

$$\int (\overset{\circ}{\cup} \overset{\circ}{\cup}) dx$$

MA 406

## General Topology

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# Lecture 1 (07-01-2021)

07 January 2021 15:04

Def. A topology on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  having the following properties: (Topology)

(1)  $\emptyset$  and  $X$  are in  $\mathcal{T}$ .

(2) The union of the elements of any subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .

(3) The intersection of the elements of any finite subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .

(Open set)

Any  $U \in \mathcal{T}$  is called an open set of  $X$  w.r.t.  $\mathcal{T}$ .

The pair  $(X, \mathcal{T})$  or just the set  $X$  is called a topological space. (abuse of notation)

Can reconcile the above with open sets in  $\mathbb{R}$ , or in general, any metric space  $X$ . That can be seen as a motivation for the definition.

## Examples

(1)  $X = \{a, b, c\}$

$\mathcal{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \rightarrow$  Can be seen (fairly easily)

$\mathcal{T}_2 = \{\emptyset, X\}$  that this is a topology

trivial (pun intended, cf. next example)

(2) If  $X$  is any set, the collection of all subsets of  $X$  is a topology on  $X$ , it is called the discrete topology.  
 $(\mathcal{T} = P(X), \text{ that is})$  (Discrete topology)

The collection  $\{\emptyset, X\}$  is also a topology on  $X$  called the indiscrete topology or trivial topology. (Indiscrete topology)  
(Trivial topology)

(3) Let  $X$  be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\emptyset\}.$$

(Finite complement topology)

Then,  $\mathcal{T}_f$  is a topology on  $X$ , called the finite complement topology on  $X$ .

- $\emptyset \in \mathcal{T}_f$  is clear.  $X \in \mathcal{T}_f$  since  $|X \setminus X| = 0 < \infty$ .
- Let  $\{U_\alpha\}_{\alpha \in \Lambda}$  be sets in  $\mathcal{T}_f$ . WLOG,  $U_\alpha \neq \emptyset \ \forall \alpha$ .

$$\begin{aligned} \text{Note } X \setminus \left( \bigcup_{\alpha} U_\alpha \right) &= X \cap \left( \bigcup_{\alpha} U_\alpha^c \right)^c \\ &= \bigcap_{\alpha} (U_\alpha^c)^c \end{aligned}$$

Note that each  $U_\alpha^c$  is finite. ( $U_\alpha \neq \emptyset$ )

Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to  $\bigcup_{i=1}^n (U_i^c)$

and conclude as earlier.

(Here, if some  $U_i$  were  $\emptyset$ , then so would be the intersection.)

(If  $X$  is finite, the  $\mathcal{T}_f = P(X)$ . Thus, we get discrete.)

- (4) Let  $X$  be a set.

Let  $\mathcal{T}_c$  be the collection of subsets such that  $X \setminus U$  is either countable or all of  $X$ .

(Generalising the previous.)

(Cocountable topology)  
Co-countable topology)

Defn

Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T}' \supset \mathcal{T}$ , we say that  $\mathcal{T}'$  is finer than  $\mathcal{T}$  and that  $\mathcal{T}$  is coarser than  $\mathcal{T}'$ .

If  $\mathcal{T}' \supsetneq \mathcal{T}$ , then the above is strictly finer and strictly

coarser, respectively.

(Finer, coarser, strictly finer, strictly coarser)

(The above gives us a way to compare two topologies)

EXAMPLE We have the usual topology on  $\mathbb{R}$ . ← strictly weaker  
We also have the discrete topology on  $\mathbb{R}$ . ← than this

Def. If  $X$  is a set, a basis for a topology on  $X$  is a collection  $B$  of subsets of  $X$  (called basis elements) such that (Basis)

- (1) for each  $x \in X$ ,  $\exists B \in \mathcal{B}$  s.t.  $x \in B$ .

(2) if  $x \in B_1 \cap B_2$  for some  $B_1, B_2 \in \mathcal{B}$ , then  
 $\exists B_3 \in \mathcal{B}$  s.t.  $x \in B_3 \subset B_1 \cap B_2$ .

Note that in the above,  $\mathcal{B}$  is just some collection of subsets of  $X$  satisfying (1) & (2). No topology is mentioned so far.

## EXAMPLES

- (1)  $X = \mathbb{R}^2$ ,  $\mathcal{B}$  is the collection of all discs w/o boundary.
  - (2)  $\mathbb{R}^n$  - rectangles -  $\mathbb{R}^n$
  - (3) Any  $X$ . The singletons form a basis.

We now get a topology out of a basis:

Defn: If  $B$  is a basis for a topology on  $X$ , the topology  $\mathcal{T}$  generated by  $B$  is described as follows:

A subset  $U$  of  $X$  is said to be open if for every  $x \in U$ , there exists  $B \in \mathcal{B}$  s.t.  $x \in B \subset U$ .

$$x \in B \subset U.$$

(By "open" in above, we mean element of  $\mathcal{T}$ . Same thing for what we see in the proof below.)

EXAMPLES (1) & (2)  $\rightarrow$  gives standard topology on  $\mathbb{R}^2$   
 (3)  $\rightarrow$  gives discrete topology on  $X$ .

We still have to show that it is topology.

Proof:

- $\emptyset \in \mathcal{T}$  vacuously  
 $X \in \mathcal{T}$  since given any  $x \in X$ ,  $\exists B \in \mathcal{B}$  s.t.  $x \in B$ .  
 $B \subset X$  is by definition
- Let  $\{U_\alpha\}_{\alpha \in I}$  be open. Let  $U := \bigcup_{\alpha} U_\alpha$ .  
 Fix  $\alpha_0 \in I$ .  
 Let  $x \in U$  be arbitrary. Then,  $x \in U_{\alpha_0} \leftarrow$  open  
 $\therefore \exists B \in \mathcal{B}$  s.t.  $x \in B \subset U_{\alpha_0} \subset U$ .  
 $\therefore U \in \mathcal{T}$ .
- Let  $U_1$  and  $U_2$  be open. Put  $U := U_1 \cap U_2$ .  
 Let  $x \in U$ .  
 Then  $x \in U_1$  and  $x \in U_2$   
 $\downarrow$   
 $\exists B_1 \in \mathcal{B}$  s.t.  $x \in B_1 \subset U_1$        $\uparrow$   
 $\exists B_2 \in \mathcal{B}$  s.t.  $x \in B_2 \subset U_2$   
 $\therefore x \in B_1 \cap B_2 \subset U_1 \cap U_2$   
 $\therefore x \in B_3 \in \mathcal{B}$  s.t.  $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U$ .  
 $\Rightarrow U \in \mathcal{T}$

By induction, any finite intersection is in  $\mathcal{T}$ . □

$$\bigcap_{i=1}^n U_i = U_n \cap \left( \bigcap_{i=1}^{n-1} U_i \right).$$

## Lecture 2 (11-01-2021)

11 January 2021 15:31

Lemma 1. Let  $B$  be a basis and  $\mathcal{T}$  the topology generated by  $B$ . Then,  $\mathcal{T}$  is the collection of all unions of elements of  $B$ .

Note that  $\emptyset$  is empty union.

Proof. Given  $\{U_n\} \subset B$ , it is clear that  $\bigcup U_n \in \mathcal{T}$  since  $\mathcal{T}$  is a topology and  $U_n$  are open. (By defn.)

Conversely, let  $V \in \mathcal{T}$ . Given any  $x \in V$ ,  $\exists B_x \in B$  st.  $x \in B_x \subset V$ . (By defn of  $\mathcal{T}$ .)

Thus,

$$\bigcup_{x \in V} B_x = V.$$

( $\subseteq$ ) since  $B_x \subset V$

( $\supseteq$ ) Each  $x \in V$  is in  $B_x$ .

(Note that if  $V = \emptyset$ , the last union is the empty union!)

[The above gives us a way of extracting a basis  $B$  if we are already given a topology  $\mathcal{T}$ . Namely, pick any subcollection  $B \subset \mathcal{T}$  such that  $\mathcal{T}$  is precisely the collection of all unions of elements of  $B$ .]

Lemma 2. Let  $B$  and  $B'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively, on  $X$ . TFAE:

- $\mathcal{T}'$  is finer than  $\mathcal{T}$ . (recall this means  $\mathcal{T} \subset \mathcal{T}'$ )
- for each  $x \in X$  and each basis element  $B \in B$  containing  $x$ ,  $\exists B' \in B'$  st.  $x \in B' \subset B$ .

Proof. (i)  $\Rightarrow$  (ii) Let  $x \in X$  and  $B \in B$  be arbitrary.

Note that  $B$  is open in  $(X, \mathcal{T})$ , i.e.,  $B \in \mathcal{T}$ .  
Then  $B \subset \mathcal{T}'$ . (by (i))

Note that  $B$  is open in  $(X, \mathcal{T})$ , i.e.,  $B \in \mathcal{T}$ .  
 Thus,  $B \in \mathcal{T}'$ . (by (i))  
 Since  $B$  is open in  $\mathcal{T}'$ ,  $\exists B' \in \mathcal{B}'$  s.t.  $x \in B' \subset B$ .  
(Def<sup>n</sup> of top. generated.)

(ii)  $\Rightarrow$  (i) Suppose  $U \in \mathcal{T}$ . We show that  $U \in \mathcal{T}'$ .  
 Let  $x \in U$ . By def<sup>n</sup> of  $\mathcal{T}$ ,  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subset U$ .  
 By (ii),  $\exists B' \in \mathcal{B}'$  s.t.  $x \in B' \subset B \subset U$ .

Since  $x$  was arbit, we see that  $U \in \mathcal{T}'$ . (By def<sup>n</sup> of  $\mathcal{T}'$ )  
 Thus,  $\mathcal{T} \subset \mathcal{T}'$ . □

Lemma 3. Let  $X$  be a topological space. Suppose  $\mathcal{C}$  is a collection of **open sets** of  $X$  s.t. for each open set  $U \subset X$  and each  $x \in U$ ,  $\exists C \in \mathcal{C}$  s.t.  $x \in C \subset U$ .

Then  $\mathcal{C}$  is a basis for the topology.

Prof. • Showing  $\mathcal{C}$  is a basis.

(i) Given any  $x \in X$ ,  $X$  is an open set containing  $x$ .  
 Thus, by hypothesis,  $\exists C \in \mathcal{C}$  s.t.  $x \in C$ .

(ii) Let  $C_1, C_2 \in \mathcal{C}$  s.t.  $x \in C_1 \cap C_2$ .

Note that  $C_1, C_2$  are open and hence,  $C_1 \cap C_2$  is open.

By hypothesis,  $\exists C_3 \in \mathcal{C}$  s.t.  $x \in C_3 \subset C_1 \cap C_2$ .

Thus,  $\mathcal{C}$  satisfies both properties of a topology.

•  $\mathcal{C}$  generates the topology.

Let  $\mathcal{T}$  denote the topology of  $X$ . Let  $\mathcal{T}'$  be the topology generated by  $\mathcal{C}$ .

Let  $U \in \mathcal{T}'$ , then  $U$  is some union of elements of  $\mathcal{C}$ .

but elements of  $\mathcal{C}$  are elements of  $\mathcal{T}$  and thus,  $U \in \mathcal{T}$ .  
 $(\mathcal{T} \text{ is topo})$

Thus,  $\mathcal{T}' \subset \mathcal{T}$ .

Conversely, let  $U \in \mathcal{T}$ . for each  $x \in U$ ,  $\exists C_x \in \mathcal{C}$  s.t.  
 $x \in C_x \subset U$ .

As earlier,

$$U = \bigcup_{x \in U} C_x \in \mathcal{T}'$$

Thus,  $\mathcal{T} \subset \mathcal{T}'$ . □

Def. Let  $\mathcal{B}$  be the collection of all bounded intervals.

That is,

$$\mathcal{B} = \{(a, b) : -\infty < a < b < \infty\}.$$

$\mathcal{B}$  is a basis and the topology generated by  $\mathcal{B}$  is called the standard topology on  $\mathbb{R}$ . (Standard topology on  $\mathbb{R}$ )

If  $\mathcal{B}'$  is the collection of all half open intervals of the form  $[a, b)$ , then  $\mathcal{B}'$  is also a basis and the topology generated by  $\mathcal{B}'$  is called the lower limit topology on  $\mathbb{R}$ . (Lower limit topology on  $\mathbb{R}$ )

Lemma 4. The lower limit topology is strictly finer than the standard topology.

Proof. Let  $\mathcal{T}$  denote the standard topology and  $\mathcal{T}'$  the lower limit.

- $\mathcal{T} \subseteq \mathcal{T}'$ . Let  $(a, b)$  be an arbit. basis element and let  $x \in (a, b)$ . Then,  $[x, b)$  is a basis element for  $\mathcal{T}'$  &  $x \in [x, b) \subset (a, b)$ .

Thus,  $T \subseteq T'$ , by Lemma 2.

$\cdot T' \neq T$ . Note that  $[0, 1] \in T'$ .  
but given  $0 \in [0, 1]$ , there is no  $(a, b) \ni 0$   
s.t.  $(a, b) \subset [0, 1]$ .  $\square$

Defn. A subbasis  $S$  for a topology is a collection of subsets of  $X$  whose union is  $X$ . (Subbasis, sub basis)

(Note that no topology given so far. Similar to what we saw for basis.)

The topology generated by the subbasis  $S$  is defined to be the collection of all unions of finite intersections of elements of  $S$ .

We need to show that the topology defined above is actually a basis.

Let  $B$  be the collection of finite intersections of elements of  $S$ . We show  $B$  is a basis. (This suffices. Why? Lemma 1.)

(i) Let  $x \in X$ . Then,  $\exists S \in S$  s.t.  $x \in S$ . ( $\because \bigcup_{S \in S} S = X$ )  
But  $S \in B$ .

(ii) Let  $B_1, B_2 \in B$  and  $x \in B_1 \cap B_2$ .  
But note that  $B_1 \cap B_2 \in B$ . (Why?)

Thus, both the conditions are satisfied.

Remark. The standard topology of  $\mathbb{R}$  is also called the order topology on  $\mathbb{R}$ , because of the order relation of  $\mathbb{R}$ .

(We will see this in general, later.)

Defn. Let  $X$  and  $Y$  be topological spaces. The product topology on  $X \times Y$  is the topology having as basis the collection  $\mathcal{B}$  of all sets of the form  $U \times V$ , where  $U \subseteq X$  and  $V \subseteq Y$  are open. (Product topology)

(Open in the respective topologies, i.e.)

Note that  $\mathcal{B}$  is a basis because:

- (i)  $X \times Y$  is itself a basis element
- (ii)  $U \times V, U' \times V' \in \mathcal{B} \Rightarrow (U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V') \in \mathcal{B}$   
↓      ↓  
intersection of  
open sets

Note  $\mathcal{B}$  itself won't be the topology. (In general.)

Thm 5. If  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}_x$  on  $X$ , and  $\mathcal{C}$  for  $\mathcal{T}_y$  on  $Y$ , then the collection

$$\mathcal{D} = \{ B \times C : B \in \mathcal{B}, C \in \mathcal{C} \}$$

is a basis for the product topology of  $X \times Y$ .

Proof. We check that the hypotheses of Lemma 3 are satisfied.

Let  $W \subseteq X \times Y$  be open and  $(x, y) \in W$ .

Then, by def<sup>n</sup> of prod. top.,  $\exists U \in \mathcal{T}_x, V \in \mathcal{T}_y$  s.t.

$$(x, y) \in U \times V \subset W.$$

Since  $\mathcal{B}$  is a basis for  $\mathcal{T}_x$ ,  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subset U$ .

11)  $\exists c \in C$  s.t.  $y \in c \subset V$ .

$\Rightarrow (x, y) \in B \times C \subset U \times V \subset W$ . 



## Lecture 3 (14-01-2021)

14 January 2021 15:28

By last lecture's discussion, we know that

$$\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$$

is a basis for the product topology on  $\mathbb{R}^2$ .  
This is called the standard topology on  $\mathbb{R}^2$ .

Defn. Given any two sets  $X$  and  $Y$ , we have the two projection maps  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  given as

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \\ \forall (x, y) \in X \times Y.$$

(Projections)

Note that  $\pi_1^{-1}(U) = U \times Y$  for any  $U \subseteq X$  and similarly  $\pi_2^{-1}(V) = X \times V$  for any  $V \subseteq Y$ .

Thm. The collection

$$S = \{\pi_1^{-1}(U) \mid U \subseteq X \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \subseteq Y \text{ open}\} \subseteq P(X \times Y)$$

is a subbasis for the product topology on  $X \times Y$ .

Proof. Let  $T_p$  denote the product topology on  $X \times Y$ .  
Let  $T_S$  ——— topology generated by  $S$ .

Note that any element of  $S$  is of the form  $U \times Y$  or  $X \times V$ .

$$\begin{matrix} U \subseteq X \\ \text{open} \end{matrix} \quad \begin{matrix} V \subseteq Y \\ \text{open} \end{matrix}$$

Thus,  $S \subseteq T_p$  since both the above are actually basis elements. Since  $T_p$  is a topology, it is closed under arbitrary unions of finite intersections. Thus,  $T_S \subseteq T_p$ .

On the other hand, consider any arbitrary basis elt. of  $\mathcal{T}_p$ .  
 It is of the form  $U \times V$ .  $U \subseteq X$ ,  $V \subseteq Y$  open.  
 Note now

$$U \times V = \underbrace{\pi_1^{-1}(U)}_{\in \mathcal{T}_S} \cap \underbrace{\pi_2^{-1}(V)}_{\in \mathcal{T}_S} \in \mathcal{T}_S$$

Thus,  $U \times V \in \mathcal{T}_S$ . Since  $\mathcal{T}_S$  is a topology,  
 arbitrary union of basis elements of  $\mathcal{T}_p$  is in  $\mathcal{T}_S$ .  
 Thus,  $\mathcal{T}_p \subseteq \mathcal{T}_S$ .  $\square$

Defn. Let  $(X, \mathcal{T})$  be a topological space and  $Y \subseteq X$ . Then,  
 the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on  $Y$ , called the **subspace topology**.

With this topology,  $(Y, \mathcal{T}_Y)$  is called a **subspace** of  $(X, \mathcal{T})$ .

(Subspace topology)

(We will often just say "Y is a subspace of X" if it is clear.)

We now check that  $\mathcal{T}_Y$  is actually a topology.

