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MA 406

General Topology

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Spring 2020-21

Lecture 1 (07-01-2021)

07 January 2021 15:04

Def^n. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) ϕ and X are in $\mathcal{T}$.
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Any $U \in \mathcal{T}$ is called an open set of X w.r.t. $\mathcal{T}$.
The pair $(X, \mathcal{T})$ or just the set X is called a topological space.

Can reconcile the above with open sets in $\mathbb{R}$, or in general, any metric space X. That can be seen as a motivation for the definition.

Examples

(1) $X = \{a, b, c\}$
$\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ → (Can be seen (fairly easily) that this is a topology
$\mathcal{T}_2 = \{\phi, X\}$
→ trivial (pun intended, cf. next example)

(2) If X is any set, the collection of all subsets of X is a topology on X, it is called the discrete topology.
($\mathcal{T} = \mathcal{P}(X)$, that is)

The collection $\{\phi, X\}$ is also a topology on X called the indiscrete topology or trivial topology.

(3) Let X be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\emptyset\}.$$

Then, $\mathcal{T}_f$ is a topology on X, called the finite complement topology on X.

- $\emptyset \in \mathcal{T}_f$ is clear. $X \in \mathcal{T}_f$ since $|X \setminus X| = 0 < \infty$.
- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be sets in $\mathcal{T}_f$. WLOG, $U_\alpha \neq \emptyset$ $\forall \alpha$.

Note $$X \setminus \left(\bigcup_\alpha U_\alpha \right) = X \cap \left(\bigcup_\alpha U_\alpha \right)^c$$
$$= \bigcap_\alpha (U_\alpha^c)$$

Note that each U_α^c is finite. $(U_\alpha \neq \emptyset)$

Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to $\bigcup_{i=1}^{n} (U_i^c)$ and conclude as earlier.

(Here, if some U_i were $\emptyset$, then so would be the intersection.)

(If X is finite, the $\mathcal{T}_f = \mathcal{P}(X)$. Thus, we get discrete.)

(4) Let X be a set.
Let $\mathcal{T}_c$ be the collection of subsets such that $X \setminus U$ is either countable or all of X.
Called the co-countable topology.
(Generalising the previous)

Def^n Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X.
If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and that $\mathcal{T}$ is coarser than $\mathcal{T}'$!
If $\mathcal{T}' \supsetneq \mathcal{T}$, then the above is strictly finer and strictly ...

coarser, respectively.

(The above gives us a way to compare two topologies)

EXAMPLE We have the usual topology on $\mathbb{R}$. ← strictly coarser than this
We also have the discrete topology on $\mathbb{R}$.

Def^n. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

(1) for each $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Note that in the above, $\mathcal{B}$ is just some collection of subsets of X satisfying (1) & (2). No topology is mentioned so far.

EXAMPLES

(1) $X = \mathbb{R}^2$, $\mathcal{B}$ is the collection of all discs w/o boundary.
(2) ——"—— ——"—— - rectangles ——"——
(3) Any X. The singletons form a basis.

We now get a topology out of a basis:

Def^n. If $\mathcal{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows:

A subset U of X is said to be open if for every $x \in U$, there exists $B \in \mathcal{B}$ s.t.
$x \in B \subset U$.

$$x \in B \subset U.$$

(By "open" in above, we mean element of *T*. Same thing for what we see in the proof below.)

EXAMPLES (1) & (2) $\rightarrow$ gives standard topology on $\mathbb{R}^2$

(3) $\rightarrow$ gives discrete topology on X.

We still have to show that it is topology.

Proof:

- $\phi \in \mathcal{T}$ vacuously

  $X \in \mathcal{T}$ since given any $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.

  $B \subset X$ is by definition.

- Let $\{U_\alpha\}_{\alpha \in \Lambda \neq \phi}$ be open. Let $U := \bigcup_\alpha U_\alpha$.

  Fix $\alpha_0 \in \Lambda$.

  Let $x \in U$ be arbitrary. Then, $x \in U_{\alpha_0}$ ← open

  $\therefore$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U_{\alpha_0} \subset U$.

  $\therefore$ $U \in \mathcal{T}$.

- Let U_1 and U_2 be open. Put $U := U_1 \cap U_2$.

  Let $x \in U$.

  Then $x \in U_1$ and $x \in U_2$

  $\Downarrow$ $\Downarrow$

  $\exists B_1 \in \mathcal{B}$ $\exists B_2 \in \mathcal{B}$

  s.t. $x \in B_1 \subset U_1$ s.t. $x \in B_2 \subset U_2$

  $$\therefore \underbrace{x \in B_1 \cap B_2}_{\Downarrow} \subset U_1 \cap U_2$$

  $$\exists B_3 \in \mathcal{B} \text{ s.t. } x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U.$$

  $\Rightarrow U \in \mathcal{T}$.

By induction, any finite intersection is in $\mathcal{T}$. $\boxtimes$

$$n \geq 3 \left(\bigcap_{i=1}^{n} U_i = U_n \cap \left(\bigcap_{i=1}^{n-1} U_i \right) \right.$$