# Algebraic Topology

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In what follows, $I$ will denote the closed interval $[0, 1] \subset \mathbb{R}$.
Whenever we talk about a map $f : X \to Y$ between topological spaces $X$ and $Y$, we will always mean a *continuous function* $f$.
A path $\sigma$ in a space $X$ is a map $\sigma : I \to X$. If $x_0 = \sigma(0)$ and $x_1 = \sigma(1)$, we write this as

$$x_0 \xrightarrow{\sigma} x_1.$$

Moreover, $x_0$ and $x_1$ are called the *end points* of $\sigma$. In particular, $x_0$ is the initial point and $x_1$ is the terminal point.
All the topological spaces are assumed to be nonempty.

# §1. Homotopy of Paths

## §§1.1. The Fundamental Group

**Definition 1.1** (Homotopy)**.** Let $\sigma$ and $\tau$ be paths in a space $X$ with the same end points, i.e., $\sigma(0) = \tau(0)$ and $\sigma(1) = \tau(1)$.
We say that $\sigma$ and $\tau$ are *homotopic with ends points held fixed* written

$$\sigma \simeq \tau \quad \text{rel } \{0, 1\}$$

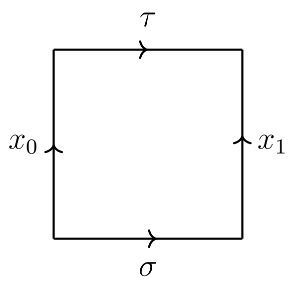
if there is a map $F : I \times I \to X$ such that

1. $F(s, 0) = \sigma(s)$ for all $s \in I$,
2. $F(s, 1) = \tau(s)$ for all $s \in I$,
3. $F(0, t) = x_0$ for all $t \in I$,
4. $F(1, t) = x_1$ for all $t \in I$.

$F$ is called a *homotopy* from $\sigma$ to $\tau$. We write

$$F : \sigma \simeq \tau \quad \text{rel } \{0, 1\}.$$

The above can be pictorially depicted as




The above picture is interpreted as follows:
Along the (bottom) line $t = 0$, $F$ agrees with $\sigma$ and along the (top) line $t = 1$, $F$ agrees with $t = 1$.
Similarly, along the (left) line $s = 0$, $F$ is identically equal to $x_0$ and along the (right) line $s = 1$, it is $x_1$.

In particular, if $\sigma$ is a *loop*, i.e., $x_0 = x_1$ and $e_{x_0}$ is the constant loop $s \mapsto x_0$ for $s \in I$, and if $\sigma \simeq e_{x_0}$ rel $\{0, 1\}$, we say that "$\sigma$ can be shrunk to a point," or is *homotopically trivial*.

**Proposition 1.2** ($\simeq$ is an equivalence relation)**.**

1. $\sigma \simeq \sigma \quad \mathrm{rel}\ \{0,1\}$,
2. $\sigma \simeq \tau \quad \mathrm{rel}\ \{0,1\} \implies \tau \simeq \sigma \quad \mathrm{rel}\ \{0,1\}$,
3. $\sigma \simeq \tau \quad \mathrm{rel}\ \{0,1\}$ and $\tau \simeq \rho \quad \mathrm{rel}\ \{0,1\} \implies \sigma \simeq \rho \quad \mathrm{rel}\ \{0,1\}$.

*Proof.* 1. Define $F(s,t) := \sigma(s)$.

2. Define $F(s,t) := F(s, 1-t)$.
3. Given $F : \sigma \simeq \tau \quad \mathrm{rel}\ \{0,1\}$ and $G : \tau \simeq \rho \quad \mathrm{rel}\ \{0,1\}$, define $H : I \times I \to X$ as

$$H(s,t) := \begin{cases} F(s,2t) & 0 \leq 2t \leq 1, \\ G(s,2t-1) & 1 \leq 2t \leq 2. \end{cases}$$

   Note that $F$ and $G$ do agree for $2t = 1$ since we have $F(s,1) = \tau(s) = G(s,0)$ for all $s \in I$. It is easy to see that $H$ is well-defined.
   Note that $H$ is continuous (by the pasting lemma) and it satisfies all the four properties of a homotopy (from $\sigma$ to $\rho$), since $F$ and $G$ do so. □

Thus, we can consider the homotopy classes $[\sigma]$ of paths $\sigma$ from $x_0$ to $x_1$ under the equivalence relation $\simeq$ . (Note very carefully that all paths in an equivalence class have the same end points.)

**Definition 1.3** (Multiplication of paths)**.** Let $\sigma$ be a path from $x_0$ to $x_1$ and $\tau$ from $x_1$ to $x_2$.
The product $\sigma * \tau$ is a path from $x_0$ to $x_2$ defined as

$$\sigma * \tau(s) := \begin{cases} \sigma(2s) & 0 \leq 2s \leq 1, \\ \tau(2s-1) & 1 \leq 2s \leq 2. \end{cases}$$

Once again, it's an easy check that $\sigma\tau$ is well-defined and continuous (using the pasting lemma).

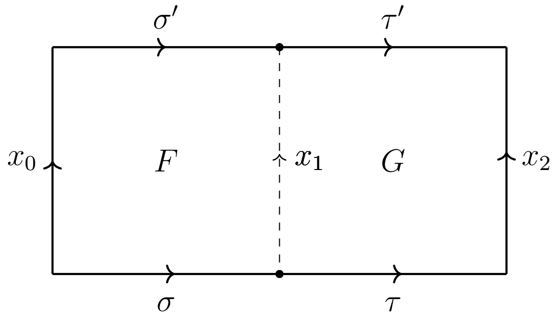
The above $\sigma * \tau$ is essentially the path from $x_0$ to $x_1$ obtained by first travelling from $x_0$ to $x_1$ via $\sigma$ and then from $x_1$ to $x_2$ via $\tau$.

We will now be lenient with notation and simply denote $\sigma * \tau$ as $\sigma\tau$ unless necessary. The next proposition shows how this product behaves with the equivalence relation.

**Proposition 1.4.**

$$\sigma \simeq \sigma' \quad \mathrm{rel}\ \{0,1\} \text{ and } \tau \simeq \tau' \quad \mathrm{rel}\ \{0,1\} \implies \sigma\tau \simeq \sigma'\tau' \quad \mathrm{rel}\ \{0,1\}.$$

*Proof.* The proof is motivated by the following diagram.




Given $F : \sigma \simeq \sigma' \quad \text{rel}\ \{0, 1\}$ and $G : \tau \simeq \tau' \quad \text{rel}\ \{0, 1\}$, define $H : I \times I \to X$ as

$$H(s,t) := \begin{cases} F(2s,t) & 0 \le 2s \le 1, \\ G(2s-1,t) & 1 \le 2s \le 2. \end{cases}$$

As earlier, $H$ is well-defined (since $F(1,t) = x_1 = G(0,t)$ for all $t \in I$) and continuous. Moreover, we have

$$H(0,t) = F(0,t) = x_0, \quad H(1,t) = G(1,t) = x_2,$$

$$H(s,0) = \begin{cases} F(2s,0) & 0 \le 2s \le 1, \\ G(2s-1,0) & 1 \le 2s \le 2 \end{cases} = \begin{cases} \sigma(2s) & 0 \le 2s \le 1, \\ \tau(2s-1) & 1 \le 2s \le 2 \end{cases} = \sigma\tau(s),$$

and similarly,

$$H(s,1) = \sigma'\tau'(s) \text{ for all } s \in I.$$

This shows that

$$H : \sigma\tau \simeq \sigma'\tau' \quad \text{rel}\ \{0,1\}.$$

□

**Definition 1.5** (Product of equivalence classes)**.** In view of the above proposition, we define

$$[\sigma] * [\tau] := [\sigma * \tau].$$

The above, of course, is defined only when the terminal point of $\sigma$ (and thus, any other representative of $[\sigma]$) equals the initial point of $\tau$ (and thus, any other representative of $[\tau]$).
As before, we shall drop the $*$ and simply write $[\sigma][\tau]$.

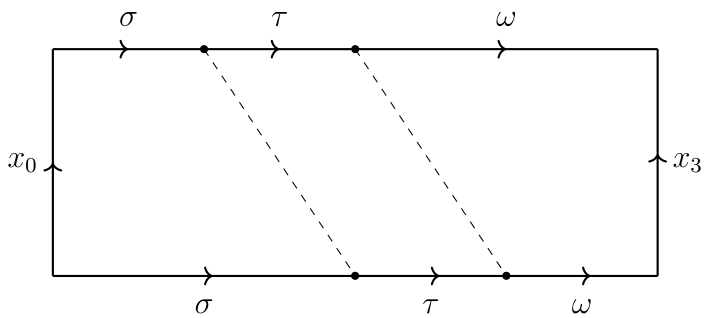
**Lemma 1.6.** Let $\sigma, \tau, \omega$ be paths such that the products $\sigma(\tau\omega)$ and $(\sigma\tau)\omega$ are defined. Then,

$$\sigma(\tau\omega) \simeq (\sigma\tau)\omega \quad \text{rel}\ \{0,1\}.$$

*Proof.* Let $x_0, x_1, x_2, x_3$ be points such that

$$x_0 \xrightarrow{\sigma} x_1 \xrightarrow{\tau} x_2 \xrightarrow{\omega} x_3.$$

We define a homotopy $F$ from $\sigma(\tau\omega)$ to $(\sigma\tau)\omega$. To motivate the definition of $F$, we may first visualise the homotopy as follows.




One can note that the top line depicts the path $(\sigma\tau)\omega$ and the bottom $\sigma(\tau\omega)$.

We define $F : I \times I \to X$ piece-wise on the three regions (from left to right) as follows:

$$F(s,t) := \begin{cases} \sigma\left(\dfrac{4s}{2-t}\right) & 0 \le s \le \dfrac{1}{4}(2-t), \\[2ex] \tau\left(4s+2-t\right) & \dfrac{1}{4}(2-t) \le s \le \dfrac{1}{4}(3-t), \\[2ex] \omega\left(\dfrac{4s+t-3}{t+1}\right) & \dfrac{1}{4}(3-t) \le s \le 1. \end{cases}$$

It is clear that $F$ is continuous on each piece. By the pasting lemma, it is continuous everywhere.
The four properties of being a homotopy are also clear, by construction. (The diagram makes it clear why.) □

**Definition 1.7** (Inverse path)**.** Given a path $\sigma$ from $x_0$ to $x_1$, its *inverse path* $\sigma^{-1}$ is a path from $x_1$ to $x_0$ given by

$$\sigma^{-1}(s) := \sigma(1-s), \qquad s \in I.$$

The above is simply "travelling backwards $\sigma$."

**Lemma 1.8.** Let $\sigma, \sigma' : I \to X$ be paths such that $\sigma \simeq \sigma' \;\; \text{rel } \{0,1\}$. Then,

$$\sigma^{-1} \simeq \sigma'^{-1} \;\; \text{rel } \{0,1\}.$$

*Proof.* Let $F : \sigma \simeq \sigma' \quad \text{rel } \{0, 1\}$ be a homotopy. Then, $F'(s, t) := F(1 - s, t)$ is a homotopy between the inverses. $\square$

**Definition 1.9** (Inverse class)**.** Let $\sigma : I \to X$ be a path. We define the inverse of the class $[\sigma]$ as

$$[\sigma]^{-1} := [\sigma^{-1}].$$

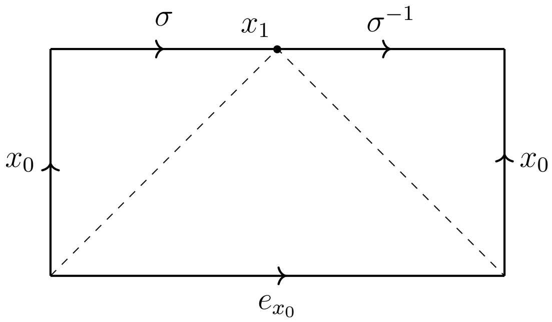
In view of the above lemma, the above definition is indeed well-defined.

