Noncommutative algebra

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Contents

Introduction

This document is a compilation of facts about noncommutative rings. The motivation is to understand facts about finite algebras over a field, e.g., $k[G]$ for a finite group G.

The facts are stated without proofs. The sequence of the theorems may seem funny. These are *not* in logical order. For example, we begin the sections on simple and semisimple rings by stating the structure theorem for these rings. The abstract facts that follow later would then be easy consequences of the structure theorem. However, in practice, one goes the other way around in proving the structure theorem (possibly even using theorems from later subsections). The hope is that the organisation here makes it an easy reference to get to the relevant facts.

We first begin by stating all the relevant definitions in Part I. It may benefit the reader to only look these up when referred to in Part II, where we list the theorems.

Part I

Definitions

Rings throughout will be unital but not necessarily commutative. All modules will also be unital. We assume familiarity with basic notions of (left/right) artinian and noetherian rings and modules. For example, a left artinian ring is left noetherian. A module has finite length iff it is both artinian and noetherian.

For this section, A will denote an arbitrary ring.

Note that a *left* A-module M is (by definition) an abelian group M with a multiplication $A \times M \to M$ satisfying certain properties. This is precisely the data of an abelian group M and a ring homomorphism

$$\rho\colon A \to \operatorname{End}_{\mathbb{Z}}(M).$$

For this reason, we may think of ρ (or M) as a representation of A. The annihilator of M is defined as $\ker(\rho)$ and denoted as $\operatorname{ann}_A(M)$.

We have that $\rho(a)$ is the left-multiplication (or homothety) map. We denote this by a_M, i.e.,

$$\begin{aligned} a_M\colon M &\to M \\ m &\mapsto a \cdot m. \end{aligned}$$

Note that $a_M \in \operatorname{End}_{\mathbb{Z}}(M)$ but this is not necessarily an A-module homomorphism.

Definition 0.1. If M is a left A-module, the commutant is defined as $A' := \operatorname{End}_A(M)$, and the bicommutant as $A'' := \operatorname{End}_{A'}(M)$.

There is a (well-defined) ring homomorphism $\lambda_M \colon A \to A''$ given by $a \mapsto a_M$.

Note that M is a left A'-module with the multiplication $A' \times M \to M$ given by $(\varphi, m) \mapsto \varphi(m)$. Thus, the bicommutant is well-defined. Moreover, each homothety a_M (for $a \in A$) is an element of the bicommutant $\operatorname{End}_{A'}(M)$ since $a_M \circ \varphi = \varphi \circ a_M$ for $a \in A$ and $\varphi \in A'$.

Definition 0.2. Let R be a commutative ring. An R-algebra is a (possibly noncommutative) ring A with a ring homomorphism $\rho \colon R \to \operatorname{center}(A)$.

In particular, whenever we talk about a k-algebra (for a field k), we have that k lies in the center of A. Note that even though we may have a natural ring homomorphism $\mathbb{C} \hookrightarrow \mathbb{H}$, the division ring $\mathbb{H}$ is *not* a $\mathbb{C}$-algebra.

Definition 0.3. A (nonzero) module is simple if it has exactly two submodules.

The two submodules are then zero and the module itself.

Definition 0.4. An A-module M is faithful if $\operatorname{ann}_A(M)$ is zero. Equivalently, if λ_M is injective.

Definition 0.5. A module M is semisimple if any of the following equivalent conditions hold:

- every submodule of N has a direct sum complement, i.e., every inclusion $N \hookrightarrow M$ splits;
- every short exact sequence of the form $0 \to N \to M \to L \to 0$ splits;
- M is the sum of its simple submodules;
- M is the direct sum of a family of simple modules.

A ring A is (left) semisimple if A is a semisimple as a left module over A.

Semisimplicity is a symmetric notion: A is left semisimple iff A is right semisimple.

Definition 0.6. A (nonzero) ring A is simple if A is is semisimple and has only one isomorphism class of simple left ideals.

The definition above is more restrictive than found in some books. Also note that being simple (or even weakly-simple) as a ring is weaker than being simple as a left-module over itself. See Theorem 1.1.

Definition 0.7. A (nonzero) ring A is weakly-simple if A has exactly two two-sided ideals.

See Theorems 1.3 and 1.4 for the implications.

Definition 0.8. A (nonzero) ring A is primitive if A has a faithful simple module.

Definition 0.9. A ring A is semiprimitive if for any nonzero $a \in A$, there is an irreducible representation ρ of A such that $\rho(a) \neq 0$.

