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MA 839

Advanced Commutative Algebra

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Spring 2020-21

A Quick Intro.

Setup: A ring is commutative with 1.
Let M be an R-module.

Observation: ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$),
we get an R-linear map $R \twoheadrightarrow M$, $a \mapsto ax$, which is onto.

Then, $M \cong R/I$ where I is the kernel.
In this case, $I = \operatorname{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R.

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$.
$= \{ax + by \mid (a, b) \in R^{\oplus 2}\}$

Then, we get an onto R-linear map $\underbrace{Re_1 \oplus Re_2}_{\{e_1, e_2\} \text{ is a basis}} \xrightarrow{\varphi} M$
$e_1 \mapsto x$
$e_2 \mapsto y$ } extend this

this lets us extend the map

In particular, $M \cong R^2/\ker \varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle\ \rangle} \oplus \frac{R}{\langle\ \rangle} \ ?$$

This has a positive answer: ① R is a field
② R is a PID

CAUTION: We won't include fields as PID.
That is, when we say "PID", we exclude fields!!

③ Suppose M is a finitely generated (f.g.) R-module.
(That is, suppose $M = \langle x_1, \dots, x_n \rangle$.)

way to get

Then, M is a quotient of $R^{\oplus n}$.

way to get this

Then, M is a quotient of $R^{\oplus n}$.

Define $R^{\oplus n} \overset{\varphi}{\twoheadrightarrow} M$ by $e_i \longmapsto x_i$.

$$M \cong R/\ker\varphi.$$

④ In general, consider a free module with "M as basis", call it $F(M)$. Then $F(M)$ maps onto M.

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$, then $F(A)$ maps onto M.

Thus, M can be written as a quotient of a free-module.

To Summarise: If M is an R-module, then M can be written as a quotient of a free R-module. Moreover, if M is f.g., then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R-module mapping onto M with kernel K. That is, $\varphi : F \twoheadrightarrow M$ is onto R-linear and $\ker\varphi = K$.

Now, $\exists$ a free R-module G and an onto map $\psi : G \to K$.

We capture this in the following diagram

← compose ψ with $K \hookrightarrow F$

$$\begin{tikzcd}
G \arrow[rr, dashed, "\psi"] \arrow[dr, "\psi"'] & & F \arrow[r, "\varphi"] & \underbrace{M \longrightarrow 0}_{\text{means } \varphi \text{ is onto}} \\
& K \arrow[ur] \arrow[dr] & & \\
0 \arrow[ur] & & 0 &
\end{tikzcd}$$

means the map $K \to F$ is 1-1

Note that $\operatorname{im}\psi = K = \ker\varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \longrightarrow 0$.

① φ is onto and $\ker\varphi = \operatorname{im}\psi$.
② G and F are free R-modules.

Note that we can repeat the above process with K instead of F.

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \varphi$, $\varphi_1 := \psi$.

$$\begin{tikzcd}
 & K_1 \arrow[dr] & & & & \\
F_2 \arrow[ur, "\varphi_2"] \arrow[rr, "\varphi_2"'] & & F_1 \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_1"'] & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
 & & & K_0 \arrow[u] \arrow[dr] & & \\
 & & & & 0 &
\end{tikzcd}$$

Thus, we get free modules $\{F_n, \varphi_n : F_n \to F_{n-1}\}$ such that $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$ written as

$$\cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

F_is are free, φ_0 is onto & $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to \underbrace{F_1 \xrightarrow{\varphi_1} F_0 \to 0}$$ as an

R free resolution of M.

im $\varphi_1 = K$, this is not exact here. φ_1 not onto (nec.)

Q: ① If M is f.g.: Can we get F_is so that $\operatorname{rank}(F_i) < \infty$ $\forall i$.

② If yes, are $\operatorname{rank}(F_i)$s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_is s.t. the maps are "described nicely"?

Q. If two modules have "isomorphic" free resolutions, are they isomorphic?

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_3"] & F_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2", "\wr"'] & F_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1", "\wr"'] & F_0 \arrow[r] \arrow[d, "\psi_0", "\wr"'] & 0 \\
\cdots \arrow[r, "\varphi_3'"] & F_2' \arrow[r, "\varphi_2'"] & F_1' \arrow[r, "\varphi_1'"] & F_0' \arrow[r] & 0
\end{tikzcd}$$

$M \simeq F_0/\operatorname{im}\varphi_1$

$M' \simeq F_0'/\operatorname{im}\varphi_1'$

$$\varphi_1' \psi_1 = \psi_0 \varphi_1 \quad (*)$$

$\underline{\text{Claim}}$. $\gamma_0(\operatorname{im}\varphi_1) = \operatorname{im}\varphi_1'$

$(\subseteq)$ clear by (*)

$(\supseteq)$ clear again since $\varphi_1' = \gamma_0 \varphi_1 \gamma_1^{-1}$

Thus, $M_0 \simeq F_0 / \operatorname{im}\varphi \simeq \dfrac{\psi_0(F_0)}{\psi_0(\operatorname{im}\varphi)} = \dfrac{F_0'}{\operatorname{im}\varphi_1'} \simeq M_0'$

Lecture 1 (11-01-2021)

11 January 2021 11:07

Free modules:

(Free modules)

As usual: R is a (commutative) ring (with 1)
M is an R-module.

Def^n. ① Let $A \subset M$. A is said to be a generating set of M (as an R-module) if $\forall x \in M$, $\exists x_1, \dots, x_n \in A$ and $(a_1, \dots, a_n) \in R^n$ s.t.

$$x = a_1 x_1 + \dots + a_n x_n.$$

(Note that A need not be finite.)

Notation: $M = \langle A \rangle$

If $A = \{x_1, \dots, x_n\}$ is finite, then $M = \langle x_1, \dots, x_n \rangle$ and M is said to be finitely generated.

② ⓐ Let $x_1, \dots, x_n \in M$. We say $\{x_1, \dots, x_n\}$ is linearly independent (over R) if for $(a_1, \dots, a_n) \in R^n$,

$$a_1 x_1 + \dots + a_n x_n = 0 \implies (a_1, \dots, a_n) = 0 \text{ in } R^n.$$

ⓑ A subset $A \subset M$ is linearly independent (over R) if every finite subset of A is linearly independent (over R).

③ A subset $A \subset M$ is a basis of M (over R) if $M = \langle A \rangle$ (over R) and A is linearly independent (over R).

④ M is free (over R) if M has a basis (over R).

REMARKS. ① Not every R-module has a basis.
② A minimal generating set need not be lin. indep.
③ A maximal lin. indep. set need not be a gen. set.

Q. If every R-module has a basis, is R a field?
(Yes. Take a non-field ring R and any non-trivial ideal $I \subsetneq R$.
Then, R/I has no lin. indep. set over R.)

Q. If an R-module M has a basis, does every basis have the same cardinality?

Ans. Yes. This is called the Invariant Basis Number (IBN) property of R.

Remark. This is not true if R is non-commutative. (That is, we can find a counterexample of a non-commutative ring.)
If R is a division ring, then again we have IBN.

Defn If M has a finite basis, say B, then we define
$$\text{rank}(M) := |B|.$$
If M is free with an infinite basis, $\text{rank}(M) := \infty$.

well-defined, by IBN

(Rank)

(When we do say "rank", we will usually mean "finite rank.")

