

$$\int (\cos^5 x) dx$$

MA 839

Advanced Commutative Algebra

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A Quick Intro.

Setup: A ring is commutative with 1.

Let M be an R -module.

Observation: ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$),
we get an R -linear map $R \rightarrow M$ which is onto.
 $a \mapsto ax$

Then, $M \cong R/I$ where I is the kernel.

In this case, $I = \text{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R .

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$.
 $= \{ax + by \mid (a, b) \in R^{02}\}$

Then, we get an onto R -linear map $R e_1 \oplus R e_2 \xrightarrow{\varphi} M$
 $e_1 \mapsto x$
 $e_2 \mapsto y$
 $\{e_1, e_2\}$ is a basis
 } extend this
 → this lets us extend the map

In particular, $M \cong R^2 / \ker \varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle \rangle} \oplus \frac{R}{\langle \rangle} ?$$

This has a positive answer: ① R is a field

② R is a PID

CAUTION: We won't include fields as PID.
That is, when we say "PID", we exclude fields.

③ Suppose M is a finitely generated (f.g.) R -module.

(That is, suppose $M = \langle x_1, \dots, x_n \rangle$.)

Then, M is a quotient of $R^{\oplus n}$.

way
to get
this

Define $R^{\oplus n} \xrightarrow{\varphi} M$ by $e_i \mapsto x_i$.

$$M \cong R / \ker \varphi.$$

④ In general, consider a free module with " M as basis", call it $F(M)$. Then $F(M)$ maps onto M .

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$,

then $F(A)$ maps onto M .

Thus, M can be written as a quotient of a free-module.

To Summarise : If M is an R -module, then M can be written as a quotient of a free R -module. Moreover, if M is f.g., then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R -module mapping onto M with kernel K . That is, $\varphi: F \rightarrow M$ is onto R -linear and $\ker \varphi = K$.

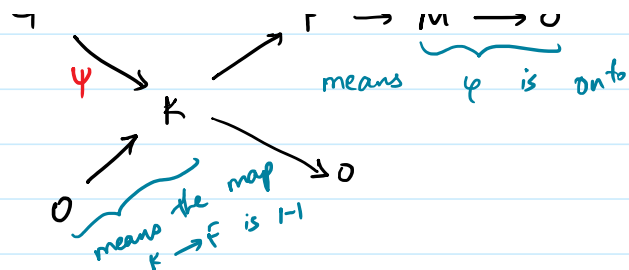
Now, \exists a free R -module G and an onto map $\psi: G \rightarrow K$

We capture this in the following diagram

$$\begin{array}{ccccc} G & \xrightarrow{\psi} & F & \xrightarrow{\varphi} & M \rightarrow 0 \\ & \searrow \psi & \nearrow & & \\ & & & & \end{array}$$

compose φ with $K \hookrightarrow F$

means φ is onto



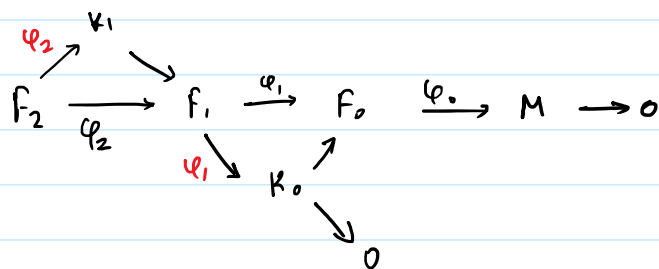
Note that $\text{im } \psi = K = \ker \varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \rightarrow 0$.

- ① φ is onto and $\ker \varphi = \text{im } \psi$.
- ② G and F are free R -modules.

Note that we can repeat the above process with K instead of F .

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \varphi$, $\varphi_1 := \psi$.



Thus, we get free modules $\{F_n, \varphi_n: F_n \rightarrow F_{n-1}\}$ such that $\ker \varphi_{n-1} = \text{im } \varphi_n$ written as

$$\dots \rightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \dots \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

F_i s are free, φ_0 is onto & $\ker \varphi_{n-1} = \text{im } \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F. : \dots \rightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \rightarrow 0 \text{ as an}$$

R free resolution of M .

$\text{im } \varphi_1 = K$, this is not exact here.
 φ_1 not onto (rec.)

Q: ① If M is f.g.:

Can we get F_i s so that $\text{rank}(F_i) < \infty \forall i$.

② If yes, are $\text{rank}(F_i)$ s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_i s s.t. the maps are "described nicely"?

Q. If two modules have "isomorphic" free resolutions, are they isomorphic?