Algebraic Topology

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In what follows, I will denote the closed interval $[0,1] \subset \mathbb{R}$.
Whenever we talk about a map $f : X \to Y$ between topological spaces X and Y, we will always mean a *continuous function* f.
A path σ in a space X is a map $\sigma : I \to X$. If $x_0 = \sigma(0)$ and $x_1 = \sigma(1)$, we write this as

$$x_0 \xrightarrow{\sigma} x_1.$$

Moreover, x_0 and x_1 are called the *end points* of σ. In particular, x_0 is the initial point and x_1 is the terminal point.
All the topological spaces are assumed to be nonempty.

Theorem 0.1 (The Eckmann-Hilton Argument)**.** Let M be a set and $*, \star$ be two unital binary operations on M with units 1_* and $1_\star$, respectively. Suppose that

$$(a * b) \star (c * d) = (a \star c) * (b \star d)$$

for all $a, b, c, d \in M$.
Then, $1_* = 1_\star$, $* = \star$, and furthermore, the operation(s) are commutative and associative.

Proof.
Claim 1. $1_* = 1_\star$.

$$\begin{aligned} 1_* = 1_* * 1_* &= (1_* \star 1_\star) * (1_\star \star 1_*) \\ &= (1_* * 1_\star) \star (1_\star * 1_*) = 1_\star \star 1_\star = 1_\star. \end{aligned}$$

Thus, we define $1 := 1_* = 1_\star$ which is the unit for both operations.

. . . .

Claim 2. $a * b = a \star b$ for all $a, b \in M$.

$$\begin{aligned} a * b &= (a \star 1) * (1 \star b) \\ &= (a * 1) \star (1 * b) = a \star b. \end{aligned}$$

Thus, we define $\cdot = * = \star$.

....

Claim 3. $a \cdot b = b \cdot a$ for all $a, b \in M$.

$$\begin{aligned} a \cdot b &= (1 \cdot a) \cdot (b \cdot 1) \\ &= (1 \cdot b) \cdot (a \cdot 1) = b \cdot a. \end{aligned}$$

....

Claim 4. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in M$.

$$\begin{aligned} (a \cdot b) \cdot c &= (a \cdot b) \cdot (1 \cdot c) \\ &= (a \cdot 1) \cdot (b \cdot c) = a \cdot (b \cdot c). \end{aligned}$$

Thus, we have proven all the claims. $\square$

§1. Homotopy of Paths

§§1.1. The Fundamental Group

Definition 1.1 (Homotopy)**.** Let σ and τ be paths in a space X with the same end points, i.e., $\sigma(0) = \tau(0)$ and $\sigma(1) = \tau(1)$.
We say that σ and τ are *homotopic with ends points held fixed* written

$$\sigma \simeq \tau \ \ \mathrm{rel}\,\{0,1\}$$

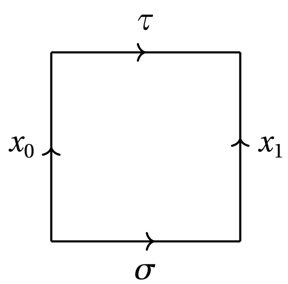
if there is a map $F : I \times I \to X$ such that

1. $F(s,0) = \sigma(s)$ for all $s \in I$,
2. $F(s,1) = \tau(s)$ for all $s \in I$,
3. $F(0,t) = x_0$ for all $t \in I$,
4. $F(1,t) = x_1$ for all $t \in I$.

F is called a *homotopy* from σ to τ. We write

$$F : \sigma \simeq \tau \ \ \mathrm{rel}\,\{0,1\}.$$

The above can be pictorially depicted as



The above picture is interpreted as follows:
Along the (bottom) line $t = 0$, F agrees with σ and along the (top) line $t = 1$, F agrees with $t = 1$.
Similarly, along the (left) line $s = 0$, F is identically equal to x_0 and along the (right) line $s = 1$, it is x_1.

In particular, if σ is a *loop*, i.e., $x_0 = x_1$ and e_{x_0} is the constant loop $s \mapsto x_0$ for $s \in I$, and if $\sigma \simeq e_{x_0}$ rel $\{0,1\}$, we say that "σ can be shrunk to a point," or is *homotopically trivial.*

Proposition 1.2 ($\simeq$ is an equivalence relation)**.**

1. $\sigma \simeq \sigma \ \operatorname{rel}\{0,1\}$,
2. $\sigma \simeq \tau \ \operatorname{rel}\{0,1\} \Longrightarrow \tau \simeq \sigma \ \operatorname{rel}\{0,1\}$,
3. $\sigma \simeq \tau \ \operatorname{rel}\{0,1\}$ and $\tau \simeq \rho \ \operatorname{rel}\{0,1\} \Longrightarrow \sigma \simeq \rho \ \operatorname{rel}\{0,1\}$.

Proof. 1. Define $F(s,t) := \sigma(s)$.

2. Define $F(s,t) := F(s,1-t)$.
3. Given $F : \sigma \simeq \tau \ \operatorname{rel}\{0,1\}$ and $G : \tau \simeq \rho \ \operatorname{rel}\{0,1\}$, define $H : I \times I \to X$ as

$$H(s,t) := \begin{cases} F(s,2t) & 0 \le 2t \le 1, \\ G(s,2t-1) & 1 \le 2t \le 2. \end{cases}$$

   Note that F and G do agree for $2t = 1$ since we have $F(s,1) = \tau(s) = G(s,0)$ for all $s \in I$. It is easy to see that H is well-defined.
   Note that H is continuous (by the pasting lemma) and it satisfies all the four properties of a homotopy (from σ to ρ), since F and G do so. □

Thus, we can consider the homotopy classes $[\sigma]$ of paths σ from x_0 to x_1 under the equivalence relation $\simeq$. (Note very carefully that all paths in an equivalence class have the same end points.)

Definition 1.3 (Multiplication of paths)**.** Let σ be a path from x_0 to x_1 and τ from x_1 to x_2.
The product $\sigma * \tau$ is a path from x_0 to x_2 defined as

$$\sigma * \tau(s) := \begin{cases} \sigma(2s) & 0 \le 2s \le 1, \\ \tau(2s-1) & 1 \le 2s \le 2. \end{cases}$$

Once again, it's an easy check that $\sigma\tau$ is well-defined and continuous (using the pasting lemma).

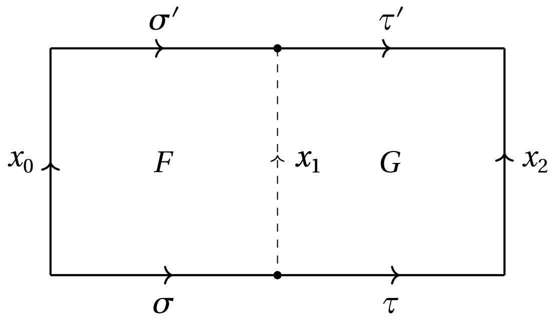
The above $\sigma * \tau$ is essentially the path from x_0 to x_1 obtained by first travelling from x_0 to x_1 via σ and then from x_1 to x_2 via τ.

We will now be lenient with notation and simply denote $\sigma * \tau$ as $\sigma\tau$ unless otherwise necessary.
The next proposition shows how this product behaves with the equivalence relation.

Proposition 1.4.

$$\sigma \simeq \sigma' \ \operatorname{rel}\{0,1\} \text{ and } \tau \simeq \tau' \ \operatorname{rel}\{0,1\} \Longrightarrow \sigma\tau \simeq \sigma'\tau' \ \operatorname{rel}\{0,1\}.$$

Proof. The proof is motivated by the following diagram.



Given $F : \sigma \simeq \sigma' \ \mathrm{rel}\,\{0,1\}$ and $G : \tau \simeq \tau' \ \mathrm{rel}\,\{0,1\}$, define $H : I \times I \to X$ as

$$H(s,t) := \begin{cases} F(2s,t) & 0 \le 2s \le 1, \\ G(2s-1,t) & 1 \le 2s \le 2. \end{cases}$$

As earlier, H is well-defined (since $F(1,t) = x_1 = G(0,t)$ for all $t \in I$) and continuous. Moreover, we have

$$H(0,t) = F(0,t) = x_0, \quad H(1,t) = G(1,t) = x_2,$$

$$H(s,0) = \begin{cases} F(2s,0) & 0 \le 2s \le 1, \\ G(2s-1,0) & 1 \le 2s \le 2 \end{cases} = \begin{cases} \sigma(2s) & 0 \le 2s \le 1, \\ \tau(2s-1) & 1 \le 2s \le 2 \end{cases} = \sigma\tau(s),$$

and similarly,

$$H(s,1) = \sigma'\tau'(s) \text{ for all } s \in I.$$

This shows that

$$H : \sigma\tau \simeq \sigma'\tau' \ \mathrm{rel}\,\{0,1\}.$$

□

Definition 1.5 (Product of equivalence classes)**.** In view of the above proposition, we define

$$[\sigma] * [\tau] := [\sigma * \tau].$$

The above, of course, is defined only when the terminal point of σ (and thus, any other representative of $[\sigma]$) equals the initial point of τ (and thus, any other representative of $[\tau]$).

As before, we shall drop the $*$ and simply write $[\sigma][\tau]$.

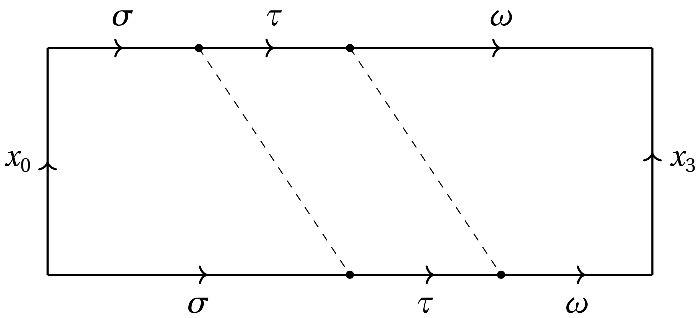
Lemma 1.6. Let σ, τ, ω be paths such that the products $\sigma(\tau\omega)$ and $(\sigma\tau)\omega$ are defined. Then,

$$\sigma(\tau\omega) \simeq (\sigma\tau)\omega \ \text{rel}\,\{0,1\}.$$

Proof. Let x_0, x_1, x_2, x_3 be points such that

$$x_0 \xrightarrow{\sigma} x_1 \xrightarrow{\tau} x_2 \xrightarrow{\omega} x_3.$$

We define a homotopy F from $\sigma(\tau\omega)$ to $(\sigma\tau)\omega$. To motivate the definition of F, we may first visualise the homotopy as follows.



One can note that the top line depicts the path $(\sigma\tau)\omega$ and the bottom $\sigma(\tau\omega)$.

We define $F : I \times I \to X$ piece-wise on the three regions (from left to right) as follows:

$$F(s,t) := \begin{cases} \sigma\left(\dfrac{4s}{2-t}\right) & 0 \le s \le \dfrac{1}{4}(2-t), \\ \tau(4s+2-t) & \dfrac{1}{4}(2-t) \le s \le \dfrac{1}{4}(3-t), \\ \omega\left(\dfrac{4s+t-3}{t+1}\right) & \dfrac{1}{4}(3-t) \le s \le 1. \end{cases}$$

It is clear that F is continuous on each piece. By the pasting lemma, it is continuous everywhere.
The four properties of being a homotopy are also clear, by construction. (The diagram makes it clear why.) □

Definition 1.7 (Inverse path)**.** Given a path σ from x_0 to x_1, its *inverse path* σ^{-1} is a path from x_1 to x_0 given by

$$\sigma^{-1}(s) := \sigma(1-s), \qquad s \in I.$$

The above is simply "travelling backwards σ."

Lemma 1.8. Let $\sigma, \sigma' : I \to X$ be paths such that $\sigma \simeq \sigma'$ rel $\{0,1\}$. Then,

$$\sigma^{-1} \simeq \sigma'^{-1} \text{ rel } \{0,1\}.$$

Proof. Let $F : \sigma \simeq \sigma'$ rel $\{0,1\}$ be a homotopy. Then, $F'(s,t) := F(1-s,t)$ is a homotopy between the inverses. □

Definition 1.9 (Inverse class)**.** Let $\sigma : I \to X$ be a path. We define the inverse of the class $[\sigma]$ as

$$[\sigma]^{-1} := [\sigma^{-1}].$$

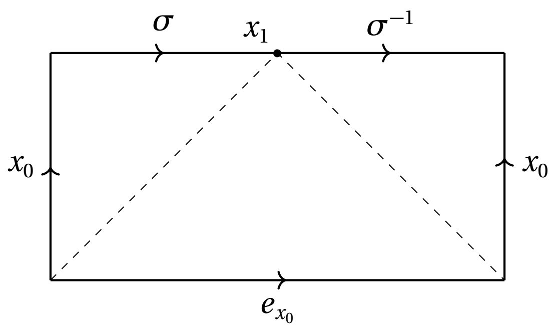
In view of the above lemma, the above definition is indeed well-defined.

Lemma 1.10. Given any path σ from x_0 to x_1, we have

$$e_{x_0} \simeq \sigma\sigma^{-1} \text{ rel } \{0,1\},$$

where e_{x_0} denotes the constant loop at x_0.

Proof. As usual, we motivate the proof with a diagram. In this case, it is the following:



The homotopy $F : I \times I \to X$ in this case, is defined as

$$F(s,t) := \begin{cases} \sigma(2s) & 0 \le 2s \le t, \\ \sigma(t) & t \le 2s \le 2-t, \\ \sigma^{-1}(2s-1) & 2-t \le 2s \le 2. \end{cases}$$

It is clear that the piecewise definitions agree on the dashed line $2s = t$. Observe that $\sigma^{-1}(2s-1) = \sigma(2-2s)$ and thus, the functions do agree on the dashed line $2s = 2-t$ as well.

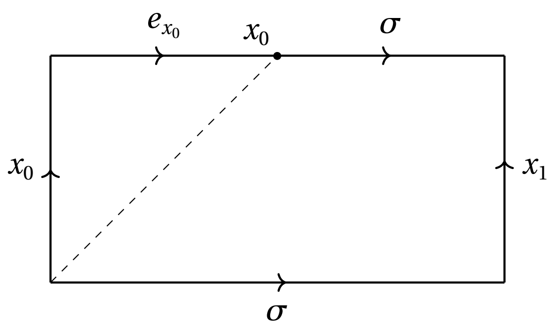
One can easily check that the four properties of the homotopy are satisfied. To see the bottom line property, note that $F(s,0)=\sigma(0)$ (using the second piece definition) and $\sigma(0)=x_0=e_{x_0}(s)$ for all $s\in I$. $\square$

Note that since $(\sigma^{-1})^{-1}=\sigma$, the above also shows that $\sigma^{-1}\sigma=e_{x_1}$.

Lemma 1.11. Let $x_0\xrightarrow{\sigma}x_1$ and e_{x_0} be the constant path at x_0. Then,

$$\sigma\simeq e_{x_0}\sigma \ \text{ rel}\,\{0,1\}.$$

Proof. The proof is motivated by this diagram.



The homotopy is $F:I\times I\to X$ defined as

$$F(s,t):=\begin{cases} x_0 & 0\le 2s\le t,\\ \sigma\left(\dfrac{2s-t}{2-t}\right) & t\le 2s\le 2.\end{cases}$$ $\square$

As one would expect, we have a lemma in the other direction as well.

Lemma 1.12. Let $x_1\xrightarrow{\sigma}x_0$ and e_{x_0} be the constant path at x_0. Then,

$$\sigma\simeq \sigma e_{x_0} \ \text{ rel}\,\{0,1\}.$$

Proof. Similar as in the last case and we omit it. $\square$

The astute reader might have sensed a group sneaking around the corner. However, note that the product of equivalence classes defined above is not a binary operation unless the endpoints are the same. Due to this, we restrict ourselves to loops in the next theorem.

Theorem 1.13. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X at x_0.
If multiplication in $\pi_1(X, x_0)$ is defined as above, $\pi_1(X, x_0)$ becomes a group, in which the neutral element is the class $[e_{x_0}]$ and the inverse of a class $[\sigma]$ is the class of the inverse $[\sigma^{-1}]$.

Proof. Interpreting Lemmas 1.6 to 1.12 as equalities of the equivalence classes shows that $\pi_1(X, x_0)$ verifies the group axioms. □

The next proposition tells us how $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are related in the case that x_0 and x_1 lie in the same path-connected component. (In the case that they do not, nothing can be said.)

Proposition 1.14. Let α be a path from x_0 to x_1. The mapping $\widehat{\alpha}$ defined by

$$[\sigma] \mapsto [\alpha^{-1}] * [\sigma] * [\alpha] = [\alpha^{-1}\sigma\alpha]$$

is an isomorphism of the group $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$.

Note that the above is well-defined since $*$ is well-defined.

Proof. We first note that if $[\sigma] \in \pi_1(X, x_0)$, then $\alpha^{-1}\sigma\alpha$ is path as follows:

$$x_1 \xrightarrow{\alpha^{-1}} x_0 \xrightarrow{\sigma} x_0 \xrightarrow{\alpha} x_1$$

and thus, $[\alpha^{-1}\sigma\alpha]$ is indeed an element of $\pi_1(X, x_1)$.
Moreover, note that

$$\begin{aligned}
\widehat{\alpha}([\sigma\sigma']) &= [\alpha^{-1}\sigma\sigma'\alpha] \\
&= [\alpha^{-1}\sigma][\sigma'\alpha] \\
&= [\alpha^{-1}\sigma][\alpha\alpha^{-1}][\sigma'\alpha] \\
&= [\alpha^{-1}\sigma\alpha][\alpha^{-1}\sigma'\alpha] \\
&= \widehat{\alpha}([\sigma])\widehat{\alpha}([\sigma']).
\end{aligned}$$

This shows that $\widehat{\alpha}$ is a homomorphism. That this is an isomorphism follows by noting that it has as inverse $\widehat{\alpha^{-1}}$. □

Corollary 1.15. If X is path-connected, the group $\pi_1(X, x_0)$ is independent of the point x_0, up to isomorphism.

Note that if C is a connected component of X containing x_0, then $\pi_1(X, x_0) = \pi_1(C, x_0)$ since any loop at x_0 must necessarily lie in C. For this reason, we might as well only work with path-connected spaces.

Definition 1.16. If X is path-connected, we write $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it *the fundamental group* of X.

Note that this group depends on x_0 in the sense that the elements of the group depend on the base point x_0 but the isomorphism class does not.

Definition 1.17 (Simply connected)**.** A space X is called simply connected if it is path-connected and its fundamental group is trivial.

Lemma 1.18. Let X be simply connected. If σ and τ are paths in X with the same initial and terminal points, then $\sigma \simeq \tau \text{ rel}\{0,1\}$.

Proof. Let the initial and terminal points be x_0 and x_1, respectively.
Consider the path $\sigma\tau^{-1}$, which is path at x_0. Since X is simply connected, we have

$$\sigma\tau^{-1} \simeq e_{x_0} \text{ rel}\{0,1\}.$$

By the previously seen properties, we see that

$$(\sigma\tau^{-1})\tau \simeq e_{x_0}\tau \text{ rel}\{0,1\}$$

or

$$\sigma \simeq \tau \text{ rel}\{0,1\}.$$

□

§§1.2. Functoriality

We now wish to turn π_1 into a functor. Since we need to take care of the base points, we look at the category of *Pointed Topological spaces.*

Definition 1.19 (Pointed Topological Spaces)**.** The category $\mathsf{Top}_\bullet$ of *pointed topological spaces* is the category whose objects and morphisms are given as follows:

- Objects: Pairs (X, x_0) where X is a topological space and $x_0 \in X$,
- Morphisms: $f : (X, x_0) \to (Y, y_0)$ such that $f : X \to Y$ is a continuous function and $f(x_0) = y_0$.

That the above is a category can be easily verified.

Definition 1.20. Let $h : (X, x_0) \to (Y, y_0)$ be a morphism. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([\sigma]) = [h \circ \sigma].$$

The map h_* is called the *homomorphism induced by* h, relative to the base point x_0.

To see that h_* is well-defined, we note that if

$$F : \sigma \simeq \sigma' \text{ rel } \{0, 1\}$$

for loops σ, σ' in X at x_0, then

$$h \circ F : h \circ \sigma \simeq h \circ \sigma' \text{ rel } \{0, 1\}.$$

That is to say, if two loops at x_0 are homotopic, then so are the loops obtained by pre-composing h.

