

$$\int (\cos^5 x) dx$$

MA 839

Advanced Commutative Algebra

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A Quick Intro.

Setup: A ring is commutative with 1.

Let M be an R -module.

Observation: ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$),
we get an R -linear map $R \rightarrow M$ which is onto.
 $a \mapsto ax$

Then, $M \cong R/I$ where I is the kernel.
In this case, $I = \text{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R .

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$.
 $= \{ax + by \mid (a, b) \in R^{\oplus 2}\}$

Then, we get an onto R -linear map $R \oplus R \xrightarrow{\varphi} M$
 $e_1 \mapsto x$
 $e_2 \mapsto y$ } extend this
 $\{e_1, e_2\}$ is a basis
→ this lets us extend the map

In particular, $M \cong R^2 / \ker \varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle \rangle} \oplus \frac{R}{\langle \rangle} ?$$

This has a positive answer: ① R is a field
② R is a PID

CAUTION: We won't include fields as PID.
That is, when we say "PID", we exclude fields.

③ Suppose M is a finitely generated (f.g.) R -module.

(That is, suppose $M = \langle x_1, \dots, x_n \rangle$.)

Then, M is a quotient of $R^{\oplus n}$.
→ R^n

Then, M is a quotient of R^n .

way to get this

Define $R^n \xrightarrow{\varphi} M$ by $e_i \mapsto x_i$.

$$M \cong R / \ker \varphi.$$

④ In general, consider a free module with " M as basis", call it $F(M)$. Then $F(M)$ maps onto M .

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$,

then $F(A)$ maps onto M .

Thus, M can be written as a quotient of a free-module.

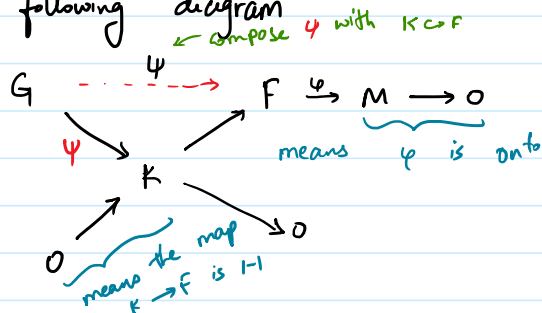
To Summarise: If M is an R -module, then M can be written as a quotient of a free R -module. Moreover, if M is fg, then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R -module mapping onto M with kernel K . That is, $\varphi: F \rightarrow M$ is onto R -linear and $\ker \varphi = K$.

Now, \exists a free R -module G and an onto map $\psi: G \rightarrow K$

We capture this in the following diagram



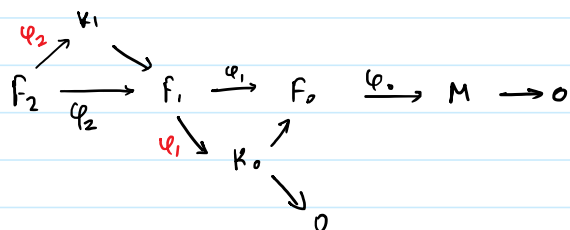
Note that $\text{im } \psi = K = \ker \varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \rightarrow 0$.

- ① φ is onto and $\ker \varphi = \text{im } \psi$.
- ② G and F are free R -modules.

Note that we can repeat the above process with K instead of F .

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \varphi$, $\varphi_1 := \psi$.



Thus, we get free modules $\{F_n, \varphi_n: F_n \rightarrow F_{n-1}\}$ such that $\ker \varphi_{n-1} = \text{im } \varphi_n$ written as

$$\dots \rightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \dots \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

F_i s are free, φ_0 is onto & $\ker \varphi_{n-1} = \text{im } \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F: \dots \rightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \rightarrow 0 \text{ as an}$$

R free resolution of M .

$\text{im } \varphi_1 = K$, this is not exact here.
 φ_1 not onto (rec.)

Q: ① If M is f.g.:

Can we get F_i s so that $\text{rank}(F_i) < \infty \forall i$.

② If yes, are $\text{rank}(F_i)$ s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_i s s.t. the maps are "described nicely".

Q: If two modules have "isomorphic" free resolutions, are they isomorphic?

