and $f = g \mu$ a.e.
 To show: $g^{-1}(A) \in \mathcal{M}$ for any measurable A .
 Let $N = \{x : f(x) \neq g(x)\}$. Note $N \in \mathcal{M}$ and $\mu(N) = 0$.
 Note $g^{-1}(A) = (f^{-1}(A) \cap N^c) \cup (g^{-1}(A) \cap N)$
 $\quad \quad \quad \downarrow \in \mathcal{M} \quad \quad \quad \downarrow \in \mathcal{M}$
 $\subseteq N \text{ and hence null}$
 $\because \in \mathcal{M} \text{ since } \mathcal{M} \text{ is complete}$
 $\Rightarrow g^{-1}(A) \in \mathcal{M}$

Conversely, if μ is not complete, pick E, N s.t.
 $E \subset N$ and $\mu(N) = 0$. (E exists since not complete.)

Define $f : X \rightarrow \mathbb{R}$ as constant 0.

$$g : X \rightarrow \mathbb{R} \text{ as } g = \mathbf{1}_E.$$

Then, $f = g$ μ a.e. but g is not measurable.

(b) Exercise. 13

Propn. Suppose (X, \mathcal{M}, μ) is a measure space and $(X, \bar{\mathcal{M}}, \bar{\mu})$ is its completion. Then, for any $\bar{\mathcal{M}}$ -measurable function f , there is an \mathcal{M} measurable function g s.t.

$$f = g \quad \bar{\mu} \text{ a.e.}$$

Proof. If $f = \mathbf{1}_E$ for $E \in \bar{\mathcal{M}}$, then it is trivial.

(Take $E = F \cup N$ for $F \in \mathcal{M}$ and $\mu(N) = 0$. Then, $g = \mathbf{1}_F$ works.)

(Thus, true for all simple functions.)

In general, write $f = f^+ - f^-$ where

$$f^+ = \max(f, 0) \text{ and } f^- = \max(-f, 0).$$

Then, $f^+, f^- \geq 0$ are measurable. We get sequences

ϕ_n^+ and ϕ_n^- converging p.wise to f^+ and f^- .

Then, $\phi_n = \phi_n^+ - \phi_n^- \rightarrow f^+ - f^- = f$ p.wise.

For each n , let ψ_n be \mathcal{M} -measurable simple s.t. $\psi_n = \phi_n$ except on some $E_n \in \bar{\mathcal{M}}$ with $\bar{\mu}(E_n) = 0$.

Choose $N \in \mathcal{M}$ s.t. $\mu(N) = 0$ and $N \supset \bigcup_n E_n$.

Put $g = \lim \mathbf{1}_{N \setminus E_n} \psi_n$. Then, g is μ -meas.

and $g = f$ on N^c .

Remark The above shows that we can actually always approximate any measurable f pointwise using simple functions. (Not just non-negative ones.)

just non-negative ones.)

INTEGRALS FOR NON-NEGATIVE VALUED FUNCTIONS

Def. Suppose (X, \mathcal{M}) is a measure space.

Define

$$\mathcal{L}^+ = \{ f: X \rightarrow [0, \infty] \mid f \text{ is } \mathcal{M}\text{-measurable} \}.$$

If $\phi = a \mathbb{1}_E \in \mathcal{L}^+$, define $(E \in \mathcal{M})$

$$\int \phi d\mu := a \mu(E).$$

For $\phi = \sum_{i=1}^n a_i \mathbb{1}_{E_i} \in \mathcal{L}^+$ where $E_i \in \mathcal{M}$ are pairwise disjoint,

define

$$\int \phi d\mu := \sum_{i=1}^n a_i \mu(E_i). \quad (0 \cdot \infty = 0, \text{ by convention})$$

For $A \in \mathcal{M}$, $\int_A \phi d\mu = \int \phi \cdot \mathbb{1}_A d\mu.$

(Note that ϕ simple $\Rightarrow \phi \cdot \mathbb{1}_A$ is simple.)

Prop. 12. Let ϕ, ψ be simple in \mathcal{L}^+ .

(i) For $c \geq 0$, $\int c \phi d\mu = c \int \phi d\mu,$ } linearity

(ii) $\int (\phi + \psi) d\mu = \int \phi d\mu + \int \psi d\mu,$

(iii) If $\varphi \leq \psi$, then $\int \varphi d\mu \leq \int \psi d\mu,$

(iv) The map $A \mapsto \int_A 1 d\mu$ is a measure on \mathcal{M} .

Remark: We would have had to talk about \int being well-defined.

Note that a function $f: X \rightarrow \mathbb{R}$ is simple iff it is measurable and $f(x)$ is finite. In this case, we define the standard representation as

$$f = \sum_{i=1}^r a_i \mathbf{1}_{E_i} \quad \text{where } f(x) = \{a_1, \dots, a_n\} \text{ and } E_i = f^{-1}(\{a_i\}).$$

Then, $\int f d\mu := \sum a_i \mu(E_i)$.

With this, all the above can be proven.

Proof. (i) trivial. (ii) Let $\varphi = \sum a_i \mathbf{1}_{E_i}$, $\psi = \sum b_j \mathbf{1}_{F_j}$ be std. reps.

$$\text{Then, } (\varphi + \psi) = \sum_{i,j} (a_i + b_j) \mathbf{1}_{E_i \cap F_j}$$

$$\begin{aligned} \Rightarrow \int (\varphi + \psi) d\mu &= \sum_{i,j} (a_i + b_j) \mu(E_i \cap F_j) \\ &= \sum_{i,j} a_i \mu(E_i \cap F_j) + \sum_{i,j} b_j \mu(E_i \cap F_j) \\ &= \sum_i a_i \mu(E_i) + \sum_j b_j \mu(F_j) = \int \varphi d\mu + \int \psi d\mu. \end{aligned}$$

Defn. Suppose $f: X \rightarrow [0, \infty]$ is in L^+ . Define

$$\int f d\mu := \sup \left\{ \int \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Note that the defn above coincides with that for simple f as earlier.

Also, it is easy to see from this defn that if $f \leq g$,

then

$$\int f d\mu \leq \int g d\mu \quad \text{and}$$

$$\int cf d\mu = c \int f d\mu \quad \text{for any } c \geq 0.$$

Thm. 11 (Monotone Convergence Theorem (MCT))

If $\{f_n\}$ is a sequence in L^+ s.t. $f_n \leq f_{n+1} \quad \forall n \in \mathbb{N}$, then

$$f = \lim_{n \rightarrow \infty} f_n$$

is in L^+ and

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

(Note that existence of \int or limit is not an issue since we work in \mathbb{R} .)

Proof: First, observe that $\left(\int f_n d\mu \right)_{n=1}^\infty \uparrow$ and thus,

does have a limit.

$$\text{Moreover, } f_n \leq f \Rightarrow \int f_n \leq \int f \quad \forall n$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu.$$

For the other inequality, pick an arbitrary $\alpha \in (0, 1)$ and let φ be a simple function with $0 \leq \varphi \leq f$.

Define

$$E_n = \{x : f_n(x) \geq \alpha \varphi(x)\}.$$

Then, $E_n \in \mathcal{M}$, $E_n \subset E_{n+1}$ and

$$\bigcup_n E_n = X.$$

(Since $\alpha \varphi(x) \leq \alpha f(x) < f(x)$ if $f(x) \neq 0$)

$$\text{Hence, } \int_{E_n} f_n \geq \int_{E_n} f_n \geq \int_{E_n} \alpha \varphi = \alpha \int_{E_n} \varphi \mathbf{1}_{E_n}.$$

$$(\text{Ex.}) \quad \int \phi \mathbf{1}_{E_n} \uparrow \int \phi$$

$$\text{Hence, } \int f_n \geq \alpha \int \varphi \mathbf{1}_{E_n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \varphi \quad \forall \alpha \in (0, 1)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int f_n \geq \int \varphi \quad \forall 0 \leq \varphi \leq f$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int f_n \geq \int f. \quad \blacksquare$$

Lecture 5

24 January 2021 18:12

A Quick CONSEQUENCE OF MCT:

Recall $\int f d\mu = \sup \left\{ \int \varphi d\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \right\}$

possibly uncountable

The MCT allows us to take a sequence $(\varphi_n)_n$ of simple functions increasing to f and computing the limit there.

(Sequences are easier to work with.)

$f_n \uparrow f$ is a necessary condition. (At least, without any other extra hypothesis.)
Else, consider $f : [0, \infty) \rightarrow \mathbb{R}$ as
 $f_n = \mathbf{1}_{[n, n+1)} \quad \text{for } n \geq 0.$

Clearly, $\lim_{n \rightarrow \infty} f_n =: f = 0$. ← zero function

Then, $\int_{\mathbb{R}} f_n dm = 1 \quad \forall n$ but $\int f = 0 \neq 1$.

Lebesgue measure

Prop. For $f, g \in \mathcal{L}^+$,

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu.$$

More generally, if $f = \sum_{n \geq 1} f_n$, with $f_n \in \mathcal{L}^+$,

then $\int f d\mu = \sum_{n \geq 1} \int f_n d\mu.$

Proof. Let $\varphi_n \uparrow f$, $\psi_n \uparrow g$ be sequences of simple functions.
Then, $\varphi_n + \psi_n \uparrow f + g$. By MCT

$$\lim_{n \rightarrow \infty} \int (\varphi_n + \psi_n) d\mu = \int (f + g) d\mu$$

|| had shown for simple functions

$$\lim_{n \rightarrow \infty} \int \varphi_n d\mu + \lim_{n \rightarrow \infty} \int \psi_n d\mu = \int f d\mu + \int g d\mu$$

By induction, for finite sums, it follows that

$$\sum_{i=1}^n \int f_i d\mu = \int \sum_{i=1}^n f_i d\mu$$

Let $N \rightarrow \infty$ and use MCT on partial sums to get

$$\sum_{i=1}^{\infty} \int f_i d\mu = \int f d\mu. \quad \square$$

Prop. If $f \in L^+$, then $\int f d\mu = 0 \Leftrightarrow f = 0 \text{ } \mu \text{ a.e.}$

Proof. The statement is obvious for simple functions

(\Leftarrow) Suppose $f = 0 \text{ } \mu \text{ a.e.}$ and $0 \leq \varphi \leq f$.

\hookrightarrow simple

Then, $\varphi = 0 \text{ } \mu \text{ a.e.}$ and thus, $\int \varphi d\mu = 0$.

Since $\int f d\mu$ is the sup over all such $\int \varphi d\mu$,

we get

$$\int f d\mu = 0.$$

(\Rightarrow) Suppose $\int f d\mu = 0$.

$$\left\{ x : f(x) > 0 \right\} = \bigcup_{n \geq 1} \left\{ x : f(x) > \frac{1}{n} \right\}$$

$$\{x : f(x) >_0\} = \bigcup_{n \geq 1} \{x : f(x) > \frac{1}{n}\}$$

|| ||
 E $E_n \in \mathcal{M}$

If $\mu(E) > 0$, then $\mu(E_n) > 0$ for some $n \in \mathbb{N}$.