EXAMPLES. ① $R^{\oplus n}$ is a free R-module of rank n
$M_{m \times n}(R)$ of rank mn
$R[x]$ of rank ∞

② Let A be a non-empty set and
$$\mathcal{F}_0(A, R) = \{ f : A \to R \mid f(a) = 0 \text{ for all but fin. many } a \in A \}.$$
Then, $\mathcal{F}_0(A, R)$ is an R-module under pointwise operations.
In fact, $\mathcal{F}_0(A, R)$ is a free R-module with basis $\{\chi_a\}_{a \in A}$, where

$$\chi_a(b) = \begin{cases} 0 & ; b \neq a \\ 1 & ; b = a \end{cases}$$

To see where the above set is generating, given any $f \in \mathcal{F}_0(A, R)$, we can write

$$f = \sum_{a \in A} f(a)\, \chi_a.$$

the sum is actually finite since $f(a)=0$ for all but finitely many.
(it is to be understood that 0s are ignored.)

Q. What if we take $\mathcal{F}(A, R)$? (All functions.)

Universal Property of free modules:

(Free R-module on A)

Def^n: Given a non-empty set A, a free R-module on A is a pair $(F(A), e)$ where (i) $F(A)$ is an R-module,
(ii) $e : A \to F(A)$ is a (set) function

satisfying :

Given an R-module M and a function $f : A \to M$, there exists a unique R-linear $\tilde{f} : F(A) \to M$ making the following diagram commute.

$$\begin{tikzcd} & F(A) \arrow[d, dashed, "\tilde{f}"] \\ A \arrow[ur, "e"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=4, "\circlearrowright"] & M \end{tikzcd}$$

(That is, $\tilde{f} e = f$.)

Remarks. ① Given $A = \phi$, a free R-module on A exists, and is unique up to isomorphism.
Moreover, $e : A \to F(A)$ is one-one and $F(A)$ is free with basis $\{e_a\}_{a \in A}$, where $e_a := e(a)$.

② If M is a free R-module, then $M \cong F(B)$, where B is a(ny) basis of M.

Thus, an R-module M is free iff $M \cong F(A)$ for some A.

What the universal property is really saying is that: given a free R-module M with basis A, every R-linear

$$M \to N \quad R\text{-module}$$

is completely determined by its action on A.

(The above is in the sense that given any assignment of values on A, we do get an R-linear map.)

EXAMPLE: Given an R-module M, such that $M = \langle A \rangle$, we can write M as a quotient of $F(A)$.
(What we did last lec.)

Lecture 2 (12-01-2021)

12 January 2021 08:35

Weyl Algebra

Ex. $\mathbb{k}$ is a field, $\mathbb{k}[x_1, \dots, x_d]$

$\partial_1, \dots, \partial_d \to$ partial diff op.

$A_d(k) = k[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$ D-modules

non-comm. How would you define product?

Tensor Product

(Tensor product)

Tensor product (of two modules) essentially converts the study of bilinear maps to linear maps.

Defn. Given R-modules M and N, the tensor product of M and N (over R) is a pair (T, θ), where T is an R-module, $\theta: M \times N \to T$ is R-bilinear satisfying:

Given (K, φ) where K is an R module, $\varphi: M \times N \to K$ is R-bilinear, there exists a unique R-linear map $\tilde{\varphi}: T \to K$ making the following diagram commute

$$\begin{tikzcd} & T \arrow[d, dashed, "\tilde{\varphi}"] \\ M \times N \arrow[ur, "\theta"] \arrow[r, "\varphi"'] & K \end{tikzcd}$$

, i.e., $\tilde{\varphi}\theta = \varphi$.

(We are using "with", but can use "and" once we prove $M \otimes_R N \cong N \otimes_R M$.)

Thm. A tensor of M with N exists and is unique, up to isomorphism.

Uniqueness follows by universal property.

Notation: $M \otimes_R N$

Construction:

Want

$$\begin{tikzcd} M\times N \arrow[rr, dashed, "\theta"] \arrow[dr, "\varphi"'] & & T \\ & K \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

$\theta(x_1+x_2,\ y) = \theta(x_1,y)+\theta_2(x,y)$

Step 1: Let $F = F(M\times N)$, the free module on the set $M\times N$.
We get a map $e : M\times N \longrightarrow F(M,N)$
$$(x,y) \longmapsto e_{(x,y)}$$

$\{e_{(x,y)} : x\in M,\ y\in M\}$ is a basis for F.

Let G be the submodule of F generated by

- $e_{(x_1+x_2,\, y)} - e_{(x_1,y)} - e_{(x_2,\,y)}$
- $e_{(x,\, y_1+y_2)} - e_{(x,\,y_1)} - e_{(x,\,y_2)}$
- $e_{(ax,\,y)} - a\,e_{(x,y)}$
- $e_{(x,\,ay)} - a\,e_{(x,y)}$

$\forall\, x, x_1, x_2 \in M$, $\forall\, y, y_1, y_2 \in M$, $\forall\, a \in R$

Step 2: Define $T = F/G$. Let $\pi : F \twoheadrightarrow T$ be the natural map.
Set $\pi\left(e_{(x,y)}\right) =: x\otimes y$.

Note that
$$\left.\begin{aligned} (x_1+x_2)\otimes y &= x_1\otimes y + x_2\otimes y \\ x\otimes(y_1+y_2) &= x\otimes y_1 + x\otimes y_2 \\ (ax)\otimes y &= a(x\otimes y) = x\otimes(ay) \end{aligned}\right\}\ \begin{aligned} &\forall\, x, \ldots \in M \\ &\forall\, y, \ldots \in N \\ &\forall\, a\in R \end{aligned}$$

Consider $\theta = \pi e : M\times N \longrightarrow T$
$$(x,y) \longmapsto x\otimes y$$

$$\begin{tikzcd} & F(M\times N) \arrow[dr, "\pi"] & \\ M\times N \arrow[ur, "e"] \arrow[rr, dashed, "\theta"] & & T \end{tikzcd}$$

(x,y) ; $e(x,y)$; $x\otimes y$

Step 3. Now, suppose we are given a bilinear
$\varphi : M\times N \longrightarrow K$. ($K$ is some R-module.)

$$\begin{tikzcd} M\times N \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & T = F/G \arrow[d, dashed, "\exists ! \tilde{\varphi}"] \\ & K \end{tikzcd}$$

Q1: Is there even a func. $\tilde{\varphi}$?

Q2: Is it R-linear?

Q3: Is it unique?

Note also

$$\begin{tikzcd} & F \arrow[d, dashed] & \\ M\times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists ! \tilde{\varphi}"] \\ & K & \end{tikzcd}$$

Note we have a set map $M\times N \xrightarrow{\varphi} K$ which induces an R linear map $\bar{\varphi} : F \to K$. (UMP of free modules)

$$\begin{tikzcd} & F \arrow[d, dashed, "\bar{\varphi}"] & \\ M\times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists ! \tilde{\varphi}"] \\ & K & \end{tikzcd}$$

We now want to show that $\bar{\varphi}$ factors through T. It would suffice to show that $G \subseteq \ker \bar{\varphi}$.

Using bilinearity of φ, it follows that all our (four types of) generators of G are in $\ker \bar{\varphi}$.

Thus, $\bar{\varphi}$ factors through quotient. That is, $\exists !$ R-linear $\tilde{\varphi} : T \to K$ s.t.

$$\begin{tikzcd} F \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi"'] & K \\ T \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes. That is, $\bar{\varphi} = \tilde{\varphi}\,\pi$.

$$\begin{tikzcd} & F \arrow[d, dashed, "\bar{\varphi}"] \arrow[dr, dashed, "\pi"] & \\ M\times N \arrow[ur, dashed, "e"] \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\tilde{\varphi}"] \\ & & \end{tikzcd}$$

$\varphi \qquad \tilde{\varphi}$

K

Can now verify $\tilde{\varphi}\,\theta = \varphi$. (Use commutation of diff. triangles.)
Can also verify that $\tilde{\varphi}$ is unique R-linear such.