In the language of modules: for every $a \neq 0$, there is a simple A-module M such that $aM \neq 0$.
This is equivalent to the existence of a faithful *semisimple* module, see Theorem 5.1.

Definition 0.10. Let L be a left ideal of A. The idealiser of L is defined as

$$(L : A) \coloneqq \{a \in A \colon aA \subseteq L\} = \operatorname{ann}_A(A/L).$$

The idealiser is the unique largest two-sided ideal contained in L.

Definition 0.11. Let A be a ring. The (Jacobson) radical is a two-sided ideal of A, denoted $\operatorname{rad}(A)$, defined as any of the following equivalent objects.

(a) the intersection $\bigcap_\rho \ker(\rho)$, where ρ varies over all irreducible representations of A;

(b) the intersection $\bigcap_M \operatorname{ann}_A(M)$, where M is a simple left A-module;

(c) the intersection $\bigcap_L L$ over all maximal left ideals L of A;

(d) the intersection $\bigcap_R R$ over all maximal right ideals R of A;

(e) the intersection $\bigcap_P P$ over all two-sided ideals P such that A/P is a primitive ring;

(f) the set of elements z such that $1 - az$ has a left inverse for every $a \in A$.

Definition 0.12. An element $a \in A$ is called left quasi-regular if $1 - a$ has a left inverse.

A right quasi-regular element is defined analogously. An element is quasi-regular if it is both left and right quasi-regular.

A left ideal is quasi-regular if all its elements are left quasi-regular. Analogously, a right ideal is quasi-regular if all its elements are right quasi-regular.

Definition 0.13. An ideal I is nilpotent if $I^k = 0$ for some k.

Definition 0.14. Let N be a submodule of the A-module M. We say that

(a) M is an essential extension of N if for every <u>nonzero</u> submodule $K \subseteq M$, we have $N \cap K \neq 0$;

(b) N is a superfluous submodule of M if for every <u>proper</u> submodule $K \subsetneq M$, we have $N + K \neq M$.

Definition 0.15. Let M be an A-module.

The socle of M is defined as

$$\begin{aligned} \operatorname{soc}(M) &:= \sum_{N \subseteq M \text{ simple}} N \\ &= \bigcap_{N \subseteq M \text{ essential}} N. \end{aligned}$$

The radical of M is defined as

$$\begin{aligned} \operatorname{rad}(M) &:= \bigcap_{N \subseteq M \text{ maximal}} N \\ &= \sum_{N \subseteq M \text{ superfluous}} N. \end{aligned}$$

The convention is that the empty sum is 0 and the empty intersection is M.

Note that $N \subseteq M$ is maximal iff M/N is simple.

Definition 0.16. Let M be an A-module.

An injective hull of M is an essential extension $\iota \colon M \hookrightarrow E$ with E injective.

A projective cover of M is a surjection $\pi \colon P \twoheadrightarrow M$ with P projective and $\ker(\pi) \subseteq P$ a superfluous submodule.

Definition 0.17. A is called

- (left) perfect if every left module has a projective cover;
- (left) semiperfect if every finitely generated left module has a projective cover.

Definition 0.18. A left principal indecomposable module (PIM) of A is an indecomposable direct summand of A. Equivalently, an indecomposable, projective, cyclic module.

Definition 0.19. A is said to be a subdirect product of $A_1, \ldots, A_k$ if there is an injective ring homomorphism $\iota\colon A \to \prod_i A_i$ such that the each induced map $\pi_i \circ \iota\colon A \to A_i$ is surjective for each i.

Definition 0.20. A (nonzero) ring is called local if its set of nonunits forms a two-sided ideal.

Definition 0.21. An A-module M is

- primordial if $\mathrm{End}_A(M)$ is a local ring;
- semi-primordial if it is the direct sum of a family of primordial submodules.

Injectives over a noetherian ring are semi-primordial.

Part II

Theorems

A summary of the main implications is given below.

$$
\begin{tikzcd}
 & \text{simple} \arrow[dl, Rightarrow] \arrow[rd, Rightarrow] & & \\
\text{weakly-simple} \arrow[ur, Rightarrow, bend left, "\text{semisimple}"] \arrow[d, Rightarrow] & & \text{semisimple} \arrow[dr, Rightarrow] \arrow[ddl, Rightarrow, bend left] & \\
\text{primitive} \arrow[dr, Rightarrow] & & & \text{artinian} \arrow[ul, Rightarrow, bend right, "\text{semiprimitive}"'] \arrow[d, Rightarrow] \\
 & \text{semiprimitive} \arrow[d, Leftrightarrow] & & \text{perfect} \arrow[d, Rightarrow] \\
 & \operatorname{rad} = 0 & \text{local} \arrow[r, Rightarrow] & \text{semiperfect}
\end{tikzcd}
$$