Check. (i) Since  $\emptyset \in \mathcal{T}$ , we get  $\emptyset = \emptyset \cap Y \in \mathcal{T}_Y$ .

$$\text{If } Y = Y \cap X \in \mathcal{T}_Y.$$

(ii,iii) Let  $\{U_i\}_{i \in I} \subseteq \mathcal{T}_Y$ . Then, we have  $\{U'_i\}_{i \in I}^2 \subseteq \mathcal{T}$  s.t.

$$U'_i \cap Y = U_i \quad \forall i \in I.$$

Then,

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} (U'_i \cap Y) = \left( \bigcup_{i \in I} U'_i \right) \cap Y$$

and similarly,

$$\bigcap_{i \in I} U_i = \left( \bigcap_{i \in I} U'_i \right) \cap Y. \quad \square$$

and similarly,

$$\bigcup_{i \in I} U_i = \left( \bigcup_{i \in I} U_i \right) \cap Y.$$

□  
if  $|I| < \infty$

Lemma 2. If  $\mathcal{B}$  is a basis for  $(X, \mathcal{T})$ , then the collection

$$\mathcal{B}_Y = \{ B \cap Y : B \in \mathcal{B} \}$$

is a basis for  $(Y, \mathcal{T}_Y)$ , i.e., the subspace topology on  $Y$ .

Proof We use Lemma 3 from Lecture 2.

Using that, it suffices to show that for any  $U \in \mathcal{T}_Y$  and any  $x \in U$ ,  $\exists B \in \mathcal{B}_Y$  s.t.  $x \in B \subset U$ .

To this end, let  $u, U$  be as given. Then,  
 $U = U' \cap Y$  for some  $U' \in \mathcal{T}$ .

Clearly,  $x \in U'$ .

Then,  $\exists B' \in \mathcal{B}$  s.t.  $x \in B' \subset U'$ . ( $\mathcal{B}$  is a basis for  $\mathcal{T}$ .)

Then,  $B = B' \cap Y \in \mathcal{B}_Y$  and  
 $x \in B \subset U$ . □

Lemma 3. Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$ , then  $U$  is open in  $X$ .

Proof.  $U = U' \cap Y$  for some  $U' \subset X$  open.

Since  $U'$  and  $Y$  are open in  $X$ , so is  $U = U' \cap Y$ . (Finite intersection of open sets.)

EXAMPLES. (i) If  $Y = [0, 1] \subset X = \mathbb{R}$  in subspace topology,  
it has a basis given as  
 $\{ (a, b) \cap Y : a < b \in \mathbb{R} \}$ .

More explicitly, here we have

1, 2, ..., 7, ... -1

$$(a, b) \cap Y = \begin{cases} (a, b) & a \in Y \ni b \\ [0, b) & a \notin Y \ni b \\ (a, 1] & a \in Y \ni b \\ \emptyset \text{ or } Y & a \notin Y \ni b \end{cases}$$

(2) Consider  $Y = [0, 1] \cup \{2\} \subseteq \mathbb{R}$ .

Note that

$$\{2\} = (1.5, 2.5) \cap Y.$$

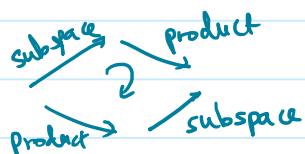
Thus,  $\{2\}$  is open in  $Y$ . (Was not open in  $\mathbb{R}$ !)

Similarly,  $[0, 1]$  is open in  $Y$  but not  $\mathbb{R}$ .

Thm 4. If  $A$  is a subspace of  $X$  and  $B$  of  $Y$ , then the product topology on  $A \times B$  is the same as the topology  $A \times B$  inherits as a subspace of  $X \times Y$ .

Note that the above tells us that the <sup>following</sup> two ways of topologising  $A \times B$  are the same:

- consider  $A$  and  $B$  as spaces by themselves and give  $A \times B$  the product topology
- consider the topological space  $X \times Y$  in product topology.  
Note that  $A \times B$  is a subset of  $X \times Y$  and hence, can be given the subspace topology.



Proof.

Note the following:

typical basis  
elt of  $X \times Y$



$$\{(U \times v) \cap (A \times B) : U \subseteq X, v \subseteq Y \text{ open}\}$$

basis for subspace topology on  $A \times B$   
by Lemma 2

$$= \{ (U \cap A) \times (V \cap B) : U \subseteq X, V \subseteq Y \text{ open} \}$$

↓                    ↓  
 a general open set  
 in the subspace  
 topology of  $A \subseteq X$   
 or  $B \subseteq Y$

basis for prod top.  
 on  $A \times B$

Thus, both the topologies have a common basis.

Defn. A subset of a topological space is said to be **closed** if its complement is open.

(Closed set)

Example. (1)  $[a, b] \subseteq \mathbb{R}$  is closed because

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty) \text{ is open.}$$

(2)  $[0, \infty) \times [0, \infty) \subseteq \mathbb{R}^2$  is closed because

$$\mathbb{R}^2 \setminus [0, \infty)^2 = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0)) \text{ is open.}$$

(3) In the discrete topology, every set is open and hence, every set is closed.

(4) Consider  $Y = [-1, 0] \cup (2, 3) \subseteq \mathbb{R}$ .

Both  $[-1, 0]$  and  $(2, 3)$  are open in  $Y$ .

$$\begin{matrix} " \\ [-2, 1] \cap Y \end{matrix}$$

Since they are complements of each other (in  $Y$ ), we have that both the sets are closed as well, in  $Y$ .

Thms. Let  $X$  be a topological space. Then,

- (i)  $\emptyset$  and  $X$  are closed,
- (ii) arbitrary intersection of closed sets is closed,
- (iii) finite union of closed sets is closed.

Proof.

$$X \setminus \emptyset = X, \quad X \setminus X = \emptyset.$$

Proof.  $X \setminus \emptyset = X, X \setminus X = \emptyset.$

$$X \setminus \left( \bigcap_{i \in I} C_i \right) = \bigcup_{i \in I} (X \setminus C_i).$$

$$X \setminus \left( \bigcup_{i=1}^n C_i \right) = \bigcap_{i \in I} (X \setminus C_i).$$

Conclude. □

Remark. The above is also a way to define a topology.

Thm 6. Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ .

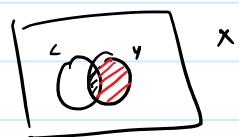
Then,  $A$  is closed in  $Y$  iff  $A$  equals the intersection of a closed set (in  $X$ ) with  $Y$ .

Proof. ( $\Leftarrow$ ) Suppose  $A = C \cap Y$  for some closed set  $C \subseteq X$ .

Then,

$$Y \setminus A = (X \setminus C) \cap Y$$

$\underbrace{\text{open in } X}_{\text{open in } Y}$



$\therefore A$  is closed in  $Y$ .

( $\Rightarrow$ ) Suppose  $A$  is closed in  $Y$ .

Then,  $Y \setminus A$  is open in  $Y$ . Thus,  $\exists U \subseteq X$  open s.t.

$$Y \setminus A = U \cap Y$$

$$\begin{aligned} \text{Then, } Y \setminus (Y \setminus A) &= Y \setminus (U \cap Y) \\ &\stackrel{A}{=} Y \cap (U \cap Y)^c \\ &= Y \cap (U^c \cup Y^c) \\ &= (Y \cap U^c) \cap (Y \cup Y^c) \end{aligned}$$

The  $(C)^c$  is  
complement  
in  $X$ .

$$\Rightarrow A = Y \cap U^c$$

Since  $U^c \subseteq X$  is closed, we are done.

Remark: A set can be both open and closed. For example,  $\emptyset$  and  $X$ .  
A less trivial example : Take  $X = [0, 1] \cup [2, 3]$ .  
Then,  $A = [0, 1] \subset X$  is both open & closed.

# Lecture 4 (18-01-2021)

18 January 2021 15:24

Defn. Given a topological space  $X$  and  $A \subset X$ , we define:

(Interior) The interior of  $A$  as the union of all open sets contained in  $A$ .

Notation:  $\text{int } A$  or  ${}^\circ A$ .  $(\in \mathcal{P}(A))$

(Closure) The closure of  $A$  as the intersection of all closed sets containing  $A$ .

Notation:  $\text{cl}(A)$  or  $\bar{A}$ .  $(A \subset X)$

Remark.  ${}^\circ A$  is an open set and  $\bar{A}$  is a closed set. Further,

$${}^\circ A \subset A \subset \bar{A}.$$

$A$  is open iff  $A = {}^\circ A$ .

$A$  is closed iff  $A = \bar{A}$ .

Defn. Let  $x \in X$ . A neighbourhood of  $x$  is any set  $A$  such that there is an open set  $U \subset X$  with  $x \in U \subseteq A$ .

(Neighbourhood)

(That is, a neighbourhood is any set containing an open set containing the point. This is different from the def<sup>n</sup> in Munkres!)

Thm 1. Let  $A$  be a subset of a topological space  $X$ .

Then,  $x \in \bar{A}$  iff every neighbourhood  $U$  of  $x$  intersects  $A$ .

Proof. ( $\Leftarrow$ )  $x \notin \bar{A} \Rightarrow U = X \setminus \bar{A}$  is a nbd of  $x$  not intersecting  $A$ .

( $\Rightarrow$ ) Suppose  $\exists$  nbd  $C$  of  $x$  s.t.  $C \cap A = \emptyset$ .

Let  $U$  be open s.t.  $x \in U \subseteq C$ . (Def<sup>n</sup> of nbd.)

Then,  $X \setminus U$  is a closed set s.t.  $A \subset X \setminus U$ .

$\Rightarrow \bar{A} \subset X \setminus U$  (why?  $\because \bar{A}$  is the inter. of all closed sets cont.  $A$ .)

$$\Rightarrow \bar{A} \cap U = \emptyset$$

$$\Rightarrow \bar{A} \cap U = \emptyset$$

$$\Rightarrow x \notin \bar{A}.$$

Q

Example

①  $X = \mathbb{R}$  and  $A = [0, 1]$ . Then,  $\bar{A} = [0, 1]$ .

However, if  $X = [0, 1] = A$ , then  $\bar{A} = A$ .

②  $X = \mathbb{R}$ ,  $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ . Then,  $\bar{B} = B \cup \{0\}$ .

③  $C = \{0\} \cup (1, 2)$ . Then,  $\bar{C} = \{0\} \cup [1, 2]$

④  $\bar{\mathbb{Q}} = \mathbb{R}$

⑤  $\bar{\mathbb{N}} = \mathbb{N}$

⑥  $\bar{\mathbb{R}_+} = \mathbb{R}_+ \cup \{0\} = [0, \infty)$ .

Def<sup>n</sup>.

Let  $X$  be a top. space and  $A \subset X$ . (Limit point)

A point  $x \in X$  is said to be a limit point of  $A$  if every neighbourhood of  $x$  intersects  $A$  in some point other than  $x$ .

Notation :  $A'$

Example

Subset of  $\mathbb{R}$

Set of limit points

$$① [1, 2] \quad [1, 2]$$

$$② \{\frac{1}{n} : n \in \mathbb{N}\} \quad \{0\}$$

$$③ \{0\} \cup (1, 2) \quad [1, 2]$$

$$④ \mathbb{Q} \quad \mathbb{R}$$

$$⑤ \mathbb{N} \quad \emptyset$$

$$⑥ \mathbb{R}_+ \quad \overline{\mathbb{R}_+}$$

Thm 2.

$$\bar{A} = A \cup A'.$$

(Proof at the end)

Corollary 3.  $A$  is closed iff  $A' \subset A$ .

Proof.  $A$  is closed  $\Leftrightarrow A = \bar{A} \xrightarrow{\text{Thm 2.}} A' \subset A$ .

Thm 2.

Def<sup>n</sup>: (Order relation or Simple order)

A relation  $C$  on set  $A$  is called an **order relation** (or a **simple order**) if it has the following properties:

- (1) (**Comparability**) For every  $x, y \in A$ ,  $x \neq y \Rightarrow x C y$  or  $y C x$ .
- (2) (**Non reflexivity**)  $\nexists x \in A$  s.t.  $x C x$
- (3) (**Transitivity**)  $x C y$  and  $y C z \Rightarrow x C z$ .

A set with a simple order is called an **ordered set**.

Example: Usual ' $<$ ' on  $\mathbb{R}$  is a simple order.

Def<sup>n</sup>: If  $X$  is a set and ' $<$ ' a simple order relation. Then,

we define " $x \leq y$ " as " $x < y$  or  $x = y$ ".

Let  $A \subset X$ . An element  $a \in A$  is said to be the **smallest element** of  $A$  if

$$a \leq x \quad \forall x \in A.$$

Similarly, we define the **largest element**.

$\left( \begin{array}{l} \text{We have used "the" since uniqueness is simple to check.} \\ \text{Existence, however, is not guaranteed. } (\mathbb{R} \text{ has no largest or smallest element. Neither does } (0, 1)). \end{array} \right)$

Def<sup>n</sup>: If  $(X, <)$  is an ordered set, then for  $a, b \in X$ , we define the **intervals**

$$(a, b) := \{x \in X : a < x < b\},$$

$$(a, b] := \{x \in X : a < x \leq b\},$$

$$[a, b) := \{x \in X : a \leq x < b\},$$

$$[a, b] := \{x \in X : a \leq x \leq b\}.$$

(Intervals)

Def<sup>n</sup>: (Order topology)

Let  $(X, \subset)$  be an ordered set. Let  $B$  be the collection

Let  $(X, \subset)$  be an ordered set. Let  $\mathcal{B}$  be the collection of sets of the form:

- (1) All  $(a, b)$  for  $a, b \in X$ .
- (2) All  $[a_0, b)$  for  $b \in X$  where  $a_0 \in X$  is the smallest element of  $X$ , if any.
- (3) All  $(a, b_0]$  for  $a \in X$  where  $b_0 \in X$  is the largest element of  $X$ , if any.

Then,  $\mathcal{B}$  is a basis (check) and the topology generated is called the **order topology** on  $X$ .

Example. The standard topology on  $\mathbb{R}$  is the order topology derived from the usual order on  $\mathbb{R}$ .

Defn. (Dictionary order)

Suppose that  $(A, \subset_A)$  and  $(B, \subset_B)$  are two ordered sets.

We can define  $\subset$  on  $A \times B$  by we will denote elements of  $A \times B$  by  $a \times b$  instead of  $(a, b)$ .

$$a_1 \times b_1 < a_2 \times b_2.$$

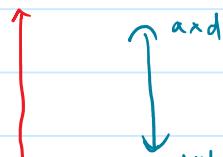
if  $a_1 \subset_A a_2$  or if  $a_1 = a_2$  and  $b_1 < b_2$ .

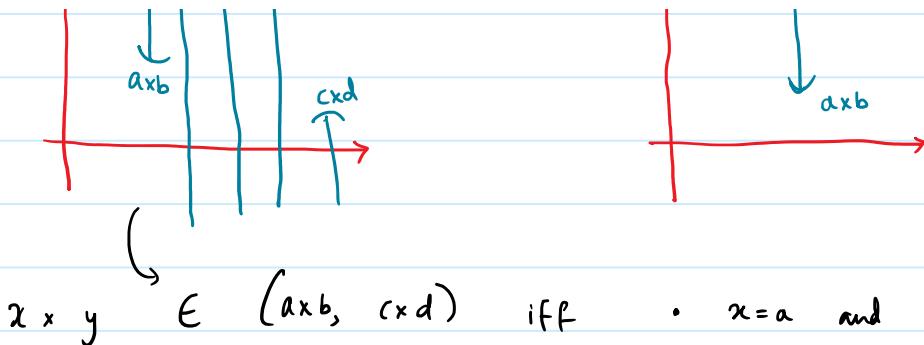
$\subset$  is a simple order on  $A \times B$ , called the **dictionary order** on  $A \times B$ .

Example  $\mathbb{R} \times \mathbb{R}$  can be given an order topology in this dict. order.

A basis will be

$$\{(a \times b, c \times d)\} \text{ where } a < c \text{ or } a = c \text{ & } b < d.$$





- $x = a$  and  $b \leq y$  or
- $a < x < c$  and  $y \in \mathbb{R}$
- $x = c$  and  $y < d$

Remark If  $Y = [0, 1) \cup \{2\}$ , then  $\{2\}$  is NOT open in the order topology.

Note that any basis element containing  $2$  is of the form  $B = (a, 2]$  with  $a \in Y$ .

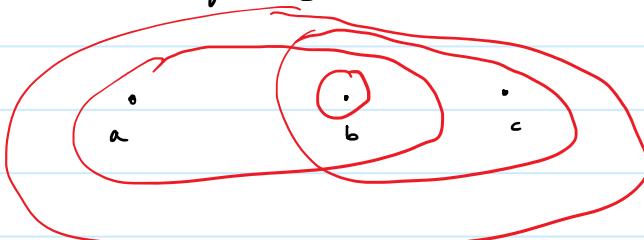
This means that  $0 \leq a < 1$  and hence,  $\frac{a+1}{2} \in B$ .

Thus, it always contains a point distinct from 2.

This shows that subspace and order topologies do not "commute"

Remark. Singletons in  $\mathbb{R}$  (or  $\mathbb{R}^n$ ) are closed. This need not be true in general.

Consider the following topologies



$$X = \{a, b, c\}$$

$$\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$$



$$X = \{a, b, c\}, \mathcal{T} = \{\emptyset, X\}$$

$\{b\}$  is not closed in either of the above since  $\{a, c\}$  is not open.

These spaces are not "nice". In fact, in the above spaces, a convergent sequence may have multiple limits. (Haven't defined this yet, though!) We restrict ourselves to "nicer" spaces.

Defn. A topological space  $X$  is called **Hausdorff** if for every distinct  $x_1, x_2 \in X$ , there exist neighbourhoods  $U_1, U_2$  of  $x_1, x_2$ , respectively such that  $U_1 \cap U_2 = \emptyset$ .

Thm 4. Every finite set in a Hausdorff space is closed.

Proof. It suffices to show the statement for singleton since finite unions of closed sets is closed.

Let  $x_0 \in X$  be arbitrary. We show  $\{x_0\}$  is closed.

(Clearly,  $\{x_0\} \subset \overline{\{x_0\}}$ . Now, consider  $y \in \{x_0\}^c$ .

That is,  $y \neq x_0$ . By Hausdorffness,  $\exists U_1, U_2$  nbd's s.t.  
 $x_0 \in U_1, y \in U_2$  and  $U_1 \cap U_2 = \emptyset$ .

Thus,  $U_2 \cap \{x_0\} = \emptyset$ . Thus,  $y \notin \overline{\{x_0\}}$ . (Thm 1)

Proof of Thm 2.  $\bar{A} = A \cup A'$ .

( $\subseteq$ ) Let  $x \in \bar{A}$ . Suppose  $x \notin A$ . We show  $x \in A'$ .

Let  $U$  be an arbit. nbd of  $x$ .

By Thm 1,  $U \cap A \neq \emptyset$ .

By assumption,  $x \notin U \cap A$ .

Thus,  $x \in A'$ , by def<sup>n</sup> of  $A'$ .

