Lecture 1 (22-08-2022)

22 August 2022 10:43

Lee's — Introduction to Smooth Manifolds.

http://www.math.utah.edu/~wortman/6510.pdf

- Weekly homeworks (Friday - Friday)
- 1 Midterm
- 1 Final (Similar to Quals)

Def^n. A (topological) n-manifold is a Hausdorff, second-countable topological space M with the following property: for every $p \in M$, $\exists$ nbd $U \subseteq M$ of p, $\exists\, V \subseteq \mathbb{R}^n$ nbd and a homeomorphism $\varphi : U \to V$.

Compatibility? Suppose that $\varphi_i : U_i \to V_i$ $(i = 1, 2)$ are homeomorphisms as mentioned above.

transition maps

Note : $\varphi_2 \circ \varphi_1^{-1}$ is a homeomorphism.

defined on $\varphi_1(U_1 \cap U_2)$

Was not part of any definition!

Later, we will need to talk about diffeomorphisms. (As of now, makes no sense to talk about φ_1, φ_2 being diffeos.)

Lecture 2 (24-08-2022)

Wednesday, August 24, 2022 10:44 AM

Recall: $F: U \to \mathbb{R}^m$ ($U \subseteq_{op} \mathbb{R}^n$) is differentiable at $x \in \mathbb{R}^n$ if $\exists L: \mathbb{R}^n \to \mathbb{R}^m$ linear s.t.

$$\lim_{v \to 0} \frac{f(x+v) - f(x) - L(v)}{\|v\|} = 0.$$

We denote $L = Df(x)$.

We say f is continuously differentiable if f is differentiable at every $x \in U$ and $x \mapsto Df(x)$ is continuous. (Note: the space of linear transforms $\mathbb{R}^n \to \mathbb{R}^m$ has a natural topology.)

Thm. F is continuously differentiable $\iff$ every partial derivative of F exists and is continuous.

Thm (Inverse Function Theorem) Suppose $U \subseteq \mathbb{R}^n$ open and $F: U \to \mathbb{R}^n$ is differentiable in a nbd of x such that Df is continuous and invertible at x.

Then, F is invertible on a nbd of x and

$$D(F^{-1})(F(x)) = (DF(x))^{-1}.$$

Defn. Let $U, V \subseteq \mathbb{R}^n$ be open sets.

$F: U \to V$ is a diffeomorphism if

- F is a bijection,
- F and F^{-1} are continuously differentiable.

Non-example: $x \mapsto x^3$ is a cont. diff. bijection from $\mathbb{R}$ to $\mathbb{R}$. But the inverse is not diff at 0.

Corollary If $U \subseteq \mathbb{R}^n$ is open, $F : U \to \mathbb{R}^n$ is a bijection onto its image, F is C^∞, and $DF(x)$ is invertible for all $x \in U$, then F is a diffeo onto its image.

Thm (Chain Rule)
If
$$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{G} \mathbb{R}^d$$
are differentiable, then $G \circ F$ is differentiable and
$$D(G \circ F)(x) = DG(F(x)) \circ DF(x),$$
for $x \in \mathbb{R}^n$.

(Can write a local version...)

Recall. A topological n-manifold is a space M which is
1. Hausdorff,
2. second-countable,
3. locally Euclidean.

Defn. Let M be a topological n-manifold.
An atlas of charts is any collection
$$\mathcal{A} = \{ \underbrace{(U_\alpha, \varphi_\alpha)}_{\text{chart}} : \alpha \}$$
Here, $U_\alpha \subseteq M$ are open and $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ are homeo onto the image.

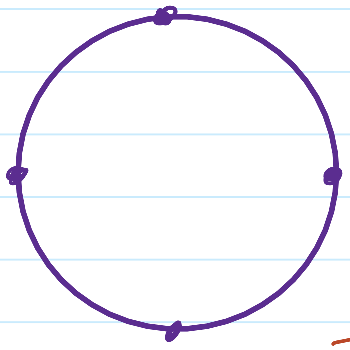
and $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ are homeo onto the image.

such that $\bigcup_\alpha U_\alpha = M$.

The atlas is smooth if $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are C^∞ (diffeos) whenever $U_\alpha \cap U_\beta \neq \emptyset$.

$\downarrow$ follows automatically

Example. The following is a smooth atlas for S^1.



$$U_{h,+} = \{x \in S^1 : x_1 > 0\},$$
$$U_{h,-} = \{x \in S^1 : x_1 < 0\},$$
$$U_{v,+} = \{x \in S^1 : x_2 > 0\},$$
$$U_{v,-} = \{x \in S^1 : x_2 < 0\}.$$

The inverse of the φs are given as:

$$\varphi_{h,+}^{-1}(t) = (\cos t, \sin t) \qquad t \in (-\pi/2, \pi/2),$$
$$\varphi_{h,-}^{-1}(t) = (\cos t, \sin t) \qquad t \in (\pi/2, 3\pi/2),$$
$$\varphi_{v,+}^{-1}(t) = (\cos t, \sin t) \qquad t \in (0, \pi),$$
$$\varphi_{v,-}^{-1}(t) = (\cos t, \sin t) \qquad t \in (\pi, 2\pi).$$

Check $\varphi_{\dots} \circ \varphi_{\dots}^{-1}$ is smooth whenever there are overlaps. They will either be the id or $\mathrm{id} \pm 2\pi$.

Defn. A smooth atlas is called maximal if it is not properly contained in any smooth atlas. A maximal atlas is also called a smooth structure.

Propn. (1.17. in Lee) If $\mathcal{A}$ is a smooth atlas on M, there is a unique smooth structure on M containing $\mathcal{A}$.

Prop. (1.17. in Lee) If $\mathcal{A}$ is a smooth atlas on M, there is a unique smooth structure on M containing $\mathcal{A}$.

Proof. Suppose $\mathcal{A}$ is a smooth atlas on M.
Let $\bar{\mathcal{A}}$ denote the set of charts (U, φ) such that if $U \cap U_\alpha \neq \phi$ (for some $U_\alpha \in \mathcal{A}$), then $\varphi_\alpha \circ \varphi^{-1}$ is an appropriate diffeo.

$\bar{\mathcal{A}}$ is the desired maximal smooth structure. (Check...) $\blacksquare$

Lecture 3 (26-08-2022)

Friday, August 26, 2022 10:42 AM

Recall: · A topological n-manifold is
(1) Hausdorff,
(2) second countable,
(3) locally n-Euclidean.

· A smooth atlas is a collection of charts covering M with C^∞ transition maps.

· A smooth structure is a maximal smooth atlas (can always be constructed uniquely from a smooth atlas).

Warning: Manifolds forget a lot of information from their natural embedding in $\mathbb{R}^n$. [One goal: any manifold can be embedded in some $\mathbb{R}^N$.]
Examples of things not remembered: distances, directions (latitude, longitude, etc.)

- What is remembered is: which functions are smooth.
- Tangent spaces

Def: Let M $= (M, \mathcal{A})$ be a smooth n-manifold. A k-foliation on M is determined by a special smooth atlas of charts $\mathcal{F} \subseteq \mathcal{A}$ such that

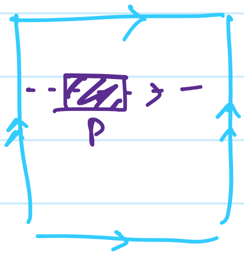
$\varphi, \psi \in \mathcal{F} \Rightarrow \forall x \in \mathbb{R}^{n-k} \ \exists y \in \mathbb{R}^{n-k}$ s.t. $(\varphi \circ \psi^{-1})(\mathbb{R}^k \times \{x\}) \subseteq \mathbb{R}^k \times \{y\}$.

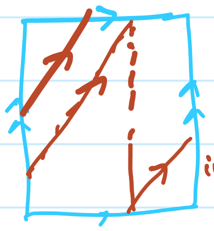
$$\left[\begin{array}{l} \varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k} \\ \psi : V \to \mathbb{R}^k \times \mathbb{R}^{n-k} \end{array}\right]$$

$\psi^{-1}(\mathbb{R}^k \times \{x\})$ is a local leaf

for $p \in M$: $L(p)$ = collection of points for $p \in M$: $L(p)$ = collection of points reached through paths contained in local leaves.

$T^2 \cong S^1 \times S^1$



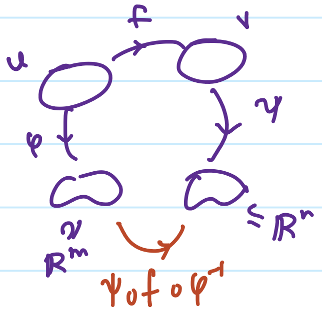


irrational slope

Lecture 4 (29-08-2022)

Monday, August 29, 2022 10:36 AM

Def^n. Let M, N be manifolds.
A continuous map $f: M \to N$ is called smooth (smooth) (or C^∞) if for any pair of charts (U, φ) and (V, ψ) for M and N, respectively such that $f(U) \subseteq V$, we have that



$\psi \circ f \circ \varphi^{-1}$ is C^∞.

(This makes sense since this map is between open subsets of Euclidean space.)

Alt. def^n. For every $p \in M$,
- $\exists$ chart (U, φ) s.t. $p \in U$,
- $\exists$ chart (V, ψ) s.t. $f(p) \in V$, $f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}$ is C^∞.

Def^n f is a diffeomorphism if
- f is bijective,
- f is C^∞,
- f^{-1} is C^∞.

Concept of isomorphism in the category of smooth manifolds

Remark. Whenever M is a smooth manifold with smooth structure $\mathcal{S}$ and $f: M \to N$ is a homeo,

$$f_*(\mathcal{S}) = \{ (f(U), \varphi \circ f^{-1}) : (U, \varphi) \in \mathcal{S} \}$$

is a smooth structure on N.

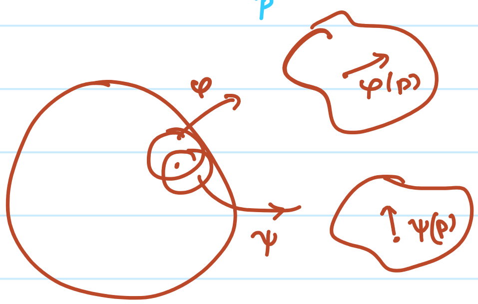
Tangent Spaces

Def^n. Let M be a smooth manifold, and $p \in M$.
Let (U, φ) be a chart containing p.

Let $T_p^{\varphi} := T_{\varphi(p)}(\mathbb{R}^n) := \mathbb{R}^n$.

Finally $T_p(M) := \coprod_{\varphi} T_p^{\varphi} M \big/ \sim$.

$$\underset{\substack{\cap \\ T_p^{\varphi}}}{v} \sim \underset{\substack{\cap \\ T_p^{\psi}}}{w} \iff D\left(\psi \circ \varphi^{-1}\right)(\varphi(p))(v) = w.$$



Remark. $T_p M$ is a vector space and isomorphic to $\mathbb{R}^n$.

Def^n. The tangent bundle is the set $TM := \coprod_{p \in M} T_p M$.

Next week's HW:
- TM has a canonical smooth manifold structure induced from M.
- TM is a "vector bundle"

induced from M.

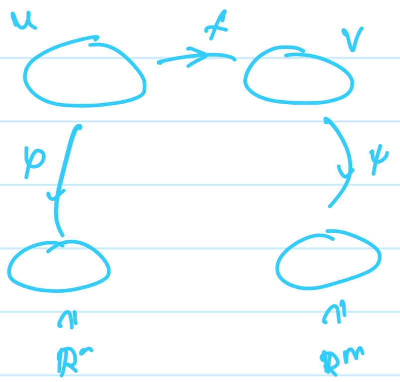
- TM is a "vector bundle".
- TM is not always diffeomorphic to $M \times \mathbb{R}^n$.
  (If so, then the manifolds are called "parallelizable".)

Defn. If $f: M \longrightarrow N$ is a C^∞ function, and $p \in M$, then

$$Df(p) : T_pM \longrightarrow T_{f(p)}N$$

is the linear transformation given by

$$Df(p) = D(\psi \circ f \circ \varphi^{-1}) \text{ acting on } T_p^{\varphi}(M).$$



Claim: $Df(p)$ is well defined.

Proof Let $\hat{\varphi}$ and $\hat{\psi}$ be different charts s.t. ...

$$v \in T_p^{\varphi}M \quad \sim D(\hat{\varphi} \circ \varphi^{-1})(\varphi(p))(v) = \hat{v}$$

To show: $D(\psi \circ f \circ \varphi^{-1})(\varphi(p))(v)$
$$= D(\hat{\psi} \circ f \circ \hat{\varphi}^{-1})(\hat{\varphi}(p))(\hat{v})$$

$$w := D(\psi \circ f \circ \varphi^{-1})(\varphi(p))(v)$$
$$= D(\psi \circ f \circ \hat{\varphi}^{-1})(\hat{\varphi}(p))(\hat{v}) \quad \text{chain rule}$$

$$\hat{w} := D(\hat{\psi} \circ \psi^{-1})(\psi(f(p)))(w) \quad \sim w$$
$$= D(\hat{\psi} \circ f \circ \hat{\varphi}^{-1})(\hat{\varphi}(p))(\hat{v}).$$

Lecture 5 (31-08-2022)

31 August 2022 10:44

Manifolds & (Discrete) Group Actions

$\Gamma \rightsquigarrow$ group (countable)
$M \rightsquigarrow$ smooth manifold

$$\alpha : \Gamma \longrightarrow \mathrm{Diff}^\infty(M)$$

$= \mathrm{Aut}(M)$ all C^∞ diffeos from M to M (composition)

group homomorphism

We denote the above by $\alpha : \Gamma \curvearrowright M$ or $\Gamma \overset{\alpha}{\curvearrowright} M$.

Ex: • $\mathbb{R}^2 \curvearrowright \mathbb{R}^2$
$v \longmapsto T_v$. $\quad T_v(x) = x + v$.

(We will restrict to countable groups though.)

• Similarly $\mathbb{Z} \curvearrowright \mathbb{R}$, $n \mapsto T_n$.

Defn If $\Gamma \curvearrowright M$, then the orbit of $x \in M$ is given by

$$\Gamma \cdot x = \{\gamma(x) : \gamma \in \Gamma\}.$$

If $\Gamma \cdot x$ is discrete $\forall x$, then the action is said to be free.

Idea: Think about points of M/Γ as orbits of Γ in M.

Defn. If $\Gamma \curvearrowright X$, then the stabiliser of $x \in M$ is the subgroup

$$\mathrm{Stab}_\Gamma(x) = \{\gamma \in \Gamma : \gamma \cdot x = x\}.$$

The action is free if $\mathrm{Stab}_\Gamma(x) = \{e_\Gamma\}$ for all $x \in X$.

Examples. The earlier examples of translation were free.

Defn. A group action $\Gamma \curvearrowright M$ is called properly discontinuous if $\forall p \in M$ $\exists U$ nbd of p s.t.

$$(\gamma \cdot U) \cap U \neq \emptyset \iff \gamma = e_\Gamma.$$

Remark. Properly discontinuous $\Rightarrow$ free + discrete
$\not\Leftarrow$ Consider $\mathbb{R}^2 \circlearrowleft \mathbb{R}^2$ as earlier.
free $\Rightarrow$ faithful.

Example. let $\mathbb{Z}/4 \circlearrowleft \mathbb{R}^2$ by rotation.
↳ Faithful, not free (origin fixed).

Q. Free + discrete $\overset{?}{\Rightarrow}$ properly discontinuous

Theorem. If $\Gamma \circlearrowleft M$ properly discontinuously, $(\Gamma \to \mathrm{Diff}^\infty M)$ then

$$M/\Gamma = \{\Gamma \cdot x \;:\; x \in M\}$$

inherits a smooth structure from M.
Furthermore, $\dim(M/\Gamma) = \dim(M)$.

Proof. Fix $[x] \in M/\Gamma$.
Choose a nbd $\mathcal{U}^{\subseteq M}$ of x as given by Γ acting prop. disc.
By shrinking $\mathcal{U}$ as necessary, we can assume $\mathcal{U}$ is the domain of a chart $(\mathcal{U}, \varphi)$.

If $[y] \cap \mathcal{U} \neq \emptyset$, then $\exists!\ y \in [y]$ s.t. $y \in \mathcal{U}$.
Thus, we get the following map:

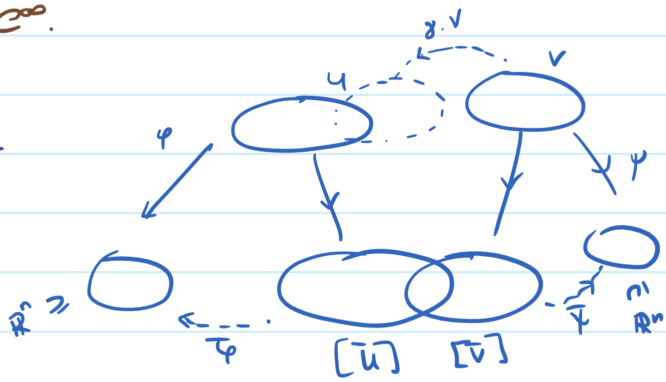
$$\bar{\varphi} : [\mathcal{U}] \longrightarrow \mathbb{R}^n \qquad (\{y\} = [y] \cap \mathcal{U})$$
$$[y] \longmapsto y$$

$$\left([\mathcal{U}] := \{[y] : [y] \cap \mathcal{U} \neq \emptyset\}\right)$$

Claim: If $\bar{\varphi}$ and $\bar{\psi}$ are charts constructed as above, and their domains $[\mathcal{U}]$ and $[V]$ intersect, then $\bar{\varphi} \circ \bar{\psi}^{-1}$ is C^∞.

$\gamma \cdot V$

and their domains $[u]$ and $[v]$ intersect, then
$\bar\varphi \circ \bar\psi^{-1}$ is C^∞.



Proof. Choose lifts U and V of $[U]$ and $[V]$ in M.
Since $[U] \cap [V] \neq \phi$,
$(\Gamma \cdot U) \cap V \neq \emptyset$.

Choose $\gamma \in \Gamma$ s.t. $(\gamma \cdot U) \cap V \neq \emptyset$.

Claim: $[\gamma \cdot (U \cap V)] = [U] \cap [V]$. ⊠

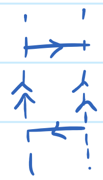
Then, $\bar\varphi \circ \bar\psi^{-1} = \varphi \circ \gamma \circ \psi^{-1}$ which is smooth. ⊠

We are now done. ⊠

Examples ① $\mathbb{R}^n / \mathbb{Z}^n \to$ n-torus

② $F : (-1,1) \times \mathbb{R} \longrightarrow (-1,1) \times \mathbb{R}$
$(t, s) \longmapsto (-t, s+1)$

$(-1,1) \times \mathbb{R} / \langle F \rangle \to$ Möbius strip



③ $S^n / \{\pm id\} = \mathbb{RP}^n$

Lecture 6 (02-09-2022)

02 September 2022 10:45

Inheriting smooth structures: Conditions on differentials

$F: M \longrightarrow N$ is a C^∞ map.

1) F is an immersion if $DF(x)$ is injective for all $x \in M$.

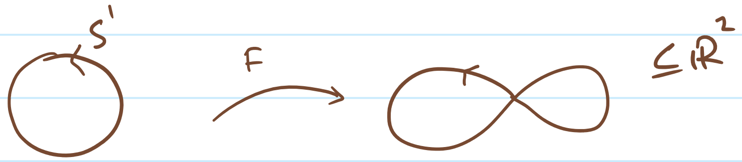
2) F is an submersion if $DF(x)$ is injective for all $x \in M$.

3) F is a local diffeomorphism if it is both.

Remark. Inv. FT gives that "local diffeos are local diffeos."

Defn A subset $M \subseteq N$ is called an immersed submanifold if it is the image of an immersion. If the map is a homeomorphism onto its image, M is called embedded.