**Lemma 1.10.** Given any path $\sigma$ from $x_0$ to $x_1$, we have

$$e_{x_0} \simeq \sigma\sigma^{-1} \quad \text{rel } \{0, 1\},$$

where $e_{x_0}$ denotes the constant loop at $x_0$.

*Proof.* As usual, we motivate the proof with a diagram. In this case, it is the following:




The homotopy $F : I \times I \to X$ in this case, is defined as

$$F(s, t) := \begin{cases} \sigma(2s) & 0 \leq 2s \leq t, \\ \sigma(t) & t \leq 2s \leq 2 - t, \\ \sigma^{-1}(2s - 1) & 2 - t \leq 2s \leq 2. \end{cases}$$

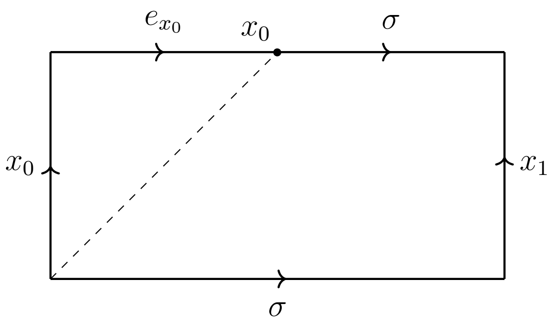
It is clear that the piecewise definitions agree on the dashed line $2s = t$. Observe that $\sigma^{-1}(2s - 1) = \sigma(2 - 2s)$ and thus, the functions do agree on the dashed line $2s = 2 - t$ as well.
One can easily check that the four properties of the homotopy are satisfied. To see the bottom line property, note that $F(s, 0) = \sigma(0)$ (using the second piece definition) and $\sigma(0) = x_0 = e_{x_0}(s)$ for all $s \in I$. $\square$

Note that since $(\sigma^{-1})^{-1} = \sigma$, the above also shows that $\sigma^{-1}\sigma = e_{x_1}$.

**Lemma 1.11.** Let $x_0 \xrightarrow{\sigma} x_1$ and $e_{x_0}$ be the constant path at $x_0$. Then,

$$\sigma \simeq e_{x_0}\sigma \quad \text{rel } \{0, 1\}.$$

*Proof.* The proof is motivated by this diagram.




The homotopy is $F : I \times I \to X$ defined as

$$F(s,t) := \begin{cases} x_0 & 0 \le 2s \le t, \\ \sigma\left(\dfrac{2s-t}{2-t}\right) & t \le 2s \le 2. \end{cases}$$ □

As one would expect, we have a lemma in the other direction as well.

**Lemma 1.12.** Let $x_1 \xrightarrow{\sigma} x_0$ and $e_{x_0}$ be the constant path at $x_0$. Then,

$$\sigma \simeq \sigma e_{x_0} \quad \text{rel } \{0,1\}.$$

*Proof.* Similar as in the last case and we omit it. □

The astute reader might have sensed a group sneaking around the corner.
However, note that the product of equivalence classes defined above is not a binary operation unless the endpoints are the same. Due to this, we restrict ourselves to loops in the next theorem.

**Theorem 1.13.** Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in $X$ at $x_0$.
If multiplication in $\pi_1(X, x_0)$ is defined as above, $\pi_1(X, x_0)$ becomes a group, in which the neutral element is the class $[e_{x_0}]$ and the inverse of a class $[\sigma]$ is the class of the inverse $[\sigma^{-1}]$.

*Proof.* Interpreting Lemmas 1.6 to 1.12 as equalities of the equivalence classes shows that $\pi_1(X, x_0)$ verifies the group axioms. □

The next proposition tells us how $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are related in the case that $x_0$ and $x_1$ lie in the same path-connected component. (In the case that they do not, nothing can be said.)

**Proposition 1.14.** Let $\alpha$ be a path from $x_0$ to $x_1$. The mapping $\widehat{\alpha}$ defined by

$$[\sigma] \mapsto [\alpha^{-1}] * [\sigma] * [\alpha] = [\alpha^{-1}\sigma\alpha]$$

is an isomorphism of the group $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$.

Note that the above is well-defined since $*$ is well-defined.

*Proof.* We first note that if $[\sigma] \in \pi_1(X, x_0)$, then $\alpha^{-1}\sigma\alpha$ is path as follows:

$$x_1 \xrightarrow{\alpha^{-1}} x_0 \xrightarrow{\sigma} x_0 \xrightarrow{\alpha} x_1$$

and thus, $[\alpha^{-1}\sigma\alpha]$ is indeed an element of $\pi_1(X, x_1)$.
Moreover, note that

$$\begin{aligned} \widehat{\alpha}([\sigma\sigma']) &= [\alpha^{-1}\sigma\sigma'\alpha] \\ &= [\alpha^{-1}\sigma][\sigma'\alpha] \\ &= [\alpha^{-1}\sigma][\alpha\alpha^{-1}][\sigma'\alpha] \\ &= [\alpha^{-1}\sigma\alpha][\alpha^{-1}\sigma'\alpha] \\ &= \widehat{\alpha}([\sigma])\widehat{\alpha}([\sigma']). \end{aligned}$$

This shows that $\widehat{\alpha}$ is a homomorphism. That this is an isomorphism follows by noting that it has as inverse $\widehat{\alpha^{-1}}$. □

**Corollary 1.15.** If $X$ is pathwise connected, the group $\pi_1(X, x_0)$ is independent of the point $x_0$, up to isomorphism.

Note that if $C$ is a connected component of $X$ containing $x_0$, then $\pi_1(X, x_0) = \pi_1(C, x_0)$ since any loop at $x_0$ must necessarily lie in $C$. For this reason, we might as well only work with pathwise connected spaces.

**Definition 1.16.** If $X$ is pathwise connected, we write $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it *the fundamental group* of $X$.

Note that this group depends on $x_0$ in the sense that the elements of the group depend on the base point $x_0$ but the isomorphism class does not.

**Definition 1.17** (Simply connected)**.** A space $X$ is called simply connected if it is pathwise connected and its fundamental group is trivial.

**Lemma 1.18.** Let $X$ be simply connected. If $\sigma$ and $\tau$ are paths in $X$ with the same initial and terminal points, then $\sigma \simeq \tau \quad \text{rel } \{0, 1\}$.

*Proof.* Let the initial and terminal points be $x_0$ and $x_1$, respectively.
Consider the path $\sigma\tau^{-1}$, which is path at $x_0$. Since $X$ is simply connected, we have

$$\sigma\tau^{-1} \simeq e_{x_0} \quad \text{rel } \{0, 1\}.$$

By the previously seen properties, we see that

$$(\sigma\tau^{-1})\tau \simeq e_{x_0}\tau \quad \text{rel } \{0, 1\}$$

or

$$\sigma \simeq \tau \quad \text{rel } \{0, 1\}.$$ □

## §§1.2. Functoriality

We now wish to turn $\pi_1$ into a functor. Since we need to take care of the base points, we look at the category of *Pointed Topological spaces*.

**Definition 1.19** (Pointed Topological Spaces)**.** The category $\mathsf{Top}_\bullet$ of *pointed topological spaces* is the category whose objects and morphisms are given as follows:

- Objects: Pairs $(X, x_0)$ where $X$ is a topological space and $x_0 \in X$,
- Morphisms: $f : (X, x_0) \to (Y, y_0)$ such that $f : X \to Y$ is a continuous function and $f(x_0) = y_0$.

That the above is a category can be easily verified.

**Definition 1.20.** Let $h : (X, x_0) \to (Y, y_0)$ be a morphism. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([\sigma]) = [h \circ \sigma].$$

The map $h_*$ is called the *homomorphism induced by* $h$, relative to the base point $x_0$.

To see that $h_*$ is well-defined, we note that if

$$F : \sigma \simeq \sigma' \quad \text{rel } \{0, 1\}$$

for loops $\sigma, \sigma'$ in $X$ at $x_0$, then

$$h \circ F : h \circ \sigma \simeq h \circ \sigma' \quad \text{rel } \{0, 1\}.$$

That is to say, if two loops at $x_0$ are homotopic, then so are the loops obtained by pre-composing $h$.

To see that $h_*$ is a homomorphism, first note that

$$(h \circ \sigma)(h \circ \sigma') = h \circ (\sigma\sigma').$$

(This follows from the definition of the product of paths.)
Then, we see that

$$h_*([\sigma\sigma']) = [h \circ (\sigma\sigma')] = [h \circ \sigma][h \circ \sigma'] = h_*([\sigma])h_*([\sigma']).$$

**Theorem 1.21** (Functoriality)**.** If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are morphisms, then

$$(k \circ h)_* = k_* \circ h_*.$$

If $i : (X, x_0) \to (X, x_0)$ is the identity map, then $i_*$ is the identity homomorphism.

*Proof.* By definition, we have

$$\begin{aligned}(k \circ h)_*([\sigma]) &= [(k \circ h) \circ \sigma] \\ &= [k \circ (h \circ \sigma)] \\ &= k_*([h \circ \sigma]) \\ &= k_*(h_*([\sigma])) \\ &= (k_* \circ h_*)([\sigma]).\end{aligned}$$

Thus, $(k \circ h)_* = k_* \circ h_*$.

Now, if $i$ is the identity map, then we have

$$i_*([\sigma]) = [i \circ \sigma] = [\sigma],$$

showing that $i_*$ is the identity map of $\pi_1(X, x_0)$. □

The above then shows that $\pi_1$ defines a functor from the category $\mathsf{Top}_*$ to $\mathsf{Grp}$.
Since functors preserve isomorphisms in general, we get the following corollary.

**Corollary 1.22.** If $h : (X, x_0) \to (Y, y_0)$ is a morphism such that $h : X \to Y$ is a homeomorphism, then

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Since we aren't discussing Category Theory, we give a proof for this special example of functors.

*Proof.* Let $h^{-1} : Y \to X$ be the inverse, which is continuous since $h$ is a homeomorphism. Moreover, $h^{-1}(y_0) = x_0$ and thus, $h^{-1} : (Y, y_0) \to (X, x_0)$ is a morphism and the inverse of $h$.
Now, note that,

$$(h_*) \circ ((h^{-1})^*) = (h \circ h^{-1})^* = (\mathrm{id}_{(Y,y_0)})^* = \mathrm{id}_{\pi_1(Y,y_0)},$$

by functoriality. Similarly, we have

$$((h^{-1})^*) \circ (h_*) = \mathrm{id}_{(X,x_0)},$$

proving the corollary. $\square$

# §2. Homotopy of Maps

In the previous section, we talked about homotopy of special types of maps. More precisely, we only considered maps $I \to X$. However, we can replace $I$ by an arbitrary topological space $Y$. In the place of endpoints, we just consider a subspace $A \subset Y$.

**Definition 2.1** (Relative homotopy)**.** Given maps $f, g : Y \to X$ such that $f|_A = g|_A$, we say $f$ and $g$ are homotopic relative to $A$ written

$$f \simeq g \quad \text{rel} \ \ A$$

if there is a map $F : Y \times I \to X$ satisfying

1. $F(y, 0) = f(y)$ for all $y \in Y$,
2. $F(y, 1) = g(y)$ for all $y \in Y$,
3. $F(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$.

This map $F$ is called a homotopy from $f$ to $g$ relative to $A$ and we write

$$F : f \simeq g \quad \text{rel} \ \ A.$$

Note that the "second coordinate" above is still $I$.
Note that (3) is satisfied vacuously if $A = \varnothing$ and we have $f|_A = g|_A$ for all maps $f, g : Y \to X$. Keeping this in mind, we have the following definition.