To see that h_* is a homomorphism, first note that

$$(h \circ \sigma)(h \circ \sigma') = h \circ (\sigma\sigma').$$

(This follows from the definition of the product of paths.)
Then, we see that

$$h_*([\sigma\sigma']) = [h \circ (\sigma\sigma')] = [h \circ \sigma][h \circ \sigma'] = h_*([\sigma])h_*([\sigma']).$$

Theorem 1.21 (Functoriality)**.** If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are morphisms, then

$$(k \circ h)_* = k_* \circ h_*.$$

If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof. By definition, we have

$$\begin{aligned}(k \circ h)_*([\sigma]) &= [(k \circ h) \circ \sigma] \\ &= [k \circ (h \circ \sigma)] \\ &= k_*([h \circ \sigma]) \\ &= k_*(h_*([\sigma])) \\ &= (k_* \circ h_*)([\sigma]).\end{aligned}$$

Thus, $(k \circ h)_* = k_* \circ h_*$.

Now, if i is the identity map, then we have

$$i_*([\sigma]) = [i \circ \sigma] = [\sigma],$$

showing that i_* is the identity map of $\pi_1(X, x_0)$. □

The above then shows that π_1 defines a functor from the category Top_* to Grp.
Since functors preserve isomorphisms in general, we get the following corollary.

Corollary 1.22. If $h : (X, x_0) \to (Y, y_0)$ is a morphism such that $h : X \to Y$ is a homeomorphism, then
$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
is an isomorphism.

Since we aren't discussing Category Theory, we give a proof for this special example of functors.

Proof. Let $h^{-1} : Y \to X$ be the inverse, which is continuous since h is a homeomorphism. Moreover, $h^{-1}(y_0) = x_0$ and thus, $h^{-1} : (Y, y_0) \to (X, x_0)$ is a morphism and the inverse of h.
Now, note that,

$$(h_*) \circ ((h^{-1})^*) = (h \circ h^{-1})^* = (\mathrm{id}_{(Y,y_0)})^* = \mathrm{id}_{\pi_1(Y,y_0)},$$

by functoriality. Similarly, we have

$$((h^{-1})^*) \circ (h_*) = \mathrm{id}_{(X,x_0)},$$

proving the corollary. □

§2. Homotopy of Maps

In the previous section, we talked about homotopy of special types of maps. More precisely, we only considered maps $I \to X$. However, we can replace I by an arbitrary topological space Y. In the place of endpoints, we just consider a subspace $A \subset Y$.

Definition 2.1 (Relative homotopy)**.** Given maps $f, g : Y \to X$ such that $f|_A = g|_A$, we say f and g are homotopic relative to A written

$$f \simeq g \ \text{ rel } A$$

if there is a map $F : Y \times I \to X$ satisfying

1. $F(y, 0) = f(y)$ for all $y \in Y$,
2. $F(y, 1) = g(y)$ for all $y \in Y$,
3. $F(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$.

This map F is called a homotopy from f to g relative to A and we write

$$F : f \simeq g \ \text{ rel } A.$$

Note that the "second coordinate" above is still I.
Note that (3) is satisfied vacuously if $A = \varnothing$ and we have $f|_A = g|_A$ for all maps $f, g : Y \to X$. Keeping this in mind, we have the following definition.

Definition 2.2 (Homotopy)**.** Maps $f, g : Y \to X$ are said to be *homotopic* if f and g are homotopic relative to $\varnothing$.
We write this more simply as

$$f \simeq g.$$

Moreover, any F as before is simply called a homotopy from f to g.
As before, we write

$$F : f \simeq g.$$

Once again, we obtain an equivalence. The homotopies defined as in the proof of Proposition 1.2 work again.

Definition 2.3 (Contractible space)**.** If X is a topological space such that the identity map on X is homotopic to a constant map on some point in X, we say that X is *contractible.*

Proposition 2.4. *X is contractible if and only if for any space Y, any two maps of Y into X are homotopic. A contractible space is path-connected.*

Proof. ($\Longrightarrow$) Let X be contractible and Y be any space. Fix any $x_0 \in X$ such that id_X is homotopic to the constant map $e_{x_0} : X \to X$.
Let $f_{x_0} : Y \to X$ denote the constant map $y \mapsto x_0$.
Now, given any map $f : Y \to X$, we show that it is homotopic to f_{x_0}.
This will prove that any two maps of Y into X are homotopic since $\simeq$ is an equivalence relation.
Let $H : \mathrm{id}_X \simeq e_{x_0}$ be any homotopy. Then, we have

$$H(x,0) = x, \quad H(x,1) = x_0; \qquad \text{for all } x \in X.$$

(Note that H is continuous.)
Now, we define $F : Y \times I \to X$ as

$$F(y,t) = H(f(y),t).$$

It is clear that F is a map. (That is, F is continuous.)
Moreover, note that

$$F(y,0) = H(f(y),0) = f(y), \quad F(y,1) = H(f(y),1) = x_0 = f_{x_0}(y); \qquad \text{for all } y \in Y.$$

This shows that $F : f \simeq f_{x_0}$, as desired.

($\Longleftarrow$) To show that X is contractible, simply consider $Y = X$ and consider the maps id_X and e_{x_0}. (Both of these are indeed continuous.)
By hypothesis, these maps are homotopic and by definition, X is contractible.

Now, we show that X is path-connected assuming that it is contractible.
Let x_0 and x_1 be any two points in X. As X is contractible, ($\Longrightarrow$) tells us that the maps e_{x_0} and e_{x_1} are homotopic.
Let F be any homotopy from e_{x_0} and e_{x_1}. Define $\sigma : I \to X$ as

$$\sigma(t) := F(x_0, t).$$

σ is clearly continuous. Moreover, we have

$$\sigma(0) = F(x_0,0) = e_{x_0}(x_0) = x_0,$$
$$\sigma(1) = F(x_0,1) = e_{x_1}(x_0) = x_1.$$

Thus, σ is path from x_0 to x_1 in X, proving the proposition. □

Example 1. Every convex subset X of Euclidean space is contractible.
Given maps $f_1, f_2 : Y \to X$, we have a homotopy $F : f_1 \simeq f_2$ given by

$$F(y,t) = t f_2(y) + (1-t) f_1(y), \quad y \in Y, t \in I.$$

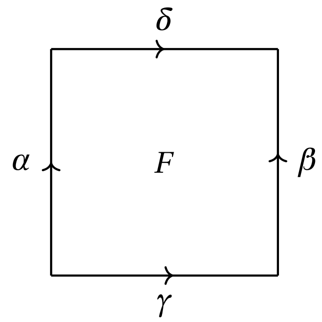
By the convexity assumption, the above F is indeed a map into X.
By the previous proposition, this shows that X is contractible.

Example 2. $\mathbb{R}^n$ is contractible for any n.
To see this, we could either appeal to the previous example or do it directly by defining a homotopy $F : e_0 \simeq \mathrm{id}_{\mathbb{R}^n}$ as

$$F(x,t) = tx.$$

We would now like to show that any contractible space is simply connected. What we do know is that any loop would be homotopic to a point. However, we do not know if this homotopy is relative to $\{0,1\}$. Indeed, to show that we do have a homotopy relative to $\{0,1\}$, we would need to use the fact that X is contractible once again.
Before proving that, we first look at a lemma.

Lemma 2.5. Let $F : I \times I \to X$ be a map. Set $\alpha(t) = F(0,t)$, $\beta(t) = F(1,t)$, $\gamma(s) = F(s,0)$, and $\delta(s) = F(s,1)$, as in the diagram



Then, $\delta = \alpha^{-1}\gamma\beta$.

Proof. The proof is quite intuitive. First, we define the paths

$$\sigma : I \to I \times I, \quad \tau : I \to I \times I$$

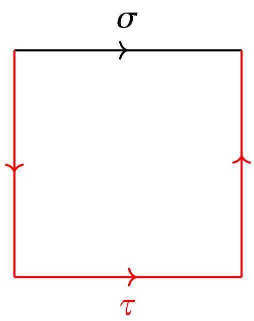
as

$$\sigma(s) := (t,1)$$

and

$$\tau(s) := \begin{cases} (0, 1-4s) & 0 \le 4s \le 1, \\ (4s-1, 0) & 1 \le 4s \le 2, \\ (1, 2s-1) & 1 \le 2s \le 2. \end{cases}$$

These paths are the following ones in I^2:



As it should be clear from the diagram (and one can easily check), we have

$$\delta = F \circ \sigma, \quad (\alpha^{-1}\gamma)\beta = F \circ \tau.$$

(Note that the bracketing in $(\alpha^{-1}\gamma)\beta$ is necessary.)
Also, since I^2 is convex, we see that σ and τ are homotopic relative to $\{0,1\}$ with $H : I \times I \to I \times I$ being a required homotopy defined as

$$H(s,t) := (1-t)\sigma(s) + t\tau(s).$$

Thus,

$$\begin{aligned} & F \circ H : F \circ \sigma \simeq F \circ \tau \ \text{ rel}\,\{0,1\} \\ \Longrightarrow\ & F \circ H : \delta \simeq (\alpha^{-1}\gamma)\beta \ \text{ rel}\,\{0,1\}, \end{aligned}$$

as desired. □

Theorem 2.6. Let X be a contractible space. Then, X is simply connected.

Proof. Note that by Proposition 2.4, we know that X is path-connected. Now we show that that $\pi_1(X)$ is trivial.
Let $x_0 \in X$ be arbitrary and $\alpha : I \to X$ be a loop at x_0 in X.
If we show that $\alpha \simeq e_{x_0}$ rel $\{0,1\}$, then we are done.

To do this, we will use the earlier lemma after constructing an appropriate F.
Using that X is contractible, we fix a homotopy $H : \mathrm{id}_X \simeq f_{x_0}$ where $f_{x_0} : X \to X$ is the constant function $x \mapsto x_0$.
(This is different from e_{x_0} since the domains are different in general.)
To recall, H has the following properties:

$$H(x,0) = x,\ H(x,1) = x_0 \quad \text{for all } x \in X.$$

Now, we define $F : I \times I \to X$ as

$$F(s,t) := H(\sigma(s),t).$$

Now, note that if we set $\alpha, \beta, \gamma, \delta$ as in the previous lemma, we have

$$\begin{aligned}\alpha(t) = F(0,t) = H(\sigma(0),t) = H(x_0,t) \\ = H(\sigma(1),t) = F(1,t) = \beta(t),\end{aligned}$$

$$\gamma(s) = F(s,0) = H(\sigma(s),0) = \sigma(s),$$
$$\delta(s) = F(s,1) = H(\sigma(s),1) = x_0.$$

In other words, we have

$$\alpha = \beta, \gamma = \sigma, \delta = e_{x_0}.$$

By the previous lemma, we know that $[\delta] = [\alpha^{-1}\gamma\beta]$, where $[.]$ is the homotopy class of a path relative to $\{0,1\}$. Thus, we have

$$\begin{aligned} [e_{x_0}] &= [\alpha^{-1}\sigma\alpha] \\ \implies [\alpha][e_{x_0}][\alpha^{-1}] &= [\sigma] \\ \implies [e_{x_0}] &= [\sigma] \\ \implies e_{x_0} &\simeq \sigma \ \text{ rel}\,\{0,1\}, \end{aligned}$$

finishing the proof. □

Proposition 2.7. Let $f, g : Y \to X$ be maps which are homotopic by means of a homotopy $F : Y \times I \to X$.
Let $y_0 \in Y$, $x_0 := f(y_0) = F(y_0, 1)$, and $x_1 := g(y_0) = F(y_0, 1)$.
Let $\alpha : I \to X$ be a path from x_0 to x_1 given by

$$\alpha(t) = F(y_0, t) \quad t \in I.$$

Then, the following diagram commutes.

$$\begin{tikzcd} \pi_1(Y, y_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(X, x_0) \arrow[d, "\widehat{\alpha}"] \\ & \pi_1(X, x_1) \end{tikzcd}$$

Proof. The diagram commuting is just saying that

$$\widehat{\alpha} \circ f_* = g_*.$$

Let $[\sigma] \in \pi_1(Y, y_0)$ be arbitrary. Showing that the above is true is equivalent to showing that

$$(\widehat{\alpha} \circ f_*)([\sigma]) = g_*([\sigma]).$$

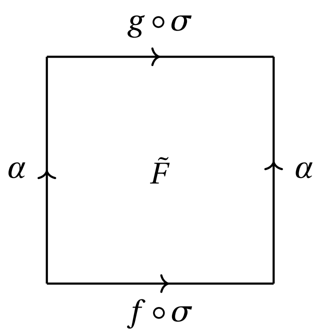
Using the definitions of $\widehat{\alpha}$ and f_*, we note that

$$\begin{aligned}
&(\widehat{\alpha} \circ f_*)([\sigma]) = g_*([\sigma]) \\
\iff\ &\widehat{\alpha}(f_*([\sigma])) = g_*([\sigma]) \\
\iff\ &\widehat{\alpha}([f \circ \sigma]) = [g \circ \sigma] \\
\iff\ &[\alpha^{-1}(f \circ \sigma)\alpha] = [g \circ \sigma].
\end{aligned}$$

Now, defining $\tilde{F} : I \times I \to X$ as

$$\tilde{F}(s, t) = F(\sigma(s), t).$$

Then, we have the following diagram as in Lemma 2.5 which proves the proposition.



To see that the sides are indeed as labeled, recall that σ is a loop at y_0 and note that

$$\begin{aligned}
\tilde{F}(0, t) &= F(\sigma(0), t) = F(y_0, t) = \alpha(t), \\
\tilde{F}(1, t) &= F(\sigma(1), t) = F(y_0, t) = \alpha(t), \\
\tilde{F}(s, 0) &= F(\sigma(s), 0) = g(\sigma(s)) = (g \circ \sigma)(s), \\
\tilde{F}(s, 1) &= F(\sigma(s), 1) = f(\sigma(s)) = (f \circ \sigma)(s).
\end{aligned}$$

By the conclusion of Lemma 2.5, we are done. $\square$

Recall that $\widehat{\alpha}$ is an isomorphism and thus, we get the following corollary.

Corollary 2.8. With the same setup as above, f_* is an isomorphism if and only if g_*.

What the above corollary says is that if f and g are homotopic, then f_* is an isomorphism iff g_* is.

Definition 2.9 (Homotopy equivalence)**.** A map $f : Y \to X$ is said to be a *homotopy equivalence* if there exists a map $f' : X \to Y$ such that

$$\begin{aligned} ff' &\simeq \mathrm{id}_X, \\ f'f &\simeq \mathrm{id}_Y. \end{aligned}$$

If such a map exists, we say that X and Y are *homotopically equivalent spaces*.

It can be checked that being homotopically equivalent is an "equivalence relation."

Corollary 2.10. If $f : Y \to X$ is a homotopy equivalence, then f_* is an isomorphism

$$\pi_1(Y, y_0) \to \pi_1(X, f(y_0))$$

for any $y_0 \in Y$.

Proof. Let $f' : X \to Y$ be as in the definition.
Then, $ff' \simeq \mathrm{id}_X$. By the previous corollary, we have that $(ff')_*$ is an isomorphism. (Since $(\mathrm{id}_X)_*$ is.)
Similarly, $(f'f)_*$ is an isomorphism. Since $(ff')_* = f_* \circ f'_*$ and $(f'f)_* = f'_* \circ f_*$, we see that f_* is a bijection and hence, an isomorphism. □

The above shows that the fundamental group of a path-connected space is a *homotopy invariant*. We had shown earlier that this was a topological invariant.
Note that being homotopically equivalent is a weaker concept than being topologically invariant (i.e., homeomorphic). Clearly, if $f : X \to Y$ is a homeomorphism, it also a homotopy equivalence with $f' = f^{-1}$.
However, the closed interval I is homotopically equivalent to the point space but clearly not homeomorphic. In fact, one can note that X is contractible if and only if it is homeomorphic to a point.

§3. Fundamental Group of the Circle

In this section, we prove a more general result. S^1 will turn out to be a special case of that. First, we need a lemma.

Lemma 3.1. Let K be a compact metric space and G a topological group. Let $V \subset G$ be open such that $1 \in V$.
If $f : K \to G$ is continuous, then there exists $\delta > 0$ such that

$$d(k, k') < \delta \implies f(k)(f(k'))^{-1} \in V.$$

The above is essentially mimicking something like "uniform continuity."

Proof.

Claim 1. There exists an open set $U \subset G$ such that

1. $1 \in U \subset V$,
2. $g, g' \in U \implies gg^{-1} \in V$.

Proof. The function $\varphi : G \times G \to G$ defined as

$$\varphi(g, g') := g(g')^{-1}$$

is continuous. Thus, $\varphi^{-1}(V)$ is open.
Note that $(1,1) \in \varphi^{-1}(V)$. Thus, there exists a basis element of the form $U_1 \times U_2$ satisfying

$$(1,1) \in U_1 \times U_2 \subset \varphi^{-1}(V).$$

Let $U := U_1 \cap U_2 \cap V$. Clearly, U is open and $1 \in U \subset V$.
Moreover,

$$g, g' \in U \implies (g, g') \in U_1 \times U_2 \subset \varphi^{-1}(V) \implies \varphi(g, g') \in V \implies g(g')^{-1} \in V,$$

as desired. □

With this, we can mimic the proof of continuous functions being uniformly continuous on compact sets. (The above U will help us use "triangle inequality" in the codomain.)
Let U be as in the above claim.

Claim 2. Given any $k \in K$, there exists $\delta_k > 0$ such that

$$d(k, k') < \delta_k \implies f(k)(f(k'))^{-1} \in U.$$

Proof. The function $f_k : K \to G$ defined by $f_k(k') = f(k)(f(k'))^{-1}$ is continuous with $f_k(k) = 1 \in U$.
Consider the open set $f_k^{-1}(U)$. Since it contains k, there exists $\delta > 0$ such that $B_\delta(k) \subset f_k^{-1}(U)$. Thus, if $k' \in B_\delta(k)$, then $f_k(k') \in U$, as desired for the first condition.

Note that we can find a suitable δ_k' for the other condition as well. Taking the minimum of the two proves the claim. □

Let $V_k = B_{\delta_k/2}(k)$. Clearly, $\{V_k\}_{k \in K}$ is an open cover of K. Since K is compact, we may extract a finite subcover.
Let $k_1, \dots, k_n$ be the indices of one such. Set

$$\delta := \min_{1 \le i \le n} \frac{1}{2} \delta_{k_i}.$$

Clearly, $\delta > 0$. Moreover, it satisfies the condition of the lemma. To see this, let $k, k' \in K$ be such that $d(k, k') < \delta$.
Since $\{V_{k_i}\}_{1 \le i \le n}$, is an open cover, k lies in V_{k_i} for some $1 \le i \le n$. That is, $2d(k, k_i) < \delta_i$. Now, using triangle inequality, note that

$$d(k', k_i) \le d(k', k) + d(k, k_i) < \delta + \frac{1}{2}\delta_i \le \frac{1}{2}\delta_i + \frac{1}{2}\delta_i = \delta_i.$$

Thus, both k and k' are at most δ_i from k_i. By the definition of δ_i (from Claim 2), we see that $f(k)(f(k_i))^{-1} \in U$ and $f(k')(f(k_i))^{-1} \in U$.
By the property of U, we have

$$\left[f(k)(f(k_i))^{-1}\right]\left[f(k')(f(k_i))^{-1}\right]^{-1} = f(k)(f(k'))^{-1} \in V,$$

as desired. □

Now, for the remainder of this section, we shall fix G as any simply connected topological group and $H \le G$ is a *discrete* normal subgroup of G. We will show that $\pi_1(G/H, 1) \cong H$.
(In the special case that $G = \mathbb{R}$ and $H = \mathbb{Z}$, we see that $\pi_1(S^1, 1) \cong \mathbb{Z}$ or simply, $\pi_1(S^1) \cong \mathbb{Z}$.)

We also fix the map $\varphi : G \to G/H$ to be the projection $g \mapsto gH$.

Lemma 3.2. There exists an open neighbourhood U of 1 in G which is mapped homeomorphically onto an open neighbourhood V of 1 in G/H be φ.

Proof. Since H is discrete, $\{1\}$ is open in H. Thus, there exists an open neighbourhood U_1 of 1 in G such that $U_1 \cap H = \{1\}$.
As in claim 1 of the previous proof, we may find an open $U \subset U_1$ such that $1 \in U$ and $g, g' \in U \implies gg'^{-1} \in U_1$. Clearly, $U \cap H = \{1\}$ as well.

> **Claim 1.** $\varphi|_U$ is injective.
>
> *Proof.* Let $g_1, g_2 \in U$ with $\varphi(g_1) = \varphi(g_2)$.
> Then, $g_1 H = g_2 H$ or $Hg_1 = Hg_2$ or $Hg_1 g_2^{-1} = H$ or $g_1 g_2^{-1} \in H$.
> Since $g_1, g_2 \in U$, we also have $g_1 g_2^{-1} \in U_1$. Since $U_1 \cap H = \{1\}$, we see that $g_1 g_2^{-1} = 1$ or $g_1 = g_2$. □

Let $V = \varphi(U)$. Clearly, φ maps U bijectively onto V, in view of the previous claim. Moreover, this must be a homeomorphism. To see this, we recall a general result.

