

# Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

# Bruns & Herzog  $\rightarrow$  Cohen-Macaulay rings  
- 1st Part

# Affine algebra



## Derived Category

$R \rightarrow$  ring (possibly noncomm.)

$R$ -complexes :

$$\dots \rightarrow M_{i+1} \xrightarrow{\delta} M_i \xrightarrow{\delta} M_{i-1} \rightarrow \dots \quad \delta^2 = 0$$

$$\text{im}(\delta_{i+1}) \subset \text{ker}(\delta_i)$$

$$H_i(M) = \text{ker}(\delta_i) / \text{im}(\delta_{i+1})$$

$$H(M) = (H_i(M))_{i \in \mathbb{Z}}$$

$C(R) =$  category of  $R$ -complexes  
*(morphisms as usual)*

If  $f: M \rightarrow N$ , we get an induced map

$$H(f): H(M) \rightarrow H(N).$$

Defn:  $f$  is a quasi isomorphism (or weak equivalence)

if  $H(f)$  is bijective.  
*(Automatically iso.)*

$W :=$  collection of weak equivalences in  $C(R)$

$$D(R) := C(R)[W^{-1}] \quad (\text{or } W^{-1}C(R))$$

- Key property:  $W$  has the 2-out-of-6 property:  
i.e. ... composable morphisms  $\dots \xrightarrow{f} \xrightarrow{g} \xrightarrow{h} \dots$

- Key property:  $W$  has the  $2\text{-out-of-}3$  property.  
 Given composable morphisms  $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$ ,  
 if  $gf$  and  $hg \in W$ , then  $f, g, h, hgf$   
 are in  $W$ .

Ex:  $\Rightarrow 2\text{-out-of-}3$  property

If  $f, g, fg$  are defined and 2 are  
 in  $W$ , then so is the third.

Concretely:

$$C(R) \rightsquigarrow K(R) \rightsquigarrow D(R).$$

$\uparrow$   
homotopy category

$M, N \rightarrow R\text{-complexes}$

$\text{Hom}_R(M, N) :=$  Hom-complex of abelian groups  
 (when  $R$  is comm this is  
 an  $R$ -complex)

$\text{Hom}_R(M, N)_n :=$  Maps of degree  $n$  from  
 $M \rightarrow N$  (no compatibility!)

$$\dots \rightarrow M_i \rightarrow M_{i+1} \rightarrow \dots = \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i, N_{i+n})$$

$\swarrow$        $\searrow$   
 $\dots \rightarrow N_{i+n} \rightarrow N_{i+n-1} \rightarrow \dots$

$$\partial: \text{Hom}_R(M, N)_{n+1} \rightarrow \text{Hom}_R(M, N)_n$$

$$\partial(f) = \partial^n f - (-)^{n+1} f \partial^m.$$

$$\underline{\text{Check: }} \partial^2 = 0.$$

Observe:  $Z_0(\text{Hom}_R(M, N)) = \text{Hom}_e(M, N).$

Def.  $f, g \in \text{Hom}_e(M, N)$  are homotopic if

$$f-g \in B_0(\text{Hom}_R(M, N)), \text{ i.e.,}$$

$$f-g = \partial h \quad \text{for some } h \in \text{Hom}_R(M, N).$$

$K(R) := \mathcal{C}(R)/\text{homotopy relation.}$

Object =  $R$ -complexes

$$\text{Hom}_K(M, N) = H_0(\text{Hom}_R(M, N)).$$

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•  $f \sim g$  in  $\mathcal{C} \Rightarrow f = g$  in  $K(R)$ .

$\Rightarrow H(f) = H(g)$

Defn.  $M$  an  $R$ -complex.

$\sum M$  (or  $M[i]$ ) is the  $R$ -complex

$$(\sum M)_i = M_{i-1}$$

with  $\partial^{\sum M} = -\partial^M$ .

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$\text{Proj } R := \text{Projective } R\text{-modules}$

$$\begin{array}{ccc} K(R) & & \\ \downarrow & \nearrow \text{localisation} & \\ K(\text{Proj } R) & \xleftarrow[-f-]{} & D(R) \\ \downarrow & & \end{array}$$

$\exists p : D(R) \rightarrow K(\text{Proj } R)$ , a full and faithful embedding  
"projective resolutions"

left adjoint to  $q$ .

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$$\text{Hom}_K(pM, N) = \text{Hom}_D(M, qN)$$

$f : M \rightarrow N$  morphism

$\text{cone}(f) := N \oplus \sum M$  with differential

$$\begin{matrix} N_i & \xrightarrow{+} & N_i \\ \oplus & & \oplus \\ M_i & \xrightarrow{f} & M_{i-1} \end{matrix}$$

$$\partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$$0 \rightarrow N \hookrightarrow \text{cone}(f) \rightarrow \sum M \rightarrow 0.$$

$f$  is w.e.  $\Leftrightarrow H(\text{cone}(f)) = 0$ .

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## Image of $P$ ?

K-projectives.

$P$  an  $R$ -complex is K-projective if given any solid diagram

$$\begin{array}{ccc} & \overset{Z}{\nearrow} & M \\ P & \xrightarrow{\alpha} & N \\ & \pi \downarrow & \end{array} \quad \text{w.e.}, \quad \exists \text{ lift } Z.$$

FACT:  $p: D(R) \xrightarrow{\sim} K\text{-Proj}(R) \subseteq K(\text{Proj } R)$ .

$\hookleftarrow$  morphism up here are homotopy

- $\text{Hom}_e(R, M) = Z_0(M)$

$$\begin{array}{ccc} 0 & \xrightarrow{\circ} & M_0 \\ R & \xrightarrow{\circ} & M_0 \xrightarrow{\circ} 0 \\ & \xrightarrow{\circ} & M_{-1} \end{array}$$

Using this,

check:  $R$  is K-projective.

(use:  
surjective + w.e.  
 $Z(M) \rightarrow Z(R)$  onto)

- $(P_\lambda)_\lambda$  family of K-projectives

Then,  $\bigoplus P_\lambda$  is also K-projective.

Conversely closed under direct summands.

- K-projectives are closed under suspensions.

$$\dots \xrightarrow{\circ} P_{i+1} \xrightarrow{\circ} P_i \xrightarrow{\circ} P_{i-1} \xrightarrow{\circ} \dots$$

is K-proj, if  $P_i$  projective  $\forall i$ .

Ex:  $0 \rightarrow P' \xrightarrow{f} P \xrightarrow{g} P'' \rightarrow 0$  exact seq.  
of complexes.

Then, if  $P'$  and  $P''$  are K-projective, so is  $P$ .

If  $P', P, P''$  are complexes of projectives, then  
any two being K-projective  $\Rightarrow$  third is K-proj.

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Corollary. Any bounded complex of projectives is K-projective.

Proof. P. :  $0 \rightarrow P_b \rightarrow \dots \rightarrow P_a \rightarrow 0$ , each  $P_i$  proj.

Induce on  $b-a$ .

$b-a=0$  done earlier.

$$0 \rightarrow P_{\leq b-1} \rightarrow P \rightarrow \sum^b P_b \rightarrow 0. \quad \blacksquare$$

" "

$$0 \rightarrow P_{b-1} \rightarrow \dots \rightarrow P_a \rightarrow 0$$


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Next: Any complex of projectives with  $P_i = 0 \forall i > a$  is K-projective.

$$P. : \dots \rightarrow P_{a+1} \rightarrow P_a \rightarrow 0$$

$P = \underset{n \geq a}{\operatorname{colim}} P_{\leq n}$ , each  $P_{\leq n}$  is projective since bounded.

$$0 \rightarrow \bigoplus_n P_{\leq n} \xrightarrow{1-\delta} \bigoplus_n P_{\leq n} \rightarrow P \rightarrow 0.$$

$\downarrow$  K-proj       $\downarrow$  projectives

Use 2-out-of-3.

Do directly...

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$$\begin{array}{ccc} M & & \\ \downarrow \pi & \swarrow \text{v.e.} & \\ N & & \end{array}$$

$$P \text{ is K-projective} \Leftrightarrow \operatorname{Hom}_K(P, M) \cong \operatorname{Hom}_K(P, N).$$

## Lecture 2 (11-01-2023)

11 January 2023 13:24

$R$  ring.

$$D(R) \simeq k\text{Proj}(R)$$

Recall:  $P \in \mathcal{C}(\text{Proj } R)$  is  $K$ -projective if

$$\begin{array}{ccc} & X & \\ P & \xrightarrow{\sim} & Y \\ & \downarrow \varepsilon & \end{array}$$

Example  $P \in \mathcal{C}(\text{Proj } R)$  with  $P_i = 0$  for all  $i < 0$ .

$$\dots \rightarrow P_{n-1} \rightarrow P_n \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

Sketch. Construct lifting one step at a time.

Suppose  $\tilde{\alpha}: P_{\leq n} \rightarrow X$  is a lifting.

Want  $\tilde{\alpha}: P_{\leq n+1} \rightarrow X$  compatibly.

Let  $s \in P_{n+1}$

We must have

$$-\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$$

$$-\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$$

$$\begin{array}{ccc} & X & \\ \tilde{\alpha} & \nearrow & \downarrow \varepsilon \\ & \approx & \end{array}$$

Check that the above can be solved.

This uses three things:  $\varepsilon$  surjective  $\Rightarrow \varepsilon$  surjective on boundaries

$H(E)$  iso  $\Rightarrow \varepsilon$  surjective on cycles

$\ker(\varepsilon)$  is acyclic.

$\Rightarrow$  Every module has a  $K$ -projective resolution.

$$F: \dots \rightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M$$

Can avoid choices  
by taking generating  
set to be  $M, \ker \varepsilon,$   
 $\ker \partial_1, \dots$

$$F \xrightarrow{\varepsilon} M.$$

Defn. A  $K$ -projective resolution of  $M \in \mathcal{C}(R)$  is a morphism

$$E: P \rightarrow M \text{ s.t.}$$

①  $E$  is a quasi iso,

(Not insisting  
surjective.)

②  $P$  is  $K$ -projective.

②  $P$  is  $K$ -projective.

This makes it functorial!

Thm  $\forall M \in \mathcal{C}(R)$ ,  $\exists$  surjective  $K$ -projective resolution:

$$P \xrightarrow{\sim} M$$

$\downarrow$   
 $K\text{-proj.}$

Defn. An  $R$ -complex  $F$  is semi-free if  $F$  admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \dots \subseteq \bigcup_{n \geq 0} F(n) = F$$

s.t. ①  $F(n) \subseteq F$  is a subcomplex

②  $\frac{F(n+1)}{F(n)}$  graded free module with  $\partial = 0$ ,  
i.e.  $\partial(F(n+1)) \subseteq F(n)$

Fact: semi-free  $\Rightarrow K\text{-proj.}$

Example.  $\dots \rightarrow F_{a+1} \rightarrow F_a \rightarrow 0$

$$F(n) = F_{\leq n} = \dots \rightarrow F_n \rightarrow \dots \rightarrow F_a \rightarrow 0$$

$$\frac{F(n+1)}{F(n)} = \dots \rightarrow F(n+1) \rightarrow 0 \rightarrow \dots$$

$$= \sum^{n+1} F_{n+1}.$$

Thm. Each  $M \in \mathcal{C}(R)$  has a surjective semi-free resolution

$$F \xrightarrow{\sim} M$$

$\uparrow$   
 $K\text{-projective}$

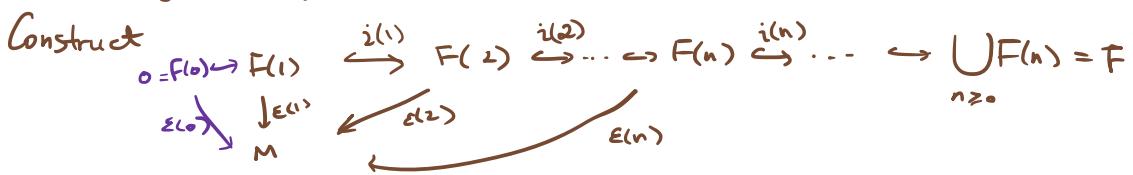
Corollary. Every  $K$ -projective is a retract of a semi-free.

Proof.  $P$  is  $K$ -projective:

$$\begin{array}{ccc} & F & \\ P & \xrightarrow{\quad i \quad} & F \\ \dashrightarrow & & \downarrow \partial \\ P & \xrightarrow{\quad id \quad} & P \end{array}$$

Sketch (Baby ver of Quillen's "small object argument".)

Construct  $\dots \rightarrow F(0) \hookrightarrow F(1) \xrightarrow{i(1)} F(2) \xrightarrow{i(2)} \dots \hookrightarrow F(n) \xrightarrow{i(n)} \dots \hookrightarrow \bigcup F(n) = F$



s.t. ①  $F(n+1)/F(n)$  is graded free with  $\partial = 0$ ,

②  $\varepsilon(1)$  is surjective on homology.

(In turn, each  $\varepsilon(n)$  is surjective on homology.)

③  $\ker(H(E(n))) \subseteq H(F(n))$  maps to 0 under  $H(i(n))$ .

This does the job. [Something 0 in column is 0 at finite stage.]

Why is  $\varepsilon$  surjective?

Remark.  $\varepsilon: X \rightarrow Y$  s.t.

$Z(\varepsilon)$  surjective +  $H(\varepsilon)$  bijective.

Then,  $\varepsilon$  is surjective.

Indeed, we have c.c.e.s:

$$\begin{array}{ccccccc}
 0 & \rightarrow & B(x) & \rightarrow & Z(x) & \rightarrow & H(x) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & B(y) & \rightarrow & Z(y) & \rightarrow & H(y) \rightarrow 0 \\
 \hline
 0 & \rightarrow & Z(x) & \rightarrow & x & \rightarrow & \sum B(x) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & Z(y) & \rightarrow & y & \rightarrow & \sum B(y) \rightarrow 0
 \end{array}$$

Snake lemma  
 $B(x) \rightarrow B(y)$   
 is epi  
 $\Downarrow$   
 $x \rightarrow y$  is surj

Construction of  $F(n), \varepsilon(n)$ :

$\varepsilon(1): F(1) \rightarrow M$  free cover of cycles.

$\underline{\text{o diff}}$  ( $\varepsilon(1)$  morphism since mapping on cycles)

Say we have constructed  $\varepsilon(n): F(n) \rightarrow M$ .

Choose cycles  $(z_n)_n \subseteq F(n)$  that map  
to a generating set of  $\ker(H(\varepsilon(n)))$

Pick  $w_x$  s.t.  $\partial(w_x) = \varepsilon(n)(z_x)$ .

Set  $F(n+1) = F(n) \oplus R\mathbb{Q}_x$   $\deg(\varepsilon_{n+1}) = \deg(z_x) + 1$ .

with  $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_n) = z_n.$$

Define  $\varepsilon^{(n+1)} : F^{(n+1)} \rightarrow M$   
 $\varepsilon^{(n+1)}|_{f(n)} = \varepsilon^{(n)},$   
 $\varepsilon^{(n+1)}(e_n) = \omega.$

□

- Remarks. ① As before, the above construction can be made functorial by avoiding choices (consider all choices!).  
② Depending on what we wish to do with the resolution, there are other constructions.

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Given a module, we have the graded homology module  $H(M) = \langle H_i(M) \rangle_{i \in \mathbb{Z}}.$

(Recall: for us, a graded module is a collection of modules.)

If  $P_\cdot \xrightarrow{\sim} H(M)$  is a projective resolution, one can "perturb" the differentials of  $P_\cdot$  to construct a  $K$ -projective resolution of  $M$ .

(Adam's resolution, Gorenstein/Gitlenberg resolution)

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Exercise.  $P_\cdot$   $K$ -proj  $\Rightarrow P_i$  projective  $\forall i$ .

Converse of above NOT true.

Example (Dold's): Let  $R = \mathbb{Z}/4\mathbb{Z}$  and consider the complex

$$P_\cdot : \dots \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} \dots$$

One way of seeing that the above is not  $K$ -projective is to do the following exercise and note that  $P_\cdot$  is acyclic but not contractible.

is to do the following exercise now:

$P$  is acyclic but not contractible.

Exercise. If  $P$  is  $K$ -proj and  $H(P) = 0$ , then  $P$  is contractible, i.e.,  $\text{id}_P \sim 0$ .  
(or:  $P$  is the mapping cone of some idc.)

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We saw we have an inclusion

$$K\text{Proj}(R) \hookrightarrow K(\text{Proj } R).$$

FACT. Let  $R$  be comm. Noetherian.

The above inclusion is an equality iff  $R$  is regular.

Examples of reg. rings:  $\mathbb{Z}$ ,  $k[x_1, \dots, x_n]$ .

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## Derived functors

Let  $M.$  be an  $R$ -complex.

FACT. If  $P. \xrightarrow{\sim} M.$  and  $Q. \xrightarrow{\sim} M.$  are  $K$ -projective resolutions, then  $P. \cong Q.$  in  $K(\text{Proj } R)$ .

Given any  $N. \in \mathcal{C}(R)$ , set

$$R\text{Hom}_R(M, N) := \text{Hom}_R(P, N),$$

where  $P.$  is a  $K$ -proj. resl" of  $M.$

The object on the right is defined in the homotopy category of abelian groups, i.e.,

$R\text{Hom}_R(-, N)$  is a functor  
 $\mathcal{C}(R) \rightarrow K(\mathbb{Z}).$

$R\text{Hom}_R(-, N)$  is a functor  
 $\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$   
 $(\exists R \text{ is comm, then } \rightarrow K(R).)$

Define  $\text{Ext}_R^i(M, N) := H^i(R\text{Hom}_R(M, N))$   
 $= H_{-i}(\text{Hom}_R(P, N))$

$\text{Ext}_R^0(M, N) = H_0(\text{Hom}_R(P, N))$   
 $= \text{morphisms } P \rightarrow N, \text{ up to homotopy}$

$\text{Ext}_R^i(M, N) = - \rightarrowtail P \rightarrow \sum^i N, \rightarrowtail$

If  $Q \xrightarrow{\sim} N$  is a  $K$ -proj "rel", then

$\text{Hom}_R(P, Q) \cong \text{Hom}_R(P, N).$   
 $\uparrow \text{ quasi-iso}$

Tensors Let  $M.$  be a chain complex of  
right  $R$ -modules.

Let  $N. \in \mathcal{C}(R).$

$M. \otimes N.$  is a complex of  $\mathbb{Z}$ -modules defined  
 by

$$(M. \otimes N.)_i = \bigoplus_{j \in \mathbb{Z}} M_j \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-)^{|m|} m \otimes \partial n$$

FACT. If  $X. \xrightarrow{\sim} Y.$  is a quasiiso,

then  $P. \otimes_R X. \xrightarrow{\sim} P. \otimes Y.$  for any  $K$ -proj  $P.$

Defn.  $M \otimes_R^L N := P \otimes_R N$ . where

$P \xrightarrow{\sim} M$  is a  
 $K$ -proj. resol<sup>n</sup>.

$$\overline{\text{Tor}}_i^R(M, N) = H_i(P \otimes_R N).$$

$$X \xrightarrow{\sim} Y \text{ quasi iso} \Rightarrow \overline{\text{Tor}}_i^R(M, X) = \overline{\text{Tor}}_i^R(M, Y).$$

# Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \rightarrow$  comm. Noetherian ring

$M \rightarrow R\text{-module}$

$r \in R$  is a zero divisor on  $M$  if  $r \cdot m = 0$  for some  $m \neq 0$ .  
 nzd = "not a zero divisor"

$$Z_R(M) = \{r \in R : r \text{ is a z.d. on } M\}$$

$$= \bigcup_{p \in \text{Ass}(M)} p.$$

( $M$  need not be finite.  
 Union need not be.)

Fix  $R, M$ . Let  $\underline{x} = x_1, \dots, x_n$  be a sequence in  $R$ .

$\underline{x}$  is weakly  $M$ -regular or a weakly regular sequence on  $M$

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \dots, x_i)M} \text{ for } 0 \leq i \leq n-1.$$

$\underline{x}$  is  $M$ -regular (or ...) if further  $M/(x)M \neq 0$ .