Basic Properties:

(1) ["Identity"] $R \otimes_R M \simeq M$

(2) [Commutativity] $M \otimes_R N \simeq N \otimes_R M$

(3) [Associativity] $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$

(4) [Distributivity] $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$

Q. How does get an R-linear map $M \otimes_R N \to L$?
A. Give an R-bilinear map $M \times N \to L$.
Pretty much the only way. $x \otimes y$ would be 0 even if $x, y \neq 0$.
Thus, checking "well-defined" ness would become quite difficult.

Q. Let M be an R-module. $R \subset S$ subring.
Can you identify a natural S-module on $S \otimes_R M$?
(Base change)

Lecture 3 (14-01-2021)

14 January 2021 09:31

Distributivity : Given R-modules L, M, N

$$L \otimes_R (M \oplus N) \cong (L \otimes_R M) \oplus (L \otimes_R N)$$

How do we show? Want something like:

$$x \otimes (y, z) \longmapsto (x \otimes y, x \otimes z)$$

Note that elements of this form only GENERATE the tensor.

We now need to show the above map is well-defined. (As a function)
To do so, we go back to $L \times (M \oplus N)$ and use the universal property.

This ϕ is well defined. Every elt. here is uniquely written in the given form.

$$\begin{tikzcd}
L \times (M \oplus N) \arrow[r, "\phi"] \arrow[d, "\theta"'] & (L \otimes_R M) \oplus L \otimes_R N \\
L \otimes (M \oplus N) \arrow[ur, dashed, "\tilde{\phi}"'] &
\end{tikzcd}$$

$$(x,(y,z)) \longmapsto (x \otimes y, x \otimes z)$$

Note that ϕ is R-bilinear, thus an R-linear map $\tilde{\phi}$ (as indicated) which makes the diagram commute does exist.

To now show that is an isomorphism, we construct an inverse

$$\psi : (L \otimes_R M) \oplus (L \otimes_R N) \longrightarrow L \otimes_R (M \oplus N)$$
$$(x \otimes y, 0) \longmapsto x \otimes (y, 0)$$
$$(0, x \otimes z) \longmapsto x \otimes (0, z)$$

verify these are well defined (Again universal property)

Can verify now that ψ is the two-sided inverse of $\tilde{\phi}$.

REMARK. Suppose M and N are R-modules.

① If $x = 0 \in M$, then $x \otimes y = 0 \quad \forall y \in N$.
However if $x \otimes y = 0$ for some $x \in M, y \in N$, we cannot conclude $x = 0$ or $y = 0$.

Example: Take $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$.
In fact, look at $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ ← this is the zero module.
Thus $x \otimes y = 0 \ \forall y \in N \not\Rightarrow x = 0$.

② $M \otimes_R N$ is generated by $\{x \otimes y : x \in M, y \in N\}$ as R-module.
In particular, if M and N are f.g., then so is $M \otimes_R N$.

Take fin. gen. sets S_M and S_N. Then
$M \otimes_R N = \langle x \otimes y : x \in S_M, y \in S_N \rangle$

③ If M and N are free, then so is $M \otimes_R N$. Identify a basis.

For finite rank: write $M = \overbrace{R \oplus \cdots \oplus R}^{m}$
$N = \underbrace{R \oplus \cdots \oplus R}$

For finite rank: write $M = \overbrace{R \oplus \cdots \oplus R}^{m}$

$N = \underbrace{R \oplus \cdots \oplus R}_{n}$

Then, $M \otimes_R N = (R \oplus \cdots \oplus R) \otimes_R (R \oplus \cdots \oplus R)$

} distribute and use $R \otimes_R R \simeq R$.

④ It is possible that $M \neq 0 \neq N$ but $M \otimes_R N = 0$.
(See 1.)

Q. Given a simple tensor $x \otimes y$, how can we determine if it's 0?
Concrete ex: Is $2 \otimes 3 \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ non-zero?

Q. Is it possible that $M \otimes_R M = 0$ even if $M \neq 0$?
Yes. Take $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$.

$$\left(\frac{a}{b} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right) = \left(\frac{a}{db} + \mathbb{Z}\right) \otimes \big(\underbrace{c + \mathbb{Z}}_{0}\big) = 0.$$

Tensor Algebra

(Tensor Algebra)

The tensor algebra of M

$$T(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$$

$= R \oplus M \oplus T_2(M) \oplus T_3(M) \oplus \cdots$

$T(M)$ is clearly an additive group. One can define multiplication by "concatenation."

$$\underbrace{(x_1 \otimes \cdots \otimes x_m)}_{m^{th}\text{ piece}} \cdot \underbrace{(y_1 \otimes \cdots \otimes y_n)}_{n^{th}\text{ piece}} = \underbrace{x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n}_{(m+n)^{th}\text{ piece}}.$$

Elements of $T(M)$ are written as formal sums: $\underset{R}{z_0} + \underset{M}{z_1} + \cdots + \underset{M^{\otimes n} = T_n(M)}{z_n}$

Identify $T(R^{\oplus n})$.

Some quotients of $T(M)$:

① Symmetric algebra (Symmetric Algebra)

Given $x, y \in M$, $x \otimes y \neq^* y \otimes x$.

($\neq^*$: not necessarily equal)

Define $$\mathrm{Sym}(M) = \frac{T(M)}{\langle x \otimes y - y \otimes x \mid x, y \in M \rangle} = R \oplus M \oplus \mathrm{Sym}_2(M) \oplus \cdots$$

$\mathrm{Sym}(M)$ is now a commutative algebra.

R and M not affected. Only $M^{\otimes 2}$ onwards.

② Exterior algebra. [wedge (M)]
(Exterior algebra)

$$\Lambda(M) = \frac{T(M)}{\langle x \otimes y + y \otimes x \rangle} = R \oplus M \oplus \Lambda^2(M) \oplus \cdots$$

Q. What are $\mathrm{Sym}(R^{\oplus n})$ and $\Lambda(R^{\oplus n})$?

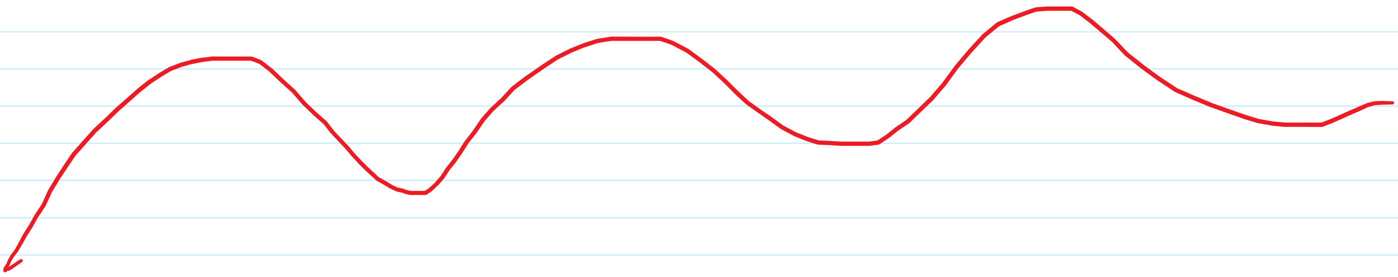
(Trivial extension)

③ $R \ltimes M = \dfrac{T(M)}{\langle x \otimes y \mid x, y \in M \rangle}$ (Trivial extension or)

③ $R \ltimes M = \dfrac{T(M)}{\langle x \otimes y \mid x, y \in M \rangle}$ (Trivial extension or idealisation.)

In this algebra, M is an ideal, with $M^2 = 0$.
This is called an idealisation of M.

