$\mathbb{R}$eal Analysis

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Contents

§0. Notations

1. $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers.
2. $\mathbb{Z}$ is the set of integers.
3. $\mathbb{Q}$ is the set of rational numbers.
4. $\mathbb{R}$ is the set of real numbers.
5. $A \subset B$ is read as "A is a subset of B." In particular, note that $A \subset A$ is true for any set A.
6. $A \subsetneq B$ is read "A is a *proper* subset of B."
7. $\supset$ and $\supsetneq$ are defined similarly.
8. Given a set S, the set $\mathcal{P}(S)$ is the *power set* of S, i.e., the set of all subsets of $\mathcal{P}(S)$.
9. Given a function $f : X \to Y$, $A \subset X$, $B \subset Y$, we define
$$f(A) = \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y,$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X.$$
(Note that this f^{-1} is different from the inverse of a function. In particular, this is always defined, even if f is not bijective. However, the f and f^{-1} above need not be "inverses." See 1 of §1.)
Remark. The above is essentially an abuse of notation since we are using $f : A \to B$ to get another function $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ which we are again denoting with f.
10. If $f : X \to Y$ is a function and $A \subset X$, then $f|_A$ is a function
$$f|_A : A \to Y$$
defined as
$$f|_A(a) = f(a), \quad a \in A.$$
11. Since Rudin follows a non-usual definition for "countable," I shall use the following, which makes it always clear:
    (a) At most countable: A set S is at most countable if there exists an injection $i : S \to \mathbb{N}$.

(b) Countably infinite: A set S is countably infinite if it is at most countable and infinite.

(c) Uncountable: A set S is uncountable if it not at most countable.

In particular, I will not use the term "countable" just by itself since Rudin uses it to mean "countably infinite" but usually people mean "at most countable."

§1. Sets and stuff

1. Given a function $f : X \to Y$, $A \subset X$, $B \subset Y$, we define

$$f(A) = \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y,$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X.$$

   (a) Show that $A \subset f^{-1}(f(A))$.
   Show that this inclusion can be proper.
   Thus, it is possible that $A \neq f^{-1}(f(A))$.
   Show that equality holds if f is injective.

   (b) Show that $B \supset f(f^{-1}(B))$.
   Show that this inclusion can be proper.
   Thus, it is possible that $B \neq f(f^{-1}(B))$.
   Show that equality holds if f is surjective.

2. Let $i : A \to B$ and $j : B \to A$ be injections.
   Show that there exists a bijection between A and B.
   Remark. This is known as the Schröder–Bernstein theorem. (The link has a proof of it as well.)

3. Show that if S is infinite, then there is an injection $i : \mathbb{N} \to S$.

4. Show that if S is infinite and if there exists an injection $j : S \to \mathbb{N}$, then S is at countably infinite.

5. Let C be a countably infinite set. Show that if S is infinite and if there exists an injection $j : S \to C$, then S is countably infinite.

6. Show that $\mathbb{Q}$ is countably infinite.

7. Show that if A is at most countable, then so is $A \times A$. Conclude that A^n is at most countable for all $n \geq 1$.

8. Show that $\mathbb{Q}^n$ is countably infinite for all $n \geq 1$.

9. Let $\{0,1\}^{\mathbb{N}}$ be the set of all sequences with entries from $\{0,1\}$.
   In other words, $\{0,1\}^{\mathbb{N}}$ is the set of all functions from $\mathbb{N}$ to $\{0,1\}$.
   Show that $\{0,1\}^{\mathbb{N}}$ is uncountable.

10. Show that $[0,1]$ is uncountable. (Hence, so is $\mathbb{R}$.)

11. Show that there exists a bijection between any two of the following sets:

$$(0,1),\ [0,1],\ (0,1],\ \mathbb{R},\ \mathbb{R} \setminus \mathbb{Q}.$$

12. Show that there exists a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$.
(You can use properties such as binary/ternary expansions.)

13. Given a set I, $\{P_\alpha\}_{\alpha \in I}$ is a shorthand for writing a set of the form $\{P_\alpha \mid \alpha \in I\}$.
(P_α is defined given the context.)