> **Claim 2.** The quotient map $\phi : G \to G/H$ is open.
>
> *Proof.* Let W be an open subset of G. The set
>
> $$WH = \{wh : w \in W, h \in H\}$$
>
> is open since $WH = \bigcup_{h \in H} Wh$, which is a union of open subsets of G since right multiplication is a homeomorphism.
> Note that $\varphi^{-1}(\varphi(W)) = WH$. Since φ is a quotient map and WH is open, we see that $\varphi(W)$ is open, as desired. □

Thus, we see that $\varphi|_U : U \to V$ is a bijective open map. In particular, it is a homeomorphism. □

For the remainder of this section, we fix $U \subset G$ and $V \subset G/H$ as above. Moreover, we fix

$$\psi := (\varphi|_U)^{-1}.$$

By our above discussion, $\psi : V \to U$ is a continuous function.
For better clarity, we shall use 1 for the identity of G/H and 1_G for the identity of G.

Now, we prove two key lemmas.

Lemma 3.3 (Lifting Lemma)**.** If σ is a path in G/H with initial point 1, there is a unique path σ' in G with initial point 1_G such that

$$\varphi \circ \sigma' = \sigma.$$

Lemma 3.4 (Covering Homotopy Lemma)**.** If τ is also a path in G/H with the initial point 1 such that

$$F : \sigma \simeq \tau \ \text{ rel}\,\{0,1\},$$

then there is a unique $F' : I \times I \to G$ such that

$$\begin{aligned} F' : \sigma' \simeq \tau' \ \text{ rel}\,\{0,1\}, \\ \varphi \circ F' = F. \end{aligned}$$

(Note that τ' above is the unique path in G as given by the Lifting Lemma.)

Proof. We prove both results together.
Let $(K, f : K \to G/H, 0 \in K)$ be either $(I, \sigma, 0 \in I)$ or $(I \times I, F, (0,0) \in I \times I)$. The first choice corresponds to Lemma 3.3 and the second to Lemma 3.4.

For the sake of less ugly notation, we shall use a/b or $\frac{a}{b}$ to denote ab^{-1} for $a, b \in G/H$. (Note that we are fixing this to mean ab^{-1} without any assumption of abelianity.)

Since K is compact, there exists $\epsilon > 0$ such that

$$|k - k'| < \epsilon \implies f(k)/(f(k')) \in V,$$

by Lemma 3.1.
In particular, for such k and k', $\psi\left(\dfrac{f(k)}{f(k')}\right)$ is defined. Fix $N \in \mathbb{N}$ large enough such that

$$|k| < N\epsilon$$

for all $k \in K$. (This can be done since K is bounded by 2.)
Now, define

$$f' : K \to G$$

by

$$\begin{aligned} f'(k) :=& \psi\left(f(k)/f\left(\frac{N-1}{N}k\right)\right) \\ &\cdot \psi\left(f\left(\frac{N-1}{N}k\right)\Big/ f\left(\frac{N-2}{N}k\right)\right) \\ &\cdots \psi\left(f\left(\frac{1}{N}k\right)\Big/ f(0)\right). \end{aligned}$$

Then, f' is continuous $K \to G$, $f'(0) = (\varphi(1))^N = 1_G$, and $\varphi \circ f' = f$. To see the last point, note that φ is a homomorphism and thus,

$$\begin{aligned}(\varphi \circ f')(k) = & \varphi\left[\psi\left(f(k)/f\left(\frac{N-1}{N}k\right)\right)\right] \\ & \cdot \varphi\left[\psi\left(f\left(\frac{N-1}{N}k\right)\Big/ f\left(\frac{N-2}{N}k\right)\right)\right] \\ & \cdots \varphi\left[\psi\left(f\left(\frac{1}{N}k\right)\Big/ f(0)\right)\right].\end{aligned}$$

Now, using that $\varphi\psi(k) = k$, we see that the fractions cancel and we are left with

$$(\varphi \circ f')(k) = f(k)/f(0) = f(k),$$

since $f(0) = 1_G$, in either case.

Now, suppose we had $f'' : K \to G$ satisfying $f''(0) = 1_G$, and $\varphi \circ f'' = f$.
Then, we would have $[\varphi \circ (f'/f'')](s) = \varphi(f'(s))/\varphi(f''(s))$, since φ is a homomorphism. However, this equals $f(s)/f(s) = 1$.
Thus, f'/f'' is a continuous map from Y into $\ker\varphi = H$.
Since Y is connected and H is discrete, f'/f'' is a constant. Since $f'(0)/f''(0) = 1_G$, we see that $f' = f''$.
This proves the uniqueness of f'.

Note that in the case of the first lemma (that is, $Y = I$), we have $f'(0) = 1_G$ and thus, f' is the required σ'.

For the case of the second lemma, we still have to prove that $F' = f'$ is the desired (relative) homotopy.
First, we show that F' is indeed a (not necessarily relative) homotopy. To see this, set $\alpha(s) := F'(s, 0)$ and $\beta(s) = F'(s, 1)$.
Note that $\varphi \circ \alpha(s) = \varphi \circ F'(s, 0) = F(s, 0) = \sigma(s)$ and $\alpha(0) = F'(0, 0) = 1_G$.
Since σ' is the unique such path, we see that $\alpha = \sigma'$.
Similarly, we can conclude $\beta = \tau$ if we show that $\beta(0) = 1_G$. By definition, we have $\beta(0) = F'(0, 1)$.
Note that F' is continuous and $\varphi \circ F'$ is 1 on $\{0\} \times I$. Thus, $F'|_{\{0\}\times I}$ maps into $\ker\varphi = H$. As before, we see that F' is constant on $\{0\} \times I$. Thus, $F'(0, 1) = F'(0, 0) = 1_G$ and hence, $\beta = \tau'$.
In fact, we have even proven that F' is constant on $\{0\} \times I$. This shows that F' is a homotopy relative to $\{0\}$. All that remains is to show that it is constant on $\{1\} \times I$ as well.
For that, we once again note that $\varphi \circ F' = F$ is constant on $\{1\} \times I$. Thus, $F'|_{\{1\}\times I}$ maps into a coset of $\ker\varphi = H$. Since the coset is homeomorphic to

H, it must be discrete as well. This proves that F' is constant on $\{1\} \times I$ as well, proving that

$$F' : \sigma' \simeq \tau' \ \operatorname{rel}\{0,1\}. \qquad \square$$

Corollary 3.5. The end point of σ' only depends on the homotopy class of σ.
In particular, if σ is a loop at 1, then $\sigma'(1) \in H$.

Proof. Let σ, τ be paths in the same homotopy class. Let $F : \sigma \simeq \tau \ \operatorname{rel}\{0,1\}$ be a (relative) homotopy.
Then, F' is a homotopy from σ' to τ' relative to $\{0,1\}$.
In particular, we have $\sigma'(1) = F(1,0) = F(1,1) = \tau'(1)$. This proves the first statement.

For the second statement, note that $\varphi \circ \sigma'(1) = \sigma(1) = 1$ and thus, $\sigma'(1) \in \ker \varphi = H$. $\square$

Now, we have the following theorem.

Theorem 3.6. If G is a simply connected topological group, H a discrete normal subgroup, then

$$\pi_1(G/H, 1) \cong H.$$

Proof. Using Corollary 3.5, we define $\chi : \pi_1(G/H, 1) \to H$ by

$$\chi([\sigma]) = \sigma'(1).$$

> **Claim 1.** χ is a homomorphism.
>
> *Proof.* Let $[\sigma], [\tau] \in \pi_1(G/H, 1)$.
> Let $h_1 = \sigma'(1)$ and $h_2 = \tau'(1)$. (Again, we see that these are well-defined and elements of H by Corollary 3.5.)
> Let τ'' be the path from h_1 to $h_1 h_2$ in G given by
>
> $$\tau''(s) = h_1 \tau'(s).$$
>
> (Note that $\tau''(0) = \tau'(0) h_1 = 1_G h_1 = h_1$ and $\tau''(1) = h_1 \tau'(1) = h_1 h_2$.)
> Note that
> $$(\varphi \circ \tau'')(s) = \varphi(\tau'(s) h_1) = \varphi(\tau'(s)) \varphi(h_1) = \tau(s).$$
> (Note that $\varphi(h_1) = 1$ since $h_1 \in H = \ker \varphi$.)

Since, $\sigma'(1) = \tau''(0) = h_1$, we can consider the path $\tau''\sigma'$ in G. Note that

$$\varphi \circ (\tau''\sigma')(s) = \begin{cases} \varphi(\sigma'(2s)) & 0 \le 2s \le 1 \\ \varphi(\tau''(2s-1)) & 1 \le 2s \le 2. \end{cases} = (\sigma\tau)(s).$$

Thus, $\tau''\sigma'$ is the unique lift of $\sigma\tau$ as given by the Lifting Lemma.
Thus,

$$\chi([\sigma][\tau]) = \chi[\sigma\tau] = (\tau''\sigma')(1) = h_1 h_2 = \chi[\sigma]\chi[\tau].$$ □

Now, we show that χ is bijective.

Claim 2. χ is injective.

Proof. It suffices to show that $\ker\chi$ is trivial.
Let $[\sigma] \in \ker\chi$. Then, $\sigma'(1) = 1_G$.
In other words, σ' is a loop at 1_G in G. Since G is simply connected, σ' is path homotopic to a constant loop. We may choose the constant loop to be e_{1_G}.
Thus, $\sigma' \simeq e_{1_G}$ rel $\{0,1\}$.
Applying φ, we get that $\sigma \simeq e_1$ rel $\{0,1\}$ or $[\sigma] = [e_1]$, the identity of $\pi_1(G/H, 1)$. □

Claim 3. χ is surjective.

Proof. Let $h \in H$ be arbitrary.
Since G is simply-connected, it is path-connected. Let σ' be path from 1_G to h in G.
Then, $\varphi \circ \sigma' : I \to G/H$ is a loop at 1 in G/H with

$$\chi[\sigma] = \sigma'(1) = h.$$ □

With that, we are done! □

Corollary 3.7. The fundamental group of S^1 is (isomorphic to) $\mathbb{Z}$.

(Since S^1 is path-connected, we need not care about base point.)
In particular, the above corollary shows that S^1 is not simply connected. This is our first example of a non-simply connected space.

Corollary 3.8. The fundamental group of a torus is (isomorphic to) $\mathbb{Z} \times \mathbb{Z}$.

Proof. The torus is (homeomorphic to) $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$. □

Note that the torus is also homeomorphic to $S^1 \times S^1$. Using this, we could've calculated the fundamental group in a different way with the help of the following proposition.

Proposition 3.9. Given spaces $X, Y, x_0 \in X, y_0 \in Y$, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. The isomorphism is obtained as follows. First, consider the maps of pointed topological spaces given by the projections

$$(X, x_0) \xleftarrow{p_X} (X \times Y, (x_0, y_0)) \xrightarrow{p_Y} (Y, y_0).$$

These maps induce the homomorphisms

$$\pi_1(X, x_0) \xleftarrow{(p_X)_*} \pi_1(X \times Y, (x_0, y_0)) \xrightarrow{(p_Y)_*} \pi_1(Y, y_0).$$

Using the universal property of product of groups, we get a homomorphism $\langle (p_X)_*, (p_Y)_* \rangle$ as follows

$$\begin{tikzcd}
\pi_1(X, x_0) & \pi_1(X \times Y, (x_0, y_0)) \arrow[l, "(p_X)_*"'] \arrow[r, "(p_Y)_*"] \arrow[d, "\langle (p_X)_*, (p_Y)_* \rangle"] & \pi_1(Y, y_0) \\
& \pi_1(X, x_0) \times \pi_1(Y, y_0) \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}$$

such that the diagram commutes. (The $\twoheadrightarrow$s are the usual projections.)
Let $\varphi := \langle (p_X)_*, (p_Y)_* \rangle$. We show that this is an isomorphism by constructing an inverse $\psi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$.

Any element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ is of the form $([\sigma], [\tau])$ for some loop σ (resp., τ) at x_0 (resp., y_0) in X (resp., Y).
We define $\psi([\sigma], [\tau])$ as the class of the loop at (x_0, y_0) in $X \times Y$ given by

$$(\sigma, \tau)(s) := (\sigma(t), \tau(t)), \quad t \in I.$$

That is, $\psi([\sigma], [\tau]) = [(\sigma, \tau)]$. One can verify that this is well-defined.
(That is, if $\sigma \simeq \sigma'$ and $\tau \simeq \tau'$, then $(\sigma, \tau) \simeq (\sigma', \tau')$, all relative to $\{0, 1\}$.)
Now, one can verify that $\varphi \circ \psi$ and $\psi \circ \varphi$ are both the respective identities.

Alternately, as a more category theoretic proof, one can verify that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X,x_0) & \pi_1(X\times Y,(x_0,y_0)) \arrow[l, "(p_X)_*"'] \arrow[r, "(p_Y)_*"] & \pi_1(Y,y_0) \\
& \pi_1(X,x_0)\times\pi_1(Y,y_0) \arrow[ul] \arrow[u, "\psi"] \arrow[ur] &
\end{tikzcd}$$

Thus, given any object and arrows $\pi_1(X,x_0) \longleftarrow Z \longrightarrow \pi_1(Y,y_0)$, we get an arrow $\eta : Z \longrightarrow \pi_1(X,x_0)\times\pi_1(Y,y_0)$ such that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X,x_0) & \pi_1(X\times Y,(x_0,y_0)) \arrow[l, "(p_X)_*"'] \arrow[r, "(p_Y)_*"] & \pi_1(Y,y_0) \\
& Z \arrow[ul] \arrow[u, "\psi\circ\eta"] \arrow[ur] &
\end{tikzcd}$$

That is, $\pi_1(X\times Y,(x_0,y_0))$ satisfies the universal mapping property of a product. Since products are unique up to isomorphism, we see that

$$\pi_1(X\times Y,(x_0,y_0)) \cong \pi_1(X,x_0)\times\pi_1(Y,y_0).$$

□

Definition 3.10 (Retract)**.** A subset Y of a topological space X is called a *retract* if there exists a map $r : X \to Y$ such that

$$ri = \mathrm{id}_Y,$$

where $i : Y \hookrightarrow X$ is the inclusion map.

Theorem 3.11. The circle S^1 is not a retract of the closed disc D^2.

Proof. We prove a stronger result that $ri \simeq \mathrm{id}_{S^1}$ is impossible for any map $r : D^2 \to S^1$.

Indeed, assume the contrary and let $r : D^2 \to S^1$ be a map such that $ri \simeq \mathrm{id}_{S^1}$.

Then, $(ri)_* = r_* i_*$ is an isomorphism, by Corollary 2.8.
However, note that

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[dr, "\mathrm{id}"'] & \pi_1(D^2) \arrow[d, "r_*"] \\ & \pi_1(S^1) \end{tikzcd}$$

Recalling that $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) = \{1\}$, we see that the above is impossible since $\mathbb{Z} \to \{1\} \to \mathbb{Z}$ cannot be an isomorphism.
(There is neither any injection $i_* : \mathbb{Z} \to \{1\}$ nor any surjection $r_* : \{1\} \to \mathbb{Z}$.) $\square$

Corollary 3.12 (Special Brouwer Fixed Theorem)**.** Any continuous map of the closed disc into itself has a fixed point.

Proof. Suppose $f : D^2 \to D^2$ has no fixed point. We define $r : D^2 \to S^1$ as follows:
Take the ray joining $f(x)$ to x and extend it until it reaches the circle S^1. Call this point on S^1 $r(x)$.
Clearly, if $x \in S^1$, then $r(x) = x$. Thus, $ri = \mathrm{id}_{S^1}$, a contradiction to the previous theorem. $\square$

Remark. This is a special case of Brouwer's fixed point theorem for $n = 2$. The case $n = 1$ is simple by considering the function $g(x) = f(x) - x$ and noting that $g(-1) \geq 0$ and $g(1) \leq 1$, thereby giving us that $g(c) = 0$ for some $c \in D^1 = [-1, 1]$.

Note that it must be justified that the r defined above is indeed continuous. This is a fairly straightforward calculation. An outline is as follows:
Consider the ray ζ_x given by $\zeta_x(t) = (1-t)f(x) + tx$ for $t \geq 0$. We need a solution $t > 0$ for $\|\zeta_x(t)\| = 1$. This turns out to be equivalent to solving

$$\|x - f(x)\|^2 t^2 + 2(\langle x, f(x)\rangle - \|f(x)\|^2) + \|f(x)\|^2 - 1 = 0.$$

By our assumption, $x \neq f(x)$ and thus, the above is a genuine quadratic expression for all x. Moreover, using $\|f(x)\|^2 \leq 1$, one can show that the above has one unique positive root, call this $t(x)$. Clearly, $x \mapsto t(x)$ is continuous. (Quadratic formula.)
Thus, $r(x) = (1 - t(x))f(x) + t(x)x$ is a continuous function of x.

Theorem 3.13. Let X be a topological space and $x_0 \in X$. Suppose $\mathscr{A}$ is an open cover of X with the following properties:

1. $A_\alpha \cap A_\beta$ contains x_0 and is path-connected for all $A_\alpha, A_\beta \in \mathscr{A}$,
2. A_α is simply connected for all $A_\alpha \in \mathscr{A}$.

Then, X is simply connected.

This is a special case of The Van Kampen Theorem which we prove in section 6.

Proof. It is clear that X is path-connected since it is the union of path-connected sets with a point in common. Thus, we just need to show that any loop is path homotopic to a constant loop. Of course, since X is path-connected, we can choose any base point of our choice. We choose the point x_0.

Let $\sigma : I \to X$ be any loop at x_0.
By the Lebesgue number lemma, there exists a subdivision

$$[\sigma] = [\sigma_1] * \cdots * [\sigma_n]$$

such that each $\sigma_i(I)$ is contained in some $A_i \in \mathscr{A}$.
Now, we define the paths $g_1, \ldots, g_{n+1}$ as follows:

- g_1 and g_{n+1} are the constant loops at x_0.
- For $1 < i \leq n$, g_i is any path joining $\sigma_i(0)$ to x_0 lying in $A_{i-1} \cap A_i$. We can do so because $\sigma_i(0) = \sigma_{i_1}(1)$ is a point in $A_{i-1} \cap A_i$. Since this intersection contains x_0 and is path-connected, we are done.

Now, note that the path $g_i^{-1} \sigma_i g_{i+1}$ is a loop that lies in A_i for all $1 \leq i \leq n$. Since A_i is simply connected, we see that $[g_i^{-1} \sigma_i g_{i+1}]$ is the constant element $[e_{x_0}] \in \pi_1(X)$.
Moreover, observe that when the product is taken over all $1 \leq i \leq n$, the g_is cancel out. That is,

$$\begin{aligned}
[\sigma] &= [\sigma_1] \cdot \cdots \cdot [\sigma_n] \\
&= \prod_{i=1}^{n} [g_i^{-1} \sigma_i g_{i+1}] \\
&= \prod_{i=1}^{n} [e_{x_0}] \\
&= [e_{x_0}],
\end{aligned}$$

as desired. (Note that $[g_1^{-1}] = [g_{n+1}] = [e_{x_0}]$ as well.) $\square$

Also, note that homotopy classes $[\sigma_i]$ in the above are classes of paths in X, not loops.

Proposition 3.14. The space S^n is simply connected for $n \geq 2$.

Proof. We apply the above theorem with $X = S^n$, $\mathscr{U} = \{U, V\}$ with $U = S^n \setminus \{(1,0,\dots,0)\}$ and $V = S^n \setminus \{(-1,0,\dots,0)\}$.
(In other words, U is S^n with one point removed and V is S^n with the opposite point removed.)
It is clear that $\mathscr{U}$ is open cover. Recall that $\mathbb{R}^n$ is homeomorphic to S^n with a point removed.
Thus, both U and V are simply connected since $\mathbb{R}^n$ is.
Moreover, $U \cap V$ is homeomorphic to $\mathbb{R}^n$ with two points removed. Since $n \geq 2$, this space is path-connected.
Thus, $\mathscr{U}$ satisfies the criterion of the previous theorem and the result follows. $\square$

§4. Covering spaces

In this section, we try to generalise the ideas of earlier. The previous section let us calculate $\pi_1(X)$ in the particular case that X was a topological group (and could be realised as a quotient group in a particular manner).
In section, we shall consider X which is not necessarily a group but represent it as a quotient space of a simply connected space $\tilde{X}$. As before, we shall work in the case that the fibers of $\tilde{X} \to X$ are discrete.
Towards this end, we have the following definition.

Definition 4.1 (Covering space). $E \xrightarrow{p} X$ is a *covering space of* X if every $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets S_i in E, each of which is mapped homeomorphically onto U by p. Such U are said to be *evenly covered,* and the S_i are called *sheets* over U.