Example:



immersion image not a manifold!

this is injective!

immersion image still not a manifold!

Submersion

Submersion Theorem: Let $f: M \longrightarrow N$ be a submersion. Then, for for all $y \in N$, $F^{-1}(y)$ is an embedded submanifold. $\dim(F^{-1}(y)) = \dim(M) - \dim(N)$. (Proof next week.)

Lecture 7 (07-09-2022)

Wednesday, September 7, 2022 10:38 AM

① $X \times Y \longrightarrow X$, $(x, y) \longmapsto x$ submersion ✓

② $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto x^2 - y^2$ not submersion (because of $(0,0)$)

③ $X = S^3 \subseteq \mathbb{C}^2 \longrightarrow Y = S^2 \subseteq \mathbb{C} \times \mathbb{R}$
$(z_1, z_2) \mapsto (2 z_1 \bar{z}_2, \; |z_1|^2 - |z_2|^2)$ also a submersion ✓ (a Hopf fibration)

Defn. Let $p : X \longrightarrow Y$ be a C^∞ map.
$y \in Y$ is a regular value if p is a submersion at x for all $x \in p^{-1}(y)$.

Theorem. If $\pi : X \longrightarrow Y$ is a submersion, then for every $y \in Y$, $\pi^{-1}(y)$ is a(n embedded) submanifold and form the leaves of a foliation.
(For just the "embedded submanifold" part, it suffices for π to be C^∞ and y a regular value.)

Proof. By working in coordinates, it suffices to prove that if $p : U \longrightarrow \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$, is C^∞ s.t. $Dp(x)$ is onto for all $x \in U$, then $\{p^{-1}(y) : y \in p(U)\}$ is a foliation of U.

Fix $x \in U$, and let $L_x = \ker(Dp(x))$.
$\because Dp(x)$ is onto, $\dim(L_x) = n - m$.
Choose a chart φ at x s.t. $L_x = \operatorname{span}\{e_{m+1}, \dots, e_n\}$.
Define $G(y) = (p(y), \pi(y))$, $G : U \longrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}$
where $\pi(y_1, \dots, y_n) = (y_{m+1}, \dots, y_n)$.

define $G(y) = (p(y), \pi(y))$

where $\pi(y_1, \dots, y_n) = (y_{m+1}, \dots, y_n)$.

$$DG(x) = \begin{array}{c|cc} & m & m-n \\ \hline m & & \\ m-n & & \end{array} \left(\begin{array}{c:c} \underset{\text{invertible}}{*} & 0 \\ \hdashline 0 & I \end{array} \right)$$

Thus, $\det(DG(x)) \neq 0$ and DG is inv. at x.

By C^∞-ness, DG is inv. on a nbd of x.

By Inverse Funct. Theorem, $\exists$ local inverse H,

which will be our desired foliation chart.

$H: V \longrightarrow \tilde{U} \subseteq \mathcal{U}$, $V \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$

<u>Claim</u>: $H(\{y\} \times \mathbb{R}^{n-m}) = \{x \in \tilde{U} : p(x) = y\}$.

<u>Proof.</u> ($\subseteq$) $x \in H(\{y\} \times \mathbb{R}^{n-m})$

$\Rightarrow G(x) \in \{y\} \times \mathbb{R}^{n-m}$

$\Rightarrow p(x) = y$.

($\supseteq$) $p(x) = y \Rightarrow G(x) = (y, *)$

$\Rightarrow \underset{\substack{\parallel \\ x}}{H(G(x))} = H((y, *))$

$\blacksquare$

Sard's Theorem

Measures of critical/regular values.

will come to this later...

Key application: immersed non-open manifold will have zero measure.

Vector Fields

$$TM = \bigsqcup_{p \in M} T_p M.$$

$\pi : TM \longrightarrow M$

$\pi(v) =$ basepoint of v.

Smooth structure on TM defined in HW.

Example. $TS^1 \overset{\text{diffeo}}{\cong} \mathbb{R} \times S^1$

$TS^2 \not\cong \mathbb{R}^2 \times S^2$

Def. If M is a manifold, a vector field is a C^∞ section $X : M \longrightarrow TM$, i.e., $\pi(X(p)) = p \quad \forall p \in M$.

Things to do with vector fields:

1. Integrate them (flows)
2. Lie Brackets (infinitesimal commutator of flows)
3. Closed subalgebras (Lie group actions)
4. Frobenius theorem (detect foliations through subbundles)

Lecture 8 (09-09-2022)

Friday, September 9, 2022 10:37 AM

Smooth Vector Bundles

Def^n A smooth n-dimensional vector bundle ξ^n is a triple

$$\xi = (\underbrace{E}_{\text{total space}}, \overbrace{B}^{\text{base}}, p),$$

where E and B are smooth manifolds, $p: E \to B$ is a smooth map, and each fiber $\xi_b = p^{-1}(b)$ is equipped with the structure of a (real) vector space of dimension n s.t. the following local triviality condition holds:

- every $b \in B$ has a nbd U (trivializing nbd) and there is a diffeomorphism

$$\Phi: p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "p|_{p^{-1}(U)}"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

commutes

and $\Phi: p^{-1}\{x\} \longrightarrow \{x\} \times \mathbb{R}^n$ is a vector space isomorphism for all $x \in U$.

Exercise p above is a submersion. (Check locally using trivializing nbd.)

trivializing nbd.)

Examples ① Trivial bundle, $E = B \times \mathbb{R}^n$.

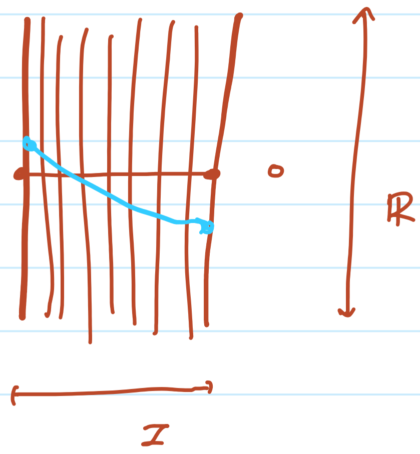
p is projection.

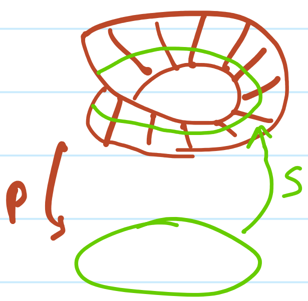
Take $U = B$ for all points.

② Start with $I \times \mathbb{R}$.

Let $E = I \times \mathbb{R} / (0,t) \sim (1,-t)$.

$B = (0,1) / 0 \sim 1 \approx S^1$.





Defn. A section of a vector bundle $\xi = (E, B, p)$ is a smooth map $s : B \to E$ so that $p \circ s = id_B$.

Example: Zero section: $s(b) := 0 \in \xi_b$.

Check this is smooth. (Local again!)

Defn Sections $s_1, \dots, s_k$ of ξ are linearly independent if for all $b \in B$: $s_1(b), \dots, s_k(b)$ are lin. indep. in ξ_b.

Example: $B \times \mathbb{R}^n \to B$ has n. lin. indep sections:

$$s_i : B \to B \times \mathbb{R}^n$$
$$b \mapsto (b, e_i).$$

Defn. Two bundles ξ, ξ' over the same base are said

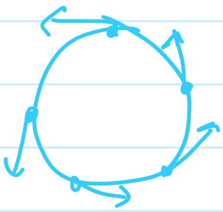
Defn. Two bundles ξ, ξ' over the same base are said to be **isomorphic** if $\exists$ diffeo $\Phi : E(\xi) \xrightarrow{\sim} E(\xi')$ s.t.

$$\Phi|_{\xi_b} : \xi_b \longrightarrow \xi'_b \quad \text{is an iso of vector spaces } \forall b \in B.$$

- A bundle is said to be **trivial** if it is isomorphic to the trivial bundle.

Lemma. $p : E \to B$ n-dim'l bundle.
p is trivial $\iff$ p has n-linearly independent sections.

Example. For 1-dim'l, this just means a nonzero section.



TS^1 is trivial.

Proof of Lemma. ($\Rightarrow$) is clear (since trivial bundle has n ...).

($\Leftarrow$) Let $s_1, \ldots, s_n$ be linearly independent sections.
Define

$$\Phi : B \times \mathbb{R}^n \longrightarrow E$$
$$(b, (t_1, \ldots, t_n)) \longmapsto \sum_{i=1}^{n} t_i s_i(b) \in \xi_b.$$

Φ is smooth. ✓
Bijection. ✓ (Isomorphism of fibers)

To check diffeo: we work in trivialising nbds.
By shrinking etc. assume B is a trivialising nbd

To check diffeo: we work in trivialising nbds.
By shrinking etc, assume B is a trivialising nbd, we may assume $E \to B$ is also trivial, i.e. $E = B \times \mathbb{R}^n$.

$$\Phi(b, v) = (b, F(b)_v),$$

where $F : B \longrightarrow GL_n(\mathbb{R})$ has columns $s_1, \dots, s_n$. So the coefficients of F are smooth functions $B \longrightarrow \mathbb{R}$.

Moreover, Φ^{-1} is smooth (we can compute it explicitly and it comes to $^{-1} : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ being smooth). ☒

Tangent Bundle

X : $(n-)$ manifold
Recall: A tangent vector v at $x \in X$ is an equivalence class of triples (U, φ, w), where (U, φ) is a chart around x, and $w \in \mathbb{R}^n$.

$(U, \varphi, w) \sim (U', \varphi', w')$ if
$$D(\varphi' \varphi^{-1})(\varphi(x))(w) = w'.$$

We write $d\varphi(x) = w$.

This collection is denoted $T_x X$. tangent space at x

Defn: $TX = \coprod T_x X$.

Define $\qquad TX = \coprod_{x \in X} T_x X.$

- $p : TX \longrightarrow X,$
  $(x, v) \longmapsto x.$ $\longrightarrow$ the tangent bundle!

Charts on TX: Homework.

Def^n A vector field on X is a smooth section of the tangent bundle.

Thm TS^2 is nontrivial; in fact, every vector field on S^2 has a zero. (So, there is not even one lin. indep. section, let alone two.)

Lecture 9 (12-09-2022)

12 September 2022 10:42

Derivations

X : smooth manifold

$C^\infty(X) = \{ f : X \to \mathbb{R} \mid f \text{ smooth} \}$ is an algebra.

Defn. A derivation at a point $p \in X$ is a linear map

$$D : C^\infty(X) \longrightarrow \mathbb{R}$$

s.t.
$$D(fg) = (Df) \cdot g(p) + f(p) \cdot (Dg),$$

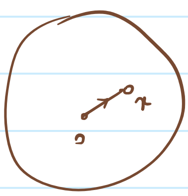
and if $f \equiv g$ on a nbd of p, then $D(f) = D(g)$.
The space of derivations is denoted $\mathrm{Der}(p)$.

Example. If $v \in T_p X$, then ∂_v is a derivation.

Theorem 1.
$$T_p X \longrightarrow \mathrm{Der}(p)$$
$$v \longmapsto \partial_v$$
is an isomorphism.

Lemma 2. Let $U \subseteq \mathbb{R}^n$ be open and convex with $0 \in U$.
Let $f : U \to \mathbb{R}$ be smooth with $f(0) = 0$.
Then,
$$f(x) = \sum_{i=1}^{n} x_i \cdot g_i(x), \quad \text{where } g_i : U \to \mathbb{R} \text{ are smooth,}$$
and $g_i(0) = \dfrac{\partial f}{\partial x_i}(0)$.

Proof.



Fix $x \in U$.
Define $g : [0,1] \longrightarrow \mathbb{R}$ by
$$g(t) = f(tx).$$
Then,
$$g(1) - g(0) = \int_0^1 g'(t)\,dt$$

Then,

$$\underset{\displaystyle f(x)}{\overset{\|}{g(1)}} - \underset{\displaystyle \cancel{f(0)}_{0}}{g(0)} = \int_0^1 g'(t)\,dt = \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)\,dt$$

$$\| $$

$$f(x) \qquad = \sum_{i=1}^n x_i \underbrace{\left[\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt \right]}_{=:\, g_i(x)}$$

$$\Rightarrow f(x) = \sum_{i=1}^n x_i\, g_i(x). \qquad \square$$

<u>Proof of Theorem 1 for $X = \mathbb{R}^n$</u>: Can assume $p = 0$.

Note $D(1 \cdot 1) = D(1) + D(1) \Rightarrow D(1) = 0$.

$\therefore D(\text{constant}) = 0$ by linearity.

Let $f \in C^\infty(X)$.

By subtracting constant, we can assume $f(0) = 0$.

Then,

$$f(x) = \sum x_i\, g(x), \quad \text{as in Lemma 2.}$$

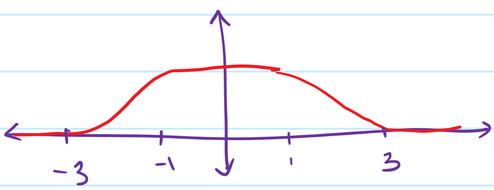
$$\Rightarrow Df = \sum_{i=1}^n \left[D(x_i)\, g_i(0) + 0 \right]$$

$$= \sum_{i=1}^n D(x_i) \cdot \frac{\partial f}{\partial x_i}(0).$$

Thus, $D = \partial_v$, where $v = \sum_{i=1}^n D(x_i) \frac{\partial}{\partial x_i}(0)$.

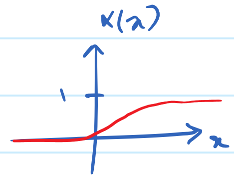
This shows surjective. Injectivity is clear. $\square$

<u>Lemma</u>: $\exists \rho : \mathbb{R} \to \mathbb{R}$ smooth, $\rho \geq 0$, $\rho \equiv 0$ outside $[-3,3]$, $\rho \equiv 1$ on $[-1,1]$.



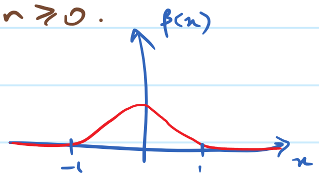
Proof. Define

$$\alpha(x) = \begin{cases} e^{-1/x} & ; \ x > 0, \\ 0 & ; \ x \le 0. \end{cases}$$

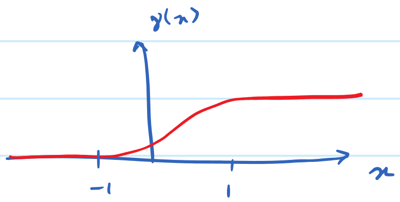


Then, α is C^∞ with $\alpha^{(n)}(x) = 0 \ \forall n \geq 0$.

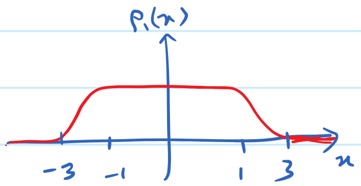
Put $\beta(x) = \alpha(1+x)\,\alpha(1-x)$.



$$\gamma(x) = \int_{-1}^{x} \beta.$$



$$\rho_1(x) = \gamma(2+x)\,\gamma(2-x)$$

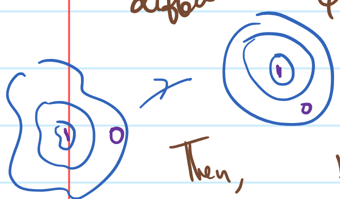


$$\rho(x) = \text{scale } \rho_1.$$

$\blacksquare$

Corollary. Let X be a manifold, $U \subseteq X$ open, $f: U \to X$ smooth. Let $p \in U$. $\exists \tilde{f}: X \to \mathbb{R}$ smooth that agrees with f on a nbd of p.

Proof. After shrinking and rescaling, we may assume that there is a diffeo $\varphi: U \to \varphi(U)$ s.t. $\varphi(U) \subseteq \mathbb{R}^n$ is an open ball of radius 4, $\varphi(p) = 0$.



Then, $\mu(x) := \rho(\|\varphi(x)\|)$ is a smooth function on U.

(Need to be careful since $\|\cdot\|$ is not smooth but here no problem since ρ is locally constant).

Now, extend μ to X by $\mu \equiv 0$ outside U.

Then, $\tilde{f} = \mu \cdot f$ works. $\blacksquare$

Partitions of Unity

Def^n X smooth manifold, $\mathcal{U}$ open cover of X.
A smooth partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\langle \phi_i : X \to \mathbb{R}_{\geq 0} \rangle_{i \in I}$ s.t.

(1) $\{ \mathrm{supp}(\phi_i) : i \in I \}$ is a locally finite collection of closed sets.

$\mathrm{supp}(\phi) = \overline{\{x : \phi(x) \neq 0\}}$.

locally finite: every $p \in X$ has a nbd that intersects only finitely many of the supports

(2) $\forall i : \exists U \in \mathcal{U}$ s.t. $\mathrm{supp}(\phi_i) \subseteq U$.

(3) $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in X$.

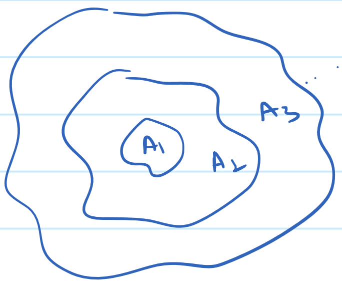
(for every x, the sum is a finite one)

Theorem. $\forall X \; \forall \mathcal{U} \; \exists$ smooth partition of unity subordinate to $\mathcal{U}$.

Sketch. Step 1. Find an exhaustion of X: compact sets $K_1 \subseteq K_2 \subseteq \cdots$ s.t. $K_i \subseteq K_{i+1}^{\circ}$, $\bigcup_{i=1}^{\infty} K_i = X$.

$\to \exists$ basis $V_1, V_2, \ldots$ s.t. $\overline{V_i}$ is compact.
Take $K_1 := \overline{V_1}$. Assume constructed till K_i.
Cover $K_i \subseteq V_1 \cup \cdots \cup V_m$ and then put
$K_{i+1} := \overline{V_1 \cup \cdots \cup V_m}$.

Step 2. Take $A_i := \overline{K_i \setminus K_{i-1}}$. (Put $K_0 := \emptyset$.)
Each A_i is compact and $\langle A_i \rangle_{i \geq 1}$ is locally finite.



Let $x \in A_i$. Choose a β-like function ϕ_x^i s.t. $\mathrm{supp}(\phi_x)$ is contained in some $U_x \in \mathcal{U}$ and is disjoint from A_j for $|j - i| > 1$.
Use compactness of A_i to find finitely many $\phi_{x_1}, \ldots, \phi_{x_{m_i}}$.

Now conclude ... ∎

Lecture 10 (14-09-2022)

Wednesday, September 14, 2022 10:35 AM

Example Cotangent bundle

$$\underset{T^*M!!}{E} = \{ (\varphi, x) \mid \varphi : T_xM \to \mathbb{R} \text{ is linear} \}.$$

vector bundle over M

$\pi : T^*M \to M$ is $(\varphi, x) \mapsto x$.

T^*M and TM are diffeomorphic, but not canonically.
(Essentially because $(\mathbb{R}^n)^* \cong \mathbb{R}^n$.)

Let $i : M \to \mathbb{R}^n$ be an immersion. Then, $\forall x \in M$,

$$\operatorname{Im}(D_i(x)) \subseteq T_{i(x)}\mathbb{R}^n = \mathbb{R}^n.$$

Let $E = \{ (x, v) : x \in M,\ v \in \operatorname{Im}(D_i(x))^\perp \}$.

E is the normal bundle (associated to i).

(n need not be $\dim(M)$.)

Thm. (Existence-Uniqueness for ODE solutions)
Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function.
Fix $x \in \mathbb{R}^n$. Then, $\exists \varepsilon > 0$ s.t.

$$\begin{cases} u(0) = x \\ u'(t) = V(u(t)) \end{cases}$$

has a unique sol^n $u : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$.

has a unique sol^n $u : (-\varepsilon, \varepsilon) \to \mathbb{R}$.