Ex.  $R = k[x_1, \dots, x_n]$ .  
 $\underline{x} := x_1, \dots, x_n$  is a regular sequence on  $R$ .

Koszul Complexes. Given  $r \in R$ ,

$$K(r; R) = 0 \rightarrow R \xrightarrow{r} R \rightarrow 0.$$

$\uparrow \deg 1 \quad \uparrow \deg 0$

$H_1(K(r; R)) = 0 \Leftrightarrow r \text{ is nzd on } R$ .

Given  $\underline{x} = x_1, \dots, x_n$ , we define

$$K(\underline{x}; R) = \bigoplus_{i=1}^n K(x_i; R).$$

$$\begin{array}{ccccccc} & & \pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \\ \end{bmatrix} & & & & \\ \text{or} & \cdots & \cdots & \rightarrow & R'' & \rightarrow \cdots \rightarrow & R'' \xrightarrow{\cdot (x_1 \cdots x_n)} R \rightarrow 0 \end{array}$$

$$K(\underline{x}; R) = 0 \rightarrow R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \\ x_n \end{bmatrix}} R'' \rightarrow \dots \rightarrow R^{\binom{n}{2}} \xrightarrow{\cdot(x_1 \dots x_n)} R^n \rightarrow R \rightarrow 0$$

$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mapsto \sum r_i x_i$

Now, given  $M \in \mathcal{C}(R)$ ,

$$K(\underline{x}; M) := K(\underline{x}, R) \otimes M.$$

$\hookrightarrow$  Koszul complex on  $\underline{x}$  with coefficients in  $M$ .

$$H_i(\underline{x}; M) := H_i(K(\underline{x}; M)). \rightarrow \text{Koszul homology}$$


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If  $M$  is simply an  $R$ -module (viewed in degree 0),

then

$$K(\underline{x}; M) :$$

$$0 \rightarrow M \rightarrow M'' \rightarrow \dots \rightarrow M^n \rightarrow M \rightarrow 0$$

"same" differentials

$$H_0(\underline{x}; M) = M / \underline{x}M,$$

$$\begin{aligned} H_n(\underline{x}; M) &= \{m \in M : x_i m = 0 \ \forall i\} \\ &= (0 :_M (\underline{x})). \end{aligned}$$


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$$\begin{aligned} ① \quad K(\underline{x}; M) &= K(x_1; R) \otimes_R K(x_2; R) \otimes_R \dots \otimes_R K(x_n; R) \otimes_R M \\ &= K(x_1; R) \otimes K(x_{\geq 2}, M) \\ &= K(x_1; K(x_{\geq 2}, M)) \end{aligned}$$

$$② \quad x, y \in \mathcal{C}(R) \rightsquigarrow x \otimes_R y \xrightarrow{\sim} y \otimes_R x \text{ as } R\text{-complexes.}$$

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

$$\therefore K(\underline{x}; R) \cong K(\underline{x}^\sigma; R) \quad \text{for any } \sigma \in S_n.$$

$$\Rightarrow K(\underline{x}; M) \cong K(\underline{x}^\sigma; M) \quad \text{---} \quad \text{H} \quad \text{---}$$

(Can apply this to Obs ①.)

2nd Perspective: "Koszul complexes are iterative"  
mapping cones.

$f: X \rightarrow Y$  morphism of complexes

$$\text{cone}(f) = (Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & -\partial^X \end{pmatrix}).$$

$$\text{s.e.s. : } 0 \longrightarrow Y \longrightarrow \text{cone}(f) \longrightarrow \Sigma X \rightarrow 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$

↓

$\begin{pmatrix} y \\ x \end{pmatrix} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow{H_i(f)} H_i(Y) \longrightarrow H_i(\text{cone}(f)) \rightarrow H_i(\Sigma X) \rightarrow \dots$$

$\downarrow$

$H_{i-1}(X)$

*connecting map*

Consider:  $x \in R$

$$f: R \xrightarrow{x} R$$

$1 \mapsto x$

$$\text{cone}(R \xrightarrow{x} R) = (R \oplus R, \begin{pmatrix} \circ & x \\ 0 & \circ \end{pmatrix})$$

$\overset{\text{deg } 0}{\uparrow} \quad \overset{\text{deg } 1}{\uparrow}$

$= K(x; R).$

Ditto: If  $x \in R$  and  $M \in \mathcal{C}(R)$  *no complex, not necessarily in  $\mathcal{C}(R)$*

$$\text{cone}(M \xrightarrow{x} M) = K(x; M)$$

$$\underline{x} = x_1, x_2, \dots, x_n$$

$$K(\underline{x}; M) = K(x_1; K(x_{\geq 2}; M)) \\ = \text{cone}\left(K(x_{\geq n}; M) \xrightarrow{x_1} K(x_{\geq 2}; M)\right)$$

on homology

iterate  
:

$$H_i(x_{\geq 2}; M) \xrightarrow{\pm x_1} H_i(x_{\geq 2}; M) \rightarrow H_i(\underline{x}; M) \\ \downarrow \\ H_{i-1}(x_{\geq 2}; M) \\ \downarrow \pm x_1 \\ \vdots$$

↓ s.e.s.

$$0 \rightarrow \frac{H_i(x_{\geq 2}; M)}{x_1 H_i(x_{\geq 2}; M)} \rightarrow H_i(\underline{x}; M) \rightarrow (0 : \frac{x_1}{H_{i-1}(x_{\geq 2}; M)}) \rightarrow 0$$

$M \rightarrow R\text{-module}$

$$K(\underline{x}; M) \rightsquigarrow H_0(\underline{x}; M) = \frac{M}{\underline{x}M}$$

$$\text{So, } K(\underline{x}; M) \rightarrow \frac{M}{\underline{x}M} \text{ is a w.e. quasi iso} \\ \Leftrightarrow H_i(\underline{x}; M) = 0 \quad \forall i > 0.$$

Defn.  $\underline{x}$  is Koszi-regular on  $M$  (or...) if

$$H_i(\underline{x}; M) = 0 \quad \forall i \geq 1.$$

Lemma. When  $\underline{x}$  is weakly  $M$ -eq,  $(M \rightarrow R\text{-mod})$

(weakly-reg)  $K(\underline{x}, M) \rightarrow M / (\underline{x}M)$  is a w.eq.  
 $\Rightarrow$  Koszi-reg

Proof.  $n=1$ :  $0 \rightarrow M \xrightarrow{\underline{x}} M \rightarrow 0$ .

$H_1(\underline{x}; M) = 0 \Leftrightarrow \underline{x}$  nad on  $M$

$n \geq 2$ :  $K(\underline{x}; M) = K(x_n; K(x_{\leq n}; M))$ .

By induction,

$$K(x_{\leq n}; M) \xrightarrow{\sim} \frac{M}{(x_{\leq n})M}.$$

Now,  $0 \rightarrow R \xrightarrow{x_n} R \rightarrow 0$   
is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x}; M) = K(x_n; R) \otimes_R K(x_{\leq n}; M)$$

$$\cong K(x_n; R) \otimes_R \frac{M}{(x_{\leq n})M}$$

Now note that  $x_n$  is a nzd  
on  $\frac{M}{(x_{\leq n})M}$  and we  
are done.  $\square$

Instead of semifree,  
can use l.e.c. of homology  
and induction.

$$\begin{aligned} &\text{(Semifree Lemma)} \\ &\Rightarrow \left( \begin{array}{l} M \cong N \text{ quasi} \\ \Downarrow \\ K(\underline{x}; M) \cong K(\underline{x}; N) \text{ quasi} \end{array} \right) \end{aligned}$$

---

Note: ①  $\underline{x}$  Koszti-reg  $\Rightarrow \underline{x}^\sigma$  is Koszti-reg  $\forall \sigma \in S_n$ .

② Not true for weakly regular.  $\rightarrow$

---

Theorem. Say  $\underline{x} \subseteq J(R)$  and  $M$  f.g.  $R$ -module.

TFAE:

- 1)  $\underline{x}$  is  $M$ -regular. ( $\equiv$  weakly  $M$ -reg. by NAK.)
- 2)  $\underline{x}$  is Koszti  $M$ -regular, i.e.,  $H_i(\underline{x}; M) = 0 \ \forall i \geq 1$ .
- 3)  $H_1(\underline{x}; M) = 0$ .

In particular, take  $R$  local and  $x_i$  nonunits.

Proof. ①  $\Rightarrow$  ②  $\Rightarrow$  ③ is clear.

Only need to prove ③  $\Rightarrow$  ①.

Already saw for  $n=1$ .

Induction:

$$K(\underline{x}; M) = K(x_n; K(x_{\leq n}; M)).$$

I.e.s.

$$0 \rightarrow \frac{H_i(x_{\leq n}; M)}{x_n H_i(x_{\leq n}; M)} \rightarrow H_i(\underline{x} ; M) \rightarrow (0 : \frac{x_n}{H_{i-1}(x_{\leq n}; M)}) \rightarrow 0. \quad (*)$$

Put  $i=1$  to get  $\frac{H_1(x_{\leq n}; M)}{x_n H_1(x_{\leq n}; M)} = 0$

$$\xrightarrow{\text{NAK}} H_1(x_{\leq n}; M) = 0.$$

(Note: K<sub>1</sub>-homology modules are f.g.  
when  $M$  is f.g.)

$\xrightarrow{\text{induction}}$   $x_1, \dots, x_{n-1}$  is  $M$ -reg. — (1)

Moreover, (\*) now tells us

$$(0 : \frac{x_n}{H_0(x_{\leq n}; M)}) = 0.$$

$$\text{ku}\left( \frac{M}{x_{\leq n} M} \xrightarrow{x_n} \frac{M}{x_{\leq n} M} \right).$$

$\therefore x_n$  is nzd on  $\frac{M}{(x_{\leq n})M}$ . — (2)

① & ② finish.  $\square$

Corollary.  $\underline{x} \subseteq J(R)$ ,  $M$  f.g., the property of  $\underline{x}$  being regular is not dependent on the order of  $x_i$ .

(Permutation of regular is regular.)

---

$$R = k[x, y, z]$$

$x, y(1-n), z(1-n)$  reg |  
 $y(1-n), z(1-n), x$  NOT |

Lemma. If  $\underline{x} = x_1, \dots, x_n \subseteq R$ ,  $M$  an  $R$ -module.

TFAE:

①  $\underline{x}$  is M-Koszul-regular.

②  $\underline{x}^a$  is M-Koszul-regular for some  $a \geq 1$ .

Proof. Sufficient to prove:

$x_1, \boxed{x_2, \dots, x_n}$  is M-KR

$\Leftrightarrow x_1^a, \boxed{x_2, \dots, x_n}$  is M-KR for some  $a \geq 1$ .  
(all)

$x_1, x_2, \dots, x_n$  KR

$\Rightarrow K(x_1; K(x_{\geq 2}; M)) \xrightarrow{\sim} K\left(x_1; \frac{M}{(x_{\geq 2})M}\right)$ .

Replacing  $M$  by  $M/\underline{(x_{\geq 2})M}$  we are reduced  
to  $n=1$ .

But

$x$  is reg on  $M$

$\Leftrightarrow x$  is nzd on  $M$

$\Leftrightarrow x^a$  is nzd on  $M$  for some  $a \geq 1$

$\Leftrightarrow x^a$  is reg on  $M$  — n —.

---

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$  and  $M$  f.g.  $R$ -module.

Let  $i \geq 0$  be s.t.  $H_i(\underline{x}; M) = 0$ .

Then,  $H_j(\underline{x}; M) = 0 \quad \forall j \geq i$ .

HW.

# Lecture 4 (23-01-2023)

Monday, January 23, 2023 1:19 PM

$R \rightarrow$  commutative Noetherian

Given complexes,  $M, N \in \mathcal{C}(R)$ .

$$R\text{Hom}_R(M, N) := \text{Hom}_R(pM, N)$$

$pM \xrightarrow{\sim} M$  is a K-proj res<sup>r</sup>

$$\text{Ext}_R^+(M, N) := H^i(R\text{Hom}_R(pM, N)).$$

Given  $M, N, P \in \mathcal{C}(R)$ , we have a morphism

$$\Theta : R\text{Hom}_R(M, N) \otimes_R^L P \longrightarrow R\text{Hom}_R(M, N \otimes_R^L P).$$

Lemma.  $\Theta$  is a w.e. when  $P$  is perfect, i.e,

$$P \simeq (0 \rightarrow P_1 \rightarrow \dots \rightarrow P_n \rightarrow 0),$$

each  $P_i$  f.g.  $R$ -module.

$$\begin{aligned} & \text{Hom}_R(pM, N) \otimes_R pP \longrightarrow \text{Hom}_R(pM, N \otimes_R pP) \\ & f \otimes x \longmapsto m \mapsto (-)^{\binom{\deg(m)}{\deg(x)}} f(m) \otimes x. \end{aligned} \quad \left. \begin{array}{l} \text{in} \\ \mathcal{C}(R) \end{array} \right\}$$

check this is a morphism.

Proof. Replace  $P$  with  $0 \rightarrow P_1 \rightarrow \dots \rightarrow P_n \rightarrow 0$ .

"Things true for  $R$  free" Then it reduces to checking for one f.g. projective  $P$ .

"for perfect." But that can be reduced to f.g. free.

That reduces to R. Obstruction  $\Rightarrow$

## Rees' Lemma:

Setup.  $\underline{x} = x_1, \dots, x_c$   $\subseteq C \subset R$  finite subset.

$N \rightarrow R\text{-module}$  s.t.  $\exists N = 0$ .

$M \rightarrow A$ -module s.t.  $x$  is  $K$

$M \rightarrow R\text{-module}$  s.t.  $\exists$  is Koszi-regular on  $M$ .

$$G^{\text{re}} \times (\underline{x}; M) \xrightarrow{\sim} M_{\mathcal{M}_M}$$

Lemma: Then,  $R\text{Hom}_R(N, M/\varphi M) \xrightarrow{\sim} R\text{Hom}_R(N, M) \otimes_R \Lambda^*(\Sigma R^c)$ .

In particular,  $\text{Ext}_R^*(N, \frac{M}{\varphi M}) \cong \text{Ext}_R^*(N, M) \otimes_R 1^*(\varSigma R^c)$ .

$$\Lambda^*(\Sigma R^c) : 0 \rightarrow R \xrightarrow{\circ} R^c \xrightarrow{\circ} \dots \xrightarrow{\circ} R \xrightarrow{\binom{c}{2}} R^c \xrightarrow{\circ} R \xrightarrow{\circ} 0$$

$\hookdownarrow \text{def}_c \quad \quad \quad \circlearrowleft$

$$\begin{aligned}
 \text{Proof. } R\text{Hom}_R(N, M/\underline{\gamma}M) &\simeq R\text{Hom}_R(N, K(\underline{x}; M)) \\
 &\simeq R\text{Hom}_R(N, M \overset{L}{\otimes}_R K(\underline{x}; R)) \\
 &\simeq R\text{Hom}_R(N, M) \overset{L}{\otimes}_R K(\underline{x}; R) \quad \text{perfect}
 \end{aligned}$$

$$\text{Since } \underline{x}^N = 0, \quad \underline{x}^T \underline{x}^N_{B^*}(n, m) = 0.$$

$$\text{Altér: } R\text{Hom}_R(N, M) \cong \text{Hom}_R(N, I) \quad \left. \begin{array}{l} \text{where } M \hookrightarrow I \text{ is} \\ \text{an injective reln.} \end{array} \right\}$$

$$\text{Now, } \underline{x} \cdot \text{Home}_k(N, I) = 0.$$

$$\cong \text{Hom}_R(N, I) \otimes_R K(\underline{x}; R)$$

$$\cong K(x; \text{Hom}_R(n, I))$$

$$\cong K\left(\underline{0}; \text{Hom}_R(N, I)\right)$$

key in c

$$\cong \text{Hom}_R(N, I) \otimes_R K(Q; R)$$

$$\begin{aligned} &\supset \text{Hom}_R(N, I) \otimes_R K(\Omega; R) \\ &= \text{Hom}_R(N, I) \otimes_R \Lambda^*(\Sigma R^c). \end{aligned}$$

Last statement means:

$$\begin{aligned} \text{Ext}_R^n\left(N, \frac{M}{\underline{\chi} M}\right) &\cong \left(\text{Ext}_R(N, M) \otimes_R \Lambda(\Sigma R^c)\right)^n \\ &= \bigoplus_i \text{Ext}_R^i(N, M) \otimes_R \left(N(\Sigma R^c)\right)^{n-i} \\ &= \bigoplus_i \text{Ext}_R^i(N, M) \otimes_R R^{\binom{c}{i-n}}. \end{aligned}$$

$V \rightarrow \mathbb{Z}$ -graded object

View  $V$  as having both an upper grading and lower grading via  $V^i = V_{-i}$ .

Notation:

$$\begin{aligned} \sup \text{V}^* &= \sup \{i : V^i \neq 0\}, \\ \inf \text{V}^* &= \inf \{i : V^i \neq 0\}. \\ \sup \text{V}_* &= \sup \{i : V_i \neq 0\} \dots \\ &= -\inf \text{V}^* \end{aligned}$$

Corollary.  $\rho := \inf \text{Ext}_R^*(N, M) = \inf \text{Ext}_R^*(N, M/\underline{\chi} M) + c.$

$$\text{Ext}_R^\rho(N, M) = \text{Ext}_R^{\rho-c}(N, M/\underline{\chi} M)$$

Defn. Fix  $I \subseteq R$  ideal.  $M \in \mathcal{C}(R)$ .

$$\text{depth}_R(I, M) := \inf \text{Ext}^*(R/I, M).$$

$I$ -depth of  $M$   $\uparrow$

Properties. ①  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  exact sequence  
of complexes on . . .

Properties. ①  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  exact sequences  
of complexes or modules).

Then,  $\text{depth}_R(I, M) \geq \min \{\text{depth}_R(I, L), \text{depth}_R(I, N)\}$ .

Similarly relations with other two using I.e.s.

$$\rightarrow \text{Ext}_R^i(R/I, L) \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \text{Ext}_R^i(R/I, N)$$

$$\rightarrow \text{Ext}_R^{i+1}(R/I, L) \rightarrow \dots$$

Thm (depth).

② Let  $\underline{x} = x_1, \dots, x_c$  be a gen. set for the ideal  $I$ .

$$\text{depth}_R(I, M) = c - \sup H_{\infty}(\underline{x}; M) \quad \forall M \in \mathcal{C}(R).$$

Will prove when  $M$  is an  $R$ -module. (recall Koszul homology)

### Koszul complexes revisited

Giving  $x_1, \dots, x_c$  in  $R$

$\leftrightarrow$  Giving a map  $f: F \rightarrow \mathbb{C}$ , where  $F$  is a free mod of rank  $c$  and chosen basis

$K(f) := (\Lambda^*(\mathbb{C}^F), \partial)$ , where  $\partial$  is

$$e_{i_1} \wedge \dots \wedge e_{i_m} \mapsto \sum_{j=1}^n (-1)^{j-1} f(e_{ij}) x_{i_1} \dots x_{i_{j-1}} \hat{x}_{ij} x_{i_{j+1}} \dots x_{i_n}$$

Lemma: Fix  $\underline{x} = x_1, \dots, x_c \subseteq R$  for any  $y \in (x)$ , we have

$$\begin{aligned} K(\underbrace{\underline{x}, y}_{c+1 \text{ Seq.}} ; M) &\cong K(\underline{x}, 0 ; M) \\ &\cong K(\underline{x} ; M) \otimes (0 \rightarrow R \rightarrow R \rightarrow 0). \end{aligned}$$

Proof.

$$\begin{array}{ccccc} R^{c+1} & \xrightarrow{[x_1 \dots x_c \ y]} & R & & \uparrow \text{deg } 0 \\ \downarrow \cong & \xrightarrow{2} & \parallel & & \\ \left[ \begin{array}{c|c} \vdots & \vdots \\ \hline r_1 & x_1 \\ \vdots & \vdots \\ r_c & x_c \end{array} \right] & & & & y = \sum r_i x_i \end{array}$$

$$\begin{array}{c} \left[ \begin{array}{c|ccccc} & x_1 & & & & \\ \hline & & x_2 & & & \\ & & & x_3 & & \\ & & & & \ddots & \\ & & & & & x_n \end{array} \right] \xrightarrow[R^{\text{can}}]{\cong} \begin{array}{c} 2 \\ \xrightarrow{\quad [x_1 \cdots x_n] \quad} \\ R \end{array} \end{array} \quad \text{if } y = \sum r_i x_i$$

In particular,

$$\sup H_{*}(\underline{x}, y; M) = 1 + \sup H_{*}(\underline{x}; M)$$

Thus,

$$c + 1 - \sup H_{*}(\underline{x}, y; M) = c - \sup H_{*}(\underline{x}; M).$$

Corollary.  $c - \sup H_{*}(\underline{x}, y; M)$  is independent of a

generating set of  $I$ . (If  $\underline{x}, \underline{y}$  generate, look at  $\underline{x}, \underline{y}$  &  $\underline{y}, \underline{x}$ .)