Proposition 4.2 (Consequences). From the above definition, the following results follow.

1. The fiber $p^{-1}(x)$ over any point is discrete;
2. p is a local homeomorphism;
3. p maps E onto X and X has the quotient topology from E.
4. If E is locally path-connected, then so is X.

Proof.

1. Let $x \in X$ and U be a neighbourhood of x which is evenly covered. Then, $p^{-1}(U) = \bigsqcup_{i \in I} S_i$.
   Let $y \in p^{-1}(x)$. Then, $y \in S_i$ for some $i_0 \in I$. Moreover, since $p : S_{i_0} \to U$ is homeomorphism, it is one-one and thus, $p(y') \neq x$ for any $y \neq y' \in S_{i_0}$.
   In other words, $S_{i_0} \cap p^{-1}(x) = \{y\}$ and thus, $\{y\}$ is open in $p^{-1}(x)$. (Since S_{i_0} was open.)
   This shows that $p^{-1}(x)$ is discrete.
2. By definition, we need to show that given any $e \in E$, there exists a neighbourhood V of e such that $p(V)$ is open in X and $p|_V : V \to p(V)$ is a homeomorphism.

   To this end, let $e \in E$ be arbitrary and let $x = p(e)$.
   Let U an evenly covered neighbourhood of x and S_{i_0} be the sheet (over U) containing e.

By definition (of covering spaces), we have that $p_{S_{i_0}}$ is a homeomorphism, as desired.

3. The fact that p is onto follows straight from the definition. (Every $x \in X$ has a neighbourhood U which is evenly covered and thus, a sheet maps onto U and in particular, something gets mapped to $x \in U$.)

   Showing that X has the quotient topology from E is the same as showing that p is a quotient map. Let $U \subset X$. We need to show that $p^{-1}(U)$ is open iff U is open. (We already know that p is surjective.)
   If U is open, then $p^{-1}(U)$ is open since p is continuous. (It is a local homeomorphism.)
   Conversely, let $p^{-1}(U)$ be open. We show that U is open. To this end, let $x \in U$. Consider any $e \in E$ such that $p(e) = x$. Then, $e \in p^{-1}(U)$. Since p is a local homeomorphism and $p^{-1}(U)$ is open, we can find a neighbourhood V of e contained in $p^{-1}(U)$ such that $p(V)$ is open.
   However, note that $x \in p(V) \subset U$. This shows that x is an interior point and thus, U is open. (Since x was arbitrary.)

4. Let $x \in X$ and U be an arbitrary neighbourhood of x.
   Choose a neighbourhood U' of x which is evenly covered and let S' be a sheet over U'. Then, $p|_{S'}$ is a homeomorphism.
   Let $W = U \cap U'$. Consider $p|_{S'}^{-1}(W)$; this is an open subset of S' and hence, of E.
   Since E is locally path-connected, we can find a path-connected neighbourhood $V \subset p|_{S'}^{-1}(W)$ of $p|_{S'}^{-1}(x) \in S'$.
   Then, its image $p_{S'}(V) \subset W \subset U$ is a neighbourhood of x and is path-connected. (Since it is homeomorphic to V.)
   This shows that X is locally path-connected. $\square$

Thus, covering spaces is the generalisation of the previous section that we described earlier.
We now give the generalisations of Lemma 3.3 and Lemma 3.4.

Theorem 4.3 (Unique lifting theorem)**.** Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering space with base points, $(Y, y_0) \xrightarrow{f} (X, x_0)$ any map. Assume that Y is connected. *If there is a map* $(Y, y_0) \xrightarrow{E, e_0}$ *such that* $pf' = f$, *then* it is unique.

(Note that this is different from Lemma 3.3 since we don't guarantee the *existence* of an f'.)
Diagrammatically, this can be depicted as

$$\begin{tikzcd} (E, e_0) \arrow[d, "p"] & \\ (X, x_0) & (Y, y_0) \arrow[l, "f"] \arrow[ul, dashed, "f'"'] \end{tikzcd}$$

Proof. With everything as in the lemma, assume that $f'' : (Y, y_0) \to (E, e_0)$ is also a map such that $pf'' = f$.
We show that $f' = f''$.

Define $A \subset Y$ as

$$A := \{y \in Y \mid f'(y) = f''(y)\}.$$

Note that $A \neq \varnothing$ since $y_0 \in A$. (Since $f'(y_0) = e_0 = f''(y_0)$, by assumption.)
We will show that A is both open and closed. Then, since Y is connected and A is nonempty, it will follow that $A = Y$. In turn, that will show that $f' = f''$.

Claim 1. A is open.

Proof. Let $y \in A$. Let U be an evenly covered neighbourhood of $f(y) = pf'(y)$.
Then, $f'(y)$ lies on some sheet S over U. Since $y \in A$, we have $f'(y) = f''(y)$. Thus, the set $B := f'^{-1}(S) \cap f''^{-1}(S)$ is an open set containing y.

Subclaim 1.1. $B \subset A$.

Proof. Let $y_1 \in B$. Then, $y \in f'^{-1}(S) \cap f''^{-1}(S)$.
That is $f'(y_1) \in S \ni f''(y_1)$.
Note that $p|_S$ is a homeomorphism and in particular, one-one. Since $pf'(y_1) = f(y_1) = pf''(y_1)$, we see that $f'(y_1) = f''(y_1)$ and hence, $y_1 \in A$. □

Thus, we have seen that given any $y \in A$, there exists an open set B with $y \in B \subset A$, showing that A is open. □

Claim 2. A is closed.

Proof. We show that $Y \setminus A$ is open. Let $y \in Y \setminus A$.
As before, let U be an evenly covered neighbourhood of $f(y) = pf'(y)$.
Since p restricted to sheets is injective and $pf'(y) = pf''(y)$, it follows

that $f'(y)$ and $f''(y)$ lie on different sheets, say S_1 and S_2, respectively. Let $B' := f'^{-1}(S_1) \cap f''^{-1}(S_2)$. Clearly, $y \in B'$.

Subclaim 2.1. $B' \subset X \setminus A$.

Proof. Let $y_1 \in B'$.
Then, $f'(y_1) \in S_1$ and $f''(y_2) \in S_2$. Since S_1 and S_2 are disjoint, the claim follows. □

The above subclaim proves that $X \setminus A$ is open, as earlier. □

Thus, we are done. □

Theorem 4.4 (Path Lifting Theorem)**.** For $(E, e_0) \xrightarrow{p} (X, x_0)$ a covering space with base points, if σ is a path in X with initial point x_0, there is a unique path σ'_{e_0} in E with initial point e_0 such that $p\sigma'_{e_0} = \sigma$.

Proof. Note that σ is actually a pointed map $(I, 0) \xrightarrow{\sigma} (X, x_0)$ and I is connected. Thus, uniqueness of σ'_{e_0} follows from Unique lifting theorem.

Special case: The whole space X is evenly covered.
Let S be the sheet (over X) containing e_0. Then, $p|_S : S \to X$ is a homeomorphism. Let $\psi : X \to S$ be the inverse to this.
Then, $\sigma'_{e_0} = \psi \circ \sigma$ is the desired map.
Note that $p\sigma'_{e_0} = p\psi\sigma = \sigma$ and $\psi\sigma(0) = \psi(x_0) = e_0$, since $p(e_0) = x_0$. Thus, σ'_{e_0} indeed is a pointed map.

General case: Note that $\sigma(I) \subset X$ is compact. Thus, we can find a finite open cover $\{U_i\}_{i=0}^{n-1}$ of $\sigma(I)$ such that each U_i is evenly covered.
Thus, by the Lebesgue number lemma, we can find a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that $\sigma([t_i, t_{i+i}])$ lies in the evenly covered neighbourhood U_i of $\sigma(t_i)$ for all $0 \le i < n$.
(Well, not exactly but we can renumber U_i wlog so that they satisfy the above condition.)
Thus, note that for each "sub-path" $s|_{[t_i, t_{i+1}]} : [t_i, t_{i+1}] \to U_i$, we can apply the first case.
In particular, for $i = 0$, we lift $s|_{[0, t_1]}$ to a path $\sigma'_1 : [0, t_i] \to E$ such that $\sigma'_1(0) = e_0$.
Assume, as induction, that we have lifted $\sigma|_{[0, t_i]}$ to a map $\sigma'_i : [0, t_i] \to E$ such that $\sigma_i(0) = e_0$. $(0 \le i < n - 1.)$

Also, observe that $p\sigma_i'(t_i) = \sigma(t_i)$.
Then, we can lift $\sigma|_{[t_i,t_{i+1}]}$ to a path $\tau_i : [t_i, t_{i+1}] \to E$ with $\tau_i(t_i) = \sigma_i'(t_i)$. (This is because for the lifting theorem, all we used was that e_0 was a point that gets mapped to x_0 under p. By our previous observation, we see that $\sigma_i(t_i) \xmapsto{p} \sigma(t_i)$ and thus, we can lift a path preserving initial points like that.)
Thus, we get a path $\sigma_{i+1}' : [0, t_{i+1}] \to E$ given by joining σ_i and τ_i.

Thus, by induction, we get a path σ_n' which is our desired σ_{e_0}'. $\square$

Theorem 4.5 (Covering Homotopy Theorem)**.** Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering map as before. Let $F : I \times I \to X$ be a map with $F(0,0) = x_0$.
There is a unique lifting of F to a continuous map

$$F' : I \times I \to E$$

such that $F'(0,0) = e_0$. Moreover, if F is a path homotopy, then F' is a path homotopy.

Proof. We first define $F'(0,0) = e_0$. We will construct F' piece-wise.
First, we use the preceding theorem to extend F to the left edge $\{0\} \times I$ and bottom edge $I \times \{0\}$.
Now, choose subdivision

$$\begin{aligned} 0 = s_0 < s_1 < \cdots < s_m = 1, \\ 0 = t_0 < t_1 < \cdots < t_n = 1 \end{aligned}$$

such that each rectangle

$$I_i \times J_i = [s_{i-1}, s_i] \times [t_{i-1}, t_i]$$

is mapped by F into an open subset of X which is evenly covered by p. (This is for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Such a subdivision exists by the Lebesgue number lemma.)
We now define the lift F' inductively. First we define it on $I_1 \times J_1$, continuing with the other rectangles $I_i \times J_1$ in the bottom row from left to right, then with the rectangles $I_i \times J_2$ in the second row from left to and right, and so on.

In general, given i_0 and j_0, we assume that F' has been defined on set

$$A = \bigcup_{\substack{j < j_0 \\ 1 \leq i \leq n}} (I_i \times J_j) \cup \bigcup_{i < i_0} (I_i \times J_{j_0}) \cup (\{0\} \times I) \cup (I \times \{0\}).$$

(That is, A is the union of the left and bottom edges along with the "previous" rectangles.)
We also assume that F' defined on A so far is a continuous lifting of $F|_A$. Using this, we define F' on $I_{i_0} \times J_{j_0}$ such that it's continuous on $A \cup (I_{i_0} \times J_{j_0})$.

Choose an open set U which is evenly covered by p and contains $I_{i_0} \times J_{j_0}$. (Such a U exists by our construction of the subdivision.)
Let $\{S_\alpha\}$ be the set of sheets, each S_α being mapped homeomorphically onto U by p.
Note that F' is already defined on the subset of $I_{i_0} \times J_{j_0}$ given by $C = A \cap (I_{i_0} \times J_{j_0})$. This subset is *connected* and hence, $F'(C)$, being connected must lie entirely in one sheet.
Let S_0 be this sheet. Let $p_0 := p|_{S_0}$. Then,

$$p_0 : S_0 \to U$$

is a homeomorphism. Moreover, for $x \in C$, we have

$$p_0(F'(x)) = p(F'(x)) = F(x),$$

since F' is a lifting of $F|_A$. Thus, for $x \in C$, we have that

$$F'(x) = p_0^{-1}(F(x)).$$

Thus, if we now define

$$F'(y) = p_0^{-1}(F(y))$$

for $y \in I_{i_0} \times J_{j_0}$, we see that F' must be continuous on $A \cup (I_{i_0}) \times (J_{j_0})$, by the pasting lemma.
Moreover, it is clearly a lift of $F|_{A \cup (I_{i_0} \times J_{j_0})}$ as well. Thus, it satisfies our inductive hypothesis and we may carry out this process and define F' on all of $I \times I$.

To see uniqueness, note that we were forced to define $F'(0,0) = e_0$. Thus, considering (Y, y_0) with $Y = I \times I$ and $y_0 = (0,0)$, appealing to the Unique lifting theorem, we see that at each step, there is a unique lift to $I_{i_0} \times J_{j_0}$. Thus, defining $F'(0,0)$ uniquely determines F'.

Now, suppose that F is a path homotopy. (Note that since we are not saying anything about the two paths between which it is a homotopy, all that matters is that F is constant on the vertical edges.)
Then, the map F carries $\{0\} \times I$ onto a singleton $\{x_0\}$. Since $pF' = F$, we must have that

$$(pF')(\{0\} \times I) = \{x_0\}.$$

In other words, F' carries $\{0\}\times I$ into $p^{-1}(x_0)$. However, note that $\{0\}\times I$ is connected whereas $p^{-1}(x_0)$ is discrete. Thus, F' must be constant on $\{0\}\times I$. Similarly, it must be constant on $\{1\}\times I$ as well, proving the result. □

Theorem 4.6 (Path homotopy lifting theorem)**.** Let $(E,e_0)\xrightarrow{p}(X,x_0)$ be a covering map as before. Let f and g be two paths in X from x_0 to x_1; let f' and g' be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then so are f' and g'. In particular, f' and g' have the same terminal point.

Proof. Let $F : f \simeq g \ \operatorname{rel}\{0,1\}$ be a path homotopy from f to g.
Let F' be given as in the preceding lemma. We wish to show that the bottom edge is f' and top g'.
To this end, define $\alpha,\beta : I \to E$ as

$$\begin{aligned}\alpha(s) &:= F'(s,0),\\ \beta(s) &:= F'(s,1).\end{aligned}$$

We show that $\alpha = f'$ and $\beta = g'$.
Note that $\alpha(0) = F'(0,0) = e_0 = F'(0,1) = \beta(0)$.
Moreover, $p(\alpha(s)) = p(F'(s,0)) = F(s,0) = f(s)$ and similarly, $p(\beta(s)) = g(s)$.
Thus, α and β are some lifts of f and g starting at e_0. By the Unique lifting theorem, we are done. □

Corollary 4.7. $p_* : \pi_1(E,e_0) \to \pi_1(X,x_0)$ is a monomorphism.

Proof. To see that p_* is a monomorphism (i.e., it is injective), it suffices to show that $\ker p_*$ is trivial.
Let $[\sigma] \in \pi_1(E,e_0)$ be an element of $\ker p_*$.
Then, σ is a loop at e_0 in E such that $p\circ\sigma$ is a loop at x_0 such that

$$p\circ\sigma \simeq e_{x_0} \ \operatorname{rel}\{0,1\}.$$

(Where e_{x_0} denotes the constant loop as usual.)
Lifting them back and using the previous theorem, we see that

$$\sigma \simeq e_{e_0} \ \operatorname{rel}\{0,1\}.$$

□

Note that if σ is a loop at x_0 in X, its lifting σ'_{e_0} in X need not be a loop at e_0. (For example, consider $(\mathbb{R},0)\xrightarrow{p}(S^1,(1,0))$ given by $p(x) = e^{2\pi i x}$. The lift of the loop σ in S^1 given by $s\mapsto e^{2\pi i s}$ is the loop σ'_0 in $\mathbb{R}$ given by $s\mapsto s$, which ends at 1.)

However, its terminal point will be a point in $p^{-1}(x_0)$. Moreover, as we saw earlier, the endpoint only depends on the homotopy class of the loop. Thus, we get a well-defined operation

$$\cdot : p^{-1}(x_0) \times \pi_1(X, x_0) \to p^{-1}(x_0)$$

given by

$$e \cdot [\sigma] = \sigma'_e(1).$$

Proposition 4.8 ($\cdot$ is a group action)**.** The above operations satisfies the following properties:

1. $e \cdot 1 = e$ for all $e \in p^{-1}(x_0)$,
2. $e \cdot ([\sigma] * [\tau]) = (e \cdot [\sigma]) \cdot [\tau]$ for all $e \in p^{-1}(x_0)$ and all $\sigma, \tau \in \pi_1(X, x_0)$.

Thus, the above $\cdot$ is a *right group action.*

Proof. Let $e \in p^{-1}(x_0)$, $[\sigma], [\tau] \in \pi_1(X, x_0)$ be arbitrary.

1. Note that $1 \in \pi_1(X, x_0)$ is simply the class of the constant loop $[e_{x_0}]$. The lift of the constant loop is again a constant loop. Thus, since $1'_e$ starts at e, it must end at e as well. In other words,

   $$e = 1'_e(1) = e \cdot 1,$$

   as desired.
2. Define $c \in p^{-1}(x_0)$ as $c := \sigma'_e(1) = e \cdot [\sigma]$.
   We wish to show that

   $$e \cdot ([\sigma * \tau]) = (e \cdot [\sigma]) \cdot [\tau].$$

   In other words, we wish to show that

   $$(\sigma * \tau)'_e(1) = \tau'_c(1).$$

   Consider the path $\sigma'_e * \tau'_c$ in E. The product is well defined since $\sigma'_e(1) = c = \tau'_c(0)$.
   Now, observe that

$$p(\sigma'_e * \tau'_c)(s) = \begin{cases} p(\sigma'_e(2s)) & 0 \le 2s \le 1, \\ p(\tau'_c(2s-1)) & 1 \le 2s \le 2 \end{cases}$$
$$= \begin{cases} \sigma(2s) & 0 \le 2s \le 1, \\ \tau(2s-1) & 1 \le 2s \le 2 \end{cases}$$
$$= (\sigma * \tau)(s).$$

In other words, $\sigma'_e * \tau'_c$ is a lift of $\sigma * \tau$ with initial point e. By uniqueness of lifts, we see that

$$(\sigma * \tau)'_e = \sigma'_e * \tau'_c.$$

Thus, we see that

$$(\sigma * \tau)'_e(1) = \sigma'_e * \tau'_c(1) = \tau'_c(1),$$

as desired. □

Proposition 4.9 (Description of stabilisers)**.** The stabiliser of a point $e_0 \in p^{-1}(x_0)$ is the subgroup

$$p_*\pi_1(E, e_0) \subset \pi_1(X, x_0).$$

Proof. Note that $[\sigma] \in \pi_1(E, x_0)$ belongs to the stabiliser S of e_0 iff $\sigma'_{e_0}(1) = e_0$. In other words, $[\sigma] \in S$ iff σ lifts to a loop at e_0.
If $\sigma = p \circ \sigma'$ for some loop σ' at e_0, then $[\sigma] \in S$.
Conversely, if $[\sigma] \in S$, then $\sigma'_{e_0}(1) = e_0$ and thus, $[\sigma'_{e_0}] \in \pi_1(E, e_0)$ with $\sigma = p \circ \sigma'_{e_0}$. □

Proposition 4.10. If E is path-connected, $\pi_1(X, x_0)$ acts transitively.

Proof. Let $e, c \in p^{-1}(x_0)$. We wish to show that there exists $[\sigma] \in \pi_1(X, x_0)$ such that $e \cdot [\sigma] = c$.
Since E is path-connected, we can find a path σ' in E from e to c. Then, $\sigma = p \circ \sigma'$ fits the bill.
To see this, note that σ' is indeed the lift of σ with initial point σ. That is, $\sigma' = \sigma'_e$. Moreover, since it ends at c, we get

$$e \cdot [\sigma] = \sigma'_e(1) = \sigma'(1) = c.$$ □

Recall from group theory that given an action $\cdot : S \times G \to S$ with $s_0 \cdot g = s_1$, we have $G_{s_0} = g G_{s_1} g^{-1}$, where G_s denotes the stabiliser of s in G.
Thus, if E is path-connected, then all the different subgroups $p_*\pi_1(E, e)$ are conjugate, as e runs over all points in $p^{-1}(x_0)$.

Corollary 4.11. If E is path-connected, the map $[\sigma] \mapsto e_0 \cdot [\sigma]$ induces a bijection of the set of all cosets $p_*\pi_1(E, e_0)[\sigma]$ onto the fiber. In particular, if $p^{-1}(x_0)$ is finite, the number of points in the fiber is equal to the index of the subgroup $p_*\pi_1(E, e_0)$.

Proof. In general, let $\cdot : S \times G \to S$ be a group action.
Let $G_s \leq G$ be the stabiliser of $s \in S$.
Then, given any $g,' \in G$ we have

$$s \cdot g = s \cdot g'$$

iff

$$g \cdot g'^{-1} \in G_s \text{ or } g \in G_s g'.$$

Thus, the map $G/G_s \to S$ given by

$$G_s g \mapsto s \cdot g$$

is well defined and an injection.
Moreover, if the action is transitive, then the above map is clearly surjective as well.
(In the above, G/G_s is just the *set* of right cosets, no assumptions of normality.) □

Exercise 4.12. If $E \xrightarrow{p} X$ is a covering map and X is connected, then all fibers have the same cardinality.