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\varphi_3} & F_2 & \xrightarrow{\varphi_2} & F_1 & \xrightarrow{\varphi_1} & F_0 \rightarrow 0 \\
 & & \downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \varphi_0 \\
 \dots & \xrightarrow{\varphi'_3} & F'_2 & \xrightarrow{\varphi'_2} & F'_1 & \xrightarrow{\varphi'_1} & F'_0 \rightarrow 0
 \end{array}$$

$$M \cong F_0 / \text{im } \varphi_1$$

$$M' \cong F'_0 / \text{im } \varphi'_1$$

$$\varphi'_1 \varphi_1 = \varphi_0 \varphi_1 \quad (*)$$

Claim. $\gamma_0(\text{im } \varphi_1) = \text{im } \varphi'_1$

(\subseteq) clear by (*)

(\supseteq) clear again since $\varphi'_1 = \gamma_0 \varphi_1 \gamma_1^{-1}$

$$\text{Thus, } M_0 \cong \frac{F_0}{\text{im } \varphi_0} \cong \frac{\varphi_0(F_0)}{\varphi_0(\text{im } \varphi)} = \frac{F_0'}{\text{im } \varphi'_1} \cong M'_0$$

Lecture 1 (11-01-2021)

11 January 2021 11:07

Free modules:

As usual : R is a (commutative) ring (with 1).
 M is an R -module.

Defⁿ. ① Let $A \subset M$. A is said to be a **generating set** of M (as an R -module) if
 $\forall x \in M, \exists x_1, \dots, x_n \in A$ and $(a_1, \dots, a_n) \in R^n$ s.t.
 $x = a_1 x_1 + \dots + a_n x_n$.

(Note that A need not be finite.)

Notation : $M = \langle A \rangle$

If $A = \{x_1, \dots, x_n\}$ is finite, then $M = \langle x_1, \dots, x_n \rangle$
and M is said to be **finitely generated**.

②ⓐ Let $x_1, \dots, x_n \in M$. We say $\{x_1, \dots, x_n\}$ is **linearly independent** (over R) if for $(a_1, \dots, a_n) \in R^n$,

$$a_1 x_1 + \dots + a_n x_n = 0 \Rightarrow (a_1, \dots, a_n) = 0 \text{ in } R^n.$$

ⓑ A subset $A \subset M$ is **linearly independent** ^(over R) if every finite subset of A is linearly independent _(over R).

③ A subset $A \subset M$ is a **basis** of M (over R) if $M = \langle A \rangle$ _(over R) and A is linearly independent _(over R).

④ M is **free** _(over R) if M has a basis _(over R).

REMARKS.

- ① Not every R -module has a basis.
- ② A minimal generating set need not be lin. indep.
- ③ A maximal lin indep. set need not be a gen. set.

Q. If every R -module has a basis, is R a field?

(Yes. Take a non-field ring R and any non-trivial ideal $I \subsetneq R$.
Then, R/I has no lin. indep. set over R .)

Q. If an R -module M has a basis, does every basis have the same cardinality?

Ans. Yes. This is called the Invariant Basis Number (IBN) property of R .

Remark. This is not true if R is non-commutative. (That is, we can find a counterexample of a non-commutative ring.)
If R is a division ring, then again we have IBN.

Defⁿ. If M has a finite basis, say B , then we define $\text{rank}(M) := |B|$. } \text{well-defined by IBN}

If M is free with an infinite basis, $\text{rank}(M) := \infty$.

(When we do say "rank", we will usually mean "finite rank".)

EXAMPLES. ① $R^{(n)}$ is a free R -module of rank n
 $M_{m \times n}(R)$ of rank mn
 $R[x]$ of rank ∞

② Let A be a non-empty set and

$$F_0(A, R) = \{f: A \rightarrow R \mid f(a) = 0 \text{ for all but fin. many } a \in A\}.$$

Then, $F_0(A, R)$ is an R -module under pointwise operations.

In fact, $F_0(A, R)$ is a free R -module with basis $\{\chi_a\}_{a \in A}$, where

$$\chi_a(b) = \begin{cases} 0 & ; b \neq a \\ 1 & ; b = a \end{cases}$$

To see where the above set is generating, given any $f \in F_0(A, R)$, we can write

$$f = \sum_{a \in A} f(a) \chi_a.$$

↑ the sum is actually finite since $f(a)=0$ for all but finitely many a .
(it is to be understood that 0s are ignored.)