**Definition 2.2** (Homotopy)**.** Maps $f, g : Y \to X$ are said to be *homotopic* if $f$ and $g$ are homotopic relative to $\varnothing$.
We write this more simply as

$$f \simeq g.$$

Moreover, any $F$ as before is simply called a homotopy from $f$ to $g$.
As before, we write

$$F : f \simeq g.$$

Once again, we obtain an equivalence. The homotopies defined as in the proof of Proposition 1.2 work again.

**Definition 2.3** (Contractible space)**.** If $X$ is a topological space such that the identity map on $X$ is homotopic to a constant map on some point in $X$, we say that $X$ is *contractible*.

**Proposition 2.4.** $X$ is contractible if and only if for any space $Y$, any two maps of $Y$ into $X$ are homotopic. A contractible space is pathwise connected.

*Proof.* ( $\Longrightarrow$ ) Let $X$ be contractible and $Y$ be any space. Fix any $x_0 \in X$ such that $\mathrm{id}_X$ is homotopic to the constant map $e_{x_0} : X \to X$.
Let $f_{x_0} : Y \to X$ denote the constant map $y \mapsto x_0$.
Now, given any map $f : Y \to X$, we show that it is homotopic to $f_{x_0}$.
This will prove that any two maps of $Y$ into $X$ are homotopic since $\simeq$ is an equivalence relation.
Let $H : \mathrm{id}_X \simeq e_{x_0}$ be any homotopy. Then, we have

$$H(x, 0) = x, \quad H(x, 1) = x_0; \qquad \text{for all } x \in X.$$

(Note that $H$ is continuous.)
Now, we define $F : Y \times I \to X$ as

$$F(y, t) = H(f(y), t).$$

It is clear that $F$ is a map. (That is, $F$ is continuous.)
Moreover, note that

$$F(y, 0) = H(f(y), 0) = f(y), \quad F(y, 1) = H(f(y), 1) = x_0 = f_{x_0}(y); \qquad \text{for all } y \in Y.$$

This shows that $F : f \simeq f_{x_0}$, as desired.

( $\Longleftarrow$ ) To show that $X$ is contractible, simply consider $Y = X$ and consider the maps $\mathrm{id}_X$ and $e_{x_0}$. (Both of these are indeed continuous.)
By hypothesis, these maps are homotopic and by definition, $X$ is contractible.

Now, we show that $X$ is pathwise connected assuming that it is contractible.
Let $x_0$ and $x_1$ be any two points in $X$. As $X$ is contractible, ( $\Longrightarrow$ ) tells us that the maps $e_{x_0}$ and $e_{x_1}$ are homotopic.
Let $F$ be any homotopy from $e_{x_0}$ and $e_{x_1}$. Define $\sigma : I \to X$ as

$$\sigma(t) := F(x_0, t).$$

$\sigma$ is clearly continuous. Moreover, we have

$$\begin{aligned} \sigma(0) &= F(x_0, 0) = e_{x_0}(x_0) = x_0, \\ \sigma(1) &= F(x_0, 1) = e_{x_1}(x_0) = x_1. \end{aligned}$$

Thus, $\sigma$ is path from $x_0$ to $x_1$ in $X$, proving the proposition. □

**Example 1.** Every convex subset $X$ of Euclidean space is contractible.
Given maps $f_1, f_2 : Y \to X$, we have a homotopy $F : f_1 \simeq f_2$ given by

$$F(y, t) = t f_2(y) + (1 - t) f_1(y), \quad y \in Y, t \in I.$$

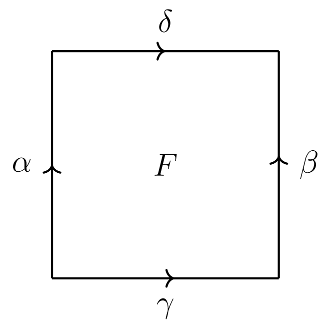
By the convexity assumption, the above $F$ is indeed a map into $X$.
By the previous proposition, this shows that $X$ is contractible.

**Example 2.** $\mathbb{R}^n$ is contractible for any $n$.
To see this, we could either appeal to the previous example or do it directly by defining a homotopy $F : e_0 \simeq \mathrm{id}_{\mathbb{R}^n}$ as

$$F(x,t) = tx.$$

We would now like to show that any contractible space is simply connected. What we do know is that any loop would be homotopic to a point. However, we do not know if this homotopy is relative to $\{0,1\}$. Indeed, to show that we do have a homotopy relative to $\{0,1\}$, we would need to use the fact that $X$ is contractible once again.
Before proving that, we first look at a lemma.

**Lemma 2.5.** Let $F : I \times I \to X$ be a map. Set $\alpha(t) = F(0,t)$, $\beta(t) = F(1,t)$, $\gamma(s) = F(s,0)$, and $\delta(s) = F(s,1)$, as in the diagram




Then, $\delta = \alpha^{-1}\gamma\beta$.

*Proof.* The proof is quite intuitive. First, we define the paths

$$\sigma : I \to I \times I, \quad \tau : I \to I \times I$$

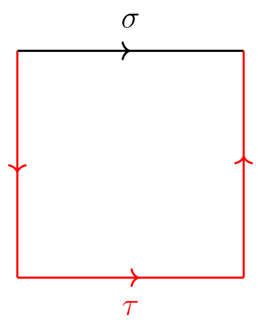
as

$$\sigma(s) := (t,1)$$

and

$$\tau(s) := \begin{cases} (0, 1-4s) & 0 \le 4s \le 1, \\ (4s-1, 0) & 1 \le 4s \le 2, \\ (1, 2s-1) & 1 \le 2s \le 2. \end{cases}$$

These paths are the following ones in $I^2$ :




As it should be clear from the diagram (and one can easily check), we have

$$\delta = F \circ \sigma, \quad (\alpha^{-1}\gamma)\beta = F \circ \tau.$$

(Note that the bracketing in $(\alpha^{-1}\gamma)\beta$ is necessary.)
Also, since $I^2$ is convex, we see that $\sigma$ and $\tau$ are homotopic relative to $\{0, 1\}$ with $H : I \times I \to I \times I$ being a required homotopy defined as

$$H(s, t) := (1 - t)\sigma(s) + t\tau(s).$$

Thus,

$$\begin{aligned} &F \circ H : F \circ \sigma \simeq F \circ \tau \quad \text{rel } \{0, 1\} \\ \implies &F \circ H : \delta \simeq (\alpha^{-1}\gamma)\beta \quad \text{rel } \{0, 1\}, \end{aligned}$$

as desired. □

**Theorem 2.6.** Let $X$ be a contractible space. Then, $X$ is simply connected.

*Proof.* Note that by Proposition 2.4, we know that $X$ is pathwise connected. Now we show that $\pi_1(X)$ is trivial.
Let $x_0 \in X$ be arbitrary and $\alpha : I \to X$ be a loop at $x_0$ in $X$.
If we show that $\alpha \simeq e_{x_0}$ rel $\{0, 1\}$, then we are done.

To do this, we will use the earlier lemma after constructing an appropriate $F$.
Using that $X$ is contractible, we fix a homotopy $H : \mathrm{id}_X \simeq f_{x_0}$ where $f_{x_0} : X \to X$ is the constant function $x \mapsto x_0$.
(This is different from $e_{x_0}$ since the domains are different in general.)
To recall, $H$ has the following properties:

$$H(x, 0) = x, \ H(x, 1) = x_0 \quad \text{for all } x \in X.$$

Now, we define $F : I \times I \to X$ as

$$F(s, t) := H(\sigma(s), t).$$

Now, note that if we set $\alpha, \beta, \gamma, \delta$ as in the previous lemma, we have

$$\begin{aligned}\alpha(t) = F(0,t) = H(\sigma(0),t) &= H(x_0,t) \\ &= H(\sigma(1),t) = F(1,t) = \beta(t),\end{aligned}$$

$$\gamma(s) = F(s,0) = H(\sigma(s),0) = \sigma(s),$$
$$\delta(s) = F(s,1) = H(\sigma(s),1) = x_0.$$

In other words, we have

$$\alpha = \beta, \gamma = \sigma, \delta = e_{x_0}.$$

By the previous lemma, we know that $[\delta] = [\alpha^{-1}\gamma\beta]$, where $[.]$ is the homotopy class of a path relative to $\{0,1\}$. Thus, we have

$$\begin{aligned} [e_{x_0}] &= [\alpha^{-1}\sigma\alpha] \\ \implies [\alpha][e_{x_0}][\alpha^{-1}] &= [\sigma] \\ \implies [e_{x_0}] &= [\sigma] \\ \implies e_{x_0} &\simeq \sigma \quad \text{rel } \{0,1\}, \end{aligned}$$

finishing the proof. □

**Proposition 2.7.** Let $f, g : Y \to X$ be maps which are homotopic by means of a homotopy $F : Y \times I \to X$.
Let $y_0 \in Y$, $x_0 := f(y_0) = F(y_0, 1)$, and $x_1 := g(y_0) = F(y_0, 1)$.
Let $\alpha : I \to X$ be a path from $x_0$ to $x_1$ given by

$$\alpha(t) = F(y_0, t) \quad t \in I.$$

Then, the following diagram commutes.

$$\begin{tikzcd} \pi_1(Y, y_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(X, x_0) \arrow[d, "\widehat{\alpha}"] \\ & \pi_1(X, x_1) \end{tikzcd}$$

*Proof.* The diagram commuting is just saying that

$$\widehat{\alpha} \circ f_* = g_*.$$

Let $[\sigma] \in \pi_1(Y, y_0)$ be arbitrary. Showing that the above is true is equivalent to showing that

$$(\widehat{\alpha} \circ f_*)([\sigma]) = g_*([\sigma]).$$

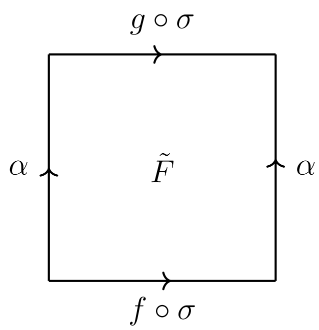
Using the definitions of $\widehat{\alpha}$ and $f_*$, we note that

$$\begin{aligned}
&(\widehat{\alpha} \circ f_*)([\sigma]) = g_*([\sigma]) \\
\iff\ &\widehat{\alpha}(f_*([\sigma])) = g_*([\sigma]) \\
\iff\ &\widehat{\alpha}([f \circ \sigma]) = [g \circ \sigma] \\
\iff\ &[\alpha^{-1}(f \circ \sigma)\alpha] = [g \circ \sigma].
\end{aligned}$$

Now, defining $\tilde{F} : I \times I \to X$ as

$$\tilde{F}(s,t) = F(\sigma(s), t).$$

Then, we have the following diagram as in Lemma 2.5 which proves the proposition.




To see that the sides are indeed as labeled, recall that $\sigma$ is a loop at $y_0$ and note that

$$\begin{aligned}
\tilde{F}(0,t) &= F(\sigma(0), t) = F(y_0, t) = \alpha(t), \\
\tilde{F}(1,t) &= F(\sigma(1), t) = F(y_0, t) = \alpha(t), \\
\tilde{F}(s,0) &= F(\sigma(s), 0) = g(\sigma(s)) = (g \circ \sigma)(s), \\
\tilde{F}(s,1) &= F(\sigma(s), 1) = f(\sigma(s)) = (f \circ \sigma)(s).
\end{aligned}$$

By the conclusion of Lemma 2.5, we are done. $\square$

Recall that $\widehat{\alpha}$ is an isomorphism and thus, we get the following corollary.

**Corollary 2.8.** With the same setup as above, $f_*$ is an isomorphism if and only if $g_*$.

What the above corollary says is that if $f$ and $g$ are homotopic, then $f_*$ is an isomorphism iff $g_*$ is.