Solution. Choose $x_0 \in X$. Let $\mathfrak{x} = |p^{-1}(x_0)|$. ($\mathfrak{x}$ is just a cardinal number, need not be finite.)
Let $A = \{x \in X : |p^{-1}(x)| = \mathfrak{x}\}$. We wish to show $A = X$. We first show that A is open.
Let $x \in A$ and let U be an evenly covered neighbourhood of x. Then, for any $x' \in U$, we must have that $|p^{-1}(x')|$ equals the (cardinal) number of sheets over U. (Since each sheet contains exactly one element of $p^{-1}(x')$ and any element of $p^{-1}(x')$ must lie in one of these sheets.)
Thus, $U \subset A$ and A is open.
By a similar argument, we see that $X \setminus A$ is also open. Since X is connected and $x_0 \in A$, we see that $A^c = \varnothing$ or $A = X$. □

Definition 4.13 (Sheeted coverings)**.** Let $E \xrightarrow{p} X$ be a covering space. If each $p^{-1}(x)$ has cardinality a finite number n, the covering is called an *n-sheeted covering.*

Definition 4.14 (Group of covering transformations)**.** Given a covering space $E \xrightarrow{p} X$, the group G of *covering transformations* is the group of all homeomorphisms of E which preserves the fibers, that is, all those φ such that $p\varphi = p$.

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Theorem 4.15. Given a covering space $(E, e_0) \xrightarrow{p} (X, x_0)$ with group of covering transformation G. If E is simply connected and locally path-connected, G is canonically isomorphic to $\pi_1(X, x_0)$.

This achieves the result we described at the beginning of the section.

Proof. First, we define a homomorphism

$$\chi : G \to \pi_1(X, x_0).$$

Let $\varphi \in G$. Since E is simply connected, all paths from e_0 to $\varphi(e_0)$ are homotopic relative $\{0, 1\}$. (By Lemma 1.18.)
Thus, if σ' is such a path, then $p_*([\sigma'])$ depends only on e_0 and $\varphi(e_0)$; we define

$$\chi(\varphi) = [p \circ \sigma'].$$

(That is, we define $\chi(\varphi)$ to be $p_*([\sigma'])$ for any path σ' from e_0 to $\varphi(e_0)$. Note that e_0 is fixed.)
Note that since $p\varphi = p$, we see that $p(\varphi(e_0)) = p(e_0) = x_0$ and hence, $p \circ \sigma'$ is indeed a *loop* at x_0. Thus, the above map χ indeed is a map from G to $\pi_1(X, x_0)$.

Claim 1. χ is a homomorphism.

Proof. Let $\varphi, \psi \in G$. Let σ' be any path from e_0 to $\varphi(e_0)$ and τ' be any path from e_0 to $\psi(e_0)$.
Define the path $\alpha' = \psi \circ \sigma'$. This is clearly a path from $\psi(e_0)$ to $\psi(\varphi(e_0))$.
In particular, $\tau' * \beta'$ is a path from e_0 to $\psi(\varphi(e_0))$.
Moreover, since $\psi \in G$, we have

$$p \circ \alpha' = p \circ \psi \circ \sigma' = p \circ \sigma'.$$

Thus, we have

$$\begin{aligned}\chi(\psi\circ\varphi) &= [p\circ(\tau'*\alpha')] \\ &= [p\circ\tau']*[p\circ\alpha'] \\ &= [p\circ\tau']*[p\circ\sigma'] \\ &= \chi(\psi)*\chi(\varphi).\end{aligned}$$

□

Claim 2. χ is injective.

Proof. By definition, it is clear that

$$\varphi(e_0) = e_0\cdot\chi(\varphi).$$

Hence, $\chi(\varphi)=1$ implies that $\varphi(e_0)=e_0\cdot 1=e_0$, i.e., φ fixes e_0.
However, note that being a covering transformation, we have that $p\varphi = p$; in other words, φ lifts p. By 4.3, there is only one lift of p which fixes e_0. Since the identity is one such, we see that $\chi(\varphi)=1 \implies \varphi = \mathrm{id}$, the identity of $\pi_1(X,x_0)$, proving that χ is injective. □

Claim 3. χ is surjective.

Proof. Let $[\sigma]\in\pi_1(X,x_0)$ be arbitrary. We construct a $\varphi\in G$ such that $\chi(\varphi)=[\sigma]$.
We define φ as follows:
Let $e\in E$, let τ' be any path from e_0 to e, and let $\tau = p\circ\tau'$. Note that τ is a path from $p(e_0)=x_0$ to $p(e)=: x$. Then, $\tau^{-1}\sigma\tau$ is a loop at x. We define

$$\varphi(e) := e\cdot[\tau^{-1}\sigma\tau],$$

where the $\cdot$ is as before. (The endpoint of the unique lift of $\tau^{-1}\sigma\tau$ in E starting at e.)
Note that the above does not depend on τ' since E is simply connected. (As earlier, we use Lemma 1.18.)
In other words, φ just depends on $[\sigma]$.
Now, taking $e=e_0$, we may take τ' as the constant map and we see that $\varphi(e_0)=e_0\cdot[\sigma]=\sigma'_{e_0}(1)$.
Thus, to compute $\chi(\varphi)$ using the definition of χ, we may take the path joining e_0 and $\varphi(e_0)$ to be σ'_{e_0} and we get

$$\chi(\varphi) = [p\circ\sigma'_{e_0}] = [\sigma],$$

as would be desired. Thus, we just need to show that $\varphi\in G$.

It is easy to see that that $p\varphi = p$. Indeed, since $\varphi(e)$ is the endpoint of a lift of a *loop* at $p(e)$, we see that that it must belong to the fiber $p^{-1}(x)$. Thus, $p(\varphi(e)) = x = p(e)$.
Moreover, φ has an inverse of the same type that is obtained by replacing σ with σ^{-1} in the definition. Thus, we just need to show that φ is continuous. (The same will show that φ^{-1} is also continuous.)

To do so, we will show the following: For every $e_1 \in E$ and every neighbourhood V' of $\varphi(e_1)$, there exists a neighbourhood V of e_1 such that $\varphi(V) \subset V'$.
To this end, let $e_1 \in E$ be arbitrary. Consider $x_1 = p(e_1) \in X$.
Let U be an open neighbourhood of x_1 which is evenly covered. Since E is locally path-connected, so is X and thus, we may assume so is U. (Or we replace U by a smaller path-connected neighbourhood, which will still be evenly covered.)
Then, $e_1 \in S_1$ and $\varphi(e_1) \in S_1'$ for some sheets S_1, S_1' over U. (Recall that e_1 and $\varphi(e_1)$ belong to the same fiber $p^{-1}(x_1)$.)
We claim that $\varphi(S_1) \subset S_1'$.
To see this, note that if $e \in S_1$, we can join e_1 to e by some path α' in S_1 (since E is locally path-connected); then, consider the path $p \circ \tau$ in X from x_1 to $p(e)$; lifting this to a path $\tau'_{\varphi(e_1)}$, we see that it is in S_1'. In particular, its end point is a point in S_1'. This end point is just $\varphi(e)$. Thus, we have that shown $\varphi(e) \in S_1'$ or that $\varphi(S_1) \subset S_1'$.

Now, given any neighbourhood V' of $\varphi(e_1)$, we can find a neighbourhood $S_1' \subset V'$ of $\varphi(e_1)$ of the above type. (That is, a neighbourhood of $\varphi(e_1)$ which is a sheet over some open neighbourhood U of $x_1 \in X$.)
This proves that φ is continuous and thus, $\varphi \in G$. □

With that, we are done! □

§§4.1. Even actions

In this subsection, we prove some more results and ways to calculate the fundamental group of a space. Following Greenberg and Fulton, we develop this theory via exercises.

Exercise 4.16. Assume only E is path-connected and locally path-connected. Let N be the normalizer of $p_*\pi_1(E, e_0)$ in $\pi_1(X, x_0)$. Modify the above argument to obtain a homomorphism of N onto G with kernel $p_*\pi_1(E, e_0)$. Thus, we get that

$$N/p_*\pi_1(E, e_0) \cong G.$$

Solution. We give an outline of the proof. The details follow in the same way as the previous proof.
Define $\chi : N \to G$ as follows:
Given $[\sigma] \in N$, define the function $\varphi_{[\sigma]} : E \to E$ by

$$\varphi_{[\sigma]}(e) = e \cdot [\tau^{-1}\sigma\tau]$$

where τ' is any path joining e_0 to e and $\tau = p \circ \tau'$. Note that $p\varphi_{[\sigma]} = p$ and that $\varphi_{[\sigma]}$ is a homeomorphism by the same arguments are earlier. (Here is where we used that E was locally path-connected.)

First, we note that this is an *anti*-homomorphism. Let $[\alpha], [\beta] \in N$, $e \in E$, and τ' be any path from e_0 to e and let $\tau = p \circ \tau'$. Then, we have

$$\begin{aligned}
\varphi_{[\alpha\beta]}(e) &= e \cdot [\tau^{-1}\alpha\beta\tau] \\
&= e \cdot \left([\tau^{-1}\alpha\tau][\tau^{-1}\beta\tau]\right) \\
&= (e \cdot [\tau^{-1}\alpha\tau]) \cdot [\tau^{-1}\beta\tau] \\
&= (\varphi_{[\alpha]}(e)) \cdot [\tau^{-1}\beta\tau] \\
&= \varphi_{[\beta]}(\varphi_{[\alpha]}(e)).
\end{aligned}$$

Thus, $\varphi_{[\alpha\beta]} = \varphi_{[\beta]} \circ \varphi_{[\alpha]}$ showing that χ is an anti-homomorphism.

To show that χ is surjective, consider an arbitrary $\varphi \in G$. Consider any path $e_0 \xrightarrow{\tau'} \varphi(e_0)$ and the path $p \circ \tau'$ in X which is a loop at x_0.
One can check that $[p \circ \tau']$ is an element of N. As before, one can check that $\chi([p \circ \tau']) = \varphi$. Thus, χ is surjective.

Lastly, we show that $\ker \chi = p_*\pi_1(E, e_0)$.
First, let $[\sigma] \in p_*\pi_1(E, e_0)$. Then, $\sigma = p \circ \sigma'$ for some loop σ' at e_0 in E.
Let $e \in E$ be arbitrary and let τ' be any path from e_0 to e.

$$\begin{aligned}
\varphi_{[\sigma]}(e) &= e \cdot [\tau^{-1}\sigma\tau] \\
&= e \cdot [p \circ (\tau'^{-1}\sigma'\tau')].
\end{aligned}$$

Recalling the definition of $\cdot$, we see that $e \cdot [p \circ (\tau'^{-1}\sigma'\tau')] = e$, since $\tau'^{-1}\sigma'\tau'$ is a lift of $p \circ (\tau'^{-1}\sigma'\tau')$ with initial point e that ends at e.
Thus, we see that $\varphi_{[\sigma]} = \mathrm{id}_E$. This shows that $\ker \chi \supset p_*\pi_1(E, e_0)$.
Conversely, let $[\sigma] \in \ker \chi$. Then, $\varphi_{[\sigma]} = \mathrm{id}_E$. In particular,

$$\varphi_{[\sigma]}(e_0) = e_0.$$

Considering τ to be the constant loop at e_0, we see that

$$e_0 = \varphi_{[\sigma]}(e_0) = e_0 \cdot [\sigma].$$

That is, σ has a lift σ' which is a loop at e_0. Thus, $[\sigma] = [p \circ \sigma'] \in p_* \pi_1(E, e_0)$, as desired. $\square$

Definition 4.17 (Even actions)**.** Let E be a topological space and G any group of homeomorphisms of E.
E is said operate *evenly* if for any $e \in E$, there is an open neighbourhood V of e such that

$$V \cap gV = \varnothing \qquad \text{for all } g \in G \setminus \{1\}.$$

Exercise 4.18. Given a space E path connected and locally path-connected. Let G be a group of homeomorphisms of E which operates evenly. Let $X = E/G$ be the space of orbits, $p : E \to X$ the map sending any e onto its orbit Ge. Then

1. $E \xrightarrow{p} X$ is a covering space (for this, we don't need any connectedness),
2. G is its group of covering transformations, and
3. $p_* \pi_1(E, e_0)$ is a normal subgroup of $\pi_1(X, x_0)$ for all $e_0 \in E$.

By the previous exercise, we see that we have

$$\pi_1(X, x_0)/p_* \pi_1(E, e_0) \cong G.$$

This exercise tells us that if we know a simply connected covering space E of X and its group G of covering transformations, then not only do we know

$$\pi_1(E/G) = \pi_1(X) \cong G,$$

but we also can recover X (up to homeomorphism) as E/G.

Solution.

1. Let $x \in X$. Then, $x = Ge$ for some $e \in E$.
   Since G acts evenly, there exists some a neighbourhood V of e such that $V \cap gV = \varnothing$ for all $1 \neq g \in G$.
   Also note that g is homeomorphism for all $g \in G$ and thus, each gV is open.
   Also, note that $S = \displaystyle\bigsqcup_{g \in G} gV$ is open and saturated with respect to q. (That is, if $s \in S$, then $p^{-1}(p(s)) \subset S$.)

To see the saturation, note that if $s \in S$, then $s \in gV$ for some $g \in G$. In particular, $s = g(v)$ for some $v \in V$. (Recall that an element g of G is a homeomorphism from E to itself.)
Now, if $s' \in p^{-1}(p(s))$, then $p(s') = p(s)$. In other words, s' and s belong to the same orbit. Thus, $s' = g'(s) = g'(g(v))$ for some $g' \in G$.
Since G is a group, we see that $g'g \in G$ and thus, $s' \in g'gV \subset S$, proving saturation.

Since q is a quotient map (we are giving X the quotient topology and thus, q is a quotient map by construction), we see that $U := p(S)$ is open in X. Moreover, U contains x.
In other words, U is an open neighbourhood of x and we have

$$p^{-1}(U) = \bigsqcup_{g \in G} gV.$$

Lastly, we need to show that p maps each gV homeomorphically onto U.

First, we make the following observation: Given $g \in G$ and $v \in V$, we have

$$p|_{gV}(g(v)) = G(gv) = (Gg)v = Gv = p|_V(v).$$

That is, $p|_{gV} \circ g = p|_V$. Thus, it suffices to show that $p|_V$ is a homeomorphism and then it would follow that each $p|_{gV}$ is, too.

One-one: Let $v, v' \in V$ with $p(v) = p(v')$. Then, we have

$$Gv = Gv'.$$

Since $1(v) = v$ is an element of the LHS, it must be one of the RHS as well. Thus, $v = gv'$ for some $g \in G$. In particular, $v \in gV$.
Hence, $v \in V \cap gV$. This forces $g = 1$ and hence, $v = v'$.

Onto: Let $u \in U$. Then, $u = p(s)$ for some $s \in S$. This is same as saying that $s \in gV$ for some $g \in G$.
Thus, $u = p(g(v))$ for some $g \in G$ and $v \in V$. Note that

$$u = p(g(v)) = G(g(v)) = Gv = p(v).$$

Thus, $u = p(v)$, showing that $p|_V$ is surjective.

Note that p is continuous by virtue of being a quotient map and hence, so is $p|_V$. Thus, we only need to show that $p|_V^{-1}$ is continuous.
This is the same as showing that $p|_V(W)$ is open in U for all open

$W \subset V$.
To this end, consider an arbitrary open subset W of V. Consider the counterparts $gW \subset gV$ for all $g \in G$. Since each g is a homeomorphism, each gW is open in gV and thus, in G.
Consider the set

$$S_W = \bigsqcup_{g \in G} gW \subset S.$$

As before, S_W is saturated and thus, $p(S_W)$ is open in U. However, note that

$$p|_V(W) = p(S_W).$$

The inclusion $\subset$ is obvious since $W \subset S_W$. For the reverse, note that if $u \in p(S_W)$, then $u = p(gw)$ for some $g \in G$ and $w \in W$. Thus, $u = G(gw) = Gw = p(w)$, showing that $u \in p|_V(W)$, and completing the proof.

2. We show that G is precisely the set of all those homeomorphism that lift p. Define

$$H := \{\varphi : E \to E \mid \varphi \text{ is a homeomorphism and } p\varphi = p\}.$$

We wish to show that $H = G$.

($\supset$) Suppose $g \in G$.
Then, g is homeomorphism from E onto itself by definition. We just need to show that $pg = p$. This too is simple. Indeed, let $e \in E$, then

$$p(g(e)) = G(ge) = (Gg)e = Ge = p(e).$$

($\subset$) Suppose $\varphi \in H$. Then, $\varphi : E \to E$ is homeomorphism such that $p\varphi = p$.
Fix any $e_0 \in E$. Then, we must have $p(\varphi(e_0)) = p(e_0)$.
Thus, $G\varphi(e_0) = Ge_0$ which tells us that

$$\varphi(e_0) = g(e_0)$$

for some $g \in G$. We now wish to claim that $\varphi(e) = g(e)$ for *all* $e \in E$.
Let $x_0 := p(g(e_0)) = p(e_0)$. Thus, we get that φ and g are lifts of p as in the following diagram:

$$\begin{tikzcd}
(E, g(e_0) = \varphi(e_0)) \arrow[d, "p"] & \\
(X, x_0) & (E, e_0) \arrow[l, "p"] \arrow[ul, dashed, "g, \varphi"']
\end{tikzcd}$$

However, E is connected! Thus, appealing to the Unique lifting theorem, we see that $\varphi = g$ and thus, $\varphi \in G$, as desired.

3. Fix any $e_0 \in E$. We wish to show that

$$p_* \pi_1(E, e_0) \lhd \pi_1(X, x_0).$$

To this end, let $[\sigma] \in p_* \pi_1(E, e_0)$ and $[\tau] \in \pi_1(X, x_0)$.
We will show that $[\tau][\sigma][\tau]^{-1} \in p_* \pi_1(E, e_0)$.

Note that $[\sigma] \in p_* \pi_1(E, e_0)$ is the same as saying that the lift σ'_{e_0} of σ starting at e_0 is a loop.
Let τ'_{e_0} be the lift of τ starting at e_0. Let the terminal point be e_1. Note that e_1 belongs to the same fiber as e_0, i.e., $p(e_1) = x_0$.
Note that this means $Ge_0 = Ge_1$ or that $e_1 = g(e_0)$ for some $g \in G$.
We may lift σ to a path σ'_{e_1} starting at e_1. In fact, $g \circ \sigma'_{e_0}$ is one such (and hence, the only one). In particular, note that σ'_{e_1} is again a loop.

Thus, we see that we have a loop $\tau'_{e_0} \sigma'_{e_1} \tau'^{-1}_{e_0}$ as follows:

$$e_0 \xrightarrow{\tau'_{e_0}} e_1 \xrightarrow{\sigma'_{e_1}} e_1 \xrightarrow{\tau'^{-1}_{e_0}} e_0$$

Moreover, we have

$$\tau \sigma \tau^{-1} = p \circ (\tau'_{e_0} \sigma'_{e_1} \tau'^{-1}_{e_0})$$

or

$$[\tau \sigma \tau^{-1}] = p_* [\tau'_{e_0} \sigma'_{e_1} \tau'^{-1}_{e_0}],$$

showing $[\tau][\sigma][\tau]^{-1} \in p_* \pi_1(E, e_0)$, as desired. □

Exercise 4.19. Projective n-space $\mathbb{R}P^n$ is defined as the quotient space of S^n obtained by identifying antipodal points. The group of covering transformations of $S^n \to \mathbb{R}P^n$ consists of the identity and the antipodal mapping only (because S^n is connected, $n > 0$), and since S^n is simply connected for $n \geq 2$, we have

$$\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}, \quad n \geq 2.$$

Solution. Fix $n \geq 2$. We shall construct G as in the previous exercise with $E = S^n$.
Let G consist of the identity homeomorphism and the antipodal homeomorphism $x \mapsto -x$.
It is easy to see that G acts evenly. Indeed, let $x \in S^n$. Take the open ball of radius 1 in $\mathbb{R}^{n+1}$ centered at x and intersect it with S^n. This intersection U

is a neighbourhood of x such that $U \cap (-U) = \varnothing$.
Moreover, orbit of any point $e \in E$ is precisely $\{e, -e\}$ which gives us that $E/G = \mathbb{R}P^n$ in this case.
Thus, by the previous exercise, we see that $p_*\pi_1(S^n, e_0)$ is a normal subgroup of $\pi_1(\mathbb{R}P^n, p(e_0))$ for any e_0. The exercise before that then tells us that

$$\pi_1(\mathbb{R}P^n, p(e_0))/p_*\pi_1(S^n, e_0) \cong G.$$

Clearly, we have that $G \cong \mathbb{Z}/2\mathbb{Z}$ since that is the only group with two elements up to isomorphism. Moreover, we know that $\pi_1(S^n, e_0) = (1)$ since S^n is simply connected for $n \geq 2$. (Proposition 3.14) Moreover, $\mathbb{R}P^n$ is path-connected, being the quotient of a path-connected space.
Thus, we see that

$$\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}.$$

□

Exercise 4.20. Show that $\mathbb{R}P^1$ is homeomorphic to S^1 and thus,

$$\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}.$$

Solution. Define $\varphi : S^1 \to S^1$ by $\varphi(z) = z^2$. (Viewing $S^1 \subset \mathbb{C}$.)
Note that φ is surjective and continuous. Moreover, S^1 is compact and Hausdorff. Thus, φ is a quotient map.
Moreover the equivalence relation induced by setting $x \sim y$ iff $\varphi(x) = \varphi(y)$ is precisely the equivalence relation that identifies the antipodal points and thus, we see that this induces a map

$$\tilde{\varphi} : \mathbb{R}P^1 \to S^1$$

which is a quotient map. (The universal property of quotient maps.)
Since this map $\tilde{\varphi}$ is one-one, we see that $\tilde{\varphi}$ is actually a homeomorphism, as desired. □

Exercise 4.21. Show that $\pi_1(\mathbb{R}P^n)$, $n \geq 2$, is generated by the composition pg where $g : I \to S^n$ is any continuous map satisfying $g(0) = -g(1)$ and $p : S^n \to \mathbb{R}P^n$ is the quotient map as above.