( $\supseteq$ )  $A \subset \bar{A}$  is clear.  $A' \subset \bar{A}$  is also clear by def<sup>n</sup> of  $A'$  and Thm 1. \square

## Lecture 5 (21-01-2021)

21 January 2021 15:36

Thm 1. Let  $X$  be a Hausdorff space,  $A \subset X$ , and  $x \in X$ .

Then,  $x \in \bar{A} \Leftrightarrow$  every nbd of  $x$  contains infinitely many points of  $A$ .

Proof. ( $\Leftarrow$ ) Trivial since infinitely many points imply one point apart from  $x$ .

( $\Rightarrow$ ) Let  $x$  be a limit point.  $\stackrel{\text{open}}{\exists} U \ni x$  for the sake of contradiction, let  $N = \cup_{i=1}^n U_i$  be a nbd of  $x$  s.t.  $A \cap (U_i \setminus \{x\}) = \{x_1, \dots, x_n\}$  is finite.

Note  $\{x_1, \dots, x_n\}$  is closed since  $X$  is Hausdorff.

Thus,  $V = U \cap (X \setminus \{x_1, \dots, x_n\})$  is a nbd of  $x$ .

But  $V \cap (A \setminus \{x\}) = \emptyset$ .  $\rightarrow \leftarrow$

$\hookrightarrow$  Note this makes sense even if  $x \notin A$ .

Recall from tutorial:

(1) Order top. is Hausdorff.

(2) Product of Hausdorff spaces is Hausdorff.

(3) Subspace of Hausdorff spaces is Hausdorff

Defn

Continuous functions

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces.

A function  $f: X \rightarrow Y$  is said to be **continuous** if  $f^{-1}(U) \in \mathcal{T}_X$  for all  $U \in \mathcal{T}_Y$ .

In other words, inverse image of open sets (in  $Y$ ) is open (in  $X$ ).

Remark

By our earlier discussions, it is easily to see that it suffices to check that inverse images of basis (or subbasis) elements are open.

$$\text{Recall } f^{-1}\left(\bigcup_{\alpha \in \Lambda} B_\alpha\right) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$$

$$f^{-1}\left(\bigcap_{\alpha \in \Lambda} B_\alpha\right) = \bigcap_{\alpha \in \Lambda} f^{-1}(B_\alpha)$$

Example. (i)  $f: \mathbb{R} \rightarrow \mathbb{R}_l$ ,  $f(x) := x$  is not continuous since the topology of  $\mathbb{R}_l$  is strictly finer.

(ii)  $g: \mathbb{R}_l \rightarrow \mathbb{R}$ ,  $g(x) := x$  is continuous.

Theorem 2: Let  $X$  and  $Y$  be top. spaces and  $f: X \rightarrow Y$ .

TFAE

(i)  $f$  is continuous.

(ii) For every  $A \subset X$ ,  $f(\bar{A}) \subset \overline{f(A)}$ .

(iii)  $f^{-1}(B)$  is closed for every closed  $B \subset Y$ .

Proof. (i)  $\Rightarrow$  (ii)

Let  $y \in f(\bar{A})$ . Then,  $y = f(x)$  for some  $x \in \bar{A}$ .

We show  $x \in \overline{f(A)}$ .

Let  $V$  be any open nbd. of  $y$ . (Want to show  $V \cap f(A) \neq \emptyset$ .)

Then,  $f^{-1}(V)$  is an open nbd. of  $x$ .

Then,  $A \cap f^{-1}(V) \neq \emptyset$ . Let  $x' \in A \cap f^{-1}(V)$ .

Then,  $f(x') \in f(A) \cap f(f^{-1}(V))$

$\Rightarrow f(x') \in f(A) \cap V$  (\*)

Thus,  $f(A) \cap V \neq \emptyset$ , as desired.

Since any nbd contains an open nbd, we are done.

(ii)  $\Rightarrow$  (iii) Let  $B \subset Y$  be closed.

Put  $A = f^{-1}(B)$ . To show:  $A$  is closed.

$A$  is closed  $\Leftrightarrow A = \bar{A} \Leftrightarrow \bar{A} \subset A$ .

$$\begin{aligned} x \in \bar{A} &\Rightarrow f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B \quad (*) \\ &\Rightarrow x \in f^{-1}(B) \\ &\Rightarrow x \in A. \end{aligned}$$

(\*)  $f(f^{-1}(B)) \subset B$ , in general. Equality if  $f$  onto.

(iii)  $\Rightarrow$  (i) Obvious since  $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ .  $\square$

Defn. Let  $X$  and  $Y$  be topological spaces and  $f: X \rightarrow Y$  be a bijection.  $f$  is said to be a **homeomorphism** if  $f$  and  $f^{-1}$  are both continuous.

$X$  and  $Y$  are said to be **homeomorphic** if there exists a homeomorphism  $f: X \rightarrow Y$ .

(Homeomorphism, homeomorphic)

A homeomorphism can also be defined as a bijection  $f: X \rightarrow Y$  s.t.  $f(U)$  is open in  $Y$  iff  $U$  is open in  $X$ .

Thus,  $f$  is not only a bijection of  $X$  and  $Y$  but also of  $T_X$  and  $T_Y$ .

Defn. Let  $f: X \rightarrow Y$  be an injective continuous function.

Let  $Z = f(X)$  be the image of  $X$  in the subspace topology. Then, the restriction

$f': X \rightarrow Z$  is a bijection.

If  $f'$  is a homeomorphism, then we say that  $f: X \rightarrow Y$  is a **topological imbedding** or an imbedding of  $X$  in  $Y$ .

(Imbedding)

Example (i)  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined  $f(x) := 2x + 4$  is a homeomorphism.

(ii)  $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$  defined  $f(x) := \tan x$  is a homeomorphism.

(iii)  $g: \mathbb{R} \rightarrow \mathbb{R}$  defined  $g(n) := n$  is bijective and continuous but not a homeomorphism.

(iv) Let  $S' := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  be in subspace topology of  $\mathbb{R}^2$ .

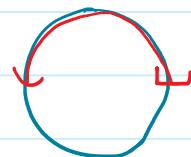
Let  $f: [0, 1] \rightarrow S'$  be defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t).$$

The  $f$  is bijective and continuous but  $f'$  is not continuous. To see the last part, consider  $U = [0, Y_2] \subseteq [0, 1]$ .

$U$  is open but  $f(U) \rightarrow$  top arc of  $S'$

$\cup$   
not open in  $S'$



note that  $1 \times 0 \in$  top arc but no basis elt around that point.

Thm 3: Let  $X, Y$ , and  $Z$  be topological spaces.

If  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are continuous, then  $g \circ f: X \rightarrow Z$  is continuous.

Proof. Use  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ . □

Defn.

Box topology, Product Topology

Let  $J$  be an indexing set and  $\{X_\alpha\}_{\alpha \in J}$  a collection of topological spaces.

Let us consider a basis for a topology on the Cartesian product

$$\prod_{\alpha \in J} X_\alpha,$$

the collection of all set of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where each  $U_\alpha$  is open in  $X_\alpha$ . The topology induced is called the **box topology**.

Let  $\pi_\beta : \prod_{\alpha \in J} X_\alpha \rightarrow X_\beta$  be the projection map

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta.$$

Let  $S_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\}$  and let

$$S = \bigcup_{\beta \in J} S_\beta.$$

Then  $S$  is a subbasis for a topology on  $\prod_{\alpha \in J} X_\alpha$ . The topology generated is called the **product topology**.

Remark: ① A typical basis elt. for prod. topology is

$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \quad \left[ \begin{matrix} \beta_1, \dots, \beta_n \\ p\text{-wise distinct} \end{matrix} \right]$$

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \left\{ U_{\beta_i} : \alpha = \beta_i \right\}$$

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \begin{cases} U_{\beta_i} & ; \alpha = \beta_i \\ X_\alpha & ; \text{else} \end{cases}$$

② If  $J$  is finite, both box and product coincide.

③ In general, box topology is finer than product.

If  $|J| = \infty$ , then it can be strictly finer.

If each  $X_\alpha = \mathbb{R}$ , then strictly finer.

If each  $X_\alpha = \{0\}$ , then not.

If each  $X_\alpha$  is in indiscrete topology, then not.

## Lecture 6 (25-01-2021)

25 January 2021 15:37

Thm1. The box topology is finer than the product topology.

Proof. Every basis element of prod. topology is also one of box.  $\square$

- Remarks
- (1) For finite products, the two are the same.
  - (2) If we simply refer to the product space, we shall mean the product topology, by default.

Thm2. Let  $f: A \rightarrow \prod_{\alpha \in J} X_\alpha$  be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where  $f_\alpha: A \rightarrow X_\alpha$  for each  $\alpha$ . Let  $\prod X_\alpha$  have the product topology. Then,  $f$  is continuous iff each  $f_\alpha$  is continuous.

Proof. Note that  $\pi_\beta: \prod X_\alpha \rightarrow X_\beta$  is continuous  $\forall \beta$  since each  $\pi_\beta^{-1}(U_\beta)$  is a subbasis element.

$\Rightarrow$  Now, suppose that  $f: A \rightarrow \prod X_\alpha$  is continuous.  
So,  $f_\alpha = \pi_\alpha \circ f$  is continuous  $\forall \alpha$ .

$\Leftarrow$  Conversely, suppose each  $f_\alpha$  is continuous.  
It suffices to show that inverse images of subbasis elements are open.

A typical subbasis elt is  $\pi_\alpha^{-1}(U_\alpha)$  for  $U_\alpha$  open in  $X_\alpha$ .

$$\text{But } f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha) \quad \square$$

$\uparrow$   
open since  $f_\alpha$  is continuous.

Remark. Above not true for box topology.

Take

$$f: \mathbb{R} \rightarrow \prod_{i=1}^{\infty} \mathbb{R} \text{ given by}$$

$t \mapsto (t, t, \dots)$  is not continuous in box.

Consider the open set  $U = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times \dots$

Defn. A metric  $d$  on a set  $X$  is a function  
 $d: X \times X \rightarrow \mathbb{R}$  satisfying

(1)  $d(x, y) \geq 0$  and  $d(x, y) = 0 \Leftrightarrow x = y$ .

(2)  $d(x, y) = d(y, x)$

(3)  $d(x, z) \leq d(x, y) + d(y, z)$

For a metric  $d$  on  $X$ , the number  $d(x, y)$  is called the distance between  $x$  and  $y$  in metric  $d$ .

Given  $\epsilon > 0$ , the set

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

is called the  $\epsilon$ -ball centered at  $x$ .

We often write  $B(x, \epsilon)$  if  $d$  is understood.

The collection  $\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$  is a basis and the topology induced is called the metric topology on  $X$ .

A topological space is called metrisable if there exists a metric on  $X$  which induces the given topology on  $X$ .

Example. (1) Given a set  $X$ , define

$$d(x, y) = \begin{cases} 1 & ; x \neq y, \\ 0 & ; x = y. \end{cases}$$

This  $d$  is a metric and the topology induced is the discrete topology, since  $B(x, 1) = \{x\}$ .  
 (Thus, singletons are open and thus, every set is.)

(2) Standard topology on  $\mathbb{R}$  is induced by  
 $d(x, y) := |x - y|$ .

Note

$$(a, b) = B_d(x, \epsilon) \text{ for } x = \frac{a+b}{2} \text{ and } \epsilon = \frac{b-a}{2}.$$

(a  $\subset$  b)

(3) On  $\mathbb{R}^n$ , we have the Euclidean metric given by  
 $d(x, y) = \|x - y\| = \left[ (x_1 - y_1)^2 + \dots + (x_n - y_n)^2 \right]^{\frac{1}{2}}$ .

Another example is the square metric

$$\rho(x, y) = \max \{|x_1 - y_1|, \dots, |x_n - y_n|\}.$$

Both the metrics induce the same topology, which is the same as the usual product topology.

Thm 3.

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and

$f: X \rightarrow Y$  a function. Then,

$f$  is continuous  $\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0$  s.t.

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \epsilon.$$

Proof. Exercise. 回

Defn. (Sequence and convergence)

Let  $X$  be a set. A sequence  $(x_n)_{n=1}^{\infty}$  is a function  
 $\mathbb{N} \rightarrow X$ . ( $n \mapsto x_n$ )

It is said to converge to  $x \in X$  if for every nbhd  $U$  of  $x$ ,  $\exists n_0 \in \mathbb{N}$  s.t.  $x_n \in U \forall n > n_0$ .

It is said to converge or be convergent if it converges to some  $x \in X$ .

Lemma:

Let  $X$  be a topological space and  $A \subset X$ .

If  $\exists$  a seq.  $(x_n)_{n=1}^{\infty} \subset A$  which converges to  $x \in X$ , then  $x \in \bar{A}$ .

The converse is true if  $X$  is metrisable.

Proof.

( $\Rightarrow$ ) Let  $(x_n)_{n=1}^{\infty}$  and  $x$  be as in Lemma. Let  $U$  be an arbitrary nbhd of  $x$ . We show  $\bigcup_{n=1}^{\infty} A \neq \emptyset$  to conclude.

By def<sup>n</sup> of convergence,  $\exists n_0 \in \mathbb{N}$  s.t.  $x_n \in U \forall n > n_0$ . Then,  $\emptyset \neq \bigcup_{n=n_0+1}^{\infty} A \subset \bigcup_{n=1}^{\infty} A$ .

( $\Leftarrow$ ) Assume  $d$  metrises  $X$  and  $x \in \bar{A}$ .

For each  $n \in \mathbb{N}$ ,  $B(x, \frac{1}{n}) \cap A \neq \emptyset$ .

For each  $n \in \mathbb{N}$ , pick  $x_n \in B(x, \frac{1}{n})$ . (Need some choice.)

Then,  $d(x, x_n) < \frac{1}{n} \rightarrow 0$  and thus,

$$x_n \rightarrow x.$$

□

(Note: An easy check that convergence of sequences in metric space coincides.)

Defn.

A space  $X$  is said to have a countable basis at  $x$  if there is a countable collection  $\mathcal{B}$  of open nbds of  $x$  s.t. each nbhd of  $x$  contains an element  $\mathcal{B}$ . A space that has a countable basis at each  $x \in X$  is said to be first countable.

Eg.  $\mathbb{R}$ ,  $\mathbb{R}^n$ , take  $\{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$  at each  $x$ .

Lemma 4.2. The converse of Lemma 4 holds even if  $X$  is first countable. More generally, if  $x \in \bar{A}$ , then only countable basis at  $x$  is required.

## Lecture 7 (28-01-2021)

28 January 2021 15:34

Thm1.

Let  $X, Y$  be topological spaces and  $f: X \rightarrow Y$  be continuous.

(Suppose  $x_n \rightarrow x$  in  $X$ . Then,  $f(x_n) \rightarrow f(x)$  in  $Y$ .)  $\leftarrow$

If  $X$  is metrisable, then  $\leftarrow$  implies continuity.

(That is, if  $f(x_n) \rightarrow f(x)$  for every convergent subsequence  $x_n \rightarrow x$ )  
for every  $x \in X$ , then  $f$  is continuous.

Proof.

Let  $x_n \rightarrow x$  in  $X$ .

Let  $U$  be an arbitrary neighbourhood of  $f(x)$ .

Then,  $f^{-1}(U)$  is a nbd of  $x$ .

Thus,  $\exists N \in \mathbb{N}$  s.t.  $x_n \in f^{-1}(U) \quad \forall n \geq N$ .

Thus,  $f(x_n) \in U \quad \forall n \geq N$  proving that  $f(x_n) \rightarrow f(x)$ .

Now, suppose that  $X$  is Hausdorff.

Assume that  $\leftarrow$  is satisfied.

It suffices to show  $f(\bar{A}) \subset \overline{f(A)}$ .

Let  $A \subset X$  be arbitrary and let  $y \in f(\bar{A})$ .

Then,  $y = f(x)$  for some  $x \in \bar{A}$ .

Thus,  $\exists (x_n) \subset A$  s.t.  $x_n \rightarrow x$ . (Lemma 4 from Lec 6,  
 $X$  is metrisable.)

By our condition,  $f(x_n) \rightarrow f(x)$  and  $f(x_n) \in f(A)$ .

Thus,  $y = f(x) \in \overline{f(A)}$ . (In general.)  $\square$

Remark.

As in Lec 6, the "metrisable" can be relaxed to first countability.

Thm2.

If  $X$  is a topological space and  $f, g: X \rightarrow \mathbb{R}$  are continuous, then  $f \pm g, f \cdot g$  are continuous.

If  $g(x) \neq 0 \quad \forall x \in X$ , then  $f/g$  is continuous.

Proof.

$+,-,\cdot : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous.

$x \mapsto y_x$  is continuous  $\mathbb{R} \setminus \{\emptyset\} \rightarrow \mathbb{R}$ .

Since  $f \times g : X \rightarrow \mathbb{R} \times \mathbb{R}$  is continuous, we are done.  $\square$

Defn.  $\mathbb{R}^\omega := \prod_{n \in \mathbb{N}} X_n$  where  $X_n = \mathbb{R}$  for all  $n \in \mathbb{N}$ .

Example (1) Let  $f: \mathbb{R} \rightarrow \mathbb{R}^\omega$  be defined by

$$f(t) := (t, t, \dots).$$

Then, (a)  $f$  is continuous if  $\mathbb{R}^\omega$  is equipped with prod topology.

*Thm 2 of Lec 6* (b)  $f$  is NOT continuous if  $\mathbb{R}^\omega$  has box topology.

Note  $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times \dots$

$= \prod_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$  is a basis elt. of  
box topology.

*Thus, Thm 2 of Lec 6 is not true* Then,  $f^{-1}(B) = \{\emptyset\}$  is not open in  $\mathbb{R}$ .

(2) Again, take  $\mathbb{R}^\omega$  in box topology.

$$\text{let } A = \{(x_1, x_2, \dots) \mid x_i > 0 \ \forall i\}.$$

Then,  $\underline{0} \in \bar{A}$ . (Here  $\underline{0} = (0, 0, \dots) \in \mathbb{R}^\omega$ )

*std basis elt. around it:*  $B = (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) \times \dots$

$$\text{Then, } \underline{x} = \left(\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}, \dots\right) \in B \cap \bar{A}.$$

Claim.  $\nexists (x_n) \subset A$  s.t.  $\underline{x}_n \rightarrow \underline{0}$ .

Proof. Assume not. Let  $(x_n) \subset A$  be s.t.  $\underline{x}_n \rightarrow \underline{0}$ .

Note that  $\underline{x}_n = (x_{1n}, x_{2n}, \dots)$  where  $x_{in} > 0 \ \forall i$ .

