Model Categories

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Notations

1. C will denote a category.
2. f, g will denote morphisms in a category.
3. Given a ring R, $\mathrm{Ch}(R)$ will denote the category of nonnegatively graded chain complexes over R, i.e., objects are of the form

$$\cdots \to M_2 \to M_1 \to M_0,$$

where the M_i are R-modules and the morphisms are the obvious ones.

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such that ri and $r'i'$ are the appropriate identity maps.

Definition 3

A model category is a category C with three distinguished classes of maps:

1. weak equivalences ($\xrightarrow{\sim}$),
2. fibrations ($\twoheadrightarrow$), and
3. cofibrations ($\hookrightarrow$),

each of which is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called an acyclic fibration (resp. acyclic cofibration).

Additionally, we require the model category axioms **MC1** - **MC5** to be satisfied, which are stated on the next slide.

Model Category Axioms

MC1 Finite limits and colimits exist in C.

MC2 Let f and g be maps such that gf is defined. If two of of the three maps f, g, gf are weak equivalences, then so is the third.

MC3 If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so if f.

MC4 Given a commutative diagram of the form $$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$, a lift exists in either of the following two situations: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

MC5 Any map f can be factored in two ways $f = pi = qj$, where i is a cofibration, p is an acyclic fibration, j is an acyclic fibration, and q is a fibration.

Fibrant and Cofibrant objects

By **MC1**, a model category C has both an initial object $\varnothing$ and a final object $*$.

Definition 4

An object $A \in \mathsf{C}$ is said to be cofibrant if $\varnothing \to A$ is a cofibration and fibrant if $A \to *$ is a fibration.

An example

The category $\mathrm{Ch}(R)$ can be given the structure of a model category by defining a map $f : M \to N$ to be

1. a weak equivalence if f induces an isomorphism on homology groups,
2. a cofibration if for each $k \geqslant 0$, the map $f_k : M_k \to N_k$ is a monomorphism with a *projective* R-module as its cokernel,
3. a fibration if for each $k \geqslant 1$, the map $f_k : M_k \to N_k$ is an epimorphism.

Note that $\varnothing$ and $*$ are both the zero chain complex. The cofibrant objects in $\mathrm{Ch}(R)$ are the chain complexes M such that each M_k is projective. On the other hand, object is fibrant.

The homotopy category $\mathrm{Ho}(\mathrm{Ch}(R))$ is equivalent to the category whose objects are these cofibrant chain complexes and whose morphisms are ordinary chain homotopy classes of maps.

Another example

The category Top of topological spaces can be given the structure of a model category by defining a map $f : M \to N$ to be

1. a weak equivalence if f is a homotopy equivalence,
2. a cofibration if f is a closed Hurewicz cofibration,
3. a fibration if f is a Hurewicz fibration.

In this case, the homotopy category Ho(Top) is the usual homotopy category of topological spaces.

Some constructions

Given a model category C, we may construct some new model categories.

Example

The opposite category C^{op} is quite naturally a model category by keeping the weak equivalences the same and switching fibrations with cofibrations.

Example

If A is an object of C, $A \downarrow \mathsf{C}$ is the category in which an object is a map $f : A \to X$ in C. A morphism in this category from $f : A \to X$ to $g : A \to Y$ is a map $h : X \to Y$ such that $hf = g$. (For example, $* \downarrow \mathsf{Top}$ is the category of pointed spaces.)
This has the structure of a model category by defining h to be a weak equivalence, fibration, or cofibration according to whether it was so in C.
An object X of $* \downarrow \mathsf{Top}$ is cofibrant iff the basepoint of X is closed and nondegenerate.

Definition 5

Let $i : A \to B$ and $p : X \to Y$ be maps such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

has a lift for any choice of horizontal arrows (that make the diagram commute). Then, i is said to have the left lifting property (LLP) with respect to p, and p is said to have the right lifting property (RLP) with respect to i.

Proposition 6

Let C be a model category.

1. The cofibrations in C are precisely the maps which have the LLP with respect to acyclic fibrations.
2. The acyclic cofibrations in C are precisely the maps which have the LLP with respect to fibrations.
3. The fibrations in C are precisely the maps which have the RLP with respect to acyclic cofibrations.
4. The acyclic fibrations in C are precisely the maps which have the RLP with respect to cofibrations.

This shows that the axioms for model category are overdetermined in some sense: more precisely, if C is a model category, then given just the classes of weak equivalences and fibrations is enough to determine the class of cofibrations.

Given a pushout diagram

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "j"'] & C \arrow[d, "j'"] \\ A \arrow[r, "i'"'] & P \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd},$$

the map i' is the cobase change of i (along j). Similarly, one may define base change.

Proposition 7

Let C be a model category.

1. The classes of fibrations and acyclic fibrations are closed under cobase change.
2. The classes of cofibrations and acyclic cofibrations are closed under base change.

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