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MA 526

Commutative Algebra

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Lecture 1 (07-01-2021)

07 January 2021 08:07

Noetherian Rings and Modules

Def (Poset) A set S with a relation $\leq$ which is
(i) Reflexive
(ii) Anti-symmetric
(iii) Transitive

A total order is a poset in which any two elements are comparable.
A subset of a poset is called a chain if it is totally ordered.

Prop. Let S be a poset.
TFAE
(1) $x_1 \leq x_2 \leq x_3 \leq \cdots \Rightarrow \exists N \in \mathbb{N}$ s.t. $x_n = x_{n+1}$ $\forall n \geq N$
(2) $T \subset S$, $T \neq \phi \Rightarrow T$ has a maximal element.

Proof. (1) $\Rightarrow$ (2)
Let $\phi \subsetneq T \subsetneq S$. Suppose, for the sake of contradiction, that T has no maximal element.

Pick any $x_1 \in T$. x_1 not maximal. $\therefore \exists\, x_2 \in T$ s.t. $x_2 > x_1$.
x_2 not maximal. $\exists\, x_3 \in T$ with $x_3 > x_2$. ...
We get a chain $x_1 < x_2 < \cdots$ which does not stabilise.

(2) $\Rightarrow$ (1) Let $x_1 \leq x_2 \leq x_3 \leq \cdots$ be a chain.
Consider $T = \{x_i : i \in \mathbb{N}\}$. This has a maximal element.
Let $N \in \mathbb{N}$ be s.t. x_N is maximal.
By assumption, $x_N \leq x_{N+1}$ but also maximal.
$\therefore x_N = x_{N+1}$.
In fact, for any $M > N$, the above argument holds. ∎

(1) is called the ascending chain condition. (a.c.c.)
(2) — " — maximal condition.

Def^n. Let R be a commutative ring with 1.
Let M be an R-module.
Let P be the poset of submodules of M (w.r.t. inclusion).
M is said to be Noetherian if P satisfies a.c.c.

(Equivalently, P satisfies maximal condition.)

If R is a Noetherian R-module, R is called a Noetherian ring.

There are the dual properties: descending chain condition (d.c.c.)
minimal condition.

Def^n. If submodules of an R-module M satisfy d.c.c., M is called an Artinian module.
Similarly, if R is Artinian as an R-module, it is called an Artinian ring.

Note that R-submodules of R are precisely ideals.
Thus, the Art./Noe. conditions are a.c.c./d.c.c. on ideals.

We shall soon see that Noe. rings are Art. but converse not true.

Examples.

(1) R PID. $R = \mathbb{Z}$ or $\mathbb{k}[x]$, for example.
Let us consider $\mathbb{Z}$.

$$0 \subsetneq (n_1) \subsetneq (n_2) \subsetneq \cdots$$

$n_2 \mid n_1$ with $n_2 \neq \pm n_1$, ...
At each stage, at least one prime is exhausted

Similar argument works in $\mathbb{K}[x]$ or any PID.

$\mathbb{Z}$ is NOT Noetherian. $(2) \supsetneq (2^2) \supsetneq (2^3) \supsetneq \cdots$
Can do the same in any PID which is not a field.

(2) $\mathbb{K}$ a field. $\mathbb{K}$ is both. $\Big\}$ have only finitely many ideals. Satisfy acc & dcc trivially.

(3) $\mathbb{Z}/n\mathbb{Z}$ $\leftarrow$ both
$n > 1$

(4) Any finite abelian group G is a $\mathbb{Z}$-module.
Only finitely many subgroups ($\mathbb{Z}$-submodules) and hence, both.

(5) $\mathbb{Q}/\mathbb{Z}$. $\quad 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$

$$\mathbb{Q}/\mathbb{Z} = \left\{ \frac{r}{s} + \mathbb{Z} \;\middle|\; r, s \in \mathbb{Z} \text{ with } s \neq 0 \right\}$$

is an infinite abelian group.