**Definition 2.9** (Homotopy equivalence)**.** A map $f : Y \to X$ is said to be a *homotopy equivalence* if there exists a map $f' : X \to Y$ such that

$$\begin{aligned} ff' &\simeq \mathrm{id}_X, \\ f'f &\simeq \mathrm{id}_Y\,. \end{aligned}$$

If such a map exists, we say that $X$ and $Y$ are *homotopically equivalent spaces*.

It can be checked that being homotopically equivalent is an "equivalence relation."

**Corollary 2.10.** If $f : Y \to X$ is a homotopy equivalence, then $f_*$ is an isomorphism

$$\pi_1(Y, y_0) \to \pi_1(X, f(y_0))$$

for any $y_0 \in Y$.

*Proof.* Let $f' : X \to Y$ be as in the definition.
Then, $ff' \simeq \mathrm{id}_X$. By the previous corollary, we have that $(ff')_*$ is an isomorphism. (Since $(\mathrm{id}_X)_*$ is.)
Similarly, $(f'f)_*$ is an isomorphism. Since $(ff')_* = f_* \circ f'_*$ and $(f'f)_* = f'_* \circ f_*$, we see that $f_*$ is a bijection and hence, an isomorphism. □

The above shows that the fundamental group of a path-connected space is a *homotopy invariant*. We had shown earlier that this was a topological invariant.
Note that being homotopically equivalent is a weaker concept than being topologically invariant (i.e., homeomorphic). Clearly, if $f : X \to Y$ is a homeomorphism, it also a homotopy equivalence with $f' = f^{-1}$.
However, the closed interval $I$ is homotopically equivalent to the point space but clearly not homeomorphic. In fact, one can note that $X$ is contractible if and only if it is homeomorphic to a point.

# §3. Fundamental Group of the Circle

In this section, we prove a more general result. $S^1$ will turn out to be a special case of that. First, we need a lemma.

**Lemma 3.1.** Let $K$ be a compact metric space and $G$ a topological group. Let $V \subset G$ be open such that $1 \in V$.
If $f : K \to G$ is continuous, then there exists $\delta > 0$ such that

$$d(k, k') < \delta \implies f(k)(f(k'))^{-1} \in V.$$

The above is essentially mimicking something like "uniform continuity."

*Proof.*

**Claim 1.** There exists an open set $U \subset G$ such that

1. $1 \in U \subset V$,
2. $g, g' \in U \implies gg^{-1} \in V$.

*Proof.* The function $\varphi : G \times G \to G$ defined as

$$\varphi(g, g') := g(g')^{-1}$$

is continuous. Thus, $\varphi^{-1}(V)$ is open.
Note that $(1, 1) \in \varphi^{-1}(V)$. Thus, there exists a basis element of the form $U_1 \times U_2$ satisfying

$$(1, 1) \in U_1 \times U_2 \subset \varphi^{-1}(V).$$

Let $U := U_1 \cap U_2 \cap V$. Clearly, $U$ is open and $1 \in U \subset V$.
Moreover,

$$g, g' \in U \implies (g, g') \in U_1 \times U_2 \subset \varphi^{-1}(V) \implies \varphi(g, g') \in V \implies g(g')^{-1} \in V,$$

as desired. $\square$

With this, we can mimic the proof of continuous functions being uniformly continuous on compact sets. (The above $U$ will help us use "triangle inequality" in the codomain.) Let $U$ be as in the above claim.

**Claim 2.** Given any $k \in K$, there exists $\delta_k > 0$ such that

$$d(k, k') < \delta_k \implies f(k)(f(k'))^{-1} \in U,$$
$$d(k, k') < \delta_k \implies f(k')(f(k))^{-1} \in U.$$

*Proof.* The function $f_k : K \to G$ defined by $f_k(k') = f(k)(f(k'))^{-1}$ is continuous with $f_k(k) = 1 \in U$.
Consider the open set $f_k^{-1}(U)$. Since it contains $k$, there exists $\delta > 0$ such that $B_\delta(k) \subset f_k^{-1}(U)$. Thus, if $k' \in B_\delta(k)$, then $f_k(k') \in U$, as desired for the first condition.

Note that we can find a suitable $\delta'_k$ for the other condition as well. Taking the minimum of the two proves the claim. □

Let $V_k = B_{\delta_k/2}(k)$. Clearly, $\{V_k\}_{k\in K}$ is an open cover of $K$. Since $K$ is compact, we may extract a finite subcover.
Let $k_1, \ldots, k_n$ be the indices of one such. Set

$$\delta := \min_{1\le i\le n} \frac{1}{2}\delta_{k_i}.$$

Clearly, $\delta > 0$. Moreover, it satisfies the condition of the lemma. To see this, let $k, k' \in K$ be such that $d(k, k') < \delta$.
Since $\{V_{k_i}\}_{1\le i\le n}$, is an open cover, $k$ lies in $V_{k_i}$ for some $1 \le i \le n$. That is, $2d(k, k_i) < \delta_i$. Now, using triangle inequality, note that

$$d(k', k_i) \le d(k', k) + d(k, k_i) < \delta + \frac{1}{2}\delta_i \le \frac{1}{2}\delta_i + \frac{1}{2}\delta_i = \delta_i.$$

Thus, both $k$ and $k'$ are at most $\delta_i$ from $k_i$. By the definition of $\delta_i$ (from Claim 2), we see that $f(k)(f(k_i))^{-1} \in U$ and $f(k_i)(f(k'))^{-1} \in U$.
By the property of $U$, we have

$$(f(k)(f(k_i))^{-1})((k_i)(f(k'))^{-1}) = f(k)(f(k'))^{-1} \in V,$$

as desired. □

Now, for the remainder of this section, we shall fix $G$ as any simply connected topological group and $H \le G$ is a *discrete* normal subgroup of $G$. We will show that $\pi_1(G/H, 1) \cong H$.
(In the special case that $G = \mathbb{R}$ and $H = \mathbb{Z}$, we see that $\pi_1(S^1, 1) \cong \mathbb{Z}$ or simply, $\pi_1(S^1) \cong \mathbb{Z}$.)

We also fix the map $\varphi : G \to G/H$ to be the projection $g \mapsto gH$.

**Lemma 3.2.** There exists an open neighbourhood $U$ of $1$ in $G$ which is mapped homeomorphically onto an open neighbourhood $V$ of $1$ in $G/H$ be $\varphi$.

*Proof.* Since $H$ is discrete, $\{1\}$ is open in $H$. Thus, there exists an open neighbourhood $U_1$ of 1 in $G$ such that $U_1 \cap H = \{1\}$.
As in claim 1 of the previous proof, we may find a subset $U \subset U_1$ such that $g, g' \in U \implies gg'^{-1} \in U_1$. Clearly, $U \cap H = \{1\}$ as well.

> **Claim 1.** $\varphi|_U$ is injective.
>
> *Proof.* Let $g_1, g_2 \in U$ with $\varphi(g_1) = \varphi(g_2)$.
> Then, $g_1 H = g_2 H$ or $Hg_1 = Hg_2$ or $Hg_1 g_2^{-1} = H$ or $g_1 g_2^{-1} \in H$.
> Since $g_1, g_2 \in U$, we also have $g_1 g_2^{-1} \in U_1$. Since $U_1 \cap H = \{1\}$, we see that $g_1 g_2^{-1} = 1$ or $g_1 = g_2$. □

Let $V = \varphi(U)$. Clearly, $\varphi$ maps $U$ bijectively onto $V$, in view of the previous claim. Moreover, this must be a homeomorphism. To see this, we recall a general result.

> **Claim 2.** The quotient map $\phi : G \to G/H$ is open.
>
> *Proof.* Let $W$ be an open subset of $G$. The set
>
> $$WH = \{wh : w \in W, h \in H\}$$
>
> is open since $WH = \bigcup_{h \in H} Wh$, which is a union of open subsets of $G$ since right multiplication is a homeomorphism.
> Note that $\varphi^{-1}(\varphi(W)) = WH$. Since $\varphi$ is a quotient map and $WH$ is open, we see that $\varphi(W)$ is open, as desired. □

Thus, we see that $\varphi|_U : U \to V$ is a bijective open map. In particular, it is a homeomorphism. □

For the remainder of this section, we fix $U \subset G$ and $V \subset G/H$ as above. Moreover, we fix

$$\psi := (\varphi|_U)^{-1}.$$

By our above discussion, $\psi : V \to U$ is a continuous function.
For better clarity, we shall use $1$ for the identity of $G/H$ and $1_G$ for the identity of $G$.

Now, we prove two key lemmas.

**Lemma 3.3** (Lifting Lemma)**.** If $\sigma$ is a path in $G/H$ with initial point $1$, there is a unique path $\sigma'$ in $G$ with initial point $1_G$ such that

$$\varphi \circ \sigma' = \sigma.$$

**Lemma 3.4** (Covering Homotopy Lemma)**.** If $\tau$ is also a path in $G/H$ with the initial point $1$ such that

$$F : \sigma \simeq \tau \quad \text{rel}\ \{0, 1\},$$

then there is a unique $F' : I \times I \to G$ such that

$$\begin{aligned} F' : \sigma' \simeq \tau' \quad \text{rel}\ \{0, 1\}, \\ \varphi \circ F' = F. \end{aligned}$$

(Note that $\tau'$ above is the unique path in $G$ as given by the Lifting Lemma.)

*Proof.* We prove both results together.
Let $(K, f : Y \to G/H, 0 \in K)$ be either $(I, \sigma, 0 \in I)$ or $(I \times I, F, (0, 0) \in I \times I)$. The first choice corresponds to Lemma 3.3 and the second to Lemma 3.4.

For the sake of less ugly notation, we shall use $a/b$ or $\frac{a}{b}$ to denote $ab^{-1}$ for $a, b \in G/H$. (Note that we are fixing this to mean $ab^{-1}$ without any assumption of abelianity.)

Since $K$ is compact, there exists $\epsilon > 0$ such that

$$|k - k'| < \epsilon \implies f(k)/(f(k')) \in V,$$

by Lemma 3.1.
In particular, for such $k$ and $k'$, $\psi\left(\dfrac{f(k)}{f(k')}\right)$ is defined. Fix $N \in \mathbb{N}$ large enough such that

$$|k| < N\epsilon$$

for all $k \in K$. (This can be done since $K$ is bounded by 2.)
Now, define

$$f' : K \to G$$

by

$$\begin{aligned} f'(k) :=& \psi\left(f(k)/f\left(\frac{N-1}{N}k\right)\right) \\ & \cdot \psi\left(f\left(\frac{N-1}{N}k\right) \Big/ f\left(\frac{N-2}{N}k\right)\right) \\ & \cdots \psi\left(f\left(\frac{1}{N}k\right)/f\left(0\right)\right). \end{aligned}$$

Then, $f'$ is continuous $K \to G$, $f'(0) = (\varphi(1))^N = 1_G$, and $\varphi \circ f' = f$. To see the last point, note that $\varphi$ is a homomorphism and thus,

$$\begin{aligned}(\varphi \circ f')(k) =& \varphi\left[\psi\left(f(k)/f\left(\frac{N-1}{N}k\right)\right)\right] \\ &\cdot \varphi\left[\psi\left(f\left(\frac{N-1}{N}k\right)\Big/ f\left(\frac{N-2}{N}k\right)\right)\right] \\ &\cdots \varphi\left[\psi\left(f\left(\frac{1}{N}k\right)/f\left(0\right)\right)\right].\end{aligned}$$

Now, using that $\varphi\psi(k) = k$, we see that the fractions cancel and we are left with

$$(\varphi \circ f')(k) = f(k)/f(0) = f(k),$$

since $f(0) = 1_G$, in either case.