However, for the class of **left artinian** rings, the following chain of implications hold.

$$
\begin{tikzcd}
\text{simple} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{weakly-simple} \arrow[r, Leftrightarrow] & \text{primitive} \\
\text{semisimple} \arrow[r, Leftrightarrow] & \operatorname{rad} = 0 \arrow[r, Leftrightarrow] & \text{semiprimitive}
\end{tikzcd}
$$

§1. Implications between some of the properties

Theorem 1.1. Let A be a ring. The following are equivalent.

(a) A is simple as a left module over A.

(b) A is a division ring.

(c) A is simple as a right module over A.

In such a case, A is a simple ring, a weakly-simple ring, and a semisimple ring.

Theorem 1.2. A semisimple ring is left and right artinian.

Theorem 1.3. There exists a weakly-simple ring that is not artinian and hence not semisimple.

Theorem 1.4. Let A be a ring. The following are equivalent:

(a) A is simple.

(b) A is weakly-simple and left artinian.

(c) A is weakly-simple and semisimple.

(d) A is weakly-simple and has exactly one isomorphism class of minimal left ideals.

All "left"s can be replaced with "right"s as well.

Theorem 1.5. A weakly-simple ring is primitive.

Theorem 1.6. A left artinian primitive ring is simple.

§2. Simple rings

We recall again that our definition of simple (Definition 0.6) is possibly stricter than the one the reader may have seen before (Definition 0.7). See Theorem 1.4.

We begin by stating the main theorems.

Theorem 2.1. Every simple ring A is of the form $\mathrm{End}_D(V)$, where D is a division ring and V is a finite-dimensional vector space over D.

In other words, A is a matrix ring $M_n(D)$ over a division ring. This D and n is uniquely determined.

See Theorem 2.6 for some description of D and V.

Theorem 2.2. Let D be a division ring, and V a finite-dimensional vector space over D. Set $A := \mathrm{End}_D(V)$. Then, A is a simple ring, M a simple A-module, and $D = \mathrm{End}_A(V)$.

Theorem 2.3. A simple ring has exactly one simple module up to isomorphism.

Specifically, the unique simple module over $M_n(D)$ is D^n.

Theorem 2.4. Let A be a simple ring. Then, A is a finite direct sum of simple left ideals.

If I and J are simple left ideals of A, then there exists $a \in A$ such that $Ia = J$.

Theorem 2.5. Let A be a simple ring, M a simple A-module and L a simple left ideal of A. Then, $LM = M$ and M is faithful.

Theorem 2.6. Let A be a simple ring. Then, there exists a faithful simple A-module M such that $D \coloneqq \mathrm{End}_A(M)$ is a division ring and $A \cong \mathrm{End}_D(M)$ via $\lambda_M \colon a \mapsto a_M$.

The module M can be chosen to be any simple left ideal of A.

Note that λ_M appears in Definition 0.1. See also Jacobson Density Theorem 9.1 and Corollary 9.2 which tells us that M is a finite D-module.

§3. Semisimple rings

Theorem 3.1. Let A be a ring. The following conditions are equivalent. Parts of the equivalence constitute the Wedderburn–Artin theorem.

(a) A is semisimple.

(b) A is a finite direct product of rings of the form $\mathrm{End}_D(V)$ where D is a division ring, and V is a finite-dimensional D-vector space.

(c) $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where D_i are division rings, and n_i are positive integers.
This decomposition is unique.

(d) A is left artinian and semiprimitive.

(e) A is left artinian and $\mathrm{rad}\, A = 0$

(f) A is left artinian and contains no nonzero nilpotent ideals.

(g) A is left artinian and the subdirect product of weakly-simple rings.

See Definition 0.19 for the definition of subdirect product.

Remark 3.2. We describe the recovery of the k, D_i, and n_i:

- The number k is the number of isomorphism classes of simple <u>right</u> A-modules.
- Let $S_1, \ldots, S_k$ be a list of the non-isomorphic simple right modules.
- We recover D_i as the endomorphism ring of S_i.
- The corresponding n_i is the number of right ideals in A that are isomorphic to S_i.

The reason for choosing right ideals and right modules is that we have $A \cong \mathrm{End}_A(A_A)$, where A_A denotes A viewed as a right module over itself.
If we had worked with left ideals, we would have to consider D_i^{op}.