Thm. Let M be a manifold and V be a vector field on M. Then, there exists an open neighbourhood U of $\{0\} \times M \subseteq \mathbb{R} \times M$ and a function

$$\varphi : U \to M$$

such that

$$\frac{\partial}{\partial t} \varphi(t, x) = V(\varphi(t, x)).$$

Furthermore, $\boxed{\varphi(t+s, x) = \varphi(t, \varphi(s, x))}$ whenever defined.

flow equation

Common notation: $\varphi(t, x) =: \varphi_t(x)$.

Then,

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$$

or

$$\varphi_{t+s} = \varphi_t \circ \varphi_s.$$

Def^n Let V and W denote vector fields on a manifold M. If φ_t^V denotes the flow generated by V, the Lie derivative of W along V is

$$[V, W] := \frac{d}{dt} (\varphi_t^V)_* (W) \Big|_{t=0}.$$

this is again a vector field

$$(\varphi_t^V)_* (W)(x) = D\varphi_t^V (W(\varphi_{-t}^V(x)))$$

Lecture 11 (16-09-2022)

Friday, September 16, 2022 10:46 AM

Push-forwards

$F : M \longrightarrow N$ diffeomorphism

X = vector field on M

We define a vector field (v.f.) $F_*(X)$ on N.

$$F_*(X)(p) = DF(F^{-1}(p))\,(X(F^{-1}(p))), \qquad p \in N.$$

- X represented by a curve γ on M

  $\Downarrow$

  $F_*(X)$ is represented by $F \circ \gamma$

- X represented by derivation:

  $$f \in C^\infty(M) \Rightarrow X \cdot f \in C^\infty(M)$$

  Given $g \in C^\infty(N)$,

  $$F_*(X) \cdot g = [X \cdot (g \circ F)] \circ F^{-1}.$$

Recall: $X, Y \rightsquigarrow$ v.f.s on M

φ_t^X = flow generated by X

$$[X, Y] := \frac{d}{dt}\Big|_{t=0} (\varphi_t^X)_* (Y)$$

$$[X,Y](p) = \frac{d}{dt}\Big|_{t=0} (\varphi_t^X)_*(Y)(p).$$

$$[X,Y](p) = \frac{d}{dt}\Big|_{t=0} (\varphi_t^X)_*(Y)(p).$$

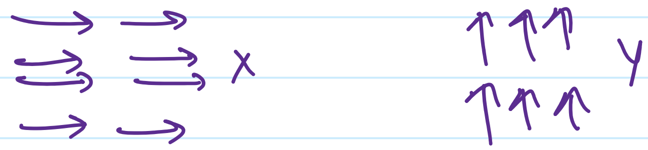
$\varphi_t^X : M \to M$ family of transformations s.t.

setting $\quad \gamma_x(t) := \varphi_t^X(x)$ gives

$$\gamma_x'(t) = X(\gamma_x(t)).$$

<u>Examples</u>. ① $X = \frac{\partial}{\partial x} \quad \left(= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ on $\mathbb{R}^2$

$Y = \frac{\partial}{\partial y} \quad \left(= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$



$$\varphi_t^X(x, y) = (x+t, y).$$

$$(\varphi_t^X)_*(Y)(x,y) = D\varphi_t^X(x-t, y)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} (\cdots) = 0.$$

$\therefore [X, Y] = 0.$ (X and Y are said to commute.)

② $X = \partial/\partial x + y\, \partial/\partial z \quad \left(= \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} \right)$ on $\mathbb{R}^3$

$Y = \partial/\partial y - x\, \partial/\partial z \quad \left(= \begin{pmatrix} 0 \\ 1 \\ \end{pmatrix} \right.$

$$Y = \partial/\partial y - x\,\partial/\partial z \quad \left(\simeq \begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix} \right)$$

$$\varphi_t^X(x, y, z) = (x+t,\ y,\ z+ty)$$

$$(\varphi_t^X)_*(Y)(x,y,z) = D\varphi_t^X(x-t,\ y,\ z-ty)\, Y(x-t,\ y,\ z-ty)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -x+t \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2t-x \end{pmatrix}$$

$$\therefore \quad [X, Y](x,y,z) = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 0 \\ 1 \\ 2t-x \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 2\frac{\partial}{\partial z}.$$

Understanding Lie derivatives via derivations:

Local coordinates:

$$Y = \sum y_i(s)\, \partial/\partial s_i$$

$$X = \sum x_i(s)\, \partial/\partial s_i$$

$$\big((\varphi_t^X)_* Y\big) \cdot f(s) = \Big(\big[Y \cdot (f \circ \varphi_t^X)\big] \circ \varphi_{-t}^X\Big)(s)$$

$$= \sum_i y_i(\varphi_{-t}(s))\, \frac{\partial}{\partial s_i}(f \circ \varphi_t^X)(\varphi_{-t}^X(s))$$

$$= \sum_i y_i(\varphi_{-t}(s)) \sum_j \frac{\partial f}{\partial s_j}(s) \frac{\partial}{\partial s_i}((\varphi_t^X)_j)(\varphi_{-t}^X(s))$$

$$= \sum_j \frac{\partial f}{\partial s_j}(s) \underbrace{\sum_i y_i(\varphi_{-t}(s)) \frac{\partial}{\partial s_i}((\varphi_t^X)_j)(\varphi_{-t}^X(s))}_{\frac{\partial}{\partial t}\big|_{t=0} \;\to\; I_i + II_i}$$

using Product rule

$$I_i = \sum_k \delta_{ij} (-x_k(s)) \frac{\partial y_i}{\partial s_k}(s)$$

$$II_i = y_i(s) \frac{\partial x_j}{\partial s_i}(s)$$

$$\therefore \frac{d}{dt}\Big|_{t=0} (Y\cdot(f \circ \varphi_t^X))(\varphi_{-t}^X(s))$$

$$= \sum_j \frac{\partial f}{\partial s_j}(s) \left(\sum_i y_i \frac{\partial x_j}{\partial s_i} - x_i \frac{\partial y_j}{\partial s_i} \right)$$

shows antisymmetry

$$\therefore [X, Y] = -[Y, X] \text{ and}$$

$$[X, Y] \cdot f = Y \cdot (X \cdot f) - X \cdot (Y \cdot f).$$

Back to earlier example: $X = \partial/\partial x + y\,\partial/\partial z$,

$Y = \partial/\partial y - x\,\partial/\partial z.$

$$Y \cdot X = \left(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial y \partial x} + \frac{\partial}{\partial z} + y\frac{\partial^2}{\partial y \partial z}$$

$$- x\frac{\partial^2}{\partial z \partial x} - xy\frac{\partial^2}{\partial z^2}$$

$$X \cdot Y = \left(\partial/\partial x + y\,\partial/\partial z\right)\left(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x \partial y} - \frac{\partial}{\partial z} - x\frac{\partial^2}{\partial x \partial z} + y\frac{\partial^2}{\partial z \partial y} - xy\frac{\partial^2}{\partial z^2}$$

Lecture 13 (21-09-2022)

Wednesday, September 21, 2022 10:39 AM

Recall: · An ℓ-distribution on M is an assignment to each $p \in M$ an ℓ-subspace $E(p) \subseteq T_p(M)$. (In a smooth manner.)

· A foliation on M is an atlas of charts $\mathcal{F}$ s.t. if $\varphi, \psi \in \mathcal{F}$, we have

$$(\varphi \circ \psi^{-1})(\mathbb{R}^\ell \times \{x\}) \subseteq \mathbb{R}^\ell \times \{\varphi(\psi^{-1}(x))\}$$

for all $x \in \operatorname{im}(\psi)$

· Theorem (Frobenius)

A distribution E is integrable (i.e., $E = T\mathcal{F}$ for some $\mathcal{F}$) $\iff$ $\forall$ v.f.s subordinate to E, $[X, Y]$ is subordinate to E.

· $T\mathcal{F}$ is the distribution given as

$$T\mathcal{F}(p) = \left\langle D\varphi^{-1}_{\varphi(p)}\left(\partial/\partial x_i\right) : i = 1, \dots, \ell \right\rangle,$$

where φ is a foliation chart.

Examples of distributions:

① $$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

- H is diffeomorphic to $\mathbb{R}^3$.

- 3 nice subgroups:

$$A = \left\{ \begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

isomorphic and diffeomorphic to $\mathbb{R}$.

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ & 1 & y \\ & & 1 \end{pmatrix} : y \in \mathbb{R} \right\}.$$

$$Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

"One-parameter subgroups"

Flow 1: $$\varphi_t^A(g) = \begin{pmatrix} 1 & t & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t+x & ty+z \\ & 1 & y \\ & & 1 \end{pmatrix}.$$

φ_t^A is generated by $V_A = \partial/\partial x + y\, \partial/\partial z$.

"^{ly} φ_t^B is defined and generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & t \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \qquad V_B = \partial/\partial y.$$

$$= \begin{pmatrix} 1 & x & z \\ & 1 & t+y \\ & & 1 \end{pmatrix}$$

$$V_Z = \partial/\partial z.$$

$E_{A,B} = \langle V_A, V_B \rangle$, $E_{A,z} = \langle V_A, V_z \rangle$, $E_{B,z} = \langle V_B, V_z \rangle$.

($\langle \cdot, \cdot \rangle$ denotes the subspace spanned.)

$E_{B,z}$ is integrable (using Frob. thm).
Just need to check

$$[f\,\partial/\partial y,\ g\,\partial/\partial z] \subseteq E_{B,z}.$$

We have

$$\left[f\,\partial/\partial y,\ g\,\partial/\partial z\right] = f\,\partial/\partial y\left(g\,\partial/\partial z\right) - g\,\partial/\partial z\left(f\,\partial/\partial y\right)$$

$$= f\frac{\partial g}{\partial y}\frac{\partial}{\partial z} + fg\frac{\partial^2}{\partial y\partial z} - fg\frac{\partial^2}{\partial y\partial z} - g\frac{\partial f}{\partial z}\frac{\partial}{\partial y}$$

$$= f\frac{\partial g}{\partial y}\frac{\partial}{\partial z} - g\frac{\partial f}{\partial z}\frac{\partial}{\partial y} \in \left\langle \partial/\partial z, \partial/\partial y \right\rangle.$$

Can check even $E_{A,z}$ is integrable by similar computation.

However, $[V_A, V_B] = -\partial/\partial z \notin \langle V_A, V_B \rangle$.

$\therefore E_{A,B}$ is NOT integrable.

Remark If $E_{A,B}$ is integrable to a foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ contain orbits of φ_X^A, φ_X^B.

$\Downarrow$

$\mathcal{F}(e) \supseteq$ subgroup generated by A, B.

Note that the Lie group commutator is

$$\left[\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ & 1 & y \\ & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & \pm xy \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in Z.$$

② $GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}$ → open subset of $\mathbb{R}$. canonical smooth structure (4-manifold)

$SL(2,\mathbb{R}) = \det^{-1}(\{1\})$. → 1 is a regular value of det.

$\therefore SL(2,\mathbb{R})$ is a 3-manifold

$$U := \left\{ \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$A := \left\{ \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$V := \left\{ \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

(U, A, V generate $SL(2,\mathbb{R})$.)

$$\varphi_t^u\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$

$$X_u = c\,\partial/\partial a + d\,\partial/\partial b.$$

$$X_v = a\,\partial/\partial c + b\,\partial/\partial d.$$

$$X_A = \frac{1}{2}\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d}\right).$$

$$\begin{pmatrix} e^{t/2} & \\ & e^{-t/2}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} ae^{t/2} & be^{t/2} \\ ce^{-t/2} & de^{-t/2}\end{pmatrix}$$

$$\frac{\partial}{\partial t}(ae^{t/2})\Big|_{t=0} = \frac{a}{2}e^{t/2}\Big|_{t=0} = a/2$$

$$[X_A, X_u] = \left[\frac{a}{2}\frac{\partial}{\partial a} + \frac{b}{2}\frac{\partial}{\partial b} - \frac{c}{2}\frac{\partial}{\partial c} - \frac{d}{2}\frac{\partial}{\partial d},\ c\frac{\partial}{\partial a} + d\frac{\partial}{\partial b}\right]$$

$$= \left(-\frac{c}{2}\frac{\partial}{\partial a} - \frac{d}{2}\frac{\partial}{\partial b}\right) - \left(\frac{c}{2}\frac{\partial}{\partial a} + \frac{d}{2}\frac{\partial}{\partial b}\right)$$

$$= -c\frac{\partial}{\partial a} - d\frac{\partial}{\partial b} = -X_u.$$

Check: $[X_A, X_v] = X_v$

$[X_u, X_v] = -2X_A$

$E_{u,v}$ not integrable.

$E_{A,u}$ and $E_{A,v}$ are, need to compute with f and g as before.

Lecture 15 (26-09-2022)

Monday, September 26, 2022 10:45 AM

Midterm exam: October 7th

Syllabus: Manifolds, tangent spaces, differentials / pushforwards, immersions / submersions, vfs, flows, Frobenius / Lie brackets, transversality.

Theorem (Frobenius)
A distribution E is integrable $\Leftrightarrow$ E is involutive.

Proof. ($\Rightarrow$) exercise.

($\Leftarrow$) [Lee: force vfs to commute
Warner: similar]

Fix a chart $\varphi_0 : U \to \mathbb{R}^n$ of M. Working in $\varphi_0(U)$, the distribution is spanned by (nonvanishing) vector fields $X_1(p), \dots, X_\ell(p)$. Fix $p_0 \in \varphi_0(U)$. After an affine transform, we may assume $p_0 = 0$ and $X_i(0) = e_i$.

Define

$$F : \underset{V}{\mathbb{R}^\ell \times \mathbb{R}^{n-\ell}} \supseteq V \longrightarrow \mathbb{R}^n$$

$$(t_1, \dots, t_\ell, s) \longmapsto \varphi_{t_\ell}^{X_\ell} \circ \dots \circ \varphi_{t_1}^{X_1}(\bar{0}, s).$$

CLAIM: $\forall (t, s) : DF(t,s)(\mathbb{R}^\ell) = E(F(t,s))$.

Proof. For a basis vector e_i, note

$$DF(t,s)(e_i) = \frac{d}{d\tau}\Big|_{\tau=0} \varphi_{t_\ell}^{X_\ell} \circ \cdots \circ \varphi_{t_i+\tau}^{X_i} \circ \cdots \varphi_{t_1}^{X_1}(0,s)$$

$$= D\varphi_{t_\ell}^{X_\ell} \circ \cdots \circ D\varphi_{t_{i+1}}^{X_{i+1}} (X_i(\varphi_{t_i}^{X_i}(\cdots(\varphi_{t_1}^{X_1}(0,s))\cdots)))$$

Fix $Y \in E(\varphi)$ and assume X is subordinate to $\mathcal{E}$.

CLAIM: $D\varphi_t^X(Y) \in E(p)$.

Proof. With fixed p, let

$$A_t = (X_1(\varphi_t^X(p)) \cdots X_\ell(\varphi_t^X(p)) \quad e_{\ell+1} \cdots e_n).$$

$$A_0 = id.$$
$\therefore A_t$ is invertible on a nbd.

Let $B_t := A_t^{-1}$.

$$\partial/\partial t \quad A_t B_t = id$$

$$A_t' B_t + A_t B_t' = 0$$

$$\Rightarrow B_t' = -A_t^{-1} A_t' B_t$$
$$\boxed{B_t' = -B_t A_t' B_t}$$

$$\text{image}(B_t') \subseteq \mathbb{R}^n.$$

Lecture 16 (28-09-2022)

Wednesday, September 28, 2022   10:36 AM

Last time : $\varphi_0 : U_0 \longrightarrow \mathbb{R}^n$

$$F : \varphi(U) \longrightarrow \mathbb{R}^n$$

$$dF_p(\mathbb{R}^\ell) = E(F(p))$$

$$\varphi = F^{-1} \circ \varphi_0 = \text{takes } E \text{ (on } M) \text{ to } \mathbb{R}^\ell$$

Consider $\mathcal{F} = \{\varphi \text{ as obtained from } \varphi_0\}$
$\overset{\text{alt}}{=} \{\text{charts } \varphi \text{ s.t. } D\varphi(E) = \mathbb{R}^\ell\}$

AIM : $\mathcal{F}$ defines a foliation.

To see : check that if $\varphi, \psi,$ then

$$(\psi \circ \varphi^{-1})(\{x\} \times \mathbb{R}^\ell) \subseteq \{\psi(\varphi^{-1}(x))\} \times \mathbb{R}^\ell. \quad \dots$$

Transversality

Def^n. · Let $f : M \longrightarrow N$ be a C^∞ map and $Q \subseteq N$ be an embedded submanifold.
f is said to be transverse to Q ($f \pitchfork Q$) if

$$\boxed{\operatorname{im}(Df(p)) + T_{f(p)}Q = T_{f(p)}N,}$$

whenever $f(p) \in Q$.

· When M_1, M_2 are embedded submanifolds, we say that they are transverse ($M_1 \pitchfork M_2$) if

· When M_1, M_2 are embedded submanifolds, we say that they are transverse $(M_1 \pitchfork M_2)$ if

$$\boxed{T_pM_1 + T_pM_2 = T_pM}$$

for all $p \in M_1 \cap M_2$.

(Same as taking $Q = M_2$, $f = i_{M_1}$ or
$Q = M_1$, $f = i_{M_2}$ in first definition.)

Theorem (Transversality Theorem)

If $f : M \longrightarrow N$ is transverse to $Q \subset N$, then $\hat{Q} = f^{-1}(Q)$ is an embedded submanifold of M.
Furthermore,

$$\operatorname{codim}(\hat{Q}) = \operatorname{codim}(Q).$$

Special cases: ① f = submersion, $Q = \{x\}$ $\longrightarrow$ submersion theorem

② $\dim M + \dim Q = \dim N$, f embedding.

$\hat{Q} \rightarrow$ discrete, countable collection of points

Lecture 17 (30-09-2022)

Friday, September 30, 2022 10:48 AM

Recall: $f : M \to N$ C^∞, $Q \subseteq N$ embedded.

$$f \pitchfork Q \iff \operatorname{im}(Df(p)) + T_{f(p)}Q = T_{f(p)}N \quad \forall p \in f^{-1}(Q).$$

Theorem (Transversality theorem)

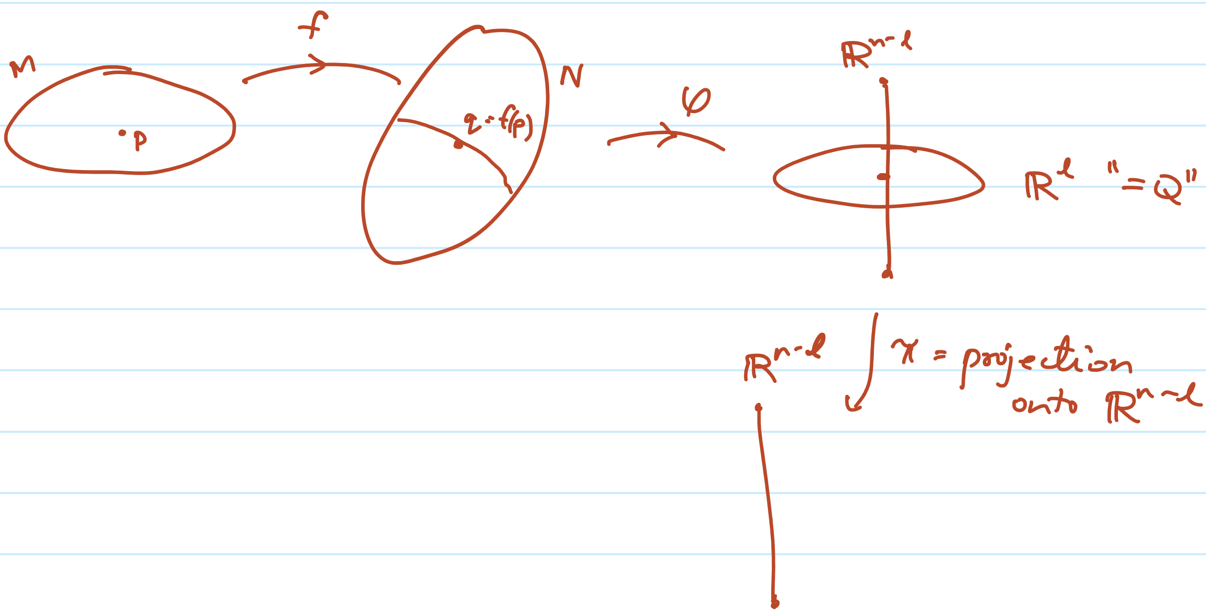
Let $f \pitchfork Q$. Then, $\hat{Q} := f^{-1}(Q)$ is an embedded submanifold of M, and $f|_{\hat{Q}}$ is an embedding whenever f is an embedding.