Define

$$B = (-y_{11}, y_{11}) \times (-y_{22}, y_{22}) \times \dots$$

Clearly,  $\underline{0} \in B$  but  $y_{in} \notin B \ \forall n$ .

Thus,  $y_{in} \not\rightarrow 0$ .

Cor 3.  $\mathbb{R}^\omega$  in box topology is not metrizable.

(Lemma 4 from Lec 6.)

Lemma 4

Let  $\rho: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  be the metric

$$\rho(x, y) = \max \{|x_i - y_i| : 1 \leq i \leq n\}.$$

Then,  $\rho$  is a metric which induces the standard (product) topology on  $\mathbb{R}^n$ .

(The proof that  $\rho$  is indeed a metric is omitted.)

Proof.

Let  $B = (a_1, b_1) \times \dots \times (a_n, b_n)$  be a std. basis elt of prod. topology. let  $x = (x_1, \dots, x_n) \in B$ .

For each  $i = 1, \dots, n$ , pick  $\epsilon_i > 0$  s.t.  $(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$ .

Then, put  $\epsilon = \min \{\epsilon_1, \dots, \epsilon_n\} > 0$ .

$$\begin{aligned} B_\epsilon(x, \epsilon) &= (x_1 - \epsilon, x_1 + \epsilon) \times \dots \times (x_n - \epsilon, x_n + \epsilon) \\ &\subseteq (x_1 - \epsilon_1, x_1 + \epsilon_1) \times \dots \times (x_n - \epsilon_n, x_n + \epsilon_n) \\ &\subseteq B. \end{aligned}$$

Conversely, each  $\epsilon$  ball in the metric topology is a basis element of the product topology. □

Defn.

(Connected, separation)

Let  $X$  be a topological space.

A separation of  $X$  is a pair  $U, V$  of non-empty disjoint open subsets of  $X$  such that  $U \cup V = X$ .

$X$  is said to be connected if no separation exists.

Lemma 5.

A space  $X$  is connected iff the only clopen (closed as well as open) subsets of  $X$  are  $\emptyset$  and  $X$ .

Proof.

$\Rightarrow$  Let  $U$  be a clopen set s.t.  $\emptyset \neq U \neq X$ .

Then,  $V = U^c$  is also clopen and nonempty.

Then  $X = U \cup V$ .  $\rightarrow \leftarrow$

$\Leftarrow$  Suppose  $X$  is not connected. let  $U, V$  be a

separation, then  $\emptyset \neq U \neq X$  and  $U = V^c$  is closed.  $\square$

(Ex.) Let  $Y$  be a subspace of  $X$  and  $A \subset Y$ .

Then  $\overline{A \cap Y}$  is the closure of  $A$  in  $Y$ .

Thm 6. A pair of disjoint non-empty sets  $A$  and  $B$  whose union is  $\gamma$  is a separation of  $\gamma$  iff neither contains a limit point of the other.

Proof. ( $\Rightarrow$ ) Then,  $A = \text{cl}_Y(A) = \bar{A} \cap Y$ .

Claim.  $A \cap B = \emptyset$

$$\underline{\text{Proof.}} \quad A = \bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B) \\ = \emptyset \cup (\bar{A} \cap B) = \bar{A} \cap B$$

Thus,  $\bar{A} \cap B \subset A$ .

$$\text{Thus, } (\bar{A} \cap B) \cap A = \bar{A} \cap B$$

$\stackrel{''}{=} \bar{A} \cap (B \cap A)$

$\stackrel{''}{=} \bar{P}$

Similarly,  $A \cap \bar{B} = \emptyset$ , as desired.

$\Leftarrow$  We <sup>only</sup> need to show that A and B are open in Y.

Equivalently, it suffices to show that A and B are closed in Y.

We know  $A \cap B = \emptyset = A \cap \bar{B}$ .

$$\text{Thus, } \bar{A} \cap Y < \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B) \\ = A \cup \emptyset = A$$

Thus,  $\text{cl}_Y(A) = A$ . Thus,  $A$  is closed in  $Y$ .

$\parallel^{rly}$ , B — u .

EXAMPLES (1) Any set in indiscrete topology is connected.

(2)  $Y = (0, 5) \cup (5, 7) \subseteq \mathbb{R}$  is not connected.

$(0, 5), (5, 7)$  form a separation.

(3)  $Y = (0, 5] \cup (5, 7) = (0, 7)$ .

$(0, 5], (5, 7)$  does NOT form a separation.

Note  $[0, 5]$  contains the limit point 5 of  $(5, 7)$ .

Aliter:  $(0, 5]$  is not open in  $Y$ .

Later, we shall see that intervals in  $\mathbb{R}$  are connected.

(4)  $\mathbb{Q}$  is not connected. Let  $I = (\sqrt{2}, \infty) \subset \mathbb{R}$ .

$I \cap \mathbb{Q}$  is clearly open in  $\mathbb{Q}$  since  $I$  is open in  $\mathbb{R}$ .

$$\text{Now, } \mathbb{Q} \setminus (I \cap \mathbb{Q}) = (\mathbb{R} \setminus I) \cap \mathbb{Q}$$

$$= (-\infty, \sqrt{2}] \cap \mathbb{Q}$$

$$= (-\infty, \sqrt{2}) \cap \mathbb{Q}$$

↳ also open.

B

(5) Let  $A = \mathbb{R} \times \{0\}$  and  $B = \{(x, y) : x > 0, y = \frac{1}{x}\}$ .

Put  $Y = A \cup B \subseteq \mathbb{R}^2$  in subspace topology.

Then,  $A$  and  $B$  are closed in  $\mathbb{R}^2$  and hence, in  $Y$ .

Since  $A \cap B = \emptyset$ , we are done. ( $A \neq \emptyset \neq B$ )

## Lecture 8 (01-02-2021)

01 February 2021 20:58

Lemma: If the sets  $C$  and  $D$  form a separation of  $X$  and if  $Y$  is a connected subspace of  $X$ , then  $Y \subseteq C$  or  $Y \subseteq D$ .

Proof. Note that  $(Y \cap C) \cup (Y \cap D) = Y$  and  $(Y \cap C) \cap (Y \cap D) = \emptyset$  with  $Y \cap C$  and  $Y \cap D$  open in  $Y$ . Thus, one must be empty.  $Y \cap D = \emptyset \Rightarrow Y \subseteq C$  and  $Y \cap C = \emptyset \Rightarrow Y \subseteq D$ .  $\square$

Proof. Let  $\{A_\alpha\}_{\alpha \in I}$  be a collection of connected spaces.

Pick  $p \in \bigcap A_\alpha$ .

Put  $Y = \bigcup A_\alpha$ . Suppose, for the sake of contradiction, that  $Y = C \cup D$  is a separation.

WLOG,  $p \in C$ . ( $\because p \notin D$ )

Now, given any  $\alpha \in I$ , we must have  $A_\alpha \subset C$ , by the previous theorem.

Thus,  $A_\alpha \subset C \subset Y$ . Thus,  $Y \subset C$  and hence,  $D = \emptyset$ .  $\rightarrow \leftarrow$

Thm 3. If  $A \subset X$  is connected and  $B \subset X$  is such that  $A \subset B \subset \bar{A}$ , then  $B$  is connected.

In particular,  $\bar{A}$  is connected.

Proof. Suppose  $B = C \cup D$  is a separation.

Then,  $A \subset C$  wlog. ( $A$  is connected.)

Thus,  $\bar{A} \subset \bar{C}$ . Moreover,  $\bar{C} \cap D = \emptyset$ , since  $(C, D)$  form a separation. Thus,  $\bar{A} \cap D = \emptyset$ .  $\sqcup$

form a separation. Thus,  $\bar{A} \cap D = \emptyset$ .  
 (Thm 5, last lec.)  $B \cap D = D$

Thus,  $D = \emptyset$ .

R

Thm 4. Let  $f: X \rightarrow Y$  be continuous. If  $X$  is connected, then  $f(X)$  is connected.

Proof. Put  $Z = f(X)$ . Then, we get a function  $f: X \rightarrow Z$ .

Moreover, this new  $f$  is still continuous. ( $Z$  in subspace topology.)

If  $V \subset Z$  is open, then  $V = V \cap Z$  for  $V$  open in  $Y$ .  
 Then,  $f^{-1}(V) = f^{-1}(V \cap Z) = f^{-1}(V) \cap f^{-1}(Z) = f^{-1}(V) \cap X = f^{-1}(V) \rightarrow \text{open.}$

We now look at the surjective map  $f: X \rightarrow Z$ .

Suppose  $Z = A \cup B$  is a separation.

Then,  $f^{-1}(Z) = f^{-1}(A) \cup f^{-1}(B)$  and  $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ .  
 " "  $\downarrow$   $\downarrow$   $\rightarrow \leftarrow$   
 non-empty and open

Thm 5. Cartesian product of finitely many connected spaces is connected.

Proof. Since  $X_1 \times \dots \times X_n \cong (X_1 \times \dots \times X_{n-1}) \times X_n$  for  $n \geq 3$ , it suffices to prove for  $n=2$ .

Let  $X$  and  $Y$  be connected, we say  $X \times Y$  is connected.

Fix a point  $(a, b) \in X \times Y$ . Let  $x \in X$  be orbit. Consider the connected sets  $\{x\} \times Y (\cong Y)$  and  $X \times \{b\} (\cong X)$ .

Moreover, the slices have  $(a, b)$  in common. Then,

$$T_x = (\{x\} \times Y) \cup (X \times \{b\})$$

is connected for each  $x \in X$ .

However, note that  $(a, b) \in T_x \quad \forall x \in X$ .

Thus,  $\bigcup_{x \in X} T_x$  is connected. But  $X \times Y = \bigcup_{x \in X} T_x$ ,  
as desired.  $\square$

Def. (Path, path-connected)

Let  $X$  be a topological space and  $x, y \in X$ .

A path from  $x$  to  $y$  in  $X$  is a function

$f: [0, 1] \rightarrow X$  s.t.  $f(0) = x$  and  $f(1) = y$ .

$X$  is said to be path-connected if for any  $x, y \in X$ ,  
there exists a path from  $x$  to  $y$ .

(Usually we may take  $[a, b]$  instead of  $[0, 1]$ .)

Fact: Intervals in  $\mathbb{R}$  are connected. (Recall from  $\mathbb{R}$  Analysis.)

Tm 6. Any path connected space is connected.

Proof. Suppose  $X$  is path-connected and  $X = A \cup B$  is a sep.

Pick  $x \in A$  and  $y \in B$ . By hypothesis,  $\exists f: [0, 1] \rightarrow X$   
s.t.  $f(0) = x$  &  $f(1) = y$ .

But  $[0, 1]$  is connected and thus, so is  $f([0, 1])$ .

Thus, by Lemma 1,  $f([0, 1]) \subset A$  or  $f([0, 1]) \subset B$ .  $\leftarrow$

Examples. (1) The unit ball  $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\} \subset \mathbb{R}^n$  is  
path-connected. (The straight line path works.)

(2)  $\mathbb{R}^n \setminus \{0\}$  is path-connected if  $n > 1$ .

Proof. Let  $x, y \in \mathbb{R}^n \setminus \{0\}$ . If  $0$  does not lie on  
the line seg. joining  $x$  and  $y$ , take that line seg.  
Else, pick  $z$  not on line and join  $x$  to  
 $z$  and  $z$  to  $y$ .  $\square$

If  $n = 1$ , then  $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$  is not even connected, let alone path-connected.

(3) For  $n \geq 2$ , define  $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \subseteq \mathbb{R}^n$ .

It is path-connected. To see this, define

$$g: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1} \text{ by}$$

$$x \mapsto x/\|x\|.$$

Then,  $g$  is continuous and maps  $\mathbb{R}^n \setminus \{0\}$  onto  $S^{n-1}$ .

The image of a path-connected space is path-connected and hence,  $S^{n-1}$  is path-connected  
 (Ex.)

(Ex.) Continuous image of path-connected space is path-connected.

P.f. Let  $g: X \rightarrow Z$  be continuous and onto.

Pick  $z_1, z_2 \in Z$ . Then,  $\exists x_1, x_2 \in X$  s.t.  $x_1 \mapsto z_1$  &  $x_2 \mapsto z_2$ .

Now,  $\exists \gamma: [0, 1] \rightarrow X$  s.t.  $x_1 \xrightarrow{\gamma} x_2$ .

Then,  $g \circ \gamma: [0, 1] \rightarrow Z$  is continuous

and  $(g \circ \gamma)(0) = z_1$  &  $(g \circ \gamma)(1) = z_2$ .  $\square$

# Lecture 9 (08-02-2021)

08 February 2021 15:32

(4) Let  $S = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$ .

The set  $\bar{S}$  is called the *topologist's sine curve*.

(Topologist's sine curve)

$$\bar{S} = S \cup \{0\} \times [-1, 1].$$

Note that  $S$  is connected, being the image of a connected set  $(0, 1]$  under a continuous map  $x \mapsto (x, \sin \frac{1}{x})$ .

As seen, this implies  $\bar{S}$  is connected.

Claim. However,  $\bar{S}$  is not path-connected.

Proof. Suppose not. Let  $f: [a, c] \rightarrow \bar{S}$  be a path from  $(0, 0)$  to  $(1, \sin 1)$ .

Let  $D = f^{-1}(\{0\} \times [-1, 1]). D \subset [a, c]$  is closed.

Thus,  $b = \sup D \in D$ .

$\therefore f: [b, c] \rightarrow \bar{S}$  has the property that  $f(b) \in \{0\} \times [-1, 1]$   
but  $f(b) \in S$  for  $x > b$ .

WLOG,  $[b, c] = [0, 1]$ . Write  $f(t) = (x(t), y(t))$ .

Claim.  $\exists (t_n) \subset (0, 1)$  s.t.  $t_n \rightarrow 0$  and  $y(t_n) = (-1)^n$ .

Proof. For  $n \in \mathbb{N}$ ,  $x(y_n) > 0$ . Thus, we can choose  $u_n$  s.t.  $0 < u_n < n(Y_n)$  and  $\sin(Y_n) = (-1)^n$ .

By IVT,  $\exists t_n$  s.t.  $0 < t_n < Y_n$  and  $x(t_n) = u_n$ .

Thus,  $y(t_n) = \sin(Y_x(t_n)) = \sin(Y_{u_n}) = (-1)^n$ .

$0 < t_n < Y_n \Rightarrow t_n \rightarrow 0$ . □

Thus,  $t_n \rightarrow 0$  and  $y(t_n)$  does not converge. Thus,  
 $y$  is not continuous. Therefore,  $f$  is not continuous.  $\rightarrow \square$  □

Def<sup>n</sup>.

(Connected components)

Given  $X$ , define the equivalence relation  $x \sim y$  if  $\exists a$

Connected subset of  $X$  containing  $x$  and  $y$ .

The equivalence classes are called the components or connected components.

Remark:  $\{x\}$  is connected.

Sym.: Trivial.

Transitive: let  $x \sim y$  and  $y \sim z$ .  $\exists A, B \subset X$  connected s.t.

$x, y \in A$  and  $y, z \in B$ . Then,  $A \cup B$  is connected  
since  $y \in A \cap B$ . But  $x, z \in A \cup B$ .  $\therefore x \sim z$ .

Thm: The components of  $X$  are connected disjoint subsets of  $X$   
whose union is  $X$ , s.t. each connected subset of  $X$   
intersects only one of them.

Proof: The part about being disjoint and union being  $X$  follows  
because  $\sim$  was an equiv. relation.

Now, suppose  $A$  is a connected set s.t.  $A$  intersects  
the components  $C_1$  and  $C_2$ . Let  $x_1 \in A \cap C_1$  and  $x_2 \in A \cap C_2$ .  
But then,  $x_1, x_2 \in A$  and hence,  $x_1 \sim x_2$ .  $\therefore C_1 = C_2$ .

This proves the second part.

We just have to prove that each component  $C$  is connected.

Fix  $x_0 \in C$ .  $\forall x \in C$ ,  $x_0 \sim x$ .  $\therefore \exists A_x$  s.t.  $x, x_0 \in A_x$

and  $A_x$  connected. By the earlier part,  $A_x \subset C$ .

$\therefore A_x \subset C \quad \forall x$

$$\Rightarrow C = \bigcup_{x \in C} A_x. \quad \text{But } \bigcap_{x \in C} A_x \ni x_0.$$

$\therefore C = \bigcup_{x \in C} A_x$  is connected. □

Def: (Cover, open cover) A collection  $\mathcal{U}$  of subsets of  $X$  is said to  
be a cover of  $X$  if  $\bigcup_{U \in \mathcal{U}} U = X$ .

If each  $U \in \mathcal{U}$  is open, then  $\mathcal{U}$  is said to be

an open cover of  $X$ .

Defn. (Compact)  $X$  is said to be compact if every open cover (of  $X$ ) has a finite sub-cover.

Example (1)  $\mathbb{R} \rightarrow$  not compact

$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2) \quad \text{but no finite subcover}$$

since  $\mathbb{R}$  is not bounded.

(2)  $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$  is compact.

Let  $\mathcal{U}$  be an open cover of  $K$ .

$\exists U_0 \in \mathcal{U}$  s.t.  $0 \in U_0$ .

Thus,  $\exists N$  s.t.  $K \cap \left(0, \frac{1}{N}\right) \subset U_0$ .

Now, for  $k = 1, \dots, N$ , choose  $U_k \in \mathcal{U}$  s.t.  $y_k \in U_k$ .

Then,  $K \subset U_0 \cup U_1 \cup \dots \cup U_N$ . B

(3)  $(0, 1]$  not compact.  $(0, 1] = \bigcup_{n \geq 2} \left(\frac{1}{n}, 1\right)$ .

Defn. If  $Y$  is a subspace of  $X$ , and  $\mathcal{C}$  a collection of subsets of  $X$ , then  $\mathcal{C}$  is said to cover  $Y$  if

$$Y \subseteq \bigcup_{C \in \mathcal{C}} C$$

Lemma 2. Let  $Y$  be a subspace of  $X$ .

Then,  $Y$  is compact (in subspace topology) iff every covering of  $Y$  by sets open in  $X$  contains a finite subcollection covering  $Y$ . B

Lemma 3. Every closed subspace of a compact space is compact.  $\square$

Thm 4 Every compact subspace of a Hausdorff space is closed.

Proof. Let  $Y \subset X$  be closed, where  $X \leftarrow$  Hausdorff.

We prove that  $X \setminus Y$  is open.

Let  $x_0 \in X \setminus Y$ . For each  $y \in Y$ ,  $\exists$  disjoint open nbd's  $U_y$  and  $V_y$  of  $x_0$  and  $y$ , resp. The collection

$$\{V_y : y \in Y\}$$

Covers  $Y$ . Thus,  $\exists y_1, \dots, y_n \in Y$  s.t.  $Y \subset V_{y_1} \cup \dots \cup V_{y_n}$ .