Proof of Thm (Depth). When  $M$  is a module

$$\begin{aligned} \text{depth}(I, M) = 0 &\Leftrightarrow \text{Hom}(R/I, M) \neq 0 \\ &\Leftrightarrow I \subseteq \text{Zdu}_R(M) \\ &\Leftrightarrow H_c(\underline{x}; M) \neq 0 \\ &\Leftrightarrow \sup H_{*}(\underline{x}; M) = c. \end{aligned} \quad \left. \begin{array}{l} \text{Actually,} \\ \left\{ \begin{array}{l} H_c(\underline{x}; M) \\ \text{Hom}(R/I, M) \end{array} \right. \end{array} \right\} \forall i$$

Can assume  $\text{depth}_R(I, M) \geq 1$ , i.e.,  $\exists y \in I$  <sub>nzd on  $M$</sub>

Then, in particular,  $y$  is Kosei-reg on  $M$ .

$$\text{By Reg, } \because \text{depth}_R(R/I) = 0 \quad \inf \text{Ext}_R^{*}(R/I, M/yM) = \inf \text{Ext}_R^{*}(R/I, M) - 1.$$

$$\therefore \text{depth}_R(I, M) = 1 + \text{depth}_R(I, \frac{M}{yM})$$

induction now applies

$$= 1 + \left( c - \sup H_{*}(\underline{x}; \frac{M}{yM}) \right)$$

$$\sup H_{*}(\underline{x}; M/yM)$$

$$\begin{aligned}
 & H^*(\underline{x}; M/y_M) \\
 \left\{ \begin{aligned} & \vdash H^*(\underline{x}; K(y; M)) \\ & \vdash H^*(K(\underline{x}, y; M)) \\ & \vdash H^*(K(\underline{x}, o; M)) \end{aligned} \right. \\
 & = C + 1 - \sup H^*(\underline{x}, o; M) \\
 & = C - \sup H^*(\underline{x}; M). \quad \square
 \end{aligned}$$

Ex. Show  $\text{depth}_R(I, M)$  is the length of longest irregular in  $I$ .

## Lecture 5 (25-01-2023)

Wednesday, January 25, 2023 1:19 PM

$R \rightarrow$  comm. noetherian ring

$I \subseteq R$  ideal

$M \rightarrow R\text{-complex}$

$$\text{depth}(I, M) = \inf \text{Ext}_R^*(R/I, M) \quad \xrightarrow{I = (x_1, \dots, x_c)}$$

$$= c - \sup H_*(\underline{x}; M)$$

When  $M$  is a module,  $\text{depth}(I, M) =$  length of any maximal   
  $M$ -Koszul-regular sequence in  $I$

(Maybe  $IM \neq M$  needed.)

$\left\{ \begin{array}{l} \text{If } M \text{ f.g. and} \\ I \subseteq \text{Jac}(R) \end{array} \right.$

= length of any maximal   
  $M$ -reg. seq in  $I$

Observation.  $\underline{x} = x_1, \dots, x_c$   
 $\underline{y} = y_1, \dots, y_d$

$$\sup H_*(\underline{x}, \underline{y}; M) \leq \sup H_*(\underline{x}; M) + d.$$

(Can use I.e.s. to see this.)

$$k(\underline{x}, \underline{y}; M) \cong k(\underline{x}; k(\underline{y}; M))$$

This implies

$$I \subseteq J \Rightarrow \text{depth}_R(I, M) \leq \text{depth}_R(J, M).$$

$$\text{Also, } \sqrt{I} = \sqrt{J} \Rightarrow \text{depth}_R(I, M) = \text{depth}_R(J, M).$$

Today.  $(R, \mathfrak{m}, k)$  is a local ring:

- $R$  is commutative Noetherian,
- $\mathfrak{m}$  is the unique maximal ideal of  $R$ ,

$$- k = R/\mathfrak{m}.$$

In this case,  $\text{depth}(M) = \text{m-depth of } M$   
 $= \inf_R \text{Ext}_R^*(k, M).$

It suffices to compute the above using  $\underline{x} = x_1, \dots, x_c$   
 s.t.  $\sqrt{(\underline{x})} = \mathfrak{m}.$

Thus we can take  $c$  minimal as  $c = \dim(R)$   
 (Then,  $\underline{x}$  is a system of parameters.)

$\therefore$  Can compute using  $\dim(R)$  elements.

## Ausland-Buchsbaum Equality

- $F \rightarrow$  an  $R$ -complex

$F$  has finite flat dimension if

$$F \cong (0 \rightarrow F_0 \rightarrow \dots \rightarrow F_a \rightarrow 0)$$

with  $F_i$  flat.

We write  $\text{flat dim}_R F < \infty$ .

Examples • Flat modules.

- Perfect complexes.
- Koszul complexes.

If  $\text{flat dim}_R F < \infty$ , then  $\text{Tor}_i(-, F) = 0 \quad \forall |i| > 0$   
 on  $\text{Mod } R$ .

$$\begin{aligned}\text{Tor}_i^R(M, F) &= H_i(M \otimes_R (0 \rightarrow F_b \rightarrow \dots \rightarrow F_a \rightarrow 0)) \\ &= 0 \quad \text{for } i \notin [a, b].\end{aligned}$$

In fact, the above characterizes flat dim<sub>R</sub> F < ∞.

Theorem (AB equality) (R, m, k) local.

F → finite flat dimension. Then,

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R M - \underbrace{\sup H_k(k \otimes_R^L F)}_{\text{Tor}_k^R(k, F)},$$

for A  $\cong$  N R-complex M.

Specialise: ① M = R.

$$\text{depth}_R(F) = \text{depth}_R(R) - \sup H_k(k \otimes_R^L F).$$

Now, let N be a f.g. R-module.

Such an N has a minimal free resolution.

$$\begin{array}{ccccccc} \cdots & R^{b_2} & \xrightarrow{\partial_2} & R^{b_1} & \xrightarrow{\partial_1} & R^{b_0} & \xrightarrow{\varepsilon \text{ minimally}} N \\ & \downarrow & \nearrow & \downarrow \min & \nearrow & \downarrow & \\ & \ker \partial_2 & \subseteq \eta_{Rb_2} & \ker \varepsilon & \subseteq \eta_{R^{b_0}} & \text{by minimality} & \end{array}$$

This gives a complex

$$G: (\dots \rightarrow R^{b_2} \xrightarrow{\partial_2} R^{b_1} \xrightarrow{\partial_1} R^{b_0} \rightarrow 0) \xrightarrow{\sim} N$$

$\partial G \subseteq \eta G$ . G turns out to be unique up to isomorphism of complexes.

up to isomorphism of complexes.

"The" minimal free resolution of  $N$ .

$$\text{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes G)_i.$$

$H_i = \text{everything}$  since  $\partial \otimes k = 0$

$$\therefore \text{Tor}_i^R(k, N) = 0 \iff G_i = 0.$$

$\therefore \text{flat dim}_R N < \infty \iff N \text{ has a finite free resolution, i.e., } N \text{ is perfect.}$

$$\sup \text{Tor}_*^R(k, N) = \text{length of } G$$

$$=: \text{proj dim}_R N.$$

② If  $N$  is a f.g.  $R$ -module with  $\text{proj dim}_R N < \infty$ ,

then

$$\text{proj dim}_R(N) + \text{depth}_R(N) = \text{depth}(R).$$

(Classical AB Equality.)

Corollary.  $\text{proj dim}_R N < \infty \Rightarrow \text{depth}(N) \leq \text{depth}(R)$ .

Furthermore, equality if  
False without assumption of  $\text{proj dim} < \infty$   
 $N$  is projective.

Note we had:

$$\text{depth}_R(F) = \text{depth}_R(R) - \sup H_*(k \otimes^L F)$$

$$\text{depth}_R(M \otimes^L F) = \text{depth}_R(M) - \sup H_*(k \otimes^L F)$$

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

Subtract:

$$\text{depth}_R(F) - \text{depth}_R(M \otimes_R^L F) = \text{depth}(R) - \text{depth}(M).$$

Note: Some terms above may be  $\infty$ .

When  $H_i(\Sigma; M) = 0 \forall i$ , then

$$\sup H_*(\Sigma; M) = -\infty.$$

### Proof of AB Equality.

$$R\text{Hom}_R(k, M \otimes_R^L F) \xleftarrow{\cong} \underbrace{R\text{Hom}_R(k, M)}_{\text{quasi iso because } k \text{ is a f.g. module}} \otimes_R^L F$$

$\therefore \exists G \rightarrowtail k$  where each  $G_i$  is finite free  
And flat dim  $F < \infty$ .

More generally:

$$\text{Hom}_R(N, M) \otimes_R \cong \text{Hom}_R(N, M \otimes_R^L F)$$

$N$  f.g.  $R$ -mod,  $F$  flat

Key observation:

$R\text{Hom}_R(k, M) \cong$  complexes of  $k$ -vector spaces  
(take injective resolution of  $M$ )

$$R\text{Hom}_R(k, M) \otimes_R^L F \cong R\text{Hom}_R(k, M) \otimes_R^L (k \otimes_R^L F)$$

$$\therefore \text{Ext}_R^*(k, M \otimes_R^L F) = \text{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^L F). \quad \square$$

Observation. Let  $M$  be an  $R$ -complex.

Suppose  $s := \sup H_*(M)$  is finite.

Then,  $\text{depth}_R M \geq -s$

Then,  $\operatorname{depth}_R M \geq -s$

with equality iff  $\operatorname{depth}_R H_s(M) = 0$ .

Note. For an  $R$ -module  $M$ ,  $\operatorname{depth} M = 0 \Leftrightarrow \inf \operatorname{Ext}(k, M) = 0$

$\Leftrightarrow \operatorname{Hom}(k, M) \neq 0$

$\Leftrightarrow k \hookrightarrow M$

$\Leftrightarrow \eta \in \operatorname{Ass}_R M$ .

→ One proof:  $M$  as above.

$$\operatorname{Ext}_R^{-s}(N, M) = \operatorname{Hom}_R(N, H_s(M))$$

$N$  any  $R$ -module.

- key.  $M \cong M'$  with  $M'_i = 0 \forall i > s$ .

$$\dots \rightarrow M_{s+1} \xrightarrow{\partial} M_s \rightarrow M_{s-1} \rightarrow \dots = M.$$

$$\downarrow \circ \quad \downarrow \quad \quad \quad \parallel \quad \quad \quad \parallel \quad \quad \quad \downarrow \circ$$

$$\dots \rightarrow 0 \rightarrow \frac{M_s}{\partial(M_{s+1})} \rightarrow M_{s-1} \rightarrow \dots = M.$$

$\therefore$  Can assume  $M_i = 0$  for  $i \geq s+1$ .

In particular,

$$0 \rightarrow \sum^s H_s(M) \hookrightarrow M \xrightarrow{\quad} M'' \rightarrow 0.$$

$\uparrow$   
iso on homology  
in degrees  $\leq s-1$

$$H_i(M'') = 0 \quad i \geq s.$$

Let  $\Sigma = x_1, \dots, x_n$  gen set for  $\eta$ .

Then,

Then,

$$H_{i+1}(\underline{x}; M'') \rightarrow H_i(\underline{x}, \sum^s H_s(M)) \rightarrow H_i(\underline{x}; M) \rightarrow H_i(\underline{x}; M'')$$

— (\*)

$$H_j(M'') = 0 \quad \forall j \geq s.$$

$$\text{So, } M'' \cong M''' \quad \text{with} \quad M_j''' = 0 \quad \text{for } j \geq s.$$

$$K(\underline{x}; M''')_j = 0 \quad \text{for } j \geq s+n+1.$$

$$\text{Thus, } H_j(\underline{x}; M''') = 0 \quad \forall j \geq s+n+1.$$

$$\Rightarrow H_i(\underline{x}; \sum^s H_s(M)) = 0 \quad \forall i \geq s+n+1.$$

Put  $i \geq n+s$  in (\*) :

$$H_j(\underline{x}; M) = 0 \quad \forall j \geq n+s+1$$

$$\Rightarrow \sup H_k(\underline{x}; M) \leq n+s$$

$$\Rightarrow -s \leq n - \sup H_k(\underline{x}; M) = \text{depth } M. \quad \square$$

Moreover,

$$H_{n+s}(\underline{x}; M) \cong H_{n+s}(\underline{x}; \sum^s H_s(M))$$

$$\cong H_n(\underline{x}, H_s(M))$$

The above is nonzero iff  $\text{depth } H_s(M) = 0$ .  $\square$

① flat  $\dim_R F < \infty$ . Then  $\forall M$

$$\text{depth}_R(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_k(k \otimes_R^L F).$$

②  $s = \sup H_k(M)$  is finite.

Then,  $\text{depth}_R(M) \geq -s$ .

Equality  $\Leftrightarrow \text{depth}(H_s(M)) = 0$ .

Application. Say

$F = 0 \rightarrow F_d \rightarrow \dots \rightarrow F_0 \rightarrow 0$   
is a finite free complex s.t.  
 $\hookrightarrow \text{minimal}$   
 $\partial F \subseteq \eta F$

$$0 < \text{length}_R H_*(F) < \infty.$$

(all  $H_i(F)$  are finite length and  
at least one is non-zero.)

Then, for any  $M$ ,

$$\text{depth}_R M = d - \sup H_*(F \otimes_R M).$$

Thus such an  $F$  is depth sensitive.

When  $\eta M \neq M$ , one can check some  $H_*(F \otimes_R M) \neq 0$ .

Then, one gets

$$d \geq \text{depth}_R(M).$$

Note: Over any local ring,  $\exists M$  s.t.  $\eta M \neq M$  and  
 $\text{depth}_R M = \dim R$ .  
M need not be f.g.

In particular,  $d \geq \dim(R)$ .  $\rightarrow$  New Intersection

Theorem

Hochster ('70)  
André (2016)  
Bhatt (2021)

Proof. Assume  $s = \sup H_*(F \otimes_R M)$ .

Take any prime  $p \neq \eta$ .

$$H_i(F \otimes_R M)_p \cong H_i(F_p \otimes_{R_p} M_p)$$

$$H_i(F_p) = 0 \quad (\because \text{length } H_i(F) < \infty)$$

i.e.  $F_p = 0$  in  $D(R_p)$ .

$$\therefore H_i(F_p \otimes_{R_p} M_p) = 0$$

Thus,  $H_i(F \otimes_R M)$  is  $\eta$ -power torsion.

(I.e., each  $a \in H_i(F \otimes_R M)$  is killed by some  $\eta^n$ .)

$$\therefore \text{depth } H_s(F \otimes_R M) = 0.$$

Thus, by previous result,

$$\text{depth}_R(F \otimes_R M) = -s$$

|| AB

$$\text{depth}(M) = \sup H(k \otimes_R^L F)$$

$$\Rightarrow \text{depth}(M) = \sup \underset{d}{\underset{\exists}{\sup}} H(k \otimes_R^L F) - s. \quad \text{D}$$

## Lecture 6 (30-01-2023)

Monday, January 30, 2023 1:26 PM

Recap.  $I \subseteq R$  Comm. Noe  
 $M$  an  $R$ -complex.

$$\text{depth}_R(I, M) = \inf \text{Ext}_R^*(R/I, M).$$

. Choose any fin gen set  $\underline{x} = x_1, \dots, x_c$  of  $I$ .

$$\text{depth}_{\underline{x}}(I, M) = c - \sup H_{\underline{x}}(\underline{x}; M).$$

(Focus on the case  $H_{\underline{x}}(M)$  bounded.)

$(R, M_R, k) \rightarrow \text{local}$

$$\text{depth}_R M := \text{depth}_R(M_R, M).$$

$$\text{depth}_R(M) \geq -\sup H_{\underline{x}}(M). \quad \text{--- (1)}$$

$$\text{Equality} \Leftrightarrow m \in \text{Ass}(H_s(M)) \\ s = \sup H_{\underline{x}}(M).$$

Exercise 1.

Suppose  $M = 0 \rightarrow M_b \rightarrow \dots \rightarrow M_a \rightarrow 0$

$$\text{depth}_R M \geq \inf \{ \text{depth}(M_i) - i : a \leq i \leq b \}.$$

$$\text{depth}_R M \geq \inf \{ \text{depth } H_i(M) - i : \inf H_s(M) \leq i \leq \sup H_{\underline{x}}(M) \}. \quad \text{--- (2)}$$

(Note (2)  $\Rightarrow$  (1).)

Setup.  $H_{\underline{x}}(M)$  bounded.  $\underline{x} = x_1, \dots, x_c$ .  $(R \text{ not necessarily local.})$

$$\sup H_{\underline{x}}(M) \stackrel{(i)}{\leq} \sup H_{\underline{x}}(\underline{x}; M) \stackrel{(ii)}{\leq} \sup H_{\underline{x}}(M) + c$$

Lemma 2. (a) Inequality (ii) always holds.

$$\text{Equality iff } \text{depth}_R(\underline{x}; H_s(M)) = 0 \quad s = \sup H_{\underline{x}}(M)$$

(b) (i) holds if  $\underline{x} \subseteq \mathfrak{J}(R)$ , each  $H_i(M)$  is f.g.

(b) (i) holds if  $\underline{x} \subseteq \mathfrak{z}(R)$ , each  $H_i(M)$  is f.g.  
 Equality holds iff  $\underline{x}$  is  $H_S(M)$ -regular.

Proof (a)  $H(\underline{x}; M) = H(x_1; k(x_2, \dots, x_c; M))$ .

Reduce to  $c=1$ .  $x := x_1$ .

In this case, we have

$$H_{i+1}(M) \xrightarrow{\cong} H_i(M) \rightarrow H_{i+1}(x; M) \rightarrow H_i(M) \xrightarrow{\cong} H_i(M)$$

for  $i \geq 3$ , we get  $H_{i+1}(x; M) = 0$ .

$\therefore$  (ii) follows.

Moreover,  $H_{i+1}(x; M) \neq 0 \Leftrightarrow x$  is a zd on  $H_i(M)$ .

$$(b) H_i(M) \neq 0 \stackrel{\text{NAK}}{\Rightarrow} H_i(x; M) \neq 0 \\ \stackrel{\cong}{\Rightarrow} \frac{H_i(M)}{xH_i(M)}$$

$$\therefore \sup H_x(x; M) \geq \sup H_x(M).$$

Moreover,

$$0 \rightarrow H_{i+1}(x; M) \rightarrow H_S(M) \xrightarrow{\cong} H_S(M). \quad \text{B}$$

Corollary 3.  $\operatorname{depth}_R(\underline{x}; M) \geq -\sup H_x(M)$ .

Equality  $\Leftrightarrow \operatorname{depth}(\underline{x}, H_S(M)) = 0$   
 $\Leftrightarrow (\underline{x}) \subseteq \mathfrak{p} \in \operatorname{Ass} H_S(M)$ .

Prop<sup>n</sup> 4.  $(R, \mathfrak{m}, k)$  local.

Let  $M$  be any bounded complex.

Then, for any  $I \subset R$ ,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

In particular, if  $M$  is a f.g. module

$$\operatorname{depth}_R(M) \leq \inf \left\{ \dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M \right\} \\ \leq \dim_R(M). \quad \hookrightarrow \sup \left\{ \dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M \right\}$$

Proof. Say  $I = (y_1, \dots, y_c)$  and  $d = \dim(R/I)$ .

