Morphisms of Schemes: Chevalley's Theorem

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- **②** Given f ∈ A, A_f will denote the localisation of A at the multiplicative set $\{1, f, f^2, ...\}$.



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The above data is required to satisfy the following conditions:

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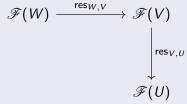
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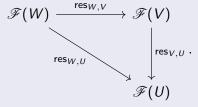


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Moreover, the "obvious diagrams" must commute.

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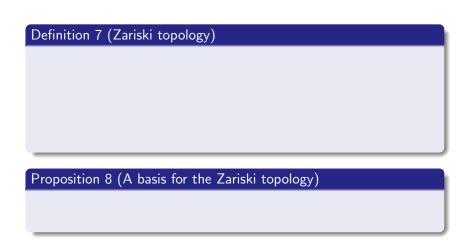
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Proposition 8 (A basis for the Zariski topology)

The collection $\{D(f): f \in A\}$ forms a basis for the above topology.

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To conclude, the only closed singleton subset of \mathbb{A}^1_k is $\{\langle 0 \rangle\}$.

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This is called the structure sheaf on Spec A.

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A scheme can be covered by affine opens.

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In fact, (it follows that) the affine opens form a basis for X.

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$$B_{g} \longrightarrow A_{\pi^{\sharp}g}$$

The above is a morphism of affine schemes. That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

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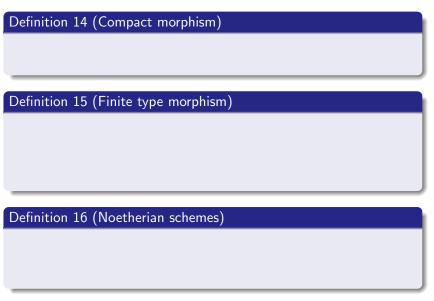
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