( $Y$  is compact)

Then,  $U_{y_1} \cap \dots \cap U_{y_n}$  is an open nbd of  $x_0$  contained in  $(V_{y_1} \cup \dots \cup V_{y_n})^c = Y^c$ .

$\therefore Y \setminus X$  is open.

Remark. The above proof shows the following:

If  $X$  is Hausdorff,  $Y \subset X$  is compact, and  $x_0 \notin Y$ , then  $\exists$  disjoint open sets  $U$  and  $V$  of  $X$  containing  $x_0$  and  $Y$ , resp.

# Lecture 10 (10-02-2021)

10 February 2021 16:05

Thm1. The continuous image of a compact set is compact.  $\square$

Thm2. Let  $f: X \rightarrow Y$  be a bijective continuous map. If  $X$  is compact and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.

Proof. We need to show  $f^{-1}$  is continuous.

Let  $K \subseteq X$  be closed. Then,  $K$  is closed, since  $X$  is compact.

Thus,  $f(K) \subseteq Y$  is compact. Then,  $f(K)$  is closed, since  $Y$

is Hausdorff. Thus,  $f$  is a closed map and hence,  $f^{-1}$   
is continuous.  $\square_2$

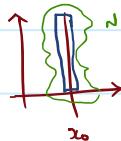
Thm3. The cartesian product of finitely many compact spaces is compact.

Proof. As for the case of connectedness, it suffices to prove  
for product of two spaces.

Proof. Let  $Y$  be a compact space.

Step 1. Suppose that  $x_0 \in X$  and  $N$  is an open set in  $X \times Y$   
containing the "slice"  $\{x_0\} \times Y$ . We show that  $\exists$  nbd  $W$  of  
 $x_0$  in  $X$  s.t.  $W \times Y \subseteq N \times Y$ .

(called a "tube")



$$N = \bigcup_{i \in I} U_i \times V_i$$

First, we cover  $\{x_0\} \times Y$  by  $\{U_i \times V_i\}$   $\rightarrow$  open sets, basis elements.

By compactness, we can cover by finitely many,  $i = 1, \dots, n$ .

WLOG, assume that  $(U_i \times V_i) \cap (\{x_0\} \times Y) \neq \emptyset$  for  $i = 1, \dots, n$ .

Then,  $W = U_1 \cap \dots \cap U_n$  is a neighbourhood of  $x_0$ .

Then,  $W \times Y \stackrel{\leq N}{\subseteq}$  is the desired tube.

Now, assume  $X$  is also compact.

Step 2. Let  $\mathcal{d}$  be an open covering of  $X \times Y$ .

Given  $x_0 \in X$ ,  $\{x_0\} \times Y$  is compact and hence covered by finitely many  $A_1, \dots, A_n \in \mathcal{d}$ . Then,

$N = A_1 \cup \dots \cup A_n$  is an open set

containing  $\{x_0\} \times Y$ .

By step 1,  $\exists$  tube  $w \times Y$  s.t.  $\{x_0\} \times Y \subseteq w \times Y \subseteq N$ .

Thus, for each  $x \in X$ ,  $\exists w_x$  s.t.  $w_x \times Y$  is covered by finitely many elements of  $\mathcal{d}$ . By compactness of  $X$ ,  $X$  is covered by finitely many  $w_{x_1}, \dots, w_{x_n}$ . Each corresponding tube is covered by finitely many elements of  $\mathcal{A}$ . □

Thm 4 (Tube Lemma) Let  $Y$  be compact and  $x_0 \in X$ . Let  $N \subseteq X \times Y$  be open such that  $\{x_0\} \times Y \subseteq N$ . Then,  $\exists$  open  $w \subseteq X$  s.t.  $\{x_0\} \times Y \subseteq w \times Y \subseteq N$ .

Proof. Step 1 of earlier. □

Remark. Compactness of  $Y$  is needed.

Take  $X = Y = \mathbb{R}$  and  $0 \in X$  and

$$N = \left\{ (x, y) \in \mathbb{R}^2 : |x| < \frac{1}{1+y^2} \right\}.$$

No tube exists!



# Lecture 11 (11-02-2021)

11 February 2021 15:36

Defn. A collection  $\mathcal{C}$  of subsets of  $X$  is said to satisfy the finite intersection condition if for every finite subcollection  $\{C_1, \dots, C_n\}$  of  $\mathcal{C}$ , the intersection  $C_1 \cap \dots \cap C_n$  is non-empty.

Thm 1. Let  $X$  be a topological space. Then  $X$  is compact iff every collection  $\mathcal{C}$  of closed sets in  $X$  satisfying the finite intersection condition satisfies  $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$ .

Proof. For  $\mathcal{C}$ , define  $\mathcal{U}_c = \{X \setminus C : C \in \mathcal{C}\}$ .

$\mathcal{C} \rightarrow$  closed sets,  $\mathcal{U}_c \rightarrow$  open sets

$\bigcap_{C \in \mathcal{C}} C = \emptyset \Leftrightarrow \mathcal{U}_c$  is an open cover.

$\mathcal{C}$  has finite inter. property  $\Leftrightarrow$  no finite subcollection of  $\mathcal{U}_c$  covers  $X$ .

Conclude!



Cor 2. If  $X$  is compact and  $X \supset C_1 \supset C_2 \supset \dots$  with  $C_n \neq \emptyset$  for all  $n \in \mathbb{N}$ , then  $\bigcap_{n \geq 1} C_n \neq \emptyset$ .

Proof.  $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$  satisfies finite intersection property.

Defn. (Limit point compact) A space  $X$  is said to be limit point compact if every infinite subset of  $X$  has a limit point.

Thm 3. Compactness  $\Rightarrow$  Limit point compactness.

Proof. Let  $X$  be compact and  $A \subset X$  be s.t.  $A$  has no limit point. We show  $A$  is finite.

Since  $A' = \emptyset$ ,  $\bar{A} = A$ ; i.e.,  $A$  is closed.

For each  $a \in A$ , we can choose an open nbhd  $U_a$  of  $a$  that does not intersect  $A \setminus \{a\}$ .

(Since  $a$  is not a lt. point of  $A$ )

Note that  $\{U_a : a \in A\}$  covers  $A$ . Since  $A$  is closed in  $X$ ,

$A$  is compact. Thus, it has a finite subcover.

But  $(U_a \cap A) = \emptyset \quad \forall a \in A$ , we see that  $A$  is finite.  $\square$

Remark. The converse of the above is not true.

Consider any set  $Y$  with two points and give it the indiscrete topology.  $\{\bar{0}, \bar{1}\}$

Consider  $X = \mathbb{N} \times Y$  in product topology.

Then, any non-empty subset of  $X$  has a lt. point.

(A basis of  $X$  is  $\{\{n\} \times Y : n \in \mathbb{N}\}$ . Thus, given any  $\emptyset \neq A \subseteq X$ , pick  $(n, x) \in A$ . Then,  $(n, 1-x)$  is in any nbhd of  $(n, x)$ .)

However,  $X$  is NOT compact. We have

$$X = \bigcup_{n \in \mathbb{N}} \{n\} \times Y. \quad \square$$

Def<sup>n</sup>. (Sequentially compact) A space  $X$  is said to be

sequentially compact if every sequence has a convergent subsequence.

Thm 1. Let  $X$  be a metrisable space. TFAE:

- (i)  $X$  is compact.
- (ii)  $X$  is limit point compact.
- (iii)  $X$  is sequentially compact.

Proof

(i)  $\Rightarrow$  (ii) Previous theorem.

(ii)  $\Rightarrow$  (iii)

Let  $(a_n)$  be a sequence in  $X$ . If  $(a_n)$  has a constant subsequence, we are done. Thus, assume not. Then,  $A = \{a_n : n \in \mathbb{N}\}$  is infinite and hence, has a limit point  $x$ .

Pick an element in  $A \cap B(x, 1)$ . It is of the form  $a_{k_i}$  for some  $k_i \in \mathbb{N}$ .

Assume we have chosen  $k_1 < k_2 < \dots < k_n$  s.t.

$$a_{k_i} \in B(x, 1_i) \cap A \quad \forall i = 1, \dots, n.$$

Now,  $B(x, 1_{n+1}) \cap A$  is infinite.

Thus, can choose  $k_{n+1} > k_n$  s.t.  $a_{k_{n+1}} \in B(x, 1_{n+1}) \cap A$ .

Then,  $a_{n_k} \rightarrow x$  as  $k \rightarrow \infty$ .

(iii)  $\Rightarrow$  (i) ① We show the following statement:

Let  $\mathcal{A}$  be an open cover of  $X$ . Then  $\exists \delta > 0$  s.t. for each subset of  $X$  having diameter less than  $\delta$ , there is an element of  $\mathcal{A}$  containing it.

$\rightarrow$  Let  $\mathcal{A}$  be an open cover for which no such  $\delta$  exists.

Taking  $\delta = 1_n$ , we get sets  $B_n$  such that

$$\text{diam}(B_n) < \delta \quad \text{and} \quad B_n \not\subset A \quad \forall A \in \mathcal{A}.$$

Choose an  $x_n \in B_n \quad \forall n$ . Then  $(x_n)$  has a convergent subsequence  $(x_{n_k})$ . Let  $x \in X$  be the limit. Choose  $\epsilon > 0$  s.t.  $B(x, \epsilon) \subset A$ .

Eventually,  $x_{n_k} \in B(x, \epsilon/2)$ .

By choose  $k$  large enough,  $B_{n_k} \subset B(x, \epsilon) \subset A$ .  $\rightarrow \leftarrow$

② We show the following:

For every  $\epsilon > 0$ ,  $\exists$  a finite covering of  $X$  by  $\epsilon$ -balls.

→ Assume  $\exists \epsilon > 0$  for which the above does not hold.

Choose  $x_1 \in X$  arbitrarily. Then,  $B(x_1, \epsilon) \neq X$ .

Choose  $x_2 \in X - B(x_1, \epsilon)$ .

Inductively, we can choose

$$x_{n+1} \in X - \bigcup_{i=1}^n B(x_i, \epsilon).$$

Then,  $(x_n)_{n=1}^\infty$  has a convergent subsequence  $(x_{n_k})$ .

But  $d(x_i, x_j) \geq \epsilon \quad \forall i \neq j \quad \therefore \text{No subseq. can conv.} \rightarrow \epsilon$

③ Now, we show  $X$  is compact.

Let  $\mathcal{A}$  be an open cover of  $X$ . Choose  $\delta > 0$

such that ① holds.

By ②,  $X$  can be covered by finitely many  $\delta/3$ -balls.

Since each have diameter  $\leq 2\delta/3$ , each of them lie

in an element of  $\mathcal{A}$ . Choose one such element for  
each ball (of which there are finitely many). These

elements form a finite subcover. □

Remark:  $\delta$  is called the Lebesgue number of  $\mathcal{A}$ .

# Lecture 12 (18-02-2021)

18 February 2021 15:36

Defn. Let  $p: X \rightarrow Y$  be a surjective function.

The map is said to be a quotient map if any

$U \subseteq Y$  is open iff  $p^{-1}(U) \subseteq X$  is open.

Can replace "open" with "closed" since  $p^{-1}(U^c) = (p^{-1}(U))^c$ .

Remarks (1) A quotient map is continuous. (quotient map)

(2) It need not be bijective.

(3) A homeomorphism is a quotient map.

(4) Quotient + Injective  $\Rightarrow$  Homeomorphism

(5) Surj. + open map  $\Rightarrow$  Quotient ( $\Leftarrow$  not true!)

(6) Surj. + closed map  $\Rightarrow$  Quotient ( $\Leftarrow$  not true!)

Defn. A subset  $C \subseteq X$  is said to be saturated w.r.t.  $p$  if

$$p^{-1}(\{y\}) \cap C \neq \emptyset \Rightarrow p^{-1}(\{y\}) \subseteq C.$$

(Saturated)

(That is, if  $C$  contains one pre-image, it contains all.)

Remark. Thus,  $p$  is a quotient map iff  $p$  is a continuous surjection that maps open saturated sets to open sets.  
(Or "closed" instead of "open")

Example. ① Let  $X = [0, 1] \cup [2, 3] \stackrel{\text{Euclidean}}{\sim}$  and  $Y = [0, 2] \stackrel{\text{Euclidean}}{\sim}$

Define  $p: X \rightarrow Y$  by

$$p(x) = \begin{cases} x & ; x \in [0, 1] \\ x-1 & ; x \in [2, 3] \end{cases}$$

$p$  is continuous and surjective. Moreover, it is closed because  $X$  is compact and  $Y$  Hausdorff.

Thus,  $p$  is a quotient map.

(Not homeomorphism since  $p(1) = p(2)$ .)

However,  $p$  is not open.  $[0, 1] \subset X$  is open but

$$p([0,1]) = [0,1] \subset Y \text{ is } \underline{\text{NOT}} \text{ open.}$$

(Note that  $[0,1]$  is NOT saturated since  $p^{-1}(\{1\}) \cap [0,1] \neq \emptyset$   
 but  $p^{-1}(\{1\}) \not\subset [0,1]$ .)

② Let  $A = [0,1] \cup [2,3] \subseteq X$ .

Define  $g: A \rightarrow Y$  by  $g = p|_A$ .

Then,  $g$  is a bijection and thus, every subset is saturated. However,  $[2,3]$  is open in  $X$  but  $g([2,3])$  is not open in  $Y$ . (Note  $g$  is continuous!)

Defn. let  $X$  be a topological space  $A$  a set. let  $p: X \rightarrow A$  be a surjective function. Then, there exists a unique topology on  $A$  which  $p$ , a quotient map. This is called the quotient topology on  $A$ .

(Quotient topology)

Proof:

Let  $\mathcal{T} = \{U \subset A : p^{-1}(U) \text{ is open in } X\}$ .

T.S.T.  $\mathcal{T}$  is a topology.

$$\textcircled{1} \quad \emptyset = p^{-1}(\emptyset) \text{ and } A = p^{-1}(X) \therefore \emptyset, A \in \mathcal{T}$$

$$\textcircled{2} \quad p^{-1}\left(\bigcup_{\alpha \in \Lambda} U_\alpha\right) = \bigcup_{\alpha \in \Lambda} p^{-1}(U_\alpha) \text{ and}$$

$$\textcircled{3} \quad p^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n p^{-1}(U_i) \text{ show}$$

closure under finite intersection and arbitrary union.

Uniqueness is clear. That  $p: X \rightarrow A$  is a quotient map is also clear.

Defn. Let  $X$  be a topological space and  $X^*$  a partition of  $X$ . Let  $p: X \rightarrow X^*$  be the natural projection map. (This is surjective) The space  $X^*$

with the quotient topology induced by  $p$  is called a quotient space of  $X$ .

Recall that partitions of a set are equivalent to (no pun intended) an equivalence relation  $\sim$ .

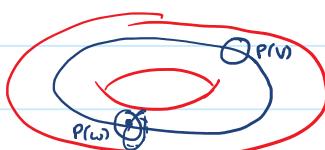
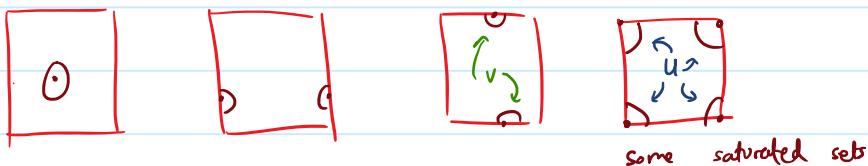
A subset  $U \subseteq X^*$  is a collection of equivalence classes and  $p^{-1}(U) \subseteq X$  is simply the union of those equivalence classes.

Example. ③ Let  $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$  be the closed unit disc in  $\mathbb{R}^2$ .

Let  $X^* = \{\{(x, y)\} : x^2 + y^2 < 1\} \cup \{S^1\}$ .  
 ( $\forall \{(x, y)\}$  for  $(x, y) \in X^\circ$  and  $S^1$ )

④  $X = [0, 1] \times [0, 1]$ .

$$\begin{aligned} X^* = & \left\{ \{(x, y)\} : (x, y) \in \overset{\circ}{X} \right\} \cup \\ & \left\{ \{(0, y), (1, y)\} : 0 < y < 1 \right\} \cup \\ & \left\{ \{(x, 0), (x, 1)\} : 0 < x < 1 \right\} \cup \\ & \left\{ \{(0, 0), (1, 0), (1, 1), (0, 1)\} \right\}. \end{aligned}$$



Thm. Let  $p: X \rightarrow Y$  be a quotient map.

Let  $A$  be a subspace which is saturated w.r.t.  $p$ .

Let  $g: A \rightarrow p(A)$  be the restriction  $p|_A$ .

Then,

(i) If  $A$  is either open or closed, then  $g$  is

a quotient map.

(2) If  $p$  is either open or closed, then  $q$  is a quotient map.

Proof

Step 1

Claim. (a)  $V \subset p(A) \Rightarrow q^{-1}(V) = p^{-1}(V)$  and

(b)  $p(V \cap A) = p(V) \cap p(A)$  if  $V \subseteq X$ .

Proof. (a) ( $\subseteq$ ) Let  $x \in q^{-1}(V)$ .

Then  $q(x) \in V \subset p(A)$ .

$\therefore q(x) \in p(A)$ . Thus,  $q(x) = p(a)$  for some  $a \in A$ .

Since  $a$  is saturated,  $x \in p^{-1}(\{p(a)\}) \subset A$ .

( $\supseteq$ ) Let  $x \in p^{-1}(V)$ . Then  $p(x) \in V \subset p(A)$ . Same argument again.

(b)  $p(V \cap A) \subset p(V) \cap p(A)$  is true in general.

( $\supseteq$ ) Let  $y \in p(V) \cap p(A)$ .

Then,  $y = p(u) = p(a)$  for some  $a \in A$  and  $u \in V$ .

$a \in p^{-1}\{y\} \cap A$ .

$\therefore p^{-1}\{y\} \subset A$  and hence,  $u \in A$ .

$\therefore u \in V \cap A$ .

Step 2. ① Suppose  $A$  is open in  $X$ .

Claim.  $q$  is a quotient map.

Proof. Let  $V \subset p(A)$ .

Suppose  $q^{-1}(V)$  is open in  $A$ .  $\Rightarrow A$  is open in  $X$ .

Then,  $q^{-1}(V)$  is open in  $X$ .  $\Rightarrow$  (a) of Step 1.

Then,  $p^{-1}(V)$  is open in  $X$ .

Then,  $V$  is open in  $Y$  since  $p$  is quotient.

In particular,  $V$  is open in  $p(A)$ .  $\square$

② Suppose  $p$  is open.