Fix a prime $p > 0$. Define $G_n \subset \mathbb{Q}/\mathbb{Z}$ as

$$G_n := \left\{ \frac{a}{p^n} + \mathbb{Z} \;\middle|\; a \in \mathbb{Z} \right\}.$$

$$G_0 = 0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$

$$\left(\frac{1}{p^n} + \mathbb{Z} \in G_n \setminus G_{n-1} \right)$$

Thus, $\mathbb{Q}/\mathbb{Z}$ is not Noetherian. (as a $\mathbb{Z}$-module)

Moreover, $G = \bigcup_{n=1}^{\infty} G_n \leq \mathbb{Q}/\mathbb{Z}$. This subgroup is also not a Noetherian $\mathbb{Z}$-module.

However, G does satisfy d.c.c.
(Ex. Every subgroup of G is of the form G_n.)

Thus, G is Artinian but not Noetherian!

(6) Hilbert Basis Theorem. $\Bbbk[x_1, \ldots, x_n]$ is Noe. ($n=1$ done above)

However, $\Bbbk[x_1, \ldots]$ is not Noetherian.
$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$

Not Artinian either. $R \supsetneq (x_1, x_2, \ldots) \supsetneq (x_2, \ldots) \supsetneq (x_3, \ldots) \supsetneq \cdots$
$(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \cdots$

(7) $0 \to \mathbb{Z} \to H \stackrel{\subseteq}{} \mathbb{Q} \to G \to 0$

$H = \left\{ \frac{m}{p^n} : m \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\} \right\}$ (p fixed prime)

Then H is not Art because $\mathbb{Z}$ is not.
H is not Noe. because G is not.

Lecture 2 (12-01-2021)

12 January 2021 14:02

Thm. Suppose R is a ring and M an R-module. Then M is Noetherian iff every submodule of M is f.g.

Proof ($\Rightarrow$) Suppose M is Noetherian and $N \le M$ a submodule.

To show: N is not f.g.

Suppose not.

Then, $N \neq \{0\}$. $\quad (\because \langle \phi \rangle = \{0\})$

$\Rightarrow \exists\, x_1 \in N$ s.t. $x_1 \neq 0$.

$N_1 = Rx_1 \subsetneq N$. Thus, $\exists\, x_2 \in N \setminus N_1$.

$N_1 \subsetneq N_2 = Rx_1 + Rx_2 \subsetneq N$.

Similarly, we can construct $x_3, \ldots$

Thus, $0 \subsetneq N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \cdots \subseteq N \le M$.

$\rightarrow\leftarrow$

Thus, N is f.g.. As N was arbitrary every submodule of M is f.g.

($\Leftarrow$) Suppose every submodule of M is f.g.

We show that a.c.c. holds

Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \le M$ be a seq. of submodules.

Put $N := \bigcup_{i=1}^{\infty} M_i$. $\leftarrow$ This is a submodule of M since $\{M_i\}_{i=1}^{\infty}$ is a chain.

Thus, N is f.g. Then, $R = \langle x_1, \ldots, x_g \rangle$ for some $x_1, \ldots, x_g \in N$.

$\because N = \bigcup_{i=1}^{\infty} M_i$, for some x_j, $\exists\, M_j$ s.t. $x_j \in M_j$.

$\therefore \; N = \bigcup_{i=1}^{\infty} M_i$, for some x_j, $\exists M_j$ s.t. $x_j \in M_j$.

However, note that $\{M_i\}$ is a chain and $\exists\, t \in \mathbb{N}$ s.t.

$$x_1, \dots, x_g \in M_t.$$

Thus, $x_1, \dots, x_g \in M_T \quad \forall\, T \geq t$.

$$\Rightarrow \quad M_t = M = M_T \quad \forall\, T \geq t.$$

Thus, M is Noetherian.

Cor. A ring is Noetherian iff every ideal of R is f.g.

Prop^n. Suppose $0 \to N \xrightarrow{f} M \xrightarrow{g} P \to 0$ is an exact sequence. (That is, $\ker f = 0$, $\operatorname{im} f = \ker g$, $\operatorname{im} f = P$.)

(i) M is Noetherian $\Leftrightarrow$ N and P are Noetherian

(ii) M is Artinian $\Leftrightarrow$ N and P are Artinian

Proof. We prove (i). (ii) is similar.