Furthermore, $\operatorname{codim}(\hat{Q}) = \operatorname{codim}(Q)$.

Lemma. If $Q \subseteq N$ is an embedded submanifold, and $q \in Q$, $\exists$ a chart $\varphi : U \to \mathbb{R}^n$ $(= \mathbb{R}^\ell \times \mathbb{R}^{n-\ell})$ s.t. $q \in U$ and

$$\varphi(Q \cap U) \underset{\text{open}}{\subseteq} \mathbb{R}^\ell. \qquad (\ell = \dim(Q))$$

Take such a chart φ at $q = f(p)$.



Claim. $F := \pi \circ \varphi \circ f$ is a submersion near p.

Pf. We have $\mathrm{im}(Df(p)) + T_{f(p)}Q = T_{f(p)}N.$

$$\Rightarrow \mathrm{Im}(D\varphi \circ Df) + \mathbb{R}^l \times \{0\} = \mathbb{R}^n$$

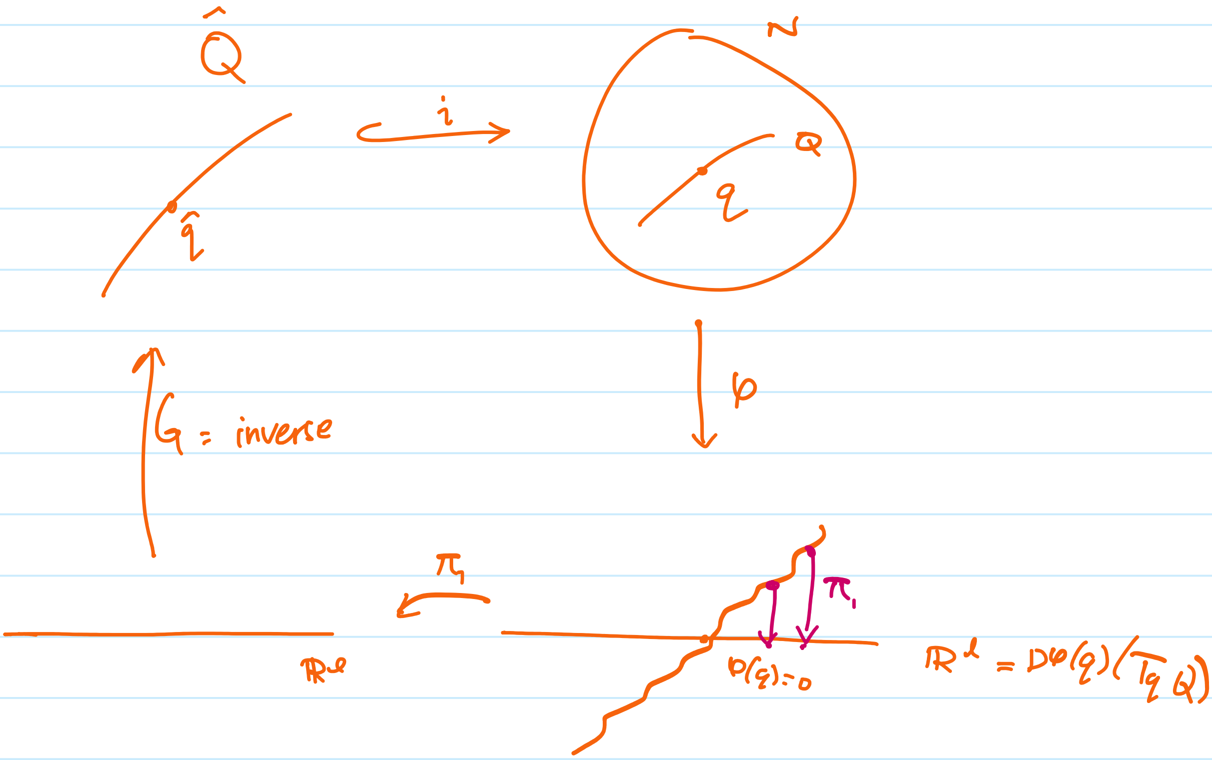
$$\Rightarrow \mathrm{Im}(D\pi \circ D\varphi \circ Df) = \mathbb{R}^{n-l}. \qquad \blacksquare$$

(Suffices to prove submersion at p, then continuity gives on a nbd.)

$F^{-1}(0) = f^{-1}(U \cap Q).$

Now use submersion theorem to conclude the transversality theorem.

Sketch of proof of lemma:



$\tau : \mathbb{R}^l \longrightarrow \mathbb{R}^{n-l}$

$\tau = \pi_2 \, \varphi \, i \, G$

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$F(x,t) = (x,\ t - \tau(x)).$ ☒

Lecture 18 (03-10-2022)

Monday, October 3, 2022 10:41 AM

Exercise. Let $f_t(x, y) = (t+x, y, x+y)$. $(f_t : \mathbb{R}^2 \to \mathbb{R}^3)$
For which values of t does $\mathrm{im}(f_t)$ have a non trivial transverse intersection with S^2.

Sol^n Expect an open interval.

$$\mathrm{Im}(D_p f) + T_{f(p)} S^2 = \mathbb{R}^3 \quad \forall p \text{ s.t. } f(p) \in S^2$$

always 2 dim'l

$\therefore$ suffices to check $\mathrm{Im}(D_p f) \neq T_{f(p)} S^2$.

Note $\mathrm{Im}(f_t) = g_t^{-1}(0)$, where
$$g_t(x, y, z) = z - y - (x - t).$$

Define $F_t(x, y, z) = (x^2 + y^2 + z^2 - 1,\ g_t(x, y, z))^T$.

$F_t : \mathbb{R}^3 \to \mathbb{R}^2$

Need to check where DF_t is full rank.

Reduced both manifolds to zero sets.

$$D_{(x,y,z)} F_t = \begin{bmatrix} 2x & 2y & 2z \\ -1 & -1 & 1 \end{bmatrix}$$

These two vectors are proportional

$$\left.\begin{array}{l} \text{if } x = y = -z. \\ x^2 + y^2 + z^2 = 1 \end{array}\right\} \Rightarrow (x, y, z) = \pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

If $g\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = 0$,

then $-\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} + \left(\frac{1}{\sqrt{3}} - t\right)$

$\Rightarrow t = \sqrt{3}$

$g\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \Rightarrow t = -\sqrt{3}$

$\therefore \left(-\sqrt{3}, \sqrt{3}\right)$ is the answer.

Midterm: No transversality II, Sard's theorem.

(Wortman, Page 80)

Let M and N be connected smooth manifolds with M compact.

Let $F_t : M \to N$ be a family of C^∞ maps s.t. $(t, x) \mapsto F_t(x)$ is C^∞ from $\mathbb{R} \times M$.

(That is, F_t varies C^∞ in t.)

Then, the sets of t for which the following hold are open (i.e., these are open properties):

1. F_t is an immersion.
2. submersion.
3. local diffeo.
4. is transverse to a fixed $Q \subseteq N$.
5. embedding.
6. diffeo.

check using full rank

(Compactness $\Rightarrow$ injection = homeomorphism onto image)

Proof. 5. WLOG, we check open-ness around 0.

(... injection + homeomorphism onto image)

Proof. ⑤ WLOG, we check open-ness around 0.

Assume F_0 is an embedding but $\exists$ sequence $t_k \searrow 0$ s.t. F_{t_k} not an embedding.

Define
$$G : \mathbb{R} \times M \longrightarrow \mathbb{R} \times N$$
$$(t, x) \longmapsto (t, F(t,x)).$$

Claim: $DG(0,x)$ is injective for all $x \in M$.

Proof.

$$DG(0,x) = \begin{array}{c c} & \begin{array}{cc} 1 & m \end{array} \\ \begin{array}{c} 1 \\ n \end{array} & \left(\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & DF_0(x) \\ 0 & \end{array} \right) \end{array}$$

$\dim M = m$
$\dim N = n$

full rank

This is still full rank. ∎

F_{t_k} fails to be an embedding by not being injective. (Immersion is open.)

$\exists p_k, q_k \in M$ s.t. $p_k \neq q_k$ and $F_{t_k}(p_k) = F_{t_k}(q_k)$

By compactness, we may assume p_k and q_k converge to p and q, resp.

By continuity $F_0(p) = F_0(q)$.
Since F_0 is an embedding, we have $p = q$.

However, G is injective on a small nbd U of $(0, p)$ but for $k \gg 0$, (t_k, p_k) and (t_k, q_k) are in U with
$$G(t_k, p_k) = (t_k, F_{t_k}(p_k)) = (t_k, F_{t_k}(q_k))$$
$$= G(t_k, q_k). \rightarrow\leftarrow \quad ∎$$

[illegible]

$-G(t_k, q_k)$. → [illegible]

Lecture 19 (17-10-2022)

Monday, October 17, 2022 10:41 AM

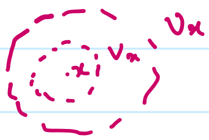
Theorem (Whitney)

If M is a C^∞ n-manifold, there exists an embedding of M into $\mathbb{R}^{2n}$.

We will prove a weaker version: Will show assuming M compact and show M embeds into <u>some</u> $\mathbb{R}^N$.

Proof (with simplifying assumption). For each $x \in M$, choose a C^∞ chart (U_x, φ_x) s.t. $x \in U_x$. For each x, choose some open $V_x \ni x$ such that $\overline{V}_x \subseteq U_x$.

s.t. $\varphi_x(U_x)$ is bdd in $\mathbb{R}^n$



By compactness, we choose a finite subcover $V_{x_1}, \ldots, V_{x_r}$.

For each i, choose C^∞ $\psi_i : M \to \mathbb{R}$ s.t. $\operatorname{supp} \psi_i \subseteq U_{x_i}$,

$\psi_i|_{V_{x_i}} \equiv 1$,

$\psi_i^{-1}(1) = V_{x_i}$.

Let $N := r \cdot n + r$.

Define $F : M \to \mathbb{R}^N$ by

$$F(x) = (F_1(x), \ldots, F_r(x), \psi_1(x), \ldots, \psi_r(x)).$$

$F_i : M \to \mathbb{R}^n$ is defined by

$$F_i(x) = \begin{cases} \varphi_{x_i}(x)\, \psi_i(x), & x \in U_{x_i} \\ 0, & \text{otherwise} \end{cases}$$

(our assumption being $\varphi_x(U_x)$ bdd tells F_i is C^∞)

Claim 1. F is injective. (Hence, homeo onto image. (Since M compact.))

↳ **Proof.** Suppose $x, y \in M$ satisfy $F(x) = F(y)$.

↳ Proof. Suppose $x, y \in M$ satisfy $F(x) = F(y)$.
Choose i s.t. $x \in V_{x_i}$.

Then, $\psi_i(x) = 1$. Consequently, $\psi_i(y) = 1$.
$\therefore y \in V_{x_i}$ as well.

But now, $\varphi_{x_i}(x) = \varphi_{x_i}(y)$ and thus,
$x = y$ ($\because \varphi_{x_r}$ is 1-1). ∎

Claim 2. F is an immersion (and hence, an embedding).

↳ Proof. Need to check $\operatorname{rank}(DF(x)) = n \ \forall x$.

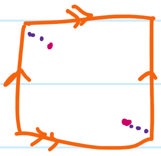
$$DF(x) = \begin{bmatrix} DF_1(x) & \cdots & DF_r(x) & D\psi_1(x) \cdots D\psi_r(x) \end{bmatrix}$$

Pick i s.t. $x \in V_{x_i}$.
On a nbd of x, $F_i \equiv \varphi_{x_i}$.
But then
$\operatorname{rank}(DF_i(x)) = \operatorname{rank}(D\varphi_{x_i}(x))$
$= n$. ∎

Thus, we are done. ∎

Thm. (Nash) $\exists C^1$-embedding of T^1_{flat} into $\mathbb{R}^3$ which is an isometry.
$\nexists C^2$

T^1_{flat} = flat torus,
distance measured 'along' torus



Defn: A (topological) n-manifold with boundary is a Hausdorff second-countable topological space s.t. for every $x \in M$, either one of the two conditions hold:

- M is locally n-Euclidean at x,
- $\exists$ nbd U containing x and a homeo $\varphi: U \to V$ s.t. V is open in $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\varphi(x) \in \{0\} \times \mathbb{R}^{n-1}$.

M has a smooth structure if transition maps are C^∞.

↳ boundary should go to boundary, smooth on interior and on $\mathbb{R}^{n-1}$.

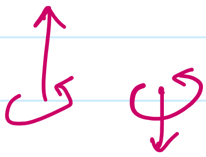
Lecture 20 (19-10-2022)

Wednesday, October 19, 2022 10:39 AM

Orientations

1-manifolds: "left" or "right" 

2-manifold: "clockwise" or "counterclockwise" 

3-manifold: "left-hand rule" or "right-hand rule" 

<u>Idea</u>: An orientation on a vector space $V \neq 0$ is an equivalence class of ordered frames (bases).

$(v_1, \dots, v_d) \sim (w_1, \dots, w_d) \iff$ the linear transform mapping $v_i \mapsto w_i$ has det > 0.

<u>Remark</u>: On every vector space $\neq 0$, $\exists$ exactly two equiv classes.

Isometries of $\mathbb{R}^2$: Rotations, Translations, $\}$ preserve
Reflections. $\to$ don't preserve

Defs. ① If M is a manifold, a pointwise orientation of M is a function which assigns to each $x \in M$ an orientation of $T_x M$.

② An orientation $\mathcal{O}$ of M is a ptwise orientation s.t.

② An orientation $\mathcal{O}$ of M is a ptwise orientation s.t. $\forall x \in M$ $\exists$ nbd $U \ni x$ and vfs $X_1, \ldots, X_d$ defined on U s.t.

$$\mathcal{O}(p) = [(X_1(p), \ldots, X_d(p))] \quad \text{for all } p \in U.$$

Examples ① $\mathbb{T}^d$ is orientable.

$\overset{\shortparallel}{\mathbb{R}^d/\mathbb{Z}^d}$

Choose the standard basis vectors at each point.

② S^n is orientable. (See next theorem.)

Theorem. Let $M \subseteq \mathbb{R}^{n+1}$ be an n-dimensional manifold. Assume $\exists$ a v.f. X on $\mathbb{R}^{n+1}$ s.t. $X(p) \notin T_pM$ $\forall p \in M$. Then, M is orientable.

Proof. Fix $p \in M$, and let $\varphi : U \longrightarrow \underset{\overset{\shortparallel}{\varphi(U)}}{\mathbb{R}^n}$ be a chart containing p
connected

Then, $(D_{\varphi(x)}\varphi^{-1}(e_1), \ldots, D_{\varphi(x)}\varphi^{-1}(e_n))$ is a local framing of T_xM at x, for $x \in U$.
We say φ is a positively oriented chart if

$$(D_{\varphi(x)}\varphi^{-1}(e_1), \ldots, D_{\varphi(x)}\varphi^{-1}(e_n), X(x)) \sim (e_1, \ldots, e_{n+1})$$
(Inside $\mathbb{R}^{n+1}$)
$\forall x \in U$.

Note that

$$x \longmapsto \det(D_{\varphi(x)}\varphi^{-1}(e_1), \ldots, D_{\varphi(x)}\varphi^{-1}(e_n), X(x))$$

is continuous and nonzero. $\therefore$ same sign.

Thus, given any such (connected) chart (U, φ), either φ or $(-\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a pos. chart.

pos., given any such chart (U, φ), either φ or $(-\varphi_1, \varphi_2, \dots, \varphi_n)$ is a pos. chart.

Thus, for any $x \in M$, we can define a +vely oriented chart.

Now, we claim that for $x \in M$, defining

$$\Theta(x) := \left[\left(D_{\varphi(x)}\varphi^{-1}(e_1), \dots, D_{\varphi(x)}\varphi^{-1}(e_n)\right)\right]$$

works (for any +ve oriented chart φ).

Need to check : well-defined.

Check : If φ, ψ are two such +ve chart, then the RHS is same. (Easy.)

Def^n: If M is a C^∞ manifold, let $\tilde{M}$ denote the set of pairs (x, Θ_x), where Θ_x is an orientation of $T_x M$.
Then, $\exists$ a 2:1 map $\pi : \tilde{M} \to M$

$$(x, \Theta_x) \mapsto x.$$

$\to$ $\tilde{M}$ can be given the structure of a C^∞ manifold s.t. $\tilde{M}$ is orientable.

Lecture 21 (21-10-2022)

Friday, October 21, 2022 10:38 AM

$\tilde{M} = \{ (x, \theta_x) : \theta_x$ is an orientation on $T_x M \}$.

$$\pi : \tilde{M} \to M$$
$$(x, \theta_x) \mapsto x$$

<u>Smooth / topological structure</u> :

$\varphi : U \to \mathbb{R}^n$ a chart on M.

Define

$$\underset{\tilde{M}}{\overset{\cap}{U_+}} := \{ (x, \underbrace{[(D_{\varphi(x)} \varphi^{-1}(e_1), \dots, D_{\varphi(x)} \varphi^{-1}(e_n))]}_{= \theta_{\varphi,+}(x)}) : x \in U \}.$$

$\varphi_+ : U_+ \to \mathbb{R}^n$ is defined by
$$\varphi_+ (x, \theta_{\varphi,+}(x)) := \varphi(x).$$

Similarly, define
$$U_- := \{ (x, -\theta_{\varphi,+}) : x \in U \}$$
and $\varphi_- : U_- \to \mathbb{R}^n$ ...

Check: topology given on $\tilde{M}$ by declaring U_+, U_- as the open sets makes $\pi : \tilde{M} \to M$ a covering map.

Let $\psi : V \to \mathbb{R}^n$ be another chart.

<u>Case</u>: $V_+ \cap U_+ \neq \emptyset$.

Then, $\exists p \in V \cap U$ s.t. $\theta_{\varphi,+}(p) = \theta_{\psi,+}(p)$.

Then, $\exists p \in V \cap U$ s.t. $\Theta_{\varphi,+}(p) = \Theta_{\psi,+}(p)$.

Then,

$$B_1 = \left(D_{\varphi(p)}\varphi^{-1}(e_1), \dots, D_{\varphi(p)}\varphi^{-1}(e_n)\right)$$
$$\sim \left(D_{\psi(p)}\psi^{-1}(e_1), \dots, D_{\psi(p)}\psi^{-1}(e_n)\right) = B_2.$$

Let A_p be the lin. transform taking B_1 to B_2.

Then, $\det(A_p) > 0$.

$\Rightarrow \det(A_x) > 0 \quad \forall x \in$ connected component...

$\Rightarrow$ this component is open.

$$\underbrace{(V \cap U)_+}_{\text{for } \varphi \text{ or } \psi} = \underbrace{V_+}_{\text{for } \psi} \cap \underbrace{U_+}_{\text{for } \varphi}$$

Remark. An orientation on M is a section of $\pi : \tilde{M} \to M$.
i.e. a (C^∞) map $\sigma : M \to \tilde{M}$ and $\pi \circ \sigma = id_M$.