Theorem 3.3 (Artin–Wedderburn for k-algebras)**.** If A is a finite-dimensional semisimple k-algebra, then in the decomposition $A \cong \prod_i M_{n_i}(D_i)$, each D_i is a finite-dimensional division algebra over k.

If k is algebraically closed, then $D_i = k$ for all i.

Note that without the algebraic closure hypothesis, we only have that the center of D_i is a finite extension of k. It could be larger than k. For example, $\mathbb{C}$ as an algebra over $\mathbb{R}$.

Remark 3.4. A semisimple ring may very well contain nonzero nilpotent *elements*! Indeed, the matrix rings contain many such elements.

Theorem 3.5. Any module over a semisimple ring is a semisimple module.

Theorem 3.6. A semisimple ring A has finitely many isomorphism classes of simple modules. Moreover, every simple left A-module is isomorphic to a (simple) left ideal.

Theorem 3.7. Let A be a semisimple ring, L a simple left ideal, and M a simple left module. If $L \ncong M$ as left modules, then $LM = 0$.

Theorem 3.8. A semisimple ring A can be written as a finite product of "simple subrings" (which are, in fact, two-sided ideals):

$$A = \prod_{i=1}^{k} A_i.$$

Moreover, if e_i is the unit of $A_i \subseteq A$, then $1_A = \sum_{i=1}^{k} e_i$ and $A_i = Ae_i$.

Some clarification is needed here: A_i is not really a subring of A since $1 \notin A_i$.
What is true is that A is a product of simple rings A_i, in which case each A_i is naturally a subset of A. Each A_i is further then a two-sided ideal generated by an idempotent e_i that acts as identity of A_i.

§4. Weakly-simple rings

Theorem 4.1. If A is weakly-simple, then every nonzero A-module is faithful.

Theorem 4.2 (Rieffel)**.** Let A be a weakly-simple ring, and L a nonzero left ideal. Let $A' := \mathrm{End}_A(L)$, $A'' := \mathrm{End}_{A'}(L)$, and $\lambda\colon A \to A''$ be as in Definition 0.1.

Then, λ is an isomorphism. In particular, L is a faithful A-module.

Theorem 4.3 (Wedderburn)**.** Let A be a ring and M a faithful simple module over A. So, $A' := \mathrm{End}_A(M)$ is a division ring, and $\lambda\colon A \hookrightarrow A''$ makes A a subring of A''.

If M is finite-dimensional over A', then λ is an isomorphism, i.e., $A = A''$.

§5. (Semi)primitive rings

Theorem 5.1. Let A be a ring. The following statements are equivalent.

(a) A is left semiprimitive.

(b) A is right semiprimitive.

(c) A admits a faithful semisimple module.

(d) A is a finite subdirect product of primitive rings.

(e) $\mathrm{rad}(A) = 0$.

See Definition 0.19 for the definition of subdirect product. See also Theorem 6.5 for a generalisation of the last result.

Theorem 5.2. Let A be a ring.

A is left primitive iff A has a maximal left ideal L that contains no non-zero two-sided ideal of A.

A is (left) semiprimitive iff $\bigcap_L (L : A) = 0$, where the intersection is taken over all maximal left ideals of A.

Corollary 5.3. Let R be a commutative ring.

R is primitive iff R is a field.

R is semiprimitive iff R is a subdirect product of fields.

§6. The radical

Since the definition is really a theorem in itself, we state it again.

Definition 6.1. Let A be a ring. The (Jacobson) radical is a two-sided ideal of A, denoted $\operatorname{rad}(A)$, defined as any of the following equivalent objects.

(a) the intersection $\bigcap_\rho \ker(\rho)$, where ρ varies over all irreducible representations of A;

(b) the intersection $\bigcap_M \operatorname{ann}_A(M)$, where M is a simple left A-module;

(c) the intersection $\bigcap_L L$ over all maximal left ideals L of A;

(d) the intersection $\bigcap_R R$ over all maximal right ideals R of A;

(e) the intersection $\bigcap_P P$ over all two-sided ideals P such that A/P is a primitive ring;

(f) the set of elements z such that $1 - az$ has a left inverse for every $a \in A$.

Theorem 6.2. Let A be a ring. The two notions of $\operatorname{rad}(A)$ coincide: namely the Jacobson radical of A as a ring, and the radical of A as a left-module over A.

Theorem 6.3. The Jacobson ideal $\operatorname{rad} A$ is quasi-regular as a left ideal.
Moreover, $\operatorname{rad} A$ contains every quasi-regular left ideal.