Q. What is $R \ltimes R^{\oplus n}$?



Lecture 4 (18-01-2021)

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Base change:

(Base change or extension of scalars)

Let R and S be rings and $\varphi : R \to S$ a ring homomorphism. Then, we say that S is an R-algebra "via φ", i.e., S has an R-module structure defined by

$$a \cdot x = \varphi(a)x \qquad \forall a \in R \ \forall x \in S.$$

Two key examples:
1. R is a subring of S (φ is 1-1)
2. S is a quotient of R (φ is onto)

(and their compositions)

Example. If $I \subset R[x_1, \dots, x_d]$ is an ideal, then

$$S = \frac{R[x_1, \dots, x_d]}{I} \text{ is an } R\text{-algebra}$$

(via the natural maps $R \hookrightarrow R[x_1, \dots, x_d] \twoheadrightarrow S$)

A consequence of the Hilbert Basis Theorem:

Thm. Every finitely generated algebra over a Noetherian ring is a Noetherian ring.

(Not saying Noetherian as an R-module! Hilbert's Basis Thm does not give that!)

Note. If S is an R-algebra (via φ) and M is an S-module, then M has a natural R-module structure (via φ).

Q. What about the reverse? If M is an R-module, is there a "natural" way to induce an S-module structure on it?

"natural" way to induce an S-module structure on it?

No, in general. Take $\mathbb{Z}$ as a $\mathbb{Z}$ module and $\varphi: \mathbb{Z} \hookrightarrow \mathbb{Q}$.
Is there any "good" $\mathbb{Q}$-module structure on $\mathbb{Z}$?

Example. Let M be an R-module and $I \trianglelefteq R$ an ideal, $A \subset R$ m.c.s.

① When is M an R/I-module?
② When is M an R_A-module?

$\Big\}$ induced multiplication

In general, we can create modules over R/I and R_A starting from M: They are M/IM and M_A, respectively.

Obs. $M/IM \simeq R/I \otimes_R M$ and $M_A \simeq R_A \otimes_R M$
(Isomorphic as R-modules.)

Base change (or extension of scalars):

Let S be an R-algebra (via φ) and M be an R-module.
The R-module

$$S \otimes_R M$$

has a natural S-module structure defined by

$$a(b \otimes x) := (ab) \otimes x. \qquad \forall a, b \in S \ \forall x \in M$$

(Note that the above is only being defined for simple tensors.)

Note: ① If $M = \langle x_1, \dots, x_n \rangle$ over R, then

$$S \otimes_R M = S\langle 1 \otimes x_i : 1 \leq i \leq n \rangle.$$

② If $M \cong R^{\oplus n}$, $S \otimes_R M \cong S^{\oplus n}$ as R-modules.

$\otimes$ distributes over $\oplus$

In fact, $S \otimes_R M \cong S^{\oplus n}$ as S-modules as well.

Q. Given an R-linear map $\psi: M \to N$, will this induce an S-linear map $\bar{\psi}: S \otimes_R M \to S \otimes_R N$?

Does this help in the above?

③ If $M = R[x]$, then $S \otimes_R M \cong S[x]$ as S-modules.

④ If M is a free R-module, then $S \otimes_R M$ is a free S-module.

Example of base change: "Mod p test for irreducibility of a polynomial in $\mathbb{Z}[x]$"

Recall the test: Given $f = a_0 + a_1 x + \dots + a_n x^n \in \mathbb{Z}[x]$

If we can find a prime p s.t. $f \bmod p$ is irred in $(\mathbb{Z}/p\mathbb{Z})[x]$, then f is irred.*

(*Need to take care of degree not falling.)

This is an example of base change with $R = \mathbb{Z}$ and $S = \mathbb{Z}/p\mathbb{Z}$.

Complexes and Homology

(Complexes and Homology)

Example Construct a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[rr] \arrow[dr] & & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\
& & & & & 2\mathbb{Z} \arrow[ur, "\ker"'] & 1 \arrow[r, mapsto] & \bar{1} &
\end{tikzcd}$$

$$x \mapsto 2x$$

$$\cong_{(-1)}$$

Q1. What is a generating set for $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$?

Ans. $\{\bar{1}\}$ ← singleton.

Thus, we map one copy of $\mathbb{Z}$ onto $\mathbb{Z}/2\mathbb{Z}$.

That is,

Thus, we map one copy of $\mathbb{Z}$ onto $\mathbb{Z}/2\mathbb{Z}$.
That is,

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ 1 &\longmapsto \bar{1} \end{aligned}$$

Q2. What is ker ?
Ans. It is $2\mathbb{Z}$.

Q3. What can map onto $2\mathbb{Z}$?
Ans. $\mathbb{Z}$. $\quad x \mapsto 2x$

Q4. What is ker ?
It is 0. The map is 1-1. This gives the diagram

Note that we could have also written

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

since $2\mathbb{Z}$ itself is a free $\mathbb{Z}$-module. The reason we did not do this is because we are sticking to writing free R-modules as copies of R.

Q. Let R be a ring and $a \in R$. Is the following a free resolution of $R/\langle a\rangle$ over R?

$$0 \to R \xrightarrow{a} R \to R/\langle a\rangle \to 0$$
$$x \mapsto ax$$

(finite free resolution) Called a finite free resolution.

Obs. • Note that if some ker is free, we can stop there.
(That is basically saying that $R^{\oplus n} \to \ker$ will be 1-1.)

• The above does happen if R is a PID and M is f.g. So, in that case, the resolution stops right away as above. (At Stage 1.)

$$\cdots \to 0 \to 0 \to 0 \to \underset{\text{stage 1}}{R^{\oplus m}} \longrightarrow \underset{\text{stage 0}}{R^{\oplus n}} \longrightarrow M \to 0 \quad (m \le n)$$

stage 0 $(m \le n)$

Thus, every f.g. module has a finite free resolution of "length" 1.
(Recall that a submodule of a f.g. free module over a PID is free.)

· Question of "optimality" of free resolution

We know $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$
is a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

So is
$$0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \xrightarrow{\begin{bmatrix} 2 & 0 \end{bmatrix}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$
$$f \mapsto e_2 \qquad e_1 \mapsto 2, \quad e_2 \mapsto 0$$

Note that we can go on and create an extra copy of $\mathbb{Z}$ and make it longer.

$$0 \to \mathbb{Z}g \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 0 \end{bmatrix}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$
$$e_1 \longmapsto 2$$
$$f_1 \longmapsto e_2 \longmapsto 0$$
$$g \longmapsto f_2 \longmapsto 0$$

Q. If $\operatorname{rank}(F_i) \geq \operatorname{rank}(F_{i-1})$, does that mean non-optimal?
I don't think so. If a column is 0, then yes. But otherwise, don't think so.

Q. What is a free resolution of R over itself

$$0 \to R \xrightarrow{id} R \to 0.$$

(Dropping the module: $0 \to R \to 0$)

In fact, the above is true for any free R-module F.

$$0 \to F \to F \to 0.$$

(Dropping the module $0 \to F \to 0$.)

↳ f.g.

Have to conclude I is free. That would imply I is principal.

INCOMPLETE.

Back to example: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$ is a $\mathbb{Z}$-free resolution of $\mathbb{Z}/2\mathbb{Z}$.

Tensor with $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}$. ← We get a sequence of $\mathbb{Z}/6\mathbb{Z}$ modules via base change.

① Note that $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}/6\mathbb{Z}$ module. ($\bar{1}_{\mathbb{Z}/6\mathbb{Z}} \mapsto \bar{1}_{\mathbb{Z}/2\mathbb{Z}}$ gives a ring homo)

② $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}/(6\mathbb{Z}+2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

↳ should be generated by $\{\bar{1} \otimes \bar{1}\}$ (recall tensor generated by tensor of gen.)

③ $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$

Thus, tensoring the free resolution with $\mathbb{Z}/6\mathbb{Z}$ gives

$$0 \to \mathbb{Z}/6\mathbb{Z} \xrightarrow[\bar{2}]{?} \mathbb{Z}/6\mathbb{Z} \xrightarrow{?} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$\bar{1} \longmapsto \bar{1}$$

$$\bar{1} \longmapsto \bar{2}$$

Is this a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/6\mathbb{Z}$?