Theorem: M is connected.
$\tilde{M}$ is not connected $\Leftrightarrow$ $\tilde{M}$ is orientable.

Proof. $(\Leftarrow)$ Assume M orientable.

Then, $\exists\, C^\infty$ sections $\sigma_\pm : M \to \tilde{M}$ defined by

$$\sigma_\pm(x) = (x, \Theta_\pm(x)).$$

Then,

$$\tilde{M} = \underbrace{\sigma_+(M)}_{\text{open since local diffeo}} \sqcup \sigma_-(M).$$

$(\Rightarrow)$ Assume $M = M_+ \sqcup M_-$ for nonempty clopen subsets.

Then, $\pi : \tilde{M} \to M$ is a 2-1 covering map.

Then, $\pi(M_+)$ is also clopen.

... is a 2-covering map.

Then, $\pi(M_+)$ is also clopen.

But M is connected. Thus, $\pi(M_+) = M$.

$\|^{rly}$ $\pi(M_-) = M$. $\because \pi$ is 2-1, $\pi|_{M_+}$ is a bijection.

But π is a local diffeo.

$\therefore \pi|_{M_+}$ is a diffeo.

$\therefore \pi|_{M_+}^{-1}$ is a section. $\blacksquare$

Remark. $\tilde{M}$ is always orientable.

θ_x is orientation at $(x, \theta_x) \in \tilde{M}$.

Let M, N be manifolds and $F: M \to N$ a diffeomorphism.
If Θ is an orientation on M, the **pushforward** of Θ is

$$F_*(\Theta)(y) = [(D_x F(v_1), \ldots, D_x F(v_n))] \qquad (x := F^{-1}(y))$$

where $[(v_1, \ldots, v_n)] = \Theta(x)$.

Exercises: $(F \circ G)_* = F_* \circ G_*$.

Def^n. Let $F: M \to M$ be a diffeo. (connected oriented manifold)
F is called **orientation preserving** if $F_*(\Theta) = \Theta$.
Orientation reversing otherwise.

$\exists$ group homomorphism $\Theta: \mathrm{Diff}^\infty(M) \longrightarrow \{-1, 1\}$

$$F \mapsto \begin{cases} 1 & ; F \text{ preserves} \\ -1 & ; F \text{ reverses} \end{cases}$$

$\mathbb{R}P^n$ orientable $\Longleftrightarrow$ n is odd.

Lecture 22 (24-10-2022)

Monday, October 24, 2022 10:48 AM

Differential Forms

Recall: The chain rule for integration:

If $U, V \subseteq \mathbb{R}^n$ are open, and $F : U \to V$ is a diffeo (think: change of coordinates)

and $\varphi \in C^\infty(V)$. Then,

$$\int_V \varphi(x)\, dx = \int_U (\varphi \circ F)(x)\, |\mathrm{Jac}(F)(x)|\, dx.$$

integration wrt standard Lebesgue measure

$$\left(\mathrm{Jac}(F)(x) = \det(D_x F). \right)$$

Note:
- Since F is a diffeo the Jacobian is of same sign (on at least connected components). This hints that orientations are important.
- The $\mathrm{Jac}(F)$ term says that any naive type of integral defined in terms of charts will in fact depend on charts.

Goal: Construct/attach extra data to keep track of the $\mathrm{Jac}(F)$ term.

Some linear algebra:

Recall when $f: \underbrace{V \times \cdots \times V}_{k\text{-fold}} \to \mathbb{R}$ is called k-linear or multilinear.

Similarly recall alternating...

Lastly, recall

$$\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n\text{-times}} \to \mathbb{R}$$

is the unique multilinear and alternating function s.t. $\det(e_1, \dots, e_n) = 1$.

Moreover, any mult. alt. f^n is a scalar multiple of det.

$\Lambda^k(V)$ = space of k-linear alternating functions.

- $\Lambda^k(V)$ is a vector space.
- $\dim(\Lambda^k V) = \binom{\dim V}{k}$. [In particular, $\Lambda^k V = 0$ for $k > \dim V$.]
- Let $n = \dim V$.

  $\Lambda^n(V)$ is one-dimensional.

  If we have a (ordered) basis for V, we get a generator of $\Lambda^n(V)$. (Think of det.)

  $(v_1, \dots, v_n) \to$ basis for V,

  $\varphi : \underbrace{V \times \cdots \times V}_{n} \longrightarrow \mathbb{R}$

  $\left(\sum a_{1i} v_i, \dots, \sum a_{ni} v_i\right) \mapsto \det [a_{ij}]$.

Elements of $\Lambda^n(V)$ are called **top forms**.

Defn. If $\omega \in \Lambda^k(V)$ and $F: W \to V$ is linear, then the **pullback of ω by F**, $F^*\omega \in \Lambda^k(W)$ is the pullback of ω by F, $F^*\omega \in \Lambda^k(W)$ is defined by

$$F^*\omega(w_1, \dots, w_k) = \omega(F(w_1), \dots, F(w_k)).$$

- F is not assumed invertible. Even $\dim W = \dim V$ is not needed.

- Pullbacks are contravariant.

$$(F_2 \circ F_1)^*\omega = F_1^*(F_2^*\omega)$$

$$\begin{tikzcd} U \arrow[r, "F_1"] \arrow[dr, "F_2 \circ F_1"'] & W \arrow[d, "F_2"] \\ & V \end{tikzcd} \qquad \begin{tikzcd} \Lambda^k(U) & \Lambda^k(W) \arrow[l, "F_1^*"'] \\ & \Lambda^k(V) \arrow[u, "F_2^*"'] \arrow[ul, "(F_2 \circ F_1)^*"] \end{tikzcd}$$

- If $F: \mathbb{R}^n \to \mathbb{R}^n$, then
$$F^*(\det) = \det(F)\det.$$

Defn. Let M be a smooth manifold. Define $\Lambda^k(TM)$ to be the vector bundle whose fibers are $\Lambda^k(T_x M)$.

(Give this a smooth structure using pullbacks...)

A differential k-form is a C^∞ section of $\Lambda^k(TM)$.

Ex: $\Lambda^1(TM)$ is the cotangent bundle.

Integration: Fix a family of charts $\varphi_k : U_k \to \mathbb{R}^n$ such that $\{U_k\}_k$ covers M.

Choose a partition of unity $\psi_k : M \to [0,1]$ s.t.

- $\operatorname{supp}(\psi_k) \subseteq U_k$,
- $\sum \psi_k \equiv 1$.

If $f: M \to \mathbb{R}$ is C^∞ and ω an n-form on M, define

define

$$\int_M f \cdot \omega = \sum_k \int_{\varphi(U_k)} \psi_k(\varphi_k^{-1}(x))\, f(\varphi_k^{-1}(x)) \left[\frac{D\varphi_k(x)^*(\det)}{\omega(\varphi^{-1}(x))} \right] dx$$

this makes sense
because
$\omega(\varphi^{-1}(x))$ spans
the 1-dim space.

Lecture 23 (26-10-2022)

Wednesday, October 26, 2022 10:46 AM

Theorem. M is orientable $\Leftrightarrow$ M has a nonvanishing top-form.

Proof. ($\Leftarrow$) Let ω be a nonvanishing top form on M. Fix $p \in M$. We choose $v_1, \dots, v_n \in T_pM$ s.t.

$$\omega(p)(v_1, \dots, v_n) = 1.$$

Make such a choice for every p. Define

$$\mathcal{O}(p) = [(v_1, \dots, v_n)].$$

This varies smoothly: follows from ω being C^∞.

($\Rightarrow$) Assume M is orientable. (Assume M compact for ease.) For each $p \in M$, let $\varphi_p : U_p \to \mathbb{R}^n$ be a truly oriented chart s.t. $p \in U_p$.

Choose finitely many such charts: $U_1, \dots, U_m$. Let $\rho_1, \dots, \rho_m : M \to [0,1]$ be a partition of unity subordinate to $U_1, \dots, U_m$.

Define the form

$$\omega(x)(v_1, \dots, v_n) = \sum_{i=1}^{m} \rho_i(x)\, \varphi_i^*(\det)(v_1, \dots, v_n).$$

$$\parallel$$

$$\sum \rho_i(x) \det\left(D\varphi_i(x) v_1, \dots, D\varphi_i(x) v_n\right).$$

$\angle \rho_i(x) \det (D\varphi_i(x) v_1, \ldots, D\varphi_i(x) v_n).$

This works - ☑

Recall: M = n-manifold (oriented)

ω = top form

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$, charts s.t. $(U_i)_i$ cover M

$\rho_i : M \longrightarrow [0,1]$ part^n of unity sub. to $(U_i)_i$

$$\int_M \omega := \sum_i \int_{\varphi_i(U_i)} \rho_i(\varphi_i^{-1}(x)) \left[\frac{\omega}{(\varphi_i)^*(\det)} \right] (\varphi_i^{-1}(x)) \, dx.$$

This is independent of ... everything (but ω)

Lecture 24 (28-10-2022)

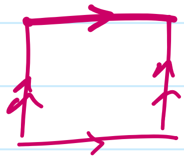
Friday, October 28, 2022 10:42 AM

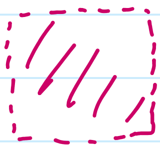
Integration in practice:

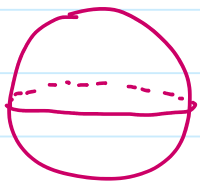
- $\varphi_i : \mathcal{U} \to \mathbb{R}^n$.
- $\mathcal{U}_i \cap \mathcal{U}_j = 0$ if $i \neq j$.
- $M \setminus \left(\bigcup_{i=1}^{n} \mathcal{U}_i \right) =$ finitely many embedded submanifolds

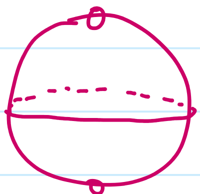
"0 measure" (Sard's)

$$\int_M \omega = \sum_{i=1}^{n} \int_{\varphi_i(\mathcal{U}_i)} \frac{\omega}{(\varphi_i)^*(\det)}.$$

Ex: ① $\mathbb{T}^2 =$ 

$\mathcal{U} =$  "$dx \wedge dy$"

② $S^2 =$ 

$\mathcal{U} =$  $= S^2 \setminus \{$ north pole, south pole $\}$

Alternating forms on $\mathbb{R}^n$

- $\Lambda^1(\mathbb{R}^n) = (\mathbb{R}^n)^*$

- $\Lambda^1(\mathbb{R}^n) = (\mathbb{R}^n)^*$

Basis : $dx_1, \ldots, dx_n$, where

$$dx_i(v) = v_i.$$

$(v = (v_1, \ldots, v_n))$

Dual to standard basis.

- $\Lambda^k(\mathbb{R}^n)$

Define $dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k(\mathbb{R}^n)$ as

$$(v^1, \ldots, v^k) \mapsto \sum_{\sigma \in S_k} (-1)^{\operatorname{sgn}(\sigma)} v_{i_1}^{\sigma(1)} \cdots v_{i_k}^{\sigma(k)}.$$

$v^1, \ldots, v^k \in \mathbb{R}^n$, $v^i = (v_1^i, \ldots, v_n^i) \in \mathbb{R}^n$.

Theorem. $\{ dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 < \cdots < i_k \le n \}$ forms a basis for $\Lambda^k(\mathbb{R}^n)$.

Wedge Product

If $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$, the wedge product $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ is defined as

$$(v^1, \ldots, v^{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^{\operatorname{sgn}(\sigma)} \alpha(v^{\sigma(1)}, \ldots, v^{\sigma(k)}) \beta(v^{\sigma(k+1)}, \ldots, v^{\sigma(k+l)}).$$

Example. $(dx_1 \wedge dx_2)\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right)$

$$= dx_1\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) dx_2\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) - dx_1\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) dx_2\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right)$$

$$= v_1 w_2 - w_1 v_2.$$

Theorem. The wedge product is associative, bilinear, and satisfies

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha,$$

$$\alpha \in \Lambda^k(V), \ \beta \in \Lambda^l(V).$$

Example. Let

$$\left.\begin{aligned} \alpha &= y\,dx - x\,dy, \\ \beta &= x^2\,dx + y\,dy. \end{aligned}\right\} \text{1-forms on } \mathbb{R}^2$$

(section of $\Lambda^1(T\mathbb{R}^2)$)

$$\alpha \wedge \beta = (y\,dx - x\,dy) \wedge (x^2\,dx + y\,dy)$$

$$= x^2 y\,\cancel{dx \wedge dx} + y^2\,dx \wedge dy - x^3\,dy \wedge dx - xy\,\cancel{dy \wedge dy}$$

$dx \wedge dx = (-1)\, dx \wedge dx$

$$= (y^2 + x^3)\,dx \wedge dy.$$

Tautological 1-form: Let M be a C^∞ manifold.

(Canonical/Liouville)

T^*M = cotangent bundle (space of 1-forms)

Consider the manifold: $N := T^*M$.

Define a one-form on N as follows:

$$\begin{array}{c} T^*N \\ \downarrow \pi \\ M \end{array}$$

If $v \in T(T^*M)$,

$$\theta_{(\alpha,p)}(v) = \alpha\left(D\pi(\alpha,p)(v)\right).$$

$p \in M$,
$\alpha \in (T_pM)^*$

<u>Example</u>. $M = \mathbb{R}^n$.

$$T^*M \cong \mathbb{R}^n \times \mathbb{R}^n$$
$$(q, \alpha_q)$$
$$\alpha_q = p$$

$\alpha_p(v) = \langle v, \omega(\alpha_p)\rangle$.
(any linear f^{n}'l is an inner product)

$$\theta_{(q,p)}(v, w) = \sum_{i=1}^{n} v_i p_i$$

$$\theta = \sum_{i=1}^{n} p_i \, dq_i.$$

Exterior derivatives

If α is a k-form on a manifold M, we define $d\alpha$ as a $(k+1)$-form on M:

$$d\alpha(x)(v^1(x), \dots, v^{k+1}(x)) = \sum_{i=1}^{k+1} (-1)^{i+1}\left[v^i \cdot \alpha(v^1, \dots, \hat{v}^i, \dots, v^{k+1})\right](x)$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha(x)\left([v^i, v^j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{k+1}\right)(x)$$

$v^1, \dots, v^{k+1}$ v.f.s

<u>In coords:</u> $d\alpha(x)\left(\partial/\partial x_1, \partial/\partial x_2\right) = +\frac{\partial}{\partial x_1} \cdot \alpha\left(\frac{\partial}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\, \alpha\left(\frac{\partial}{\partial x_1}\right).$

In coords: $d\alpha(x)\left(\partial/\partial x_1, \partial/\partial x_2\right) = + \frac{\partial}{\partial x_1} \cdot \alpha\left(\frac{\partial}{\partial x_2}\right) - \frac{\partial}{\partial x_2} \alpha\left(\frac{\partial}{\partial x_1}\right).$

If $\alpha = f\,dx_1 + g\,dx_2$ then,

$$d\alpha(x) = + \frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2}.$$

$$\therefore d(f\,dx_1 + g\,dx_2) = \left(\frac{-\partial f}{\partial x_2} + \frac{\partial g}{\partial x_1}\right) dx_1 \wedge dx_2.$$

Formulae for exterior derivative:

1) If $f: M \to \mathbb{R}$ is a 0-form (i.e., C^∞ f'n), df is the usual differential.

2) $d^2\alpha = 0$ for all forms α.

3) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$. $\quad p = \deg(\alpha)$

Using these:

$d(f\,dx_1 + g\,dx_2)$

$= df \wedge dx_1 + dg \wedge dx_2$

$= \ldots$

Lecture 25 (02-11-2022)

Wednesday, November 2, 2022 10:38 AM

Theorem (Stoke's Theorem)

Let M be an oriented n-manifold with boundary and ω be a compactly supported $(n-1)$-form on M. Then,

$$\int_M d\omega = \int_{\partial M} \omega$$

where ∂M has the induced orientation.

Proof. Case 1. ω supported on a chart contained in the interior of M.

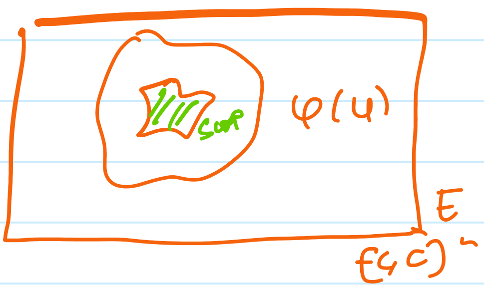
Want to check $\int_M d\omega = 0$.

In coordinates:

$$\omega = \sum_{i=1}^{n} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

$$d\omega = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

$$= \underbrace{\left(\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right)}_{\text{each individually integrates to } 0:} dx_1 \wedge \cdots \wedge dx_n$$



$$\int_{\varphi(U)} \frac{\partial f_1}{\partial x_1} dx_1 \cdots dx_n$$

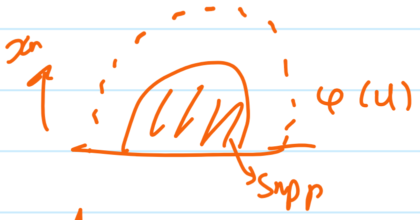
$$= \int_{[-c,c]^n} \frac{\partial f}{\partial x_1} dx_1 \, dx_2 \cdots dx_n$$

$[-c,c]^n$

$$= \int_{[-c,c]^{n-1}} \left(\int_{-c}^{c} \frac{\partial f}{\partial x_1}\, dx_1 \right) dx_2 \dots dx_n$$

$$= 0. \quad \big)_{=\tau c}$$

Case 2. ω supported on a chart of ∂M.



(φ +vely oriented)

$$\omega = \sum f_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

$$d\omega = \left(\sum (-1)^{j-1} \frac{\partial f}{\partial x_i} \right) dx_1 \wedge \dots \wedge dx_n .$$

As before, for all but the last term, integrating over M is 0.

$$\therefore \int_M d\omega = \int_M (-1)^{n-1} \frac{\partial f}{\partial x_n} \, dx_1 \wedge \dots \wedge dx_n$$

$$= (-1)^{n-1} \int_{[-c,c]^{n-1}} \left[\int_0^c \frac{\partial f}{\partial x_n} \, dx_n \right] dx_1 \dots dx_{n-1}$$

$$= (-1)^{n-1} \int_{[-c,c]^{n-1}} \left[\overset{0}{\cancel{f(x_1, \dots, x_{n-1}, c)}} - f(x_1, \dots, x_{n-1}, 0) \right] dx_1 \dots dx_{n-1}$$

$$= \int_{\varphi(\partial M)} (-1)^n f(x_1, \dots, x_{n-1}, 0) \, dx_1 \dots dx_{n-1}$$

$$= \int_{\partial M} \omega .$$

de Rham Cohomology

$\Omega^k(M)$ = k-forms on M.

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$.

$d \circ d = 0$.

closed k-forms

$$H^k_{dR}(M) = \frac{\ker(\Omega^k \longrightarrow \Omega^{k+1})}{\operatorname{im}(\Omega^{k-1} \longrightarrow \Omega^k)}.$$

exact k-forms

Thm. $H^k_{dR}(M)$ is always finite dim'l. (M is connected.)

Thm. If M is a compact, connected, oriented n-manifold, (w/o boundary)
$H^n_{dR}(M) \cong \mathbb{R}$.

Sketch. $\int_M : H^n_{dR}(M) \longrightarrow \mathbb{R}$ is an iso.

Thm. $H^k_{dR}(M \times N) \cong H^k_{dR}(M) \times H^k_{dR}(N)$.

Cor. $H^1_{dR}(\mathbb{T}^n) \cong \mathbb{R}^n$.

Defn $\Omega^k_c(M)$: Compactly supported k-forms.

$$\Omega^0_c \xrightarrow{d} \Omega^1_c \xrightarrow{d} \Omega^2_c \xrightarrow{d} \cdots$$

$$H^k_{c,dR}(M) = \frac{\ker(\Omega^k_c \to \Omega^{k+1}_c)}{\operatorname{im}(\Omega^{k-1}_c \to \Omega^k_c)}$$

In general, $H^n_{c,dR}(M) \cong \mathbb{R}$, M connected, oriented, n-manifold.