Theorem 6.4. Any nilpotent element is left quasi-regular.
Thus, any nilpotent left ideal is quasi-regular.

Theorem 6.5. Let A be a ring, and I a two-sided ideal.

- A is semiprimitive iff $\operatorname{rad}(A) = 0$.
- If A/I is semiprimitive, then $\operatorname{rad}(A) \subseteq I$.
- $A/\operatorname{rad}(A)$ is a semiprimitive ring. Hence, $\operatorname{rad}(A/\operatorname{rad} A) = 0$.

Theorem 6.6. Let M be an A-module, and N a submodule.

- $\operatorname{rad}(M/\operatorname{rad} M) = 0$.
- $\operatorname{rad}(M/N) = 0$ implies $N \supseteq \operatorname{rad} M$.

Theorem 6.7. For any A-linear map $f\colon M \to N$, one has $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.

Theorem 6.8 (Radical and semisimplicity)**.** Let M be an A-module.

- If M is semisimple, then $\operatorname{rad}(M) = 0$ and $\operatorname{rad}(\operatorname{End}_A(M)) = 0$.
- If M is artinian and $\operatorname{rad}(M) = 0$, then M is semisimple.

Theorem 6.9. Let A be a left artinian ring, and M a left A-module. Then,

$$\operatorname{rad}(M) = \operatorname{rad}(A) \cdot M.$$

§7. Local rings

Theorem 7.1. Let A be a ring. Let A^* denote its set of units. The following are equivalent.

(a) A is local, i.e., $A \setminus A^*$ is an ideal.

(b) $A \setminus A^*$ is closed addition.

(c) $A \setminus A^* \subseteq \operatorname{rad}(A)$.

(d) $A \setminus A^* = \operatorname{rad}(A)$.

(e) $A \setminus \operatorname{rad}(A) = A^*$.

(f) A has a unique maximal right ideal.

(g) A has a unique maximal left ideal.

Theorem 7.2. If A is local, then A is semiperfect.

Theorem 7.3 (Kaplansky)**.** Any projective module over a local ring is free.

Theorem 7.4. Let A be a ring in which every element is either invertible or nilpotent. Then, A is local.

§8. Indecomposable-esque modules and locality

Recall that a module is primordial if its endomorphism ring is local.

Theorem 8.1. A simple module is primordial.
A primordial module is indecomposable.
An indecomposable module of finite length is primordial.

$$\begin{tikzcd}
\text{simple} \arrow[r, Rightarrow] & \text{primordial} \arrow[rr, Rightarrow, bend left] & & \text{indecomposable} \arrow[ll, Rightarrow, bend left, "\text{finite length}"]
\end{tikzcd}$$

Lemma 8.2 (Fitting)**.** Let M be an A-module of finite length. Given $u \in \mathrm{End}_A(M)$, we have

$$M = \ker(u^\infty) \oplus \mathrm{im}(u^\infty).$$

Further, u restricted to $\ker(u^\infty)$ is nilpotent and u restricted to $\mathrm{im}(u^\infty)$ is an isomorphism.

To clarify notation: the sequences of submodules $(\ker(u^N))_N$ and $(\mathrm{im}(u^N))_N$ both stabilise. The stable value is denoted by $\ker(u^\infty)$ and $\mathrm{im}(u^\infty)$ respectively.

Corollary 8.3. If E is an indecomposable A-module of finite length, then any map in $\mathrm{End}_A(E)$ is either nilpotent or invertible. Thus, $\mathrm{End}_A(E)$ is local.

Theorem 8.4 (Schur)**.** Any nonzero map between simple modules is an isomorphism.

If E is a simple A-module, then $\mathrm{End}_A(E)$ is a division ring.

Theorem 8.5 (Krull–Remak–Schmidt)**.** Let M be an A-module of finite length. Then, M can be *uniquely* decomposed as a (necessarily finite) direct sum of indecomposable modules.

Uniqueness means the following: if we have

$$M \cong E_1 \oplus \cdots \oplus E_r \cong F_1 \oplus \cdots \oplus F_s$$

for indecomposable modules E_i and F_j, then $r = s$, and there exists a permutation σ of $[r]$ such that $E_i \cong F_{\sigma(i)}$.

Corollary 8.6. Let A be a left artinian ring, and M a finitely generated left A-module. Then, M is the finite direct sum of indecomposable modules (in a unique way).

In particular, A is the direct sum of principal indecomposable modules.

