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MA 408

Measure Theory

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Contents

Lecture 1.

Theorem 1.1 (Non existence of ideal measure)**.** There is no map $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. $\mu(E) = \mu(x + E)$ for all $x \in \mathbb{R}$ and $E \in \mathcal{P}(\mathbb{R})$,
   where $x + E := \{x + y \mid y \in E\}$,
3. for any disjoint countable collection $\{E_i\}_i^\infty$ of subsets of $\mathbb{R}$, we have
$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i),$$
4. $\mu([0, 1]) = 1$.

Note that the last is a "normalisation" property. Otherwise $\mu \equiv 0$ or $\mu(X) = \begin{cases} 0 & X = \emptyset, \\ \infty & \text{otherwise} \end{cases}$ would also satisfy and give us "useless" functions.

Replacing "countable union" with "finite union" also won't do the trick in general due to the Banach-Tarski "paradox" (theorem).

Both the above required a use of the Axiom of Choice.

Definition 1.2 (Algebra)**.** Let X be a non-empty set.
An algebra ("field") on X is a non-empty collection $\mathcal{F} \subset \mathcal{P}(X)$ satisfying

1. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
2. $A_1, \ldots, A_n \in \mathcal{F} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{F}$.

Definition 1.3 (σ-algebra)**.** Let X be a non-empty set.
A σ-algebra ("σ-field") on X is a non-empty collection $\mathcal{F} \subset \mathcal{P}(X)$ satisfying

1. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
2. $A_1, A_2, \ldots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Example 1.4 (Countable-cocountable σ-algebra)**.** Let $X \neq \emptyset$. Then,
$$\mathcal{F} = \{E \subset X \mid E \text{ or } E^c \text{ is countable}\}$$
is a σ-algebra on X.

Definition 1.5 (σ-algebra generated by a set)**.** Let $\mathcal{E} \subset \mathcal{P}(X)$. Then,

$$\mathcal{M}(\mathcal{E}) := \bigcap_{\substack{\mathcal{E} \subset \mathcal{B} \\ \mathcal{B} \text{ is a } \sigma-\text{algebra}}} \mathcal{B}$$

is a σ-algebra. Moreover, it is the smallest σ-algebra containing $\mathcal{B}$.

This is called the σ-algebra generated by $\mathcal{B}$.

Definition 1.6 (Borel σ-algebra)**.** Let $(X, \mathcal{T})$ be a topological space. The σ-algebra generated by $\mathcal{T}$ is called the Borel σ-algebra on X, denoted $\mathcal{B}(X)$.

In other words, $\mathcal{B}(X)$ is the σ-algebra generated by the open sets of X.

Proposition 1.7. All of the following are contained in $\mathcal{B}(\mathbb{R})$:

1. All closed sets.
2. All open sets.
3. All F_σ and G_δ sets.

Recall that an F_σ set is a set which can be written as countable union of closed sets. Similarly, G_δ as countable intersection of open sets.

Proposition 1.8. $\mathcal{B}(\mathbb{R})$ is generated by any of the following collections.

1. $\{(a,b) \mid a < b\}$ or $\{[a,b] \mid a < b\}$,
2. $\{(a,b] \mid a < b\}$ or $\{[a,b) \mid a < b\}$,
3. $\{(a,\infty) \mid a \in \mathbb{R}\}$ or $\{(-\infty,b) \mid b \in \mathbb{R}\}$,
4. $\{[a,\infty) \mid a \in \mathbb{R}\}$ or $\{(-\infty,b] \mid b \in \mathbb{R}\}$.

Definition 1.9 (Product of σ-algebrae)**.** Let $\{(X_i, \mathcal{M}_i)\}_{i=1}^n$ be a finite collection of sets and σ-algebrae.

Put $X := \prod_{i=1}^n X_i$ and let $\pi_i : X \to X_i$ denote the projection onto the i-th coordinate.

Let

$$\mathcal{B} = \{\pi_i^{-1}(E) \mid E \in \mathcal{M}_i,\ i = 1, \dots, n\}.$$

Then, $\mathcal{M} = \mathcal{M}(\mathcal{B})$ is the product σ-algebra induced by $\{M_i\}_{i=1}^n$ which we (misleadingly) denote by $\prod_{i=1}^n \mathcal{M}_i$.

With the above, we get two (possibly different) σ-algebrae on $\mathbb{R}^n$. One is the Borel σ-algebra on it, by virtue of it being a topological space, i.e., $\mathcal{B}(\mathbb{R}^n)$ and the other is the product of σ-algebra, i.e., $\prod_{i=1}^n \mathcal{B}(\mathbb{R})$. As it turns out, both are equal.

Theorem 1.10. $\mathcal{B}(\mathbb{R}^n) = \prod_{i=1}^n \mathcal{B}(\mathbb{R})$.

Remark 1.11. In general, the above can be generalised to a product of separable metric spaces. (Note that the product of metric spaces in the product topology is metrisable.)

Definition 1.12 (Measure)**.** Suppose X is a non-empty set and $\mathcal{M}$ a σ-algebra on X. A measure on X is a map

$$\mu : X \to [0, \infty]$$

satisfying

1. $\mu(\emptyset) = 0$,
2. if $\{E_i\}_1^\infty \subset \mathcal{M}$ are pairwise disjoint, then

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

$(X, \mathcal{M}, \mu)$ is called a measure space.

Note that $\mu\left(\bigsqcup E_i\right)$ makes sense because $\mathcal{M}$ is a σ-algebra and hence $\bigsqcup E_i \in \mathcal{M}$.