But then, $\int f \geq \int \frac{1}{n} \cdot 1_{E_n} = \frac{1}{n} \mu(E_n) > 0.$

→ ↵

Cor. If $f_n \in L^+$ and $f_n \uparrow f$ a.e., then $\lim_{n \rightarrow \infty} \int f_n = \int f$.

Proof. Let $E = \{x : \lim_{n \rightarrow \infty} f_n(x) = f(x)\}$. measurable since
 $g = \lim f_n$ is measurable
and then, so is $g-f$
Then, $\mu(E^c) = 0$.

Thus, $f - f \cdot 1_E = 0$ a.e. and hence,

$$(\star) \quad \int f d\mu = \underbrace{\int f \cdot 1_E d\mu}_{\text{above prop}}$$

Also, $f_n - f_n \cdot 1_E = 0$ a.e. and hence,

$$(*) \quad \int f_n d\mu = \int f_n \cdot 1_E d\mu \quad \forall n \in \mathbb{N}.$$

Note that $f_n \cdot 1_E \uparrow f \cdot 1_E$. Thus, by MCT,

$$\lim_{n \rightarrow \infty} \int f_n \cdot 1_E d\mu = \int f \cdot 1_E d\mu$$

|| (*) || (*)

$$\lim_{n \rightarrow \infty} \int f_n d\mu \quad \int f d\mu$$

Recall that we needed $f_n \uparrow f$ in MCT, as seen

in example. However, the following does hold.

Thm. Fatou's Lemma

If $\{f_n\} \subseteq L^+$, then

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$$

Proof. Consider for each $k \in \mathbb{N}$,

$$\inf_{n \geq k} f_n(x) \leq f_j(x) \quad \forall j \geq k$$

$$\Rightarrow \int \inf_{n \geq k} f_n d\mu \leq \int f_j d\mu \quad \forall j \geq k$$

$$\Rightarrow \int \inf_{n \geq k} f_n d\mu \leq \liminf_{j \geq k} \int f_j d\mu \quad (*)$$

Note that $\left(\inf_{n \geq k} f_n \right)_k$ is a sequence in L^+ increasing to

$$\liminf_{n \rightarrow \infty} f_n.$$

By the MCT, taking $k \rightarrow \infty$ in $(*)$ gives

$$\int \liminf_{n \rightarrow \infty} f_n d\mu \leq \lim_{k \rightarrow \infty} \left(\liminf_{j \geq k} \int f_j d\mu \right)$$

$$\leq \lim_{j \rightarrow \infty} \int f_j. \quad \square$$

INTEGRABILITY OF ALL KINDS OF FUNCTIONS

(Not necessarily non-negative.)

For $f: X \rightarrow \mathbb{R}$, define $f^+(x) := \begin{cases} f(x) & ; f(x) \geq 0, \\ 0 & ; f(x) < 0, \end{cases}$
 and $f^-(x) := \begin{cases} -f(x) & ; f(x) \leq 0, \\ 0 & ; f(x) > 0. \end{cases}$

Hence, $f = f^+ - f^-.$

Note that $f^+, f^- \geq 0$. $f^+ = \max\{f, 0\}$ and
 $f^- = \min\{f, 0\}.$

Thus, if f is measurable, so are f^+, f^- . Moreover,
 $f^+, f^- \in \mathcal{L}^+$.

Defn. (Integrable)

We say that f is integrable on (X, M, μ) if both

$$\int f^+ d\mu < \infty \text{ and } \int f^- d\mu < \infty.$$

(More generally, one can ask for at least one to be finite but we shall stick with the above.)

Defn. (L^1)

$$L^1(X, \mu) := \{f \mid f \text{ is integrable on } (X, \mu)\}.$$

Shall also write $L^1(\mu)$ or \mathcal{L}^1 if context is clear.

Note that $|f| = f^+ + f^-$.

So,

$$L^1(\mu) = \{f \mid \int |f| d\mu < \infty\}.$$

Defn. If f is integrable, i.e., $f \in L^1$, then

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu.$$

Prop. $L'(\mu)$ is a vector space and \int is linear on this space. Furthermore, for $f \in L'$,

$$\left| \int f d\mu \right| \leq \int |f| d\mu.$$

Proof. For $f, g \in L'$ and $\alpha, \beta \in \mathbb{R}$, we have

$$|\alpha f + \beta g| \leq |\alpha| |f| + |\beta| |g|.$$

have finite integrals

Thus, $\alpha f + \beta g \in L'$. To show linearity:

Let $h = f + g$, note

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$

or

$$h^+ + f^- + g^- = h^- + f^+ + g^+$$

and thus, (each function is in L')

$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- + \int f^+ + \int g^+.$$

Rearrange to conclude-

Also,

$$\int \alpha f = \alpha \int f \text{ is easy to check.}$$

(Take $\alpha > 0$ and $\alpha < 0$.)

For the last part:

$$\begin{aligned} \left| \int f d\mu \right| &= \left| \int f^+ d\mu - \int f^- d\mu \right| \leq \left| \int f^+ d\mu \right| + \left| \int f^- d\mu \right| \\ &= \int f^+ d\mu + \int f^- d\mu \end{aligned}$$

.....

$$= \int f^+ d\mu - \int f^- d\mu$$

$$= \int |f| d\mu.$$

□

Thm.

(Dominated Convergence Theorem, DCT)

Suppose (f_n) is a sequence in $L^1(\mu)$ and $f_n \rightarrow f$ a.e.

If furthermore $\exists g \in L^1$ s.t.

$$|f_n| \leq |g| \quad \text{a.e., then}$$

(i) $f \in L^1$, and

$$(ii) \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

part of thm that limit does exist

Proof.

(i) It follows from Prop 8 and 9 from Lecture 4 that f is measurable after a possible redefinition on a null set. Also, $|f| \leq |g|$ and thus, $f \in L^1$.

(ii) Note that $g + f_n \geq 0$ and $g - f_n \geq 0$ a.e.

By Fatou,

$$\int g + \int f \leq \liminf \int g + f_n = \int g + \liminf \int f_n$$

$$\Rightarrow \int f \leq \liminf \int f_n \quad \text{--- (1)}$$

$$\text{and } \int g - \int f \leq \liminf \int g - f_n = \int g - \limsup \int f_n$$

$$\Rightarrow \int f \geq \limsup \int f_n \quad \text{--- (2)}$$

From (1) and (2), we are through.

(3)

Lecture 6 (25-01-2021)

25 January 2021 14:01

CONSEQUENCES OF DCT

Prop. Suppose (f_i) is a sequence in $L^1(\mu)$ such that

$$\sum_{i=1}^{\infty} \int |f_i| < \infty. \quad \text{Then,}$$

$\sum_{i=1}^{\infty} f_i$ converges a.e. to a function in L' , and

$$\int \sum f_i = \sum \int f_i.$$

Proof. Define $g := \sum_{i=1}^{\infty} |f_i|$. As a consequence of MCT and hypothesis, $g \in L'$.

Thm. (Differentiating + Taking limits within integral)

Suppose that $f: X \times [a, b] \rightarrow \mathbb{R}$ and $f(\cdot, t) \rightarrow \mathbb{R}$ is integrable for each $t \in [a, b]$.

Let

$$F(t) = \int_X f(x, t) d\mu(x).$$

(a) Suppose that there exists $g \in L^1(\mu)$ such that
 $|f(x, t)| \leq g(x)$ for all x, t .

If

$$\lim_{t \rightarrow t_0} f(x, t) = f(x, t_0) \quad \text{for every } x,$$

$$\text{the } \lim_{t \rightarrow t_0} F(t) = F(t_0);$$

in particular, if $f(x, \cdot)$ is continuous for each x ,
then F is continuous.

(b) Suppose that $\frac{\partial f}{\partial t}$ exists and $\exists g \in L'(\mu)$ s.t.

$$|\left(\frac{\partial f}{\partial t}\right)(x, t)| \leq g(x) \quad \forall x, t.$$

Then, F is differentiable and

$$F'(x) = \int \left(\frac{\partial f}{\partial t}\right)(x, t) d\mu(x).$$

Proof

(a) Let $t_n \rightarrow t_0$ and let $F_n(x) = F(x, t_n)$. By hypothesis, $|F_n(x)| \leq g(x)$ and $F_n(x) \rightarrow f(x, t_0)$. So, we are done, by DCT.

(b) Let $t_n \rightarrow t_0$ be an arbitrary sequence. ($t_n \neq t_0 \forall n$)
let

$$h_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}.$$

h_n are measurable and $\lim_{n \rightarrow \infty} h_n(x) = \frac{\partial f}{\partial t}(x, t_0)$.

So, $\frac{\partial f}{\partial t}(x, t_0)$ is measurable. Moreover

$$|h_n(x)| \leq \sup_{t \in [a, b]} \left| \frac{\partial f}{\partial t}(t, x) \right| \leq g, \text{ by assumption.}$$

mean value
thm

Thus, we are again through, by PCT.

(Riemann integrability vs Lebesgue Integrability)

Riemann Integrability vs Lebesgue Integrability

Q. If f is Riemann integrable on $[a, b]$, is it (Lebesgue)-integrable? If yes, are the integrals equal?

If \mathcal{P} is a partition

$$a = t_0 < t_1 < \dots < t_n = b,$$

$$L(f, \mathcal{P}) = \sum_{i=1}^n m_i (t_i - t_{i-1}), \quad m_i = \inf_{[t_{i-1}, t_i]} f$$

$$U(f, \mathcal{P}) = \sum_{i=1}^n M_i (t_i - t_{i-1}), \quad M_i = \sup_{[t_{i-1}, t_i]} f$$

Notation: $\int_a^b f(x) dx \leftarrow \text{Riemann integral}$

Consider, $G_p := \sum M_i \mathbf{1}_{(t_{i-1}, t_i]}$,

$$g_p := \sum m_i \mathbf{1}_{(t_{i-1}, t_i)}.$$

Note $g_p \leq f \leq G_p \quad \forall \mathcal{P}$.

If $f \in R[a, b]$, then \exists a sequence of refined partitions (\mathcal{P}_n)

s.t.

$$L(f, \mathcal{P}_n) \uparrow \int_a^b f(x) dx \quad \text{and} \quad U(f, \mathcal{P}_n) \downarrow \int_a^b f(x) dx.$$

Let $g_n := g_{\mathcal{P}_n}$ and $G_n := G_{\mathcal{P}_n}$.

Consider

$$g(x) := \lim_{n \rightarrow \infty} g_n(x) \quad \text{and} \quad G(x) := \lim_{n \rightarrow \infty} G_n(x).$$

furthermore, since $g_{p_n} \leq f \leq G_{p_n}$, it follows that

$g \leq f \leq G$. Furthermore since f is bdd and $[a, b]$ is a bounded interval, by DCR

$$\lim_{n \rightarrow \infty} \int g_n dm = \int g dm$$

↳ Lebesgue measure

Also, $\lim_{n \rightarrow \infty} \int G_n =$

Remark: The above is for proper Riemann integrals.

Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) := \begin{cases} \frac{\sin x}{x} & ; x \neq 0, \\ 1 & ; x = 0. \end{cases}$$

$\int_{-\infty}^{\infty} f(x) dx$ exists, as an improper Riemann integral but

$\int |f| dm = \infty$ and hence, f is not Lebesgue integrable

Modes Of Convergence

(Modes of Convergence)

Suppose we have a sequence of functions $\{f_n\}$ on

say $[0, 1]$. Want to talk about limit.

POINTWISE. For each n , $\{f_n(n)\}$ must be convergent.

Not very good with overall functional properties,
as we have seen.

NOTIONS OF CONVERGENCE = 'NEW' METRIC SPACES.

UNIFORM CONVERGENCE: $\{f_n\}$ converges uniformly on X

\Leftrightarrow given $\epsilon > 0$, $\exists N(\epsilon)$ s.t. $\forall x \quad \forall n \geq N(\epsilon)$:

$$|f_n(x) - f(x)| < \epsilon.$$

$C[0, 1] := \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$.

$C[0, 1]$ is a metric space wrt. sup norm metric,

$$d(f, g) := \sup_{x \in [0, 1]} |f(x) - g(x)|.$$

(exists since $f-g$ is cts.)

Now, uniform convergence is precisely convergence in this metric.
(of continuous functions)

Moreover, continuous functions converge uniformly to cts functions.

Thus, $C[0, 1]$ is a complete metric function.

Thm.

Suppose (X, \mathcal{M}, μ) is a complete measure space.

$$L'(\mu) = \{f : X \rightarrow \mathbb{R} \mid \int |f| d\mu < \infty\}.$$

L' is a \mathbb{R} -vector space. Moreover

$$\textcircled{1} \quad \int af d\mu = a \int f d\mu \quad \text{for } a \in \mathbb{R}, f \in L'.$$

$$\textcircled{2} \quad \int (f+g) d\mu = \int f d\mu + \int g d\mu \quad \text{for } f, g \in L'.$$

NORMED SPACES

(Normed spaces)

Defⁿ: Suppose V is an \mathbb{R} -vector space. A norm is a function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- $\|x\| = 0 \iff x = 0$,
- $\|\alpha x\| = |\alpha| \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

Example: $V = \mathbb{R}^n$, $\|x\|_2 := \sqrt{x_1^2 + \dots + x_n^2}$
 $\|x\|_1 := |x_1| + \dots + |x_n|$
 are norms.

FACT: If V has a norm $\|\cdot\|$, then this induces a metric.
 $d(x, y) := \|x - y\|$.

Consider a complete measure space (X, \mathcal{M}, μ) .
 For $f \in L^1(\mu)$, define

$$\|f\|_1 := \int |f| d\mu.$$

(doesn't satisfy norm axioms, $\int |f| d\mu \neq f \equiv 0$)

Consider an equivalence relation: $f \sim g$ if $f \neq g$ only on a set of measure 0.

We know that if $f = g$ μ a.e., then

$$\int f d\mu = \int g d\mu.$$

Thus, \int makes sense on equivalence classes of \sim .

With that, $L^1(\mu)/\sim$ becomes a normed space.

By abuse of notation, we shall continue denoting it $L^1(\mu)$.

Ihm. $L'(\mu)$ is a complete metric space wrt. the metric induced by $\|\cdot\|_1$.

(Special case of Riesz-Fischer Theorem.)

Defn. A normed vector space which is complete wrt. the metric induced by the norm is called a **Banach Space**.

(Thus, L' is a Banach space)

Proof. Want to show: Cauchy sequence converge in L' .

Suppose $\{f_n\}$ is Cauchy, i.e., given $\epsilon > 0$, $\exists N(\epsilon)$ s.t.
 $\forall m, n \geq N(\epsilon)$

$$\|f_n - f_m\|_1 < \epsilon.$$

WTS: $\exists f \in L'(\mu)$ s.t. $\{f_n\}$ converges to f .
↳ in $\|\cdot\|$

Get a sequence $\{n_i\}_i$ s.t.

$$\|f_{n_{i+1}} - f_{n_i}\|_1 < \frac{1}{2^i} \quad \text{for } i \geq 1.$$

(Follows straightforwardly from Cauchy.)

Define $g_k : X \rightarrow \mathbb{R}$ by

$$g_k := \sum_{i=1}^k |f_{n_{i+1}} - f_{n_i}| \quad \text{and}$$

$$g := \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|.$$

$$\|g_k\|_1 < 1 \quad \forall k. \quad g(\omega) = \lim_{k \rightarrow \infty} g_k(\omega).$$

So, by Fatou's Lemma

$$\int \liminf_{n \rightarrow \infty} g_n \leq \liminf_{n \rightarrow \infty} \int g_n \leq 1$$

$$\int \liminf_n g_n \leq \liminf_n \int g_n \leq 1$$

$\int g d\mu$

Since $g \geq 0$, we get $g \in L^1$. ($g < \infty$ μ-a.e.)

$$\text{Let } f(x) = \begin{cases} f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x)) & \text{when the sum } < \infty \\ 0 & \text{otherwise} \end{cases}$$

f is our candidate for the limit.

$$f_N := f_{n_1} + \sum_{i=1}^N f_{n_{i+1}} - f_{n_i}$$

$$= f_{n_{N+1}}$$

Thus, $f_{n_{N+1}}$ converges μ-a.e. as $N \rightarrow \infty$

To complete the proof, wts: $\|f_n - f\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

Note that $|f - f_n| = \liminf_n |f_{n_{N+1}} - f_n|$

So again, by Fatou

$$\begin{aligned} \|f - f_n\|_1 &= \int |f - f_n| d\mu \leq \liminf_n \underbrace{\int |f_{n_{N+1}} - f_n| d\mu}_{= \|f_{n_{N+1}} - f_n\|_1} \\ &= \|f_{n_{N+1}} - f_n\|_1 \end{aligned}$$

Now, given $\epsilon > 0$, pick $N(\epsilon)$ large enough so that $\forall m, n \geq N(\epsilon)$, $\|f_m - f_n\|_1 \leq \epsilon$.

Picking N large gives the result.

Picking N large gives the result.

This shows convergence in L^1 !

$$\hookrightarrow f \in L^1 \text{ follows since } \int |f| \leq \int |f-f_n| + \int |f_n| \xrightarrow{\downarrow} 0$$

Def.

(Convergence in measure, Cauchy in measure)

Given a sequence $(f_n)_n$ of measurable functions in (X, \mathcal{M}, μ) , we say that (f_n) is Cauchy in measure if for any $\epsilon > 0$,

$$\mu \{ x : |f_n(x) - f_m(x)| > \epsilon \} \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

$f_n \rightarrow f$ in measure if $\forall \epsilon > 0$,

$$\mu \{ x : |f_n(x) - f(x)| > \epsilon \} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Lecture 7 (28-01-2021)

28 January 2021 14:06

Another perspective of convergence in measure:

Consider the sequence in \mathbb{R} : $x_n = \begin{cases} 1 & ; n \text{ is a square} \\ 0 & ; \text{otherwise} \end{cases}$

Clearly, (x_n) does not converge. However, x_n is "mostly" 0.

The squares are "sparse" in \mathbb{N} . We would want the sequence to converge to 0. We don't want irregularities on sparse sets to make a difference.

Some remarks:

① $f_n \rightarrow f$ and $f_n \rightarrow g$ in measure.

Then, $f = g \mu \text{ a.e.}$

$$\{x : |f(x) - g(x)| > \varepsilon\} \subseteq \{x : |f(x) - f_n(x)| > \varepsilon/2\}$$

$$\cup \{x : |g(x) - f_n(x)| > \varepsilon/2\}.$$

② $f_n \rightarrow f$ a.e. $\not\Rightarrow f_n \rightarrow f$ in measure or $f_n \rightarrow f$ in L^1

a) $X = \mathbb{R}$, $M = \mathcal{B}(\mathbb{R})$, $\mu = m$.

$f_n(x) = \mathbf{1}_{(n, n+1]}$. Clearly $f_n \rightarrow 0$ pointwise everywhere.

But $\int f_n dm = 1$. Thus, $f_n \not\rightarrow 0$ in L^1 .

$$\text{Also, } \mu \{x : |f_n(x) - 0| > \varepsilon_2\} = \mu (n, n+1) = 1 \quad \forall n.$$

Thus, $f_n \not\rightarrow f$ in measure.

b) Consider: $f_1 = \mathbf{1}_{[0, 1]}$

$$f_2 = \mathbf{1}_{[0, y_2]} ; f_3 = \mathbf{1}_{[y_2, 1]}$$

$$f_4 = \mathbf{1}_{[0, y_4]} ; f_5 = \mathbf{1}_{[y_4, 1]} ; \dots$$

Note:

for any $\varepsilon > 0$, we have $\mu \{x : |f_n(x)| > \varepsilon\} \rightarrow 0$.

Thus, $f \rightarrow 0$ in measure.

Also, $\int f_n \rightarrow 0$. Thus, $f_n \rightarrow 0$ in L^1 .

However, for any $x \in [0, 1]$, $\{f_n(x)\}_n$ does not converge!

Prop. $f_n \rightarrow f$ in $L^1(\mu)$, then $f_n \rightarrow f$ in measure.

(Variant of Markov inequality)

Proof. Fix $\epsilon > 0$. For $n \geq 1$, define

$$E_n := \{x : |f_n(x) - f(x)| > \epsilon\}.$$

Then, $\int_{E_n} |f_n - f| d\mu \geq \int_{E_n} 1 d\mu = \mu(E_n) \geq \epsilon \int_{E_n} 1 d\mu = \epsilon \int_{E_n} 1 d\mu = \epsilon \mu(E_n) \geq 0$

↓
0
 $\therefore \mu(E_n) \rightarrow 0$. ②

Converse is not true! Consider the earlier example modified:

$$f_1 = 1_{[0,1]}, \quad f_2 = 2 \cdot 1_{[0, \frac{1}{2}]}, \quad f_3 = 2 \cdot 1_{[\frac{1}{2}, 1]}, \\ f_4 = 4 \cdot 1_{[0, \frac{1}{4}]}, \dots$$

(Make all integrals 1.) $f_n \rightarrow 0$ in measure but not in L^1 .

Tm. ① Suppose (f_n) is Cauchy in measure, then

$\exists f$ measurable s.t. $f_n \rightarrow f$ in measure.

② There is a subsequence (f_{n_j}) s.t. $f_{n_j} \rightarrow f$ a.e.