Since we cannot have infinite direct sums in a noetherian or artinian module, we still have an existence theorem:

Theorem 8.7. Let M be an A-module that is either artinian or noetherian. Then, M is a finite direct sum of indecomposables, not necessarily in a unique way.

We also have a uniqueness theorem for semi-primordial modules.

Theorem 8.8. Let A be a ring. Suppose we have an isomorphism

$$E_1 \oplus \cdots \oplus E_r \cong F_1 \oplus \cdots \oplus F_s$$

for primordial A-modules E_i and F_j. Then, $r = s$, and there exists a permutation σ of $[r]$ such that $E_i \cong F_{\sigma(i)}$.

§9. Jacobson Density Theorem

Theorem 9.1 (Jacobson Density Theorem)**.** Let M be a semisimple A-module. Let A' denote the commutant of M, and A'' the bicommutant. Let $\lambda\colon A \to A''$ be the ring homomorphism $a \mapsto a_M$.

For any $\psi \in A''$ and any finite subset $S \subseteq M$, there exists $a \in A$ such that ψ and a_M agree on S.

If M is finitely generated over A', then λ is surjective.

Corollary 9.2. Let A be a left artinian ring, and M a semisimple A-module. If $A' :=$

$\mathrm{End}_A(M)$ is a division ring (e.g., M is simple), then M is a finite-dimensional vector space over A'. Hence, $\lambda(A) = A''$.

Corollary 9.3. Let V be a finite-dimensional vector space over an algebraically closed field k, and A a subalgebra of $\mathrm{End}_k(V)$.

If V is a simple A-module, then $A = \mathrm{End}_k(V)$.

Corollary 9.4. Let V be a finite-dimensional vector space over an algebraically closed field k, and G a submonoid of $\mathrm{GL}(V)$.

If V is a simple G-module, then $k[G] = \mathrm{End}_k(V)$.

Note that here $k[G]$ is the subalgebra of $\mathrm{End}_k(V)$ generated by $G \subseteq \mathrm{GL}(V) \subseteq \mathrm{End}_k(V)$.

§10. Projective modules

Throughout this section, A will denote a ring, $J \coloneqq \mathrm{rad}(A)$ the Jacobson radical, set $\overline{A} \coloneqq A/J$, and $\overline{M} \coloneqq M/JM$ for a left A-module M. This makes $\overline{(-)}$ an additive functor in the obvious way. All modules are considered to be left modules.

In particular, given A-modules M and N, we have a map of abelian groups

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{N})$$

and a map of rings

$$\mathrm{End}_A(M) \to \mathrm{End}_{\overline{A}}(\overline{M}).$$

We study when these maps are injective or surjective.

Theorem 10.1. Let P be a projective A-module, and N an arbitrary A-module.

Then, $\overline{P}$ is a projective $\overline{A}$-module, and $\mathrm{Hom}_A(P, N) \twoheadrightarrow \mathrm{Hom}_{\overline{A}}(\overline{P}, \overline{N})$ is onto.

Theorem 10.2. Suppose A is (left) artinian, and P is a projective A-module. Then, the map $\mathrm{End}_A(P) \twoheadrightarrow \mathrm{End}_{\overline{A}}(\overline{P})$ induces an isomorphism

$$\frac{\mathrm{End}_A(P)}{\mathrm{rad}(\mathrm{End}_A(P))} \cong \mathrm{End}_{\overline{A}}(\overline{P}).$$

Corollary 10.3. Suppose A is artinian, and P, Q are projective A-modules. Then,

$$P \cong Q \Leftrightarrow \overline{P} \cong \overline{Q}.$$

Theorem 10.4. Suppose A is artinian, and P is a direct summand of A as a left module. Then,

$$P \text{ is a PIM} \Leftrightarrow P \text{ is indecomposable} \Leftrightarrow \overline{P} \text{ is indecomposable} \Leftrightarrow \overline{P} \text{ is simple.}$$

Note that $\overline{A}$ is semisimple in the above situation, giving the last $\Leftrightarrow$.

§11. Superfluous extensions and (semi)perfect rings

Theorem 11.1. The zero submodule is always superfluous, and a nonzero module is never a superfluous submodule of itself.

Theorem 11.2 (Nakayama's lemma)**.** Let M be a finitely generated left A-module. Then, $\mathrm{rad}(A)M$ is a superfluous submodule of M.

Theorem 11.3. Injective hulls and projective covers are unique up to isomorphism.

Injective hulls always exist, but projective covers may not exist.

Theorem 11.4. If M is a projective module, then its projective cover is M.

Theorem 11.5. Let A be a semiprimitive ring (i.e., $\mathrm{rad}(A) = 0$), and M an A-module.