$\dots \dots \dots \mathfrak{p}$  c.e. they form a Sop in  $R/I$

Proof. Say  $I = (y_1, \dots, y_c)$  and  $d := \dim(R/I)$ .  
Let  $x_1, \dots, x_d \in R$  s.t. they form a SGP in  $R/I$ .

$$\text{Then, } \sqrt{(y, x)} = \sqrt{M_R}.$$

Apply Lemma (2)(b) to  $k(y; M)$  to get

$$\sup H_k(y; M) \leq \sup H_k(x; k(y; M)) \\ \sup H_k(x, y; M).$$

$$\therefore d + c - \underbrace{\sup H_k(y; M)}_{\geq d+c - \sup H_k(x, y; M)} \geq \underbrace{\sup H_k(x, y; M)}$$

$$\Rightarrow d + \text{depth}(I, M) \geq \text{depth } M. \quad \blacksquare$$

Local case:

AB Equality.  $F$  an  $R$ -complex with  $\text{flatdim}_R F < \infty$ .

Then, for any  $R$ -complex  $M$ ,

$$\text{depth}(M \otimes_R^L F) = \text{depth}_R(M) - \sup H_k(k \otimes_R^L F)$$

$$\text{depth}_R M \geq -\sup H_k(M) = s$$

with equality iff  $\text{depth}(H_s(M)) = 0$ .

Propn.  $R \rightarrow$  comm. Noe.

$I \subseteq R$  ideal.

$$\text{depth}(I, M) = \inf \left\{ \text{depth}_{R_p} M_p : p \in V(I) \right\}.$$

Proof.  $I \subset p \Rightarrow \text{depth}_R(I, M) \leq \text{depth}_R(p, M) \leq \text{depth}_{R_p}(pR_p, M_p)$

$\downarrow$

check using  
Koszul

This gives  $\leq$ .

We now construct  $p$  achieving  $\text{depth}_R(I, M)$ .

Let  $T = (x_1, \dots, x_c)$ . Let  $s := \sup H_k(T; M)$ .

We now construct  $p$  achieving  $\text{depth}_R(\mathbb{I}, M)$ .

Let  $\mathbb{I} = (x_1, \dots, x_c)$ . Let  $s := \sup H_s(\underline{x}; M)$ .

Pick  $\mathfrak{p} \in \text{Ass } H_s(\underline{x}; M)$ . (Maybe even minimal.)

Then,  $\text{depth}_{R_p} H_s(\underline{x}; M)_{\mathfrak{p}} = 0$ .

Consider  $K(\underline{x}; M)_{\mathfrak{p}} = K(\underline{x}; M_p)$ .

$$\sup H_{\mathfrak{p}}(K(\underline{x}; M_p)) = \sup H_{\mathfrak{p}}(\underline{x}; M_p) = s$$

$$\text{depth}_{R_p} K(\underline{x}; M_p) = -s \quad \begin{matrix} \leftarrow \\ \backslash \end{matrix}$$

$$\text{depth}_{R_p}(K(\underline{x}; R_p) \otimes M_p) \\ \parallel AB$$

$$\text{depth}(M_p) = \sup H_{\mathfrak{p}}(k(p) \otimes_{R_p} K(\underline{x}; R_p))$$

$$\geq \text{depth}(M_p) - c$$

$$\Rightarrow -s \geq \text{depth}(M_p) - c$$

$$\therefore c - s \geq \text{depth}(M_p)$$

$$\text{depth}_R(\mathbb{I}, M)$$

Remark. The proof says

$$\text{depth}_R(\mathbb{I}, M) = \text{depth}_{R_p}(M_p)$$

$$\forall \mathfrak{p} \in \text{Ann } H_s(\underline{x}; M)$$

□

Thm.  $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$  local map.

(i.e.,  $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$ )

Let  $M$  be an  $R$ -complex,  $N$  an  $S$ -module s.t.

$N$  is flat as an  $R$ -module.

Then,

$$\text{depth}_S(N \otimes_R M) = \text{depth}_R(M) + \text{depth}_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N).$$

Corollary When  $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$  is flat. Then  $\binom{m=R}{n=S} \text{ gives}$

$$\text{depth}_S(S) = \text{depth}_R(R) + \text{depth}_{S/\mathfrak{m}_RS}(S/\mathfrak{m}_RS).$$

↳ "fiber"

$$R \xrightarrow{\quad} S \xrightarrow{\quad} \frac{S}{\mathfrak{m}_R} S$$

$$\begin{array}{ccc} \text{Spec}(R/\mathfrak{m}_RS) & \hookrightarrow & \text{Spec } S \\ \downarrow & & \downarrow \\ \{\mathfrak{m}_R\} & \hookrightarrow & \text{Spec}(R) \end{array}$$

Under the same hypothesis, we have

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_RS).$$

Proof.  $\underline{x} = x_1, \dots, x_c \in R$  s.t.  $\mathfrak{m}_R = (\underline{x})$ .

Pick  $\underline{y} = y_1, \dots, y_d \in S$  s.t.  $\underline{y}(\frac{S}{\mathfrak{m}_RS})$  is the

maximal ideal of  $\frac{S}{\mathfrak{m}_RS}$   
(i.e.,  $\mathfrak{m}_S/\mathfrak{m}_RS$ ).

Then,  $(\underline{x}S, \underline{y}) = \mathfrak{m}_S$ .

$$K(\underline{x}S, \underline{y}; N \otimes_R M) \cong K(\underline{y}; N) \otimes_R K(\underline{x}; M)$$

$\downarrow$  assoc. of  $\otimes$

$N$  flat over  $R \Rightarrow K(\underline{y}; N)$  has fin flat dim  $k$

$$\begin{aligned} \text{depth}_R K(\underline{x}, \underline{y}; N \otimes_R M) &= \text{depth}_R K(\underline{x}; M) \\ &\quad - \sup H_{\underline{x}}(k \otimes K(\underline{y}; N)) \end{aligned}$$

$(k = R/\mathfrak{m}_R)$

Note.  $(\underline{x}, \underline{y}) \cdot H_{\underline{x}}(\underline{x}, \underline{y}; N \otimes_R M) = 0$

$$\begin{aligned} \mathfrak{m}_S H_{\underline{x}}(\underline{x}, \underline{y}; N \otimes_R M) &= 0 \\ \Rightarrow \mathfrak{m}_R H_{\underline{x}}(\underline{x}, \underline{y}; N \otimes_R M) &= 0 \end{aligned}$$

Similarly  $\mathfrak{m}_R H_{\underline{x}}(\underline{x}; M) = 0$ .

$$-\sup H_X(x, y; N \otimes_R M) = -\sup H_X(x, M)$$

$-\sup H_X(y; \frac{N}{M \otimes_R N})$

Add  $c+d$  ↗  $\text{depth}_S(N \otimes_R M)$

---

Exercise R local. M f.g. R-module

$$\text{depth } R - \dim_R M \leq \text{grade}_R M \leq \text{codim}_R M$$

$$\leq \dim R - \dim M \leq \text{pdim}_R M.$$

↑ use intersection thm

height(ann M)  
"  $\inf \{\dim R_p : p \in \text{ann} M\}$

$\text{depth}_R(\text{ann}_R M, R)$

Recall. New Intersection Theorem (P. Roberts '86)

$$0 \rightarrow F_d \rightarrow \dots \rightarrow F_0 \rightarrow 0 \quad \text{finite free}$$

$$0 < \text{length } H_X(F) < \infty.$$

Then,  $d \geq \dim \dim(R)$ . \\$

"Simple consequence" of AB + Existence of big CM modules

Corollary. (Intersection theorem)

R local.  
M  $\neq 0$  f.g. R-module s.t.  $\text{pdim}_R M < \infty$ .

Then, for any f.g. R-module N,

$$\dim_R(N) - \dim_R(M \otimes_R N) \leq \text{pdim}_R(M).$$

Inspired by: R regular local (e.g.  $k[[x_1, \dots, x_n]]$ )  
(Serre)  
M, N any finite R-module.

(Serre)  $M, N$  any finite  $R$ -module.

$$\dim_R N - \dim_R (M \otimes_R N) \leq \dim R - \dim M.$$

# Lecture 7 (01-02-2023)

Wednesday, February 1, 2023 1:24 PM

$R \rightarrow$  Commutative noetherian.

$I \subseteq R$  ideal

Invariance of domain

Let  $R \rightarrow S$  finite map. ( $S$  is a finitely  $R$ -module.)

Let  $M$  be an  $S$ -complex.

Then,

$$\text{depth}_R(I, M) = \text{depth}_S(IS, M).$$

[Only need  $S$  noetherian for this.]

↳ Obvious using Koszul.

$$I = (x)$$

$$\begin{aligned} K(x; M) &= K(x; R) \otimes_R^R M \\ &= (K(x; R) \otimes_R^S S) \otimes_S M \\ &= K(x; S) \otimes_S M. \end{aligned}$$

$$\dim_R M = \dim_S M. \quad [\text{finiteness needed here.}]$$

Say  $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$  is finite and local.

$$\begin{aligned} \text{depth}_R(M) &= \text{depth}(\mathfrak{m}_R, M) \\ &= \text{depth}(\mathfrak{m}_R S, M) \\ &= \text{depth}(\mathfrak{m}_S, M) \\ &= \text{depth}_S(M). \end{aligned}$$

↗ finiteness  $\Rightarrow \sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$

Special case.  $R, M$  an  $R$ -module.

Take  $S = R/\text{ann}_R(M)$ .

Recall.  $(R, \mathfrak{m}_R)$  local,  $M \stackrel{\neq 0}{\sim}$  a f.g.  $R$ -module.

$$1. \text{ if } \dim(M) \stackrel{\textcircled{1}}{\leq} \text{grade}(M) \stackrel{\textcircled{2}}{\leq} \text{codim}(M) \stackrel{\textcircled{3}}{\leq} \dim R - \dim M$$

$$\text{depth}(R) - \dim(M) \stackrel{\textcircled{1}}{\leq} \text{grade}(M) \stackrel{\textcircled{2}}{\leq} \text{codim}(M) \stackrel{\textcircled{3}}{\leq} \dim R - \dim M$$

$\stackrel{\textcircled{4}}{\leq} \text{pdimp } M.$

$$\text{depth}(M) \leq \text{depth}_R(I, M) + \dim(R/I).$$

Using invariance of domain, can tighten the above to

$$\text{depth}_R M \leq \text{depth}_R(I, M) + \underbrace{\dim(M/I M)}_{= \dim(R/I + \text{ann } M)}$$

Example. ①  $R = \frac{k[x, y]}{(x^2, xy)}$

$$m_R = (x, y).$$

$$\dim(R) = 1 : \sqrt{(x^2, xy)} = (x)$$

$$\therefore \dim R = \dim \frac{k[x, y]}{(x^2)} = \dim k[x] = 1.$$

$$\text{depth}(R) = 0, \text{ i.e., } \text{Hom}_R(k, R) \neq 0$$

i.e.,  $(0 : m_R) \xleftarrow{\text{socle}} \neq 0.$

Taking  $M = R$  shows  
 $\text{depth } R - \dim M < \text{grade}(M)$  is strict.

② Take same  $R$ .

$$M = R/m. \text{ Then, } \text{grade } M < \text{codim } M.$$

③  $k[x, y, z]/(x \cap (y, z)) = R.$



$\text{Spec } R = V(x) \cup V(y, z)$  Pick  $M = R/\mathfrak{p}$  where  $\mathfrak{p}$  is a min't prime  
but  $\dim(R/\mathfrak{p}) < \dim(R).$

$$\mathfrak{p} = (y, z).$$

④ Say  $\text{pdimp } M < \infty.$

$$\dim R - \dim M \leq \text{pdimp } M$$

↑                      ↗ AB

Now, take  $R = k[x_1, y_1]$  and  $M = \frac{R}{(x_1^a, y_1^b)}$ .

Then, LHS of (5) is 0 but RHS is not.

Def'n  $(R, M)$  local,  $M \xrightarrow{f_0} f.g.$  over  $R$ .

$$\text{Cmd}_R(M) := \dim_R M - \text{depth}_R M.$$

 Cohen-Macaulay defect

If  $\text{Gnd}_R(M) = 0$ , i.e.,  $\dim_R(M) = \text{depth}_R(M)$ , then

$M$  is Cohen-Macaulay (CM).

$M$  is maximal Cohen-Macaulay (MCM) if  $\text{depth}(M) = \dim(R)$ .

It is also considered CM and MCM.

$\hookrightarrow \text{depth } M \geq \dim M$  for  $M = 0$ .  
 " " " " $\infty$  - " $\infty$  One can define  $\dim$  etc for complexes as well.

$(R, m_R)$  local.  $M$   $\xrightarrow{\text{to}}$  f.g. module.

$$\operatorname{depth}_R(M) \leq \inf \{\dim R/p : p \in \operatorname{Ass}_R M\} \leq \dim_R M.$$

Corollary.  $M \text{ MCM} \Rightarrow \text{Ann}_R M = \text{Min}_R M$   
 and  $\dim(R/p) = \dim(M)$   
 $\forall p \in \text{Min}(M).$

Geometrically:  $\text{Supp}(M)$  has no "embedded components" and all components have the same dimension.

Prop.  $(R, \mathfrak{m})$  local,  $M$  f.g. and CM.

$$\textcircled{1} \quad \text{depth}_R(I, M) = \dim M - \dim(M/IM) \quad \forall I \subseteq R$$

\textcircled{2} Given  $\underline{x} = x_1, \dots, x_c \in \mathfrak{m}$ , then

$$\underline{x} \text{ is } M\text{-regular} \iff \dim(M) - \dim(M/\underline{x}M) = c.$$

Note  $\iff$  is always true.

$(\Leftarrow)$  Apply \textcircled{1} to  $I = (\underline{x})$ .

The hypothesis gives

$$\text{depth}_R(\underline{x}, M) = c$$

$$\iff H_i(\underline{x}; M) = 0 \quad \forall i \geq 1$$

$\Rightarrow \underline{x}$  is regular. \(\blacksquare\)

\textcircled{3}:  $\underline{x}$  is part of an SOP for  $M \hookrightarrow_{\text{SOP}}^{\text{for}} R/\text{ann}(M)$   
 $\iff \underline{x}$  is  $M$ -regular.

Thm.  $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$  local map.

$\text{ok}$   $M$  f.g.  $R$ -module,  $N$  f.g.  $S$ -module, flat over  $R$ .

Then,

$$\text{Cnd}_S(N \otimes_R M) = \text{Cnd}_R(M) + \text{Cnd}_{(S/\mathfrak{m}_R S)}(N/\mathfrak{m}_R N).$$

Proof: We saw the above for depth instead of Cnd.  
 Same holds for dim. \(\blacksquare\)

Corollary. Under same hypothesis,

$$N \otimes_R M \text{ is CM/S} \iff M \text{ CM/R} + N/\mathfrak{m}_R N \text{ CM/}_{S/\mathfrak{m}_R S}$$

Special case:  $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_R)$  flat-local.

$S$  CM  $\iff R$  and  $S/\mathfrak{m}_R S$  are CM.



Defn: A local ring  $R$  is CM if it is so as a module over itself.

Key Consequence: For any f.g.  $R$ -module  $M$ ,

$$\text{grade}_R(M) = \text{codim}_R(M) = \dim(R) - \dim(M).$$

Exercise:  $(R, \mathfrak{m})$  M f.g.

$$M \text{ CM} \Rightarrow M_p \text{ CM over } R_p \quad \forall p \in \text{Supp } M.$$

Defn:  $R \rightarrow$  comm. no.e.

$M \rightarrow$  f.g.  $R$ -module

Then,  $M$  is CM if  $M_p \text{ CM}/R_p$  for all  $p \in \text{Supp}_R M$ .

Equivalently, for all  $\mathfrak{m} \in \text{Max}(\text{Supp}_R M)$ .

$R$  is CM if ...

Examples:  $\cdot k[x_1, \dots, x_c]$  is CM. (How?)

$\cdot k[x_1, \dots, x_c]$  is CM.  $\because \frac{\dim = c}{\text{depth} = c}$  since  $x_1, \dots, x_c$ .

$\cdot R \text{ CM} \Leftrightarrow \Lambda^I R$  is CM for some ( $=$  all)  $I$ .

$\hookrightarrow$  completion wrt  $I$

If  $R$  local,

$(R, \mathfrak{m}) \rightarrow \Lambda^I R$  "is flat and local."

Check  $S/\mathfrak{m}_R S = R/\mathfrak{m}_R \rightarrow$  field (CM)

$\therefore \Lambda^I R$  is CM  $\Leftrightarrow R$  is CM.

Thm:  $(R, \mathfrak{m})$  CM local.

$M$  f.g.  $R$  module s.t.  $\text{pd}_{R, \mathfrak{m}} M < \infty$

Then,

$M$  is CM  $\Leftrightarrow \text{pd}_{R, \mathfrak{m}} M = \text{grade}_R M$ .

Proof.  $\operatorname{pd}_{\text{R}} M = \operatorname{grade} N$



$$\operatorname{depth} R - \operatorname{depth} M = \dim R - \dim M$$



$$\operatorname{depth} M = \dim M.$$

□

## Lecture 8 (06-02-2023)

Monday, February 6, 2023 1:25 PM

# Regular Rings

For now,  $(R, \mathfrak{m}, k)$  is a comm. noetherian local ring.

Recall:  $\text{depth}(R) \stackrel{\textcircled{1}}{\leq} \dim(R) \stackrel{\textcircled{2}}{\leq} \text{edim}(R).$   
 $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$

When equality holds in ①, then  $R$  is Cohen-Macaulay.  
 (By defn.)

② holds by Krull's height theorem.

Notation: (embedding) codepth of  $R$  is

$$\text{codepth}(R) = \text{edim}(R) - \text{depth}(R).$$

Def:  $R$  is regular if  $\text{codepth}(R) = 0$ .

Exercise:  $R$  is regular  $\Leftrightarrow \mathfrak{m}$  is generated by a regular sequence

$$\Leftrightarrow \text{edim}(R) = \dim(R)$$

Example:

①  $k[[x_1, \dots, x_n]]$  with  $k$  a field is regular  
 since  $\mathfrak{m}$  is gen by a reg. seq.

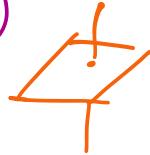
②  $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$  ——————.

③  $\mathbb{Z}_{(p)}$  similar.

④ Regular  $\Rightarrow$  CM. (Non CM  $\Rightarrow$  Non reg.)

$$\frac{k[x, y, z]}{(xz, yz)}$$

not reg.



⑤  $R = \frac{k[x, y]}{(xy)}$ . codepth = 1 > 0.  
NOT regular.

## Flat Maps

Let  $\varphi: (R, \mathfrak{m}) \rightarrow (S, \eta)$  be a local flat extension.

Recall

$S$  is CM  $\Leftrightarrow R$  is CM and  $S/\mathfrak{m}S$  are CM.

Analog fails for regularity. ( $\Leftarrow$ ) does hold.

Comida  $R \xrightarrow{\varphi} k[[x^2]] \hookrightarrow k[[x]]$  (flat ext since  $S$  is free)  
 $\swarrow$   $S/\mathfrak{m}S = k[[x]]/(x^2)$   
 regular  $\searrow$  not regular  
 codepth = 1

Also, if  $R$  is CM, then  $R_p$  is CM  $\forall p$ .

Question. [30s, Krull, Zariski, ...]

Does  $R$  regular imply  $R_p$  regular  $\forall p \in \text{Spec } R$ ?

Homological characterisation of regularity. (Used to prove!)

Recall: for a f.g.  $R$ -module  $N$ , a minimal free resolution over  $(R, \mathfrak{m}, k)$  is a free resolution

$$F \xrightarrow{\sim} M$$

with  $\partial(F) \subseteq \eta F$ .

Recall/Prove. Minimal resolutions exist and are unique up to isomorphism of complexes.