$H^1_{dR}(\mathbb{R}) = 0$, $H^1_{c,dR}(\mathbb{R}) \cong \mathbb{R}$.

Lecture 26 (04-11-2022)

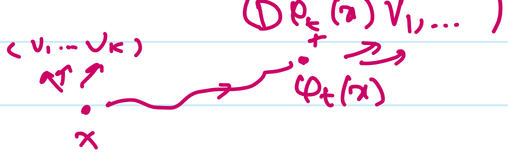
Friday, November 4, 2022 10:41 AM

Lie Derivatives

$X \to$ vf, $\omega \to k$-form on M

$$\mathcal{L}_X \omega = \frac{d}{dt}\Big|_{t=0} (\varphi_t^X)^* \omega$$

(Zero form is a C^∞ function f.
$\mathcal{L}_X f := X \cdot f$.)



$$(\mathcal{L}_X \omega)(x)(v) = \lim_{t \to 0} \frac{\omega(\varphi_t(x))(D\varphi_t(x) v) - \omega(x)(v)}{t}.$$

(written above for 1-form, just for notational ease.)

$\omega \to k$-form, $i_X \omega \to k-1$ form

$$(i_X \omega)(x)(v_1, \dots, v_{k-1}) = \omega(X(x), v_1, \dots, v_{k-1}).$$

Theorem (Cartan's Magic Formula)

$$\mathcal{L}_X = i_X \circ d + d \circ i_X.$$

Proof. To show:

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega).$$
$$\forall k \quad \forall \omega \in \Lambda^k(TM).$$

Induct on k:

$k=0$: $\mathcal{L}_X(f) = X\cdot f$

$i_X(df) = df(X) = X\cdot f$

$d(i_X f) = d(0) = 0.$

Note that all terms in the formula are lemma.

Let us prove it for a form that looks like

$$\omega = du \wedge \beta, \quad u \in C^\infty, \ \beta \in \Omega^{k-1}.$$

(A general form is an $\mathbb{R}$-lin. combination.)

$$\mathcal{L}_X(du)(x)(Y) = \lim_{t\to 0} \frac{1}{t}\left[(\varphi_t^X)^* du(x)(Y) - du(x)(Y)\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[du(\varphi_t^X(x))(D\varphi_t^X(x)Y) - du(x)(Y)\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[D\varphi_t^X(x)Y\cdot u(x) - Y\cdot u\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[Y(u\circ\varphi_t^X) - Y\cdot u\right]$$

$$= Y\cdot\left(\lim_{t\to 0}\frac{1}{t}\left((u\circ\varphi_t^X)(x) - u(x)\right)\right)$$

$$= Y\cdot(X\cdot u)$$

I $\quad \mathcal{L}_X(du\wedge\beta) \overset{\text{HW}}{=} \mathcal{L}_X(du)\wedge\beta + du\wedge(\mathcal{L}_X\beta)$

by above

$$= d(X\cdot u)\wedge\beta + du\wedge(\mathcal{L}_X\beta)$$

induction

$$= d(X\cdot u)\wedge\beta + du\wedge(i_X(d\beta) + d(i_X\beta)).$$

II $\quad i_X(d(du\wedge\beta)) \overset{ddu=0}{=} i_X(-du\wedge d\beta)$

$$= -(X\cdot u)\,d\beta + du\wedge i_X d\beta$$

III $\quad d(i_X(du \wedge \beta)) = d((X\cdot u)\beta - du \wedge i_X \beta)$
$$= d(X\cdot u)\wedge\beta + (X\cdot u)d\beta + du \wedge d(i_X\beta)$$

$$\text{I} = \text{II} + \text{III}.$$

Poincaré Lemma

$$H^k(M\times\mathbb{R}) \cong H^k(M) \qquad \forall k \geq 1.$$

Recall: $\quad H^1(\mathbb{R}) = 0.$

If α is a (closed) one-form : $\alpha = f\,dt$

Then, $\alpha = dF$, when $F(t) = \int_0^t f$.

Motivation:

On $M \times \mathbb{R}_x$, the v.f. $\partial/\partial x$ integrates to the flow

$$\varphi_t^{\partial/\partial x}(p,y) = (p, y+t).$$

$$\omega(p,t) = \omega(\varphi_t(p,0)) \longleftrightarrow \varphi_t^* \omega$$

$$\omega(p,t) = \int_0^t \frac{d}{ds}(\varphi_s^*\omega)ds = \int_0^t \mathcal{L}_{\partial/\partial x}\,\omega \; ds$$

linearity of $\mathcal{L}_{\partial/\partial x}$

$$\text{"}\approx\text{"} \; \mathcal{L}_{\partial/\partial x}\left(\int_0^t \omega(p,s)\,ds\right)$$

Define

$$P : \Omega^k(M\times\mathbb{R}) \longrightarrow \Omega^{k-1}(M\times\mathbb{R})$$

$P : \Omega^k(M \times \mathbb{R}) \longrightarrow \Omega^{k-1}(M \times \mathbb{R})$

$$P(\omega)(q,t) = \int_0^t \mathcal{L}_{\partial/\partial x}\, \omega(q,s)\, ds.$$

From Cartan's Magic Formula:

$$Pd + dP = \int_0^t \mathcal{L}_{\partial/\partial x}\, \omega(q,s)\, ds$$

$$= \int_0^t \frac{\partial}{\partial x}\Big|_{x=0} \left(\varphi_x^{\partial/\partial x}\right)^* \omega(q,s)\, ds$$

$$= \int_0^t \frac{\partial}{\partial x}\Big|_{x=s} \left(\varphi_x^{\partial/\partial x}\right)^* \omega(q,0)\, ds$$

FTC something something

$$= \omega(q,t) - \omega(q,0)$$

We have $M \times \mathbb{R} \underset{i_0}{\overset{\pi}{\rightleftarrows}} M.$

$$\begin{tikzcd} H^k(M) \arrow[d, shift left, "i_0^*"] \\ H^k(M \times \mathbb{R}) \arrow[u, shift left, "\pi^*"] \end{tikzcd}$$

$$(i_0 \circ \pi)(q,t) = (q,0)$$

$$\pi \circ i_0 = id.$$

$$dP + Pd = id - (i_0 \circ \pi)^*.$$

Thus, $id \simeq (i_0 \circ \pi)^*$ on homology.

$(\pi \circ i_0)^* = id^* = id.$

Lecture 27 (07-11-2022)

Monday, November 7, 2022 10:42 AM

Recall: $H^k(M \times \mathbb{R}) \cong H^k(M)$; $k \geq 1$. (Poincaré lemma)

Porism. If M and N are homotopy equivalent, then $H^k(M) \simeq H^k(N)$.

Theorem. If M is a compact orientable connected n-manifold, $H^n(M) \simeq \mathbb{R}$.

$$\left([\omega] \longmapsto \int_M \omega \text{ is an iso.} \right)$$

Need to show: if ω is an n-form s.t.

$$\int_M \omega = 0, \quad \text{then } \omega = d\eta \text{ for some } \eta.$$

Lemma $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$.

Proof.

Choose charts $\{(U_i, \varphi_i)\}_{i=1}^m$ s.t. $\varphi_i(U_i)$ is a ball in $\mathbb{R}^n$, and $M = \bigcup_{i=1}^m U_i$.

Let $\{\rho_i\}_{i=1}^n$ partition of unity wrt ...

$$\omega_i := \rho_i \omega.$$

$$\omega = \sum \omega_i$$

and ω_i is supported in U_i.

Note $\operatorname{supp} \omega_i$ is compact ($\because$ closed, use φ_i)

$\Rightarrow (\varphi_i^{-1})^* \omega_i$ is compactly supported (in $\varphi_i(U_i)$)

$\Rightarrow (\varphi_i^{-1})^* \omega_i$ is compactly supported (in $\varphi_i(U_i)$)

↳ integral need not be zero, subtract appr constant

...

Application of Differentials Forms

(Hamiltonian flows)

$M \to$ manifold with 2-form ω which is closed, nondegenerate.

$\forall x \in M, \forall X \in T_x M \setminus \{0\}$
$\exists Y \in T_x M$
s.t. $\omega(x)(X,Y) \neq 0$.

Given $v \in T_x M$, $i_v \omega$ is a functional on $T_x M$.

Thm. Given $\theta \in (T_x M)^*$ $\exists!$ θ^* s.t. $i_{\theta^*} \omega = \theta$. (Just linear alg.)

Let $H: M \to \mathbb{R}$ be a C^∞ function. (ω is the fixed symplectic form.)
Let X_H be the vf defined by

$$i_{X_H} \omega = dH. \quad \left(\text{I.e., } X_H = (dH)^*.\right)$$

X_H is called a Hamiltonian vf. and its flow is a Hamiltonian flow.

Ex. ① $\theta \to$ Liouville form
$d\theta \to$ symplectic

$\therefore$ [illegible] θ – [illegible] form

$d\theta \to$ symplectic

$$\omega := d\theta$$

$$= \sum_{i=1}^{n} dq_i \wedge dp_i \quad \text{on } \mathbb{R}^{2n}.$$

$H \in C^\infty$.

$$X_H(x) = (\underbrace{v(x)}_{q \text{ coords}}, \underbrace{w(x)}_{p \text{ coords}})$$

want to v, w.

$a \to q$ coords
$b \to p$ coords

$$(i_{X_H}\omega)(a,b) = dH(a,b)$$

$$= \sum \frac{\partial H}{\partial q_i} a_i + \sum \frac{\partial H}{\partial p_i} b_i$$

$$(i_{X_H}\omega)(a,b) = \omega\big((v(x), w(x)), (a,b)\big) = \sum (v_i b_i - w_i a_i)$$

$$\therefore\ v_i = \frac{\partial H}{\partial p_i}, \quad w_i = -\frac{\partial H}{\partial q_i}.$$

$\Big\}$ flow

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

② $M \to$ arb. manifold.

$N = T^* M$, $\quad \omega \to$ Liouville 2-form.

$$H(\theta) = \|v\|^2$$

$\hookrightarrow \|\cdot\|$ is some Riemannian metric on M,
and v is s.t. $\theta_x(w) = \langle v(x), w \rangle$

Integral curves of X_H project to geodesics on M.

(X_H generates the geodesic flow)

Again, back to $\mathbb{R}^n$:

Again, back to $\mathbb{R}^n$:

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i .$$

$$H(q,p) = \frac{1}{2} \sum p_i^2$$

$$\Rightarrow \quad dH = \sum_{i=1}^{n} p_i \, dp_i$$

Thus, $X_H = \sum p_i \, \partial/\partial q_i$.

Integral curve

$$\gamma_{(q,p)}(t) = (q + tp, p).$$

Key properties:

① Hamiltonian flows preserve energy levels

$$X_H \cdot H = dH(X_H) = \omega(X_H, X_H) = 0.$$

② Hamiltonian flows preserve ω. Consequently, preserve $\underbrace{\omega \wedge \cdots \wedge \omega}_{n}$.

$$\mathcal{L}_{X_H}(\omega) \overset{\text{Cartan}}{=} \underbrace{d\, i_{X_H}(\omega)}_{\text{defn of } X_H} + \underbrace{i_X\, d(\omega)}_{\omega \text{ is closed}}$$

$$= d(dH) + 0$$

$$= 0.$$

Lecture 28 (09-11-2022)

Wednesday, November 9, 2022 10:39 AM

Theorem. If M is a compact, connected, orientable n-manifold, then $H^n_{dR}(M) \cong \mathbb{R}$.

Goal: (*) ω is compacted and if $\int \omega = 0$, then ω is exact.

Lemma. Let $M \to$ oriented, connected n-manifold, $N_1, N_2 \subseteq M$ are open submanifolds s.t. (*) holds for N_1, N_2 with $N_1 \cap N_2 \neq \phi$.
Then, (*) holds for $N_1 \cup N_2$.

Proof: Assume $M = N_1 \cup N_2$.
Let ω be compactly supported on M with $\int_M \omega = 0$; and fix an n-form θ cpty. supported on $N_1 \cap N_2$ s.t. $\int_M \theta = 1$.

Choose a part^n of 1 sub. to $\{N_1, N_2\}$.
$\{\varphi, 1-\varphi\}$.
$\text{supp } \varphi \subseteq N_1$, $\text{supp } (1-\varphi) \subseteq N_2$.

Let $\alpha_1 := \varphi \cdot \omega$, $\alpha_2 := (1-\varphi) \cdot \omega$.
Put $c = \int_M \varphi \omega$.

$\beta_1 := \varphi\omega - c\theta$, $\beta_2 = \alpha_2 + c\theta$.

Then, $\int_M \beta_1 = \int_M \beta_2 = 0$.

But $\text{supp } \beta_i \subseteq N_i$. $\therefore \int_{N_i} \beta_i = 0$.

By (*), $\beta_i = d\eta_i$ on N_i.
η_i compactly supp on N_i.

by ..., $\beta_i = d\eta_i$ on N_i.

η_i compactly supp on N_i.
Can extend to M by 0.

Now, $\omega = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = d(\eta_1 + \eta_2)$. ⊠

Sard's Theorem

Thm. Let $F: M \to N$ be a C^∞ map of manifolds.
Let $R_0(F) \subseteq N$ denote the set of regular values of F.
Then, $R_0(F)$ has full measure in N.
(That is, $\varphi(R_0(F))$ is full measure is every chart on N.)

Recall: $n \in N$ is regular
$\iff$ $DF(x)$ is onto for all $x \in F^{-1}(n)$.

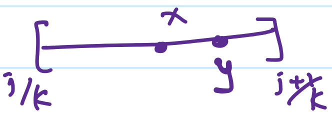
Cor. If $\dim(M) < \dim(N)$, then $\operatorname{im}(M)$ has zero measure under C^∞ maps.

Case. $M = N = [0,1]$.

$A := \sup\{F'(x), F''(x) : x \in [0,1]\}$.

For $k \in \mathbb{N}$, $0 \leq j \leq k-1$ define $I_j^k := \left[\frac{j}{k}, \frac{j+1}{k}\right] \subseteq [0,1]$.

Assume $C =$ critical points satisfies
$C \cap I_j^k \neq \emptyset$.
(pick x here



Taylor's thm: $|F(x) - F(j/k)| \leq A\left|x - \frac{j}{k}\right|^2$

$^{1/k}$ $\quad$ $^{1/k}$

↳ pick x here

Taylor's thm : $\quad |F(x) - F(j/k)| \leq A\left|x - \frac{j}{k}\right|^2$

(expand around x) $\quad |F(x) - F(j/k)| \leq A\left|x - \frac{j+1}{k}\right|^2$

$$\Rightarrow \quad \mathrm{Im}(I_j^k) \subseteq B\left(F(x), \frac{A}{k^2}\right).$$

$$\ell(F(C)) \leq \underbrace{k}_{\#\{j \,:\, C \cap I_j^k \neq \phi\}} \cdot \, 2A/k^2 = 2A/k.$$

Let $k \to \infty$. $\quad \square$

How to adapt for higher? Replace I_j^k by

$$I_{\underline{j}}^k = \left[\frac{j_1}{k}, \frac{j_1+1}{k}\right] \times \cdots \times \left[\frac{j_m}{k}, \frac{j_m+1}{k}\right].$$

Need vanishing of all $< L$ order derivatives. $(*)$

Then,

$$|F(x) - F(a)| \leq A\,|x-a|^L \quad \forall a \in \ldots$$

L to be fixed.

Then, $\quad F(I_{\underline{j}}^k) \subseteq B(F(x), A/k^L).$

$$\Rightarrow \mathrm{vol}_n(F(C)) \leq k^m \cdot c \cdot \left(\frac{A}{k^L}\right)^n \leq CA^n \frac{k^m}{k^{Ln}}.$$

Need $L > m/n$. ✓

How do we get $(*)$? Stronger assumption than critical point.

Last trick: $C := \{\text{all critical pts}\}$

$C_k := \{x : \text{all partials of order} < k \text{ vanish at } x\}$

Lemma: $\mathrm{vol}_n(F(C \setminus C_1)) = 0, \dots, \mathrm{vol}_n(F(C_{k+1} \setminus C_k)) = 0.$

↳ Induction.

Lecture 29 (11-11-2022)

Friday, November 11, 2022 10:31 AM

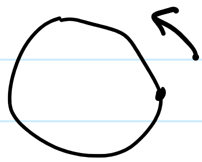
Sard's Theorem

If $F: M \to N$ is a C^∞ map between manifolds, and $\mathcal{C}(F)$ denotes the set of critical points of F, then $F(\mathcal{C}(F))$ has measure zero in N.

($\mathcal{C}(F)$ could be large.)

Degree

$f: S^1 \longrightarrow S^1$. $\deg(f) = \#$ of windings

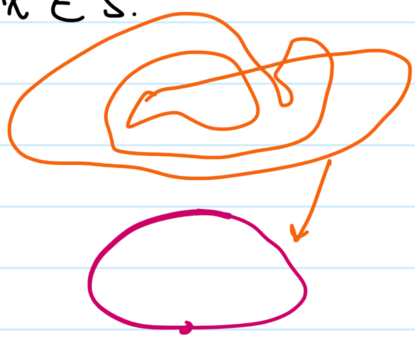


$S^1 = \mathbb{R}/\mathbb{Z}$.

Pick lift $\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}$.

$\deg(f) = \tilde{f}(1) - \tilde{f}(0)$.

2 other ways: ① Choose a regular point $x \in S^1$.



For each point $y \in \pi^{-1}(x)$, look at $df: T_y S^1 \longrightarrow T_x S^1$.

Assign + or − depending on orientation rev/prese.

Then, Σ up.

② Fix any volume form ω.

Let $\deg(f) = \int f^* \omega$.

Let $\deg(f) = \dfrac{\int f^*\omega}{\int \omega}$.

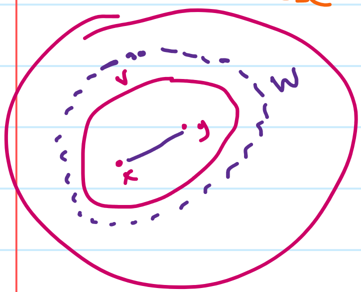
Theorem. If M is a connected manifold, $\mathrm{Diff}^\infty(M)$ acts transitively on M.

(I.e., given $x, y \in M$, $\exists f: M \to M$ diffeo s.t. $f(x) = y$.)

Proof. First let $M = \overline{B_d(0,1)}$.

Will show that for every $x, y \in \mathrm{int}(M)$, $\exists$ diffeo $f: M \circlearrowleft$ s.t. $f(x) = y$ and $f \equiv \mathrm{id}$ on a nbd of S^1.

Choose $V \subseteq_{op} M$ s.t. $[x, y] \subset V$ and $\bar{V} \subset \mathrm{int}(M)$.



Choose a nbd $W \supseteq \bar{V}$ s.t. $\bar{W} \subset \mathrm{int}(M)$.

Choose bump f^n φ s.t. $\varphi|_V \equiv 1$ and $\mathrm{supp}\,\varphi \subseteq W$.

Let v_0 be the constant v.f. $y - x$.

Let

$$v = \varphi \cdot v_0.$$

Let φ_t be the flow gen. by v, and define $f = \varphi_1$. This does the job.

General: Join $x \rightsquigarrow y$ with balls.

Lecture 30 (14-11-2022)

Monday, November 14, 2022 10:46 AM

Standing assumptions (SA): M is compact, oriented, connected.
N is oriented, connected, $\dim(M) = \dim(N)$.