Proof. Pick $n_1 < n_2 < \dots$ s.t. $g_i = f_{n_i}$ satisfies the

$$\mu(E_i) \leq 2^{-i} \text{ where } E_i := \{x : |g_i(x) - g_{i+1}(x)| \geq 2^{-i}\}.$$

(Can pick using Cauchy.)

$$\text{Let } E^{(k)} = \bigcup_{j \geq k} E_j. \quad \text{Then, } E^{(k)} \leq \frac{1}{2^{k-1}}.$$

Thus, if $x \in E^{(k)}$, then $x \in E_j \forall j \geq k$.

In particular, for $m > n \geq k$,

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq \sum_{j=n}^{m-1} |g_{j+1}(x) - g_j(x)| \\ &\leq \sum_{j=n}^{\infty} \frac{1}{2^j} \leq \frac{1}{2^{n-1}} \end{aligned}$$

Thus, for any $x \notin E^{(k)}$, $\{g_n(x)\}$ is Cauchy.

Now, if we set $E = \bigcap_{k \geq 1} E^{(k)}$, then

(a) $\mu(E) = 0$

(b) For $x \notin E$, $\{g_n(x)\}$ converges.

Define

$$f(x) = \begin{cases} \lim g_n(x) & ; x \notin E \\ 0 & ; x \in E \end{cases}$$

Then, f is measurable and $g_j \rightarrow f$ a.e.

Since $\mu(E^{(k)}) \rightarrow 0$, it follows $g_j \rightarrow f$ in measure.

It also follows that $f_n \rightarrow f$ in measure.

$$\begin{aligned} \mu\{x: |f_n(x) - f(x)| \geq \epsilon\} &\leq \mu\{x: |f_n(x) - g_j(x)| \geq \epsilon/2\} \\ &\quad + \mu\{|g_j - f| \geq \epsilon/2\}. \quad \text{R} \end{aligned}$$

Thm. (Egoroff's Theorem) Suppose $\mu(X) < \infty$ and suppose $f_n \rightarrow f$ ^{measurable} μ a.e.

Then, given any $\epsilon > 0$, $\exists E \subseteq X$ s.t. $\mu(E) < \epsilon$ s.t.
 $f_n \rightarrow f$ UNIFORMLY on E^c .

Proof. (Sketch) We may assume $f_n \rightarrow f$ on X .

For $n, k \in \mathbb{N}$, define

$$E_{n,k} := \{x: |f_m(x) - f(x)| > \frac{1}{k} \text{ for some } m \geq n\}.$$

Know: $f_n(x) \rightarrow f(x) \forall x \in X$

Thus, for a fixed k , we have

$$\bigcap_{n \in \mathbb{N}} E_{n,k} = \emptyset.$$

Since $\mu(x) < \infty$, we have $\mu(E_{n,k}) < \infty$.

This gives us that $\lim_{n \rightarrow \infty} \mu(E_{n,k}) = 0$ for each fixed k .
 $(E_{1,k} \supseteq E_{2,k} \supseteq \dots)$

Pick n_k large so that

$$\mu(E_{n_k, k}) < \frac{\epsilon}{2^k} \quad \text{and let}$$

$$E = \bigcup_{k \geq 1} E_{n_k, k}.$$

Then, $\mu(E) < \epsilon$ and for $x \notin E$, $|f_n(x) - f(x)| < \frac{1}{k}$
 whenever $n > n_k$.

In particular, the convergence is uniform on E^c .

PRODUCT MEASURES

(Product measure)

We will now use \otimes instead of Π for product of σ -fields.

Suppose (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are measure spaces.

Had defined $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$ as

$$\langle E \times F : E \in \mathcal{M}, F \in \mathcal{N} \rangle.$$

Defn: A set of the form $E \times F$ for $E \in \mathcal{M}$, $F \in \mathcal{N}$ is called a rectangle.

Want a product measure on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$.

Would want $\rho(E \times F) := \mu(E) \nu(F)$ on rectangles.

Consider $A = \text{finite disjoint union of rectangles}$

A is indeed an algebra.

Extend ρ as:

$$\cdot \rho\left(\bigcup_{i=1}^{\infty} E_i \times F_i\right) := \sum_{i=1}^{\infty} \mu(E_i) \nu(F_i); \quad \rho(\emptyset) = 0.$$

So, if ρ is a pre-measure on \mathcal{A} , then the general theory from before allows us to extend ρ to a measure on $M \otimes N$. (Since $\langle M \rangle = M \otimes N$.)

Moreover, if μ, ν are σ -finite extensions, then the extension is unique.

Thus, we now wish to show that ρ is a pre-measure.

Suppose $A \times B$ is a rectangle and

$$A \times B = \bigcup_{i=1}^{\infty} (A_i \times B_i)$$

Q. Is $\mu(A) \nu(B) = \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) ?$

Consider $f(x, y) = \mathbf{1}_A \cdot \mathbf{1}_B$ on $X \times Y$.

Then, $\mathbf{1}_{x \in A} \mathbf{1}_{y \in B} = \sum_i \mathbf{1}_{x \in A_i} \mathbf{1}_{y \in B_i}$

Fix y ; this is integrable, so

$$\int \mathbf{1}_{x \in A} \mathbf{1}_{y \in B} d\mu(x) = \mathbf{1}_{y \in B} \mu(A)$$

By MCT, $\mathbf{1}_{y \in B} \mu(A) = \sum_i \mathbf{1}_{y \in B_i} \mu(A_i)$

Again, by MCT, and integrating w.r.t. y

$$\begin{aligned} \mu(A) \int \mathbf{1}_{y \in B} d\nu(y) &= \sum_i \mu(A_i) \int \mathbf{1}_{y \in B_i} d\nu(y) \\ &\stackrel{!}{=} \mu(A) \nu(B) \quad \sum_i \mu(A_i) \nu(B_i) \end{aligned}$$

Thus, we get the product measure, denoted $\mu \otimes \nu$.

Defn. Let $E \subseteq X \times Y$, $x \in X$, $y \in Y$ with $E \in \mathcal{M} \otimes \mathcal{N}$.

x -section of E (E_x) := $\{y \in Y : (x, y) \in E\} \subseteq Y$,

y -section of E (E^y) := $\{x \in X : (x, y) \in E\} \subseteq X$.

Prop. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$.

Proof. Let $R \subseteq \mathcal{P}(X \times Y)$ be the set of all those $E \subseteq X \times Y$

s.t. $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$.

Clearly, all rectangles are in R .

$$\hookrightarrow (A \times B)_x = \begin{cases} B & ; x \in A \\ \emptyset & ; x \notin A \end{cases}$$

Moreover, R is a σ -algebra since $(\bigcup_{i=1}^{\infty} E_i)_x = \bigcup_{i=1}^{\infty} (E_i)_x$
and $(E^c)_x = (E_x)^c$

and likewise for y .

Thus, $\mathcal{M} \otimes \mathcal{N} \subseteq R$, as desired.

Cor. Given $f: X \times Y \rightarrow \mathbb{Z}$ measurable, we can define the
slice functions $f_x: Y \rightarrow \mathbb{Z}$ for $x \in X$ and
 $f^y: X \rightarrow \mathbb{Z}$ for $y \in Y$ by
 $f_x(y) = f(x, y) = f^y(x)$.

Then, f_x is \mathcal{N} -measurable for all $x \in X$ and
 f^y is \mathcal{M} -measurable for all $y \in Y$.

Lecture 8 (01-02-2021)

01 February 2021 13:58

Recall:

- Suppose $(f_n)_n$ is a seq. of meas. f^n which is Cauchy in measure. Then:

- ① $\exists f$ measurable s.t. $f_n \rightarrow f$ in measure.
- ② $\exists (n_k)_k$ s.t. $f_{n_k} \rightarrow f$ μ a.e.

Idea of proof: Get an f s.t. ② holds. Show ① for that f .

↙ How? Get $n_1 < n_2 < \dots$ s.t. $g_k = f_{n_k}$ satisfy.

$$\mu \{x : |g_{k+1}(x) - g_k(x)| \geq \frac{1}{2^k}\} \leq \frac{1}{2^k}.$$

- If $f_n \rightarrow f$ μ a.e. and $\mu(X) < \infty$, then $f_n \rightarrow f$ in measure. Moreover, given $\epsilon > 0$, $\exists E$ s.t. $\mu(E) < \epsilon$ and on E^c , $f_n \rightarrow f$ uniformly.

Fubini's Theorem

$$\lambda = \mu \otimes \nu$$

$$\int_{X \times Y} f \, d\lambda = \int_X \left(\int_Y f(x,y) \, d\nu(y) \right) \, d\mu(x)$$

↙ "WISHFUL THINKING" THEOREM

The above is not true, in general. Even with infinite sums.

$$\begin{array}{ccccccc} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \end{array}$$

Each column sum = 0.

First row sum = 1, later ones = 0.

$$\sum (\text{column sums}) = 0 \neq 1 = \sum (\text{row sums})$$

Prototype for Fubini (Tonelli's Theorem) :

Suppose $f \geq 0$, $f \rightarrow \text{measurable}$ on $\mathcal{M} \otimes \mathcal{N}$

$$\int f d\lambda \stackrel{?}{=} \int_x \underbrace{\int_y f(x,y) d\nu(y)}_{\text{function of } x} d\mu(x)$$

(Recall from last lec:
 $f(x, \cdot)$ was measurable all for x)

Is this \mathcal{M} measurable?

Let us make the following assumption: $\mu(x) < \infty$, $\nu(y) < \infty$.

Prop. Suppose (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are σ -finite measure spaces. Then, for any $E \in \mathcal{M} \otimes \mathcal{N}$:

- ① The functions $g = (x \mapsto \nu(E_x))$ and $h = (y \mapsto \mu(E^y))$ are measurable. (\mathcal{M} and \mathcal{N} meas., resp.)
- ② $(\mu \otimes \nu)_E = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y).$

In particular, we have proven Fubini for $f = \mathbf{1}_E$.

Proof. Let $\mathcal{F} = \{E \in \mathcal{M} \otimes \mathcal{N} : ① \text{ and } ② \text{ hold for } E\}$.

Suppose $\mu(x) < \infty$ and $\nu(y) < \infty$.

Note that if $E = A \times B$ for $A \in \mathcal{M}$ and $B \in \mathcal{N}$,

then $E \in \mathcal{F}$. Thus, all rectangles are in \mathcal{F} .

$g(x) = \mathbf{1}_{x \in A} \cdot \nu(B)$ is clearly \mathcal{M} -measurable and

$$\int g(x) d\mu(x) = \nu(B) \int \mathbf{1}_{x \in A} d\mu(x) = \mu(A)\nu(B).$$

$$= (\mu \otimes \nu)(A \times B).$$

Similarly for y .

Similarly for y .

Now, it suffices to show that \mathcal{F} is a σ -algebra.

Unfortunately, showing that is not too easy.