Now, suppose we had $f'' : K \to G$ satisfying $f''(0) = 1_G$, and $\varphi \circ f'' = f$.
Then, we would have $[\varphi \circ (f'/f'')](s) = \varphi(f'(s))/\varphi(f''(s))$, since $\varphi$ is a homomorphism. However, this equals $f(s)/f(s) = 1$.
Thus, $f'/f''$ is a continuous map from $Y$ into $\ker\varphi = H$.
Since $Y$ is connected and $H$ is discrete, $f'/f''$ is a constant. Since $f'(0)/f''(0) = 1_G$, we see that $f' = f''$.
This proves the uniqueness of $f'$.

Note that in the case of the first lemma (that is $Y = I$), we have $f'(0) = 1_G$ and thus, $f'$ is the required $\sigma'$.

For the case of the second lemma, we still have to prove that $F' = f'$ is the desired (relative) homotopy.
First, we show that $F'$ is indeed a (not necessarily relative) homotopy. To see this, set $\alpha(s) := F'(s, 0)$ and $\beta(s) = F'(s, 1)$.
Note that $\varphi \circ \alpha(s) = \varphi \circ F'(s, 0) = F(s, 0) = \sigma(s)$ and $\alpha(0) = F'(0, 0) = 1_G$.
Since $\sigma'$ is the unique such path, we see that $\alpha = \sigma'$.
Similarly, we can conclude $\beta = \tau$ if we show that $\beta(0) = 1_G$. By definition, we have $\beta(0) = F'(0, 1)$.
Note that $F'$ is continuous and $\varphi \circ F'$ is $1$ on $\{0\} \times I$. Thus, $F'|_{\{0\}\times I}$ maps into $\ker\varphi = H$.
As before, we see that $F'$ is constant on $\{0\} \times I$. Thus, $F'(0, 1) = F'(0, 0) = 1_G$ and hence, $\beta = \tau'$.
In fact, we have even proven that $F'$ is constant on $\{0\} \times I$. This shows that $F'$ is a homotopy relative to $\{0\}$. All that remains is to show that it is constant on $\{1\} \times I$ as well.

For that, we once again note that $\varphi \circ F' = F$ is constant on $\{1\} \times I$. Thus, $F'|_{\{1\}\times I}$ maps into a coset of $\ker \varphi = H$. Since the coset is homeomorphic to $H$, it must be discrete as well. This proves that $F'$ is constant on $\{1\} \times I$ as well, proving that

$$F' : \sigma' \simeq \tau' \quad \text{rel } \{0, 1\}.$$

□

**Corollary 3.5.** The end point of $\sigma'$ only depends on the homotopy class of $\sigma$.
In particular, if $\sigma$ is a loop at $1$, then $\sigma'(1) \in H$.

*Proof.* Let $\sigma, \tau$ be paths in the same homotopy class. Let $F : \sigma \simeq \tau \quad \text{rel } \{0, 1\}$ be a (relative) homotopy.
Then, $F'$ is a homotopy from $\sigma'$ to $\tau'$ relative to $\{0, 1\}$.
In particular, we have $\sigma'(1) = F(1, 0) = F(1, 1) = \tau'(1)$. This proves the first statement.

For the second statement, note that $\varphi \circ \sigma'(1) = \sigma(1) = 1$ and thus, $\sigma'(1) \in \ker \varphi = H$. □

Now, we have the following theorem.

**Theorem 3.6.** If $G$ is a simply connected topological group, $H$ a discrete normal subgroup, then

$$\pi_1(G/H, 1) \cong H.$$

*Proof.* Using Corollary 3.5, we define $\chi : \pi_1(G/H, 1) \to H$ by

$$\chi([\sigma]) = \sigma'(1).$$

> **Claim 1.** $\chi$ is a homomorphism.
>
> *Proof.* Let $[\sigma], [\tau] \in \pi_1(G/H, 1)$.
> Let $h_1 = \sigma'(1)$ and $h_2 = \tau'(1)$. (Again, we see that these are well-defined and elements of $H$ by Corollary 3.5.)
> Let $\tau''$ be the path from $h_1$ to $h_1 h_2$ in $G$ given by
>
> $$\tau''(s) = h_1 \tau'(s).$$
>
> (Note that $\tau''(0) = \tau'(0)h_1 = 1_G h_1 = h_1$ and $\tau''(1) = h_1\tau'(1) = h_1 h_2$.)
> Note that
> $$(\varphi \circ \tau'')(s) = \varphi(\tau'(s)h_1) = \varphi(\tau'(s))\varphi(h_1) = \tau(s).$$
> (Note that $\varphi(h_1) = 1$ since $h_1 \in H = \ker \varphi$.)

Since, $\sigma'(1) = \tau''(0) = h_1$, we can consider the path $\tau''\sigma'$ in $G$. Note that

$$\varphi \circ (\tau''\sigma')(s) = \begin{cases} \varphi(\sigma'(2s)) & 0 \le 2s \le 1 \\ \varphi(\tau''(2s-1)) & 1 \le 2s \le 2. \end{cases} = (\sigma\tau)(s).$$

Thus, $\tau''\sigma'$ is the unique lift of $\sigma\tau$ as given by the Lifting Lemma.
Thus,

$$\chi([\sigma][\tau]) = \chi[\sigma\tau] = (\tau''\sigma')(1) = h_1h_2 = \chi[\sigma]\chi[\tau].$$ □

Now, we show that $\chi$ is bijective.

**Claim 2.** $\chi$ is injective.

*Proof.* It suffices to show that $\ker\chi$ is trivial.
Let $[\sigma] \in \ker\chi$. Then, $\sigma'(1) = 1_G$.
In other words, $\sigma'$ is a loop at $1_G$ in $G$. Since $G$ is simply connected, $\sigma'$ is path homotopic to a constant loop. We may choose the constant loop to be $e_{1_G}$.
Thus, $\sigma' \simeq e_{1_G} \quad \text{rel } \{0,1\}$.
Applying $\varphi$, we get that $\sigma \simeq e_1 \quad \text{rel } \{0,1\}$ or $[\sigma] = [e_1]$, the identity of $\pi_1(G/H, 1)$. □

**Claim 3.** $\chi$ is surjective.

*Proof.* Let $h \in H$ be arbitrary.
Since $G$ is simply-connected, it is pathwise connected. Let $\sigma'$ be path from $1_G$ to $h$ in $G$.
Then, $\varphi \circ \sigma' : I \to G/H$ is a loop at $1$ in $G/H$ with

$$\chi[\sigma] = \sigma'(1) = h.$$ □

With that, we are done! □

**Corollary 3.7.** The fundamental group of $S^1$ is (isomorphic to) $\mathbb{Z}$.

(Since $S^1$ is pathwise connected, we need not care about base point.)
In particular, the above corollary shows that $S^1$ is not simply connected. This is our first example of a non-simply connected space.

**Corollary 3.8.** The fundamental group of a torus is (isomorphic to) $\mathbb{Z} \times \mathbb{Z}$.

*Proof.* The torus is (homeomorphic to) $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$. □

Note that the torus is also homeomorphic to $S^1 \times S^1$. Using this, we could've calculated the fundamental group in a different way with the help of the following proposition.

**Proposition 3.9.** Given spaces $X, Y, x_0 \in X, y_0 \in Y$, we have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

*Proof.* The isomorphism is obtained as follows. First, consider the maps of pointed topological spaces given by the projections

$$(X, x_0) \xleftarrow{\ p_X\ } (X \times Y, (x_0, y_0)) \xrightarrow{\ p_Y\ } (Y, y_0).$$

These maps induce the homomorphisms

$$\pi_1(X, x_0) \xleftarrow{\ (p_X)_*\ } \pi_1(X \times Y, (x_0, y_0)) \xrightarrow{\ (p_Y)_*\ } \pi_1(Y, y_0).$$

Using the universal property of product of groups, we get a homomorphism $\langle (p_X)_*, (p_Y)_* \rangle$ as follows

$$\begin{tikzcd}
\pi_1(X, x_0) & \pi_1(X \times Y, (x_0, y_0)) \arrow[l, "(p_X)_*"'] \arrow[r, "(p_Y)_*"] \arrow[d, "\langle (p_X)_*, (p_Y)_* \rangle"] & \pi_1(Y, y_0) \\
& \pi_1(X, x_0) \times \pi_1(Y, y_0) \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}$$

such that the diagram commutes. (The $\twoheadrightarrow$s are the usual projections.)
Let $\varphi := \langle (p_X)_*, (p_Y)_* \rangle$. We show that this is an isomorphism by constructing an inverse $\psi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ .

Any element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ is of the form $([\sigma], [\tau])$ for some loop $\sigma$ (resp., $\tau$) at $x_0$ (resp., $y_0$) in $X$ (resp., $Y$).
We define $\psi([\sigma], [\tau])$ as the class of the loop at $(x_0, y_0)$ in $X \times Y$ given by

$$(\sigma, \tau)(s) := (\sigma(t), \tau(t)), \quad t \in I.$$

That is, $\psi([\sigma], [\tau]) = [(\sigma, \tau)]$. One can verify that this is well-defined.
(That is, if $\sigma \simeq \sigma'$ and $\tau \simeq \tau'$, then $(\sigma, \tau) \simeq (\sigma', \tau')$, all relative to $\{0, 1\}$.)
Now, one can verify that $\varphi \circ \psi$ and $\psi \circ \varphi$ are both the respective identities.

Alternately, as a more category theoretic proof, one can verify that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X,x_0) & \pi_1(X\times Y,(x_0,y_0)) \arrow[l, "(p_X)_*"'] \arrow[r, "(p_Y)_*"] & \pi_1(Y,y_0) \\
& \pi_1(X,x_0)\times\pi_1(Y,y_0) \arrow[ul] \arrow[u, "\psi"] \arrow[ur] &
\end{tikzcd}$$

Thus, given any object and arrows $\pi_1(X,x_0) \longleftarrow Z \longrightarrow \pi_1(Y,y_0)$, we get an arrow $\eta : Z \longrightarrow \pi_1(X,x_0)\times\pi_1(Y,y_0)$ such that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X,x_0) & \pi_1(X\times Y,(x_0,y_0)) \arrow[l, "(p_X)_*"'] \arrow[r, "(p_Y)_*"] & \pi_1(Y,y_0) \\
& Z \arrow[ul] \arrow[u, "\psi\circ\eta"] \arrow[ur] &
\end{tikzcd}$$

That is, $\pi_1(X\times Y,(x_0,y_0))$ satisfies the universal mapping property of a product. Since products are unique up to isomorphism, we see that

$$\pi_1(X\times Y,(x_0,y_0)) \cong \pi_1(X,x_0)\times\pi_1(Y,y_0). \qquad \square$$

**Definition 3.10** (Retract)**.** A subset $Y$ of a topological space $X$ is called a *retract* if there exists a map $r : X \to Y$ such that

$$ri = \mathrm{id}_Y,$$

where $i : Y \hookrightarrow X$ is the inclusion map.

**Theorem 3.11.** The circle $S^1$ is not a retract of the closed disc $D^2$.

*Proof.* We prove a stronger result that $ri \simeq \mathrm{id}_{S^1}$ is impossible for any map $r : X \to Y$. Indeed, assume the contrary and let $r : X \to Y$ be a map such that $ri \simeq \mathrm{id}_{S^1}$. Then, $(ri)_* = r_*i_*$ is an isomorphism, by Corollary 2.8.
However, note that

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r, "i_*"] \arrow[dr, "\mathrm{id}"'] & \pi_1(D^2) \arrow[d, "r_*"] \\
& \pi_1(S^1)
\end{tikzcd}$$

Recalling that $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) = \{1\}$, we see that the above is impossible since $\mathbb{Z} \to \{1\} \to \mathbb{Z}$ cannot be an isomorphism.
(There is neither any injection $i_* : \mathbb{Z} \to \{1\}$ nor any surjection $r_* : \{1\} \to \mathbb{Z}$.) $\square$

**Corollary 3.12** (Special Brouwer Fixed Theorem)**.** Any continuous map of the closed disc into itself has a fixed point.