No! The map $\bar{1} \mapsto \bar{2}$ is not injective.

$3\mathbb{Z}/6\mathbb{Z}$ is the kernel!

↪ not free :(

After base change, the free resolution does not remain free.

Lecture 5 (19-01-2021)

19 January 2021 11:31

Recall: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is a $\mathbb{Z}$-free resolution of $\mathbb{Z}/2\mathbb{Z}$

Tensoring with $\mathbb{Z}/6\mathbb{Z}$ does not give a $\mathbb{Z}/6\mathbb{Z}$-free resolution of $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

It gives a free "complex" with "homology".

Note. If S is an R-algebra (via φ), $M \xrightarrow{\psi} N$ is R-linear, then ψ induces a natural S-linear map $S \otimes_R M \to S \otimes_R N$.

Def^n. A complex of R-modules is a sequence (finite or countable) of R modules with maps between them

$$\cdots \to M_{n+1} \xrightarrow{\varphi_{n+1}} M_n \xrightarrow{\varphi_n} M_{n-1} \to \cdots \quad \text{such that}$$

$$\ker \varphi_n \supset \operatorname{im} \varphi_{n+1}, \quad \text{i.e.,} \quad \varphi_n \circ \varphi_{n+1} = 0 \quad \forall n.$$

(Finite / infinite in one or both directions)

The complex is exact at n^{th} stage if $\ker \varphi_n = \operatorname{im} \varphi_{n+1}$.

$\mathbb{Z}$-modules: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$ (dropping the modules)

$\mathbb{Z}/6\mathbb{Z}$-modules (after tensoring): $0 \xrightarrow{\varphi_2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\varphi_0} 0$

$\bar{1} \longmapsto \bar{2}$; $\bar{1} \longmapsto 0$

0^{th} stage

$\underbrace{\begin{array}{l} \ker \varphi_0 = \mathbb{Z}/6\mathbb{Z} \\ \operatorname{im} \varphi_1 = 2\mathbb{Z}/6\mathbb{Z} \end{array}}_{\text{NOT EXACT!}}$

1^{st} stage

$\underbrace{\begin{array}{l} \ker \varphi_1 = 3\mathbb{Z}/6\mathbb{Z} \\ \operatorname{im} \varphi_2 = 0 \end{array}}_{\text{NOT EXACT!}}$

In both cases, we do have im $\subset$ ker. Thus, it is indeed a complex.

Q: Note that $0 \to M \xrightarrow{\varphi} N \to 0$ is always a complex.
When is it exact?

A: Precisely when φ is an iso. (0^{th} stage exact $\Leftrightarrow$ φ is onto
1^{st} stage exact $\Leftrightarrow$ φ is 1-1)

Remark $0 \to M \xrightarrow{\varphi} N$ is exact $\Leftrightarrow$ φ is 1-1
$M \xrightarrow{\varphi} N \to 0$ is exact $\Leftrightarrow$ φ is onto

---

Def^n A complex $\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \cdots$ is **exact** if it is exact at each stage n.

Remark. Thus, if it is not exact, then it is not exact at some n.
That is, $\text{im } \varphi_{n+1} \subsetneq \ker \varphi_n$.
Since we do have containment, the quotient $\ker \varphi_n / \text{im } \varphi_{n+1}$ makes sense. $\text{im } \varphi_{n+1} = \ker \varphi_n \Leftrightarrow \ker \varphi_n / \text{im } \varphi_{n+1} = 0$.

Def^n Given a complex $C_\cdot$, we define the **n^{th} homology** of $C_\cdot$ as

$$H_n(C_\cdot) = \frac{\ker \varphi_n}{\text{im } \varphi_{n+1}}.$$

Thus, the homology is an "obstruction" to the complex being exact!

(Familiar examples: $\ker \phi$ is an obstruction to ϕ being 1-1.
$\text{coker } \phi$ for onto. $[G, G]$ for G being abelian.)

Ex. Find the homologies in the previous examples.

Example. If $F_\cdot : \cdots \to F_2 \to F_1 \to F_0 \to 0$ is a free resolution of M, then

$$H_n(F_\cdot) = \begin{cases} 0 & ; \quad n > 0, \\ M & ; \quad n = 0. \end{cases}$$

(This was a complex, by construction; similar reason for $n > 0$ stages being exact.)

Remark If S is an R-algebra, $C_\cdot$ is an exact complex of R-modules, $S \otimes_R C_\cdot$ is a complex but not necessarily exact.

Not^n. An exact complex $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is called a short exact sequence (s. e.s).

This gives φ is 1-1, ψ is onto.
↳ can think of L as a submodule of M
$\operatorname{im} \varphi = \ker \psi$ gives $M/L \cong N$.

Properties of submodules and quotients can be transferred to s.e.s's usually.
E.g. M is Noe (Art) $\Leftrightarrow$ N and L are Noe. (Art.)

Functorial properties of $\otimes$

① Fix an R-module K. Define $T(M) = K \otimes_R M$ for all R-modules M.

② Note: For any R-module M, $T(M)$ is also an R-module.

③ Given $\varphi : M \to N$, R-linear, we get an induced map
$$T(\varphi) : T(M) \to T(N)$$

$$\left[\begin{array}{l} T(\varphi) : K \otimes_R M \longrightarrow K \otimes_R N \quad \text{defined by} \\ \qquad [T(\varphi)](x \otimes y) = x \otimes \varphi(y). \end{array} \right]$$
need to verify

(Not^n: $T(\varphi) = \mathrm{id} \otimes \varphi$ or φ_*.)

This assignment T has the following properties:

This assignment T has the following properties:

4
(a) $T(\psi \circ \varphi) = T(\psi) \circ T(\varphi)$ $\qquad M \xrightarrow{\varphi} N \xrightarrow{\psi} L$

$T(M) \xrightarrow[T(\varphi)]{} T(N) \xrightarrow[T(\psi)]{} T(L)$

(b) $T(\mathrm{id}_M) = \mathrm{id}_{T(M)}$

(c) $T(0) = 0$ $\quad$ [0 module or 0 map? $\quad$ Yes.]

(d) $T(M \oplus N) = T(M) \oplus T(N)$

T above is a functor by (1) - (4). Denoted by $K \otimes_R -$.
It is a covariant functor, since arrows are in same direction.

Furthermore, $T(0) = 0$ show that if $C.$ is a complex of R-modules, then so is $T(C.)$.
(Since compositions go to compositions and 0 goes to 0.)

Q. Does $K \otimes_R -$ preserve exactness?
No! We already have examples. Easy one: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is exact.
Apply $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$ to get answer.

Note that if S is an R-algebra (via φ), then base change (i.e., $S \otimes_R -$) gives a functor from R-modules to S-modules. (Covariant or contravariant.)

Q. If $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is a s.e.s. of R-modules and K is a fixed R-module, what can you say about

$$K \otimes_R (0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0).$$

Examples of functors: ① Localisation : R-modules $\to$ R_A - modules.
② Forgetful functors : Group/Rings/etc. $\to$ Set
③ Identity/inclusion functor
④ Linearisation : Set $\to$ R-modules $\quad (A \mapsto F(A))$

⑤ Fundamental group : $\mathbf{Top} \longrightarrow \mathbf{Grp}$

$\mathbf{Top} \longrightarrow \mathbf{Ring}$

$X \longmapsto \{\text{continuous } f : X \to \mathbb{R}\}$

⑥ $\mathbb{k}$ a field.

Field extensions of $\mathbb{k} \longrightarrow \mathbb{k}$-vector spaces

Is this a functor?

(Note that we have to think of the morphisms as well.)