Claim.  $q$  is a quotient map.

Proof. Let  $V \subseteq p(A)$  be s.t.  $q^{-1}(V)$  is open (in A).

Then,  $p^{-1}(V) = q^{-1}(V)$  is open.

$p^{-1}(V)$  is open in A and hence,

$$p^{-1}(V) = U \cap A \text{ for } U \subseteq X \text{ open.}$$

$$\begin{aligned} \xrightarrow{\text{onto}} & p(p^{-1}(V)) = p(U \cap A) \xrightarrow{\text{b)} \text{from Step 1.}} \\ \Rightarrow & V = p(U) \cap p(A) \end{aligned}$$

But  $U$  is open and  $p$  "an open map".

Thus,  $V$  is open in  $p(A)$ .

Thus,  $q$  is a quotient map.

Step 3. Do the same as prev. step by replacing

"open" with "closed."

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## Lecture 13

21 February 2021 11:19

Prop<sup>n</sup> 1 Let  $p: X \rightarrow Y$ ,  $q: Y \rightarrow Z$  be quotient maps.

Then  $q \circ p: X \rightarrow Z$  is a quotient map.

Proof. Clearly,  $q \circ p$  is surjective and continuous.

Let  $U \subseteq Z$  be s.t.  $(q \circ p)^{-1}(U)$  is open.

That is,  $p^{-1}(q^{-1}(U))$  is open.

Since  $p$  is quotient,  $q^{-1}(U)$  is open. Since  $q$  is quotient,  $U$  is open.  $\square$

Thm 2. Let  $p: X \rightarrow Y$  be a quotient map. Let  $Z$  be a topological space. Let  $g: X \rightarrow Z$  be a map such that  $g$  is constant on each  $p^{-1}(\{y\})$  for  $y \in Y$ .

In other words,  $p(x) = p(x') \Rightarrow g(x) = g(x')$ .

Let us refer to this as " $g$  respects  $p$ ".

Then,  $g$  induces a map  $f: Y \rightarrow Z$  s.t.

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ p \downarrow & \dashv f & f \circ p = g \quad \text{and} \\ Y & & \end{array}$$

(i)  $f$  is continuous iff  $g$  is continuous.  
(ii)  $f$  is a quotient map iff  $g$  is a quotient map.

Proof

Since  $g$  respects  $p$  and  $p$  is onto, we get a unique well-defined map  $f: Y \rightarrow Z$  defined by  $f(p(x)) = g(x)$ .

(Since each  $y \in Y$  is of the form  $p(x)$  and if  $p(x) = p(x')$ , then  $g(x) = g(x')$ .)

(i) If  $f$  is continuous, then  $g = f \circ p$  is continuous, being the composition of continuous maps.

Conversely, suppose  $g$  is continuous.

Let  $U \subseteq Z$  be open.

Then,  $p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U) = g^{-1}(U)$  is open

since  $g$  is cont. But  $p$  is quotient. Thus,  $f^{-1}(U)$  is open. Hence,  $f$  is continuous.

(ii) If  $f$  is quotient, then  $g = f \circ p$  is, by Prop 1

Conversely, let  $g$  be a quotient map.

$\therefore g$  is onto and hence, so is  $f$ .

Also,  $g$  is continuous and hence, so is  $f$ , by (i).

Now, let  $U \subseteq Z$  be s.t.  $f^{-1}(U)$  is open.

Is:  $U$  is open.  $\xrightarrow{p \text{ cont.}}$

Note  $f^{-1}(U)$  open  $\xrightarrow{p \text{ cont.}} p^{-1}(f^{-1}(U))$  is open  $\xrightarrow{g = f \circ p} g^{-1}(U)$  is open  $\xrightarrow{g \text{ quotient}} U$  is open  $\square$

Cor 3. Let  $g: X \rightarrow Z$  be a surjective continuous map

Let  $X^*$  be the partition induced by

the equivalence relation  $\sim$  on  $X$  given by

$$x \sim x' \Leftrightarrow g(x) = g(x').$$

$$(X^* = \{g^{-1}(\{z\}) : z \in Z\})$$

Consider  $X^*$  with the quotient topology induced by the natural  $p: X \rightarrow X^*$ .

(a)  $g$  induces a bijective continuous map  $f: X^* \rightarrow Z$ , which is a homeomorphism iff  $g$  is a quotient map.

(b) If  $Z$  is Hausdorff, then so is  $X^*$ .

Proof.

Here,  $Y = X^*$  and  $p$  is the canonical map.

By construction,  $f$  is bijective. (It was already onto, it is 1-1, since  $X^*$  is precisely the partition based on fibers of  $g$ .)

By the previous theorem,  $f: X^* \rightarrow Z$  is continuous and bijective.

(a) Now,  $f$  is a homeo  $\Leftrightarrow f$  is quotient  $\Downarrow$  Thm 2 (ii)  
 $\Downarrow$   
 $g$  is quotient

(b) Let  $Z$  be Hausdorff.

Let  $x, y \in X^*$  be s.t.  $x \neq y$ . Since  $f$  is 1-1,  $f(x) \neq f(y)$  in  $Z$ .

$\therefore \exists U \ni f(x), V \ni f(y)$  open s.t.  $U \cap V = \emptyset$ .

Then,  $f^{-1}(U) \cap f^{-1}(V) = \emptyset$  and are

neighborhoods of  $x$  and  $y$ , resp.

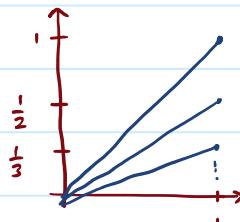
Example

$$\text{Let } X = \bigcup_{n \in \mathbb{N}} ([0, 1] \times \{n\}) \subseteq \mathbb{R}^2$$



be in subspace topology and let

$$Z = \left\{ (x, \frac{x}{n}) : x \in [0, 1], n \in \mathbb{N} \right\} \subseteq \mathbb{R}^2$$



also be in subspace top.

Define  $g: X \rightarrow Z$  by

$$g(x, n) = \left( x, \frac{x}{n} \right).$$

Now, if  $z \in Z \setminus \{(0, 0)\}$ , then  $g^{-1}(\{z\})$  is

a singleton. But if  $z = (0, 0) \in \mathbb{Z}$ , then

$$g^*(\{z\}) = \{(0, n) : n \in \mathbb{N}\}.$$

Now, take  $X^* = \{g^*(\{z\}) : z \in \mathbb{Z}\}$ . give quotient topology

By the earlier, we have a bijective continuous map

$$f: X^* \longrightarrow \mathbb{Z}.$$

Q. Is  $f$  a homeomorphism?

A. No.

Consider the set  $A = \left\{ \left( \frac{1}{n}, n \right) \in X : n \in \mathbb{Z} \right\}$ .

$A$  is closed since  $A' = \emptyset$ . Moreover  $A$  is saturated w.r.t.  $g$ . However,

$$g(A) = \left\{ \left( \frac{1}{n}, \frac{1}{n^2} \right) : n \in \mathbb{N} \right\}$$

does have a limit point outside  $g(A)$ .

Thus,  $g(A)$  is not closed. Thus,  $g$  is not a quotient map and hence,  $f$  is not a homeo.

Q

## Lecture 14 (03-03-2021, 04-03-2021)

03 March 2021 16:10

Def<sup>n</sup> A space is said to be second countable if it has a countable basis.  
(second countable)

Remark: Metric spaces need not be second countable. (They are first countable, though.) Take  $X$  uncountable with discrete.

Ex.  $\mathbb{R}$  with standard topology.  $\{(a, b) : a, b \in \mathbb{Q}\}$  is a countable basis.  
Even  $\mathbb{R}^n$ .  $\{(a_1, b_1) \times \dots \times (a_n, b_n) : a_i, b_i \in \mathbb{Q}\}$  is —.  
The space  $\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} \mathbb{R}$  (in prod. top.) is also second countable.

↪ A countable basis:  $\left\{ \prod_{n \in \mathbb{N}} U_n : U_n = (a_n, b_n) \text{ for } a_n, b_n \in \mathbb{Q} \text{ for } n \text{ many } n. U_n = \mathbb{R} \text{ else.} \right\}$

Def<sup>n</sup>:  $A \subset X$  is said to be dense in  $X$  if  $\bar{A} = X$ . (dense)

Def<sup>n</sup>: (a) A space for which every open covering contains a countable subcovering is called a Lindelöf space. (Lindelof, Lindelöf)

(b) A space having a countable dense subset is said to be separable.  
(separable)

Thm 1: Let  $X$  be second countable.

Then,

- (a)  $X$  is Lindelöf.
- (b)  $X$  is separable.

Proof: Let  $\{B_n : n \in \mathbb{N}\}$  be a countable basis for  $X$ .

- (a) Let  $\mathcal{C}$  be an open covering of  $X$ .

Let  $I = \{n \in \mathbb{N} : B_n \subset C \text{ for some } C \in \mathcal{C}\}$ .

For each  $n \in I$ , pick some  $C \in \mathcal{C}$  s.t.  $B_n \subset C$  and call it  $C_n$ . (Such a  $C$  exists by choice of  $I$ .)

Let  $\mathcal{C}' = \{C_n : n \in I\}$ . Clearly  $\mathcal{C}'$  is countable.

Claim:  $\mathcal{C}'$  covers  $X$ .

Proof: Let  $x \in X$ . Let  $C \in \mathcal{C}$  be an element s.t.  $x \in C$ .  $\because C$  is open and  $\{B_n\}_{n \in \mathbb{N}}$  a basis,  $\exists n \in \mathbb{N}$  s.t.  $x \in B_n \subseteq C \therefore n \in I$ . Thus,  $x \in C_n \in \mathcal{C}'$ .  $\square$

(b) WLOG, assume  $B_n \neq \emptyset \forall n$ .

For each  $B_n$ , pick some  $x_n \in B_n$ .

Let  $D = \{x_n : n \in \mathbb{N}\}$ .

Claim:  $D$  is dense in  $X$ .

Proof: Let  $x \in X$ . So,  $\exists$  a basis elt.  $B$ , containing  $x$ . But  $B \cap D \neq \emptyset$ .  $\square \quad \square$

Example:  $\mathbb{R}_e$  is first countable because for each  $x \in \mathbb{R}_e$ , there is a countable basis  $\{[x, x+y_n) : n \in \mathbb{N}\}$ .

$\mathbb{R}_e = \overline{\mathbb{Q}}$  and thus, separable.

We now see that  $\mathbb{R}_e$  is not second countable.

Let  $\mathcal{B}$  be a basis of  $\mathbb{R}_e$ . We show  $\mathcal{B}$  is uncountable.

Choose for each  $x \in \mathbb{R}_e$ , an element  $B_x \in \mathcal{B}$  s.t.

$x \in B_x \subseteq [x, x+1)$ . Because  $x = \inf B_x$ , we

see that  $B_x \neq B_y$  for  $x \neq y$ . Thus,  $\mathcal{B}$  is uncountable.

Remark:  $\mathbb{R}_e$  is Lindelöf.

Second countable  $\Rightarrow$  First countable, in general.

Prop 2. Every separable metrisable space is second countable.

Proof. Let  $D \subseteq X$  be a countable dense subset.

Consider the countable set

$$\mathcal{B} = \{ B(d, r) : d \in D, r \in \mathbb{Q}^+ \}.$$

Claim.  $\mathcal{B}$  is a basis.

Proof. Let  $x \in X$  and let  $U \ni x$  be open.

Let  $\epsilon > 0$  be s.t.  $B(x, \epsilon) \subseteq U$ .

To show:  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq U$ .

Now, pick  $a \in D \cap B(x, \epsilon/3)$ .

Pick  $r \in \mathbb{Q}^+$  s.t.  $d(a, x) < r < \epsilon/3$ .

(note  $d(a, x) < \epsilon/3$  since  $a \in B(x, \epsilon/3)$ )



Then,  $x \in B(a, r) \subseteq B(x, 2\epsilon/3) \subseteq U$ .  $\square$

Ex 3.  $\mathbb{R}_e$  is not metrisable.

Proof.  $\mathbb{R}_e$  is separable but not second countable.  $\square$

Defn. Let  $X$  be a space s.t.  $\{x\}$  is closed for all  $x \in X$ .

①  $X$  is said to be **regular** if for every pair  $(x, B)$  with  $x \in X \setminus B$  and  $B \subseteq X$  closed, there exist disjoint open sets containing  $x$  and  $B$ .

② The space  $X$  is said to be **normal** if for each pair  $(A, B)$  of closed subsets  $A, B \subseteq X$ ,  $\exists$  disjoint open sets containing  $A$  and  $B$ .

Remark. Normal  $\Rightarrow$  Regular  $\Rightarrow$  Hausdorff.

Lemma 4. Let  $X$  be a topological space s.t. singletons are closed.

(a)  $X$  is regular iff given any point  $x$  and open  $U \ni x$ , then  $\exists V \ni x$  open s.t.  $\overline{V} \subseteq U$ .

(b)  $X$  is normal iff given any closed set  $C$  and open  $U \ni c$ ,

Then  $\exists V$  open s.t.  $V \subset U$ .

Proof. (a)  $\Rightarrow$  Assume  $X$  regular.

Let  $x \in X$  and  $U \ni x$  be open.

Consider  $B = X \setminus U$ . Then,  $B$  is closed.

$\therefore \exists V^x, V^{2B}$  open s.t.  $V \cap V' = \emptyset$ .

Thus,  $V \subseteq V^c \leftarrow$  closed and thus,  $\bar{V} \subseteq V^c \subseteq B^c = U$ .

( $\Leftarrow$ ) Now, let  $x \in X$  and  $B \ni x$  be closed.

Let  $U = X \setminus B$ . Then,  $x \in U$ .

$\therefore \exists V \ni x$  open s.t.  $\bar{V} \subset U$ .

Now,  $X \setminus \bar{V} \supset X \setminus U = B$ .

Then,  $V$  and  $X \setminus \bar{V}$  are sets which prove regular!

(b) The same type of arguments work. □

Thm 5. Let  $\{X_\alpha : \alpha \in I\}$  be an indexed family of spaces.

Let  $A_\alpha \subset X_\alpha$  for each  $\alpha \in I$ .

Let  $\prod X_\alpha$  be equipped by either of product or box topology. Then,

$$\prod_{\alpha \in I} \overline{A_\alpha} = \overline{\prod_{\alpha \in I} A_\alpha}.$$

Proof ( $\subseteq$ ) Let  $x = (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \overline{A_\alpha}$ . ( $x_\alpha \in \overline{A_\alpha} \forall \alpha$ )

Let  $U = \prod_{\alpha \in I} U_\alpha$  be a basis for box/prod top.

Then, each  $U_\alpha \subset X_\alpha$  is open. Then, we get

$A_\alpha \cap U_\alpha \neq \emptyset \forall \alpha$  since  $x_\alpha \in U_\alpha$  and  $x_\alpha \in \overline{A_\alpha}$ .

$$\Rightarrow U \cap (\prod_{\alpha \in I} A_\alpha) \neq \emptyset$$

$$\therefore x \in \overline{\prod_{\alpha \in I} A_\alpha}.$$

( $\supseteq$ ) Let  $x \in \overline{\prod_{\alpha \in I} A_\alpha}$ .

Fix  $\alpha_0 \in I$ . We show  $x_{\alpha_0} \in \overline{A_{\alpha_0}}$ .

Let  $U_{\alpha_0} \subset X_{\alpha_0}$  be open. Consider  $U = \prod_{\substack{\alpha \in I \\ \alpha \neq \alpha_0}} X_{\alpha} \times U_{\alpha_0}$ .

Then,  $U$  is open in both topologies and contains  $x$ .

Thus,  $U \cap \prod A_{\alpha} \neq \emptyset$ .

$$\Rightarrow U_{\alpha} \cap A_{\alpha} \neq \emptyset \quad \forall \alpha$$

$$\Rightarrow U_{\alpha_0} \cap A_{\alpha_0} \neq \emptyset. \quad \square$$

# Lecture 15 (05-03-2021)

05 March 2021 21:17

Thm 1. A subspace  $Y$  of a regular space  $X$  is regular.

A product of regular spaces is regular.

Proof. Clearly, one point sets are closed in  $Y$ .

Let  $x \in Y$  and  $B \subseteq Y$  be closed with  $x \notin B$ .

Then,  $\bar{B} \cap Y = B$ .

$\hookrightarrow$  closure in  $X$        $\hookrightarrow$  closure in  $Y$

Since  $x \notin B$  and  $x \in Y$ , we see that  $x \notin \bar{B}$ .

Use regularity of  $X$ ,  $\exists U, V$  open disjoint nbds of  $x$  and  $\bar{B}$ .

Then,  $U \cap Y$  and  $V \cap Y$  are the required open sets in  $Y$ .

Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of regular spaces.

Note that each  $X_\alpha$  is then Hausdorff.

Thus,  $\prod X_\alpha$  is Hausdorff. Thus, singletons are closed.

Let  $x = (x_\alpha)_{\alpha \in J} \in X$  and let  $U$  be a basis elt of  $x$  in  $X$ .

We will use lemmas 4 and 5 from last lecture to show that

$\exists V$  open s.t.  $x \in V \subset \overline{U} \subset U$  and then use  $\prod V_\alpha = \overline{\prod U_\alpha}$ .

Write  $U = \prod U_\alpha$ . Use regularity for each  $\alpha$  to get

$\forall \alpha \exists x_\alpha$  s.t.  $x_\alpha \in V_\alpha \subseteq \overline{U}_\alpha \subseteq U_\alpha$ .

If  $U_\alpha = X_\alpha$ , then take  $V_\alpha = X_\alpha$  instead.

Then,  $\prod V_\alpha = V$  is an open bbd of  $x$ . Moreover,

$$\overline{V} = \overline{\prod V_\alpha} = \prod \overline{V_\alpha} \subseteq \prod U_\alpha = U. \quad \square$$

Example. (i) The space  $\mathbb{R}_K$  is Hausdorff but no regular.

( $\mathbb{R}$  Hausdorff  $\Rightarrow \mathbb{R}_K$  Hausdorff)

But consider  $x = 0$  and  $K = \{1, 1/2, \dots\}$ .

Note  $K$  is closed in  $\mathbb{R}_K$ .

Let us assume  $\mathbb{R}_K$  is regular.

Let  $U \ni x$  be a basis elt and  $V \supseteq K$  be open disjoint from  $U$ .

Note  $U$  must be of the form  $(a, b) \setminus K$ .

Choose  $n \in \mathbb{N}$  s.t.  $\frac{1}{n} \in (a, b)$ .

Then, choose a basis element  $(c, d) \subset V$  containing  $y_n$ .