Suppose $E_n \in \mathcal{F}$ and $E_n \uparrow$. (That is, $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$)

Let g_n and h_n be the corresponding slice functions:

$$g_n(x) = \nu(E_n)_x \text{ and}$$

$$h_n(y) = \mu((E_n)^y).$$

It is easy to see that $g_n \uparrow$. ($g_n(x) \in g_{n+1}(x) \forall x$).
Similarly, $h_n \uparrow$.

Let $g = \lim g_n$ and $h = \lim h_n$.

Put $E := \bigcup_{n \geq 1} E_n$. Easy to see that $g(x) = \nu(E_x)$ and
 $h(y) = \mu(E^y)$.

Moreover, g and h are measurable. Thus, ① holds
for E .

For ② : By MCT,

$$\begin{aligned} \int h(E^y) d\nu &= \lim_{n \rightarrow \infty} \int \mu((E_n)^y) d\nu \quad \begin{matrix} \text{MCT} \\ E_n \in \mathcal{F} \text{ and} \\ \text{the} \\ \text{two} \\ \text{is} \\ \text{true} \\ \text{for } E_n \end{matrix} \\ &= \lim_{n \rightarrow \infty} (\mu \otimes \nu)(E_n) \quad \begin{matrix} \uparrow \\ E_n \uparrow E \end{matrix} = (\mu \otimes \nu)(E). \end{aligned}$$

Similarly, we have the other equality for x .

Thus, $E \in \mathcal{F}$ and hence, \mathcal{F} is closed under increasing unions.

(To be continued.)

Def. ① A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a **monotone class**

if \mathcal{C} is closed under countable increasing unions

and countable decreasing intersections.

② Given an algebra \mathcal{A} , $\mathcal{C}(\mathcal{A})$ is the smallest

monotone class containing \mathcal{A} ; called the monotone class generated by \mathcal{A} .

(Monotone class)

Lemma (Monotone class Lemma)

If \mathcal{A} is an algebra on X , then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$. That is, the monotone class generated by \mathcal{A} is the σ -alg. generated by \mathcal{A} .

Continuing We showed that \mathcal{F} is closed under 1 otble unions.

Now, suppose $\{E_n\} \subseteq \mathcal{F}$ and $E_n \downarrow E$.

Note again that $g(x) = \nu(E_x)$

$$= \lim_{n \rightarrow \infty} \nu((E_n)_x)$$

and $h(y) = \lim_{n \rightarrow \infty} \mu((E_n)^y)$

General Q. If $(f_n); f_n \geq 0$ and $f_n \downarrow f$ in (X, \mathcal{M}, μ) , is it true that $\int f_n d\mu \rightarrow \int f d\mu$?

NO. Not in general.

Stuff like this requires finiteness of measure.

Here, we assumed that μ, ν are finite.

So, $g_n \leq g_1$ and DCT helps us.
↳ measurable with finite

Hence, if μ and ν are finite, then \mathcal{F} is a monotone class and \mathcal{F} contains all rectangles. Moreover, (by linearity) \mathcal{F} contains all finite disjoint unions of rectangles.

Thus, the new algebra \mathcal{A} which generates $\mathcal{M} \otimes \mathcal{N}$.

Thus, by monotone class lemma, $\mathcal{F} = \mathcal{M} \otimes \mathcal{N}$ and thus, we have Fubini for functions which are indicators of measurable sets.

For σ -finite split $X = \bigcup X_i$ and $Y = \bigcup Y_j$ and
conclude.

Lecture 9 (04-02-2021)

04 February 2021 14:03

Thm. let \mathcal{A} be an algebra on X . Then, if $\mathcal{C} = \mathcal{C}(\mathcal{A})$ is the monotone class generated by \mathcal{A} , and $\mathcal{M} = \mathcal{M}(\mathcal{A})$ is the σ -algebra generated by \mathcal{A} , then $\mathcal{C} = \mathcal{M}$.

Proof. Note that if \mathcal{C} itself is an algebra, then we are through.
(Countable unions will follow from finite unions since \mathcal{C} is a monotone class.)

Thus, we only need to show that \mathcal{C} is an algebra.

For $E \in \mathcal{C}$, define

$$\mathcal{C}(E) := \{ F \in \mathcal{C} : E \cap F, E \setminus F, F \setminus E \in \mathcal{C} \}.$$

Note that :

- $E \in \mathcal{C}$
- $\emptyset \in \mathcal{C}$ (Note $\emptyset \in \mathcal{C}$ since $\emptyset \in \mathcal{A} \subset \mathcal{C}(\mathcal{A}) = \mathcal{C}$)
- $E \in \mathcal{C}(F) \Leftrightarrow F \in \mathcal{C}(E)$ for $E, F \in \mathcal{C}$.

Suppose $E \in \mathcal{A}$. Then, for any $F \in \mathcal{A}$, then $F \in \mathcal{C}(E)$.
[Since $E \cap F, E \setminus F, F \setminus E \in \mathcal{A} \subset \mathcal{C}$.]

Thus, $\mathcal{A} \subseteq \mathcal{C}(E)$. — (*)

Verify : $\mathcal{C}(E)$ is a monotone class.

Thus, $\mathcal{C} = \mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(E)$ [by (*)]

$$\Rightarrow \mathcal{C} = \mathcal{C}(E) \quad \forall E \in \mathcal{A}$$

In particular, for $F \in \mathcal{C}$, $F \in \mathcal{C}(E)$. (if $E \in \mathcal{A}$.)

By reciprocity : $\forall E \in \mathcal{A} \quad \forall F \in \mathcal{C} : E \in \mathcal{C}(F)$.

$$\Rightarrow \mathcal{A} \subseteq \mathcal{C}(F) \quad \forall F \in \mathcal{C}$$

$$\Rightarrow \mathcal{C} = \mathcal{C}(F) \quad \forall F \in \mathcal{C}$$

Since $X \in \mathcal{A} \subseteq \mathcal{C}$, for any $E \in \mathcal{C}$

$x \setminus E, x \cap E, x \cup E \in \mathcal{C}$; in particular, $x \setminus E \in \mathcal{C}$.

For $E, F \in \mathcal{C}$, $E \cap F \in \mathcal{C}$.

Thus, \mathcal{C} is closed under complement and finite intersection.

Hence, hence also, a sigma-algebra. Thus, \mathcal{C} is an algebra.

$$\therefore \mathcal{M} \subseteq \mathcal{C}.$$

OTOH, \mathcal{M} is itself a monotone class. Thus, $\mathcal{C} \subseteq \mathcal{M}$.

□

Thus, we are done.

Thm.

(Fubini-Tonelli Theorem)

Suppose (x, \mathcal{M}, μ) , (y, \mathcal{N}, ν) are σ -finite measures.

(i) (Tonelli's Thm.) If $f \in L^+(x \times y)$, then the slice functions

$$g(x) = \int_y f(x, y) d\nu(y) \quad \text{and}$$
$$h(y) = \int_x f(x, y) d\mu(x) \quad \text{are}$$

in $L^+(x)$ and $L^+(y)$, resp.

Moreover,

$$\int f d(\mu \otimes \nu) = \int_x \left(\int_y f(x, y) d\nu \right) d\mu$$

$$= \int_y \left(\int_x f(x, y) d\mu \right) d\nu$$

(ii) (Fubini's Thm.) If $f \in L'(\mu \otimes \nu)$, then $f_x \in L'(\nu)$ a.e. x (w.r.t. μ) and $f_y \in L'(\mu)$ a.e. y (w.r.t. ν) and

$$\int f d(\mu \otimes \nu) = \int_x \left(\int_y f(x, y) d\nu \right) d\mu$$

$$= \int_y \left(\int_x f(x, y) d\mu \right) d\nu$$

Proof.

The proposition from previous lecture shows that (i) holds if $f = \mathbb{1}_E$ for $E \in \mathcal{M} \otimes \mathcal{N}$.

By linearity, the statement holds for finite linear combinations.

i.e., taking increasing limit function in MCT_f^+ , (i) follows.

In fact, the same proof shows that if $f \in L^+$ and
 $\int f d(\mu \otimes \nu) < \infty$, then $f < \infty$ a.e. w.r.t. $\mu \otimes \nu$.

Consequently, the functions $g < \infty$ μ a.e. and

$h < \infty$ ν a.e. and

Finally same Fubini conclusion follows by arguing with f' and f . \square

Remarks:

1. The measure spaces $(X_i, \mathcal{M}_i, \mu_i)$, of infinitely many know what $\bigoplus_{i=1}^{\infty} \mathcal{M}_i$ is. Fubini holds there with $\bigoplus_{i=1}^{\infty} \mu_i$.
2. If f is not in $L'(\mu \otimes \nu)$, then Fubini may fail.
(Recall the array of 1_s and -1_s .)
3. In general, if μ and ν are not σ -finite, then it may not hold.

Qn. Non σ -finite measures?

Take $y = [0, 1]$, $\mathcal{N} = \mathcal{B}(y)$ and $\nu = \text{counting measure}$.

(In some sense, this typifies non σ -finite measures.)

Let $X = Y$, $\mathcal{M} = \mathcal{N}$ and $\mu = m$. (Lebesgue measure)

Let $D = \{(x, y) : x \in [0, 1]\} \subset X \times Y$. (Why is it measurable?)

Put $f = \mathbb{1}_D$.

$$\int (\int f d\mu) d\nu = \int 0 d\nu = 0$$

Closed in $X \times Y$
(Borel)

$$\int_x^y \left(\int_y f d\nu \right) d\mu = \int_x^y 1 d\mu = 1$$

But $(\mu \otimes \nu)(D) = \infty$.

If we consider any open $U \supseteq D$, then it contains a ball which contains a rectangle which have $\mu \otimes \nu = \infty$.

Thus, $(\mu \otimes \nu)(D) = \inf \{(\mu \otimes \nu)(U) : U \text{ open}, U \supseteq D\}.$
 $= \infty$.

(There are some conditions such that it still goes through)
with σ -finite. Exercise in follow.

Digressive remarks:

Random variable : Measurable function on a probability space.
 $(\mu(\Omega)=1)$

- (Ω, \mathcal{M}, P) , $X_i : \Omega \rightarrow \mathbb{R}$ r.v.s

What does it mean to say X_1, \dots, X_n are indep.?

In general, for any (X, \mathcal{M}, μ) , $f : X \rightarrow \mathbb{R}$ with
 $f \in L^1 \cap L^{\infty}$, define $\nu(A) := \int f \cdot \mathbf{1}_A d\mu$.
not necessarily finite

Then, ν is also a measure on X .
finite!

Something something marginals and constructing joint measure.

- Probabilities allow us to define infinite products in a non-trivial way.

The n-dimensional Lebesgue integral

Note. The product measure $\mu \otimes \nu$ is (usually) not complete

even if μ and ν are.

Defn. By L^n , we mean the completion of $\bigoplus_{i=1}^n (R, \mathcal{B}_i, m)$ on \mathbb{R}^n . We shall use m again to denote the measure.