The  $i$ th Betti number of  $M$  is

$$\begin{aligned}\beta_i^R(M) &= \text{rank}_R(F_i) \\ &= \text{rank}_k(\text{Tor}_i^R(M, k)) \\ &= \text{rank}_k(\text{Ext}_R^i(M, k)).\end{aligned}$$

Example. ①  $R = k[[x_1, \dots, x_n]]$ .  $f \in \eta \setminus \{0\}$ .

$$M := R/(f).$$

Then the min'l free res<sup>n</sup> of  $M$  is

$$0 \rightarrow R \xrightarrow{f} R \rightarrow 0.$$

↪ Koszul

$$\beta_i(M) = \begin{cases} 1 & ; i=0, \\ 0 & ; \text{else} \end{cases}$$

The min'l res<sup>n</sup> of  $k$  is  $K(x)$ .

$$\beta_i(k) = \binom{n}{i} \quad \text{for } i \geq 0.$$

$$\textcircled{2} \quad R = \frac{k[[x, y]]}{(xy)}.$$

$M = R/\langle \rangle$  has min'l free res<sup>n</sup>:

$$\dots \xrightarrow{\cdot x} R^3 \xrightarrow{\cdot x} R^2 \xrightarrow{\cdot y} R^1 \xrightarrow{\cdot x} R^0 \rightarrow 0$$

$$\therefore \beta_i(M) = 1 \quad \forall i.$$

$\text{Refl}^n \text{ for } k: \quad R = R/(x,y)$

$$\text{repeat } \rightarrow R \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x,y)} R \rightarrow 0$$

$$\beta_i(M) = \begin{cases} 1 & j; i=0 \\ 2 & ; i \geq 1 \end{cases}$$

FACT.  $0 \neq M$  f.g.:

$$\text{projdim}_R(M) = \sup_{\parallel} \{ i \geq 0 : \beta_i^R(M) \neq 0 \}.$$

length of min'l free refl<sup>n</sup>  
 $\inf_{\parallel} (\text{length}(\text{free refl}^n))$

Theorem. [Auslander-Buchsbaum-Serre, '50s]  
 (ABS)

$(R, m, k)$  local. TFAE:

①  $R$  is regular.

②  $\text{projdim}_R(M) < \infty$  for all f.g.  $M$ .

③  $\text{projdim}_R(k) < \infty$ .

Proof. ①  $\Rightarrow$  ②  $\underline{x_1, \dots, x_d}$   
 Let  $\underline{x}$  be a min'l gen set for  $m$ . (Hence reg. seq.)  
 Then,  $K(\underline{x})$  is a min'l free refl for  $k$ .

$$\begin{aligned} \beta_i^R(M) &= \text{rank}_k(\text{Tor}_i^R(M, k)) \\ &= \text{rank}_k H_i(M \otimes_R K(\underline{x})) \\ &= 0 \quad \text{for } i > d. \end{aligned}$$

$$\therefore \text{projdim}_R(M) < \infty.$$

②  $\Rightarrow$  ③. —

$\textcircled{2} \Rightarrow \textcircled{3}$ . —

$\textcircled{3} \Rightarrow \textcircled{1}$ . Some's proof of ABS Theorem relies on:

Lemma:  $(R, m, k)$  local.

Then,  $\beta_i^R(k) \geq \binom{\text{edim}(R)}{i}$  for  $i \geq 0$ .

Proof of Lemma. Let  $F \rightarrow k$  be a mil rel.  
 $(\because \beta_i^R(k) = \text{rk}_k F_i)$

Let  $\underline{x} = x_1, \dots, x_e$  be a

min gen set for  $m$ .

$$\begin{array}{ccc} k & := & K(\underline{x}) \xrightarrow{\quad} k \\ & & \uparrow \cong \\ & \dashrightarrow & \uparrow \\ & \varphi \quad \text{a map} & \downarrow F \\ & \text{from complexes} & (\because \text{perfect}) \end{array}$$

Claim:  $\varphi_i$  is a split injection. (Then it is clear.)

$$\left[ \begin{array}{l} \varphi_i: K_i \rightarrow F_i \text{ is split injective} \\ \Leftrightarrow k \otimes_R K_i \rightarrow k \otimes_R F_i \\ \text{NAK} \quad \text{is an injection.} \end{array} \right]$$

By induction, we show  $\varphi_i$  is split inj.

$$i=0 \vee R \cong R$$

$i > 0$ : Let  $a \in K_i$  with  $\varphi_i(a) \in m F_i$ .  
 $(\text{we are using the NAK result.})$

WTS:  $a \in m K_i$ .

$$\therefore \partial^*(a) \in m^2 F_{i-1}$$

$$\begin{matrix} \parallel \\ \varphi(\partial a) \end{matrix}$$

By induction,

$$\partial^* a \in m^2 K_{i-1}$$

$$\begin{array}{ccccc} a \in K_i & \xrightarrow{\quad} & K_{i-1} & \xrightarrow{\quad} & \\ \downarrow & & \downarrow & & \\ F_i & \xrightarrow{\quad} & F_{i-1} & \xrightarrow{\quad} & \\ \varphi(a) \in m F_i & \nearrow & \nearrow & \nearrow & m^2 F_{i-1} \\ \partial(F) \subseteq m F & & & & \cap \end{array}$$

Notice that by def of  $\partial^*$  and  
since  $\underline{x}$  is a min gen set for  $m$ ,

Notice that  $\mathfrak{m}$  is a zero-dimensional ideal since  $\mathfrak{X}$  is a minimal set for  $\mathfrak{m}$ ,

(\*)  $\Rightarrow \mathfrak{a} \in \mathfrak{m} K_i$ . This does it.  $\blacksquare$

Back to:  $\text{projdim}_R(k) < \infty \Rightarrow R$  is regular.

From Serre's inequality:

$$\text{projdim}_R k \geq \text{edim } R.$$

$\parallel \rightsquigarrow$  By AB Equality, since  $\text{projdim } k < \infty$ .

$$\text{depth}(R)$$

But  $\text{depth}(R) \leq \text{edim}(R)$  always true.  $\blacksquare$

The above now solves the localisation problem.

Corollary.  $R$  regular  $\Rightarrow R_p$  is regular for all  $p \in \text{Spec } R$ .

Proof.  $R$  is reg.

$$\Rightarrow \text{projdim}_R R/\mathfrak{p} < \infty$$

$$\Rightarrow \text{projdim}_{R_p}(k(p)) < \infty \Rightarrow R_p \text{ is reg. } \blacksquare$$

Prop<sup>n</sup>.  $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{m}_S)$  local flat.

①  $R$  and  $S/\mathfrak{m}_S$  reg  $\Rightarrow S$  is reg.

②  $S$  is regular  $\Rightarrow R$  is regular.

Proof. ①  $S = S/\mathfrak{m}_S \xrightarrow{\text{flat}}$   
 $\text{depth } S = \text{depth}(R) + \text{depth}(\bar{S})$  ) regular hypothesis  
 $= \text{edim}(R) + \text{edim}(\bar{S})$

There is always a right exact sequence

more -  $\oplus$   $\cup$

$$\frac{m_R}{m_R^2} \otimes k \rightarrow \frac{m_S}{m_S^2} \rightarrow \frac{m_{\bar{S}}}{m_{\bar{S}}^2} \rightarrow 0$$
$$l = S/m_S$$

$$\therefore \text{cdim } S \leq \text{cdim } R + \text{cdim } (\bar{S}) = \text{depth}(S).$$

② Let  $F \xrightarrow{F} k$  min'l, since  $\varphi$  flat AND local,

$$F \otimes_R S \rightarrow \bar{S} \quad \text{min'l.}$$

$$\beta_i^R(k) = \beta_i^S(\bar{S}) = 0 \quad \text{for } i > 0.$$

$\therefore \text{projdim}_R k < \infty.$

$\Rightarrow R$  is reg. □

## Lecture 9 (08-02-2023)

Wednesday, February 8, 2023 1:26 PM

Recall:  $R$  local is regular if  $\operatorname{codepth}(R) = 0$ ,  
i.e.,  $\mathfrak{m}$  generated by a regular sequence.

We proved Auslander - Buchsbaum Theorem:

$(R, \mathfrak{m}, k)$  TFAE:

①  $R$  is regular,

②  $\operatorname{projdim}_R M < \infty$  for all f.g.  $M$

③  $\operatorname{projdim}_R(k) < \infty$ .

①  $\Rightarrow$  ②  $\Rightarrow$  ③ was ok.

③  $\Rightarrow$  ①: showed  $\operatorname{projdim}_R(k) \geq \operatorname{edim}(R)$  and then AB Equality.

Sketch of second proof.

Theorem [Nagata]. Let  $(R, \mathfrak{m}, k)$  be local and  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$  be u.zd. Set  $\bar{R} = R/x$ .

Then, for any f.s.  $R/\mathfrak{m}$ -module  $M$ , we have

$$\operatorname{Tor}_*^R(M, k) \cong \operatorname{Tor}_*^{\bar{R}}(M, k) \otimes_{\bar{k}} \Lambda(\sum k).$$

$$= \circ \rightarrow \bar{k} \rightarrow \bar{k} \rightarrow \circ$$

In particular,  $\beta_i^R(M) = \beta_i^{\bar{R}}(M) + \beta_{i-1}^{\bar{R}}(M)$ .

In fact, one can show the min'l  $R$ -free res' of  $M$

has the form

$$\cdots \rightarrow G_3 \xrightarrow{\begin{pmatrix} \alpha_3 & x \\ \beta_3 & -\alpha_2 \end{pmatrix}} G_2 \xrightarrow{\begin{pmatrix} \alpha_2 & x \\ \beta_2 & -\alpha_1 \end{pmatrix}} G_1 \xrightarrow{\begin{pmatrix} \alpha_1 & x \\ \beta_1 & -\alpha_0 \end{pmatrix}} G_0$$

multiplication  
by  $x$

$G_i \rightarrow \text{free}$

and  $\cdots \rightarrow \bar{G}_3 \xrightarrow{\alpha_3} \bar{G}_2 \xrightarrow{\alpha_2} \cdots$  is the  
 min'l  $\bar{R}$ -resol' of  $M$ .  
 (Stronger result but not proving this.)

Example.  $R = k[[x,y]]/(xy)$  (non regular)

Minimal  $R$ -free resol' of  $k$ :

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & xy \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \rightarrow 0$$

Consider  $y-x^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$  (mod on  $R$ ).

By row/col operations, the min'l resol' is iso to,

$$\cdots \xrightarrow{\begin{pmatrix} x & y-x^2 \\ -y & \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & y-x^2 \\ -x & \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y-x^2 \end{pmatrix}} R \rightarrow 0$$

Now, going mod  $y-x^2$  [using the result]  
 we get

$$\cdots \rightarrow \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{y} \bar{R} \xrightarrow{x} \bar{R} \rightarrow 0$$

$\bar{R} = k[[x,y]]$

$$\dots \rightarrow \bar{R} \rightarrow R$$

$$= \dots \rightarrow \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{x^2} \bar{R} \xrightarrow{x} \bar{R} \rightarrow 0$$

$$\begin{aligned}\bar{R} &= \frac{k[[x,y]]}{(xy, y-x^2)} \\ &\cong \frac{k[[x]]}{(x^3)}\end{aligned}$$

Proof of Nagata's Thm.

If  $a \in R$  is nzd, and set  $\bar{R} = R/a$ .

$M \rightsquigarrow \bar{R}$ -module.

There is a long exact sequence

$$\begin{array}{ccccccc} x & \hookrightarrow & \text{Tor}_{n-1}^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_n^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_{n+1}^{\bar{R}}(M, k) \\ & & \curvearrowright & & \curvearrowright & & \curvearrowright \\ x & \hookrightarrow & \text{Tor}_{n-2}^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_{n-1}^{\bar{R}}(M, k) & \rightarrow & \text{Tor}_{n+1}^{\bar{R}}(M, k) \end{array}$$

Specialising to our setting, we wish to show  
that each  $x$  is 0.

One can compute  $x$  in the following way:

$$\bar{F} \rightsquigarrow M \text{ min'l } \bar{R}-\text{red}.$$

Lift this to a sequence of free  $R$ -modules:  
(not saying complex!)

$$\dots \rightarrow F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \rightarrow \dots$$

$\partial^2$  may not be 0 but  
 $\dots \rightarrow F_i \xrightarrow{\partial} F_{i-1} \rightarrow \dots$

$\partial^2$  may not be 0 even

$$\partial^2 = x \Theta \text{ where}$$

$$\Theta = \{ \Theta_i : F_i \rightarrow F_{i-2} \}$$

is a chain map

and the following diagram commutes: (up to signs)

$$\begin{array}{ccc}
 F_i \otimes_R k & \xrightarrow{\Theta_i \otimes_R k} & F_{i-2} \otimes_R k \\
 \downarrow \cong & & \downarrow \cong \\
 \widetilde{F}_i \otimes_{\bar{R}} k & & \\
 \downarrow \cong & & \\
 \mathrm{Tor}_i^{\bar{R}}(M, k) & \xrightarrow{x} & \mathrm{Tor}_{i-2}^{\bar{R}}(M, k)
 \end{array}
 \quad (*)$$

Now, since  $x$  is linear and  $\partial^2 = x \Theta$ , we have that  $\Theta(F) \subseteq \eta F$ .

$\therefore$  the top map in  $(*)$  is zero.  $\therefore x = 0$ .  $\blacksquare$

Second proof of  $\textcircled{3} \Rightarrow \textcircled{1}$  in ABT.

Given  $\mathrm{projdim}_R(k) < \infty$ . WTS:  $R$  is regular.

Induct on  $d := \mathrm{depth}(R)$ .

$d=0$ : AB Equality gives  $\mathrm{projdim}_R(k) = 0$ .

$\therefore R = k$  is a field.  $R$  is reg.

$d>0$ : By prime avoidance,  $\exists x \in m \setminus \eta^2$  nzd.

$\therefore \mathrm{codeth}(R) = \mathrm{codeth}(R/x)$ .

By Nagata's thm,

$$\mathrm{ordim}_n k = \mathrm{projdim}_R k - 1 < \infty.$$

by Nagata's Thm,

$$\text{projdim}_{R/x} k = \text{projdim}_R k - 1 < \infty.$$

But  $R/x$  has smaller depth. \(\square\)

---

$(R, \mathfrak{m}, k)$  local

Propn.  $x \in \mathfrak{m}$  nzd.

$R$  is CM  $\Leftrightarrow R/x$  is CM.

$(\text{CMD}(R) = \text{CMD}(R/\mathfrak{x}).)$

$(\text{CMD} = \text{dim} - \text{depth.})$

Proposition.  $x \in \mathfrak{m}$  nzd.

① Suppose  $R$  is regular.

$R/x$  is regular  $\Leftrightarrow x \notin \mathfrak{m}^2$ .

② If  $R/x$  is regular, then  $R$  is regular.  
(Hence,  $x \notin \mathfrak{m}^2$ .)

Proof.

$$\text{codepth}(R/\mathfrak{x}) = \begin{cases} \text{codepth}(R) & x \notin \mathfrak{m}^2, \\ \text{codepth}(R) + 1 & x \in \mathfrak{m}^2. \end{cases}$$
\(\square\)

---

## Global Setting

$R \rightarrow$  comm. noetherian, not necessarily local  
or of finite Krull dim.

Dfn.  $R$  is regular if  $R_p$  is a regular local ring  
for all  $p \in \text{Spec}(R)$

$(\Leftrightarrow \forall \mathfrak{m} \in \text{max Spec}(R)).$

$(\Leftrightarrow \forall m \in \text{mSpec}(R)).$

(Since regularity localizes for local rings, the above makes sense.)

Exercise.  $R$  is regular  $\Leftrightarrow R[x]$  is regular  
 $\Leftrightarrow R[[x]]$  is regular

Example ①  $k[x_1, \dots, x_n]$  is regular. ( $k = \text{field}$ )

②  $\mathbb{Z}[x_1, \dots, x_n] \dashrightarrow$   $\mathbb{Z}$  is regular since every localization is a field or a DVR.

③ Nagata's example. (Infinite Krull dimension but regular)

Thm. [Bass - Murthy '60s].

$R \rightarrow \text{comm noetherian}$   
 $M \rightarrow \text{f.g.}$

$\text{projdim}_R M < \infty \Leftrightarrow \text{projdim}_{R_p} M_p < \infty$   
for all  $p \in \text{Spec } R$ .

Corollary.  $R \rightarrow \text{comm noe.}$

$R$  is regular  $\Leftrightarrow \text{projdim}_R M < \infty$   
for all  $M$  f.g.

Proof.

$(\Rightarrow) -$

$(\Leftarrow)$  Let  $F \xrightarrow{\sim} M$  be a free res<sup>f</sup> of

$M$ .  $F_i$  f.g. for all  $i \geq 0$ .

For  $n \geq 0$ , define

For  $n \geq 0$ , define

$$D_n := \{p \in \text{Spec } R : \text{proj dim}_{R_p} M_p \leq n\}$$
$$= \{p \in \text{Spec } R : \text{im}(\partial_n^F)_p \text{ is free over } R_p\}.$$

Since  $\text{im}(\partial_n^F)$  is f.g., the above set is  
open in  $\text{Spec}(R)$ . "free locus is open"

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$$

$$\text{and } \text{Spec } R = \bigcup_{n \geq 0} D_n.$$

Since  $R$  is noetherian,  $\text{Spec } R$  is a  
noe. top space.

$$\therefore \text{Spec } R = D_n \text{ for some } n.$$

$\Rightarrow \text{im}(\partial_n^F)$  is locally free.

$\Rightarrow \text{im}(\partial_n^F)$  is projective.

Now,  $0 \rightarrow \text{im}(\partial_n^F) \hookrightarrow F_n \rightarrow \dots \rightarrow F_0 \rightarrow 0$

is a proj. resl.

□

# Lecture 10 (13-02-2023)

13 February 2023 13:26

$R \rightarrow$  comm noetherian

$$\varphi: F_1 \longrightarrow F_0 \quad \text{free modules of finite rank}$$

$$\begin{matrix} 2^{11} & 2^{11} \\ R^s & \xrightarrow{(a_s)} R^r \end{matrix}$$

$I_c(\varphi) =$  ideal generated by  $c \times c$  minors of  $(a_{ij})$ .

$$I_0(R) = R \supseteq I_1(R) = (a_{11}, \dots, a_{rs}) \supseteq \dots \supseteq I_i(R) \supseteq I_{i+1}(R) = 0$$

$i = \min(r, s)$

---

$M$  f.g.  $R$ -module

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \rightarrow 0$$

$\hookdownarrow \text{rank } r$

finite free presentations

$$G_1 \xrightarrow{\psi} G_0 \longrightarrow M \rightarrow 0$$

$\hookdownarrow \text{rank } s$

Then,  $I_{r-c}(\varphi) = I_{s-c}(\psi)$  for all  $c$ .  
 (Exercise.)

---

Def.  $\text{Fitt}_c^R(M) := I_{r-c}(\varphi)$ ,  $c^{\text{th}}$  fitting ideal of  $M$ .

We get  $\text{Fitt}_0(M) \subseteq \text{Fitt}_1(M) \subseteq \dots$  eventually  $R$ .

$\text{Fitt}_0(M) \neq 0$  only if  $r \leq s$

$\text{Fitt}_c(M) = R$  for  $c > r$ .

---

Examples.

①  $R = k$  a field,  $\vee$  some vector space (f.g.) of rank  $n$ .

$0 \rightarrow k^n$  is a presentation

$$\text{Fitt}_c(V) = k \Leftrightarrow c = n.$$

②  $R$  a PID.

$$0 \rightarrow R^s \xrightarrow{\quad} R^r \rightarrow M \rightarrow 0$$

$\begin{pmatrix} d_1 & & \\ \ddots & & \\ 0 & & d_r \end{pmatrix}$

(can assume  $s \leq r$ )

↓ can put in normal form

$$\text{Fitt}_0(M) \neq 0 \Leftrightarrow r = s \Leftrightarrow M \text{ is torsion.}$$

In this case,  $\text{Fitt}_0(M) = (d_1 \dots d_r)$ .

Properties.

$$R^s \xrightarrow{\varphi} R^r \rightarrow M \rightarrow 0.$$

① If  $R \rightarrow S$  is any map of rings.

$$\text{Fitt}_c^S(S \otimes_R M) = S \cdot \text{Fitt}_c^R(M).$$

$$② (\text{ann}_R(M))^r \subseteq \text{Fitt}_0(M) \subseteq \text{ann}_R(M).$$

Proof. Pick an  $(r \times r)$ -minor ' $a$ ' in  $\varPhi$ .

