

Fourier Inversion for L^1 Functions

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- 1 Recap
- 2 Notations and Setup
- 3 Proof of the Main Theorem
- 4 The Stronger Theorem

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- ② Using DCT, we let $t \rightarrow 0$ in the above to conclude that

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for those $x \in \mathbb{R}^n$ for which (\star) holds.

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We will actually prove the result for a broader class of approximate identities.

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for all $x \in \text{Leb}(f)$.

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It is now clear that proving the Main Theorem will show that (\star) holds for $x \in \text{Leb}(f)$.

Some final notation

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$$\int_{B(x, r)} 1 = V_n r^n$$

for $x \in \mathbb{R}^n$ and $r > 0$.

Recap on Polar Coordinates

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$$|(f * \varphi_t)(x) - f(x)| = \left| \int_{\mathbb{R}^n} f(x - u) \varphi_t(u) \, du - f(x) \right|$$

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Thus, for all $t > 0$, we have

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Using this, we first show that $I_2(t) \xrightarrow{t \rightarrow 0} 0$.

$$l_2(t)$$

$$I_2(t) = \left| \int_{|u| \geq \delta} [f(x-u) - f(x)] \varphi_t(u) \, du \right|$$

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Taking l_2 down

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l_2 down, l_1 to go

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With these notations, we do some more calculations.

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This completes the proof.

The Stronger Theorem

- 1 Recap
- 2 Notations and Setup
- 3 Proof of the Main Theorem
- 4 The Stronger Theorem

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Suppose $\varphi \in L^1(\mathbb{R}^n)$. Let $\psi(y) = \operatorname{ess\,sup}_{\|z\| \geq \|y\|} \varphi(z)$ and for $t > 0$, let $\varphi_t(y) = t^{-n} \varphi(y/t)$. If $\psi \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $\lim_{t \rightarrow 0} (f * \varphi_t)(x) = f(x) \int_{\mathbb{R}^n} \varphi(t) \, dt$ whenever $x \in \operatorname{Leb}(f)$.

Concluding Remark

The theorem which I have proven is actually a weaker version of something more general. Forget all the notation and hypothesis that we had so far.

Theorem (General Theorem)

Suppose $\varphi \in L^1(\mathbb{R}^n)$. Let $\psi(y) = \operatorname{ess\,sup}_{\|z\| \geq \|y\|} \varphi(z)$ and for $t > 0$, let $\varphi_t(y) = t^{-n} \varphi(y/t)$. If $\psi \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $\lim_{t \rightarrow 0} (f * \varphi_t)(x) = f(x) \int_{\mathbb{R}^n} \varphi(t) \, dt$ whenever $x \in \operatorname{Leb}(f)$.

Reference: *Introduction to Fourier Analysis on Euclidean Spaces* by Stein and Weiss