($\Rightarrow$) $N \simeq f(N)$ as f is injective.
Enough to prove $f(N)$ is Noetherian. But $f(N) \leq M$.
Thus, any chain in $f(N)$ is also in M. Thus, $f(N)$ is Noetherian because M is so.

$P \simeq M/\ker g$. Note any submodule of $M/\ker g$ is of the form $L/\ker g$ for some $L \leq M$ with $\ker g \leq L$.
Conclude.

sufficient to show this is Noetherian

$(\Leftarrow)$ Let N and P be Noetherian modules.

Let $M_0 \leq M_1 \leq \cdots \leq M$ be an icreasing sequence.

$\Rightarrow \quad f^{-1}(M_0) \leq f^{-1}(M_1) \leq \cdots \leq N.$

N is Noe., thus $\exists\, n \in \mathbb{N}$ s.t. $f^{-1}(M_{n+i}) = f^{-1}(M_n)$ $\forall i \geq 0$.

Similarly,

$$g(M_0) \leq g(M_1) \leq \cdots \leq P$$

$\Rightarrow \exists\, m \in \mathbb{N}$ s.t. $g(M_m) = g(M_{m+i})$ $\forall i \geq 0$
with $m \geq n$

Then, $\left.\begin{array}{l} f^{-1}(M_m) = f^{-1}(M_{m+i}) \\ g(M_m) = g(M_{m+i}) \end{array}\right\} \forall i \geq 0$

Claim. $M_m = M_{m+i}$ $\forall i \geq 0$.

$(\subseteq)$ is given

$(\supseteq)$ Let $x \in M_{m+i}$. $g(x) \in g(M_{m+i}) = g(M_m)$

$\Rightarrow g(x) = g(y)$ for some $y \in M_m$

$\Rightarrow x - y \in \ker g = \operatorname{im} f \cap M_{n+i}$

$\Rightarrow x - y = f(z)$ for some $z \in N$

$\Rightarrow z \in f^{-1}(M_{n+i}) = f^{-1}(M_n)$

$\Rightarrow \underset{x-y}{\overset{\shortparallel}{f(z)}} \in M_n$

$\Rightarrow x - y \in M_n$ but $y \in M_n$

$\therefore x \in M_n$, as desired.

Cor. Let $M_1, \dots, M_n$ be R-modules.
Then

$$\bigoplus_{i=1}^{n} M_i \text{ is Noe} \iff M_i \text{ is Noe } \forall i.$$

Similar statetement holds for Artinian.

Proof. ($\Rightarrow$) $\pi_i : \bigoplus_{j=1}^{n} M_j \twoheadrightarrow M_i$ is onto.

$$0 \to \ker \pi_i \xrightarrow{\text{incl}} \bigoplus_{j=1}^{n} M_j \xrightarrow{\pi} M_i \to 0$$

shows M_i is Noe. (or Art).

($\Leftarrow$) Induction on n. $n=1$ true. Assume for n. Then,

$$0 \to M_{n+1} \xrightarrow{\text{ind}} \bigoplus_{i=1}^{n+1} M_i \twoheadrightarrow \bigoplus_{i=1}^{n} M_i \to 0$$

Noetherian (assumption) Noetherian (induction)

$\therefore \bigoplus_{i=1}^{n+1} M_i$ is Noe. ☒

Cor. Let R be a Noetherian (resp. Artinian) ring and M a f.g. R-module. Then, M is Noetherian (resp. Artinian).

Proof. Since M is f.g., we can write M as a quotient of $R^{\oplus n}$. (*)
But $R^{\oplus n}$ is Noe. (resp Art.) since R is.
Thus, so is M.

(*) Let $M = Rm_1 + \cdots + Rm_n$ for $m_1, \dots, m_n \in M$

$$0 \to \ker f \to \bigoplus_{i=1}^{n} Re_i \xrightarrow{f} M \to 0$$ is an exact sequence.

$$e_i \mapsto m_i$$

[Note that for Noe., it is necessary that M be f.g. Thus, it is necessary & suff. if R is Noetherian.
However, for Art., M need not be f.g.]

Remark Subrings of Noetherian rings need not be Noetherian.

