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MA 526

Commutative Algebra

Notes By: Aryaman Maithani

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Lecture 1 (07-01-2021)

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Noetherian Rings and Modules

Def^n (Poset) A set S with a relation $\leq$ which is
(i) Reflexive
(ii) Anti-symmetric
(iii) Transitive

A total order is a poset in which any two elements are comparable.
A subset of a poset is called a chain if it is totally ordered.

Prop^n. Let S be a poset.
TFAE
(1) $x_1 \leq x_2 \leq x_3 \leq \cdots \Rightarrow \exists N \in \mathbb{N}$ s.t. $x_n = x_{n+1} \ \forall n \geq N$
(2) $T \subset S$, $T \neq \phi \Rightarrow T$ has a maximal element.

Proof. (1) $\Rightarrow$ (2)
Let $\phi \subsetneq T \subsetneq S$. Suppose, for the sake of contradiction, that T has no maximal element.

Pick any $x_1 \in T$. x_1 not maximal. $\therefore \exists x_2 \in T$ s.t. $x_2 > x_1$.
x_2 not maximal. $\exists x_3 \in T$ with $x_3 > x_2$. ...
We get a chain $x_1 < x_2 < \cdots$ which does not stabilise.

(2) $\Rightarrow$ (1) Let $x_1 \leq x_2 \leq x_3 \leq \cdots$ be a chain.
Consider $T = \{x_i : i \in \mathbb{N}\}$. This has a maximal element.
Let $N \in \mathbb{N}$ be s.t. x_N is maximal.
By assumption, $x_N \leq x_{N+1}$ but also maximal.
$\therefore x_N = x_{N+1}$.
In fact, for any $M > N$, the above argument holds. ∎

(1) is called the ascending chain condition. (a.c.c.)
(2) — " — maximal condition.

Def^n. Let R be a commutative ring with 1.
Let M be an R-module.
Let P be the poset of submodules of M (w.r.t. inclusion).
M is said to be Noetherian if P satisfies a.c.c.

(Equivalently, P satisfies maximal condition.)

If R is a Noetherian R-module, R is called a Noetherian ring.

There are the dual properties: descending chain condition (d.c.c)
minimal condition.

Def^n. If submodules of an R-module M satisfy d.c.c., M is called an Artinian module.
Similarly, if R is Artinian as an R-module, it is called an Artinian ring.

Note that R-submodules of R are precisely ideals.
Thus, the Art./Noe. conditions are a.c.c./d.c.c. on ideals.

We shall soon see that Noe. rings are Art. but converse not true.

Examples.

(1) R PID. $R = \mathbb{Z}$ or $\mathbb{K}[x]$, for example.
Let us consider $\mathbb{Z}$.

$$0 \subsetneq (n_1) \subsetneq (n_2) \subsetneq \cdots$$

$n_2 \mid n_1$ with $n_2 \neq \pm n_1$, ...

At each stage, at least one prime is exhausted

Similar argument works in $\mathbb{k}[x]$ or any PID.

$\mathbb{Z}$ is NOT Noetherian. $(2) \supsetneq (2^2) \supsetneq (2^3) \supsetneq \cdots$

Can do the same in any PID which is not <u>a field</u>.

(2) $\mathbb{k}$ a field. $\mathbb{k}$ is both.

(3) $\mathbb{Z}/n\mathbb{Z}$ $\leftarrow$ both
$n > 1$

have only finitely many ideals. Satisfy acc & dcc trivially.

(4) Any finite abelian group G is a $\mathbb{Z}$-module.
Only finitely many subgroups ($\mathbb{Z}$-submodules) and hence, both.

(5) $\mathbb{Q}/\mathbb{Z}$. $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

$$\mathbb{Q}/\mathbb{Z} = \left\{ \frac{r}{s} + \mathbb{Z} \;\middle|\; r, s \in \mathbb{Z} \text{ with } s \neq 0 \right\}$$

is an infinite abelian group.

Fix a prime $p > 0$. Define $G_n \subset \mathbb{Q}/\mathbb{Z}$ as

$$G_n := \left\{ \frac{a}{p^n} + \mathbb{Z} \;\middle|\; a \in \mathbb{Z} \right\}.$$

$$G_0 = 0 \lneq G_1 \lneq G_2 \lneq \cdots$$

$$\left(\frac{1}{p^n} + \mathbb{Z} \in G_n \setminus G_{n-1} \right)$$

Thus, $\mathbb{Q}/\mathbb{Z}$ is not Noetherian. (as a $\mathbb{Z}$-module)

Moreover, $G = \bigcup_{n=1}^{\infty} G_n \leq \mathbb{Q}/\mathbb{Z}$. This subgroup is also not a Noetherian $\mathbb{Z}$-module.

However, G does satisfy d.c.c.
(Ex. Every subgroup of G is of the form G_n.)

Thus, G is Artinian but not Noetherian!

(6) Hilbert Basis Theorem. $\mathbb{k}[x_1, \ldots, x_n]$ is Noe. ($n=1$ done above)

However, $\mathbb{k}[x_1, \ldots]$ is not Noetherian.
$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$

Not Artinian either. $R \supsetneq (x_1, x_2, \ldots) \supsetneq (x_2, \ldots) \supsetneq (x_3, \ldots) \supsetneq \cdots$
$(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \cdots$

(7) $0 \to \mathbb{Z} \to H \to G \to 0$

$H = \left\{ \frac{m}{p^n} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\} \right\}$ (p fixed prime)