WTP:  $a \cdot R^r \subseteq \text{im}(\varPhi)$

Can assume

$$\varPhi = \left( \begin{array}{c|c} \diagdown & | \\ \diagup & | \\ \vdots & | \\ \hline & r \times r \end{array} \right)_{r \times s}^=$$

$\det = a$

Take the  $s \times r$  matrix

$$\left( \begin{array}{c|c} \text{Signed} & \\ \text{co-factor matrix} & \\ \hline & 0 \\ & r \times r \\ & s \times r \end{array} \right)^= = \beta$$

Then,  $A\beta = \begin{pmatrix} a & \dots & 0 \\ 0 & \dots & a \end{pmatrix}.$

$\therefore \alpha R^n \subseteq \text{im } \varphi.$  proves  $\text{Fitt}_0 \subseteq \text{ann}.$

Next:  $\text{ann}(M) \subseteq \text{Fitt}_0(M).$

Fix  $a_1, \dots, a_r \in \text{ann}(M).$

$$\begin{array}{ccc} & R^r & \\ \swarrow & \downarrow & \\ R^s & \xrightarrow{\varphi} & R^r \\ & (a_1, \dots, a_r) & \end{array}$$

Apply  $\Lambda^r(\cdot)$  to the above to get

$$\begin{array}{ccc} & R & \\ \searrow & \downarrow & \\ R^s & \xrightarrow{\Lambda^r(\varphi)} & R \\ & (a_1, \dots, a_r) & \end{array}$$

□

(3) Fix  $c \geq 0$  and  $p \in \text{Spec}(R).$

TFAE:

i)  $\text{Fitt}_c(M) \notin p.$

ii)  $\text{im}(\varphi)_p$  contains a free summand of  $R_p^r$  of rank  $\geq r-c.$

iii)  $\nu_{R_p}(M_p) \geq c.$

$\underbrace{\quad}_{\text{min}\# \text{ gen}}$

Sketch. Can assume  $(R, p, k)$  local.  $m := p.$

$\text{Fitt}_c(M) \notin m \Leftrightarrow \text{Fitt}_c(R) = 0 \Leftrightarrow \text{Fitt}_c(k \otimes_R M) \neq 0$

$\nu_n(M) \leq c \Leftrightarrow \nu_n(k \otimes_R M) \leq c \stackrel{\text{NAK}}{\Leftrightarrow} \text{first example}$

(4) Fix  $c \geq 0$  and  $p \in \text{Spec } R.$  TFAE

i)  $\text{Fitt}_{c-1}(M)_p = 0$  and  $\text{Fitt}_c(M)_p = R_p.$

ii)  $\text{im}(\varphi)_p$  is a free summand of  $R_p^r$  of rank  $r-c.$

$$\textcircled{iii} \quad M_p \cong (R_p)^c.$$

Deduce from ③.

⑤  $c \geq 0$ .  $M$  is projective of rank  $c$ .

$$\Leftrightarrow \text{Fitt}_{c-1}(M) = 0 \text{ and } \text{Fitt}_c(M) = R.$$

### Hilbert - Birch Theorem

$R \rightarrow$  comm noetherian

Given  $I \subseteq R$  with free resolution

$$0 \rightarrow R^n \xrightarrow{\varphi} R^{n+1} \rightarrow I \rightarrow 0.$$

Then,  $\exists$  nzd  $a \in R$  s.t.  $I = a \cdot \text{In}(\varphi)$ .

Moreover, if  $I$  is projective then  $I$  is principal.

If  $\text{projdim}(I) = 1$ , then  $\text{depth}(\text{In}(\varphi), R) \geq 2$ .

Conversely, if  $\varphi: R^n \rightarrow R^{n+1}$  is -matrix s.t.  $\text{depth}(\text{In}(\varphi), R) \geq 2$ ,

$$\text{then } 0 \rightarrow R^n \rightarrow R^{n+1} \rightarrow \text{In}(\varphi) \rightarrow 0$$

is a free resolution.

Regular Local Rings are VDFs.

Thm.  $M \rightarrow$  projective module

If  $M$  has a finite free res<sup>n</sup>, then it must have a free res<sup>n</sup> of length 1.

$$\text{Say } 0 \rightarrow F_1 \rightarrow F_2 \rightarrow \dots \rightarrow F_n \rightarrow 0.$$

$\downarrow \cong$

$M$  projective  $\rightarrow \exists (I_i)$  proj  $\forall i \geq 1 \dots R$

S-, Projective + FFR  $\Rightarrow$  stably free.  
 $\hookrightarrow$  finite free resolution

Corollary.  $I \subseteq R$  proj + FFR  $\Rightarrow I$  principal.

Proof.  $0 \rightarrow R^n \rightarrow R^{n+1} \rightarrow I \rightarrow 0.$   
 $I$  proj  $\Rightarrow I = (a).$  □

Thm.  $R$  comm. noe. domain s.t. each f.g.  $R$ -module has FFR. Then,  $R$  is a UFD.

Corollary. Regular local rings are UFDs.  
 $\hookrightarrow$  not true otherwise

$\rightarrow$  Proof. Suppose  $R$  is local. ( $\because$  Regular.)

- Induction on  $\dim R$ .
  - $\dim R \leq 1$  is clear.

Pick  $w \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$ . Then,  $R/w$  is also RLC and hence a domain.

$\therefore w$  is prime.

Suffice to verify  $R_w$  is a UFD.

Note  $\dim R_w < \dim R$ .

Pick  $p \in \text{Spec}(R_w)$  with  $\text{ht } p = 1$ .

Suffice to show  $p$  is principal.

Note.  $p$  has a FFR. (For  $p$  comes from  $R$ .)

It now suffices to prove it is projective.

(Earlier corollary.)

This can be tested locally. But localisations  
are RLRs of  $\dim < \dim R$ .

$\therefore$  UFD.

$\therefore p$  is locally free  
and hence projective.

For general  $R$ , again fix  $p$  of ht = 1.

Want:  $p$  is principal.

Since  $p$  has ffr, suffices to prove projective.

This is local.  $\square$

---

$R \rightarrow \text{PID}$ ,  $M$  torsion module.

$$0 \rightarrow R^r \rightarrow R^r \rightarrow M \rightarrow 0$$
$$\begin{pmatrix} d_1 & 0 \\ 0 & \ddots & d_r \end{pmatrix} \quad d_i \neq 0$$

$$\text{Fit}_0(M) = (\pi_{d_i}) \subseteq \text{ann}_R(M).$$

$$\text{length}_R(M) = \text{length}(R/\text{Fit}_0(M))$$

# Lecture 11 (15-02-2023)

Wednesday, February 15, 2023

1:18 PM

## Complete Intersections

### Cohen Structure Theory

$(R, \mathfrak{m}, k)$  local

- $R$  is equicharacteristic if  $R$  contains a field (as a subring).
  - $\mathbb{Z} \rightarrow R$  structure map.  
If  $\text{char } R = p > 0$  prime, then  $\text{char } k = p$ .
  - When  $\text{char } k = 0$ ,  $\mathbb{Q} \hookrightarrow R$ .
- Mixed characteristic :  $\mathbb{Z} \hookrightarrow R$  and  $\text{char } k = p > 0$ .

When  $R$  is  $\mathfrak{m}$ -adically complete :

- If  $R$  equicharacteristic, then  $R$  contains a copy of  $k$  as a subring.
- If  $R$  mixed char., then  $R$  contains a complete DVR  $\mathcal{O}$

s.t.  $\mathcal{O} \hookrightarrow R \rightarrow k$   
*field of fractions*

$$R = k[[x_1, \dots, x_n]]/\mathfrak{I} \quad (\text{equi. case})$$

$$R = \mathcal{O}[[x_1, \dots, x_n]]/\mathfrak{I}.$$

Thus,  $R$  is of the form  $S/I$ , where  $S$  is a regular local ring.

We call this a Cohen presentation of  $R$ .  
 $(S/I, S \text{ where } S \text{ is an RLR.})$

Say  $R = S/I$  is a Cohen presentation.

$I \subseteq \mathfrak{m}_S$ . Suppose  $I \neq \mathfrak{m}_S^2$ . Pick  $x \in I \setminus \mathfrak{m}_S^2$ .

$$\begin{array}{ccc} S & \longrightarrow & R \\ & \searrow & \nearrow \\ & S/xS & \hookrightarrow \text{regular local} \end{array}$$

Thus, we can assume that  $I \subseteq \mathfrak{m}_S^2$ .

Equivalently,  $\text{edim } R = \text{edim } S$ .

$$\begin{array}{ccc} \text{rank}_k \left( \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \right) & & \text{rank}_k \left( \frac{\mathfrak{m}_S}{\mathfrak{m}_S^2} \right) \\ " & & " \\ \text{rank}_k \left( \frac{\mathfrak{m}_S}{I + \mathfrak{m}_S^2} \right) & & \end{array}$$

In this case, we say that  $S/I$  is a minimal Cohen presentation.

Dfn.  $(R, \mathfrak{m}, k)$  local is said to be a complete intersection (c.i.) if in some Cohen presentation

$$\widehat{R} = S/I$$

$\mathfrak{m}$ -adic

$\kappa = -/\mathbb{I}$   
 m-adic  
 Completion of  $R$

and  $I$  can be generated by a regular sequence in  $S$ .

[Not every c.i. is a quotient of a regular ring.]

Intrinsic characterisation?

Let  $K^R := \text{Koszul complex on a minimal gen for } m_R$ .  
 (Well defined up to iso of complexes  
 (with multiplicative structure).)

$$- \hat{R} \otimes_R K^R = K^{\hat{R}} \quad (\because m_{\hat{R}} = m_R)$$

-  $R \rightarrow \hat{R}$  induces

$$K^R \longrightarrow \hat{R} \otimes_R K^R = K^{\hat{R}}.$$

Then,  $H(K^R) = H(\hat{R} \otimes_R K^R)$   
 || flatness of  $R \rightarrow \hat{R}$   
 $\hat{R} \otimes_R H(K^R)$

Since  $m_H(K^R) = 0$ , the map

$$H(K^R) \longrightarrow \hat{R} \otimes_R H(K^R)$$

is an iso.

- Thus  $K^R \xrightarrow{\sim} K^{\hat{R}}$ .  
 quasi

Say,  $R = S/I$  is a minimal Cohen presentation.  
 $(I \subseteq M_S^2)$

$$\underline{\text{Lemma.}} \quad \text{rank}_k \left( H_i(R) \right) = \beta_i^s(R).$$

In particular, the RHS is independent of the min'l  
column presentation.

Proof. Let  $K^S$  → Koszul complex on  $S$ .

$S$  regular, so  $K^S \xrightarrow{\sim} k$ .

$$\text{Since } I \subseteq \mathfrak{m}_S^2,$$

$$I \subseteq \eta_s^L, \quad R \otimes_s K^s = K^R. \quad \left( \begin{array}{l} \text{min'l gen set for } \eta_s \\ \text{gives a min'l gen set for } \eta_R. \end{array} \right)$$

$$\therefore H_k(K^R) = H_*(R \otimes_{\mathbb{Z}} K^S) \xrightarrow{\quad} \because K^S \cong k$$

$$= \text{Tor}_{k_*}^S(R, k).$$

四

$R = S/I \rightarrow \text{min'l Cohen presentation}$

$$\operatorname{ht} I \leq \operatorname{v}_s(I)$$

↗ # gens of  $I$  as an  $S$ -module  
(Kunz)

$\operatorname{depth}_S(I_S) =$ , since  $S$  reg.

Recall:  $I$  is generated by a regular seq iff equality holds.

Thus,  $I$  is gen by regular sequence

$$\Leftrightarrow V_S(I) = \dim(S) - \dim(S/I) \quad \text{regular}$$

$$\begin{aligned}\beta_1^S(R)'' &= e \dim(S) - \dim(R) \\ \beta_1'' &= e \dim(R) - \dim(R)\end{aligned} \quad \begin{matrix} \curvearrowleft \\ \because \text{min'l} \end{matrix}$$

$$\text{rank}_R H_i(K^R)$$

Therefore, if  $R = S/I$  min'l Cohen presentation, then

$I$  is gen. by a regular sequence

$$\Leftrightarrow \text{rank}_k H_1(k^R) = \text{edim}(R) - \dim(R).$$

Corollary.  $R$  is a local ring.

$R$  is c.i. iff

$$\text{rank}_k H_1(k^R) = \text{edim}(R) - \dim(R).$$

Proof. All the numbers above are same for  $R$  and  $\hat{R}$ .  $\blacksquare$

Remark. Say  $\hat{R} = S/I$  is some Cohen presentation where  $I = \langle \text{reg. seq.} \rangle$ .

Then, we can reduce it to a min'l Cohen presentation where ideal  $= \langle \text{reg. seq.} \rangle$ .

Proof.

$$S \longrightarrow \hat{R}$$

$$\downarrow \quad \quad \quad \uparrow$$

$$S/\pi S = S'$$

$x \in I \setminus \mathfrak{m}_S^2$ .

Then,  $x \in \mathfrak{m}_{S'}I$  and so can be extended to a min gen set for  $I$ .  
( $x, y$ )

Then,  $y$  is a regular seq...

Theorem.  $R$  c.i.  $\Rightarrow R_p$  c.i.  $\forall p \in \text{Spec } R$ .

Nontrivial!!!

This is clear if  $R$  itself has a Cohen presentation.

$R = S/I$ ,  $I$  gen. by reg. seq.

$p \in \text{Spec } R \subseteq \text{Spec } S$ .  $R_p \cong S_p / IS_p \dots$

$$(R, \mathfrak{m}, k): R \longrightarrow \hat{R}^n$$

↓                  ↓?

$$R_p \longrightarrow \hat{R}_p^{n_{R_p}}$$

↑ not a localisation of  $\hat{R}^n$ .

Pick  $q \in \text{Spec } \hat{R}$  s.t.  $q \cap R = p$ .

$$\begin{array}{ccc} R & \longrightarrow & \hat{R} \\ \downarrow & & \downarrow \\ R_p & \longrightarrow & (\hat{R})_q \\ & \uparrow & \\ & \text{flat + local} & \end{array}$$

Theorem.  $\varphi: R \rightarrow S$  flat local map.

Then,  $S$  c.i.  $\Leftrightarrow R$  and  $S/\mathfrak{m}_R S$  are c.i.

Fact.  $R$  c.i.  $\Leftrightarrow R[\![x_1, \dots, x_n]\!]$  c.i.

$(R, \mathfrak{m}, k)$  local

$R = S/I$  min'l Cohen presentation (say it exists)

Write  $\mathfrak{m} = (x_1, \dots, x_n)$  with  $n = \text{edim}(I)$ .

$I = (f_1, \dots, f_c)$ , min'l gen set.

$$\mathfrak{m}_R = \underline{\overline{x}} \cdot R.$$

$$K^R : \longrightarrow \overset{n}{\oplus} R e_i \longrightarrow R \rightarrow 0$$

$$K^R : \dots \longrightarrow \bigoplus_{i=1}^n R e_i \longrightarrow R \rightarrow 0$$

$$\partial e_i = x_i$$

Since  $I \subseteq M_s^2$ , can write

$$f_j = \sum_{i=1}^n s_{ij} x_i, \text{ where } s_{ij} \in \mathcal{M}.$$

$$\text{Let } z_j = \sum_{i=1}^n \bar{s}_{ij} e_i \in \bigoplus_{i=1}^n R e_i.$$

$$\text{Note } \partial(z_j) = \sum_{i=1}^n \bar{s}_{ij} \bar{x}_i = \bar{f}_j = 0.$$

$\therefore z_1, \dots, z_c$  are cycles.

Claim. The classes  $[z_1], \dots, [z_c] \in H$ , form a  $k$ -basis.  $\square$

Tate's results.  $(R, \mathcal{M}, k)$

$$R^n \xrightarrow{f} R \quad \begin{matrix} \text{im}(f) = \mathcal{M} \\ n = \dim(R) \end{matrix}$$

$$K^R = (\Lambda^*(\sum R^n), \partial).$$

$\leadsto K^R$  is a dg (= differential graded)  $R$ -algebra.  
(Even graded commutative.)

$\leadsto H(K^R)$  is a graded-commutative  $k$ -algebra.

Universal property gives:

$$\chi^R : \Lambda_* \left( \sum H_*(K^R) \right) \longrightarrow H(K^R)$$

$\sum$

map of  $k$ -algebras

Theorem (Tate)  $R$  is c.i.  $\Leftrightarrow \chi^R$  is an isomorphism.

(Asmno)  $\Leftrightarrow \Lambda^2 H_1(K^R) \longrightarrow H_2(K^R).$

# Lecture 12 (22-02-2023)

22 February 2023 13:26

$(R, \mathfrak{m}_R, k) \rightarrow \text{local ring}$

$R$  is c.i.  $\equiv \widehat{R} \cong S/I$ , where  $S$  regular  
 I gen by neg seq.  
 Can assume  $I \subseteq \mathfrak{m}_S^2$ .

Intrinsic characterisation:

$$\text{rank}_k H_i(k^R) = \text{edim}(R) - \dim(R)$$

(In general,  $\geq$  holds.)

Key:  $H_i(k^R) \cong I/\mathfrak{m}_S I$ .

$k^R$  is an exterior algebra with  $\partial$  satisfying Leibniz rule.

$k^R$  is a (commutative) differential graded  $R$ -algebra.

$\therefore H(k^R)$  is also graded-commutative  $\quad (ab = (-)^{\text{deg } a \text{ deg } b} ba \text{ for homog. } a, b)$   
 $k$ -alg

By the universal prop. of exterior algebra,

$$H_i(k^R) \hookrightarrow H(k^R)$$

induces

$$\Lambda_k(\sum H_i(k^R)) \longrightarrow H(k^R) \text{ of graded } k\text{-algebra.}$$

- Image is the  $k$ -subalgebra of  $H(k^R)$  generated by  $H_i(k^R)$ .

Theorem (Tate-Ausma)  $(R, \mathfrak{m}_R, k)$  local. TFAE:

①  $R$  is complete intersection.

②  $\Lambda_k(\sum H_i(k^R)) \xrightarrow{\cong} H(k^R)$ , i.e.,  $H(k^R)$  is the exterior algebra on  $H_i(k^R)$ .

③ The map above is surjective in degree two, i.e.,

$$H_1(k^R)^{12} = H_2(k^R).$$

Can assume  $R$  is complete. Let  $R = S/I$  min'l Cohen presentation.

$$\text{Then } H_*(K^R) \cong \text{Tor}_*^S(k, R) \quad \text{as } k\text{-algebras.}$$

-  $S$  regular, so  $\mathfrak{m}_S$  is generated by a regular sequence.

Now, suppose  $S$  is any local ring and  $I \subseteq J \subseteq \mathfrak{m}_S$  are ideals generated by regular sequences.

Want to compute  $T_{\text{or}}^S(S/J, S/I)$ .

(In the case we care about,  
 $J = m_s$ .)