$R = \mathbb{k}[x, y]$ $\qquad$ $\mathbb{k}$ field ; $\qquad$ x, y indeterminate

R is Noetherian. (Hilbert's basis theorem)

$S = \mathbb{k}[x, xy, xy^2, \ldots]$ is a subring of R.

Note that

$$\langle x \rangle \subsetneq \langle x, xy \rangle \subsetneq \langle x, xy, xy^2 \rangle \subsetneq \cdots$$

are strictly increasing ideals in S.

Note that in R, $\langle x \rangle = \langle x, xy \rangle$ since $y \in R$.

Thus, S is not Noetherian even though R is.

EXAMPLE. Let $X = [0, 1]$. $\mathcal{C}(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous} \}$ is a comm. ring with 1. (Pointwise operations.)

$\mathcal{C}(X)$ is not Noetherian.

Define $F_n := \left[0, \frac{1}{n}\right]$ for $n \in \mathbb{N}$.

$$F_1 \supset F_2 \supset F_3 \supset \cdots$$

Define

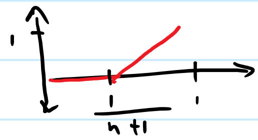
$$I_n = \{ f \in \mathcal{C}(X) : f(F_n) = 0 \}.$$

Note I_n is an ideal. Moreover

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

($\subset$) is clear because $f_{n+1} \subset f_n$

($\neq$) because



Thus, R is not Noetherian.

R : Noetherian ring, I is an ideal

$\Rightarrow R/I$ is Noetherian (as a ring)

(What NOT to do: $0 \to I \to R \to R/I \to 0$

This only shows R/I is a Noe. R-module, not as ring.)

(However, this can be improved.)

See note.

Proof $\Rightarrow$ let $K \trianglelefteq R/I$ be an ideal. Then, $K = J/I$ for some $I \subseteq J \trianglelefteq R$.

R is Noe $\Rightarrow J$ is f.g. $\Rightarrow I$ is f.g. $\blacksquare$

NOTE. Let M be an R-module.

$\text{ann}\, M := \{ r \in R : rm = 0 \ \forall m \in M \}$.

(E.g. R/I is an R-module and $\text{ann}\,(R/I) = I$.)

M is also an $R/\text{ann}\, M$ - module with operation

$$(r + \text{ann}\, M)\, m = rm. \qquad \text{(well-defined)}$$

Thus, the module structure is the "same". This shows that the previous argument actually works.

Thm. (Hilbert Basis Theorem) (Hilbert's Basis Theorem)
Let R be a Noetherian ring and X an indeterminate.
Then $R[X]$ is Noetherian.

Remark. Note the converse is trivial since $R \cong \frac{R[X]}{\langle X \rangle}$.

($x = X$.)

Proof. Suppose $R[x]$ is not Noetherian.
Then, $\exists\, I \trianglelefteq R[x]$ s.t. I is not f.g.
In particular, $I \neq 0$. $\exists f_1 \in I \setminus \{0\}$

Pick f_1 of least degree. (May be many such f_1. Does not matter.)

$f_1 = a_1 x^{d_1} + (\text{smaller terms})$

($d_1 = \deg f_1$)

$I \neq (f_1)$. Choose $f_2 \in I \setminus (f_1)$ of least degree. (d_2)

$f_2 = a_2 x^{d_2} + (\text{smaller terms})$

$I \neq (f_1, f_2)$. Continue picking $f_3, f_4, \ldots$ similarly

Note $a_1 \neq 0$, $a_2 \neq 0$, ...

Consider the following ideals of R:

$$(a_1) \subseteq (a_1, a_2) \subseteq (a_1, a_2, a_3) \subseteq \cdots$$

R is Noetherian. Thus, the above chain stabilises

$$\Rightarrow (a_1, \ldots, a_k) = (a_1, \ldots, a_k, \ldots, a_{k+i}) \quad \forall i \geq 0$$

$$a_{k+1} = b_1 a_1 + \cdots + b_k a_k \quad \text{for some } b_1, \ldots, b_k \in R.$$

$$f_1 = a_1 x^{d_1} + (\cdots)$$
$$\vdots$$
$$f_k = a_k x^{d_k} + (\cdots)$$
$$f_{k+1} = a_k x^{d_{k+1}} + (\cdots)$$