§2. Topology

1. Let X be a metric space and let $U \subset X$. Define the *boundary* of U as
$$\partial U = \bar{U} \cap \overline{(U^c)}.$$
Show that $\partial U = U \setminus U^\circ$.

2. Prove or disprove that
$$(\partial U)^\circ = \varnothing$$
for any subset U of any metric space X.
HIDDEN: Disprove it. Even in the case that $X = \mathbb{R}^n$.

3. Construct a set $A \subset [0,1] \times [0,1]$ such that A contains at most one point on each horizontal and vertical line but $\partial A = [0,1] \times [0,1]$.
HIDDEN: It suffices to ensure that A contains points in each quarter of the square $[0,1] \times [0,1]$ and also in each sixteenth, et cetera.

4. Let (X, d) be a metric space and $x \in X$. Let $\delta > 0$. Define the following sets:
$$B_\delta(x) := \{y \in X \mid d(x,y) < \delta\},$$
$$C_\delta(x) := \{y \in X \mid d(x,y) \leq \delta\}.$$
Show that $\overline{B_\delta(x)} \subset C_\delta(x)$.
Can this inclusion be proper?
HIDDEN: Not if you stay in $\mathbb{R}^n$. Think about other spaces.

5. **Topological Nim**
You and your friend want to play Topological Nim. Here's how it works:
Let X be your favourite compact metric space and $r > 0$ your favourite (positive) real number.
Each player removes an open disk of radius r from the space on their turn (only the center of the disk must not have been removed in a prior move), until one player—the winner—removes what remains of the space on his turn.

Show that no matter what moves are played, the game stops after a finite number of moves. (In other words, there is no infinite sequence of legal moves.)

Bonus: Fix $n \in \mathbb{N}$ and $r > 0$. Assuming optimal play, who will win the game if
$$X = S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

with the standard metric?
(The answer will depend on r.)

Credits: https://puzzling.stackexchange.com/questions/99859/

6. Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be a collection of disjoint nonempty open intervals.
   That is, for each $\alpha \in \Lambda$, I_α is a nonempty open interval.
   Moreover, if $\beta \in \Lambda$ with $\beta \neq \alpha$, then $I_\alpha \cap I_\beta = \varnothing$.
   Show that Λ is at most countable.
   HIDDEN: Each interval contains a rational. Construct an *injection* $\Lambda \to \mathbb{Q}$.

7. Let $I \subset \mathbb{R}$ be such that every $x \in I$ is an isolated point.
   Show that I is at most countable.
   HIDDEN: Try to use the previous result.

8. Show that every open set U in $\mathbb{R}$ can be written as a disjoint union of nonempty open intervals. Moreover, show that this set of open intervals is at most countable.
   HIDDEN: Consider an equivalence relation $\sim$ on U where $x \sim y$ iff $[x, y] \subset U$.

9. Let K be a compact subset of $\mathbb{R}^n$. Fix a constant $r > 0$.
   Show that there exists a finite collection of points $x_1, \ldots, x_k \in K$ such that the collection of open balls $\{B(x_i, 2r)\}_{i=1}^{k}$ forms an open cover of K while $B(x_i, r)$ are mutually disjoint.

§3. Continuity

1. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the first projection map, that is,
$$\pi_1(x, y) = x.$$
Show that π_1 is an *open map,* that is, $\pi_1(U)$ is open in $\mathbb{R}$ if U is open in $\mathbb{R}^2$. Is it a closed map?
HIDDEN: No.

2. **Pasting lemma.**
Let X be a metric space and $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X.
Let Y be an arbitrary metric space. Suppose that for each $\alpha \in I$, we have a continuous function
$$f_\alpha : U_\alpha \to Y.$$
Moreover, assume that whenever $x \in U_\alpha \cap U_\beta$, then $f_\alpha(x) = f_\beta(x)$. (That is, the functions agree on their common domains.)