Lecture 6 (21-01-2021)

21 January 2021 09:30

Q. What can we say about $K \otimes_R (0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0)$?

A. We do get a complex

$$0 \longrightarrow K \otimes_R L \xrightarrow{\varphi_*} K \otimes_R M \xrightarrow{\psi_*} K \otimes_R N \longrightarrow 0$$

$\varphi_* = id_K \otimes \varphi$, $\psi_* = id_K \otimes \psi$

Do we have exactness at all three points?

1. Is φ_* 1-1?
2. Is ψ_* onto?
3. Is $\operatorname{im} \varphi_* = \ker \psi_*$?

① No. $K \otimes_R -$ does not take injective maps to injective maps.
$\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} (0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z})$ was seen.

② Yes. $K \otimes_R N$ is generated by $\{x \otimes z \mid x \in K, z \in N\}$
If ψ is surjective, $\forall z \in N$, $\exists y \in M$ s.t. $\psi(y) = z$.
Hence,

$$x \otimes z = x \otimes \psi(y) = \psi_*(x \otimes y).$$

Thus, $K \otimes_R -$ takes surjections to surjections.

③ We already know it is a complex. Thus, $\operatorname{im}(\varphi_*) \subset \ker(\psi_*)$.

(⊇) Let $\sum x_i \otimes y_i \in \ker(\psi_*)$. ← Doing this will almost never work.

We prove the natural map $\dfrac{K \otimes_R M}{\operatorname{im} \varphi_*} \xrightarrow{\pi} \dfrac{K \otimes_R M}{\ker \psi_*}$ is an isomorphism.

This would prove $\operatorname{im} \varphi_* = \ker \psi_*$.

this is the natural onto obtained because im ⊂ ker is known

To do this, we prove that the map

dual

$K \otimes_R M \quad \bar{\psi}_* \pi \to K \otimes \dots$

map induced by quotient

$\bar{\psi}_* : \frac{K \otimes_R M}{\ker \psi_*} \to K \otimes_R N$

$\frac{K \otimes_R M}{\operatorname{im} \varphi_*} \xrightarrow{\bar{\psi}_* \pi} K \otimes_R N \to 0$ is invertible.

Construct the inverse as follows

$\forall x \in K, z \in N$, choose $y \in M$ s.t. $\psi(y) = z$.

(Ex 1.) Verify $x \otimes z \mapsto x \otimes y + \operatorname{im} \varphi_*$ is a well-defined R-linear map
This is an inverse of $\bar{\psi}_* \pi$.

Thus, we have shown: If $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is an s.e.s, then
$K \otimes_R L \xrightarrow{\varphi_*} K \otimes_R M \xrightarrow{\psi_*} N \otimes_R M \to 0$ is exact
and injective maps need not go to injective maps.

We say that $K \otimes_R -$ is not an exact functor but a right exact functor. In fact, our proof did not need injectivity of φ.

(Right exact functor, exact functor)

This is the def^n of right exactness. → Thus, $L \to M \to N \to 0$ exact $\Rightarrow K \otimes_R (L \to M \to N \to 0)$ exact.
Note the lack of $0 \to$ here.

Hom as a functor

Note that $K \otimes_R -$ and $- \otimes_R K$ are the same due to commutativity. However, as Hom is different.

Ex. Given R-modules M and N, is $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(N, M)$?
No. Take $R = M = \mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z}$.

(1) Fix an R-module K.
Given an R-module M, $\operatorname{Hom}_R(M, K)$ is an R-module.
Given an R-linear map $\varphi : M \to N$, we get a function

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, dashed] & N \arrow[d, dashed] \\ & K \end{tikzcd}$$

$$\mathrm{Hom}(N, K) \xrightarrow[\varphi_*]{\mathrm{Hom}(\varphi, K)} \mathrm{Hom}(M, K)$$
$$\alpha \longmapsto \alpha \circ \varphi$$

Note the reversal!

This association respects compositions (reverse), identity, and zero.

Thus, $\mathrm{Hom}_R(-, K)$ is a contravariant functor.

(Contravariant functor)

Since it preserves compositions and zeroes, it preserves complexes. That is, given an s.e.s

$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$, we get a complex

$$0 \to \mathrm{Hom}_R(N, K) \xrightarrow{\psi_*} \mathrm{Hom}_R(M, K) \xrightarrow{\varphi_*} \mathrm{Hom}_R(L, K) \to 0$$

At what place(s) is it exact?

(2) Fix K. Given M, $\mathrm{Hom}_R(K, M)$ is an R-module and given $\varphi : M \to N$, we get a map

$$\varphi_* = \mathrm{Hom}(K, \varphi) : \mathrm{Hom}(K, M) \longrightarrow \mathrm{Hom}(K, N).$$

(Ex2.) Verify that $\mathrm{Hom}_R(-, K)$ is a contravariant functor, and $\mathrm{Hom}_R(K, -)$ is a covariant functor. Both are left exact.

(Ex1.) First, we show $\Phi : K \times N \to \dfrac{K \otimes_R M}{\mathrm{im}\,\varphi_*}$

$(x, z) \mapsto x \otimes y + \mathrm{im}\,\varphi_*$ is well-defined

$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$

Note that if $\psi(y_1) = \psi(y_2)$, then $y_1 - y_2 \in \ker \psi = \mathrm{im}\,\varphi$.

Thus, $\quad y_1 = y_2 + \varphi(x')$ for some $x' \in L$.

$\Rightarrow \quad x \otimes y_1 = x \otimes y_2 + x \otimes \varphi(x')$

$\Rightarrow \quad x \otimes y_1 = x \otimes y_2 + \varphi_*(x \otimes x')$

Thus, $\quad x \otimes y_1 + \operatorname{im} \varphi_* = x \otimes y_2 + \operatorname{im} \varphi_*$.

Thus, Φ is a well-defined map. That it is bilinear is clear.
Thus we get a well-defined map

$$\tilde{\Phi} : K \times N \longrightarrow \frac{K \otimes_R M}{\operatorname{im} \varphi^*} \quad \text{defined by}$$

$$x \otimes z \longmapsto x \otimes y + \operatorname{im} \varphi^*$$

It suffices to show that it is the left inv of $\overline{\psi}_* \pi$. ← already know it is onto

$$\frac{K \otimes_R M}{\operatorname{im} \varphi_*} \xrightarrow[\pi]{} \frac{K \otimes_R M}{\ker \psi_*} \xrightarrow[\overline{\psi}_*]{} K \otimes_R N \xrightarrow[\tilde{\Phi}]{} \frac{K \otimes_R M}{\operatorname{im} \varphi_*}$$

on generators: $\quad x \otimes y + \operatorname{im} \varphi_* \longmapsto x \otimes y + \ker \psi_* \longmapsto x \otimes \psi(y) \longmapsto x \otimes y + \operatorname{im} \varphi_*$. ✓

(Ex2.) $\quad L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0 \quad$ exact

$\Downarrow$

$0 \longrightarrow \operatorname{Hom}(N, K) \xrightarrow{\psi_*} \operatorname{Hom}(M, K) \xrightarrow{\varphi_*} \operatorname{Hom}(L, K) \quad$ exact

- ψ_* 1-1

$$\begin{tikzcd} N \arrow[r, "\alpha"] & K \\ M \arrow[u, "\psi"] \arrow[ur, "\psi_*\alpha = \alpha\circ\psi"'] & \end{tikzcd}$$

Let $\alpha \in \operatorname{Hom}(N, K)$ s.t. $\psi_* \alpha = 0$.