Note $d_1 \leq d_2 \leq \cdots$

Thus, $d_{k+1} \geq d_k \geq \cdots$

Now, look at
$$g = b_1 f_1 x^{d_{k+1} - d_1} + \cdots + b_k a_k f x^{d_{k+1} - d_k} - f_{k+1}$$

Note : $\deg g < \underset{\deg f_{k+1}}{\underset{\shortparallel}{d_{k+1}}}$ but $g \notin (f_1, \ldots, f_k)$.

else $f_{k+1} \in (f_1, \ldots, f_k)$ $\rightarrow\leftarrow$

Thus, $R[x]$ is Noetherian.

<u>Cor.</u> R Noetherian $\Rightarrow$ $R[x_1, \ldots, x_n]$ is Noetherian.
Moreover, quotients are also Noetherian.

<u>Cor.</u> R Noetherian $\Rightarrow$ any f.g. R-alg is Noetherian.

$$S = R[s_1, \ldots, s_n] \simeq \frac{R[x_1, \ldots, x_n]}{I}.$$

<u>Remark</u>. Analogous result <u>NOT</u> true for Artinian. $\mathbb{k}$ & $\mathbb{k}[x]$.

Lecture 3 (15-01-2021)

15 January 2021   14:03

Lemma. Let $I \trianglelefteq R$ be an ideal and $b \in R$ be s.t.

$$I : b = \{ r \in R \mid rb \in I \}$$ and

$\langle I, b \rangle$ are finitely generated. Then, I is also f.g.

Proof.

$$\begin{tikzcd} I:b \arrow[dr, dash] & & \langle I, b\rangle \arrow[dl, dash] \\ & I & \end{tikzcd}$$

$$\langle I, b \rangle = \{ x + yb \mid x \in I, y \in R \}$$

Generators of $\langle I, b\rangle$ can be of the form $a_1, \dots, a_r \in I$, b.

$$\langle I, b \rangle = \langle a_1, \dots, a_r, b \rangle.$$

$$(I:b) = (c_1, \dots, c_s) \Rightarrow c_i b \in I \quad \forall i$$

Put $J = \langle a_1, \dots, a_r, c_1 b, \dots, c_s b \rangle \subseteq I$.

We show $I \subseteq J$ and conclude. ($\because J$ is f.g.)

Let $a \in I \subseteq \langle I, b \rangle = \langle J, b \rangle$. Then, $a = c + rb$, $c \in J, r \in R$

$\Rightarrow rb = a - c \in I$

$\Rightarrow r \in I : b$

Thus, $r = d_1 c_1 + \dots + d_s c_s$ $\quad (I : b = \langle c_1, \dots, c_s \rangle)$

$$\Rightarrow a = c + rb = \underbrace{c}_{\in J} + d_1 \underbrace{bc_1}_{\in J} + \dots + d_s \underbrace{bc_s}_{\in J}$$

$\therefore a \in J$. ∎

Thm. (Cohen's Theorem)

If prime ideals of a commutative ring are f.g., then the ring is Noetherian.

Proof. We show that all ideals are f.g.
Suppose not. Define

$$\Sigma = \{ I \mid I \trianglelefteq R \text{ s.t. } I \text{ is not f.g.} \}$$

$\Sigma \neq \phi$ by hypothesis. Σ is a poset, under $\subseteq$.

Suppose $\{I_\alpha\}_{\alpha \in \Lambda}$ is a chain of ideals in Σ.
We show that

$$I = \bigcup_{\alpha \in \Lambda} I_\alpha \quad \text{is not f.g.}$$

(That it is an ideal is clear.)

This is simple for if $I = \langle x_1, \ldots, x_r \rangle$, then one can find a suitable $\alpha \in \Lambda$ s.t. $I_\alpha \ni x_1, \ldots, x_r$. ($\because \{I_\alpha\}$ is a chain)
In that case

$$I = \langle x_1, \ldots, x_r \rangle \subseteq I_\alpha \subseteq I.$$

Thus, $I_\alpha = \langle x_1, \ldots, x_r \rangle$ is f.g. $\rightarrow\leftarrow$

Thus, Σ has a maximal element, by Zorn's Lemma.
Let J be a maximal element of Σ.
Since $J \in \Sigma$, J is not f.g and hence, not prime.