*Proof.* Suppose $f : D^2 \to D^2$ has no fixed point. We define $r : D^2 \to S^1$ as follows: Take the ray joining $f(x)$ to $x$ and extend it until it reaches the circle $S^1$. Call this point on $S^1$ $r(x)$.
Clearly, if $x \in S^1$, then $r(x) = x$. Thus, $ri = \mathrm{id}_{S^1}$, a contradiction to the previous theorem. $\square$

*Remark.* This is a special case of Brouwer's fixed point theorem for $n = 2$. The case $n = 1$ is simple by considering the function $g(x) = f(x) - x$ and noting that $g(-1) \geq 0$ and $g(1) \leq 1$, thereby giving us that $g(c) = 0$ for some $c \in D^1 = [-1, 1]$.

Note that it must be justified that the $r$ defined above is indeed continuous. This is a fairly straightforward calculation. An outline is as follows:
Consider the ray $\zeta_x$ given by $\zeta_x(t) = (1 - t)f(x) + tx$ for $t \geq 0$. We need a solution $t > 0$ for $\|\zeta_x(t)\| = 1$. This turns out to be equivalent to solving

$$\|x - f(x)\|^2 t^2 + 2(\langle x, f(x)\rangle - \|f(x)\|^2) + \|f(x)\|^2 - 1 = 0.$$

By our assumption, $x \neq f(x)$ and thus, the above is a genuine quadratic expression for all $x$. Moreover, using $\|f(x)\|^2 \leq 1$, one can show that the above has one unique positive root, call this $t(x)$. Clearly, $x \mapsto t(x)$ is continuous. (Quadratic formula.)
Thus, $r(x) = (1 - t(x))f(x) + t(x)x$ is a continuous function of $x$.

**Theorem 3.13.** Let $X$ be a topological space and $x \in X$. Suppose $\mathcal{U}$ is an open cover of $X$ with the following properties:

1. $U_i \cap U_j$ contains $x$ and is pathwise connected for all $U_i, U_j \in \mathcal{U}$,

2. $U$ is simply connected for all $U \in \mathcal{U}$.

Then, $X$ is simply connected.

*Proof.* It is clear that $X$ is pathwise connected since it is the union of pathwise connected sets with a point in common. Thus, we just need to show that any loop is path homotopic to a constant loop. Of course, since $X$ is pathwise connected, we can choose any base point of our choice. We choose the point $x$.

Let $\sigma : I \to X$ be any loop at $x$.
By the Lebsgue number lemma, there exists a subdivision

$$[\sigma] = [\sigma_1] * \cdots * [\sigma_n]$$

such that each $\sigma_i(I)$ is contained in some $U_i \in \mathcal{U}$.
Now, we define the paths $\tau_1, \ldots, \tau_{n+1}$ as follows:

- $\tau_1$ and $\tau_{n+1}$ are the constant loops at $x$.
- For $1 < i \leq n$, $\tau_i$ is any path joining $\sigma_i(0)$ to $x$ lying in $U_{i-1} \cap U_i$.
  We can do so because $\sigma_i(0) = \sigma_{i_1}(1)$ is a point in $U_{i-1} \cap U_i$. Since this intersection contains $x$ and is pathwise connected, we are done.

Now, note that the path $\tau_i^{-1}\sigma_i\tau_{i+1}$ is a loop that lies in $U_i$ for all $1 \leq i \leq n$. Since $U_i$ is simply connected, we see that $[\tau_i^{-1}\sigma_i\tau_{i+1}]$ is the constant element $[e_x] \in \pi_1(X)$.
Moreover, observe the following product taken over all $1 \leq i \leq n$ telescopes. That is,

$$\begin{aligned}
[\sigma] &= [\sigma_1] \cdots [\sigma_n] \\
&= \prod_{i=1}^{n} [\tau_i^{-1}\sigma_i\tau_{i+1}] \\
&= \prod_{i=1}^{n} [e_x] \\
&= [e_x],
\end{aligned}$$

as desired. (Note that $[\tau_1^{-1}] = [\tau_{n+1}] = [e_x]$ as well.) $\square$

**Proposition 3.14.** The space $S^n$ is simply connected for $n \geq 2$.

*Proof.* We apply the above theorem with $X = S^n$, $\mathcal{U} = \{U, V\}$ with $U = S^n \setminus \{(1, 0, \ldots, 0)\}$ and $V = S^n \setminus \{(-1, 0, \ldots, 0)\}$.
(In other words, $U$ is $S^n$ with one point removed and $V$ is $S^n$ with the opposite point removed.)
It is clear that $\mathcal{U}$ is open cover. Recall that $\mathbb{R}^n$ is homeomorphic to $S^n$ with a point removed.
Thus, both $U$ and $V$ are simply connected since $\mathbb{R}^n$ is.
Moreover, $U \cap V$ is homeomorphic to $\mathbb{R}^n$ with two points removed. Since $n \geq 2$, this space is path connected.
Thus, $\mathcal{U}$ satisfies the criterion of the previous theorem and the result follows. $\square$

# §4. Covering spaces

In this section, we try to generalise the ideas of earlier. The previous section let us calculate $\pi_1(X)$ in the particular case that $X$ was a topological group (and could be realised as a quotient group in a particular manner).
In section, we shall consider $X$ which is not necessarily a group but represent it as a quotient space of a simply connected space $\tilde{X}$. As before, we shall work in the case that the fibers of $\tilde{X} \to X$ are discrete.
Towards this end, we have the following definition.

**Definition 4.1** (Covering space)**.** $E \xrightarrow{p} X$ is a *covering space of* $X$ if every $x \in X$ has an open neighbourhood $U$ such that $p^{-1}(U)$ is a disjoint union of open sets $S_i$ in $E$, each of which is mapped homeomorphically onto $U$ by $p$. Such $U$ are said to be *evenly covered,* and the $S_i$ are called *sheets* over $U$.

**Proposition 4.2** (Consequences)**.** From the above definition, the following results follow.

1. The fiber $p^{-1}(x)$ over any point is discrete;
2. $p$ is a local homeomorphism;
3. $p$ maps $E$ onto $X$ and $X$ has the quotient topology from $E$.
4. If $E$ is locally pathwise connected, then so is $X$.

*Proof.*

1. Let $x \in X$ and $U$ be a neighbourhood of $x$ which is evenly covered. Then, $p^{-1}(U) = \bigsqcup\limits_{i \in I} S_i$.
   Let $y \in p^{-1}(x)$. Then, $y \in S_i$ for some $i_0 \in I$. Moreover, since $p : S_{i_0} \to U$ is homeomorphism, it is one-one and thus, $p(y') \neq x$ for any $y \neq y' \in S_{i_0}$.
   In other words, $S_{i_0} \cap p^{-1}(x) = \{y\}$ and thus, $\{y\}$ is open in $p^{-1}(x)$. (Since $S_{i_0}$ was open.)
   This shows that $p^{-1}(x)$ is discrete.
2. By definition, we need to show that given any $e \in E$, there exists a neighbourhood $V$ of $e$ such that $p(V)$ is open in $X$ and $p|_V : V \to p(V)$ is a homeomorphism.

   To this end, let $e \in E$ be arbitrary and let $x = p(e)$.
   Let $U$ an evenly covered neighbourhood of $x$ and $S_{i_0}$ be the sheet (over $U$) containing $e$.
   By definition (of covering spaces), we have that $p_{S_{i_0}}$ is a homeomorphism, as desired.

3. The fact that $p$ is onto follows straight from the definition. (Every $x \in X$ has a neighbourhood $U$ which is evenly covered and thus, a sheet maps onto $U$ and in particular, something gets mapped to $x \in U$.)

   Showing that $X$ has the quotient topology from $E$ is the same as showing that $p$ is a quotient map. Let $U \subset X$. We need to show that $p^{-1}(U)$ is open iff $U$ is open. (We already know that $p$ is surjective.)
   If $U$ is open, then $p^{-1}(U)$ is open since $p$ is continuous. (It is a local homeomorphism.)
   Conversely, let $p^{-1}(U)$ be open. We show that $U$ is open. To this end, let $x \in U$. Consider any $e \in E$ such that $p(e) = x$. Then, $e \in p^{-1}(U)$. Since $p$ is a local homeomorphism and $p^{-1}(U)$ is open, we can find a neighbourhood $V$ of $e$ contained in $p^{-1}(U)$ such that $p(V)$ is open.
   However, note that $x \in p(V) \subset U$. This shows that $x$ is an interior point and thus, $U$ is open. (Since $x$ was arbitrary.)

4. Let $x \in X$ and $U$ be an arbitrary neighbourhood of $x$.
   Choose a neighbourhood $U'$ of $x$ which is evenly covered and let $S'$ be a sheet over $U'$. Then, $p|_{S'}$ is a homeomorphism.
   Let $W = U \cap U'$. Consider $p|_{S'}^{-1}(W)$; this is an open subset of $S'$ and hence, of $E$. Since $E$ is locally pathwise connected, we can find a pathwise connected neighbourhood $V \subset p|_{S'}^{-1}(W)$ of $p|_{S'}^{-1}(x) \in S'$.
   Then, its image $p_{S'}(V) \subset W \subset U$ is a neighbourhood of $x$ and is pathwise connected. (Since it is homeomorphic to $V$.)
   This shows that $X$ is locally pathwise connected. □

Thus, covering spaces is the analogue of the previous section that we described earlier. We now give the analogues of Lemma 3.3 and Lemma 3.4.

**Theorem 4.3** (Unique lifting theorem)**.** Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering space with base points, $(Y, y_0) \xrightarrow{f} (X, x_0)$ any map. Assume that $Y$ is connected. *If* there is a map $(Y, y_0) \xrightarrow{E,e_0}$ such that $pf' = f$, *then* it is unique.

(Note that this is different from Lemma 3.3 since we don't guarantee the *existence* of an $f'$.)

*Proof.* With everything as in the lemma, assume that $f'' : (Y, y_0) \to (E, e_0)$ is also a map such that $pf'' = f$.
We show that $f' = f''$.

Define $A \subset Y$ as
$$A := \{y \in Y \mid f'(y) = f''(y)\}.$$

Note that $A \neq \varnothing$ since $y_0 \in A$. (Since $f'(y_0) = e_0 = f''(y_0)$, by assumption.)
We will show that $A$ is both open and closed. Then, since $Y$ is connected and $A$ is nonempty, it will follow that $A = Y$. In turn, that will show that $f' = f''$.

**Claim 1.** $A$ is open.

*Proof.* Let $y \in A$. Let $U$ be an evenly covered neighbourhood of $f(y) = pf'(y)$.
Then, $f'(y)$ lies on some sheet $S$ over $U$. Since $y \in A$, we have $f'(y) = f''(y)$.
Thus, the set $B := f'^{-1}(S) \cap f''^{-1}(S)$ is an open set containing $y$.

**Subclaim 1.1.** $B \subset A$.

*Proof.* Let $y_1 \in B$. Then, $y \in f'^{-1}(S) \cap f''^{-1}(S)$.
That is $f'(y_1) \in S \ni f''(y_1)$.
Note that $p|_S$ is a homeomorphism and in particular, one-one. Since $pf'(y_1) = f(y_1) = pf''(y_1)$, we see that $f'(y_1) = f''(y_1)$ and hence, $y_1 \in A$. □

Thus, we have seen that given any $y \in A$, there exists an open set $B$ with $y \in B \subset A$, showing that $A$ is open. □

**Claim 2.** $A$ is closed.

*Proof.* We show that $Y \setminus A$ is open. Let $y \in Y \setminus A$.
As before, let $U$ be an evenly covered neighbourhood of $f(y) = pf'(y)$.
Since $p$ restricted to sheets is injective and $pf'(y) = pf''(y)$, it follows that $f'(y)$ and $f''(y)$ lie on different sheets, say $S_1$ and $S_2$, respectively.
Let $B' := f'^{-1}(S_1) \cap f''^{-1}(S_2)$. Clearly, $y \in B'$.