Let  $I = (\underline{a})$ , where  $\underline{a} = a_1, \dots, a_m \in M_s$  reg,  
 $J = (\underline{b})$ , where  $\underline{b} = b_1, \dots, b_n \in M_s$ .

$$\begin{aligned} K(\underline{a}; S) &\xrightarrow{\sim} S/I \quad \text{and} \\ K(\underline{b}; S) &\xrightarrow{\sim} S/J. \end{aligned} \quad \left. \right\} \text{dg algebra maps}$$

$$\begin{aligned} \text{Tor}_*(S/J, S/I) &= H_* \left[ k(\underline{b}; S) \otimes_S S/I \right] \quad \text{as algebras.} \\ &= H_* \left( k(\underline{b}; S/I) \right) \end{aligned}$$

$$K(\underline{b}; S) \otimes_s S/I \xleftarrow{\sim} K(\underline{b}; S) \otimes_s K(\underline{a}; S) \xrightarrow{\text{map of dg-}} \\ \text{quasi iso } : K(\underline{a}; S) \xrightarrow{\sim} S/I \downarrow \begin{matrix} \text{and} \\ \cong \end{matrix} \quad K(\underline{b}; S) \text{ fin. free}$$

$$S/J \otimes_S k(\underline{a}; S) = k(\underline{a}; S/J)$$

$k(0; S/J)$

$$H(K(O; S/\mathbb{Z})) = K(O; S/\mathbb{Z}) = \bigwedge \underbrace{\Sigma(S/\mathbb{Z})^m}_{H_1(K(O; S/\mathbb{Z}))}$$

$$\text{Therefore : } H(b; S/I) \cong \Lambda \sum H_i(b; S/I)$$

$$\text{In summary : } \text{Tor}^S(S/J, S/I) \cong \bigwedge \sum \text{Tor}_i^S(S/J, S/I)$$

$$\text{Tor}_i^S(S/J, S/I) = \frac{I \cap J}{IJ} = I/J$$

$\curvearrowright \because I \subseteq J$

1

$$\therefore I \subseteq J$$

Thus:

Lemma. If  $I \subseteq J$  are gen. by a regular sequence, then

$$\text{Tor}_*^S(S/J, S/I) \cong \Lambda(\sum I/IJ).$$

Moreover,  $I/IJ$  free  $S/J$ -module.

Specialising to our c.i. case, we get  $\textcircled{1} \Rightarrow \textcircled{2}$  of Tate-Azumaya ...

$\textcircled{2} \Rightarrow \textcircled{3}$  ok.

Proof of  $\textcircled{3} \Rightarrow \textcircled{1}$ : Hypothesis:

$$\Lambda(\sum H_1(k^n)) \longrightarrow H_2(k^R).$$

Equivalently,

$$\Lambda \sum \text{Tor}_1^S(k, R) \longrightarrow \text{Tor}_2^S(k, R). \quad (R = S/I \text{ often pres.})$$

Let  $I = (\underline{a})$  with  $\underline{a}$  min'l gen set.

Let  $F \xrightarrow{\sim} R$  be a min'l free resolution of  $R$  over  $S$ .

$$\begin{array}{ccccccc} F: & \dots & \rightarrow S^n & \longrightarrow S^m & \longrightarrow S & \rightarrow 0 \\ & & & \underbrace{(\underline{a}, \dots, \underline{a})}_{\text{same as Koszul}} & & & \\ & & & \nearrow \text{lift of identity on } R. & & & \\ k(\underline{a}; S) & \dashrightarrow & F & & & & \text{A morphism of } S\text{-complexes.} \\ & \downarrow & & \downarrow \simeq & & & \\ R = S/\underline{a}S & = & R & & & & \end{array}$$

Want to deduce:  $H_1(k(\underline{a}; S)) = 0$

$\therefore \underline{a}$  reg.

Can ensure that the lifting is id in degrees 0, 1.

$$\begin{array}{ccccccc} k(\underline{a}; S): & \dots & \rightarrow S^{n \choose 2} & \longrightarrow S^m & \longrightarrow S & \rightarrow 0 \\ & & \downarrow \times & \parallel & \parallel & & \\ F: & \dots & \rightarrow S^n & \longrightarrow S^m & \longrightarrow S & \rightarrow 0 \end{array}$$

ETP. ( $\Rightarrow$ ) is onto. Then a diagram chase shows  $H_1 = 0$ .  
 $\Rightarrow \Delta$  is neg.  
 $\Rightarrow R$  is c.i.

By NAK, suffices to prove

$$K(\underline{a}; S) \otimes_S k \rightarrow F \otimes_S k \text{ is onto}$$

Equivalently, surjectivity on  $H_2$ . ( $\because$  differential becomes zero)

$$K(\underline{a}; S) \otimes_S k \longrightarrow F \otimes_S k.$$

In homology:

$$H_*(K(\underline{a}; k)) \xrightarrow{\cong} H_*(F \otimes_S k) = \text{Tor}_*^S(R, k)$$

$$\wedge \sum H_1(\underline{a}; k) \longrightarrow \text{Tor}_*^S(R, k) \quad (+)$$

$$\underline{\text{Note}}. \quad H_1(\underline{a}; k) \xrightarrow{\cong} \text{Tor}_1^S(R, k).$$

Hypothesis:  $\text{Tor}_2^S$  generated by  $\text{Tor}_1^S$ .

$\therefore (+)$  is onto.

#

$(R, \mathfrak{m}_R, k)$  local ring.

$$\sup \{ i \geq 0 : H_i(K^R) \neq 0 \} = \text{edim}(R) - \text{depth}(R).$$

$$\therefore H_s(K^R) = 0 \quad \text{for } s > \text{edim}(R) - \text{depth}(R).$$

Theorem of Wiebe:  $R$  is c.i.  $\Leftrightarrow H_s(K^R) \neq 0$   
 $\text{for } s = \text{edim}(R) - \text{depth}(R)$ .

Proof ( $\Rightarrow$ ) By Tate.

$$\text{Indeed, } R \text{ c.i.} \Rightarrow H(K^R) = \wedge \sum H_1(K^R)$$

$$H_1(KR) \cong k^c, \quad c = \text{edim } R - \text{depth } R \\ = \text{edim } R - \text{depth } R.$$

$\Leftarrow$  Can reduce to the case  $\text{depth}(R) = 0$ .

Ex, we can choose a min gen set  $x_1, \dots, x_n$  ( $n = \text{edim } R$ ) for  $M_R$  s.t.  $x_1, \dots, x_d$  is a reg seq. on  $R$ ,  $d = \text{depth } R$ .

$$K(x; R) = K(x_1, \dots, x_d; R) \otimes K(x_{d+1}, \dots, x_n; R)$$

$$\downarrow \cong$$

$$\underbrace{R/(x_1, \dots, x_d)R}_{R'} \otimes_R K(x_{d+1}, \dots, x_n; R)$$

$$\therefore K(x; R) \cong K(x_{d+1}, \dots, x_n; R')$$

$$\begin{matrix} \parallel \\ K^R \end{matrix} \qquad \qquad \qquad \begin{matrix} \parallel \\ K^{R'} \end{matrix}$$

$$\therefore H(KR) \cong H(K^{R'}) \text{ as } k\text{-algebras.}$$

$$\text{edim } R - \text{depth} = \text{edim } R' \quad (\& \text{depth } R' = 0).$$

$$R \text{ is c.i.} \iff R' \text{ is c.i.}$$

Thus, can assume  $\text{depth } R = 0$ . The hypothesis is

$$H_1(KR) \stackrel{\text{edim}(R)}{\neq} 0.$$

The desired conclusion is that  $R$  is c.i.

This hypothesis is equivalent to  $\text{Fitt}_0(M) \neq 0$ .

Fitting ideals and Koszul homology

Say  $J \subseteq R$  is an ideal. ( $R$  noe., possibly not local)

$$R^n \xrightarrow{(a_{ij})} R^m \xrightarrow{(r_1, \dots, r_m)} J \rightarrow 0 \quad \text{presentation}$$

$$\text{Fit}_{\text{to}}(J) = I_{m \times m}((a_{ij})).$$

Koszul  $\propto$  on  $\Sigma = r_1, \dots, r_m$ .

$$K(\Sigma; R) : \cdots \rightarrow R^{\binom{m}{2}} \xrightarrow{\quad} R^m \xrightarrow[\substack{(r_1, \dots, r_m)}]{\oplus \text{Rei}} R \rightarrow 0.$$

Cycles in  $K_1(\Sigma; R)$  are precisely the syzygies of  $J$ .

$$\left[ \partial(\sum b_i e_i) = 0 \Leftrightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \text{im}(a_{ij}) \right]$$

In particular

$$z_j := \sum_{i=1}^m a_{ij} e_i \quad \text{are cycles in } K_1(\Sigma; R) \quad \text{for } j = 1, \dots, n.$$

$$\langle z_{i_1} \wedge \dots \wedge z_{i_m} : i_1 < i_2 < \dots < i_m \rangle = I_{m \times m}((a_{ij})) \cdot K_m(\Sigma; R)$$

$$\text{In summary: } H_1(\Sigma; R)^m = \text{Fit}_{\text{to}}(J) \cdot K_m(\Sigma; R).$$

$$\text{Recall: } H_m(\Sigma; R) = \text{ann}_R(J).$$

$$\text{So, } H_1(\Sigma; R)^m \subseteq H_m(\Sigma; R) \text{ reflects}$$

$$\text{Fit}_{\text{to}}(J) \subseteq \text{ann}(J).$$

One can restate (depth 0 case) of Wiebe as:

$$\begin{array}{l} R \text{ is c.i.} \\ \text{with } \dim R = 0 \end{array} \Leftrightarrow \text{Fit}_{\text{to}}(m) \neq 0.$$

## Lecture 13 (27-02-2023)

Monday, February 27, 2023 1:26 PM

# Wiebe's Theorem

$S \rightarrow$  comm ring

$I \subseteq J$  where

$$I = (\alpha_1, \dots, \alpha_n),$$

$$J = (x_1, \dots, x_n), \quad \text{where } \alpha_i, x_j \text{ are reg.}$$

We can write  $\underline{\alpha} = U \underline{x}$ .

Lemma.  $(I : J) = I + (\det U),$   
and  $\det U \notin I$ .

Example.  $R = k[x_1, \dots, x_n].$

$$(x_1^t, \dots, x_n^t) \subseteq (x_1, \dots, x_n).$$

$$U = \begin{bmatrix} x_1^{t-1} & & & 0 \\ & \ddots & & \\ 0 & & \ddots & x_n^{t-1} \end{bmatrix}.$$

$$\left( (x_1^t, \dots, x_n^t) : (x_1, \dots, x_n) \right) = (x_1^t, \dots, x_n^t) + x_1^{t-1} \cdots x_n^{t-1}.$$

Proof of Lemma: Equivalent:

$$\left( 0 :_{S/I} J/I \right) = (\det U) \cdot S/I.$$

Note  $(0 :_{S/I} S/I) = H_n(\underline{x}, S/I).$

Also,

$$\begin{array}{ccc} S & \xrightarrow{(a_1, \dots, a_n)} & S \\ u \downarrow & & \parallel \\ S & \xrightarrow{(x_1, \dots, x_n)} & S \end{array}$$

This induces

$$\Lambda^* u: K(\underline{a}; S) \rightarrow K(\underline{x}; S).$$

The top degree map is  $K_n(\underline{a}; S) \xrightarrow{\text{det } u} K_n(\underline{x}; S).$

This induces

$$\Lambda^* u: K(\underline{a}; S/I) \rightarrow K(\underline{x}; S/I).$$

(all differentials 0 here  
 $\therefore H_n(\underline{a}) = 0$  for this)

Need to prove  $H_n(\Lambda^* u)$  is onto.

(The map is still  $\xrightarrow{\text{det } u}.$ )

Moreover:  $H_*(\underline{a}; S/I) \rightarrow H_*(\underline{x}; S/I)$

$$\cong \uparrow \qquad \qquad \uparrow \cong \text{Tate}$$

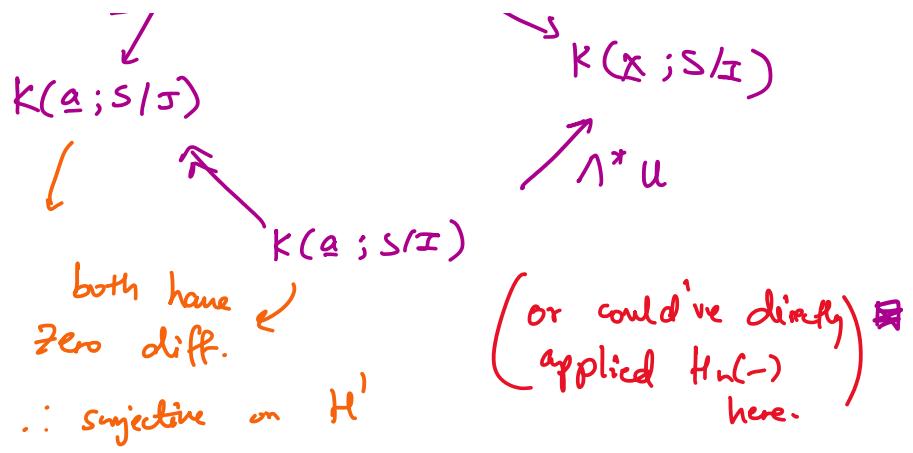
$$\Lambda^* H_*(\underline{a}; S/I) \rightarrow \Lambda^* H_*(\underline{x}; S/I)$$

Suffices to now prove surjectivity at degree one.

$$H_1(\underline{a}; S/I) \rightarrow H_1(\underline{x}; S/I)$$

$$K(\underline{a}; S) \otimes K(\underline{x}; S)$$

$$\begin{array}{ccc} & \swarrow \cong & \searrow \cong \\ k(\underline{a}; S/I) & & K(\underline{x}; S/I) \end{array}$$



Special Case.  $R = S/I$ ,  $S$  is RLR,

(Assume  $R$  complete.)

$$I \subseteq \mathfrak{m}_S^2.$$

$R$  is c.i.

- $\dim R = 0$
  - $I = (a_1, \dots, a_n) \subseteq (\underbrace{x_1, \dots, x_n}_{\text{regular}}) = \mathfrak{m}_S$ .
- $n = \text{edim } R.$

By Lemma:  $(0 :_R \mathfrak{m}_R) = (\det u).$

$x$   
 $\circ$

Def.  $(R, \mathfrak{m}_R, k)$  local.

$$\text{soc}(R) = (0 :_R \mathfrak{m}_R) \cong \text{Hom}_R(k, R).$$

socle

Note.  $\mathfrak{m}$  annihilates  $\text{soc}(R)$  by defn.  
 $\therefore \text{soc}(R)$  is a  $k$ -vector space.  $\text{rank}_k(\text{soc}(R)) =: \text{type}(R).$

Corollary.  $R$  c.i. with  $\dim R = 0$ , then

$$(0 :_R \mathfrak{m}_R) \subseteq \text{any nonzero ideal.}$$

(If  $\dim \neq 0$ , then  $\text{soc} = 0$  anyway.)

"c.i. rings are gorenstein"

Corollary.  $\varphi : (R, \mathfrak{m}_R) \longrightarrow T$ .

Suppose  $R$  is c.i.

If  $\varphi(s \in R) \neq 0$ , then  $\varphi$  is an iso.

Weibel's Thm.  $(R, \mathfrak{m}_R)$  local

If  $\text{image}(\wedge^c H_1(K) \rightarrow H_c(K)) \neq 0$ ,  
where  $c = \text{edim } R - \text{depth } R$

then  $R$  is c.i.

We showed that we can reduce to  $\text{depth } R = 0$ .

That is,

$$\wedge^n H_1(K^R) \xrightarrow{\quad} Z_n(K^R)$$

$\wedge^n H_1(K^R) \rightarrow H_n(K^R)$  non-zero

$\Rightarrow R$  is c.i.  $\begin{pmatrix} \text{necessarily} \\ \text{clim } R = 0 \end{pmatrix}$

( $= \text{Fitt}_0(\mathfrak{m}_R) \neq 0$ )

Can assume  $R$  is complete. Write

$R = S/I$  w/ Lichten presentation.

$$\frac{I}{\mathfrak{m}_S I} \longleftrightarrow H_1(K^R).$$

Hypothesis,  $\exists z_1, \dots, z_n \in Z_1(K^R) \leftarrow$

$z_1 \wedge \dots \wedge z_n \neq 0 \quad \text{in } K^R$ .

Let  $a_1, \dots, a_n \in I \setminus \mathfrak{m}_S I$  be some "rep"  
of  $z_1, \dots, z_n$ .

Case 1. Suppose  $\dim(R) = 0$ , i.e.,  $\text{height}(I) = n$ .

Then,  $\text{depth}_S(I, S) = n$  ( $\because S$  regular).

Exercise.  $\exists a'_1, \dots, a'_n \in \mathfrak{m}_S I$  s.t.

$a_1 + a'_1, \dots, a_n + a'_n$  is a reg. seq. in  $S$ .

$\therefore$  Can assume  $a_1, \dots, a_n$  is a reg. seq.

$$\begin{array}{ccc} S/(a) & = R' \longrightarrow R & \text{induces} \\ & & \downarrow \text{soc}(R') \\ \Lambda^n H_1(K^{R'}) & \longrightarrow H_n(K^{R'}) \subseteq R' & \cong \\ \text{image of this} \\ \text{map contains} \\ z_1 \wedge \dots \wedge z_n & \downarrow \text{not } 0 \Rightarrow \downarrow \neq 0 & \downarrow \\ \Lambda^n H_1(K^R) & \xrightarrow{\pi} H_n(K^R) \subseteq R & \cong \\ \text{not } 0 & & \end{array}$$

$\therefore \text{soc}(R') \neq 0$ .

$\therefore R' = R$  by earlier part.

$\therefore R = S/(a)$  with  $a$  regular!

Case 2  $\dim R > 0$  (will prove that this cannot be).

$z_1 \wedge \dots \wedge z_n \neq 0$  in  $K^R \cong R$ .

$\therefore \exists s > 0$  s.t.  $z_1 \wedge \dots \wedge z_n \notin \mathfrak{m}_R^s$ . by Krull int. thm.

$$\begin{array}{ccccc} \Lambda^n H_1(R) & \xrightarrow{\quad} & H_n(R) & \xrightarrow{\quad} & R \\ \downarrow & & \downarrow & & \downarrow \\ z_1 \wedge \dots \wedge z_n & \in & \mathfrak{m}_R^s & \subset & R \end{array}$$

$$\begin{array}{ccccc}
 \Lambda^n H_i(R) & \longrightarrow & H_n(R) & z_1, \dots, z_m & R \\
 \downarrow & & \downarrow & & \downarrow \\
 \Lambda^n H_i(K^{R/\mathfrak{m}_R^{s+1}}) & \xrightarrow{\textcircled{1}} & H_n(R/\mathfrak{m}_R^{s+1}) & & R/\mathfrak{m}_R^{s+1} \\
 \downarrow & \not\cong & \downarrow & & \downarrow \\
 \Lambda^n H_i(K^{R/\mathfrak{m}^s}) & \xrightarrow{\textcircled{2}} & H_n(R/\mathfrak{m}^s) & \not\cong & R/\mathfrak{m}_R^s
 \end{array}$$

$\therefore \textcircled{1} \neq 0.$

$\Rightarrow R/\mathfrak{m}_R^{s+1}$  is c.i. by  $\textcircled{1}.$

But  $\textcircled{2} \neq 0 \Rightarrow R/\mathfrak{m}^{s+1} \rightarrow R/\mathfrak{m}^s$  is an iso (by socle corollary).

$\therefore \mathfrak{m}^s = \mathfrak{m}^{s+1}.$

But then NAK forces  $\mathfrak{m}^s = 0.$

But then  $\dim = 0.$   $\blacksquare$

Wiebe.

$$\Lambda^i H(K^R) \rightarrow H_i(K^R)$$

Tak:  $R$  c.i.  $\Rightarrow$  iso for all  $i.$

Asmous: Onto for  $i = 2 \Rightarrow R$  c.i.

Wiebe: Non zero for  $i = \operatorname{edim} R - \operatorname{depth} R \Rightarrow R$  c.i.

Bruns: Map is zero for  $i > \operatorname{edim} R - \operatorname{dim} R.$

# Lecture 14 (01-03-2023)

Wednesday, March 1, 2023 1:28 PM

$A \rightarrow$  any ring (possibly noncomm.)

$E \rightarrow A\text{-module (left)}$

Def<sup>n</sup>.  $E$  is injective if  $\text{Hom}_A(-, E)$  is exact.

I.e., given any solid diagram

$$\begin{array}{ccc} N & \hookrightarrow & M \\ f \downarrow & \swarrow \tilde{f} & \\ E & & \end{array}$$

a dotted extension exists.