Def^n. Let $F : M \to N$ be C^∞, and $y \in N$ be a regular value. Define

$$\deg(F) = \sum_{p \in F^{-1}(y)} \sigma(DF(p)),$$

($F^{-1}(y)$ is a 0-dim'l submanifold and hence finite.)

where

$$\sigma(A) = \begin{cases} +1, & \text{if } A \text{ preserves orientation,} \\ -1, & \text{if } A \text{ reverses orientation.} \end{cases}$$

Thm. deg is well-defined (i.e., independent of y).
deg is locally constant in the C^1-topology and invariant under homotopy.
($\deg F \overset{?}{=} \deg G$, if F and G close enough)

Remark. If F is not onto, then $\deg(F) = 0$.

Ex. Let $M = N = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $F : \mathbb{T}^n \circlearrowleft$.
Can show: $\exists A \in M_n(\mathbb{Z})$ $\exists \varphi : \mathbb{T}^n \to \mathbb{R}^n$ s.t.
$$F(x) = Ax + \varphi(x). \qquad (A \text{ unique.})$$
Then, $\deg(F) = \det(A)$.

$\hookrightarrow$ If A not invertible, use remark.
$\hookrightarrow$ Else, $Ax = y \pmod 1$
$\iff x = A^{-1}y + A^{-1}m$ for some $m \in \mathbb{Z}^n$

$[\mathbb{Z}^n : A\mathbb{Z}^n] = |\det(A)|. \quad \ldots$

Cor. ① If f is a diffeo, then $A \in GL(n, \mathbb{Z})$. ($\det(A) = \pm 1$.)
② If f not onto, then $\det(A) = 0$.

Prop^n 2. Let M, N satisfy the standing assumption.
Let $H : I \times M \to N$ be a C^∞ homotopy.
Assume y is a regular value of both H_0 and H_1.
$H(0,-)$ $H(1,-)$

Then,

$$\sum_{p \in H_0^{-1}(y)} \sigma(DH_0(p)) = \sum_{p \in H_1^{-1}(y)} \sigma(DH_1(p))$$

Proof of well-definedness using Prop^n 2.:

Let $F : M \to N$ be as before.
Let $y, y' \in M$ be regular values.
As seen last time, $\exists$ flow φ_t s.t. $\varphi_1(y) = y'$.

Define $H : I \times M \to N$ by
$H(t, x) = \varphi_{-t}(x)$.

Now use the previous prop^n. ☒

Prop^n also shows that deg is homotopy invariant.
(Use Sard's to find a common reg. value.)

We now try to prove the prop^n.

Lemma 2. Let M, N satisfy (SA), and $F : M \to N$ is C^∞.
Let $y \in N$ be regular value of F.
Then, $\exists$ nbd U of y and nbds V_p of every $p \in F^{-1}(y)$
such that $F|_{V_p}$ is a diffeo onto U.
Also,

Also,

$$F^{-1}(U) = \bigsqcup_{p \in F^{-1}(y)} V_p.$$

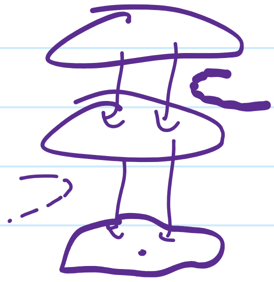
Proof. For each $p \in F^{-1}(y)$, D_pF is an isomorphism.
$\therefore \exists$ nbd W_p of p and U_p of y s.t. $F|_{W_p}$ is a diffeo onto U_p.

WLOG, W_p's are disjoint (Hausdorff, only finitely many p.)

Let $U' = \bigcap_p U_p$. This is open since finite intersection.

Set $V_p' = (F|_{W_p})^{-1}(U)$.

Now, we know
$$F^{-1}(U') \supseteq \bigsqcup_p V_p'.$$



But equality is not known.
Need compactness to shrink U' further.

Choose a compact nbd K of y, contained in U'.

Then, $F^{-1}(K) \overset{\subseteq M}{}$ is closed and hence compact.
Then, $F^{-1}(K) \setminus \bigsqcup_p V_p'$ is again closed and compact.

Moreover, does not contain any point of $F^{-1}(y)$.

Then, $C := F\left(F^{-1}(K) \setminus \bigsqcup_p V_p'\right)$ is a compact and hence, closed subset of U' which does not contain y.

Now, by separation, $\exists\, U \subseteq U'$ s.t. $U \cap C = \emptyset$ and $y \in U$.

U now does the job... $\blacksquare$

Lecture 31 (16-11-2022)

Wednesday, November 16, 2022 10:37 AM

<u>Standing assumptions</u>: $M, N \to$ connected, oriented

M compact

$\dim M = \dim N$

$F : M \to N$ is C^∞.

$$\deg(F) := \sum_{p \in F^{-1}(y)} \sigma(DF(p)),$$

y is any regular value.

<u>Prop^n</u> If $F : I \times M \to N$ is a homotopy between f_0 and f_1, then

$$\sum_{p \in f_0^{-1}(y)} \sigma(Df_0(p)) = \sum_{p \in f_1^{-1}(y)} \sigma(Df_1(p)),$$

where y is a common reg. value.

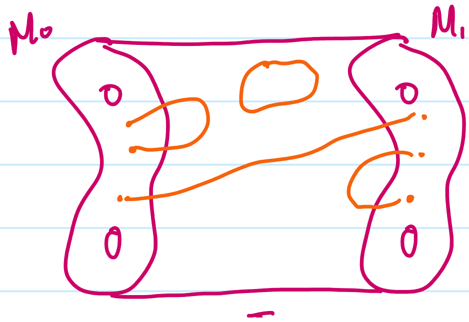
<u>Last time</u>: Under (SA), the regular values are open, and F is a local covering map at regular values.

<u>Idea</u>: We assume y is a reg. value for f_0 and f_1.

By perturbing the nbd, we may assume y is a regular value for F. (By Sard's.)

Apply regular value/submersion theorem to understand $F^{-1}(y)$.

$F^{-1}(y)$ = 1-manifold with boundary



$F^{-1}(y)$

$\to$ union of closed intervals and circles.

Moreover, endpoints of

and circles.
Moreover, endpoints of closed intervals are in $\{0\}\times M$ or $\{1\}\times M$.

Lemma. If $p, q \in F_0^{-1}(y) \cup F_1^{-1}(y)$ then

$$\sigma\left(DF_{i(p)}(p)\right)(-1)^{i(p)} = -(-1)^{i(q)}\sigma\left(DF_{i(q)}(q)\right),$$

whenever $(i(p), p)$ and $(i(q), q)$ are endpoints of a closed interval in $F^{-1}(y)$.

Proof. Let γ be a parameterisation of the closed interval connecting p, q in $I\times M$. $\gamma(0) = (i(p), p)$, $\gamma(1) = (i(q), q)$.

$\gamma'(t) \in T_{\gamma(t)}(I\times M)$ in $\ker DF(y)$. $\left(F(\gamma(t)) = y \ \forall t\right)$

But $\nearrow$ is 1-dim'l.

$\therefore \ \ker DF|\gamma(t)) = \langle \gamma'(t)\rangle$.

Remark 1. The bundle $T(I\times M)$ is trivial over $\operatorname{im}(\gamma)$.
Thus, can pick a continuously varying distribution $E_t \subseteq T_{\gamma(t)}(I\times M)$ which is complementary to $\langle\gamma'(t)\rangle$.

Fix an orientation $\mathcal{O}_t$ on E_t.

Note: $DF_{\gamma(t)}(\mathcal{O}_t)$ is either always +vely or always −vely oriented.

$$\Rightarrow F_*(\mathcal{O}_0) = F_*(\mathcal{O}_1).$$

Similarly, if $(w_1(t), \dots, w_n(t))$ is a +vely oriented framing of E_t, then $(\gamma'(t), w_1(t), \dots, w_n(t))$ is either always +vely or always −vely oriented.

of v_0, then $(\gamma'(t), w_1(t), \dots, w_{n-1})$ is either always +vely or always −vely oriented.

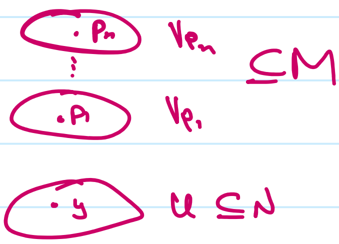
Hence, ∂_0 and ∂_1 induce opposite orientations on M, since $\gamma'(t)$ points in at 0 and out at 1.

Degree using differential forms.

Theorem. Assume (SA).
Let $\omega \in \Omega^n(N)$ be compactly supported. Then,

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

Proof Case: $\operatorname{supp}(\omega) \subseteq \mathcal{U}$, where $\mathcal{U}$ is a nbd of a regular value with the covering property.



$$\Rightarrow \int_M f^*\omega = \sum_{p \in f^{-1}(y)} \int_{V_p} f^*\omega$$

$$= \sum_{p \in f^{-1}(y)} \int \operatorname{sign}(Df|_{V_p}) \cdot \omega$$

$$= \deg(f) \cdot \int_N \omega.$$

General: Cover and partition of unity.
Need to worry about critical values.
For any $y' \in N$, choose a flow φ_t such that $\varphi_1(y) = y'$. Then, $\varphi_1(\mathcal{U}) =: \mathcal{U}'$ is a nbd of y'.

Assume $\operatorname{supp}\omega \subseteq \mathcal{U}'$. Then, $\operatorname{supp}((\varphi_1)^*\omega) \subseteq \mathcal{U}$.

$$\Rightarrow \int_N f^*(\varphi_1^* \omega) = \deg(f) \int_N \varphi_1^* \omega = \deg(f) \int_N \omega. \quad \ldots \quad \boxtimes$$

$$\overset{\text{Lemma}}{=} \int_M f^* \omega$$

Lemma. If $f : I \times M \to N$ is a C^∞ homotopy from f to g, then

$$\int_M f^* \omega = \int_M g^* \omega,$$

for all closed forms ω.

Proof. $\displaystyle \int_M g^* \omega - \int_M f^* \omega = \int_{\{0\} \times M} g^* \omega - \int_{\{1\} \times M} f^* \omega$

$$= \int_{I \times M} d F^* \omega = \int_{I \times M} F^* d\omega = 0. \quad \boxtimes$$

Lecture 32 (18-11-2022)

Friday, November 18, 2022 10:41 AM

Lie Groups

Defn. A Lie Group is a smooth manifold G with C^∞ functions
$$m: G \times G \to G,$$
$$i : G \to G,$$
and a distinguished element $e \in G$ s.t. m, I, e satisfy the group axioms.

Example / Theorem Closed subgroups of matrix groups are Lie groups.

Theorem These are (almost) all Lie groups.
(i.e, every Lie Group is locally isomorphic to a matrix group.)
(i.i.e, $\forall$ Lie Groups G $\exists$ matrix group H, discrete group Γ and a s.e.s. $1 \to \Gamma \to G \to H \to 1$.)

Won't prove/use the above theorems.

Fix a Lie group G, and define maps

$$L_g : G \to G \qquad \text{and} \qquad R_g : G \to G$$
$$h \mapsto \underset{m(g,h)}{\overset{\shortparallel}{g \cdot h}} \qquad\qquad h \mapsto h \cdot g, = m(h,g)$$
for $g \in G$.

The above maps are diffeomorphisms.

Defn. A vector field X on G is called right invariant if : $(R_g)_* X = X$ for all $g \in G$.

$Lie(G)$ = space of right-invariant vfs.

Note: $(R_g)_*$ is linear.

Thus, the space of right-invariant vfs form a vector space.

Thm. The map

$$ev : Lie(G) \longrightarrow T_e(G)$$
$$X \longmapsto X(e)$$

is an isomorphism of vector spaces.

In particular,

$$\dim_{\mathbb{R}}(Lie(G)) = \dim(G).$$

Proof Suppose $v \in T_e(G)$. Define

$$X_v(g) := DR_g(e)(v).$$

Claim 1: X_v is a (C^∞) v.f. on G. (Check Lee.)

Claim 2. X_v is right invariant.

Pf. Let $h \in G$.

$$\begin{aligned}(R_h)_*(X_v)(g) &= DR_h(gh^{-1})(X_v(gh^{-1})) \\ &= DR_h(gh^{-1})(DR_{gh^{-1}}(e)(v)) \quad \text{Chain Rule} \\ &= DR_g(e)(v) \\ &= X_v(g).\end{aligned}$$

$\boxtimes$

Check that $v \longmapsto X_v$ is inverse to ev.

Thm Let X be a right-invariant v.f. on a Lie group G, and φ_t^X be the flow generated by X.
Then,

$$f_X : \mathbb{R} \longrightarrow G$$
$$t \longmapsto \varphi_t^X(e)$$

is a group homomorphism.
The image is a one-parameter subgroup.

Furthermore, $\varphi_t^X(g) = f_X(t)\, g$.

$$\overset{\shortparallel}{g\,\varphi_t^X(e)}$$

Defn The map $\exp : \mathrm{Lie}(G) \longrightarrow G$ defined by $X \mapsto f_X(1)$ is called the exponential map.

Prop. $D(\exp)(0) : T_0(\mathrm{Lie}\, G) \longrightarrow T_e G$ is the identity.

$$T_0(\mathrm{Lie}\,G) \overset{?}{=} \mathrm{Lie}\,G, \qquad T_e G \cong \mathrm{Lie}\,G$$

Lecture 33 (21-11-2022)

Monday, November 21, 2022 10:45 AM

A **Riemannian metric** (on M) is a section of a vector bundle which assigns each $x \in M$ an inner product on $T_x M$ $\langle , \rangle_x$.

Thm. If X is a right invariant vector field on a Lie group G, and φ_t^X is the flow generated by X, then

$$f_X(t) := \varphi_t^X(e)$$

is a homomorphism $\mathbb{R} \to G$.

Moreover,

$$\varphi_t^X(g) = f_X(t) \cdot g.$$

Proof. Fix $h \in G$. Then, $(R_h)_* X = X$.

Then,

$$R_h \circ \varphi_t^X = \varphi_t^X \circ R_h.$$

Thus, $\varphi_t^X(e) h = \varphi_t^X(h)$ for all $h \in G$. — (1)

Thus,
$$f_X(t+s) = \varphi_{t+s}^X(e)$$
$$= \varphi_t^X(\varphi_s^X(e))$$
$$\overset{(1)}{=} \varphi_t^X(e)\, \varphi_s^X(e).$$

$\therefore f$ is a homomorphism.

Lastly, $\varphi_t^X(g) = \varphi_t^X(e) \cdot g = f_X(t) \cdot g.$ ∎

Example. Find the right invariant vector fields for

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

s.t. $$X(e) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And their corresp 1-parameter subgroup.

$$T_e H = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \right\rangle.$$

$$\gamma(t) = \begin{pmatrix} 1 & t & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad \text{is a homomorphism.}$$

The vf is then
$$X(g) = DR_g\, \gamma'(0)$$

$$= \frac{d}{dt}\Big|_{t=0} \left[\gamma(t) g \right] = \frac{d}{dt}\Big|_{t=0} \left[R_g \circ \gamma \right]$$

If $G \subseteq GL(d, \mathbb{R})$ is a matrix group and X is a right invariant vector field,

$$f_X(t) = \sum_{k=0}^{\infty} \frac{(tX(e))^k}{k!}.$$

Defn If $Lie(G)$ is the space of right-invariant vector

Defn. If $Lie(G)$ is the space of right-invariant vector fields, define

$$\exp : Lie(G) \longrightarrow G$$
$$X \longmapsto f_X(e).$$

We've show that $\exp|_L$ is a homomorphism whenever L is a line through the origin.

Adjoint Representation

$$Ad : G \longrightarrow GL(Lie(G))$$
$$g \longmapsto D_e(h \mapsto ghg^{-1})$$

$Ad(g) : Lie(G) \to Lie(G)$

$Lie(G) \nearrow T_eG$, then $D_e(h \mapsto ghg^{-1})$

$Lie(G) \searrow RIVF$, DL_g

Adjoint rep detects how non-abelian the group is.

Defn. Let $X \in Lie(G)$ be a RIVF. Define

$$ad(X) = \frac{d}{dt}\Big|_{t=0} Ad(f_X(t)).$$

$ad(X) : Lie(G) \longrightarrow Lie(G).$

Theorem $ad(X)Y = [X, Y] \quad \forall X, Y \in Lie(G).$

Remark. X, Y RIVfs: $(R_g)_*[X, Y] = [R_{g*}X, R_{g*}Y]$
$= [X, Y]$.
$\therefore [X, Y]$ is again r-inv.

Theorem. If G, H are connected Lie groups, G simply connected, and $\phi: Lie(G) \to Lie(H)$ is a Lie algebra homomorphism.
(That is, $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$.)

Then, $\exists!$ homomorphism $\tilde{\phi}: G \to H$ s.t. $\tilde{\phi}_*|_{Lie(G)} = \phi$.

Theorem Assume M is a connected C^∞-manifold and $G \to$ simply connected
$\exists\, \phi: Lie(G) \to \mathfrak{X}^\infty(M)$ C^∞ vfs on M

s.t. $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

Then, $\exists$ unique group action $\tilde{\phi}: G \times M \to M$
s.t. $\forall$ 1-parameter subgroup of G,

$$\frac{\partial}{\partial t}\left(\tilde{\phi}(f_X(t), x)\right) = \phi(X)(x).$$

Lecture 33 (21-11-2022)

Wednesday, November 23, 2022 10:42 AM

Lemma If $G \subseteq GL(d, \mathbb{R})$ is a Lie subgroup, $g \in G$, and X is a RIVF on G, then

$$(Ad(g)X)(e) = g\,X(e)\,g^{-1}.$$

$GL(d,\mathbb{R}) \subseteq \mathbb{R}^{d^2}$ open

identify $T_e GL(d, \mathbb{R}) \cong \mathbb{R}^{d^2}$.

Proof.
$$(Ad(g)X)(e) = ((L_g)_* X)(e)$$

def^n of $(L_g)_*$

$$= DL_g(g^{-1})(X(g^{-1}))$$

since X is RI

$$= DL_g(g^{-1})\big(DR_{g^{-1}}(e)(X(e))\big)$$

$$= DC_g(e)(X(e)) \qquad (C_g(h) = ghg^{-1})$$

derivative of multi is itself

$$= g\,X(e)\,g^{-1}. \qquad \square$$

Lemma. Under the same setup. If $X, Y \in Lie(G)$ are RIVFs, then

① $ad(X)(Y) = [X, Y]$ (no need for matrix groups)

and

② $[X, Y](e) = X(e)Y(e) - Y(e)X(e).$

Proof. ①
$$ad(X)(Y) = \frac{d}{dt}\Big|_{t=0} \big[Ad(\exp(tX))\,Y\big]$$

$$= \frac{d}{dt}\Big|_{t=0} \big(L_{\exp(tX)}\big)_* Y$$

$$= \frac{d}{dt}\Big| (\varphi_t^X)_* Y = [X, Y].$$

$$= \frac{d}{dt}\Big|_{t=0} (\varphi_t^X)_* Y = [X, Y].$$

② $$(\operatorname{ad}(X)\, Y)(e) = \frac{d}{dt}\Big|_{t=0} (\operatorname{Ad}(\exp(tX))\, Y)(e)$$

↓ prev. lemma

$$= \frac{d}{dt}\Big|_{t=0} \left[\exp(tX)\, Y(e)\, \exp(-tX)\right]$$

($\exp(tX) = \varphi_t^X(e)$

$\therefore \frac{d}{dt}\Big|_{t=0} \exp(tX) = X(e)$)

↓ product rule

$$= X(e)\,Y(e) - Y(e)\,X(e). \quad \square$$

$$\ker(\operatorname{Ad}) = Z(G). \qquad (G \text{ connected.})$$

Thm 1. Let $\Phi : G \to H$ be a C^∞ group homom. of Lie groups. Then, $\exists!$ linear map $\phi : \operatorname{Lie}(G) \to \operatorname{Lie}(H)$ s.t.

$$D\Phi(g)\,(X(g)) = (\phi(X))\,(\Phi(g)).$$

(for all RIVF $\phi \in \operatorname{Lie}(G)$)

Furthermore,

$$\underbrace{[\phi(X_1), \phi(X_2)] = \phi([X_1, X_2])}_{\text{Any linear } \phi \text{ satisfying this is called a Lie Algebra homomorphism.}} \quad \forall X_1, X_2 \in \operatorname{Lie}(G).$$

Theorem 2. Let G be a connected, simply-connected, and H be connected.