Q. What if we take $F(A, R)$? (All functions.)

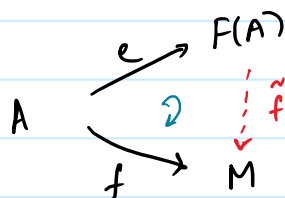
Universal Property of free modules:

Defn:

Given a non-empty set A , a free R -module on A is a pair $(F(A), e)$ where (i) $F(A)$ is an R -module, (ii) $e: A \rightarrow F(A)$ is a (set) function

satisfying:

Given an R -module M and a function $f: A \rightarrow M$, there exists a unique R -linear $\tilde{f}: F(A) \rightarrow M$ making the following diagram commute.



(That is, $\tilde{f}e = f$.)

REMARKS. ① Given $A = \emptyset$, a free R -module on A exists, and is unique up to isomorphism.

Moreover, $e: A \rightarrow F(A)$ is one-one and $F(A)$ is free with basis $\{e_a\}_{a \in A}$, where $e_a := e(a)$.

② If M is a free R -module, then $M \cong F(B)$, where B is any basis of M .

Thus, an R -module M is free iff $M \cong F(A)$ for some A .

What the universal property is really saying is that:
given a free R -module M with basis A , every R -linear
 $M \rightarrow N \rightarrow R\text{-module}$

is completely determined by its action on A .

(The above is in the sense that given any assignment of values on A , we do get an R -linear map.)

EXAMPLE: Given an R -module M , such that $M = \langle A \rangle$, we can write M as a quotient of $F(A)$.
(What we did last lec.)

Lecture 2 (12-01-2021)

12 January 2021 08:35

Weyl Algebra

Ex.

k is a field, $k[x_1, \dots, x_d]$

$\partial_1, \dots, \partial_d \rightarrow$ partial diff op.

$Ad(k) = k[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$ D -modules

\uparrow non-comm. How would you define products?

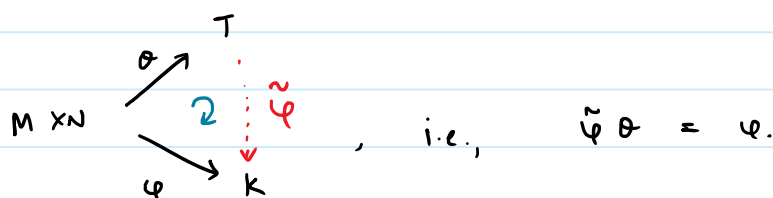
Tensor Product

Tensor product (of two modules) essentially converts the study of bilinear maps to linear maps.

Defn.

Given R -modules M and N , the **tensor product** of M and N (over R) is a pair (T, θ) , where T is an R -module, $\theta: M \times N \rightarrow T$ is R -bilinear satisfying:

Given (K, φ) where K is an R module, $\varphi: M \times N \rightarrow K$ is R -bilinear, there exists a unique R -linear map $\tilde{\varphi}: T \rightarrow K$ making the following diagram commute



(We are using "with", but can use "and" and we prove $M \otimes_R N \cong N \otimes_R M$.)

Thm. A tensor of M with N exists and is unique, up to isomorphism.

Uniqueness follows by universal property.

Notation: $M \otimes_R N$

Construction:

Want $M \times N \xrightarrow{\theta} T$ $\theta(x_1 + x_2, y) = \theta(x_1, y) + \theta(x_2, y)$

$\varphi \searrow \swarrow \tilde{\varphi}$
 $\quad \quad \quad K$

Step 1: Let $F = F(M \times N)$, the free module on the set $M \times N$.

We get a map $e: M \times N \rightarrow F(M, N)$
 $(x, y) \mapsto e(x, y)$

$\{e(x, y) : x \in M, y \in N\}$ is a basis for F .

Let G be the submodule of F generated by

- $e(x_1 + x_2, y) - e(x_1, y) - e(x_2, y)$
- $e(x, y_1 + y_2) - e(x, y_1) - e(x, y_2)$
- $e(ax, y) - a e(x, y)$
- $e(x, ay) - a e(x, y)$

$\forall x, x_1, x_2 \in M, \forall y, y_1, y_2 \in N, \forall a \in R$

Step 2: Define $T = F/G$. Let $\pi: F \rightarrow T$ be the natural map.
 Set $\pi(e(x, y)) =: x \otimes y$.