M has a projective cover iff M is projective.

Theorem 11.6. Any left artinian ring is right-and-left perfect.

Any local ring is right-and-left semiperfect.

Theorem 11.7 (Characterisation of perfect rings)**.** Let A be a ring. The following are equivalent.

(a) A is <u>left</u> perfect.

(b) Every left A-module has a projective cover.

(c) A satisfies the descending chain condition on principal <u>right</u> ideals.

Theorem 11.8 (Characterisation of semiperfect rings)**.** Let A be a ring. The following are equivalent.

(a) A is left semiperfect.

(b) A is right semiperfect.

(c) Every finitely generated left A-module has a projective cover.

(d) Every simple left A-module has a projective cover.

(e) $A/\operatorname{rad} A$ is semisimple and idempotents lift modulo $\operatorname{rad} A$.

Theorem 11.9. Over a semiperfect ring, every indecomposable projective module is a PIM, and every finitely generated projective module is a direct sum of PIMs.

§12. Artinian rings

We restate the theorem from earlier.

Theorem 12.1. For a left artinian ring, one has:

$$\begin{array}{ccccc} \text{simple} & \Longleftrightarrow & \text{weakly-simple} & \Longleftrightarrow & \text{primitive} \\ \Downarrow & & & & \\ \text{semisimple} & \Longleftrightarrow & \operatorname{rad} = 0 & \Longleftrightarrow & \text{semiprimitive} \end{array}$$

Theorem 12.2. Let A be left artinian. Then, $\operatorname{rad} A$ is nilpotent.

Theorem 12.3. Let A be left artinian. Then, $A/\operatorname{rad} A$ is semisimple.

Sketch. Use Theorems 3.1 and 6.5. $\square$

Theorem 12.4. Let A be a left artinian ring, and M a left A-module. Then,

$$\operatorname{rad}(M) = \operatorname{rad}(A) \cdot M.$$

Theorem 12.5. If A is left artinian, then A is right-and-left perfect.

Theorem 12.6. If A is left artinian, then any finitely generated left A-module is the direct sum of finitely many indecomposables.

In particular, A is a finite direct sum of PIMs.

Theorem 12.7. Over an artinian ring, every indecomposable projective module is a PIM, and every finitely generated projective module is a direct sum of PIMs.

Sketch. Let P be a projective indecomposable. Write $P \oplus Q = F$ for some free module F. Since the ring is artinian, we have that $F \cong \bigoplus_i P_i$ for PIMs P_i. Thus, $P \oplus Q = \bigoplus_i P_i$. By uniqueness of decomposition, we see that $P \cong P_i$ for some i, i.e., P is a PIM. $\square$

Theorem 12.8. Let A be left artinian. Every nonzero homomorphic image of an indecomposable left projective module is again indecomposable.

That is, if $P \twoheadrightarrow N$ with P an indecomposable project (not necessarily a PIM), then N is indecomposable (or zero).

Theorem 12.9. Let A be left artinian and local. Then, (the left module) A is the only PIM.

Theorem 12.10. If A is a left artinian ring, then A has finitely many simple left A-modules (up to isomorphism). More precisely, setting $J \coloneqq \operatorname{rad}(A)$, one has a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{simple } A\text{-modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{simple } A/J\text{-modules} \end{array} \right\}.$$

Sketch. If M is a simple A-module, then $\operatorname{rad}(A) \subseteq \operatorname{ann}_A(M)$, i.e., $\operatorname{rad}(A) \cdot M = 0$.

This gives us the isomorphism-preserving bijection. But $A/\operatorname{rad} A$ is a semisimple ring, so

Theorem 3.6 gives us the finiteness. □

Theorem 12.11. If A is a left artinian ring, and $J \coloneqq \operatorname{rad}(A)$ its Jacobson radical, then the map $P \mapsto P/JP$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{principal indecomposable } A\text{-modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{simple modules (over } A \text{ or } A/J) \end{array} \right\}.$$

In particular, the left set is finite. The module P/JP is called the head of P. The other direction is given by mapping a simple A/J-module to its projective cover (viewed as an A-module).

Sketch. Theorem 10.4 tells us that P/JP is indeed simple when P is a PIM. Corollary 10.3 shows that the map $P \mapsto P/JP$ is injective.

In the other direction, let S be a simple A-module, and $\pi\colon P \twoheadrightarrow S$ its projective cover. Write $P = \bigoplus_i P_i$ as a sum of PIMs. The induced maps $P_i \hookrightarrow P \twoheadrightarrow S$ cannot all be zero. If $P_1 \to S$ is nonzero, then P_1 is a projective cover of S. By uniqueness, $P = P_1$ is a PIM.