If $\phi : \operatorname{Lie}(G) \to \operatorname{Lie}(H)$ is a Lie algebra homomorphism, and ... is connected.

If $\phi : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ is a Lie algebra homomorphism, then

$$\exists!\ \Phi : G \longrightarrow H$$

such that ϕ is the map induced by Φ (as per Thm 1).

Proof. The PLAN: Construct Φ by finding its graph $\Gamma \subseteq G \times H$.
- Γ should be a dim(G)-dim'l submanifold.
- Γ is a subgroup.
- Γ is the leaf of a coset foliation.
- Use Frobenius to build foliation.
  Take Γ to be leaf through origin.

Fix $x \in \mathrm{Lie}(G)$, and let $\tilde{X}$ be defined on $G \times H$ by

$$\tilde{X}(g,h) = \left(X(g),\ \phi(X)(h) \right).$$

ϕ is a v.space homomorphism

$\Downarrow$ $\tilde{x} + \tilde{y} = \widetilde{x+y}$, $c\tilde{x} = \widetilde{cx}$.

$$\mathcal{D} := \mathrm{span}_{\mathbb{R}} \{ \tilde{X} : x \in \mathrm{Lie}(G) \}$$

is a dim(G)-dim'l distribution on $G \times H$.

ϕ is a Lie Algebra homom $\Rightarrow$ $\mathcal{D}$ is involutive.

$$\hookrightarrow [\tilde{x}, \tilde{y}] = [(x, \phi(x)), (y, \phi(y))]$$
$$= ([x,y], [\phi(x), \phi(y)])$$
$$= ([x,y], \phi([x,y])) \in \mathcal{D}$$

Thus, $\exists!$ foliation $\mathcal{F}$ s.t. $T\mathcal{F} = D$.

Let Γ be the leaf containing (e_G, e_H).

$\therefore$ Γ is a dim(G)-dim'l (immersed) submanifold.

Notice : $\forall (g, h) \in G \times H$, $\quad D\pi_G(g,h)\left(\tilde{X}(g,h)\right) = X(g)$.

$\Rightarrow \pi_G|_{\Gamma} : \Gamma \to G$ is a submersion.
By dimension, it is a local diffeo.

* $\Rightarrow \pi_G|_{\Gamma}$ is a covering map

$\Rightarrow \pi_G|_{\Gamma}$ is a diffeo, since G is simply connected.

Now define $\Phi : G \longrightarrow H$
$g \longmapsto \pi_H \circ (\pi_G|_{\Gamma})^{-1}$.

Lecture 34 (28-11-2022)

Monday, November 28, 2022 10:41 AM

Theorem. Let G be a connected, simply-connected Lie Group, and H be a Lie Group. If $\phi: \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ is a Lie algebra homomorphism, then $\exists!\ \Phi: G \longrightarrow H$ s.t. $(\Phi)_* = \phi$.

Pf. Last time: $\mathcal{D} = \{ (X, \phi(X)) : X \in \mathrm{Lie}(G) \}$ is an involutive distribution on $G \times H$.

$\Gamma \rightarrow$ leaf through e.

$\pi_G|_\Gamma$ is a local diffeo, $(*)$ hence diffeo.

Claim. $\Phi = \pi_H \circ (\pi_G|_\Gamma)^{-1}$ is a homomorphism.

Pf of Claim. Step 1. If $X \in \mathrm{Lie}(G)$, then

$$\Phi(\exp(tX)) = \exp(t\,\phi(X)).$$

$(\varphi, \phi, \Phi$ different$)$

$$\Phi(\exp(tX)) = \Phi(\varphi_t^X(e)) = \pi_H\left(\varphi_t^X(e), \varphi_t^{\phi(X)}(e)\right) = \varphi_t^{\phi(X)}(e) = \exp(t\,\phi(X))$$

Then,

$$\Phi(\exp((t+s)X)) = \Phi(\exp(tX))\,\Phi(\exp(sX)).$$

Step 2. If g is sufficiently close to the identity. Then, $\Phi(gh) = \Phi(g)\,\Phi(h)$ for all h.

Proof. Write $g = \exp_G(X)$.

Define $\gamma: \mathbb{R} \longrightarrow H$ by

$$\gamma(t) = \exp_H(-t\,\phi(X))\,\Phi(\exp_G(tX)h).$$

$\ldots(tX))X = 0.$

$\gamma(t) = \exp_H(-t\phi(x)) \, \Phi(\exp_G(tx)h)$.

$\left(\begin{array}{l} Ad(\exp(tx))X = 0. \\ \gamma'(s) \\ = -\phi(x)(\gamma(s)) \\ \quad + \phi(x)(\gamma(s)) \end{array}\right)$

Then, $\gamma'(t) = 0$. (Compute.) $\therefore \gamma$ constant.

$$\gamma(0) = \exp_H(0)\, \Phi(\exp_G(0)h) = \Phi(h).$$

$$\gamma(1) = \exp_H(-\phi(x))\, \Phi(\exp_G(x)h) = (\exp_H(\phi(x)))^{-1}\, \Phi(gh)$$

$$= \Phi(g)^{-1}\Phi(gh).$$

$$\therefore \quad \Phi(g)\Phi(h) = \Phi(gh).$$

Step 3. If $g = g_1 \cdots g_k$ and each g_i is sufficiently close to the identity, then $\Phi(gh) = \Phi(g)\Phi(h)$.

Clear.

But now any g can be written as such a products since G is connected.

(Lemma. If $U \subseteq G$ is an open nbd of e, then $\langle U \rangle = G$.)

This proves the Claim. ∎

$(\Phi)_* = \varphi$ is clear. (Uniqueness left...) ∎

$$D\Phi(g)X(g) = D\pi_H\left((D\pi_G|_\Gamma)^{-1}(X(g))\right)$$
$$= D\pi_H\left(X(g), \phi(X)(\Phi(g))\right)$$
$$= \phi(X)(\Phi(g)).$$

(*) Why was $\pi_G|_\Gamma$ above a covering map?

Once we get a nbd of one point, we can translate using group

on point, we can recover
using group
structure.

Remark. This same technique builds group action on manifolds whenever

C^∞ vfs on M

$$\exists \phi : \mathrm{Lie}(G) \longrightarrow \mathcal{X}^\infty(M) \text{ s.t.}$$
$$\phi([X,Y]) = [\phi(X), \phi(Y)].$$

Homogeneous Spaces

If $G \curvearrowright M$ is transitive, $x \in M$, and $H = \mathrm{Stab}(x) \subseteq G$, then "$G/H \cong M$".

diffeo

Lecture 35 (30-11-2022)

Wednesday, November 30, 2022 10:38 AM

Thm. Let G and H be connected Lie groups, G simply-connected, and $\phi: Lie(G) \to Lie(H)$ be a Lie algebra homomorphism.

Then, $\exists! \ \Phi: G \to H$ s.t. $(\Phi)_* = \phi$.

Cor. If $H, G \to$ simply-connected are s.t. $Lie(G) \cong Lie(H)$, then $G \cong H$.

Cor. Every (center-free) simply-connected Lie group is a discrete extension of a matrix group.

Proof. $ad: Lie(G) \to End(Lie(G))$ is an iso onto its image.

Defn. A subspace $\mathfrak{h} \leq \mathfrak{g}$ of Lie algebra is called a subalgebra if $\forall X, Y \in \mathfrak{h}$, $[X,Y] \in \mathfrak{h}$.

Thm Let G be a group, and $\mathfrak{g} = Lie(G)$.

- If $H \leq G$ is a Lie subgroup, then $Lie(H)$ is a subalgebra of $\mathfrak{g}$.
- If $\mathfrak{h} \leq \mathfrak{g}$ is a Lie subalgebra, $\exists$ a Lie group H with $Lie(H) = \mathfrak{h}$ and a homomorphism $i: H \to G$ s.t. $i_* = id_{\mathfrak{h}}$.

Two examples of exponentiation:

$G = \mathbb{T}^2$, $\mathfrak{h} = \mathbb{R}^2 \rightsquigarrow \phi: \mathbb{R}^2 \to \mathbb{T}^2$

$G = \mathbb{T}^2$, $\mathfrak{h} = \langle v \rangle$, v has irrational slope.

Defn. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ of Lie algebra is called an *ideal* if $\forall X \in \mathfrak{g}, \forall Y \in \mathfrak{h}$, $[X,Y] \in \mathfrak{h}$.

Thm. If G is a connected Lie group and $H \subseteq G$ is a connected Lie subgroup, then H is normal $\iff$ Lie(H) is an ideal.

Proof. $(\Rightarrow)$ Let $X \in \text{Lie}(G)$, $Y \in \text{Lie}(H)$.

$$[X, Y] \overset{(e)}{=} ad(X)\, Y(e)$$
$$= \frac{d}{dt}\Big|_{t=0} \left[Ad(\exp(tX))\, Y_{(e)} \right]$$
$$= \frac{d}{dt}\Big|_{t=0} \left[D\, C_{\exp(tX)}(e)(Y)_{(e)} \right]$$

$C \to$ conjugation

$$\underbrace{\qquad}_{\in\, T_e(H)}$$

$$\therefore\ [X, Y] \in \text{Lie}(H).$$

Sketch.

$(\Leftarrow)$ Assume Lie(H) is an ideal.
Fix $g \in G$, $h \in H$ both close to identity.
Write $g = \exp(X)$, $h = \exp(Y)$ for $X \in \text{Lie}(G)$, $Y \in \text{Lie}(H)$.

Now,

$$\exp(X)\exp(Y)\exp(-X) = C_{\exp(X)}(\exp(Y))$$
$$\overset{\text{formula}}{=} \exp(Ad(\exp(X))\,Y)$$
$$= \exp\Big(\underbrace{\sum_{k=0}^{\infty} \frac{ad(X)^k}{k!}\, Y}_{\in\, \mathfrak{h}}\Big) \in H.$$

Thus group is "locally normal". Connectedness proves the result. $\boxtimes$

Defn $\mathfrak{g}$ is called simple if it has no nontrivial ideals and non abelian.

These are classified. Done by analyzing "maximal diagonal abelian subgroups" and how they act on $Lie(G)$

Lecture 36 (02-12-2022)

Friday, December 2, 2022 10:47 AM

1. Find the connected Lie subgroups of $SL(2,\mathbb{R})$ which contain $\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} : t \in \mathbb{R}\right\}$.

Lie algebra of

Sol^n Pick such a subgroup H.

Consider $\mathrm{Lie}(H) \subseteq \mathfrak{sl}(2,\mathbb{R}) = \left\{\begin{pmatrix}a & b\\ c & -a\end{pmatrix} : a,b,c \in \mathbb{R}\right\}$.

Note $U = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \in \mathrm{Lie}(H)$. $\mathrm{ad}(U)Y = UY - YU$.

Since $\mathrm{Lie}(H)$ is a subalgebra, $\mathrm{ad}(U)(\mathrm{Lie}(H)) \subseteq \mathrm{Lie}(H)$.

Let $X = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $Y = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. $\{U, X, Y\} \to$ basis of $\mathfrak{sl}(2,\mathbb{R})$.

Let us write $\mathrm{ad}(U) : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})$ as a matrix wrt above basis.

$$\mathrm{ad}(U) = \begin{array}{c c} & \begin{array}{ccc} U & X & Y \end{array} \\ \begin{array}{c} U \\ X \\ Y \end{array} & \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$\left(UX - XU = \begin{pmatrix}0 & -2\\ 0 & 0\end{pmatrix}\right)$

$\left(UY - YU = \begin{pmatrix}1 & \\ & -1\end{pmatrix}\right)$

$\therefore$ only inv. subspaces that contain U are $\mathrm{span}\{U\}$, $\mathrm{span}\{U, X\}$, $\mathrm{span}\{U, X, Y\}$.

If two connected subgroups have the $\underbrace{\text{same Lie algebra}}_{\text{equal as subalgebras of } \mathrm{Lie}(G)}$, then they are equal.

$\therefore \exists$ exactly 3 connected subgroups

group itself ? $SL(2,\mathbb{R})$

group itself

$SL(2,\mathbb{R})$

$\exp\left(\begin{pmatrix} a & b \\ & -a \end{pmatrix}\right)$

$\begin{pmatrix} e^{a} & e^{a}b \\ & e^{-a} \end{pmatrix} = \left\{\begin{pmatrix} s & t \\ & 1/s \end{pmatrix} : s>0, t \in \mathbb{R}\right\}.$

2. Compute $\mathrm{Lie}(SL(2,\mathbb{R}))$.

Sol^n. $\dim(SL(2,\mathbb{R})) = 3.$ $\left[\det : M(2,\mathbb{R}) \to \mathbb{R}.\right]$
1 regular value.

Let $\gamma : (-\varepsilon, \varepsilon) \to SL(2,\mathbb{R})$ be any curve s.t. $\gamma(0) = id$.
(Recall: $\mathrm{Lie}(G) = T_{id}(G)$.)

$\Rightarrow \det(\gamma(t)) = 1 \quad \forall t$

$\Rightarrow \frac{d}{dt}\det(\gamma(t)) = 0$ $D_{id}(\det) = Tr.$

$\Rightarrow Tr(\gamma'(t)) = 0$

$\therefore \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

check that their exponentials lie in SL ...

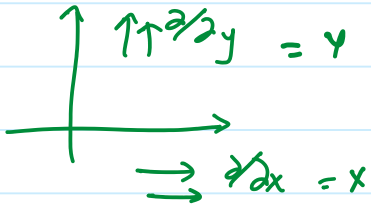
3. What is $\mathrm{Lie}(\mathrm{Isom}(\mathbb{R}^2))$?

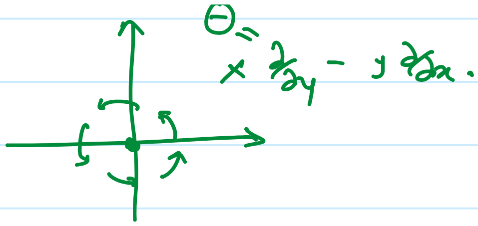
$\mathrm{Isom}^+(\mathbb{R}) \to$ orientation preserving.

Translations Rotation about a point around p

individual subgroups (after fixing p)

$\uparrow\ \uparrow\uparrow \partial/\partial y = Y$ $\uparrow$ $\Theta = x\,\partial/\partial y - y\,\partial/\partial x.$





X, Y, Θ are enough to give arbitrary rotations.

$$\left.\begin{aligned} [X, Y] &= 0 \\ [\Theta, X] &= -Y \\ [\Theta, Y] &= X \end{aligned}\right\}$$

Turns out to be centerfree

$\downarrow$

$\therefore$ adjoint rep is faithful

$$aX + bY + c\Theta \rightsquigarrow \begin{array}{c|c} & \begin{array}{ccc} X & Y & \Theta \end{array} \\ \begin{array}{c} X \\ Y \\ \Theta \end{array} & \begin{pmatrix} & c & -b \\ -c & & a \\ & & \end{pmatrix} \end{array}$$

$\mathrm{Isom}^+(\mathbb{R}) \cong SO(2) \ltimes \mathbb{R}^2$

$\mathrm{Lie}(\mathrm{Isom}^+(\mathbb{R}))$ $\longrightarrow$ this set of matrices

$$\begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

Lecture 37 (05-12-2022)

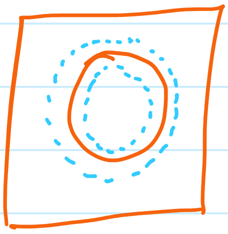
Monday, December 5, 2022 10:42 AM

Tubular Neighbourhood Theorem

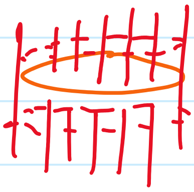
$M \subseteq \mathbb{R}^n$ (embedded) submanifold

$N_x M := \{ v \in \mathbb{R}^n : \langle v, w \rangle = 0 \ \forall\, w \in T_x M \}$ $= (T_x M)^\perp$

$NM := \coprod_{x \in M} N_x M.$ ↳ normal bundle (n-dimensional)



$\mathbb{R}^n \supseteq U \supseteq M$



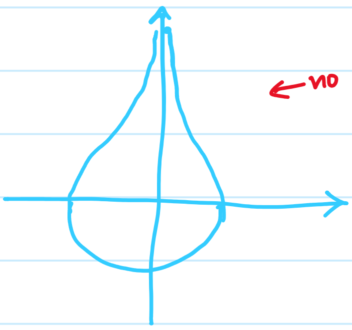
$NM \supseteq U \supseteq M.$ $M = \{0_x : x \in M\}$

$$\pi : NM \longrightarrow M$$
$$v_x \longmapsto x$$

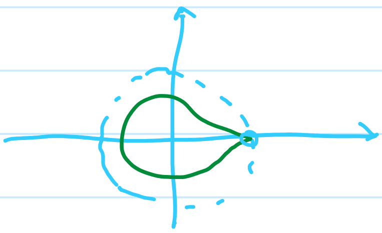
Theorem U ($\subseteq NM$) and V ($\subseteq \mathbb{R}^n$) can be chosen s.t. $\exists$ diffeo $F : U \to V$ s.t. $F|_{M_0} = \pi$.

Cor. If the normal bundle is trivial, $\exists$ nbd of M which is diffeomorphic $B(1, \underset{\sim}{0}\,;\, \mathbb{R}^{n-\dim(M)}) \times M$.

Ex.



← no "uniform thickening"



Remark. The same is true when $\mathbb{R}^n$ is replaced with $\mathbb{Q}$. Need to do some more work to define NM...

Proof in compact case:

$$\begin{tikzcd} & NM \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\ \mathbb{R}^n & & M \end{tikzcd}$$

$\pi(v_x) = x.$
$i(v_x) = v.$

$$\phi_x : T_{v_x}(NM) \longrightarrow \mathbb{R}^n$$

$$\phi_x(w) := D\pi(w) + Di(w).$$

Claim: ϕ_x is an isomorphism. (Show surjectivity and then use dim.)

Define $G : NM \to \mathbb{R}^n$ by
$v_x \longmapsto x + v.$ $\quad (G = \pi + i.)$

Then, $DG(v_x) = \phi_x$ is an iso. $(\forall x \ \forall v_x)$

$\therefore$ G is locally invertible at every 0_x.

For $k \geq 1$, let $U_k \subseteq NM$ denote the set of normal vectors v_x s.t. $\|v_x\| < 1/k$.

Claim. $\exists k$ s.t. $G|_{U_k}$ is injective.

Proof. Suppose not. $\exists (v_k)_{k \geq 1}, (v_k')_{k \geq 1}$ s.t.

we are suppressing base points for notational sake

$v_k, v_k' \in U_k,$ $\quad \forall k \geq 1$
$v_k \neq v_k',$ $\quad G(v_k) = G(v_k').$

we base points for notational sake

But v_k, v_k' can have diff base points!

$v_k, v_k' \in U_k$, $\quad \forall k \geq 1$

$v_k \neq v_k'$, $\quad G(v_k) = G(v_k')$.

By compactness of $\overline{U}_1$, we may pass to subsequences and assume

$$v_k \to v \text{ and } v_k' \to v'$$

for some $v, v' \in \bigcap U_k = M_0$.

But $G|_{M_0}$ is a diffeo. So, $v = v' = 0_x$.

But this is a contradiction since G is a local diffeo around 0_x and $v_k, v_k' \in$ nbd for high k. ☒

This finishes the proof since G is a local diffeo. ☒

(Collar Neighbourhood Theorem)

Thm. Let M be a manifold with boundary. Then, $\exists$ nbd U of ∂M s.t. U is diffeomorphic to $\partial M \times [0,1)$, and the diffeo restricted to ∂M is "id".