(\mathbb{R}^n, L^n, m) is the Lebesgue measure

Prop. If $E \in L^n$, then

$$(1) \quad m(E) = \inf \{m(U) : U \text{ open, } U \supseteq E\}$$
$$= \sup \{m(K) : K \text{ compact, } K \subseteq E\}.$$

(2) Suppose $m(E) < \infty$. Then, $\forall \epsilon > 0$, \exists a finite collection

$\{R_i\}_{i=1}^n$ of p.w. disjoint rectangles whose sides are open intervals s.t.

$$\mu(E \Delta \bigcup_{i=1}^n R_i) < 3\epsilon.$$

Rectangle: $E = E_1 \times \dots \times E_n$, E_i we called the sides of E .

Another important property of m :

Propn. m is translation invariant, i.e., for any $a \in \mathbb{R}^n$:

$$E \in L^n \Rightarrow (E+a) \in L^n \text{ and } m(E+a) \in L^n.$$

Moreover, it is rotation invariant.

Lecture 10 (08-02-2021)

08 February 2021 14:02

Defn. Let $\Omega \subseteq \mathbb{R}^n$ be open.

Suppose $G: \Omega \rightarrow \mathbb{R}^n$. G is called a C^1 -diffeomorphism

if G is injective, $D_x G$ is continuous and $D_x G \in GL(n, \mathbb{R})$

$$\forall x \in \Omega$$

Thm. Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $G: \Omega \rightarrow \mathbb{R}^n$ is a C^1 -diffeomorphism. Let $f: G(\Omega) \rightarrow \mathbb{R}$.

(i) If f is Lebesgue m'ble on $G(\Omega)$, then $f \circ G: \Omega \rightarrow \mathbb{R}$ is m'ble.

(ii) If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f dm = \int_{\Omega} (f \circ G)(x) |\det D_x G| dm$$

Proof. Suppose Q is a regular cube, i.e., Q is a rectangle whose sides are closed intervals. (Not necessarily of same length!)

We say that Q is centered at a if

$$Q_a(h) := Q = \{\underline{x} : \|\underline{x} - \underline{a}\| \leq h\} \text{ where for } \underline{x} \in \mathbb{R}^n$$

$$\text{we define } \|\underline{x}\| = \|\underline{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Suppose $G = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$. By MVT on each g_i , we have

$$g_i(\underline{x}) - g_i(\underline{a}) = \nabla g_i(S_i) \cdot (\underline{x} - \underline{a}) \text{ where } S_i$$

is on the line seg. joining \underline{a} and \underline{x} .

In other words,

$$g_i(\underline{x}) - g_i(\underline{a}) = \sum_j \left(\frac{\partial g_i}{\partial x_j}(S_i) \right) (x_j - a_j)$$

$$\text{So, } G(\underline{x}) - G(\underline{a}) = \begin{pmatrix} (\nabla g_1)(S_1) & (x_1 - a_1) \\ \vdots & \vdots \\ (\nabla g_n)(S_n) & (x_n - a_n) \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ (\nabla g_n)(\underline{s}_n) \end{pmatrix}$$

Define for a matrix $T = (T_{ij})$,

$$\|T\| := \max_{1 \leq i \leq n} \sum_j |T_{ij}|.$$

$$\text{Then, } \|T_x\|_\infty \leq \|T\| \cdot \|x\|_\infty.$$

Now, we get

$$\|G(x) - G(a)\|_\infty \leq h \cdot \left(\sup_{y \in Q} \|D_y G\| \right) \quad \text{for } x \in Q.$$

$Q_a(h)$

Consequently, $G(Q)$ is contained in cube centered at

$$G(a) \text{ with side length } h \cdot \left(\sup_{y \in Q} \|D_y G\| \right).$$

$$\text{Hence, } m(G(Q)) \leq \left(\sup_{y \in Q} \|D_y G\| \right)^n \cdot m(Q). \quad \text{--- (*)}$$

Fix $\epsilon > 0$: $D_x G$ is continuous on Q , so there exists $\delta > 0$

s.t. whenever $\|x - y\| < \delta$, then $D_x G \approx D_y G$. (??)

$$|\det((D_y G)^{-1}(D_x G))| \leq 1 + \epsilon$$

We now prove the following propⁿ:

Prop. Suppose $T \in GL(n, \mathbb{R})$. Then, if $f: \mathbb{R}^n \rightarrow \mathbb{R}$ measurable,

so is $f \circ T$. If $f \geq 0$ or $f \in L^1(\mathbb{R}^n)$, then

$$\int f dm = |\det T| \int (f \circ T) dm.$$

Proof. If the theorem holds for $S, T \in GL(n, \mathbb{R})$. Then it holds for $ST = S \circ T$.

Thus, we may only consider the elementary transformations:

$$\textcircled{1} \quad T_1(\underline{x}_1, \dots, \underline{x}_i, \dots, \underline{x}_n) = (\underline{x}_1, \dots, c\underline{x}_i, \dots, \underline{x}_n); c \neq 0,$$

$$\textcircled{2} \quad T_2(\underline{x}_1, \dots, \underline{x}_i, \dots, \underline{x}_j, \dots, \underline{x}_n) = (\underline{x}_1, \dots, \underline{x}_j, \dots, \underline{x}_i, \dots, \underline{x}_n),$$

$$\textcircled{3} \quad T_3(\underline{x}_1, \dots, \underline{x}_i, \dots, \underline{x}_j, \dots, \underline{x}_n) = (\underline{x}_1, \dots, \underline{x}_i + \underline{x}_j, \dots, \underline{x}_j, \dots, \underline{x}_n).$$

All are simple consequences of Fubini.

$T_1, T_2 \rightarrow$ use the fact for 1-dim integral

All are simple consequences of w.m.

$T_1, T_2 \rightarrow$ use the fact for 1-dim integral

$T_2 \rightarrow$ direct

B

We shall now use the above prop to prove the general theorem.

Lecture 11 (11-02-2021)

11 February 2021 14:06

Had seen:

$$m(G(Q)) \leq \left(\sup_{y \in Q} \|D_y G\| \right)^n \cdot m(Q). \quad \text{--- (*)}$$

Since the theorem holds for linear maps, replacing G with $T^{-1}G$ in $(*)$, we get

$$\begin{aligned} m(G(Q)) &= |\det T| \cdot m((T^{-1}G)Q) \quad (\text{thm. holds for } T) \\ &\leq |\det T| \cdot \left(\sup_{y \in Q} \|T^{-1}D_y G\| \right)^n \cdot m(Q) \end{aligned}$$

Let $\epsilon > 0$ be given.

Since G is a C^1 -diff ($D_y G$ is cont. on Ω), there

exists $\delta > 0$ s.t. if $\|x - y\| < \delta$, then

$$\|(D_x G)^{-1}(D_y G)\|^n \leq 1 + \epsilon$$

Cut Q into small cubes of length $< \delta$, with

centres $\underline{x}_1, \dots, \underline{x}_n$. Call these cubes Q_1, \dots, Q_n , resp.

Hence,

$$\begin{aligned} m(G(Q)) &\leq \sum_i^n m(G(Q_i)) \\ &\leq \sum_i^n |\det D_{\underline{x}_i} G| \left(\sup_{y \in Q_i} \|(D_{\underline{x}_i} G)^{-1}(D_y G)\| \right)^n m(Q_i) \\ &\leq (1 + \epsilon) \sum_i^n |\det D_{\underline{x}_i} G| m(Q_i) \end{aligned}$$

The last expression is the integral of $\sum_{i=1}^n |\det D_{\underline{x}_i} G| \cdot \mathbb{1}_{Q_i(\epsilon)}$

But this last function converges to $\int_Q |\det D_x G| dm$ as $\epsilon, \delta \rightarrow 0$.

Hence, we have

$$\boxed{\int_Q m(G(Q)) \leq \int_Q |\det D_x G| dm} \quad \text{--- holds for cubes } Q$$

FACT. Any open set is a countable union of cubes whose interiors

are pairwise disjoint.

Let U be open. $U = \bigcup_{i=1}^{\infty} Q_i$, where Q_i are cubes with p -wise disjoint interiors.

So,

$$\begin{aligned} m(G(U)) &\leq \sum_{i=1}^{\infty} m(G(Q_i)) \leq \sum_{i=1}^{\infty} \int_{Q_i} |\det D_x G| dm \\ &= \int_U |\det D_x G| dm \end{aligned}$$

MCT

Now, suppose E is Borel, suppose $m(E) < \infty$. for the Lebesgue measure,

$$m(E) = \inf \{m(U) : U \supseteq E, U \text{ open}\}.$$

In particular, $\exists \{U_j\}_j$ of \downarrow open sets such that $E \subseteq U_j \forall j$ and $m(\bigcap_{j=1}^{\infty} U_j \setminus E) = 0$.

$$m(G(E)) \leq m(G(U_j)) \leq \int_{U_j} |\det D_x G| dm \quad \forall j$$

Taking $j \rightarrow \infty$ gives

$$\begin{aligned} m(G(E)) &\leq \lim_j \int_{U_j} |\det D_x G| dm \\ &= \int_E |\det D_x G| dm \end{aligned}$$

DCR

If $m(E) = \infty$, we σ -finiteness to conclude

$$m(G(E)) \leq \int_E |\det D_x G| dm \quad \text{for all borel sets } E.$$

Now, suppose $f = \sum_{j=1}^{\infty} a_j \mathbb{1}_{E_j}$, where E_j are Borel

and $a_j > 0$. (f defined on $G(\mathbb{R}^n)$.)

$$\int_{G(\mathbb{R}^n)} f dm = \sum_j a_j m(E_j) \leq \sum_j a_j \int_{G(E_j)} |\det D_x G| dm$$

$$\int_{G(\Omega)} f dm \leq \int_{\Omega} (f \circ G)(x) |\det D_x G| dm. \quad (+)$$

for f simple.