Pick  $\epsilon < \frac{1}{n}$  s.t.  $\epsilon > \max\{c, y_{n+1}\}$ . Then,  $\epsilon \in U \cap V \rightarrow$

(2) We show  $\mathbb{R}_e$  is normal (and hence, regular).

Note singletons are closed since they are closed in  $\mathbb{R}$ .

Now, let  $A$  and  $B$  disjoint in  $\mathbb{R}_e$ .

For each  $a \in A$ , choose  $U_a = [a, x_a)$  s.t.  $U_a \cap B = \emptyset$ .

<sup>why</sup> take  $V_b = [b, x_b)$  for each  $b \in B$  s.t.  $V_b \cap A = \emptyset$ .

Put  $U = \bigcup_{a \in A} [a, x_a)$  and  $V = \bigcup_{b \in B} [b, x_b)$ .

Claim:  $[a, x_a) \cap [b, x_b) = \emptyset \quad \forall a \in A \quad \forall b \in B$

Proof. wlog  $a \leq b$ . Then  $b < x_a$ .

But then  $b \in [a, x_a) \cap B$ .

Thus,  $U \cap V$  is empty.

□

# Lecture 16 (08-03-2021)

08 March 2021 15:38

- Recall : Normal  $\Rightarrow$  Regular  $\Rightarrow$  Hausdorff  
 $\nLeftarrow$   $\nLeftarrow$   
 will see now:

Def.  $\mathbb{R}_e \times \mathbb{R}_e$  is called the Sorgenfrey plane. (Sorgenfrey plane)

- Note  $\mathbb{R}_e^2$  is regular since product of reg. spaces is regular.  
 We will show it is not normal. Recall  $\mathbb{R}_e$  was normal.  
 Thus, product of normal spaces needn't be normal. Moreover,  
 regular  $\not\Rightarrow$  normal

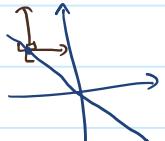
Thm.  $\mathbb{R}_e^2$  is not normal.

Proof. Assume  $\mathbb{R}_e^2$  is normal. Consider  $L = \{(x, -x) : x \in \mathbb{R}\}$ .

$L$  is closed in  $\mathbb{R}^2$  and hence, in  $\mathbb{R}_e^2$ .

The set  $[x, y) \times [-x, z)$  is open in  $\mathbb{R}_e^2$  and its intersection with  $L$  is  $\{(x, -x)\}$ .

Thus,  $L$  has discrete topology.



Thus, given any  $A \subset L$ ,  $A$  and  $L-A$  are closed in  $L$  and hence, in  $\mathbb{R}_e^2$ . (Since  $L$  is closed in  $\mathbb{R}_e^2$ .)

By (assumption of) normality of  $\mathbb{R}_e^2$ ,  $\exists U_A$  and  $V_A$  open in  $\mathbb{R}_e^2$  disjoint s.t.  $U_A \ni A$  and  $V_A \ni L-A$ .

(Fix such  $U_A$  and  $V_A$  & A.)

Let  $D = \{(x, y) \in \mathbb{R}_e^2 : x, y \in \mathbb{Q}\}$ .

On considering the basis  $\{[a, b) \times [c, d) : a < b, c < d \in \mathbb{R}\}$  of  $\mathbb{R}_e^2$ , it is clear that  $D$  is dense in  $\mathbb{R}_e^2$ .

Define  $\theta : \mathcal{P}(L) \rightarrow \mathcal{P}(D)$  by

$\begin{cases} f(x) \rightarrow \text{pw. set} \\ \text{of } x \end{cases}$

Define  $\Theta : \mathcal{P}(L) \rightarrow \mathcal{P}(D)$  by  
 $\Theta(A) = D \cap U_A$  for  $\phi \neq A \neq L$ ,  
 $\Theta(\phi) = \phi$ ,  
 $\Theta(L) = D$ .

$(\mathcal{P}(X) \rightarrow \text{pow. set})$   
 $\text{of } X$

Claim 1.  $\Theta$  is injective.

Proof. If  $\emptyset \neq A \subsetneq L$ , then  $U_A \neq \emptyset$  and denseness of  $D$  gives  $\Theta(A) \neq \emptyset$ . Moreover,  $\Theta(A) \neq D$  because  $D \cap V_A \neq \emptyset$  and thus,  $D \setminus \Theta(A) \neq \emptyset$ .

Now, suppose  $\emptyset \neq B \subsetneq L$  with  $A \neq B$ .

Wlog,  $A \setminus B \neq \emptyset$ . Pick  $x \in A \setminus B$ . Then,  $x \in L \setminus B$  and thus,  $x \in V_B \cap U_A$ .  $\leftarrow$  thus, non-empty (and open)  
 Thus,  $D \cap (V_B \cap U_A) \neq \emptyset$ .  
 $\Rightarrow \Theta(A) \cap V_B \neq \emptyset$ .  
 OTOT,  $\Theta(B) \cap V_B = \emptyset$ . Thus,  $\Theta(A) \neq \Theta(B)$ .  $\square$

However,  $|L| = |\mathbb{R}|$  and  $D$  is countably infinite.

Thus,  $|\mathcal{P}(D)| = |L| < |\mathcal{P}(L)|$ . Claim 1 is a contradiction.

Then,  $\mathbb{R}_1^2$  is not normal.  $\square$

Thm 1. Every second countable regular space is normal.

Proof. Let  $X$  be a regular space with a countable basis  $\mathcal{B}$ .

Let  $A$  and  $B$  be disjoint closed subsets of  $X$ .

Since  $B$  is closed,  $\exists$  an open nbhd  $V_x$  of each point  $x \in A$  s.t.  $U_x \cap B = \emptyset$ . Since  $X$  is regular,  $\exists$  nbhd  $V_x \ni x$  s.t.  $V_x \subset U_x$ .

So,  $\exists$  a basis element  $C_x$  of  $\mathcal{B}$  s.t.  $x \in C \subseteq V_x$ .

Now,  $A \subseteq \bigcup_{x \in A} C_x$ . Note  $\overline{C_x} \cap B = \emptyset$ .

Thus,  $\exists$  a countable covering  $\{C_n : n \in \mathbb{N}\}$  s.t.  $\overline{C_n} \cap B = \emptyset \forall n$ .  
 $\text{(by A)}$

W<sup>lly</sup>  $\{w_n : n \in \mathbb{N}\} \subset B$  s.t.  $B \subset \bigcup w_n$  &  $A \cap w_n = \emptyset \forall n$ .

(However,  $\bigcup w_n$  and  $\bigcup c_n$  need not be disjoint.)

Define  $c_n' = c_n - \bigcup_{i=1}^n \bar{w_i}$  and  $w_n' = w_n - \bigcup_{i=1}^n \bar{c_i}$ .

Clearly,  $c_n'$  and  $w_n'$  are open in  $X \quad \forall n \in \mathbb{N}$ .

Moreover,  $A \subset \bigcup_{n \in \mathbb{N}} c_n'$  since  $A \cap \bigcup_{n \geq 1} \bar{w_i} = \emptyset$ .

Similarly,  $B \subset \bigcup_{n \in \mathbb{N}} w_n'$ . However, now  $(\bigcup c_n') \cap (\bigcup w_n') = \emptyset$ .

Suppose  $x \in c_n' \cap w_m'$  for  $n, m \in \mathbb{N}$ .  
WLOG  $n \leq m$ . Then,  $w_m' = w_m - \bigcup_{i=1}^m \bar{c_i}$ .  
 $\uparrow$   
 $c_n$  appears as a subset

Thus,  $\bigcup_{n \in \mathbb{N}} c_n'$  and  $\bigcup_{n \in \mathbb{N}} w_n'$  have the desired properties.  $\square$

Thm 2. Every metrisable space is normal.

Proof. Let  $(X, d)$  be a metric space. (Singletons are closed.)

Let  $A$  and  $B$  disjoint closed subsets of  $X$ .

For each  $a \in A$ , choose  $\epsilon_a > 0$  s.t.  $B(a, \epsilon_a) \cap B = \emptyset$ .

Similarly,  $\forall b \in B$ , choose  $\epsilon_b > 0$  s.t.  $B(b, \epsilon_b) \cap A = \emptyset$ .

Define  $U = \bigcup_{a \in A} B(a, \frac{\epsilon_a}{2})$  and  $V = \bigcup_{b \in B} B(b, \frac{\epsilon_b}{2})$ .

Then,  $U$  and  $V$  are open sets containing  $A$  and  $B$ , resp.

Claim.  $U \cap V = \emptyset$ .

Proof. Suppose  $z \in U \cap V$ .

Then,  $z \in B(a, \frac{\epsilon_a}{2}) \cap B(b, \frac{\epsilon_b}{2})$  for some  $a \in A, b \in B$ .

By triangle inequality

$$d(a, b) < \frac{\epsilon_a + \epsilon_b}{2} \leq \epsilon_a. \quad (\text{wlog})$$

$$\therefore b \in B(a, \epsilon_a) \cap B \rightarrow \leftarrow.$$

Thm<sup>3</sup> Every compact Hausdorff space is normal.

Proof. Seen in tutu and midsem.  $\square$

# Lecture 17 (11-03-2021)

11 March 2021 15:35

Tm! (Urysohn Lemma) Let  $X$  be a normal and  $A, B \subset X$  be closed and disjoint. Let  $a, b \in \mathbb{R}$  with  $a < b$ . Then, there exists a continuous map

$$f: X \rightarrow [a, b]$$

such that  $f(x) = a \quad \forall x \in A$  and  $f(y) = b \quad \forall y \in B$ .

Proof. Wlog,  $[a, b] = [0, 1]$ .

Enumerate  $P = Q \cap [0, 1]$  as  $\{1, 0, x_3, x_4, \dots\}$ .

Step 1. We define open  $U_p \subset X$  for  $p \in P$  s.t.

$$\overline{U_p} \subset U_q \quad \text{whenever } p < q. \quad (*)$$

Let  $U_1 = X \setminus B$ . By normality,  $\exists U_0$  s.t.

$$A \subset U_0 \subset \overline{U_0} \subset U_1.$$

let  $P_n = \{x_1, \dots, x_n\}$ .

Suppose  $U_{x_1}, \dots, U_{x_n}$  have been defined. ( $n \geq 2$ )

We now define  $U_{x_{n+1}}$ .

$$P_{n+1} = P_n \cup \{x_{n+1}\}.$$

Since  $x_{n+1} \neq 0, 1$  it has an immediate successor and predecessor. Let  $s, p \in P_{n+1}$  be these, resp.

Then,  $\overline{U_p} \subset U_s$ . By normality,  $\exists V$  <sup>open</sup> s.t.

$$\overline{U_p} \subset V \subset \overline{V} \subset U_s.$$

Call this  $U_{x_{n+1}}$ . Then,  $(*)$  is still maintained.

Thus, we have constructed the family  $\{U_p\}_{p \in \mathbb{Q}}$  as desired.

Step 2. We now define  $U_p$  for all  $p \in \mathbb{Q}$ .

$$\text{Put } U_p = \begin{cases} \emptyset & \text{for } p < 0, \\ X & \text{for } p \geq 1. \end{cases}$$

Note that  $(*)$  still holds.

Step 3. For  $x \in X$ , define

$$Q(x) := \{p \in \mathbb{Q} : x \in U_p\}.$$

Note that  $Q(x)$  is bounded below by 0 and every rational  $> 1$  is in  $Q(x)$ .

Thus, for each  $x \in X$ ,  $\inf Q(x)$  exists and is in  $[0, 1]$ .

Then, defining  $f(x) := \inf Q(x)$  gives a map  
 $f: X \rightarrow [0, 1]$ .

We now show that  $f$  has the desired property.

- If  $x \in A$ , then  $0 \in Q(x)$ .  $\therefore f(x) = 0$ .
- If  $x \in B$ , then  $Q(x) = (1, \infty) \cap \mathbb{Q}$ .  
 $\therefore f(x) = 1$ .

Now, we prove continuity.

By  $(*)$ , ① if  $x \in \bar{U}_r$ , then  $x \in U_s \forall s > r$ .

Thus,  $f(x) \leq r$ .

$$\textcircled{1} \quad x \notin U_r \Rightarrow f(x) \geq r$$

$$\Downarrow \qquad \Uparrow \\ x \notin U_s \quad \forall s \leq r$$

Now, let  $x_0 \in X$  and  $(c, d) \ni f(x_0)$ . We show that  
 $\exists U \ni x_0$  s.t.  $f(U) \subset (c, d)$ . ( $d > 1$  or  $c < 0$  is allowed.)

<sup>open</sup> Choose rationals  $p, q$  s.t.  $c < p < f(x_0) < q < d$ .

Claim.  $U = U_q \setminus \bar{U}_p$  has the property.

Proof. Clearly,  $U$  is open in  $X$ .

Since  $p < f(x_0)$ , we have  $x_0 \notin \bar{U}_p$ .

Since  $f(x_0) < q$ , we have  $x_0 \in U_q$ .

Thus,  $x_0 \in U_q \setminus \bar{U}_p$ .

Now, let  $x \in U$ .

Then,  $f(x) \geq p$  since  $x \notin \bar{U}_p$ . Similarly

$f(x) \leq q$ .

Thus,  $f(x) \in [p, q] \subset (c, d)$ . □

The above shows that  $f$  is continuous. □

Def. If  $A, B \subset X$  are such that  $\exists$  a continuous function  
 $f: X \rightarrow [0, 1]$  s.t.  $f(A) = \{0\}$  and  $f(B) = \{1\}$ ,  
then we say that  $A$  and  $B$  can be separated by a continuous function.

Remark. UL says that disjoint closed sets in a normal space can be separated by a continuous function.

The converse is true too, as can be seen by considering  $f^{-1}[0, 1]$  and  $f^{-1}(1, 1]$ .

# Lecture 18 (15-03-2021)

15 March 2021 15:38

Thm 1

(Tietze extension theorem)

Let  $X$  be a normal space and  $A \subset X$  be closed.

(a) Any continuous function  $f: A \rightarrow [a, b]$  can be extended to  $X \rightarrow [a, b]$ .

(b) Any continuous function  $f: A \rightarrow \mathbb{R}$  can be extended to  $X \rightarrow \mathbb{R}$ .

Proof.

Step 1. If  $f: A \rightarrow [-r, r]$  is continuous, then

$\exists$  a continuous function  $g: X \rightarrow [-r, r]$  s.t.

$$|g(x)| \leq \frac{1}{3}r \quad \forall x \in X \text{ and}$$
$$|g(a) - f(a)| \leq \frac{2}{3}r \quad \forall a \in A.$$

Proof. Divide  $[-r, r]$  into the following intervals of length  $\frac{2r}{3}$ :

$$I_1 = [-r, -\frac{r}{3}], \quad I_2 = [-\frac{r}{3}, \frac{r}{3}], \quad I_3 = [\frac{r}{3}, r].$$

Put  $B = f^{-1}(I_1)$ ,  $C = f^{-1}(I_3)$ . Then,  $B$  and  $C$  are disjoint closed subsets of  $A$  and hence of  $X$ .

By Urysohn's lemma,  $\exists g: X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$  continuous

$$\text{s.t. } g(B) = \{-\frac{r}{3}\} \quad \text{and} \quad g(C) = \{\frac{r}{3}\}.$$

By construction,  $|g(x)| \leq \frac{r}{3} \quad \forall x \in X$ .

Now, let  $a \in A$ .

$$\therefore a \in A \setminus (B \cup C) \Rightarrow f(a), g(a) \in I_2 \Rightarrow |f(a) - g(a)| \leq \frac{2r}{3}.$$

$$\therefore a \in B \Rightarrow g(a) = -\frac{r}{3}, \quad f(a) \in I_1 \Rightarrow |f(a) - g(a)| \leq \frac{2r}{3}.$$

$$\therefore a \in C \Rightarrow |f(a) - g(a)| \leq 2\frac{r}{3}.$$

This finishes step 1.

Step 2. We prove (a) now.

Assume  $[a, b] = [-1, 1]$ , wlog.

By step 1,  $\exists g_1 : x \rightarrow [-\gamma_3, \gamma_3]$  s.t.

$$|g_1(a) - f(a)| \leq \frac{2}{3} \quad \forall a \in A.$$

Thus,  $f - g_1$  maps  $A$  into  $[-\frac{2}{3}, \frac{2}{3}]$ .

Use Step 1 again to get  $g_2 : x \rightarrow [-\frac{2}{3}, \frac{2}{3}]$  and so on.

We get  $g_1, \dots, g_n : x \rightarrow \mathbb{R}$  s.t.

$$|f(a) - g_1(a) - \dots - g_n(a)| \leq \left(\frac{2}{3}\right)^n \quad \forall a \in A.$$

Apply Step 1 to get  $g_{n+1} : x \rightarrow \mathbb{R}$  s.t.

$$|g_{n+1}(x)| \leq \frac{1}{3} \left(\frac{2}{3}\right)^n \quad \forall x \in X \text{ and}$$

$$|f(a) - g_1(a) - \dots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \quad \forall a \in A.$$

By induction, we get  $\{g_n\}_{n \in \mathbb{N}}$ .

Since  $\sum \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^n < \infty$ , Weierstrass M-test gives

$$s_n = \sum_{i=1}^n g_i \text{ converges uniformly.}$$

Let  $g = \lim_n s_n$ . Then,  $g$  is continuous, since  $g_i$  were. We see that  $g(a) = f(a) \quad \forall a \in A$ .

Moreover,

$$|g(x)| \leq \sum_{i=1}^{\infty} |g_i(x)| \leq \sum \left(\frac{2}{3}\right)^n \cdot \frac{1}{3} = 1.$$

$\therefore g$  maps into  $[-1, 1]$ .

Step 3. We now prove (b).

Wlog, replace  $\mathbb{R}$  with  $(-1, 1)$ . (Both are homeomorphic.)

Thus, we have  $f: A \rightarrow (-1, 1)$ .

By Step 2, we may extend it to  $g: A \rightarrow [-1, 1]$ .

Put  $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$ .

$D$  is closed in  $A$  and hence, in  $X$ .

Since  $g(A) = f(A)$ , it follows that  $D \cap A = \emptyset$ .

By Urysohn's lemma,  $\exists \phi: X \rightarrow [0, 1]$  s.t.  $\phi(D) = \{0\}$   
and  $\phi(A) = \{1\}$ .

Let  $h(x) = \phi(x) g(x)$ .

$h$  is continuous and

$$h(a) = \phi(a) g(a) = g(a) = f(a) \quad \forall a \in A$$

$$\text{and } h(d) = \phi(d) g(d) = 0 \quad \forall d \in D.$$

Thus,  $h: X \rightarrow \mathbb{R}$  maps into  $(-1, 1)$  (why?)

and agrees with  $f$  on  $A$ .