Thm. (Baer's Criterion)

$E$  is injective  $\Leftrightarrow I \hookrightarrow A$  has the extension property for all left ideals  $I$ .

$$\begin{array}{ccc} I & \hookrightarrow & A \\ \downarrow & \nearrow 1 & \downarrow \\ E & & \end{array}$$

Thus: any map  $I \rightarrow E$  is multiplication by some  $x \in E$

Prof. ( $\Rightarrow$ ) Clear.

$$(\Leftarrow) \quad N \hookrightarrow M$$

$$\begin{array}{ccc} & & \\ f \downarrow & \swarrow & \\ E & & \end{array}$$

Consider all submodules  $U$  s.t.  
 $N \leq U \leq M$

$$f \downarrow_E \dashv f_u$$

... Take max'l extension  $(U, f_u)$ .

If  $U = M$ , done.

Else pick  $x \in M \setminus U$ .

Extend to  $U + Ax\dots$

Thus, enough to check extension along  $I$ .

Now, suppose  $R$  is a commutative domain.

①  $E$  injective  $\Rightarrow E$  divisible.

② Converse holds if  $R$  is a PID.

Equivalently,

$$\begin{array}{ccc} R & \xrightarrow{r} & R \\ \downarrow & \dashv & \end{array}$$

Given  $x \in E$ ,  
 $r \in R \setminus \{0\}$ ,  
 $\exists y \in E$  s.t.  
 $ry = x$ .

has extension property  
 for all  $r \neq 0$ .

Example: Over  $\mathbb{Z}$ ,  $\mathbb{Q}$  is divisible and  $\therefore$  injective.

Lemma: Any  $\mathbb{Z}$ -module can be embedded into an injective  $\mathbb{Z}$ -module.

Proof: ① divisible  $\Rightarrow \oplus \mathbb{Q}$  divisible

$\Rightarrow$  Any quotient of  $\oplus \mathbb{Q}$  is also divisible

$\Rightarrow$  Any quotient of  $\oplus \mathbb{Q}$  is also divisible

$\Rightarrow$  Any quotient of  $\oplus \mathbb{Q}$  is injective.

Now, any free  $F$  can be embedded  $F \hookrightarrow \oplus \mathbb{Q}$ .

Now if  $U \subseteq F$  submodule, then

$$F/U \hookrightarrow \oplus \mathbb{Q}/U.$$

□

Lemma. Let  $A \rightarrow B$  map of rings.

$E$ : injective  $A$ -module.

Then,

$$\text{Hom}_A(B, E)$$

viewed as a left  $B$ -module, is injective.

[ If  $f: B \rightarrow E$  is  $A$ -linear, and  $b \in B$  then  
 $(b \cdot f)(x) = f(x \cdot b)$ . Check  $b \cdot f$  is  $A$ -linear  
and this is an action. ]

Proof. Need to check

$\text{Hom}_B(-, \text{Hom}_A(B, E))$  is exact.  
on mod- $B$

But

$$\text{Hom}_A(-, E).$$

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Prop.  $A \rightarrow$  any ring  
Any  $A$ -module embeds into an injective  $A$ -module.

Proof.  $\mathbb{Z} \rightarrow A$  structure map.

$M \rightarrow A\text{-module}$

$M \downarrow_{\mathbb{Z}} \rightarrow$  think of  $M$  as a  $\mathbb{Z}\text{-module}$

$M \downarrow_{\mathbb{Z}} \hookrightarrow E$  injective  $\mathbb{Z}\text{-module}$

$\text{Hom}_{\mathbb{Z}}(A, M \downarrow_{\mathbb{Z}}) \hookrightarrow \text{Hom}_{\mathbb{Z}}(A, E).$

$\downarrow_{A\text{-modules}}$  The map is also  $A$ -linear.

Suffice to embed

$M \hookrightarrow \text{Hom}_{\mathbb{Z}}(A, M \downarrow_{\mathbb{Z}}).$

$$\lambda(x)(a) := a \cdot x.$$

IV

## Essential Extensions.

Defn. An extension  $N \hookrightarrow M$  is **essential** if

$\forall U \neq 0 \subseteq M$  submodule

if  $U \cap N \neq 0$ .

It is **proper** if  $N \neq M$ .

Lemma. An  $A$ -module  $E$  is injective

$\Leftrightarrow E$  admits no proper essential extensions.

Proof. ( $\Rightarrow$ )  $E$  injective. If  $E \hookrightarrow_{M, i}^i$  essential then  $i$  splits.

$$F \oplus E = M. \quad \text{But } F \cap E = 0.$$

$\therefore F = 0$  by essentiality.

$$\therefore M = E.$$

( $\Leftarrow$ ) Say  $E$  admits no proper essential inclusions.

Given  $N \hookrightarrow M$

$f \downarrow$ , consider pushforward  
 $E$

$$\begin{array}{ccc} N & \hookrightarrow & M \\ f \downarrow & & \downarrow \\ E & \hookrightarrow & W = \overline{M \oplus E} \\ & & \langle (n, f(n)) : n \in N \rangle \end{array}$$

In  $W$ , let  $U \subseteq W$  be a max'l submodule  
 s.t.  $U \cap E = 0$ .

But now  $E \xrightarrow{\quad} W/U$  is injective  
 and essential.

$$\begin{array}{ccc} N & \hookrightarrow & M \\ f \downarrow & & \downarrow g \\ E & \hookrightarrow & W \\ \varphi \downarrow \cong & \nearrow p & \\ w/u & & \end{array} \quad \therefore \text{Isomorphism.}$$

Define  $M \rightarrow E$  as  
 $\varphi^{-1} \circ p \circ g$ . Desired extension.  $\blacksquare$

Defn.  $M$  any  $A$ -module.

An injective hull of  $M$  is an essential extension  
 $M \hookrightarrow E$

with  $E$  injective.

(This  $E$  is unique up to iso.)

Prop^n. Let  $M$  be an  $A$ -module.

① Take  $M \hookrightarrow I$  with  $I$  injective.

Then, any maximal essential extension of  $M$  in  $I$  is injective. (And hence an essential hull of  $M$ .)

②  $M \hookrightarrow I$  with  $I$  injective, and  $M \hookrightarrow E$  is an injective hull, then  $\exists$  mono  $E \hookrightarrow I$  s.t.

$$\begin{array}{ccc} M & \xhookrightarrow{f} & E \\ & \downarrow & \downarrow \\ & & I \end{array}$$

commutes.

$\therefore E$  is a "minimal injective module containing  $M$ ".

③  $M \hookrightarrow E$  and  $M \hookrightarrow E'$  injective hulls, then

$\exists$  comm diagram

$$\begin{array}{ccc} M & & \\ \downarrow & \swarrow & \searrow \\ E & \xrightarrow{\cong} & E' \end{array}$$

Proof. ②, ③ OK

①. let  $M \hookrightarrow E \hookrightarrow I$ .

$\downarrow$  max'l ess. ext'n of  $M$  in  $I$ .

ETP:  $E$  admits no proper essential extension.

Say  $E \hookrightarrow U$  essential.

Then,

$$\begin{array}{ccc} E & \hookrightarrow & U \\ & \nwarrow & \downarrow i \\ & I & \end{array}$$

Moreover, by (i)  $N_E = 0$ .

$\therefore N_i = 0$ .

$\Rightarrow i$  is injective.

But then  $E \subset U \subset I$  in  $I$ .

But the  $E \subseteq U \subseteq I$  in  $I$ .

maximality  $\Rightarrow E = U$ . ■

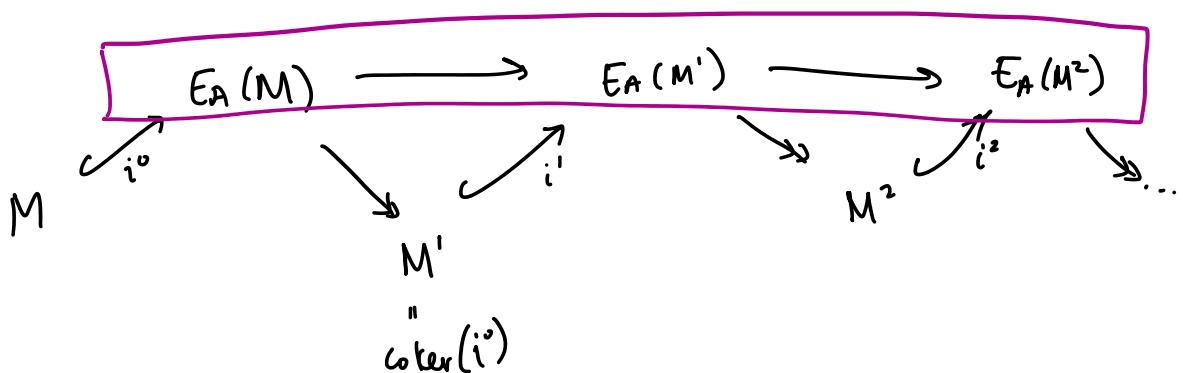
$(M \hookrightarrow E \text{ ess} + E \hookrightarrow U \text{ ess} \Rightarrow M \hookrightarrow U \text{ ess.})$

### Key Consequence.

$A \rightarrow$  any ring

$M \rightarrow$  any module.

Then,  $M$  admits a unique min'l injective resolution  
which is unique up to iso (of complexes). ■



$$M \simeq 0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$$

$$\text{Ext}_A^n(-, M) = H^n(\text{Hom}_A(-, M)).$$

Remark:  $\{E_i\}$  any family of injectives,  $\prod E_i$  is injective.

Baer's Criterion  $\Rightarrow$  If  $A$  is (left) Noetherian, then  
 $\bigoplus E_i$  is injective.

## Lecture 15 (13-02-2023)

Monday, March 13, 2023 1:24 PM

# Structure of Injectives

$$R : M, N$$

$$\text{Ext}_R^i(M, N) = H^i(\text{Hom}_R(P, N)).$$

$$\text{where } P \xrightarrow{\sim} M$$

$$E \text{ is injective} \Leftrightarrow \text{Ext}_R^i(-, E) = 0 \text{ on Mod } R$$

$$\Leftrightarrow \text{Ext}_R^i(R/I, E) = 0 \text{ for all ideals } I \subseteq R.$$

$$\Leftrightarrow \text{Ext}_R^i(-, E) = 0 \text{ on Mod } R \quad \forall i \geq 1$$

Note.  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is exact in  $\text{Mod}_R$

$$\begin{array}{c} \xrightarrow{\text{gives}} \\ 0 \rightarrow \text{Hom}(Z, E) \rightarrow \text{Hom}(Y, E) \rightarrow \text{Hom}(X, E) \\ \downarrow \\ \text{Ext}^i(Z, E) \rightarrow \dots \end{array}$$

---

R commutative noetherian.

$$\textcircled{1} \quad \left\{ E_\lambda \right\}_\lambda \text{ injective} \Rightarrow \bigoplus_\lambda E_\lambda \text{ injective.}$$

$\textcircled{2}$  Any injective can be written as a direct sum of indecomposable injectives.

$\hookrightarrow$  indecomposable injectives.

③ The indecomposable injectives are precisely those of the form

$$E_R(R/p)$$

for  $p \in \text{Spec } R$ . (These are pairwise non-iso.)

④  $E_R(R/p)$  is  $p$ -local and  $p$ -power torsion.

$M$  is  $p$ -local if  $M \xrightarrow{\sim} M_p$   
is an iso.

every  $x \in M$  is  
annihilated by  $p^n$   
 $i.e.,$

$\Leftrightarrow M \xrightarrow{r} M$  is an  
iso for  $r \in R \setminus p$

$$\bigcup_{n \geq 0} (0 :_{R/p^n}) = M$$

$\Leftrightarrow R$  action on  $M$  factors  
through  $R \rightarrow R_p$

!!

$$F_{R/p}(M)$$

Lemma:  $S := U' R$ , where  $U \subseteq R$  mult. closed.

$$R \rightarrow S.$$

Not true for  
arbit. flat  
maps.  $R \rightarrow \widehat{R}$  is an example.

①  $E$  injective  $R$ -module  $\Rightarrow S \otimes_R E$  injective  $S$ -module.

② Any injective  $S$ -module is injective over  $R$ .

③  $M \hookrightarrow N$  is essential over  $R$

$$\Rightarrow S \otimes_R M \hookrightarrow S \otimes_R N \text{ is essential.}$$

In particular,  $S \otimes_R F_R(M) = E_S(S \otimes_R M)$ .

Proof. ② Holds for any flat map  $R \rightarrow S$ .

Let  $I$  be inj  $S$ -module.

$$\text{Ext}_S^1(M, I) \cong \text{Ext}_R^1(S \otimes_R M, I) = 0. \quad \checkmark$$

$$\text{Ext}_R^1(M, I) \cong \text{Ext}_S^1(S \otimes_R M, I) = 0. \quad \checkmark$$

?   
 ; I S-module  
 and  $R \rightarrow S$  flat

① Any f.g. S-module is of the form  $S \otimes_R M$  with  
M f.g. over R.

(“clear denominators of generators”)

$$\begin{aligned} \text{Ext}_S^1(S \otimes_R M, S \otimes_R E) &\cong \text{Ext}_R^1(M, S \otimes_R E) \\ &\cong S \otimes_R \text{Ext}_R^1(M, E) = 0. \end{aligned}$$

We used the following  
key property of localisation:

$$\begin{array}{ccc} R \longrightarrow S \text{ flat} & & \left. \begin{array}{l} \text{defines} \\ \text{“absolutely flat maps”} \end{array} \right\} \\ S \otimes_R S \xrightarrow{\cong} S & \text{iso.} & \end{array}$$

③ WTS  $\bar{U}'M \hookrightarrow \bar{U}N$  is essential.

$$\text{Equiv: } S\left(\frac{x}{u}\right) \cap \bar{U}'M \neq 0 \quad \text{for } \frac{x}{u} \neq 0. \quad (x \in u, u \in U)$$

Can even just check when  $u = 1$ .

Of course,  $x \neq 0$  in N.

Consider  $\{\text{ann}_R(ux) : u \in U\} =: \Sigma$

Note  $ux \neq 0$  since  $\frac{x}{1} \neq 0$ .

Noetherian  $\Rightarrow \Sigma$  has maximal element(s).  
(proper ideals.)

Replace  $x$  by a suitable  $ux$  to ensure  
that  $\text{ann}_R(x)$  is a maximal el't of  $\Sigma$ .

$$0 \neq M \cap Rx = (r_1x, \dots, r_nx), \quad r_i \in R.$$

Say  $r \in R$  is s.t.  
 $rx = 0$  in  $U^*N$ .

$\Rightarrow ux = 0$  for some  $u \in U$ , i.e.,

$$\Rightarrow r \in \text{ann}_R(ux) \quad \dots$$

$T$  always contains  $\text{ann}_R(x)$ .

$$\therefore r \in \text{ann}_R(x).$$

Now, pick  $r_i \in \{r_1, \dots, r_n\}$  s.t.  $r_i x \neq 0$ .

Then,  $r_i x \in U^*M \cap S(\underline{x}) \setminus \{0\}$ .  $\blacksquare$

Lemma: Fix  $\mathfrak{p} \in \text{Spec } R$ .

①  $E_R(R/\mathfrak{p})$  is indecomposable.

②  $\text{Ass}_R(E_R(R/\mathfrak{p})) = \{\mathfrak{p}\}$ .

③  $E_{\mathfrak{p}}(R/\mathfrak{p})$  is  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -power torsion.

④  $\text{Hom}_{R/\mathfrak{p}}(k(\mathfrak{p}), E(R/\mathfrak{p})) \cong k(\mathfrak{p})$ .

Proof:

① A stronger property holds: any two nonzero modules in  $E_{\mathfrak{p}}(R/\mathfrak{p})$  has nontrivial intersection.  
 $(\equiv$  any submodule is indecomposable)

Let  $M, N \neq 0$  be nonzero submodules.

$R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p})$  essential.

So,  $M \cap R/\mathfrak{p} \neq 0$ ,

$$N \cap R/p \neq 0.$$

$$(M \cap R/p) \cap (N \cap R/p) = 0$$

$\hookrightarrow$  no more ideals in domain

$$\therefore M \cap N = 0.$$

This proves ①

$$\begin{array}{ccccccc} \cdot & R/p & \hookrightarrow & k(p) & \hookrightarrow & E_{R/p}(k(p)) \\ & \text{ess} & & \text{ess} & & \hookrightarrow \text{inj } R/p \\ & & & & & \Rightarrow \text{inj } R \end{array}$$

$$\therefore E_R(R/p) = E_{R/p}(k(p)) \supset \text{previous}$$

$$\cong R_p \otimes_R E_R(k(p)).$$

① follows from:  $\forall M \in \text{Mod } R$

$$\text{Ass}_R(E_R(M)) = \text{Ass}_k(M)$$

(2) true since  $M \hookrightarrow E_R(M)$ .

( $\subseteq$ ) Let  $p \in \text{Ass}_R(E_R(M))$ .

$R/p \hookrightarrow M$ . Let  $U \cong R/p$  be the image.

$$U \cap M \neq 0.$$

$\therefore \text{Ass}(U \cap M)$  has an associated prime.

But  $\text{Ass}(U \cap M) \subseteq \text{Ass}(U) = \{p\}$ .

$\therefore p \in \text{Ass}(U \cap M) \subset \text{Ass}(M)$ .  $\blacksquare$

This also implies  $p$ -power torsion.

④ May assume  $R$  local and  $p$  max'l.

④ May assume  $R$  local and  $\mathfrak{p}$  max'l.

$$\text{Ntp} \quad \text{Hom}_R(k, E_{R(k)}) \cong k.$$

The Hom is  $\neq 0$  since  $i: k \hookrightarrow E_R(k)$   
is in the Hom.

Moreover,  $\text{Hom}_R(k, E_R(k))$  is a  $k$ -subspace

of  $E_R(k)$ . But it's indecomposable  
by our earlier statement.

$\therefore$  it is  $k$ .  $\blacksquare$

---

Thm. Any injective can be written as a direct sum  
of indecomposable injectives.

Proof  $E \rightarrow$  inj.  $R$ -module.

Pick  $\mathfrak{p} \in \text{Ass}_R(E)$ .  $\therefore R/\mathfrak{p} \hookrightarrow E$ . This factors:

$$R/\mathfrak{p} \hookrightarrow E_{R(R/\mathfrak{p})} \hookrightarrow E.$$

Now ...  $\blacksquare$

This finishes proof all of the main properties!

---

Uniqueness?

Suppose

$$E \cong \bigoplus_{\mathfrak{p} \in \text{Spec } R} E(R/\mathfrak{p})^{v(\mathfrak{p})}.$$

$\not\cong$  iff  
 $\mathfrak{p} \in \Delta \subset I$

Then,  $v(p) = \text{rank}_{k(p)} \text{Hom}_{R_p}(k(p), E_p)$   $\xrightarrow{\text{iff}} p \in \text{Ass}_R E$ .  
 and hence independent of decomposition.

Key:  $\text{Hom}_{R_p}(k(p), E(R/q)_p) = \begin{cases} k(p) & p = q \\ 0 & \text{else} \end{cases}$   $\xrightarrow{\text{by earlier}}$

Say  $q \neq p$ .

If  $q \neq p$ ,  $E(R/q)$  is  $q$ -pow torsion  
 but  $\exists a \in q \setminus p$ , so  
 $a$  is inverted but kills  
 $E(R/q)$ .

Use the following:

$$\text{Hom}_{R_p}(k(p), M) \neq 0 \iff p \in \text{Ass}(M).$$

$$M \hookrightarrow E_R(M) = \bigoplus_{p \in \text{Ass}_R(M)} E(R/p)^{\mu(p)}$$

$$\mu(p) = \text{rank}_{R_p} \text{Hom}_{R_p}(k(p), E_R(M)_p)$$

$$= \text{rank}_{k(p)} \text{Hom}_{R_p}(k(p), M_p).$$

In particular, if  $M$  is f.g., then  $\mu(p)$  is finite.

Defn.  $M \in \text{Mod } R$ . The  $i^{\text{th}}$  Bass number of  $M$  is wrt  $p$

$$\mu_e^i(n; M) := \text{rank}_{k(n)} \text{Ext}_R^i(k(p), M_p).$$

$$\mu_R^i(p; M) := \text{rank}_{k(p)} \text{Ext}_{R_p}^i(k(p), M_p).$$

Let  $M \xrightarrow{\sim} I$  be a min'l inj. resolution.

Then

$$I^i \cong \bigoplus E(R/p)^{\mu^i(p; M)}.$$