Solution. Let g be any path as given. Let $x_0 := g(0)$. Note that pg is indeed a loop in $\mathbb{R}P^n$ since $p(x_0) = p(-x_0)$.
Since $\pi_1(\mathbb{R}P^n, p(x_0)) = \mathbb{Z}/2\mathbb{Z}$, all we need to show is that $[pg]$ isn't the identity element.
Suppose not. Then, we have that

$$pg \simeq e_{p(x_0)} \text{ rel } \{0, 1\}.$$

Lifting both the paths back to S^n with initial point x_0 and using Theorem 4.6, we see that

$$g \simeq e_{x_0} \ \text{rel}\,\{0,1\}$$

which is plainly wrong since g is not even a loop. □

Exercise 4.22. Let G be the subgroup of the group of homeomorphisms of the plane to itself generated by the translation $(x, y) \mapsto (x+1, y)$ and by the mapping $(x, y) \mapsto (-x, y+1)$. Show that this action of G on $\mathbb{R}^2$ is even, and identify $\mathbb{R}^2/G$ with the Klein bottle.

Solution. First we show that if $\varphi \in G$ is not the identity map, then φ has no fixed points.

> **Claim 1.** Any element of G can be written as
>
> $$(x, y) \mapsto ((-1)^m x + n, y + m)$$
>
> for some integers n and m. Denote the map above as $\varphi_{m,n}$.
>
> *Proof.* Let φ_1 and φ_2 denote the first and second map in the question, respectively.
> Note that φ_1 and φ_2 are indeed of the above form. We have $\varphi_1 = \varphi_{0,1}$ and $\varphi_2 = \varphi_{1,0}$. One can similarly check that the same is true for their inverses and the identity map is $\varphi_{0,0}$.
>
> Since every element of G is some product of φ_1, φ_2, and their inverses, it suffices to show that the product (composition) of any of these four maps with a map of the form $\varphi_{m,n}$ is again a map for the same form. Indeed, we have, in general that
>
> $$\varphi_{m_1,n_1} \circ \varphi_{m_2,n_2} = \varphi_{m_1+m_2,n_1+(-1)^{m_1} n_2}.$$
>
> The above is just a straightforward check. Since each of the four desired maps can be written as φ_{m_1,n_1} for an appropriate choice of (m_1, n_1), we are done. □

From the above claim, it clearly follows that if $\varphi \in G$ has a fixed point, then $\varphi = \varphi_{0,0} = \mathrm{id}_{\mathbb{R}^2}$.
Thus, let $\varphi \in G \setminus \{1\}$ and $x \in \mathbb{R}^2$. Since $\varphi \neq 1$, $\varphi(x) \neq x$. Moreover, note that φ is an isometry. Thus, the ball of radius $1/2$ at x gets mapped to the ball of radius $1/2$ at $\varphi(x)$. Since the distance between x and $\varphi(x)$ is at least 1,

these neighbourhoods are disjoint.
Thus, G acts evenly on $\mathbb{R}^2$.

We now prove a converse to Claim 1.

Claim 2. Given any $(m,n)\in\mathbb{Z}\times\mathbb{Z}$,

$$\varphi_{m,n}\in G.$$

Proof. We need to show that $\varphi_{m,n}$ can be written as a product of (possibly negative) powers of φ_1 and φ_2.
It is easy to see that

$$(\varphi_2)^m = \varphi_{m,0}$$

and

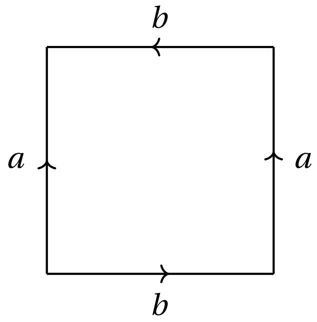
$$(\varphi_1)^n = \varphi_{0,n}.$$

Finally, this gives us

$$(\varphi_1)^n \circ (\varphi_2)^m = \varphi_{m,n}.$$

□

At this point, we remark that G is *not* $\mathbb{Z}^2$ since the product is not commutative in general.
Let $S = I \times I$, denote the unit square. It is easy to see that S contains a point from the orbit of each element. Indeed, given any $(x,y)\in\mathbb{R}^2$, first choose $m = -\lfloor y \rfloor$, and then $n = -\lfloor (-1)^m x \rfloor$.
Thus, the covering map $p : \mathbb{R}^2 \to \mathbb{R}^2/G$ restricts to a surjection $q : S \to \mathbb{R}^2/G$.
Moreover, note that if $0 < x < 1$ and $0 < y < 1$, then the orbit of (x,y) intersects S at exactly one point.
Now, suppose $0 < y < 1$, then $(0,y)$ and $(1,y)$ belong to the same orbit and nothing else from S does.
Similarly, if $0 < x < 1$, then $(x,0)$ and $(1-x,1)$ belong to the same orbit and nothing else from S does.
Finally, we see that the four corners of S belong to the same orbit and nothing else from S does.
Note that one may identity the Klein Bottle K as a quotient of S as follows:



We have shown above that the surjection $q : S \to \mathbb{R}^2/G$ factors through a map $\tilde{q} : K \to \mathbb{R}^2/G$. Moreover, q can be verified to be a quotient since $\mathbb{R}^2/G$ is Hausdorff. Thus, we see that $\tilde{q}$ is also a quotient map. By our observation earlier, $\tilde{q}$ is also one-one and thus, a homeomorphism. □

Exercise 4.23. If a finite group G acts on a Hausdorff space Y, and there are no fixed points (i.e., no $y \in Y$ is fixed by any $g \in G$ except the identity element), show that the action is even.

Solution. Let $1 = g_1, \ldots, g_n$ be the elements of G. (Assume $n \geq 2$.)
Fix i such that $2 \leq i \leq n$. That is, $g_i \neq 1$.
Thus, given any $y \in Y$, we have $g_i(y) \neq y$.
Thus, we may find disjoint neighbourhoods U_i' and V_i of y and $g_i(y)$ respectively.
Since g_i is continuous, we may find an open neighbourhood U_i'' of y such that $g_i(U_i'') \subset V_i$. Finally, setting $U_i := U_i' \cap U_i''$, we see that $U_i \cap g_i(U_i) = \varnothing$.

Now, the intersection
$$U := \bigcap_{i=2}^{n} U_i$$
is a neighbourhood of y. (Finite intersection of open sets is open.)
Moreover, we see that $g_i(U) \cap U = \varnothing$ for all $2 \leq i \leq n$. □

Exercise 4.24. Let $G = \mu_n$ be the group of nth roots of unity. The odd dimensional sphere $S^{2m-1} \subset \mathbb{R}^{2m}$ can be viewed as

$$S^{2m-1} := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_1|^2 + \cdots + |z_m|^2 = 1\}.$$

The group G acts on S^{2m-1} by

$$\zeta \cdot (z_1, \ldots, z_m) = (\zeta z_1, \ldots, \zeta z_m).$$

Show that this action is even. When n is prime, the quotient space S^{2m-1}/μ_n is called a *Lens space*.
Compute the fundamental group of the Lens spaces.

Solution. First we show that this action is even.
Note that S^{2m-1} is a Hausdorff space and G is finite. Thus, it suffices to show that G acts without fixed points. (Recall Exercise 4.23.)
This is clear, for if $z \in S^{2m-1}$, then one component z_i is nonzero. Thus, we have $z_i = \zeta z_i$ or $\zeta = 1$.
This shows that the action is even.

If $m \geq 2$, by Exercise 4.18, we see that the fundamental group is

$$\pi_1(S^{2m-1}/\mu_n) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z}.$$

(Since S^{2m-1} is then simply connected.)
If $m = 1$, then S^1/μ_n is homeomorphic to S^1. The proof is similar to the one in Exercise 4.20 by considering the map $z \mapsto z^n$. (Indeed, $\mathbb{R}P^1$ is just the special case of $n = 2$.)
Thus, we then have

$$\pi_1(S^1/\mu_n) \cong \mathbb{Z}.$$

□

Exercise 4.25. If G acts evenly on a space Y, and H is a subgroup of G, show that H also acts evenly. Show that the natural map from Y/H to Y/G is a covering mapping. If n is the index of H in G, this is an n-sheeted covering.

Solution. That H acts evenly on Y is obvious. Indeed, let $h \in H \setminus \{1\}$ and let $y \in Y$ be arbitrary. Since G acts evenly and $h \in G$, there exists a neighbourhood U of y such that $U \cap hU = \varnothing$, as desired.

Now, we see what the natural map $p : Y/H \to Y/G$ is. Note that any element of Y/H is an orbit of the form Hy. Given $y, y' \in Y$ such that $Hy = Hy'$, we clearly have $Gy = Gy'$. Thus, we get a well defined map $p : Y/H \to Y/G$ given by

$$Hy \mapsto Gy.$$

Also, note that the quotient (and covering) map $p_G : Y \to Y/G$ and the quotient (and covering) map $p_H : Y \to Y/H$ form the following commutative diagram.

$$\begin{tikzcd} Y \arrow[d, "p_H"'] \arrow[dr, "p_G"] & \\ Y/H \arrow[r, "p"'] & Y/G \end{tikzcd}$$

We wish to show that p is also a covering map. We had seen a detailed proof of p_G being a covering map in Exercise 4.18. We do not go into the details here and give the outline.
Since G acts evenly on Y, given any $y \in Y$, we can find a neighbourhood V of y which satisfies $g_1V \cap g_2V = \varnothing$ if $g_1 \neq g_2 \in G$. (We had shown that $\{gV\}$ are sheets over $p_G(V)$, an evenly covered neighbourhood of Gy.)
Then, it follows that if $p_H(gV) \cap p_H(g'V) \neq \varnothing$ for some $g, g' \in G$, then $p_H(gV) = p_H(g'V)$.
Thus, consider the set

$$\{p_H(gV) : g \in V\}.$$

We have shown that distinct elements in the above set are disjoint. This is the disjoint partition into sheets. (Why are they open?)
It follows that p is a covering map.

To show that this is an n-sheeted covering: Let $g_1, \ldots, g_n \in G$ be (right) coset representatives of H in G.
We show that given any $y \in Y$, the precise pre-image $p^{-1}(Gy)$ is given as

$$p^{-1}(Gy) = \{Hg_1y, \ldots, Hg_ny\}. \quad (*)$$

$(\subset)$ Suppose $Hy' \in p^{-1}(Gy)$. Then, $Gy = p(Hy') = Gy'$.
Thus, $y' = gy$ for some $g \in G$ or $y' = hg_iy$ for some $1 \leq i \leq n$ and $h \in H$.
Thus, we have $Hy' = Hhg_iy = Hg_iy$, as desired.
$(\supset)$ This is obvious for $p(Hg_1y) = Gg_1y = Gy$.

Lastly, we show that all the elements in the set in $(*)$ are actually distinct.
Suppose that $Hg_iy = Hg_jy$ for some $1 \leq i, j \leq n$.
Then, since g_iy is an element of the LHS, it is one of the RHS. Thus,

$$g_iy = hg_jy$$

for some $h \in H$. Since G acts evenly on Y, we must have $g_i = hg_j$. Since $\{g_k\}$ was a set of coset representatives, we have $i = j$, as desired. $\square$

§5. A Lifting Criterion

Unless otherwise stated, all the spaces in this section will be path-connected and locally path-connected.

Theorem 5.1. Consider the situation

$$\begin{tikzcd} (E,e_0) \arrow[d, "p"'] & \\ (X,x_0) & (Y,y_0) \arrow[l, "f"] \arrow[ul, dashed, "f'"'] \end{tikzcd}$$

where p is a covering map and f is an arbitrary map. There exists a lifting f' as depicted if and only if

$$f_*\pi_1(Y,y_0) \subset p_*\pi_1(E,e_0).$$

Proof. The "only if" part is clear since that follows from the functorality. $(f = pf' \implies f_* = p_*f'_* \implies f_*\pi_1(Y,y_0) = p_*f'_*\pi_1(Y,y_0) \subset p_*\pi_1(E,e_0).)$

Conversely, suppose the inclusion given above holds. We construct a lift $f' : (Y,y_0) \to (E,e_0)$ as follows:
Before doing so, we first observe what the above inclusion really says: It says that if a loop α at x_0 in X can be written as $f\circ\beta$ for some loop β at y_0 in Y, then it can also be written as $p\circ\alpha'$ for some loop α' at e_0 in E.
Now, for any $y \in Y$, choose any path σ from y_0 to y. Then, $f\circ\sigma$ is a path from x_0 to $x := f(y)$. Set

$$f'(y) = (f\circ\sigma)'_{e_0}(1).$$

> **Claim.** The above does not depend on σ.
> If τ is another path from y_0 to y, then $\sigma * \tau^{-1}$ is a loop at y_0.
> Then, $f\circ(\sigma*\tau^{-1}) = (f\circ\sigma)*(f\circ\tau^{-1})$ is a loop at x_0 and thus, has a lift which is a loop at e_0. Let this lift be α.
> Let $\beta : I \to E$ be the "second half" of α defined as:
>
> $$\beta(s) = \alpha\left(\frac{1}{2}(s+1)\right).$$

One can note that

$$\begin{aligned}(p\circ\beta)(s) &= (p\circ\alpha)\left(\frac{1}{2}(s+1)\right)\\ &= (f\circ(\sigma*\tau^{-1}))\left(\frac{1}{2}(s+1)\right)\\ &= (f\circ\tau^{-1})(s).\end{aligned}$$

In other words, β is a lift of $f\circ\tau^{-1}$ with initial point $\beta(0)=(f\sigma)'_{e_0}(1)$ and terminal point e_0.
Thus, β^{-1} is a lift of τ^{-1} with initial point e_0 and terminal point $(f\sigma)'_{e_0}(1)$.
In other words,

$$(f\circ\sigma)'_{e_0}(1)=(f\circ\tau)'_{e_0}(1),$$

as claimed.

It is clear that $pf'=f$. Indeed, if $y\in Y$, then $f\circ\sigma$ path was a path ending at $f(y)$. The projection $p\circ(f\circ\sigma)'_{e_0}$ of any lift must obviously end at the same point again. (Definition of a lift.)

Thus, all that is left to be shown is the continuity of f'. First, we see how we can write f' by removing the dependency of y_0.
For any $y_1\in Y$, let $e_1=f'(y_1)$ and let τ be any path from y_1 to y. Then, we have

$$f'(y)=(f\circ\tau)'_{e_1}(1).$$

(To see this, take any path σ_1 from y_0 to y_1 and then consider $\sigma=\tau*\sigma_1$, a path from y_0 to y and use the definition and the fact that $\cdot$ is a group action.)

Now, we show that f' is continuous at y_1. We will follow the same line of reasoning as in the proof of 4.15.
Let $e_1=f'(y_1)$, $x_1=p(e_1)=f(y_1)$ and V be any neighbourhood of e_1. Let $S\subset V$ be a neighbourhood of e_1 such that S is a sheet over a neighbourhood U of x_1.
Now, note that if σ is any path in U, then the lift of σ with initial point in S must lie completely in S.
Let $W'=f^{-1}(U)\subset Y$ (note that $y_1\in W$) and consider a path-connected neighbourhood $W\subset W'$ of y. (Can do so since Y is locally path set.)
Now, if we show that $f'(W)\subset V$, then we are done.
To see this, let $y\in W$ and $y_1\xrightarrow{\tau}y$ be any path *in* W. Then, the path $f\circ\tau$ lies in the neighbourhood U of x and in turn, its lift $(f\circ\tau)'_{e_1}$ lies in V. In particular, we have

$$f'(y)=(f\circ\tau)'_{e_1}(1)\in V.$$

□

Exercise 5.2. If, in the situation of the above theorem, $f : Y \to X$ is also a covering space, and f' exists, then $f' : Y \to E$ is a covering space.

Solution. Before proving the exercise, we prove a small lemma.

> **Lemma.** Let $E \xrightarrow{p} X$ be a covering space and $U \subset X$ be a path-connected open set which is evenly covered by p. Then, the sheets over U are the path-connected components of $p^{-1}(U)$.
>
> *Proof.* Let $\{S_i\}$ be the set of sheets. Put
>
> $$S := \bigsqcup_i S_i.$$
>
> Clearly, each sheet is path-connected since it is homeomorphic to U which is path-connected.
>
> Now, we show that these are *components*. That is to say, if $e_1 \in S_{i_1}$ and $e_2 \in S_{i_2}$ belong to different sheets, then there's no path *in* S from e_1 to e_2. (Note the emphasis.)
> One can note that the sheets are actually clopen in S. (Since each S_i is open in E and hence, in S. Since the disjoint union makes up S, each sheet is closed as well.)
> Thus, S_{i_1} and S_{i_2} form a separation of $S_{i_1} \cup S_{i_2}$ and hence, the image of any path must lie entirely within one sheet. □

The above was for a general covering space, so of course it works for $Y \xrightarrow{f} X$ as well.
Now, to show that $Y \xrightarrow{f'} E$ is a covering space:
Let $e \in E$. Consider $x = p(e)$. Since X is locally path-connected, we can find a neighbourhood U of x which is path-connected and evenly covered by path f and p.
Let $\{S_i\}$ be the slices of $p^{-1}(U)$ and $\{V_j\}$ of $f^{-1}(U)$.
Let S_{i_0} be the sheet containing e. Since

$$p \circ f' = f,$$

we have

$$f'^{-1}(p^{-1}(U)) = f^{-1}(U).$$

Since $S_{i_0} \subset p^{-1}(U)$, we have

$$f'^{-1}(S_{i_0}) \subset \bigsqcup_j V_j$$

Let V_j be an arbitrary sheet that contains a point of $f'^{-1}(S_{i_0})$.
Since V_j is path-connected (by the lemma), so is $f'(V_j) \subset \bigsqcup S_i$. Since the sheets are the path-connected components, we see that $f'(V_j)$ must be completely contained in a sheet and thus, $f'(V_j) \subset S_{i_0}$.
Thus, we have

$$f|_{V_j} = p \circ (f'|_{V_j}) = \left(p|_{S_{i_0}}\right) \circ \left(f'|_{V_j}\right).$$

Since $p|_{S_{i_0}}$ is a homeomorphism, we get that

$$f'|_{V_j} = \left(p|_{S_{i_0}}\right)^{-1} \circ f|_{V_j}.$$

Thus, not only do we get that f' maps V_j *onto* S_{i_0} but that it does so homeomorphically.
This completes the proof. □

Exercise 5.3. Let $E \xrightarrow{p} X$ be a covering space, where X is path-connected and locally path-connected, but E is not path-connected. Let C be a connected (and hence, path-connected) component of E. Then $p|_C : C \to X$ is a covering space.

Solution. Let $x \in X$. Since $E \xrightarrow{p} X$ is a covering space there exists a path-connected neighbourhood U of x which is evenly covered by p.
Let $\{S_i\}$ be one such collection of sheets. We show that given any sheet, it is either contained completely inside C or is disjoint from C.
Let S_i be an arbitrary sheet such that $S_i \cap C \neq \varnothing$. We show that $S_i \subset C$.
As before, each S_i is path-connected and in particular, connected. Since C is a connected component of E, S_i must lie completely in C. (Otherwise, $S_i \cap C$ and $S_i \cap (E \setminus C)$ would be a separation of S_i.)