□

## Lecture 19 (18-03-2021)

18 March 2021 15:49

Recall:  $X$  is compact iff every collection  $\mathcal{C}$  of closed subsets having the finite intersection property (FIP) satisfies

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

Let  $\{X_\alpha : \alpha \in J\}$  be an arbitrary family of compact sets and  $X = \prod_{\alpha \in J} X_\alpha$  in product topology.

Let  $\mathcal{A}$  be a collection of closed subsets of  $X$  having FIP. For each  $\beta \in J$ , let  $\pi_\beta : X \rightarrow X_\beta$  denote the projection. Then  $\{\overline{\pi_\beta(A)} : A \in \mathcal{A}\}$  also has the FIP for each  $\beta \in J$ .

Since  $X_\beta$  is compact, for each  $\beta \in J$ , the set  $\bigcap_{A \in \mathcal{A}} \overline{\pi_\beta(A)}$  is non-empty.

But if we choose  $x_\beta \in \bigcap_{A \in \mathcal{A}} \overline{\pi_\beta(A)}$ , it need not be the case that

$$x = (x_\beta)_{\beta \in J} \in \bigcap_{A \in \mathcal{A}} A.$$

To deal with this, we expand  $\mathcal{A}$ . Since  $\pi_\beta$  are not closed maps, the set  $\pi_\beta(A)$  need not be closed even if  $A$  is. So we need not assume  $\mathcal{A}$  is a collection of closed sets.

Lemma 1. Let  $X$  be a set.  $\mathcal{A}$  be a collection of subsets of  $X$  having FIP. Then,  $\exists$  a collection  $\mathcal{D}$  of subsets of  $X$  s.t. (1)  $\mathcal{A} \subset \mathcal{D}$ ,  
(2)  $\mathcal{D}$  has FIP,  
(3)  $\mathcal{D}$  is maximal (w.r.t.  $\subseteq$ ) with the above properties.

Proof. We use Zorn's Lemma.

Let  $\mathcal{A} = \{B \subseteq P(X) : \emptyset \subseteq B \text{ and } B \text{ has FIP}\}.$

$\mathcal{A} \neq \emptyset$  since  $\emptyset \in \mathcal{A}.$

Now, given a chain  $B \subseteq \mathcal{A}$ , put

$$\ell = \bigcup_{B \in B} B.$$

Then,  $\emptyset \subseteq \ell$ , clearly. Moreover  $\ell$  has FIP since

given  $c_1, \dots, c_n \in \ell$ ,  $\exists B_i \in B$  containing  $c_i$  for  $i=1, \dots, n$ .

Since  $B$  is a chain,  $\exists k$  s.t.  $c_1, \dots, c_n \in B_k$ .

$\therefore B_k$  has FIP,  $c_1 \cap \dots \cap c_n \neq \emptyset$ .

$\therefore \ell \in \mathcal{A}$ . Clearly  $\ell$  is an upper bound of  $B$ .

Thus, every chain has an upper bound and thus,  $\mathcal{A}$  has a maximal element, as desired.  $\square$

Lemma 2. Let  $X$  be a set and  $\mathcal{D}$  be a collection of subsets of  $X$  that is maximal w.r.t. FIP.

Then,

- (1) Any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ .
- (2) If  $A \subseteq X$  is s.t.  $A \cap D \neq \emptyset \forall D \in \mathcal{D}$ , then  $A \in \mathcal{D}$ .

Proof.

- (1) Let  $D_1, D_2 \in \mathcal{D}$ . Put  $D = D_1 \cap D_2$ .

Claim.  $\mathcal{D} \cup \{D\}$  has FIP.

Proof. Let  $E_1, \dots, E_n \in \mathcal{D} \cup \{D\}$ .

If  $E_i \neq D \ \forall i$ , then  $E_i \in \mathcal{D} \ \forall i \ \& \ \cap E_i \neq \emptyset$ .

$\therefore$  assume  $E_i = D$ . Then,  $\cap E_i = D \cap D_2 \cap E_2 \cap \dots \cap E_n \neq \emptyset$ .

By maximality,  $D \in \mathcal{D}$ . By induction, all finite intersections are in  $\mathcal{D}$ .

- (2) Claim.  $\mathcal{D} \cup \{A\}$  has FIP.

Proof. Let  $E_1, \dots, E_n \in \mathcal{D}$ . Then,  $E_1 \cap \dots \cap E_n \in \mathcal{D}$  by earlier.

$\therefore A \cap (E_1 \cap \dots \cap E_n) \neq \emptyset$ , by assumption.  $\square$

$\therefore A \in \mathcal{Q}$ , by maximality.

(2)

# Lecture 20 (22-03-2021)

22 March 2021 21:07

Thm.

(Tychonoff Theorem) An arbitrary product of compact spaces is compact.

PF.

Let  $\{X_\alpha\}_{\alpha \in J}$  be a collection of compact spaces and put

$$X = \prod_{\alpha \in J} X_\alpha, \quad \text{in product topology.}$$

Let  $\mathcal{A}$  be any collection of  $X$  having FIP.

To show  $X$  is compact, it suffices to show that  $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$ .

By Lemma 1 (Lec 19),  $\exists \mathcal{D} \supseteq \mathcal{A}$  maximal with FIP.

Enough to show  $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$ .

Fix  $\beta \in J$ . Consider  $\pi_\beta : X \rightarrow X_\beta$ . The collection

$\{\pi_\beta(D) : D \in \mathcal{D}\}$  has FIP (had seen earlier).

Thus,  $\underline{\bigcap_{D \in \mathcal{D}}} \pi_\beta(D) : D \in \mathcal{D}$ . Now, since  $X_\beta$  is compact,

$$\bigcap_{D \in \mathcal{D}} \pi_\beta(D) \neq \emptyset.$$

Now, for each  $\beta \in J$ , we can choose  $x_\beta \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\beta(D)}$ .

$$\text{Put } x = (x_\beta)_{\beta \in J} \in X.$$

Claim.  $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$ .

Proof. Let the subbasic element  $\pi_\beta^{-1}(U_\beta)$  contain  $x$ .

Thus,  $x_\beta \in U_\beta \leftarrow \text{open. Then, } U_\beta \cap \pi_\beta(D) \neq \emptyset \text{ for any } D \in \mathcal{D}. \text{ Thus, } \exists y \in D \text{ s.t. } \pi_\beta(y) \in U_\beta \text{ or } y \in \pi_\beta^{-1}(U_\beta) \cap D.$

By Lemma 2 (Lec 19), we get  $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$ .

Again, using above lemma, we get that every basis element

belong to  $\mathcal{D}$ .

Now, given any  $D \in \mathcal{D}$  and any basis ell.  $U \ni x$ , we have  $U \in \mathcal{D}$  and hence,  $D \cap U \neq \emptyset$ , by FIP.  
Thus,  $x \in \overline{D} \wedge D \in \mathcal{D}$ .  $\square$

The claim proves the result.  $\square$

Def. (Locally compact)  $X$  is said to be locally compact at  $x \in X$  if  $\exists$  a compact neighbourhood of  $x$ . (Recall our nbs only contain an open set containing  $x$ . Not necessarily open itself.)

$X$  is said to be locally compact if  $X$  is locally compact at  $x \forall x \in X$ .

Examples. (1)  $\mathbb{R}$  is locally compact.

(2)  $\mathbb{Q}$  is not locally compact.

(3)  $\mathbb{R}^n$  is locally compact.

(4) The countable product  $\mathbb{R}^\omega$  is not locally compact.

Let  $\bar{\Omega} \in \mathbb{R}^\omega$ . Assume  $K$  is a compact nbd of  $\bar{\Omega}$ .

Then,  $\exists \epsilon_1, \dots, \epsilon_n > 0$  s.t.  $U = (-\epsilon_1, \epsilon_1) \times \dots \times (-\epsilon_n, \epsilon_n) \times \mathbb{R} \times \mathbb{R} \times \dots$

Then,  $\bar{U} \subset \bar{K} = K$  since compact is closed in  $\mathbb{R}^\omega$ .

But  $\bar{U}$  is not compact.  $\square$

Thm<sup>2</sup>. Let  $X$  be a space. Then  $X$  is a locally compact Hausdorff space iff  $\exists$  space  $Y$  s.t.

(i)  $X$  is a subspace of  $Y$ ,

(ii)  $Y - X$  is a singleton,

(iii)  $Y$  is a compact Hausdorff space.

Moreover, if there is another such  $Y'$ , then  $\exists f: Y \rightarrow Y'$  homeo s.t.  $f|_X = \text{id}_X$ .

Proof

Step 1. We show uniqueness first.

$$\text{Proof. } Y = X \cup \{p\}, \quad Y' = X \cup \{p'\}.$$

Define  $h: Y \rightarrow Y'$  by

$$x \mapsto \begin{cases} x & ; x \in X, \\ p' & ; x = p. \end{cases}$$

Clearly,  $f$  is a bijection. Suffices to show  $h$  is an open map.

If  $U \subseteq Y$  is open and  $U \subseteq X$ , then we are done.

Suppose  $p \in U$ . Now,  $C = Y \setminus U \stackrel{\subseteq X}{\subseteq}$  is closed in  $Y$  and hence, compact. Thus,  $h(C) = C$  is compact and hence, closed.

Thus,  $h(U) = Y' \setminus C$  is open. Hausdorff

This proves Step 1. B

Step 2. Suppose  $X$  is LCH.

Put  $Y = X \cup \{\infty\}$  and define a topology  $\mathcal{T}$  on  $Y$  as:

- (i)  $U$  open in  $X$  or
- (ii)  $\{Y - C : C \subseteq X \text{ compact}\}$

$$\left. \begin{array}{l} \\ \end{array} \right\} \Leftrightarrow \mathcal{T}$$

Claim.  $\mathcal{T}$  is indeed a topology.

(a)  $\emptyset \rightarrow$  type (i) open set.

$Y = Y - \emptyset$  and  $\emptyset$  compact  $\rightarrow$  type (ii)

(b)  $U_1 \cap U_2 \rightarrow$  open ✓

$$(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2) \xrightarrow{\text{compact}} \text{open} \checkmark$$

$$U \cap (Y - C) = U \cap (X - C) \xrightarrow{\text{open}} \text{type (i)} \checkmark$$

$$(c) \bigcup_{\alpha} U_{\alpha} \cup \bigcup (Y - C_{\beta}) = U \cup (Y - \bigcap_{\beta} C_{\beta})$$

$$= U \cup (Y - C) = U \cup (X - C) \xrightarrow{\text{open}} \text{open} \checkmark$$

Thus,  $\mathcal{T}$  is indeed a topology on  $Y$ .  
 $X$  does have the subspace topology, clearly.  
Moreover,  $Y - X$  is indeed a singleton.

Claim.  $Y$  is Haus.

Proof. If  $y_1 \neq y_2 \in Y$ , done.

Suppose  $x \in X$ . To show:  $\infty$  and  $\omega$  have disjoint nbds.

Since  $X$  is locally compact,  $\exists U, C$  s.t.  $x \in U \subseteq C$ .

Thus,  $Y - C$  is a nbhd of  $\infty$  disjoint from  $U \ni x$ .  $\square$

Claim.  $Y$  is compact.

Proof. Take a cover  $\{U_\alpha\}$ . Then,  $\exists x_0$  s.t.  $\infty \in U_{x_0}$ .

Thus, it contains a nbhd of the form  $Y - C$ .

Cover  $C$  by finitely many.  $\square$

We are done now.

Step 3. Given such a  $Y$ ,  $X$  is LCH. ( $H$  is clear.)

Proof.  $Y = X \cup \{\infty\}$ . Let  $x \in X$ . Let  $U \ni x, V \ni \infty$  be disjoint. Then,  $V = Y - C$  for some compact  $C$ .

Then,  $x \in U \subset C$ .  $\square$

# Lecture 21 (25-03-2021)

25 March 2021 15:30

Defn. If  $Y$  is a topological space and  $X \subsetneq Y$  a proper subspace, and  $\overline{X} = Y$ , then  $Y$  is called a **compactification** of  $X$ .

If  $Y - X$  is a singleton, then  $Y$  is called a **one point compactification** of  $X$ .

(compactification, one point compactification)

Remark. If  $X$  itself was compact + Hausdorff, then the point  $\infty$  in our construction of  $Y$  was isolated. Indeed  $Y - X = \{\infty\}$  would then be open. In particular  $\overline{X} = X \neq Y$  and thus,  $Y$  is NOT a one pt. compactification of  $X$ .

On the other hand, if  $X$  is not compact, then  $Y - X$  is not open and thus  $\overline{X} \supsetneq X$ .  $\therefore Y = \overline{X}$  and  $Y$  is the one point compactification.

( $Y$  here is  $\infty$  as in the theorem earlier.)

Recall from Real Analysis.

- Cauchy sequences
- Complete metric spaces : Every Cauchy sequence converges.

Example. ①  $\mathbb{R}^n$  is complete  $\forall n \in \mathbb{N}$ .

②  $\mathbb{Q}$  is not complete. Take  $x_n = \frac{\lfloor \sqrt{2} 10^n \rfloor}{10^n} \in \mathbb{Q}$ .

Then,  $x_n \rightarrow \sqrt{2}$  in  $\mathbb{R}$ .

Thus,  $(x_n)_n$  is Cauchy but does not converge in  $\mathbb{Q}$ .  
(Limit in  $\mathbb{R}$  is unique.)

③  $(-1, 1)$  is not complete. Take  $x_n = 1 - \frac{1}{n}$ . Then,

$$|x_n - x_m| \leq \left| \frac{1}{n} - \frac{1}{m} \right| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Interesting! ④ Consider  $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ .

$X$  is not complete w.r.t. the metric  $d(x, y) = |x - y|$ .

But it is complete w.r.t. the discrete metric  $d'$ .

However,  $d$  and  $d'$  induce the same (discrete) topology!

Thus, completeness is not preserved by homeomorphism.

Also recall:

Lemma 1. A metric space  $X$  is complete if every Cauchy sequence in  $X$  has a convergent subsequence.

(Show that the limit of the subsequence is of the sequence)

Corollary 2. A compact metric space is complete.

Def'n. A metric space  $(X, d)$  is called totally bounded if for every  $\epsilon > 0$ ,  $\exists$  a finite covering of  $X$  by  $\epsilon$  balls.

Remark. Totally bounded  $\Rightarrow$  Bounded.

$(\Leftarrow)$   $\bar{d}(a, b) = \min \{|1|, |a - b|\}$  on  $\mathbb{R}$  defines a bounded metric on  $\mathbb{R}$ . But taking  $\epsilon = \gamma_2$  shows it's not totally bounded.

Example. ①  $\mathbb{R}$  is std. Topology is complete but not totally bounded.

②  $(-1, 1)$  is totally bounded but not complete.

③  $[-1, 1]$  is both.

Thm 3

A metric space  $(X, d)$  is compact iff it is complete and totally bounded.

Proof. ( $\Rightarrow$ ) Already saw that compact  $\Rightarrow$  complete.

To show: totally bounded.

Let  $\epsilon > 0$  be given.  $\{B(x_i, \epsilon)\}_{x_i \in X}$  is an open cover.  
(conclude.)

( $\Leftarrow$ ) Let  $X$  be complete and totally bounded.

We shall prove that  $X$  is sequentially compact.

(This is sufficient since  $X$  is a metric space.)

Let  $(x_n)_{n=1}^{\infty}$  be an arbitrary seq. We show it has a Cauchy subsequence. Completeness ensures convergence

$$\{B_i\}_{i=1}^{\infty}$$

• Cover  $X$  by finitely many balls of radius 1.

$\exists B_1$  s.t.  $B_1 \ni x_n$  for infinitely many  $n$ .

$$J_1 = \{n \in \mathbb{N} : x_n \in B_1\}.$$

• Cover by balls of radius  $y_2$ .

$\exists B_2 \rightarrow$  contains infinitely many  $x_n$  for  $n \in J_1$ .

Create  $J_2, J_3, \dots$  so on...

Then, pick  $n_1 \in J_1, n_1 < n_2 \in J_2, n_2 < n_3 \in J_3, \dots$

$(x_{n_k})$  is Cauchy.



## Lecture 22 (31-03-2021)

31 March 2021 16:14

Thm) Let  $C_1 \supset C_2 \supset C_3 \supset \dots$  be closed sets in a complete metric space  $X$ . If  $\text{diam}(C_n) \rightarrow 0$ , then  $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ .

Proof. Choose  $x_n \in C_n$  for each  $n$ .

Let  $\varepsilon > 0$  be given. Then,  $\exists N \in \mathbb{N}$  s.t.  $\text{diam}(C_n) < \varepsilon \forall n \geq N$ .

Thus, if  $m, n \geq N$ , then

$$d(x_m, x_n) \leq \text{diam}(C_N) < \varepsilon.$$

Thus,  $x_n \rightarrow x \in X$ .

Now, given any  $N \in \mathbb{N}$ ,  $\{x_n\}_{n \geq N} \subseteq C_N$ .

Since  $C_N$  is closed,  $x \in C_N$ . This is true for all  $N$ .

Thus,  $x \in C$ .

Defn) A space  $X$  is said to be a **Baire space** if the following holds:

Given any countable collection  $\{C_n\}_{n \in \mathbb{N}}$  of closed sets with  $C_n^\circ = \emptyset \forall n \in \mathbb{N}$ , it is the case that  $(\bigcup_{n \in \mathbb{N}} C_n)^\circ = \emptyset$ .

(Recall  $A^\circ = \text{int}(A)$ ) (Baire space)

Example (1)  $\mathbb{Q}$  is NOT a Baire space.

All singletons in  $\mathbb{Q}$  are closed with empty interior.

However, the (countable!) union of all is  $\mathbb{Q}$  but interior of  $\mathbb{Q}$  (in  $\mathbb{R}$ ) is not empty.

(2)  $\mathbb{N}$  is vacuously a Baire space since  $A^\circ = \emptyset \Leftrightarrow A = \emptyset$ .

Defn) A subset  $A$  of a space  $X$  is said to be of **first category** in  $X$  if it is contained in a countable union of closed sets with empty interior.

Otherwise it is said to be of second category.

(first category, second category)

Remark. A space  $X$  is a Baire Space iff every non-empty open subset of  $X$  is of second category.

Prop<sup>n</sup> 2. TFAE:

- (1)  $X$  is a Baire space.
- (2) If  $\{U_n\}_{n \in \mathbb{N}}$  is a collection of open, dense sets, then  $\bigcap_{n \in \mathbb{N}} U_n$  is dense in  $X$ .

Proof. Note that given  $A \subseteq X$ , TFAE:

- (i)  $A$  is closed and has empty interior.
- (ii)  $A^c$  is open and dense in  $X$ .

Conclude now using De Morgan's laws. □