This shows that we do have a well-defined map in the backwards direction.

Moreover, $P \twoheadrightarrow S$ induces $\overline{P} \twoheadrightarrow \overline{S} = S$. Since $\overline{P}$ is simple, this map is an isomorphism.

This shows that (going mod J) is a left inverse for (taking projective cover). Since the former is known to be injective, this finishes the proof. □

§13. Representation theory

For a finite group G and a field k, we let $k[G]$ denote the group algebra. This is a k-algebra of dimension equal to $|G|$.

Recall that a representation of G on a k-vector space V is a group homomorphism $\rho\colon G \to GL(V)$. This is precisely the same data as a $k[G]$-module V.

Note that $k[G]$ is (left and right) artinian, being finite-dimensional over a field. The modular case refers to the situation when $\operatorname{char}(k) \mid |G|$.

Recall that we have the augmentation map

$$\begin{aligned} \varepsilon\colon k[G] &\to k \\ \sum_{g \in G} a_g g &\mapsto \sum_{g \in G} a_g. \end{aligned}$$

Theorem 13.1. The group algebra $k[G]$ is semisimple iff $\operatorname{char}(k) \nmid |G|$.

Sketch. ($\Rightarrow$) Suppose $\operatorname{char}(k) \mid |G|$. Consider $z := \sum_{g \in G} g$. Then, kz is an ideal of $k[G]$. We have $z^2 = |G|z = 0$. Thus, kz is a nonzero nilpotent ideal. Now apply Theorem 3.1.

($\Leftarrow$) Suppose $\operatorname{char}(k) \nmid |G|$. Any inclusion $W \hookrightarrow V$ of $k[G]$-modules has a k-linear splitting $\pi \colon V \to W$. This gives rise to the $k[G]$-linear splitting given by

$$\frac{1}{|G|} \sum_{g \in G} g_V \circ \pi \circ g_V^{-1}. \qquad \square$$

Thus, we get the following.

Theorem 13.2. Let k be a field and G a finite group such that $\operatorname{char}(k) \nmid |G|$. Then,

- the indecomposable and irreducible representations of G coincide;
- there are finitely many irreducible representations of G;
- every representation decomposes as a direct sum of irreducible representations of G.

Regardless, since $k[G]$ is artinian even in the modular case, we have the following.

Theorem 13.3. Let k be a field, and G be any finite group. Then,

- the ring $k[G]$ has finitely many simple modules;
- the group G has finitely many irreducible representations over k;
- any finite-dimensional representation is (uniquely) the direct sum of finitely many indecomposable representations;
- every $k[G]$-module has a projective cover;
- the projective cover of each simple module is indecomposable and cyclic, i.e., a PIM;
- if $J := \operatorname{rad}(k[G])$, then we have a one-to-one correspondence of isomorphism classes

$$\begin{aligned} \{\text{PIMs}\} &\leftrightarrow \{\text{simple modules}\} \\ P &\mapsto P/JP, \end{aligned}$$

  with the inverse being given by taking projective covers.

In the modular case, the issue is that there exist representations that cannot be written as a (direct) sum of irreducible representations.

Theorem 13.4. Let k be a field of characteristic $p > 0$, and G a finite group.

The ring $k[G]$ is local iff G is a p-group.

In this case, the ideal of nonunits, which is equal to the Jacobson radical, is given by

$$\operatorname{rad}(k[G]) = \left\{ \sum_g a_g g : \sum_g a_g = 0 \right\} = \ker(\varepsilon).$$

§14. Symmetric and asymmetric properties

We list the properties we have considered in this document and note whether they are symmetric. A property P of a ring is symmetric if the following is true: A has P iff A^{op} has P.
Equivalently, since our properties are often listed stated as "A is left P", being symmetric is saying: A is left P iff A is right P.

(a) Artinian: not symmetric.

(b) Noetherian: not symmetric.

(c) Finite length: not symmetric.
Moreover, even if A has finite length as a left and right module, these lengths could be different.

(d) Local: symmetric by definition.

(e) Semisimple: symmetric by classification theorem.

(f) Simple: symmetric by classification theorem.

(g) Weakly-simple: symmetric by definition.

(h) Semiprimitive: symmetric since characterised by $\operatorname{rad} = 0$.

(i) Primitive: **not** symmetric!

(j) Semiperfect: symmetric.

(k) Perfect: **not** symmetric!