By MCT, $(+)$ holds for all $f \geq 0$ measurable.
 Since $(+)$ holds for all f and all "nice" G , replace (f, G) in $(+)$ with $(f \circ G, G^{-1})$ to get

$$\begin{aligned} \int_{\Omega} (f \circ G)(x) |\det D_x G| dm &\leq \int_{G(\Omega)} (f \circ G \circ G^{-1})(x) |\det D_x G| |\det D_x G| dm \\ &= \int_{G(\Omega)} f(x) dm \quad (+) \end{aligned}$$

$(+)$ and $(+)$ together give the equality. If $f \in L^1$, do the same for f^+ and f^- . \square

Example: $(r, \theta) \xrightarrow{G} (r \cos \theta, r \sin \theta)$

$$(0, \infty) \times S^1 \xrightarrow{\text{unit circle can think as } [0, 2\pi]} \mathbb{R}^2$$

$$D_{(r, \theta)} G = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$\det D_{(r, \theta)} G = r$$

$$\text{So, } \int_{\Omega} f(r \cos \theta, r \sin \theta) r dr d\theta = \underbrace{\int_{G(\Omega)} f(x, y) dm}_{\text{what measure?}}$$

An induced measure on S^{n-1} :

$$\begin{aligned} \Phi: \mathbb{R}^n \setminus \{0\} &\longrightarrow (0, \infty) \times S^{n-1} \\ \underline{x} &\mapsto (r, \underline{x}') \quad \text{where} \quad r = |\underline{x}| = \|\underline{x}\|_2, \\ &\quad \underline{x}' = \frac{\underline{x}}{r}. \end{aligned}$$

$$\Phi^{-1}(r, x') = rx'. \quad \Phi \text{ is a homeomorphism.}$$

This suggests that we can pull the Borel measure on \mathbb{R}^n onto $(0, \infty) \times S^{n-1}$ via the pull back:

If $E \subseteq (0, \infty) \times S^{n-1}$ is Borel (^{use} product topology),

define

$$m_*(E) := m(\Phi^{-1}(E)).$$

It is easy to check that m_* is a measure on $(0, \infty) \times S^{n-1}$.

Thm. There exists a unique Borel measure σ on S^{n-1} such that

$m_* = \int x \cdot \sigma$ where ρ on $(0, \infty)$ is defined by

$$\rho(E) = \int_E r^{n-1} dr. \quad \text{If } f \text{ is Borel on } \mathbb{R}^n \text{ and } f \geq 0 \text{ or } f \in L^1(\mu), \\ \text{then } \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{S^{n-1}} f(rx') r^{n-1} d\sigma(x') dr.$$

Recall that if $E \subseteq \mathbb{R}^n$ is Lebesgue meas., then for any $a > 0$,

$$m(aE) = a^n m(E).$$

$$\text{Let } B = \{x \in \mathbb{R}^n : |x| \leq 1\}, \quad m(B) = C_n$$

$$\text{Hence, } B_r = \{x \in \mathbb{R}^n : |x| \leq r\} \text{ has } m(B_r) = C_n \cdot r^n.$$

The shell at radius r and thickness $\delta < r$ has

$$\begin{aligned} \text{Lebesgue vol} &= C_n \cdot r^n - C_n \cdot (r - \delta)^n \\ &= C_n \cdot n r^{n-1} + \delta (\quad) \end{aligned}$$



Proof (of thm.) We construct such a σ . Let $E \subseteq S^{n-1}$ be Borel.

For $a > 0$, let

$$E_a := \Phi^{-1}((0, a] \times E)$$

$$\begin{aligned} m(E_a) &= m_*((0, a] \times E) \\ &\stackrel{\text{def.}}{=} \rho((0, a]) \times \sigma(E) \end{aligned}$$

→ forced, if we want $m = \rho \times \sigma$

$$\text{Def: } \mathcal{G} = \rho([0, 1] \times \sigma(\epsilon))$$

$\frac{\sigma(\epsilon)}{n}$

Thus, we define $\sigma(E) := nm(E)$.

Note that for any E Borel in S^{n-1} , $E \mapsto E_+$ is

Borel-preserving and commutes under \cup , \cap , complements.

Thus, this DOES define a measure on S^{n-1} .

Now, we need to show $m_+ = \rho \times \sigma$ on all Borel subsets of $(0, 1) \times S^{n-1}$. For this, it suffices to show that for all $A \subseteq (0, \infty)$ Borel and $E \subseteq S^{n-1}$ Borel.

$$m_* (B) = (\rho_{X^{\sigma}}) (B) \quad - (\#)$$

$(B \subseteq (0, \infty) \times S^{n-1} \text{ is Borel.})$

So far, we have:

- (i) $\sigma(\varepsilon)$ is defined
 - (ii) (#) holds if $A = (0, 1]$

it follows that

$$m(E_a) = a^n \cdot m(E_i) = \left(\int_{(0,a]} r^{n-1} dr \right) \cdot \sigma(E).$$

So, (#) holds for $A = (0, a)$.

If $0 < a < b$,

$$m_x((a, b] \times E) = m(E_b | E_a) = \frac{b-a}{n} \sigma(E) - \frac{a^n}{n} \sigma(E)$$

$\Rightarrow (\#)$ for $(a, b] \times E$ as well.

Let \mathcal{A}_E = finite disj. union of $(a, b] \times E$

(\hookrightarrow) algebra on $(0, \infty)$ $\times E$

$$\text{Generates } \mathcal{M}_E = \{A \times E : A \in \mathcal{B}_{(n,m)}\}.$$

We have $\sigma_x = \rho \times \sigma$ on ∂E and hence on M_E .

$\bigcup_{E \in \mathcal{B}(S^{n-1})} M_E = \text{set of Borel rect.s on } (0, \infty) \times S^{n-1}$

Thus, $m_E = \rho \times \sigma$ on all Borel sets. B

Cor. If $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is radial, i.e., $|x| = |y| \Rightarrow g(x) = g(y)$,

then

$$\int_{\mathbb{R}^n} g(x) dm = \sigma(S^{n-1}) \int_0^\infty g(r) r^{n-1} dr.$$

Ex. $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, where $\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx$.

Lecture 12 (15-02-2021)

15 February 2021 14:05

Differentiation

Suppose (X, \mathcal{M}, μ) is a measure space. Suppose $f \in L^+$.

Consider $v : \mathcal{M} \rightarrow [0, \infty]$ defined by

$$v(E) = \int_E f d\mu = \int f \cdot \mathbf{1}_E d\mu.$$

Easy to see that v is also a measure.

Not all measures can be obtained this way though.

Ex On $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, put the pre-measure

$$\rho(\{k\}) := \begin{cases} 1/2^k & \text{for } k \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

ρ gives rise to a measure ρ (by abuse of notation) which is "singular w.r.t. m ".

Let $E = \mathbb{N}$ and $F = \mathbb{R} \setminus E$.

$$m(E) = 0 = \rho(F).$$

Another suggestive point: If $f \in L'$ (not necessarily ≥ 0), we can still define

$$v(E) := \int_E f d\mu \quad \text{is a function}$$

$v : \mathcal{M} \rightarrow \mathbb{R}$ satisfying

① $v(\emptyset) = 0$,

② If $\{E_i\}_{i=1}^\infty$ is a countable disjoint collection of sets in \mathcal{M} , then $v\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty v(E_i)$. (from D(T))

Signed measures

Let (X, \mathcal{M}) be a set with a σ -alg. Then, a singed measure is a map $v: \mathcal{M} \rightarrow [-\infty, \infty]$ satisfying

$$\textcircled{1} \quad v(\emptyset) = 0,$$

\textcircled{2} v takes at most one of $-\infty, \infty$,

\textcircled{3} If $\{E_i\}_{i=1}^{\infty}$ is a countable disjoint collection, then

$$v(\bigcup E_i) = \sum_{i=1}^{\infty} v(E_i). \quad \text{Moreover, if this is finite, then the series converges absolutely.}$$

Remark: We shall, by "measure", mean the usual measures, which we sometimes call "positive measures".

\textcircled{1} A more general example for signed measures goes thus.

Suppose $f: X \rightarrow \mathbb{R}$ is s.t. at least one of $\int f^+$ or $\int f^-$ is finite. Then again,

$$v(E) := \int_E f d\mu \text{ defines a signed measures.}$$

\textcircled{2} Suppose μ_1, μ_2 are two (positive) measures and suppose at least one of $\mu_1(x), \mu_2(x) < \infty$.

Define

$$v(E) := \nu_1(E) - \nu_2(E).$$

Then, v is another example.

\textcircled{1} and \textcircled{2} are "effectively" the only ones.

Thm: Jordan decomposition theorem

If v is a signed measure, then there exist UNIQUE

positive measures ν^+ and ν^- s.t.

$$\nu = \nu^+ - \nu^- \text{ and } \nu^+ \perp \nu^-.$$

Defn. We say that (signed) measures μ, ν are singular with respect to each other if there exist E and F s.t.

$$E \cup F = X \text{ and } E \text{ is null for } \mu \text{ and } F \text{ for } \nu. \\ (E \cap F = \emptyset) \\ (\forall A \subseteq E, \mu(A) = 0)$$

This is denoted $\mu \perp \nu$.

$(A, E, F \in \mathcal{M})$

Defn. Suppose ν is a signed measure on (X, \mathcal{M}) .

A set $E \in \mathcal{M}$ is positive for ν , if for all $A \subseteq E$ with $A \in \mathcal{M}$, $\nu(A) \geq 0$.

Similarly, we define negative sets

(Positive set, negative set)

Thm.

Hahn Decomposition Theorem

If ν is a signed measure, there exist $P, N \in \mathcal{M}$

s.t. P is positive and N is negative for ν with

$P \cap N = \emptyset$ and $P \cup N = X$.

If (P', N') is another such pair, then $P \Delta P'$ and $N \Delta N'$ are null for ν .

Proof.

Wlog, suppose ν does not take $-\infty$.

Let $D \in \mathcal{M}$ be s.t. $\nu(D) \leq 0$.

Claim. $\exists A \subseteq D$ s.t. A is negative for ν .

Proof. Let $A_0 = D$.

Having defined $A_n \subset A_{n-1} \subset \dots \subset D$, let

$$t_n = \sup \left\{ \nu(B) : B \in \mathcal{M}, B \subseteq A_n \right\}.$$

$t_n \geq 0$ since $\emptyset = B$ is here.

Let $B_n \subseteq A_n$ be s.t. $\nu(B_n) \geq \min \left\{ \frac{t_n}{2}, 1 \right\}$.

Set $A_{n+1} = A_n \setminus B_n$ and finally consider

$$A = D \setminus \left(\bigcup B_n \right).$$

First, note that $\{B_n\}$ are disjoint. Thus, by defⁿ of measure,

$$\nu(A) = \nu(D) - \sum_{n=1}^{\infty} \nu(B_n).$$

We now show that A is negative.

If not, suppose $A_0 \subseteq A$ with $\nu(A_0) > 0$.

Then, note that $t_n \geq \nu(A_0)$ or

$$\frac{t_n}{2} \geq \frac{\nu(A_0)}{2} \leftarrow \text{fixed.}$$

But then, $\nu(A) = \nu(D) - \sum_{n=1}^{\infty} \nu(B_n) = -\infty$.

But this was ruled by assumption.

This proves the Claim.

We now construct a negative set N . let $N_0 = \emptyset$.

Having constructed N_n , let

$$s_n := \inf \{ \nu(D) : D \in \mathcal{M}, D \subseteq x \setminus N_n \}.$$

As before $s_n \leq 0$. $\because \emptyset \supseteq$

Again as before, let $D_n \subseteq x \setminus N_n$ s.t.

$$\nu(D_n) \leq \max \left\{ \frac{s_n}{2}, -1 \right\} \leq 0.$$

Let $A_n \subseteq D_n$ be as in prev. part, i.e., $A_n \subset D_n$ is

negative ala the previous construction.