    Show the following:

    (a) There exists a unique function $f : X \to Y$ such that
    $$f|_{U_\alpha} = f_\alpha \quad \text{for all } \alpha \in I.$$
    (Recall 10 from §0.)

    (b) The above function f is continuous.

3. Show that the above is not true if we replace "open" with "closed."
(In particular, observe very carefully where you used open-ness of U_α.)

4. Show that the above becomes true once again after replacing "open" with "closed" if we further impose that I be finite.

5. Show that the above is equivalent to the following formulation:
Let $f : X \to Y$ be a function between metric spaces.
Let $X = C_1 \cup \cdots \cup C_n$ where each C_i is closed in X.
Assume that $f|_{C_i} : C_i \to Y$ is continuous for all $1 \leq i \leq n$.
Then, f is continuous.
(Write the above formulation for open sets as well.)

    Remark. The above lemma for closed sets makes it especially easy to directly verify the continuity of "piece-wise" defined functions, if each piece is closed in the ambient space. (cf. 8)

6. Give a counterexample if we further drop "closed" completely, even if I is finite. (In fact, you can give one with $X = \mathbb{R}$ and $|I| = 2$.)

7. Given an example of a continuous bijection $f : X \to Y$ such that $f^{-1} : Y \to X$ is not continuous.

8. Justify that the following is an example for the above question:
   $f : [0, 1] \cup (2, 3] \to [0, 2]$ defined by

$$f(x) := \begin{cases} x & x \in [0,1] \\ x-1 & x \in (2,3] \end{cases}.$$

9. Let $f : X \to Y$ be a function between metric spaces.

   (a) f is said to be *open continuous* if $f^{-1}(U)$ is open in X whenever U is open in Y.

   (b) f is said to be *closed continuous* if $f^{-1}(U)$ is closed in X whenever U is closed in Y.

   Show that f is continuous iff f is open continuous iff f is closed continuous.

10. Let K be a compact metric space and Y an arbitrary metric space.
    Assume that $f : K \to Y$ is a continuous bijection.

    (a) Let $C \subset K$ be closed. Show that C is compact.

    (b) Show that $f(C)$ is compact.

    (c) Show that $f(C)$ is closed.

    Conclude that $f^{-1} : Y \to K$ is continuous.

11. The following question appeared on a test:

    > Given an example of a continuous bijection $f : X \to Y$ such that
    > $f^{-1} : Y \to X$ is not continuous.

    The lazy TA sees that a student has started their answer as

    > The following is example:
    > Let $f : S^1 \to S^1$ be defined as...

    The TA sees that and marks it wrong straight away. Was the TA justified (mathematically, not morally) in doing so? Why?

12. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ be continuous. We know that if I is compact, then f is bounded and it achieves (both) its bounds.
    Show that if I is not compact, then one can always construct:
    (a) a continuous f which is not bounded,
    (b) a continuous f which is bounded but fails to achieve one (or both) of its bounds.

13. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ be continuous. We know that if I is compact, then f is uniformly continuous.
    Can we again do something like the previous case?
    That is: if I is not compact, then can one always construct a continuous f which is *not* uniformly continuous?
    HIDDEN: No. Show that every function $f : \mathbb{Z} \to Y$ is not only continuous but uniformly continuous.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that
    $$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x)$$
    both exist and are finite.
    Show that f is bounded.

15. Suppose f is continuous on $[0,1]$ with $f(0) = f(1) = 0$. For all $x \in (0,1)$, there exists $h > 0$ with $0 \le x - h < x < x + h \le 1$ such that
    $$f(x) = \frac{f(x+h) + f(x-h)}{2}.$$
    Show that $f(x) = 0$ for all $x \in [0,1]$.
    (Note that given any x, the above only says that there's *one* particular h with the given property.)

§4. Derivatives

1. Prove or disprove:
   Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. If $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$, the there exists an interval I containing x_0 such that f is increasing on I.
   HIDDEN: Prove.

