

# Fourier Inversion for $L^1$ Functions

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- 1 Recap
- 2 Notations and Setup
- 3 Proof of the Main Theorem
- 4 The Stronger Theorem

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$$h_t(x) = \frac{1}{t^{n/2}} \exp\left(-\frac{\pi}{t} \|x\|^2\right).$$

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$$\textcircled{1} \quad (f * h_t)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t \|\xi\|^2} e^{2\pi i x \cdot \xi} d\xi \text{ for } \underline{\text{all}} \ x \in \mathbb{R}^n.$$

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- ② Using DCT, we let  $t \rightarrow 0$  in the above to conclude that

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for those  $x \in \mathbb{R}^n$  for which  $(\star)$  holds.

# Conclusion

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We will actually prove the result for a broader class of approximate identities.

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Note that the above  $\text{Leb}(f)$  is actually a superset of the  $\text{Leb}(f)$  we defined it in class. So, we shall prove a stronger result.

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It is now clear that proving the Main Theorem will show that  $(\star)$  holds for  $x \in \text{Leb}(f)$ .

# Some final notation

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$$\int_{B(x, r)} 1 = V_n r^n$$

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# Recap on Polar Coordinates

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$$\int_{\mathbb{R}^n} f(x) \, dx = \omega(S^{n-1}) \int_0^\infty r^{n-1} g(r) \, dr.$$

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Note that for all  $t > 0$ , we have  $\int_{\mathbb{R}^n} \varphi_t = \int_{\mathbb{R}^n} \varphi = \int_{\mathbb{R}^n} |\varphi| = 1$ .

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Using this, we first show that  $I_2(t) \xrightarrow{t \rightarrow 0} 0$ .

$$l_2(t)$$

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$l_2$  down,  $l_1$  to go

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With these notations, we do some more calculations.

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Integrate by parts



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# The Green Integral

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This completes the proof.

# The Stronger Theorem

- 1 Recap
- 2 Notations and Setup
- 3 Proof of the Main Theorem
- 4 The Stronger Theorem

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Reference: *Introduction to Fourier Analysis on Euclidean Spaces* by Stein and Weiss