**Subclaim 2.1.** $B' \subset X \setminus A$.

*Proof.* Let $y_1 \in B'$.
Then, $f'(y_1) \in S_1$ and $f''(y_2) \in S_2$. Since $S_1$ and $S_2$ are disjoint, the claim follows. □

The above subclaim proves that $X \setminus A$ is open, as earlier. □

Thus, we are done. □

**Theorem 4.4** (Path Lifting Theorem)**.** For $(E, e_0) \xrightarrow{p} (X, x_0)$ a covering space with base points, if $\sigma$ is a path in $X$ with initial point $x_0$, there is a unique path $\sigma'_{e_0}$ in $E$ with initial point $e_0$ such that $p\sigma'_{e_0} = \sigma$.

*Proof.* Note that $\sigma$ is actually a pointed map $(I, 0) \xrightarrow{\sigma} (X, x_0)$ and $I$ is connected. Thus, uniqueness of $\sigma'_{e_0}$ follows from Unique lifting theorem.

*Special case*: The whole space $X$ is evenly covered.
Let $S$ be the sheet (over $X$) containing $e_0$. Then, $p|_S : S \to X$ is a homeomorphism.
Let $\psi : X \to S$ be the inverse to this.
Then, $\sigma'_{e_0} = \psi \circ \sigma$ is the desired map.
Note that $p\sigma'_{e_0} = p\psi\sigma = \sigma$ and $\psi\sigma(0) = \psi(x_0) = e_0$, since $p(e_0) = x_0$. Thus, $\sigma'_{e_0}$ indeed is a pointed map.

*General case*: Note that $\sigma(I) \subset X$ is compact. Thus, we can find a finite open cover $\{U_i\}_{i=0}^{n-1}$ of $\sigma(I)$ such that each $U_i$ is evenly covered.
Thus, by the Lebesgue number lemma, we can find a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that $\sigma([t_i, t_{i+i}])$ lies in the evenly covered neighbourhood $U_i$ of $\sigma(t_i)$ for all $0 \leq i < n$.
(Well, not exactly but we can renumber $U_i$ wlog so that they satisfy the above condition.)
Thus, note that for each "sub-path" $s|_{[t_i,t_{i+1}]} : [t_i, t_{i+1}] \to U_i$, we can apply the first case.
In particular, for $i = 0$, we lift $s|_{[0,t_1]}$ to a path $\sigma'_1 : [0, t_i] \to E$ such that $\sigma'_1(0) = e_0$.
Assume, as induction, that we have lifted $\sigma|_{[0,t_i]}$ to a map $\sigma'_i : [0, t_i] \to E$ such that $\sigma_i(0) = e_0$. $(0 \leq i < n - 1.)$
Also, observe that $p\sigma'_i(t_i) = \sigma(t_i)$.
Then, we can lift $\sigma|_{[t_i,t_{i+1}]}$ to a path $\tau_i : [t_i, t_{i+1}] \to E$ with $\tau_i(t_i) = \sigma'_i(t_i)$. (This is because for the lifting theorem, all we used was that $e_0$ was a point that gets mapped to $x_0$ under $p$. By our previous observation, we see that $\sigma_i(t_i) \overset{p}{\mapsto} \sigma(t_i)$ and thus, we can lift a path preserving initial points like that.)
Thus, we get a path $\sigma'_{i+1} : [0, t_{i+1}] \to E$ given by joining $\sigma_i$ and $\tau_i$.

Thus, by induction, we get a path $\sigma'_n$ which is our desired $\sigma'_{e_0}$. □

**Theorem 4.5** (Covering Homotopy Theorem)**.** Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering map as before. Let $F : I \times I \to X$ be a map with $F(0, 0) = x_0$.
There is a unique lifting of $F$ to a continuous map

$$F' : I \times I \to E$$

such that $F'(0, 0) = e_0$. Moreover, if $F$ is a path homotopy, then $F'$ is a path homotopy.

*Proof.* We first define $F'(0, 0) = e_0$. We will construct $F'$ piece-wise.
First, we use the preceding theorem to extend $F$ to the left edge $\{0\} \times I$ and bottom edge $I \times \{0\}$.
Now, choose subdivision

$$0 = s_0 < s_1 < \cdots < s_m = 1,$$
$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that each rectangle

$$I_i \times J_i = [s_{i-1}, s_i] \times [t_{i-1}, t_i]$$

is mapped by $F$ into an open subset of $X$ which is evenly covered by $p$. (This is for all $1 \le i \le m$ and $1 \le j \le n$. Such a subdivision exists by the Lebesgue number lemma.)
We now define the lift $F'$ inductively. First we define it on $I_1 \times J_1$, continuing with the other rectangles $I_i \times J_1$ in the bottom row from left to right, then with the rectangles $I_i \times J_2$ in the second row from left to and right, and so on.

In general, given $i_0$ and $j_0$, we assume that $F'$ has been defined on set

$$A = \bigcup_{\substack{j<j_0 \\ 1\le i\le n}} (I_i \times J_j) \cup \bigcup_{i<i_0} (I_i \times J_{j_0}) \cup (\{0\} \times I) \cup (I \times \{0\}) \, .$$

(That is, $A$ is the union of the left and bottom edges along with the "previous" rectangles.)
We also assume that $F'$ defined on $A$ so far is a continuous lifting of $F|_A$. Using this, we define $F'$ on $I_{i_0} \times J_{j_0}$ such that it's continuous on $A \cup (I_{i_0} \times J_{j_0})$.

Choose an open set $U$ which is evenly covered by $p$ and contains $I_{i_0} \times J_{j_0}$. (Such a $U$ exists by our construction of the subdivision.)
Let $\{S_\alpha\}$ be the set of sheets, each $S_\alpha$ being mapped homeomorphically onto $U$ by $p$.
Note that $F'$ is already defined on the subset of $I_{i_0} \times J_{j_0}$ given by $C = A \cap (I_{i_0} \times J_{j_0})$. This subset is *connected* and hence, $F'(C)$, being connected must lie entirely in one sheet.
Let $S_0$ be this sheet. Let $p_0 := p|_{S_0}$. Then,

$$p_0 : S_0 \to U$$

is a homeomorphism. Moreover, for $x \in C$, we have

$$p_0(F'(x)) = p(F'(x)) = F(x),$$

since $F'$ is a lifting of $F|_A$. Thus, for $x \in C$, we have that

$$F'(x) = p_0^{-1}(F(x)).$$

Thus, if we now define

$$F'(y) = p_0^{-1}(F(y))$$

for $y \in I_{i_0} \times J_{j_0}$, we see that $F'$ must be continuous on $A \cup (I_{i_0}) \times (J_{j_0})$, by the pasting lemma.

Moreover, it is clearly a lift of $F|_{A\cup(I_{i_0}\times J_{j_0})}$ as well. Thus, it satisfies our inductive hypothesis and we may carry out this process and define $F'$ on all of $I \times I$.

To see uniqueness, note that we were forced to define $F'(0,0) = e_0$. Thus, considering $(Y, y_0)$ with $Y = I \times I$ and $y_0 = (0,0)$, appealing to the Unique lifting theorem, we see that at each step, there is a unique lift to $I_{i_0} \times J_{j_0}$. Thus, defining $F'(0,0)$ uniquely determines $F'$.

Now, suppose that $F$ is a path homotopy. (Note that since we are not saying anything about the two paths between which it is a homotopy, all that matters is that $F$ is constant on the vertical edges.)
Then, the map $F$ carries $\{0\} \times I$ onto a singleton $\{x_0\}$. Since $pF' = F$, we must have that

$$(pF')(\{0\} \times I) = \{x_0\}.$$

In other words, $F'$ carries $\{0\} \times I$ into $p^{-1}(x_0)$. However, note that $\{0\} \times I$ is connected whereas $p^{-1}(x_0)$ is discrete. Thus, $F'$ must be constant on $\{0\} \times I$.
Similarly, it must be constant on $\{1\} \times I$ as well, proving the result. □

**Theorem 4.6.** Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering map as before. Let $f$ and $g$ be two paths in $X$ from $x_0$ to $x_1$; let $f'$ and $g'$ be their respective liftings to paths in $E$ beginning at $e_0$. If $f$ and $g$ are path homotopic, then so are $f'$ and $g'$. In particular, $f'$ and $g'$ have the same terminal point.

*Proof.* Let $F : f \simeq g \quad \text{rel } \{0,1\}$ be a path homotopy from $f$ to $g$.
Let $F'$ be given as in the preceding lemma. We wish to show that the bottom edge is $f'$ and top $g'$.
To this end, define $\alpha, \beta : I \to E$ as

$$\begin{aligned} \alpha(s) &:= F'(s,0), \\ \beta(s) &:= F'(s,1). \end{aligned}$$

We show that $\alpha = f'$ and $\beta = g'$.
Note that $\alpha(0) = F'(0,0) = e_0 = F'(0,1) = \beta(0)$.
Moreover, $p(\alpha(s)) = p(F'(s,0)) = F(s,0) = f(s)$ and similarly, $p(\beta(s)) = g(s)$.
Thus, $\alpha$ and $\beta$ are some lifts of $f$ and $g$ starting at $e_0$. By the Unique lifting theorem, we are done. □

**Corollary 4.7.** $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is a monomorphism.

*Proof.* To see that $p_*$ is a monomorphism (i.e., that is is injective), it suffices to show that $\ker p_*$ is trivial.

Let $[\sigma] \in \pi_1(E, e_0)$ be an element of $\ker p_*$.
Then, $\sigma$ is a loop at $e_0$ in $E$ such that $p \circ \sigma$ is a loop at $x_0$ such that

$$p \circ \sigma \simeq e_{x_0} \quad \text{rel } \{0, 1\}.$$

(Where $e_{x_0}$ denotes the constant loop as usual.)
Lifting them back and using the previous theorem, we see that

$$\sigma \simeq e_{e_0} \quad \text{rel } \{0, 1\}.$$

□

Note that if $\sigma$ is a loop at $x_0$ in $X$, its lifting $\sigma'_{e_0}$ in $X$ need not be a loop at $e_0$. (For example, consider $(\mathbb{R}, 0) \xrightarrow{p} (S^1, (1, 0))$ given by $p(x) = e^{2\pi i x}$. The lift of the loop $\sigma$ in $S^1$ given by $s \mapsto e^{2\pi i s}$ is the loop $\sigma'_0$ in $\mathbb{R}$ given by $s \mapsto s$, which ends at $1$.)

However, its terminal point will be a point in $p^{-1}(x_0)$. Moreover, as we saw earlier, the endpoint only depends on the homotopy class of the loop. Thus, we get a well-defined operation

$$\cdot : p^{-1}(x_0) \times \pi_1(X, x_0) \to p^{-1}(x_0)$$

given by

$$e \cdot [\sigma] = \sigma'_e(1).$$

**Proposition 4.8** ($\cdot$ is a group action)**.** The above operations satisfies the following properties:

1. $e \cdot 1 = e$ for all $e \in p^{-1}(x_0)$,
2. $e \cdot ([\sigma] * [\tau]) = (e \cdot [\sigma]) \cdot [\tau]$ for all $e \in p^{-1}(x_0)$ and all $\sigma, \tau \in \pi_1(X, x_0)$.

Thus, the above $\cdot$ is a *right group action.*

*Proof.* Let $e \in p^{-1}(x_0)$, $[\sigma], [\tau] \in \pi_1(X, x_0)$ be arbitrary.

1. Note that $1 \in \pi_1(X, x_0)$ is simply the class of the constant loop $[e_{x_0}]$.
   The lift of the constant loop is again a constant loop. Thus, since $1'_e$ starts at $e$, it must end at $e$ as well. In other words,

   $$e = 1'_e(1) = e \cdot 1,$$

   as desired.