Thus, we see that

$$p|_C^{-1}(U) = p^{-1}(U) \cap C = \bigsqcup_{S_i \subset C} S_i,$$

which has the desired properties. □

Corollary 5.4. Consider the situation as in Theorem 5.1.
If Y is simply connected, then the lifting f' always exists.

Proof. Since $\pi_1(Y, y_0)$ is the trivial group, $f_*\pi_1(Y, y_0)$ is clearly contained $p_*\pi_1(E, e_0)$. □

Corollary 5.5. If $(E, e_0) \xrightarrow{p} (X, x_0)$ and $(E', e'_0) \xrightarrow{p'} (X, x_0)$ are both simply-connected covering spaces of X, then there is a unique homeomorphism

$$\varphi : (E', e'_0) \to (E, e_0)$$

such that $p\varphi = p'$.

Proof. By the previous corollary a lift φ' does exist. Then, by the Unique lifting theorem, uniqueness follows. □

Definition 5.6 (Universal coverings)**.** Call two coverings *equivalent* if there is a homeomorphism as in Corollary 5.5. We have shown that if (X, x_0) has a covering space $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ such that $\tilde{X}$ is simply connected, then $(\tilde{X}, \tilde{x}_0)$ is unique up to equivalence.
We call it the *universal covering space* of (X, x_0), since all other coverings lie "below it," in the sense of 5.2 and 5.4.

Definition 5.7 (Semi-locally simply connected)**.** A space X with is called *semi-locally simply connected* if it has the following property:
For any $xinX$, there is a neighborhood U such that any loop in U based at x can be shrunk in X to x. (In the process of shrinking the loop, we may have to go outside of U.)

The universal covering space need not exist, in general, for X is locally homeomorphic to $\tilde{X}$, and so all "small" loops in X can be shrunk to a point. Thus, we get the following proposition.

Proposition 5.8. A necessary condition for X to have a universal covering space is that X be *semi-locally simply connected.*

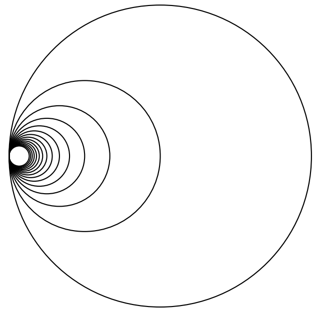
Proof. Suppose X has a universal covering space.
Let $x \in X$ and $(\tilde{X}, \tilde{x})$ be a universal covering space.
Let U be an evenly covered neighbourhood of x and let σ be any loop at x contained in U.
Consider any lift of σ' of σ. Since σ lies completely within U, σ' is again a loop. Since $\tilde{X}$ is simply connected, σ' is homotopic to the constant loop at $\sigma'(0)$. Any homotopy between them projects down to a homotopy between σ and e_x. □

What is surprising, is that the above has a converse as well. (Within the standing assumptions of this section.)

Before that, let us look at an example of a space which is *not* semi-locally simply connected.

Example 3. For a positive integer n, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ centered at $\left(\frac{1}{n}, 0\right)$. Let C be the union of all the circles C_n.

This space C is not semi-locally path connected.



A depiction of $\bigcup_{n=1}^{15} C_n$

To see this, consider the point $(0,0) \in C$. Given any $\delta > 0$, there exists $n \in \mathbb{N}$ such that C_n is completely contained in the δ neighbourhood of $(0,0)$.
Consider the loop σ starting at $(0,0)$ and looping around C_n once. Now, we need to show that σ cannot be contracted to a point, *even if we allow the loop to go outside* U.
To do this, we consider the map $p : (C, (0,0)) \to (S^1, (1,0))$ which maps C_n to S^1 in the natural way and collapses maps everything else to $(1,0)$.
(It can be verified that this is a continuous function.)
Now, we see that $p_* : \pi_1(C, (0,0)) \to \pi_1(S^1, (1,0))$ maps $[\sigma]$ to the generator of $\pi_1(S^1, (1,0))$. In particular, $[\sigma]$ is not trivial (in C, not just in U).

§§5.1. Constructing the universal covering space

Theorem 5.9. If X is a semi-locally simply connected (and path-connected and locally path-connected) space, then X has a universal covering.

In this subsection, we give a proof of this theorem by constructing a universal covering. We shall fix X to be as in the theorem.

Proof.
STEP 1. **Constructing the set** $\tilde{X}$.
Choose $x_0 \in X$. We consider the set S all paths in X with initial point x_0. Define $\sim$ on S by $\alpha \sim \beta$ if $\alpha \simeq \beta$ rel $\{0,1\}$. (In particular, $\alpha(1) = \beta(1)$.)

Let $\langle\alpha\rangle$ be the equivalence class of α. We define

$$\tilde{X} = S/\sim = \{\langle\alpha\rangle \mid \alpha \in S\}.$$

....

STEP 2. **Giving $\tilde{X}$ a topology - defining a basis.**
Let V be a neighbourhood of $\alpha(1)$. We define $\langle a, V\rangle$ to be the set of all $\langle\alpha\beta\rangle$ where β is a path in V with initial point $\alpha(1)$.
Let $\mathscr{B}$ be the set of all $\langle\alpha, V\rangle$s. We show that $\mathscr{B}$ is a base.
Note that $\langle\alpha\rangle \in \langle\alpha, X\rangle$ and thus, every element of $\tilde{X}$ does indeed belong to an element of $\mathscr{B}$.
Now, suppose $\langle\alpha''\rangle \in \langle\alpha, V\rangle \cap \langle\alpha', V'\rangle$. In particular, $\langle\alpha''\rangle \in \langle\alpha, V\rangle$ which gives that $\alpha''(1) \in V$. Thus, V is a neighbourhood of $\alpha''(1)$ as well. Moreover, if

$$\alpha'' \simeq \alpha\beta \text{ rel}\{0,1\}$$

for some path β in V, then

$$\alpha \simeq \alpha''\beta^{-1} \text{ rel}\{0,1\}$$

with β' again being in V.
Thus, we see that $\langle\alpha'', V\rangle = \langle\alpha, V\rangle$.
Similarly, $\langle\alpha'', V'\rangle = \langle\alpha', V'\rangle$. Now,

$$\langle\alpha, V \cap V'\rangle \subset \langle\alpha'', V\rangle \cap \langle\alpha'', V'\rangle$$

or

$$\langle\alpha''\rangle \in \langle\alpha'', V \cap V'\rangle \subset \langle\alpha, V\rangle \cap \langle\alpha', V'\rangle,$$

proving that $\mathscr{B}$ is a basis.
Of course, we now give $\tilde{X}$ the topology generated by $\mathscr{B}$.

....

STEP 3. **Defining the map p.**
Define

$$p : \tilde{X} \to X$$

as

$$p\langle\alpha\rangle = \alpha(1).$$

(Clearly, this is well-defined.)
To show that this is a map, that is, it is continuous, note that given any

$\langle\alpha\rangle$ and any open set V containing $\alpha(1)$, the set $p\langle\alpha, V\rangle$ is the path component of $\alpha(1)$ in V. (Note that the path component of $\alpha(1)$ in V is precisely the set of all those points x such that there's a path $\alpha(1) \xrightarrow{\beta} x$. Then, $x = p\langle\alpha\beta\rangle \in p\langle\alpha, V\rangle$.)
Thus, given any $\langle\alpha\rangle \in \tilde{X}$ and neighbourhood V of $p\langle\alpha\rangle$, we have that the neighbourhood $\langle\alpha, V\rangle$ of $\langle\alpha\rangle$ gets mapped in V. This shows that p is continuous. In fact, this also shows that p is open since basis elements get mapped to open sets. (Path components are connected components (since X is locally path-connected) which are open.)

....

STEP 4. **Showing that $\tilde{X} \xrightarrow{p} X$ is a covering map.**
Given $x \in X$, choose a *path-connected* neighbourhood V of x such that any loop at x in V can be shrunk to x in X. (We use the fact that X is locally path-connected and semi-locally simply connected.)
We show that V is evenly covered.
Let α be any path starting at x_0 such that $p\langle\alpha\rangle \in V$. (Such a path does exist since X is path-connected and so, there exists $x_0 \xrightarrow{\alpha} x \in V$.)

> **Claim 1.** $p\langle\alpha, V\rangle = V$.
>
> *Proof.* It follows by our previous observation that $p\langle\alpha, V\rangle \subset V$. We have equality this time since V is path-connected. □

We show that p maps $\langle\alpha, V\rangle$ homeomorphically onto V.
Well, we have already shown that it is *onto*. We had also shown that this is continuous and open. Thus, all we need to show that this one-one. For then, it would follow that it is a bijection and that the inverse is also continuous. (By virtue of it being open.)
Suppose that $p\langle\alpha\beta\rangle = p\langle\alpha\beta'\rangle$. (Note that $\langle\alpha\beta\rangle$ is a typical element of $\langle\alpha, V\rangle$ where β is a path in V starting at $\alpha(1)$.)
Then, β and β' have the same terminal points (and of course, initial points as well). Note that $\beta\beta'^{-1}$ is a loop at x. By choice of V, we have

$$\beta\beta^{-1} \simeq e_x \ \text{rel}\,\{0,1\}$$

or

$$\alpha\beta \simeq \alpha\beta' \ \text{rel}\,\{0,1\}$$

giving

$$\langle\alpha\beta\rangle = \langle\alpha\beta'\rangle,$$

as desired.

Moreover, note that the complete preimage of V is the disjoint union of all $\langle\alpha, V\rangle$ such that $p\langle\alpha\rangle \in V$. (That this is the complete preimage is obvious.)
To see that the union is disjoint, suppose α and α' are paths such that $\langle\alpha, V\rangle \cap \langle\alpha', V\rangle \neq \varnothing$. Then, for $\langle\alpha''\rangle$ in the intersection, we have

$$\langle\alpha, V\rangle = \langle\alpha'', V\rangle = \langle\alpha', V\rangle,$$

as earlier, showing that the union is disjoint.
Thus, this is the decomposition of $p^{-1}(V)$ into sheets, as desired.

. . . .

STEP 5. $\tilde{X}$ **is path-connected.**
Let $\tilde{x}_0 = \langle e_{x_0}\rangle$, the class of the constant loop at x_0. We show that we can join any point $\langle\alpha\rangle \in \tilde{X}$ to $\tilde{x}_0$ which would show that $\tilde{X}$ is path-connected.
Given a path $x_0 \xrightarrow{\alpha} x$ in X, define

$$\alpha_s(t) = \alpha(st), \quad s, t \in I.$$

Thus, for each $s \in I$, α_s is a path in X such that $\alpha_s(0) = \alpha(0) = x_0$. That is, each α_s is a path starting at x_0.
Now, define $\tilde{\alpha} : I \to \tilde{X}$ as

$$\tilde{\alpha}(s) := \langle\alpha_s\rangle.$$

Note that α_0 is the constant loop at x_0 and $\alpha_1 = \alpha$. Thus, we have that

$$\tilde{x}_0 \xrightarrow{\tilde{\alpha}} \langle\alpha\rangle$$

is a path in $\tilde{X}$, provided we show that $\tilde{\alpha}$ is continuous.
To see this, let $s_0 \in I$ be arbitrary and consider a basis neighbourhood $\langle\alpha_{s_0}, V\rangle$ of $\tilde{\alpha}(s) = \alpha_s$.
Note that $\alpha_{s_0}(1) \in V$, that is, $\alpha(s_0) \in V$. Since α is continuous, there exists a δ-neighbourhood U around s_0 such that $\alpha(U) \subset V$.
We show that $\tilde{\alpha}(U) \subset \langle\alpha_{s_0}, V\rangle$.
To see this, let $s \in U$. Then,

$$p\langle\alpha_s\rangle = \alpha_s(1) = \alpha(s) \in V.$$

Let $s_M = \max\{s, s_0\}$ and $s_m = \min\{s, s_0\}$.
Then, note that the α_{s_M} is a path which can be seen as a product of the path α_{s_m} with a path joining the point $\alpha(s_m)$ to $\alpha(s_M)$, the latter lying completely in V since $\alpha(U) \subset V$.
Thus, we see that $\langle\alpha_{s_M}\rangle \in \langle\alpha_{s_m}, V\rangle$ and vice-versa. Since $\{s_m, s_M\} = \{s, s_0\}$, we see that

$$\tilde{\alpha}(s) = \langle\alpha_s\rangle \in \langle\alpha_{s_0}, V\rangle,$$

as desired. This shows that $\tilde{\alpha}$ is continuous and thus, $\tilde{X}$ is path connected. Moreover, we see that

$$(p \circ \tilde{\alpha})(s) = p\langle \alpha_s \rangle = \alpha_s(1) = \alpha(s),$$

that is, $\tilde{\alpha}$ lifts α.

....

STEP 6. X **is simply connected.**
We show that $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial.
Let τ be a loop in $\tilde{X}$ at $\tilde{x}_0$, and let $\alpha = p \circ \tau$. By uniqueness of lifts, we have

$$\tau = \tilde{\alpha},$$

where $\tilde{\alpha}$ is defined as earlier. (Uniqueness since both τ and $\tilde{\alpha}$ have initial point $\tilde{x}_0$.)
In particular, $\tilde{\alpha}$ is a loop at $\tilde{x}_0$ (since so was τ).
Thus, we have

$$\langle \alpha \rangle = \langle \alpha_1 \rangle = \tilde{\alpha}(1) = \tilde{x}_0 = \langle e_{x_0} \rangle.$$

(The last equality was the definition of $\tilde{x}_0$.)
Thus, we have

$$\langle \alpha \rangle = \langle e_{x_0} \rangle$$

or that

$$\alpha \simeq e_{x_0} \ \text{rel}\, \{0,1\}.$$

By the Path homotopy lifting theorem, we see that

$$\tilde{\alpha} \simeq \widetilde{e_{x_0}} \ \text{rel}\, \{0,1\}.$$

Since we have $\tilde{\alpha} = \tau$ and $\widetilde{e_{x_0}} = e_{\tilde{x}_0}$, we are done! □

The above then finishes our construction as we have shown that

$$(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$$

is a covering space where X is simply connected.

We will now prove a result about topological groups, before which we prove a lemma.

Lemma 5.10. Let X be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(X, x_0)$ denote the set of all loops at x_0 in X. If $f, g \in \Omega(X, x_0)$, we define a loop $f \otimes g$ at x_0 by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

1. This operation makes $\Omega(X, x_0)$ into a group.
2. This operation induces a group operation $\otimes$ on $\pi_1(X, x_0)$.
3. The two group operations $*$ and $\otimes$ on $\pi_1(X, x_0)$ are the same. (Recall that $*$ was the usual product of paths, in this case, loops.)
4. $\pi_1(X, x_0)$ is abelian.

Proof.

1. This is a simple check. $\otimes$ is associative since $\cdot$ is. Moreover, e_{x_0}, the constant loop at x_0 acts as the identity as can be easily checked.
   Lastly, given $f \in \Omega(X, x_0)$, we see that $g : I \to X$ defined as
   $$g(s) = (f(s))^{-1}$$
   is an element of $\Omega(X, x_0)$ and is the (two-sided) inverse of f with respect to $\otimes$.
   Thus, $\Omega(X, x_0)$ is a group under $\otimes$.
2. In other words, we need to show that if $f \simeq f'$ and $g \simeq g'$, both $\operatorname{rel}\{0,1\}$, then
   $$f \otimes g \simeq f' \otimes g' \operatorname{rel}\{0,1\}.$$
   To see this, let $H : f \simeq f' \operatorname{rel}\{0,1\}$ and $H' : g \simeq g' \operatorname{rel}\{0,1\}$ be path homotopies. We define a new path homotopy
   $$H \otimes H' : I \times I \to X$$
   given as
   $$(H \otimes H')(s,t) = H(s,t) \cdot H'(s,t).$$
   One can note that $(H \otimes H')(0,t) = x_0 \cdot x_0$ and similarly for $(1,t)$.
   Likewise, we have
   $$(H \otimes H')(s,0) = H(s,0) \cdot H'(s,0) = f(s) \cdot g(s) = (f \otimes g)(s)$$
   and similarly for $(s,1)$.
   This shows that $\otimes$ induces a group operation on $\pi_1(X, x_0)$.
3. To do this and the next part, we just show that
   $$([f] \otimes [g]) * ([\sigma] \otimes [\tau]) = ([f] * [\sigma]) \otimes ([g] * [\tau])$$

for all $f, g, \sigma, \tau \in \Omega(X, x_0)$. The result will then follow from The Eckmann-Hilton Argument.
Since both $*$ and $\star$ are compatible with $[\cdot]$, the above is equivalent to

$$[\underbrace{(f \otimes g) * (\sigma \otimes \tau)}_{=:\alpha}] = [\underbrace{(f * \sigma) \otimes (g * \tau)}_{=:\beta}].$$

Thus, if we show that $\alpha \simeq \beta \text{ rel}\{0,1\}$, then we are done. In fact, we will show that $\alpha = \beta$.
Indeed, we have

$$\begin{aligned}
\alpha(s) &= \big((f \otimes g) * (\sigma \otimes \tau)\big)(s) \\
&= \begin{cases} (f \otimes g)(2s) & 0 \le 2s \le 1, \\ (\sigma \otimes \tau)(2s-1) & 1 \le 2s \le 2 \end{cases} \\
&= \begin{cases} f(2s) \cdot g(2s) & 0 \le 2s \le 1, \\ \sigma(2s-1) \cdot \tau(2s-1) & 1 \le 2s \le 2. \end{cases}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\beta(s) &= \big((f * \sigma) \otimes (g * \tau)\big)(s) \\
&= \big((f * \sigma)(s)\big) \cdot \big((g * \tau)(s)\big) \\
&= \begin{cases} f(2s) \cdot g(2s) & 0 \le 2s \le 1, \\ \sigma(2s-1) \cdot \tau(2s-1) & 1 \le 2s \le 2. \end{cases}
\end{aligned}$$

Thus, we see that $\alpha = \beta$, as desired. □

Theorem 5.11. If X is a topological group with operation $\cdot$, then for any covering space $E \xrightarrow{p} X$ and point e_0 in the fiber of the neutral element x_0 of X, there is a unique structure of topological group on E for which e_0 is the neutral element and p is a homomorphism.

Proof. Let $m : X \times X \to X$ be the map $(x_1, x_2) \mapsto x_1 \cdot x_2$. We wish to lift the red map $m \circ (p \times p)$ to a map m' as shown.

$$\begin{tikzcd}
(E \times E, (e_0, e_0)) \arrow[r, dashed, "m'"] \arrow[d, "p \times p"'] & (E, e_0) \arrow[d, "p"] \\
(X \times X, (x_0, x_0)) \arrow[r, "m"'] & (X, x_0)
\end{tikzcd}$$

Once we do that, we would define $\cdot$ on E as $e_1 \cdot e_2^{-1} = m'(e_1, e_2)$.
Note that any other structure would also make the above diagram commute (since p is a homomorphism) and thus, m' (if it exists) is unique. (By Unique lifting theorem.)
The criterion for its existence is

$$m_*(p \times p)_* \pi_1(E \times E, (e_0, e_0)) \subset p_* \pi_1(E, e_0),$$

as given by Theorem 5.1.
Let us first examine m_*. Given $[\alpha] \in \pi_1(X \times X, (x_0, x_0))$, we have

$$m_*([\alpha]) = [m \circ \alpha].$$

Note that any loop α at (x_0, x_0) in $X \times X$ looks like

$$\alpha = \alpha_1 \times \alpha_2$$

for some loops α_1 and α_2 at x_0 in X. Thus, we have

$$(m \circ \alpha)(t) = \alpha_1(t) \cdot (\alpha_2(t)), \quad t \in I.$$

By the previous lemma, we conclude that

$$m \circ \alpha = \alpha_1 * \alpha_2,$$

where the $*$ is the usual path (in this case, loop) product.
Similarly, if σ is a loop at (e_0, e_0) in $E \times E$, then $\sigma = (\sigma_1, \sigma_2)$ for some loops σ_1, σ_2 at e_0 in E. Moreover, we have

$$(p \times p) \circ \sigma = (p \circ \sigma_1) \times (p \circ \sigma_2).$$

Thus,

$$\begin{aligned} m \circ (p \times p) \circ \sigma &= (p \circ \sigma_1) * (p \circ \sigma_2) \\ &= p \circ (\sigma_1 * \sigma_2) \end{aligned}$$

or

$$m_*(p \times p)_*[\sigma] = p_*[\sigma_1 * \sigma_2],$$

showing that

$$m_*(p \times p)_* \pi_1(E \times E, (e_0, e_0)) \subset p_*(E_1, e_0),$$

fulfilling the lifting criterion.