Proposition 1.13. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. All sets mentioned below are in $\mathcal{M}$. Then,

1. $E \subset F \implies \mu(E) \leq \mu(F)$,
2. $\mu\left(\bigcup_1^\infty E_i\right) \leq \sum_1^\infty \mu(E_i)$,
3. If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \cdots$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_i).$$

Definition 1.14 (Null set)**.** A null set in a measure space $(X, \mathcal{M}, \mu)$ is a set N such that $N \subset F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.

Note that N need not necessarily be in $\mathcal{M}$. Of course, F in the above is also a null set.

Definition 1.15 (Completion)**.** Given a measure space $(X, \mathcal{M}, \mu)$, the completion of $\mathcal{M}$, denote $\overline{\mathcal{M}}$ is the collection of all subsets of the form $E \cup N$ where $E \in \mathcal{M}$ and N is a null set.

Clearly, $\mathcal{M} \subset \overline{\mathcal{M}}$ since $\emptyset$ is a null set.

Proposition 1.16 (Extension to completion)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. $\overline{\mathcal{M}}$ is a σ-algebra.
2. There is a unique measure
$$\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$$
such that $\overline{\mu}|\mathcal{M} = \mu$.

Lecture 2.

Definition 2.1 (Outer measure)**.** An outer measure on X is a map

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

satisfying

1. $\mu^*(\varnothing) = 0$,
2. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$,
3. $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Note that we don't demand equality even if disjoint.

Proposition 2.2 (A construction of an outer measure)**.** Suppose $\mathcal{F} \subset \mathcal{P}(X)$ and $\rho : \mathcal{F} \to [0, \infty]$ is a map such that

1. $\varnothing, X \in \mathcal{F}$,
2. $\rho(\varnothing) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \mid E_i \in \mathcal{F},\ E \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

Then, μ^* is an outer measure.

Note that the above had just the bare minimum requirement for both ρ and $\mathcal{F}$ and still gave us that μ^* is an outer measure.

Definition 2.3 (μ^*-measurable)**.** Given an outer measure μ^* on a set X, a set $A \subset X$ is said to be μ^*-measurable if for all $E \subset X$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Definition 2.4 (Complete measure)**.** A measure μ on $(X, \mathcal{M})$ is said to be complete if $\mathcal{M}$ contains all null sets of $(X, \mathcal{M}, \mu)$.

Theorem 2.5 (Carathéodory)**.** Let μ^* be an outer measure on X. Let

$$\mathcal{M} := \{E \subset X \mid E \text{ is } \mu^*\text{-measurable}\}.$$

Then,

1. $\mathcal{M}$ is a σ-algebra.
2. $\mu^*|\mathcal{M}$ is a complete measure.

Definition 2.6 (Pre-measure)**.** Suppose $\mathcal{F}$ is an algebra on X. A map

$$\mu_0 : \mathcal{F} \to [0, \infty]$$

is called a pre-measure if

1. $\mu_0(\emptyset) = 0$,
2. if $\{A_i\}_{i=1}^\infty \subset \mathcal{F}$ are pairwise disjoint such that $\bigsqcup_{i=1}^\infty A_i \in \mathcal{F}$, then

$$\mu_0\left(\bigsqcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu_0(A_i).$$

Note that by putting all but finitely many $A_i = \emptyset$, the above equality holds for finite unions as well. (The finite union *will* be in $\mathcal{F}$ since it's an algebra.)

Proposition 2.7. Suppose μ_0 is a pre-measure on an algebra $\mathcal{F}$. Then, if μ^* is the outer measure as defined in Proposition 2.2 (with $\rho = \mu_0$), then

1. $\mu^*|\mathcal{F} = \mu_0$,
2. every set in $\mathcal{F}$ is μ^*-measurable.

Theorem 2.8. Suppose $\mathcal{F} \subset \mathcal{P}(X)$ is an algebra and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{F}$.

Let μ_0 be a pre-measure defined on $\mathcal{F}$ and let μ^* be the outer measure as before. Then

1. $\mu^*|\mathcal{M}$ is a measure on $(X, \mathcal{M})$. Put $\mu = \mu^*|\mathcal{M}$ for the next part.
2. If ν is any measure extending μ_0, then

   $$\nu(E) = \mu(E)$$

   whenever $\mu(E) < \infty$.

Definition 2.9. A half-interval is a subset of $\mathbb{R}$ of one of the following forms:

1. $(a, b]$ for $-\infty \leq a < b < \infty$,
2. (a, ∞) for $-\infty \leq a < \infty$,
3. $\emptyset$.

Proposition 2.10. The collection of all finite unions of half-intervals is an algebra on $\mathbb{R}$.

Proposition 2.11. Let $\mathcal{F}$ be the algebra consisting of finite unions of half-intervals. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing an right continuous function. Define

$$\mu_0 \left(\bigsqcup_{i=1}^{n} (a_j, b_j] \right) := \sum_{i=1}^{n} [F(b_j) - F(a_j)],$$

and let $\mu_0(\emptyset) = 0$.

Then, μ_0 is a well-defined pre-measure on $\mathcal{F}$.

Lecture 3.

Definition 3.1 (Borel measure)**.** A measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called a Borel measure on $\mathbb{R}$.

Definition 3.2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. If $\mu(X) < \infty$, then μ is called finite.
2. If $X = \bigcup_{j=1}^{\infty} E_j$, where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, then μ is called σ-finite.
3. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, then μ is called semifinite.

Theorem 3.3. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) := \begin{cases} \mu\left((0, x]\right) & x > 0, \\ 0 & x = 0, \\ -\mu\left((-x, 0]\right) & x < 0, \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.