2. Prove or disprove:
   Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$, the there exists an interval I containing x_0 such that f is increasing on I.
   HIDDEN: Disprove.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $\lim\limits_{x\to\infty} f(x)$ exists and is finite.
   Prove or disprove:
   $$\lim_{x\to\infty} f'(x) = 0.$$
   HIDDEN: The limit need not exist.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim\limits_{x\to\infty} f(x)$ exists and is finite. Further assume that f' is uniformly continuous.
   Prove or disprove:
   $$\lim_{x\to\infty} f'(x) = 0.$$
   HIDDEN: Prove.

5. Let I be an open interval and $f : I \to \mathbb{R}$ be differentiable. Show that f' need not be continuous.
   Show that f' has the intermediate value property. That is, if $a, b \in I$ with $f'(a) < r < f'(b)$, then there exists $c \in (\min\{a, b\}, \max\{a, b\})$ such that $f'(c) = r$.
   This is known as Darboux's Theorem.

6. Let I be an open interval and $f : I \to \mathbb{R}$ be differentiable.
   Prove that f' is continuous if and only if the inverse image under f' of any point is a closed set.

7. Let (X, d) be a complete metric space. (That is, every Cauchy sequence in X converges.)
   Let $f : X \to X$ be a function with the following property:
   There exists $0 < K < 1$ such that
   $$d(f(x), f(y)) \leq Kd(f(x), f(y)) \quad \text{for all } x, y \in X.$$

Show that:

(a) f is (uniformly) continuous.

(b) f has a fixed point.
(That is, $f(x) = x$ for some $x \in X$.)

(c) f has a unique fixed point.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$, where $K < 1$ is some fixed positive constant.
Show that $\mathbb{R}$ has a unique fixed point.

9. Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $|f'(x)| < 1$ such that f has no unique fixed point.

   Contemplate on how this is different from the earlier question.

10. Show that $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \exp\left(-\cos^2(x)\right)$$

has a unique fixed point.
(How would you calculate it numerically? Was your proof of 7b "constructive"?)

§5. Integrals

1. Does there exist a function $f : [0,1] \to \mathbb{R}$ such that it takes only a finitely many values and is Riemann Integrable on $[0,1]$ but is not locally constant?
   HIDDEN: Yes. Find/show the existence of one.

§6. Sequence and series of functions

1. **(Non-)converse of Weierstrass M-test**
   Construct an example of a family $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\sum f_n$ converges uniformly but $\sum M_n$ does not, where $M_n := \sup\limits_{x\in\mathbb{R}} |f_n(x)|$.
   HIDDEN: Consider f_n such that f_n takes value $1/n$ at n and 0 otherwise.

2. Recall that if $f : K \to \mathbb{R}$ is a continuous function and K is compact, then there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials such that $P_n \to f$ uniformly on K.
   Show that this need not be true if K is not compact.
   HIDDEN: Consider $K = \mathbb{R}$ and $f = \exp$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials such that $P_n \to f$ **pointwise** on $\mathbb{R}$.

4. Let $K \subset \mathbb{R}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions $f_n : K \to \mathbb{R}$.
   Suppose that $f_n \to f$ uniformly on compact subsets of K.
   Show that f is continuous.
   Show that it is not necessary that f is differentiable (anywhere).
   HIDDEN: Consider K to be compact and f to be a Weierstrass type function.

   Remark. The above is different from the case in $\mathbb{C}$omplex Analysis where one has the following theorem:

   > **Montel's Theorem**.
   > Let Ω be an open set in $\mathbb{C}$ and (f_n) a sequence of (complex) differentiable functions $f_n : \Omega \to \mathbb{C}$.
   > Suppose that $f_n \to f$ uniformly on compact subsets of Ω.
   > Then, f is also (complex) differentiable.
   > Further, $f_n' \to f'$ uniformly on compact subsets of Ω.

   This is just one example of how much "better" things behave in $\mathbb{C}$ Analysis as compared to $\mathbb{R}$. In $\mathbb{R}$, not only can f fail to be differentiable but it can differentiable *nowhere*.

5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_n(x) := \left(1 + \frac{z}{n}\right)^n.$$

   Show that f_n does not converge uniformly.