$\therefore \exists a, b \in R$ s.t. $a \notin J$, $b \notin J$ but $ab \in J$.

$ab \in J \Rightarrow a \in J : b \supsetneq J$ since $a \notin J$
Also, $\langle J, b \rangle \supsetneq J$ since $b \notin J$.

Since J is maximal, $(J : b), \langle J, b \rangle \notin \Sigma$.
Thus, both are f.g. By the earlier lemma,

so is J.

Thus, we have a contradiction.

Thus, all ideals are f.g. and hence, R is Noetherian. ∎

Cor. R is Noetherian $\Rightarrow$ $R[\![x_1, \dots, x_n]\!]$ is Noetherian.

Proof. Enough to prove for $n = 1$.
Using Cohen's, it is sufficient to show that prime ideals in $R[\![x]\!]$ are f.g.

Consider the evaluation map
$$\phi : R[\![x]\!] \longrightarrow R$$
$$f(x) \mapsto f(0)$$

Let $\mathfrak{p} \in \operatorname{Spec}(R[\![x]\!])$. Then, $\phi(\mathfrak{p})$ is an ideal of R (ϕ is onto) and hence, $\phi(\mathfrak{p})$ is f.g. (since R is Noetherian)

$$\phi(\mathfrak{p}) = \langle a_1, \dots, a_r \rangle$$ ← ideal of all constant terms in $\mathfrak{p}$.

Case 1. $x \in \mathfrak{p}$.
Let $f(x) \in \mathfrak{p}$ be arbitrary

Write $f(x) = b_0 + b_1 x + \cdots = b_0 + x(b_1 + b_2 x + \cdots)$
Then, $b_0 \in \phi(\mathfrak{p})$. $\langle b_0, x \rangle$

$$b_0 = c_1 a_1 + \cdots + c_r a_r$$

$$f(x) \in \langle a_0, \dots, a_r, x \rangle \subset \mathfrak{p}$$

$\therefore$ $\mathfrak{p} = \langle a_0, \dots, a_r, x \rangle$ is f.g.

Case 2. $x \notin \mathfrak{p}$

$\phi(\mathfrak{p}) = \langle a_1, \dots, a_r \rangle$

For each $i = 1, \dots, r$, we have $f_i(x) \in \mathfrak{p}$

s.t.

$$f_i(x) = a_i + x\, g_i(x) \quad ; \qquad g_i(x) \in R[[x]].$$

Claim. $\mathfrak{p} = \langle f_1, \dots, f_r \rangle$. (2) is obvious.

Proof. Let $g(x) \in \mathfrak{p}$.

Write $g(x) = b + x h(x)$, $\quad h(x) \in R[[x]]$.

$$b = \sum_{i=1}^{r} b_i a_i$$

$$g - \sum b_i f_i = [b + x h(x)] - \sum b_i (a_i + x g_i(x))$$

$$\underbrace{\underset{\mathfrak{p} \ni}{g} - \sum \underset{\in \mathfrak{p}}{b_i f_i}}_{\in \mathfrak{p}} = \underset{\notin \mathfrak{p}}{x} \Big[\underbrace{h(x) - \sum_{i=1}^{r} b_i g_i(x)}_{\therefore \in \mathfrak{p}} \Big]$$

call this $h_1(x)$

$$g(x) = \sum b_i f_i + x h_1(x)$$

Can repeat the process on $h_1(x) \in \mathfrak{p}$ to give

$$h_1(x) = \sum c_i f_i + x h_2(x) \qquad \text{for } h_2(x) \in R[[x]].$$

$$g(x) = \sum b_i f_i + x \sum c_i f_i + x^2 h_2(x)$$

Can continue so on to get $g(x) \in \langle f_1, \dots, f_r \rangle$.

$$g(x) = f_1 (b_1 + x c_1 + x^2 d_1 + \dots)$$
$$+ \,.$$

$$\vdots$$

$$+ f_r \left(b_r + x \left(c_r + x^2 d_r + \cdots \right) \right.$$