Note that $\left. \begin{aligned} (x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\ (ax) \otimes y &= a(x \otimes y) = x \otimes (ay) \end{aligned} \right\} \begin{aligned} &\forall x, \dots \in M \\ &\forall y, \dots \in N \\ &\forall a \in R \end{aligned}$

Consider $\theta = \pi e: M \times N \rightarrow T$
 $(x, y) \mapsto x \otimes y$

$\begin{array}{ccc} & e(x, y) & \\ & \uparrow e & \searrow \pi \\ M \times N & \xrightarrow{\theta} & T \\ (x, y) & & x \otimes y \end{array}$

Step 3. Now, suppose we are given a bilinear $\varphi: M \times N \rightarrow K$. (K is some R -module.)

$$M \times N \xrightarrow{\theta} T = F/G$$

$$\varphi \searrow \swarrow \exists! \tilde{\varphi}$$

Q1: Is there even a func. $\tilde{\varphi}$?

Q2: Is it R -linear?

Q3: Is it unique?

Note also

$$\begin{array}{ccc} & & F \\ & \nearrow e & \downarrow \\ M \times N & \xrightarrow{\theta} & T = F/G \\ & \searrow \varphi & \downarrow \\ & & K \end{array} \quad \swarrow \exists! \tilde{\varphi}$$

Note we have a set map $M \times N \xrightarrow{\theta} K$ which induces an R linear map $\bar{\varphi} : F \rightarrow K$. (UMP of free modules)

$$\begin{array}{ccc} & & F \\ & \nearrow e & \downarrow \\ M \times N & \xrightarrow{\theta} & T = F/G \\ & \searrow \varphi & \downarrow \\ & & K \end{array} \quad \swarrow \exists! \tilde{\varphi}$$

We now want to show that $\bar{\varphi}$ factors through T . It would suffice to show that $G \in \ker \bar{\varphi}$.

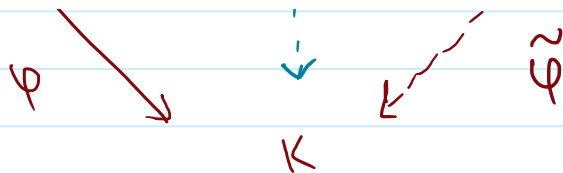
Using bilinearity of φ , it follows that all our (four types of) generators of G are in $\ker \bar{\varphi}$.

Thus, $\bar{\varphi}$ factors through quotient. That is, $\exists! R$ -linear $\tilde{\varphi} : T \rightarrow K$ s.t.

$$\begin{array}{ccc} F & \xrightarrow{\bar{\varphi}} & K \\ \downarrow \pi & \nearrow \tilde{\varphi} & \\ T & & \end{array}$$

commutes. That is, $\bar{\varphi} = \tilde{\varphi} \pi$.

$$\begin{array}{ccc} & & F \\ & \nearrow e & \searrow \pi \\ M \times N & \xrightarrow{\theta} & T = F/G \\ & \searrow \varphi & \downarrow \\ & & K \end{array} \quad \swarrow \exists! \tilde{\varphi}$$



Can now verify $\tilde{\varphi} \circ \theta = \varphi$. (Use commutation of diff. triangles.)
 Can also verify that $\tilde{\varphi}$ is unique R -linear such.

Basic Properties:

- (1) ["Identity"] $R \otimes_R M \cong M$
- (2) [Commutativity] $M \otimes_R N \cong N \otimes_R M$
- (3) [Associativity] $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$
- (4) [Distributivity] $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$

Q. How does get an R -linear map $M \otimes_R N \rightarrow L$?

A. Give an R -bilinear map $M \times N \rightarrow L$.

Pretty much the only way. $x \otimes y$ could be 0 even if $x, y \neq 0$.
 Thus, checking "well-defined"ness would become quite difficult.

Q. Let M be an R -module. $R \subset S$ subring.

Can you identify a natural S -module on $S \otimes_R M$?
 (Base change)