Set $N_{n+1} = N_n \cup A_n$ and let $N = \bigcup_{n=1}^{\infty} A_n$.

Note that $\mu(A_n) \leq 0$, and moreover A_n are p-wise disjoint.

If $E \subseteq N$, $E \in \mathcal{M}$, $E = \bigcup_{n=1}^{\infty} (E \cap A_n)$ and so,

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap A_n) \leq 0.$$

$\Rightarrow N$ is negative.

Let $P = X \setminus N$. We now claim that P is positive wrt ν .

Suppose not. Let $E \subseteq P$ be s.t. $\nu(E) < 0$.

Again, in the defⁿ of the inf of s_n , E is candidate set. Thus, $s_n \leq \mu(E)$ $\forall n$.

$$\text{Now, } \nu(N) = \sum_{n=1}^{\infty} \nu(A_n) \leq \sum_{n=1}^{\infty} \max\left(\frac{\nu(E)}{2}, -1\right) = -\infty.$$

$\rightarrow \Leftarrow$

Hence, P is positive.

Finally, if (P', N') is another such pair for ν , then note that $P \cap N'$, $P' \cap N$ are both +ve & -ve sets.

Thus, ν is null on these sets and hence on their union, which is $P \cup P'$. □

Corollary. Jordan decomposition. (Given $\nu: \exists \nu^+, \nu^-$ s.t. $\nu = \nu^+ - \nu^-$)

Proof. Given ν , get P and N as above.

For $E \in \mathcal{M}$, define $\nu^+(E) := \nu(E \cap P)$ and $\nu^-(E) := -\nu(E \cap N)$.

Easy to see that ν^+ , ν^- are (positive) measures and that for any E , $\nu(E) = \nu^+(E) - \nu^-(E)$.

Note that if μ^+ and μ^- are s.t. $\nu = \mu^+ - \mu^-$, then their positive-negative sets give another Hahn

decomposition (P^+, N^-) for ν , and so by the
 (essential) uniqueness of Hahn-decomposition, the uniqueness
 for ν^+ , ν^- follows.

Finally, we need to check $\nu^+ \perp \nu^-$ but that follows
 directly. (3)

Defn. Given signed measures μ and ν on (X, \mathcal{M}) , we say
 that ν is absolutely continuous with respect to μ if
 $\mu(E) = 0 \Rightarrow \nu(E) = 0$. This is denoted as
 $\nu \ll \mu$.

Recall the signed meas. $\nu(E) := \int_E f d\mu$.

For this, we have $\nu \ll \mu$.

Tm. Radon-Nikodym Theorem

Let ν be a ^{σ -finite} signed measure and suppose μ is σ -finite
 positive on (X, \mathcal{M}) . Then, there exist σ -finite signed
 measures λ, ρ on (X, \mathcal{M}) s.t.

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, there exists $f: X \rightarrow \mathbb{R}$ which is μ -integrable
 (in the extended sense) s.t.

$$\rho(E) = \int_E f d\mu.$$

Lastly, this decomposition is unique μ a.e.

Lecture 13 (18-02-2021)

18 February 2021 14:05

A digressive remark: Consider $([0, 1], \mathcal{L}, \mu)$. It is obvious that an arbitrary union of \mathcal{L} -null sets need not be null? But what about an arbitrary increasing family? Is their union null?

Ans. Not necessary.

Proof. Suppose not. Suppose every chain of null sets has null union. Consider

$$\mathcal{N} = \{E \subseteq [0, 1] : E \text{ is null}\}.$$

Then our assumption gives us that \mathcal{N} has a maximal element E_0 . (Zorn's Lemma)

This is absurd. ($E_0 \neq [0, 1]$ clearly but then $\bigcup_{x \in [0, 1] \setminus E_0} E_0 \supseteq E_0$)

Prop. Suppose μ, ν are finite (positive) measures on (X, \mathcal{M}) .

Then either $\nu \perp \mu$ or $\exists E \in \mathcal{M}$ and $\epsilon > 0$ s.t.

$$\nu \geq \epsilon \mu \text{ on } E \text{ and } \mu(E) > 0.$$

Prof. For $n \in \mathbb{N}$, consider $\nu_n := \nu - \frac{1}{n} \mu$. Let (P_n, N_n) be the Hahn Decomposition for ν_n .

$$\text{Let } N = \bigcap_{n=1}^{\infty} N_n \text{ and } P = N^c.$$

Note that for $E \subseteq N$, $\left(\nu - \frac{1}{n} \mu\right) E \leq 0 \quad \forall n$

and thus, $\nu(E) \leq 0$ but ν is a pos. measure and hence, $\nu|_P = 0$.

So, if $\mu(P) = 0$, then $\nu \perp \mu$.

Else, $\mu(P) > 0$

Since $P_n \uparrow$, $\mu(P_n) > 0$ for some n .

$\Rightarrow P_n$ is a +ve set for ν_n .

□

Proof (of Radon-Nikodym)

Assume first that ν and μ are finite, positive.

Consider

$$\mathcal{F} := \left\{ f : x \rightarrow [0, \infty] : \int_E f d\mu \leq \nu(E) \forall E \in \mathcal{M} \right\}.$$

$\mathcal{F} \neq \emptyset$ since $f \equiv 0$ is in \mathcal{F} .

FACT. If $f, g \in \mathcal{F}$, then $f \vee g := \max\{f, g\}$ is also in \mathcal{F} .

Proof. Let $A := \{x : f(x) > g(x)\}$

$$\int_E (f \vee g) d\mu = \int_{E \cap A} f + \int_{E \cap A} g = \int_E f + \int_E g$$

$$\leq \nu(E \cap A) + \nu(E \cap A^c) = \nu(E).$$

Let $a = \sup \left\{ \int_E f d\mu : f \in \mathcal{F} \right\} \leq \nu(X) < \infty$.

$$(\int_E f d\mu \leq \nu(X) \quad \forall f \in \mathcal{F})$$

Let $\{f_n\} \subseteq \mathcal{F}$ s.t. $\int_E f_n d\mu \rightarrow a$.

Let $g_n := f_1 \vee \dots \vee f_n$. Then,

$$\textcircled{1} \quad g_n \in \mathcal{F}$$

$$\textcircled{2} \quad g_n \uparrow \text{pointwise} \rightarrow f := \sup f_n.$$

Note that $\int_E g_n d\mu \rightarrow a$ by Sandwich. By NCT,

$$\int_E f d\mu = \lim_n \int_E g_n d\mu = a.$$

(In particular, $f < \infty$ μ -a.e.)

Define $\rho(E) := \int_E f d\mu$ and let $\lambda = \mu - \rho$.

Note $\lambda \geq 0$ by construction of \mathcal{F} .

Claim. $\lambda \perp \mu$.

Proof. Suppose not. Then, by the prev. propn, $\exists E \in \mathcal{M}, \epsilon > 0$

s.t. $\mu(E) > 0$ and $\lambda \geq \epsilon \mu$ on E .

Claim Consider the function $g = f + \varepsilon \mathbb{1}_E$. Then $g \in F$ and

$$\int g d\mu = \int f d\mu + \varepsilon \mu(E) > a, \text{ giving a contradiction.}$$

Proof. To show $g \in F$: $\forall F \in M$, want to show:

$$\int_F g d\mu \leq v(F).$$

By defⁿ of E , $v(E) = v(E) - \int_E f d\mu$ satisfies

$\lambda \geq \varepsilon \mu$ and so,

$$v(E) \geq \int_E f d\mu + \varepsilon.$$

The conclusion now follows by splitting the integral over

$F \cap E$ and $F \cap E^c$.

◻

Thus, $\lambda \perp \mu$.

◻

UNIQUENESS Suppose $v = \lambda + f d\mu$
 $= \lambda' + f' d\mu$ s.t.

$\lambda, \lambda' \perp \mu$ and f, f' are integrable.

But note that if $\lambda, \lambda' \perp \mu$, then so is $\lambda - \lambda' \perp \mu$.

But $\lambda - \lambda' = (f' - f) d\mu$. (1)

$$\Rightarrow \lambda - \lambda' \ll \mu \quad \text{--- (2)}$$

$$\text{(1) and (2)} \Rightarrow \lambda = \lambda' \Rightarrow f = f' \mu \text{ a.e.}$$

This establishes Radon-Nik. for $\mu, v \geq 0$ and finite.

Now, suppose $\mu, v \geq 0$ with μ, v σ -finite.

Get a sequence $\{A_j\}$ of p-wise disjoint sets s.t.

$$\bigcup_j A_j = X \text{ and both } \mu, v < \infty \text{ on } A_j \setminus V_j.$$

Define $v_j := v|_{A_j}$ and $\mu_j = \mu|_{A_j}$. Then, by the earlier

part, gives f_j s.t.

$$\nu_j = \lambda_j + f_j d\mu_j \quad \forall j.$$

Take $\lambda = \sum \lambda_j$ and $f = \sum f_j$ and check that it works. The main points here are the following:

① If $\{\lambda_j\} \rightarrow$ +ve measures and $\mu > 0$ and $\lambda_j \perp \mu$, so is $\sum \lambda_j$.

② $\lambda_j \ll \mu \quad \forall j \Rightarrow \sum \lambda_j \ll \mu$.

Finally, if ν is signed, write $\nu = \nu^+ - \nu^-$ from the Jordan decomposition and argue individually.

Notation: If $\lambda \ll \mu$, then $\lambda(E) = \int_E f d\mu$ for some f which is μ -integrable.

We shall denote this f by $\frac{d\lambda}{d\mu}$.

This is called a Radon-Nikodym derivative.

(Radon-Nikodym derivative)

Remark: If (say) ν is not σ -finite, the theorem is not true.

Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}[0, 1]$, $\mu = m$ (Lebesgue)
 $\nu =$ counting

Then, clearly, $m \ll \nu$ but $m \neq f d\nu$ for any f which is ν integrable.

Prop: Suppose ν is σ -finite, signed and suppose μ, λ are σ -finite +ve measures. Suppose $\nu \ll \mu$, $\mu \ll \lambda$. Then,

① if $g \in L'(\nu)$, then

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

$$\int g \, d\nu = \int g \frac{d\nu}{d\mu} \cdot d\mu$$

② $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}. \quad (\text{Chain rule})$$

DIFFERENTIATION À LA LEBESGUE

We shall restrict ourselves to the Lebesgue measure m on $(\mathbb{R}^n, \mathcal{B})$ (or $(\mathbb{R}^n, \mathcal{L})$).

Suppose $\nu \ll m$. So, $\frac{d\nu}{dm}$ exists.

1-dim case.

$$F(x) = \int_a^x f(t) dt \leftarrow \begin{matrix} \text{think of this} \\ \text{as } \nu([a, x]) \end{matrix}$$

$$\lim_{x \rightarrow a} \frac{\nu([a, x])}{m([a, x])} = f(a).$$

2-dim?