2. Define $c \in p^{-1}(x_0)$ as $c := \sigma'_e(1) = e \cdot [\sigma]$.
   We wish to show that
   $$e \cdot ([\sigma * \tau]) = (e \cdot [\sigma]) \cdot [\tau].$$
   In other words, we wish to show that
   $$(\sigma * \tau)'_e(1) = \tau'_c(1).$$
   Consider the path $\sigma'_e * \tau'_c$ in $E$. The product is well defined since $\sigma'_e(1) = c = \tau'_c(0)$. Now, observe that
   $$\begin{aligned} p(\sigma'_e * \tau'_c)(s) &= \begin{cases} p(\sigma'_e(2s)) & 0 \leq 2s \leq 1, \\ p(\tau'_c(2s-1)) & 1 \leq 2s \leq 2 \end{cases} \\ &= \begin{cases} \sigma(2s) & 0 \leq 2s \leq 1, \\ \tau(2s-1) & 1 \leq 2s \leq 2 \end{cases} \\ &= (\sigma * \tau)(s). \end{aligned}$$
   In other words, $\sigma'_e * \tau'_c$ is a lift of $\sigma * \tau$ with initial point $e$. By uniqueness of lifts, we see that
   $$(\sigma * \tau)'_e = \sigma'_e * \tau'_c.$$
   Thus, we see that
   $$(\sigma * \tau)'_e(1) = \sigma'_e * \tau'_c(1) = \tau'_c(1),$$
   as desired. □

**Proposition 4.9** (Description of stabilisers)**.** The stabiliser of a point $e_0 \in p^{-1}(x_0)$ is the subgroup
$$p_* \pi_(E, e_0) \subset \pi_1(X, x_0).$$

*Proof.* Note that $[\sigma] \in \pi_1(E, x_0)$ belongs to the stabiliser $S$ of $e_0$ iff $\sigma'_{e_0}(1) = e_0$.
In other words, $[\sigma] \in S$ iff $\sigma$ lifts to a loop at $e_0$.
If $\sigma = p \circ \sigma'$ for some loop $\sigma'$ at $e_0$, then $[\sigma] \in S$.
Conversely, if $[\sigma] \in S$, then $\sigma'_{e_0}(1) = e_0$ and thus, $[\sigma'_{e_0}] \in \pi_1(E, e_0)$ with $\sigma = p \circ \sigma'_{e_0}$. □

**Proposition 4.10.** If $E$ is pathwise connected, $\pi_1(X, x_0)$ acts transitively.

*Proof.* Let $e, c \in p^{-1}(x_0)$. We wish to show that there exists $[\sigma] \in \pi_1(X, x_0)$ such that $e \cdot [\sigma] = c$.
Since $E$ is pathwise connected, we can find a path $\sigma'$ in $E$ from $e$ to $c$. Then, $\sigma = p \circ \sigma'$ fits the bill.
To see this, note that $\sigma'$ is indeed the lift of $\sigma$ with initial point $\sigma$. That is, $\sigma' = \sigma'_e$. Moreover, since it ends at $c$, we get
$$e \cdot [\sigma] = \sigma'_e(1) = \sigma'(1) = c.$$
□

Recall from group theory that given an action $\cdot : S \times G \to S$ with $s_0 \cdot g = s_1$, we have $G_{s_0} = gG_{s_1}g^{-1}$, where $G_s$ denotes the stabiliser of $s$ in $G$.
Thus, if $E$ is pathwise connected, then all the different subgroups $p_*\pi_1(E, e)$ are conjugate, as $e$ runs over all points in $p^{-1}(x_0)$.

**Corollary 4.11.** If $E$ is pathwise connected, the map $[\sigma] \mapsto e_0 \cdot [\sigma]$ induces a bijection of the set of all cosets $p_*\pi_1(E, e_0)[\sigma]$ onto the fiber. In particular, if $p^{-1}(x_0)$ is finite, the number of points in the fiber is equal to the index of the subgroup $p_*\pi_1(E, e_0)$.

*Proof.* In general, let $\cdot : S \times G \to S$ be a group action.
Let $G_s \leq G$ be the stabiliser of $s \in S$.
Then, given any $g,' \in G$ we have

$$s \cdot g = s \cdot g'$$

iff

$$g \cdot g'^{-1} \in G_s \text{ or } g \in G_s g'.$$

Thus, the map $G/G_s \to S$ given by

$$G_s g \mapsto s \cdot g$$

is well defined and an injection.
Moreover, if the action is transitive, then the above map is clearly surjective as well.
(In the above, $G/G_s$ is just the *set* of right cosets, no assumptions of normality.) ☐

**Definition 4.12** (Group of covering transformations)**.** Given a covering space $E \xrightarrow{p} X$, the group $G$ of *covering transformations* is the group of all homeomorphisms of $E$ which preserves the fibers, that is, all those $\varphi$ such that $p\varphi = p$.

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

**Theorem 4.13.** Given a covering space $(E, e_0) \xrightarrow{p} (X, x_0)$ with group of covering transformation $G$. If $E$ is simply connected and locally pathwise connected, $G$ is canonically isomorphic to $\pi_1(X, x_0)$.

This achieves the result we described at the beginning of the section.

*Proof.* First, we define a homomorphism

$$\chi : G \to \pi_1(X, x_0).$$

Let $\varphi \in G$. Since $E$ is simply connected, all paths from $e_0$ to $\varphi(e_0)$ are homotopic relative $\{0, 1\}$. (By Lemma 1.18.)
Thus, if $\sigma'$ is such a path, then $p_*([\sigma'])$ depends only on $e_0$ and $\varphi(e_0)$; we define

$$\chi(\varphi) = [p \circ \sigma'].$$

(That is, we define $\chi(\varphi)$ to be $p_*([\sigma'])$ for any path $\sigma'$ from $e_0$ to $\varphi(e_0)$. Note that $e_0$ is fixed.)
Note that since $p\varphi = p$, we see that $p(\varphi(e_0)) = p(e_0) = x_0$ and hence, $p \circ \sigma'$ is indeed a *loop* at $x_0$. Thus, the above map $\chi$ indeed is a map from $G$ to $\pi_1(X, x_0)$.

**Claim 1.** $\chi$ is a homomorphism.

*Proof.* Let $\varphi, \psi \in G$. Let $\sigma'$ be any path from $e_0$ to $\varphi(e_0)$ and $\tau'$ be any path from $e_0$ to $\psi(e_0)$.
Define the path $\alpha' = \psi \circ \sigma'$. This is clearly a path from $\psi(e_0)$ to $\psi(\varphi(e_0))$. In particular, $\tau' * \beta'$ is a path from $e_0$ to $\psi(\varphi(e_0))$.
Moreover, since $\psi \in G$, we have

$$p \circ \alpha' = p \circ \psi \circ \sigma' = p \circ \sigma'.$$

Thus, we have

$$\begin{aligned} \chi(\psi \circ \varphi) &= [p \circ (\tau' * \alpha')] \\ &= [p \circ \tau'] * [p \circ \alpha'] \\ &= [p \circ \tau'] * [p \circ \sigma'] \\ &= \chi(\psi) * \chi(\varphi). \end{aligned}$$ □

**Claim 2.** $\chi$ is injective.

*Proof.* By definition, it is clear that

$$\varphi(e_0) = e_0 \cdot \chi(\varphi).$$

Hence, $\chi(\varphi) = 1$ implies that $\varphi(e_0) = e_0 \cdot 1 = e_0$, i.e., $\varphi$ fixes $e_0$.
However, note that being a covering transformation, we have that $p\varphi = p$; in other words, $\varphi$ lifts $p$. By 4.3, there is only one lift of $p$ which fixes $e_0$. Since the identity

is one such, we see that $\chi(\varphi) = 1 \implies \varphi = \mathrm{id}$, the identity of $\pi_1(X, x_0)$, proving that $\chi$ is injective. □

**Claim 3.** $\chi$ is surjective.

*Proof.* Let $[\sigma] \in \pi_1(X, x_0)$ be arbitrary. We construct a $\varphi \in G$ such that $\chi(\varphi) = [\sigma]$.
We define $\varphi$ as follows:
Let $e \in E$, let $\tau'$ be any path from $e_0$ to $e$, and let $\tau = p \circ \tau'$. Note that $\tau$ is a path from $p(e_0) = x_0$ to $p(e) =: x$. Then, $\tau^{-1}\sigma\tau$ is a loop at $x$. We define

$$\varphi(e) := e \cdot [\tau^{-1}\sigma\tau],$$

where the $\cdot$ is as before. (The endpoint of the unique lift of $\tau^{-1}\sigma\tau$ in $E$ starting at $e$.)
Note that the above does not depend on $\tau'$ since $E$ is simply connected. (As earlier, we use Lemma 1.18.)
In other words, $\varphi$ just depends on $[\sigma]$.
Now, taking $e = e_0$, we may take $\tau'$ as the constant map and we see that $\varphi(e_0) = e_0 \cdot [\sigma] = \sigma'_{e_0}(1)$.
Thus, to compute $\chi(\varphi)$ using the definition of $\chi$, we may take the path joining $e_0$ and $\varphi(e_0)$ to be $\sigma'_{e_0}$ and we get

$$\chi(\varphi) = [p \circ \sigma'_{e_0}] = [\sigma],$$

as would be desired. Thus, we just need to show that $\varphi \in G$.

It is easy to see that that $p\varphi = p$. Indeed, since $\varphi(e)$ is the endpoint of a lift of a *loop* at $p(e)$, we see that that it must belong to the fiber $p^{-1}(x)$. Thus, $p(\varphi(e)) = x = p(e)$.
Moreover, $\varphi$ has an inverse of the same type that is obtained by replacing $\sigma$ with $\sigma^{-1}$ in the definition. Thus, we just need to show that $\varphi$ is continuous. (The same will show that $\varphi^{-1}$ is also continuous.)

To do so, we will show the following: For every $e_1 \in E$ and every neighbourhood $V'$ of $\varphi(e_1)$, there exists a neighbourhood $V$ of $e_1$ such that $\varphi(V) \subset V'$.
To this end, let $e_1 \in E$ be arbitrary. Consider $x_1 = p(e_1) \in X$.
Let $U$ be an open neighbourhood of $x_1$ which is evenly covered. Since $E$ is locally pathwise connected, so is $X$ and thus, we may assume so is $U$. (Or we replace $U$ by a smaller pathwise connected neighbourhood, which will still be evenly covered.)
Then, $e_1 \in S_1$ and $\varphi(e_1) \in S'_1$ for some sheets $S_1, S'_1$ over $U$. (Recall that $e_1$ and

$\varphi(e_1)$ belong to the same fiber $p^{-1}(x_1)$.)
We claim that $\varphi(S_1) \subset S_1'$.
To see this, note that if $e \in S_1$, we can join $e_1$ to $e$ by some path $\alpha'$ in $S_1$ (since $E$ is locally pathwise connected); then, consider the path $p \circ \tau$ in $X$ from $x_1$ to $p(e)$; lifting this to a path $\tau'_{\varphi(e_1)}$, we see that it is in $S_1'$. In particular, its end point is a point in $S_1'$. This end point is just $\varphi(e)$. Thus, we have that shown $\varphi(e) \in S_1'$ or that $\varphi(S_1) \subset S_1'$.

Now, given any neighbourhood $V'$ of $\varphi(e_1)$, we can find a neighbourhood $S_1' \subset V'$ of $\varphi(e_1)$ of the above type. (That is, a neighbourhood of $\varphi(e_1)$ which is a sheet over some open neighbourhood $U$ of $x_1 \in X$.)
This proves that $\varphi$ is continuous and thus, $\varphi \in G$. $\square$

With that, we are done! $\square$