Thus, $\quad \alpha \circ \psi = 0$ map

$\Rightarrow \quad (\alpha \circ \psi)(m) = 0 \quad \forall m \in M$

$\Rightarrow \quad \alpha(\psi(m)) = 0 \quad \forall m \in M$

$\Rightarrow \alpha(n) = 0 \quad \forall n \in N \quad (\because \psi$ is onto)

$\Rightarrow \alpha = 0$ map $\quad \boxtimes$

- $\operatorname{im} \psi_* = \ker \varphi_*$

$$\begin{tikzcd} N \arrow[dr, dashed, "\alpha"] & \\ M \arrow[u, "\psi"] \arrow[r, "\beta"] & K \\ L \arrow[u, "\varphi"] \arrow[ur, "\beta\circ\varphi"'] & \end{tikzcd}$$

($\subseteq$) since complex. $\quad \in \operatorname{Hom}(M, K)$

(2) Let $\beta \in \ker \varphi_*$. Then, $\varphi_* \beta = 0$ map

$\Leftrightarrow \beta \circ \varphi = 0$ map

$\Rightarrow \beta \circ \varphi(l) = 0 \quad \forall\, l \in L$

$\boxed{\beta \in \operatorname{im} \psi_* \Rightarrow \beta = \alpha \circ \psi}$ $\Rightarrow$ $\ker \beta \supset \operatorname{im} \varphi = \ker \psi$

Thus, if $\beta(m_1) = \beta(m_2)$, then $\psi(m_1) = \psi(m_2)$.

Thus, by UMP of quotients, $\alpha(n) = \alpha(\psi(m)) = \beta(m)$

is well defined and R-linear.

Thus, $\beta = \alpha \circ \psi = \psi_* \alpha \in \operatorname{im} \psi_*$. ∎

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \quad \text{exact}$$

$$\Downarrow$$

$$0 \longrightarrow \operatorname{Hom}(K, L) \xrightarrow{\varphi_*} \operatorname{Hom}(K, M) \xrightarrow{\psi_*} \operatorname{Hom}(K, N) \quad \text{exact}$$

- φ_* is 1-1

Let $\alpha \in \ker \varphi_*$. $\varphi_* \alpha = 0 \Rightarrow \varphi \circ \alpha = 0$

$\Rightarrow (\varphi \circ \alpha)(k) = 0 \quad \forall\, k \in K$

$\Rightarrow \alpha(k) = 0 \quad \forall\, k \in K \quad (\because \varphi \text{ is 1-1})$

$\Rightarrow \alpha = 0$. ∎

- $\operatorname{im} \varphi_* = \ker \psi_*$

$$\begin{tikzcd} & L \arrow[d, "\varphi"] \\ K \arrow[ur, dashed, "\alpha"] \arrow[r, "\beta"] \arrow[dr, "\psi \circ \beta"'] & M \arrow[d, "\psi"] \\ & N \end{tikzcd}$$

($\subseteq$) clear.

($\supseteq$) Let $\beta \in \ker \psi_*$. Then, $\psi \circ \beta = 0$

$\Rightarrow (\psi \circ \beta)(k) = 0 \quad \forall\, k$

$\Rightarrow \beta(k) \in \ker \psi = \operatorname{im} \varphi \quad \forall\, k$

$\Rightarrow \forall\, k\ \exists\, l_k \in L$ s.t. $\varphi(l_k) = \beta(k)$.

$\because \varphi$ is 1-1, $\exists!\, l_k$

Moreover, $\alpha = (k \mapsto l_k)$ is R-linear.

Thus, $\alpha \in \operatorname{Hom}(K, L)$ and

$\beta(k) = \varphi(l_k) = (\varphi \circ \alpha)(k) \quad \forall\, k$.

Thus, $\beta = \varphi \circ \alpha = \varphi_* \alpha \in \operatorname{im} \varphi_*$. ∎

Lecture 7 (25-01-2021)

25 January 2021 10:35

Covariant Hom is left exact, i.e.,

$$0 \to L \to M \to N \text{ exact} \Rightarrow 0 \to Hom_R(K,L) \to Hom_R(K,M) \to Hom_R(K,N) \text{ exact}$$

Contra variant Hom is also left exact, i.e.,

$$L \to M \to N \to 0 \text{ exact} \Rightarrow 0 \to Hom_R(L,K) \to Hom_R(M,K) \to Hom_R(N,K) \text{ exact}$$

(Note the assumption of $\to 0$ on LHS.)

Usually we will be more relaxed and start with an s.e.s. to begin with.

Q. Is $K \otimes_R -$ exact? No. $K = \mathbb{Z}/6\mathbb{Z}$ over $R = \mathbb{Z}$.

Is $K \otimes_R -$ exact for some K over some R? Yes, $K = R$ for any R.

(1) Over any R, can you find a K s.t. $K \otimes_R -$ is not exact.

(2) Can you find a class of examples of K s.t. (a) $K \otimes_R -$ is not exact? (b) $K \otimes_R -$ is exact?

Q. Can ask and (try to) answer similar questions about both the Hom functors.

Def^n. A s.e.s. $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is split exact if $\exists$ a map $\chi : N \to M$ which is a splitting, i.e., $\psi \circ \chi = id_N$.

(Split exact sequences)

(map will always refer to the appropriate morphisms.)

Ex. (1) $M = \varphi(L) \oplus \chi(N)$

② x is injective and hence $M \simeq L \oplus N$.

① Split exact sequence captures the notion of $\oplus$.
(s.e.s. captures submodule and quotient.)

If $M = L \oplus N$, then $0 \to L \to M \to N \to 0$
is split exact with the natural maps.

② Let $\mathcal{F}$ be a functor from R-modules to R-modules which
is additive.
(Definition at end)

If

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0 \quad \text{is split exact,}$$

the what can one say about

$$0 \to \mathcal{F}(L) \xrightarrow{\varphi_*} \mathcal{F}(M) \xrightarrow{\psi_*} \mathcal{F}(N) \to 0 \, ?$$

→ The "splitness" is preserved. $\psi x = id_N \Rightarrow \psi_* x_* = id_{\mathcal{F}(N)}$.
In particular, ψ_* remains surjective.

Is φ_* injective?

$\varphi(L) \oplus x(N)$

Actually, splitting gives a map $\pi : M \to L$ as well s.t.
$\pi \circ \varphi = id_L$

Thus, $\pi_* \circ \varphi_* = id_{\mathcal{F}(L)}$ and thus, φ_* is injective.

$$0 \rightleftarrows L \underset{\pi}{\overset{\varphi}{\rightleftarrows}} M \underset{x}{\overset{\psi}{\rightleftarrows}} N \rightleftarrows 0$$

The below seq. is exact as well.

We also can say:

$$0 \to N \xrightarrow{x} M \xrightarrow{\pi} L \to 0 \quad \text{is split exact}$$

with $\varphi : L \to M$ being the splitting map.

To conclude:

$$0 \longrightarrow \mathcal{F}(L) \xrightarrow{\varphi_*} \mathcal{F}(M) \xrightarrow{\psi_*} \mathcal{F}(N) \longrightarrow 0$$

is split exact. (Verify at middle point. Do we need additivity?)

Thus, if $E = (0 \to L \to M \to N \to 0)$ is split exact, then, for any K: $K \otimes_R E$, $\mathrm{Hom}_R(K, E)$, $\mathrm{Hom}_R(E, K)$ are all split exact.

Def^n. (Additive functor)
A covariant functor $\mathcal{F} : R\text{-Mod} \to R\text{-Mod}$ is called additive if

$$\mathcal{F} : \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(\mathcal{F}(M), \mathcal{F}(N))$$ is

a group homomorphism.

Free resolutions (II)

$\mathbb{Z}$ $\quad 2\mathbb{Z}$ $\quad \mathbb{Z}$ $\quad \mathbb{Z}/\mathbb{Z} = 0$
$\mathbb{Z}/2\mathbb{Z} \to$

Q! Given an s.e.s. $0 \to L \to M \to N \to 0$ of R-modules, if we know free resolutions of two of the three, can construct a free resolution for the third?

M, L_1, L_2

(Given our experience, we might expect that a resolution of M gives for both. However, this is "hopeless".)