Thus, the lift m' exists. We now show that it gives E the structure of a topological group such that e_0 is the neutral element and p is homomorphism.
Define $\cdot : E \times E \to E$ as $e_1 \cdot e_2 = m'(e_1, e_2)$.
We verify associativity using the following diagram:

$$\begin{tikzcd}
E \times E \times E \arrow[r, "m' \times \mathrm{id}_E"] \arrow[d, "\mathrm{id}_E \times m'"'] & E \times E \arrow[d, "m'"] & \\
E \times E \arrow[r, "m' \times \mathrm{id}_E"'] & E \arrow[dr, "p"] & \\
& & X
\end{tikzcd}$$

(Note that these are actually maps preserving the obvious base-points.)
Associativity amounts to showing that the square above commutes. However, using associativity in X, we know that the blue maps composed with p equals the red maps composed with p. Then, by the Unique lifting theorem, we see that these two must be equal.

We now show that e_0 is the identity. Consider the map $i : E \to E$ defined as

$$i(e) = e \cdot e_0 = m'(e, e_0).$$

It follows that $i(e_0) = e_0$. We wish to show that $i = \mathrm{id}_E$.
Note that

$$(p \circ i)(e) = p(m'(e, e_0)) = m(p(e), p(e_0)) = m(p(e), x_0) = p(e).$$

In other words, i is a lift of p which agrees with id_E at e_0. By uniqueness of lifts, we see that $i = \mathrm{id}_E$ as desired. Similarly, we also get that e_0 is the left identity.

To construct inverses, we shall lift the inversion map $i_X : X \to X$ similar to how we had lifted m. By a similar Eckmann-Hilton type argument, we see that the lifting criterion is satisfied and thus, there exists a map $i_E : E \to E$ such that $p \circ i_E = i_X \circ p$ and $i_E(e_0) = e_0$.
Now, define $j : E \to E$ as

$$j(e) = e \cdot i_E(e) = m'(e, i_E(e)).$$

Note that $j(e_0) = e_0$. We wish to show that j is the constant map $e \mapsto e_0$. The usual trick works. Indeed, we note

$$p(j(e)) = pm'(e, i_E(e)) = p(e) \cdot p(i_E(e)) = p(e) \cdot i_X(p(e)) = x_0.$$

Thus, j is a lift of the constant map $e \mapsto x_0$ and so is the constant map e_{e_0}. Since j and e_{e_0} at e_0, they must be equal. (Similarly, this acts as a left inverse as well.)

m' and i_E are lifts by construction and thus, are continuous. Thus, E is a topological group with e_0 as identity. The fact that p is a homomorphism also follows from construction since we have $p \circ m' = m \circ (p \times p)$. $\square$

§6. Van Kampen's Theorem

§§6.1. Free Product of Groups

We briefly describe the free product of groups and fix some notation. We shall not prove the basic facts about free groups.
Let $\{G_\alpha\}_{\alpha\in A}$ be a collection of (disjoint) groups. We denote the free product of the groups as

$$*_\alpha G_\alpha.$$

(This is slight abuse of notation since we don't mention A but there won't be any confusion.)

As a set, the free product $*_\alpha G_\alpha$ consists of words of the form

$$g_1 \cdots g_m$$

of arbitrary finite length $m \geq 0$ satisfying the following conditions:

1. each letter g_i belongs to a group G_{α_i},
2. g_i is not the identity element of G_{α_i},
3. adjacent letters belong to different groups, i.e., $\alpha_i \neq \alpha_{i+1}$.

Words satisfying these conditions are called *reduced.* The idea being that an arbitrary word using with letters from G_α can be reduced to this type of word by combining letters and discarding the trivial (identity) letters.
The group operation of this group is juxtaposition, followed by reduction. The identity of this group is the empty word. (That is, the unique word of length $m = 0$.)

Given the free product $*_\alpha G_\alpha$, each group G_α is naturally identified with the subgroup $*_\alpha G_\alpha$ that contains the empty word and non-identity one-letter words $g \in G$.
Under this identification, we now state the universal property of a free product.

Theorem 6.1 (Universal property)**.** Given any collection $\{G_\alpha\}$ of groups and collection of (group) homomorphisms

$$\varphi_\alpha : G_\alpha \to H,$$

there exists a unique homomorphism

$$\varphi : *_\alpha G_\alpha \to H,$$

such that

$$\varphi|_{G_\alpha} = \varphi_\alpha,$$

for each α.

In others words, the homomorphisms $\varphi_\alpha : G_\alpha \to H$ *extend* uniquely to a homomorphism $\varphi : *_\alpha G_\alpha \to H$.
This homomorphism is defined (on *reduced* words) in the obvious manner by defining

$$\varphi(g_1 \cdots g_m) := \varphi_{\alpha_1}(g_1) \cdot \cdots \cdot \varphi_{\alpha_m}(g_m).$$

§§6.2. The Van Kampen Theorem

Let X be a topological space such that $X = \bigcup A_\alpha$, where A_α are path-connected open subsets of X, each of which contain a base-point $x_0 \in X$. (We shall fix this base-point and not mention it when writing the fundamental groups.)
By the universal property in the previous subsection, we see that the homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $(A_\alpha) \hookrightarrow (X)$ extend to a homomorphism

$$\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X).$$

Note, that the groups are not necessarily disjoint but we formally treat them to be disjoint in the free product.
To elaborate, if

$$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$$

is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$, then

$$j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha},$$

for both are the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$.
Thus, we would need Φ to agree on $i_{\alpha\beta}(w) \in \pi_1(A_\alpha)$ and $i_{\beta\alpha}(w) \in \pi_1(A_\beta)$ for $w \in \pi_1(A_\alpha \cap A_\beta)$. This tells us that the kernel of Φ should contain $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$.
Van Kampen's theorem asserts that under reasonable hypothesis, Φ is surjective and the above gives a complete description of the kernel.

Theorem 6.2 (The Van Kampen Theorem)**.** If X is the union of path-connected open sets A_α each containing the base-point $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism

$$\Phi : *_\alpha \pi_1(A) \to \pi_1(X)$$

is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$, and so Φ induces a homomorphism

$$\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N.$$

Note that Theorem 3.13 from earlier was a special case of the above general theorem. In fact, surjectivity of Φ was almost given in the proof there. We shall write some of it again in order to fix the notation consistently for this scenario.

Proof. Given a loop $f : I \to X$ at the base-point x_0, we claim (as usual) that there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of I such that each subinterval $[s_{i-1}, s_i]$ is mapped by f into a single A_α. This, of course, follows from compactness of I.
Denote that A_α containing $f([s_{i-1}, s_i])$ by A_i, and let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$ and reparameterising appropriately. As in the proof of Theorem 3.13, we see that we can write

$$[f] = [f_1 g_1^{-1}][g_1 f_2 g_2^{-1}] \cdots [g_{m-1} f_m],$$

where each g_i is a path in $A_i \cap A_{i+1}$ from x_0 to the point $f(s_i) \in A_i \cap A_{i+1}$.
The above then shows that f is homotopic to a product of *loops*, each of which lie in a single A_i. Hence, $[f]$ is in the image of Φ and Φ is surjective. (Recall the extension of homomorphisms as given by the universal property.)

. . . .

Now, we prove the harder part, namely that N is the kernel of the homomorphism Φ.
Firstly, we note that N must be contained in $\ker \Phi$. Given any α, β and $w \in \pi_1(A_\alpha) \cap \pi_1(B_\alpha)$, we recall we had seen that

$$(j_\alpha \circ i_{\alpha\beta})(w) = (j_\beta \circ i_{\beta\alpha})(w).$$

Thus, we must have $\Phi(i_{\alpha\beta}(w)) = \Phi(i_{\beta\alpha}(w))$. (Since Φ was extending the j_αs.)
Thus, $\ker \Phi$ is a normal subgroup that contains elements of the form

$$i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}.$$

Since N was the smallest such, we see that $N \le \ker \Phi$.

The above then gives us that Φ induces a map

$$Q = *_\alpha G_\alpha / N \to \pi_1(X).$$

To show that N is exactly the kernel, we will show that the above map is injective. We now introduce some terminology.
By a *factorisation* of an element $[f] \in \pi_1(X)$, we shall mean a formal product $[f_1] \cdots [f_k]$ where

1. Each f_i is a loop in some A_α with base-point x_0, and $[f_i] \in \pi_1(A_\alpha)$ is the homotopy class of f_i.
2. The loop f is homotopic to $f_1 * \cdots * f_k$ in X.

In other words, $[f_1]\cdots[f_k]$ is a word in $*_\alpha \pi_1(A_\alpha)$, *not necessarily reduced.*
(Note that for each factorisation, we are keeping track of which group $[f_i]$ lies in. In particular, even if f_i lies in some intersection $A_\alpha \cap A_\beta$, we get two different factorisations by considering it as $[f_i]$ lying in two different groups.)
Moreover, it is a word which gets mapped to $[f]$ under Φ. The proof of surjectivity earlier shows that each $[f]$ does have a factorisation.
Now, we introduce an *equivalence* of factorisations. Call two factorisations of $[f]$ *equivalent* if they are related by a finite sequence of following two sorts of moves or their inverses:

1. Combine adjacent terms $[f_i][f_{i+1}]$ into a single term $[f_i * f_{i+1}]$ if $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(A_\alpha)$.
2. Regard the term $[f_i] \in \pi_1(A_\alpha)$ as lying in the group $\pi_1(A_\beta)$ rather than $\pi_1(A_\alpha)$ if f_i is a loop in $A_\alpha \cap A_\beta$.

The first move (and its inverse) clearly does not change the element viewed as an element of $*_\alpha \pi_1(A_\alpha)$ since that is just applying the group operation on it.
The second move does change it as an element of $*_\alpha \pi_1(A_\alpha)$. However, it does *not* change it as an element of the *quotient group* $Q = *_\alpha \pi_1(A_\alpha)/N$.
In other words, equivalent factorisations give same elements of Q.

Note that we had shown that Φ induces a homomorphism $Q \to \pi_1(X)$. If we show that any two factorisations of $[f]$ are equivalent, we shall get that

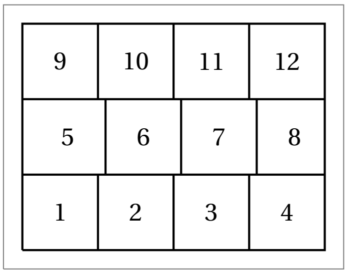
$$\Phi(a) = \Phi(b) \implies a \text{ and } b \text{ are equivalent.}$$

In other words, a and b are equal modulo N. In yet other words, the map $Q \to \pi_1(X)$ is injective and thus, N is precisely the kernel of Φ.

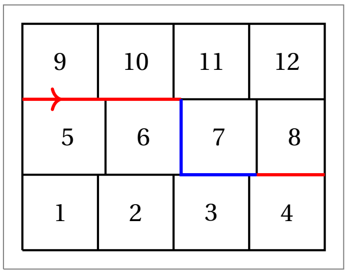
Let $[f_1]\cdots[f_k]$ and $[f_1']\cdots[f_l']$ be two factorisations of $[f]$. The composed paths $f_1 * \cdots * f_k$ and $f_1' * \cdots * f_l'$ must then be homotopic to f and hence, to each other.
Let $F : I \times I \to X$ be a path homotopy from the former to the latter. As usual, there exist partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into a single A_α, which we denote by A_{ij}.
Moreover, we may assume that the s-partition further subdivides the parti-

tions that give the products $f_1 * \cdots * f_k$ and $f_1' * \cdots * f_l'$. That is, each $f([s_{i-1}, s_i])$ lies completely in some A_α in which one of the f_j or f_j' lie.

Since each R_{ij} is compact and gets mapped into A_{ij}, there is a neighbourhood of R_{ij} which gets mapped into A_{ij}. Thus, we may slightly perturb the vertical edges of the rectangles R_{ij} so that each point of $I \times I$ lies in at most three R_{ij}s. We may also assume that $n \geq 3$, so that there are at least three rows of rectangles. In this case, we may perturb each row in a way that the topmost and bottommost rows remain unperturbed. We relabel these new rectangles as $R_1, R_2, \ldots, R_{mn}$ and ordering them as in the figure. (For which we have $m = 4, n = 3$.)

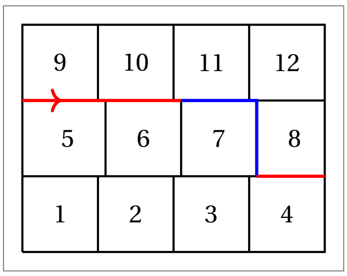


If γ is a path in $I \times I$ from the left edge to the right edge, then $F \circ \gamma$ is a loop in X with basepoint x_0. (Recall that F was a path homotopy and thus, maps both the left and right edges to x_0.)
Let γ_r be the path in $I \times I$ (from the left edge to the right) separating the first r rectangles from the rest. (γ_6 has been depicted in the figure.)



In particular, γ_0 is the bottom edge of $I \times I$ and γ_{mn} the top edge.
Also, note that we move from γ_r to γ_{r+1} by "pushing" across the rectangle R_{r+1}. (See the next figure.)

We shall call the corners of R_rs *vertices*. For each vertex v, let g_v be a path from x_0 to $F(v)$. By our construction $F(v)$ and x_0 both lie in the intersection of the two or three A_{ij}s corresponding to the R_rs containing v. Thus, we can choose g_v to also lie in this intersection since the theorem's hypothesis said that two or three of these intersections are path-connected. (This is why we had done the perturbation.)



Thus, given the *loop* $F \circ \gamma_r : I \to X$, we may insert the paths $g_v^{-1} g_v$ at successive vertices, as in the proof of surjectivity. This gives us a factorisation of $[F \circ \gamma_r]$, where each of the factor corresponds to a vertical or horizontal segment along with the g_vs padded on either side to make it a loop.
This factor can be regarded as an element of either of the $\pi_1(A_{ij})$ containing

it. Different choices will still give equivalent factorisations.

The more important observation now is that factorisations associated to γ_r and γ_{r+1} are also equivalent. When changing the factors from $F \circ \gamma_r$ to $F \circ \gamma_{r+1}$, the paths changed are homotopic via a homotopy in R_{r+1}. Thus, this is change will be equivalent as this is just a change obtained by the group operation in $\pi_1(A_{ij})$, where A_{ij} is the set corresponding to R_i.
(To see this clearly, note the two blue paths (viewed as paths in X) in the figures are homotopic (including the g_v paddings), using Lemma 2.5.)

Thus, we get that all the factorisations associated to each γ_r are equivalent. We now show that can we can associate a factorisation to γ_0 which is equivalent to $[f_1]\cdots[f_k]$.
First, we identity the bottom edge $I \times \{0\}$ with I in the natural way. Consider the path $f_1 * \cdots * f_k$ with domain I.
Note that each vertex in the bottom edge lies in at most two rectangles. Thus, for each such v, we can choose a path g_v as earlier to not only lie in the corresponding A_{ij}s of the rectangles but also in the A_α corresponding to the f_i containing the v in its domain. Thus, as earlier, we split $[f_1]\cdots[f_k]$ as $[f_1 g_1^{-1}]\cdots[g_{k-1} f_k]$.
This is a factorisation associated to γ_0 which is equivalent to $[f_1]\cdots[f_k]$.
Similarly, we get a factorisation associated to γ_{mn} which is equivalent to $[f'_1]\cdots[f'_l]$. By our earlier observations, we are done. $\square$

§7. Loop Spaces and Higher Homotopy Groups

Let X be an arbitrary topological space and X^I be the set of all maps from I to X. In other words, X^I is the set of all paths in X. (Recall that by "maps" and "paths," we always mean *continuous* functions.)
We wish to turn X^I into a topological space. We wish to do this with no assumptions on X. For this reason, we first define the sets

$$[K, U] = \{\sigma \in X^I : \sigma(K) \subset U\},$$

where $K \subset I$ is compact and $U \subset X$ is open.
Note that the collection of all such $[K, U]$ form a subbasis for a topology on X^I. (Recall that this just means that the union of all such sets equals X^I.) To see this, take $K = I$ and $U = X$ itself.

Definition 7.1 (Compact open topology)**.** The topology on X^I generated by the subbasis

$$\{[K, U] \subset X^I : K \subset I \text{ compact, } U \subset X \text{ open}\}$$

is called the *compact-open topology*.

Recall that the above means that the sets which are open in X^I are precisely those which can be written as a union of finite intersections of elements of the form $[K, U]$.

The main property that we shall by using of this topology is the following:

Proposition 7.2 (Evaluation is continuous)**.** The evaluation map $\omega : X^I \times I \to X$ given by

$$\omega(\sigma, t) = \sigma(t)$$

is continuous.

Proof. Let $(\sigma_0, t_0) \in X^I \times I$ be arbitrary and $V \subset X$ be a neighbourhood of $\omega(\sigma_0, t_0) = \sigma_0(t_0)$.
We wish to find a neighbourhood of (σ_0, t_0) that is mapped into V via ω.
Since σ is continuous and V is open, there exists an open interval U of t_0 such that $\sigma(U) \subset V$. We may find a smaller bounded open interval such that $\overline{W} \subset U$. Note that $\overline{W}$ is compact.
Note that $[\overline{W}, V]$ is one of the subbasis elements of X^I and hence, is open. Thus, $T = [\overline{W}, V] \times W$ is a neighbourhood of (σ_0, t_0).
Now, if $(\sigma', t') \in T$, then $\sigma'(t') \in V$ since $\sigma' \in [\overline{W}, V]$ maps W into V. In other words, $\omega(\sigma', t') \in V$, as desired. ($T$ is the desired neighbourhood that gets mapped in V by ω.) $\square$

Proposition 7.3 (A bijective correspondence)**.** Let X and Y be topological spaces. There is a bijective correspondence between maps

$$f : Y \to X^I \quad \text{and} \quad F : Y \times I \to X.$$

The bijection is the one induced by relating f and F as

$$f(y)(t) = F(y,t).$$

Proof. It is clear that the above relation gives a bijection between *functions* $Y \xrightarrow{f} X^I$ and $Y \times I \xrightarrow{F} X$. Now, we show that it restricts to continuous functions.

Firstly, assume that f is continuous. Note that F can be factorised as

$$I \times I \xrightarrow{f \times \mathrm{id}_I} \Omega_{x_0} \times I \xrightarrow{\omega} X.$$

By the earlier proposition, we see that F is continuous.

Conversely, assume that F is continuous. To show that f is continuous, it suffices to show that $f^{-1}([K,U])$ is open for any arbitrary subbasis element $[K,U] \subset X^I$.
Choose $y \in f^{-1}([K,U])$.
This means that given any $k \in K \subset I$, we have $f(y)(k) \in U$ or $F(y,k) \in U$. Thus, we see that $F(\{y\} \times K) \subset U$.
Since F is continuous, there exist open sets $V \subset Y$ and $W \subset X$ such that $y \in V$ and $K \subset W$ with the property that $F(V \times W) \subset U$.
Now, note that if $y' \in V$ and $k \in K \subset W$, then $f(y')(k) = F(y,k) \in U$. In other words

$$f(y') \in [K,U].$$

This shows that f maps V into $[K,U]$ proving that $f^{-1}([K,U])$ is open. $\square$

We will now turn our attention to a specific subspace of X^I.

Definition 7.4. Let X be a topological space and $x_0 \in X$. The subspace

$$\Omega_{X,x_0} = \Omega_{x_0}$$

is the subspace consisting of all *loops* at x_0 in X.

(The X in the subscript is omitted since the ambient space will be usually be clear from context.)

Proposition 7.5 (Characterising the path connected components)**.** $\sigma, \tau \in \Omega_{x_0}$ are in the same path-connected components iff $\sigma \simeq \tau \ \mathrm{rel}\,\{0,1\}$.

Proof. Given a path $f : I \to X^I$, we get a function $F : I \times I \to X$ and vice-versa by the relation

$$f(s)(t) = F(s,t).$$

As we saw in Proposition 7.3, f is continuous iff F is.

Now, if f is a path from σ to τ, then F can be verified to be a homotopy. Indeed, one notes that

$$F(0,t) = f(0)(t) = \sigma(t), \qquad \text{for all } t \in I.$$

Same follows for $F(1,t)$.
Moreover, since $f(s) \in \Omega_{x_0}$ for each $s \in I$, we see that

$$F(s,0) = f(s)(0) = x_0 = f(s)(1) = F(s,1), \qquad \text{for all } s \in I.$$

Thus, $F : \sigma \simeq \tau \ \mathrm{rel}\,\{0,1\}$.

Conversely, if F is a path homotopy, then f defined above is seen to clearly satisfy $f(0) = \sigma$ and $f(1) = \tau$. Moreover, one notes that $f(s)(0) = x_0 = f(s)(1)$ for all $s \in I$. This shows that f indeed maps into Ω_{x_0}, completing the proof. $\square$