Example. Take N as any module, we know that $\exists$ free F s.t. $F \to N \to 0$.
Take $0 \to \ker \to F \to N \to 0$.
know for this!

As an example
$$0 \to I \to R \to R/I \to 0.$$
this can't tell for both ends

Q2. Suppose M is f.g. and F a finite rank free module s.t. $M \simeq F/K$. Is K f.g.?

No. Take R to be non-noe., let $I \trianglelefteq R$ be a non-f.g. ideal. Then, $F = R$, $M = R/I$, $K = I$ is a counterexample.

Q3. With same notation, give a condition of R which forces K to be f.g.

Ans. R is Noetherian. Then, $F = R^{\oplus n}$ ← Noetherian and hence, $K \trianglelefteq F$ is f.g.

In fact, we can say more

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & & K_1 \arrow[dl] & \text{f.g. again} \\
0 & M \arrow[l] & F_0 = R^{\oplus n_0} \arrow[l] & & R^{\oplus n_1} \arrow[ll] \arrow[dll, dashed] & R^{\oplus n_2} \arrow[l] \arrow[ul, dashed] & \cdots \arrow[l] \\
 & & & K_0 \arrow[ul] & & 0 \arrow[ll] \\
 & & \text{f.g.} \arrow[ur] & & &
\end{tikzcd}$$

Thus, over a Noetherian ring R, a f.g. module M has a free resolution of the form

$$F_\bullet : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to 0 \quad \text{where}$$

$$F_i \simeq R^{\oplus n_i}.$$

Hence, fixing bases for F_i, we can write φ_i as matrices.

Def^n. (Presentation)

A matrix representation of φ_1 is called a presentation of M.

Q4. If M is f.g and $M \simeq F_1/K_1 \simeq F_2/K_2$, where $F_1 \simeq R^{\oplus n_1}$ and $F_2 \simeq R^{\oplus n_2}$, then

(a) is $n_1 = n_2$? No. Have seen already. Can always pad more Rs.

(b) Is it necessary that $K_1 \simeq K_2$? $\longrightarrow$ $n_1 = n_2 = 1$, we know

(c) How are f_1 and f_2 related?

Think about examples: $\mathbb{Z}$, $\mathbb{k}[x]$, quotients, $\frac{\mathbb{k}[x,y]}{I}$ where

$\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$

$\cong$

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

Think of as rings or $\mathbb{Z}$-modules

construct modules over it

$I = \langle x, y \rangle, \langle x^2, y^2 \rangle, \langle x^2, xy \rangle, \langle x^2, xy, y^2 \rangle, \langle x \rangle$

$\frac{\mathbb{k}[x,y,z]}{I}$; $I = \langle x, y \rangle, \langle x, y, z \rangle, \langle x^2, xy, y^2 \rangle, \ldots$

Lecture 8 (26-01-2021)

26 January 2021 11:37

Optimality of free resolutions

Example. Consider $M = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module.
$$\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

Then, $M = \langle (1,1) \rangle = \langle (1,0), (0,1) \rangle$.

(i) $0 \leftarrow M \leftarrow \mathbb{Z} \xleftarrow{6} \mathbb{Z} \leftarrow 0$

$(1,1) \mapsfrom 1$

(ii) $0 \leftarrow M \leftarrow \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \xleftarrow{\begin{bmatrix} 2 & \\ & 3 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \leftarrow 0$

$(1,0) \mapsfrom e_1$

$(0,1) \mapsfrom e_2$

Which is more optimal? How could we make the first step optimal? Pick gen. set which is least in cardinality?

(Note that both the gen sets above are minimal, since that is w.r.t. inclusion.)

Does this guarantee optimality in second step?

Recall. If $(R, \mathfrak{m}, \Bbbk)$ is local and M a f.g. module, then every minimal generating set of M has the same cardinality, denoted $\mu(M)$, where $\mu(M) = \dim_{\Bbbk}(M/\mathfrak{m}M)$.

[Follows from Nakayama.]

(\mu(M))

We would also want the kernels to be f.g.

Thus, we work in the following setting:

$(R, \mathfrak{m}, \Bbbk)$ is Noetherian local, M f.g.

(In fact, some authors : local includes Noetherian and non-Noetherian local is quasi-local for them.)

A way to ensure optimality : Let $\mu(M) = b_0$
Map $F_0 = R^{\oplus b_0}$ onto M.

$$0 \leftarrow M \leftarrow R^{\oplus b_0} \leftarrow K_0 \leftarrow 0$$

$$M = \langle x_1, \ldots, x_{b_0} \rangle \qquad x_i \longleftarrow\!\shortmid\ e_i$$

Put $b_1 := \mu(K_0)$ and map $F = R^{\oplus b_1}$ onto K_0 and continue.

$$0 \leftarrow R^{\oplus b_0} \xleftarrow{\varphi_1} R^{\oplus b_1} \xleftarrow{\varphi_2} R^{\oplus b_2} \leftarrow \cdots$$

Here, $b_0 = \mu(M)$ and $b_i = \mu(\operatorname{im} \varphi_i)$ for $i \geq 1$.

This is called a minimal free resolution of M over R.

(Minimal free resolution)

Q. If $\langle y_1, \ldots, y_{b_0} \rangle = M$ and K_0' is the kernel obtained by mapping $e_i \mapsto y_i$, how are K_0 and K_0' related?

Is $\mu(K_0) = \mu(K_0')$. $\longrightarrow$ This guarantees b_1 is well-defined. Doesn't guarantee anything for b_2, however.

Would like to see : $K_0 \cong K_0'$? If yes, then everything would go well ad infinitum.

$(n = b_0)$

Note that $y_j \in \langle x_1, \ldots, x_n \rangle \quad \forall j$ and
$x_i \in \langle y_1, \ldots, y_n \rangle \quad \forall i.$

$$y_j = a_{j1} x_1 + \cdots + a_{jn} x_n \qquad ; \; j = 1, \dots, n$$

That is,
$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Similarly,
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

where $A, B \in M_n(R)$.

Note
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = BA \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Idea: Try to show A is invertible. → hope to show $K_0 \cong K_0'$ with this.

Note that: modulo m, $\overline{BA} = \overline{id}$ since $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a basis and there is a unique way to express it in terms of itself.

Thus, $\det(\overline{BA}) \neq 0 \mod m$

$\Rightarrow \det BA \notin m$ $\quad$ (R, m) local

$\Rightarrow \det BA$ is a unit in R

$\Rightarrow BA$ is invertible in $M_n(R)$

$\Rightarrow A$ and B are invertible in $M_n(R)$.

Lecture 9 (28-01-2021)

28 January 2021 09:16

Setup: $(R, \mathfrak{m}, k)$ local Noetherian

Would like the following: ① A minimal free resolution of M over R (say $F.$) is truly minimal in the following sense: If $G.$ is a free resolution of M, then

$$\operatorname{rank}(F_i) \le \operatorname{rank}(G_i) \quad \forall i.$$

② If $F.$ is of the form $\cdots F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$

and $G.$ is of the form $\cdots G_1 \xrightarrow{\psi_1} G_0 \to M \to 0$

We would like to relate $\ker \varphi_1$ and $\ker \psi_1$. Would like $\ker \varphi_1 \hookrightarrow \ker \psi_1$.

All relations here are also here

③ For the next step:

$$\begin{array}{ccccccc} F_2 & \xrightarrow{\varphi_2} & F_1 & \longrightarrow & \ker \varphi_1 & \longrightarrow & 0 \\ G_2 & \xrightarrow[\psi_2]{} & G_1 & \longrightarrow & \ker \psi_1 & \longrightarrow & 0 \end{array}$$

What now? We only expect $\ker \varphi_1 \hookrightarrow \ker \psi_1$.

How do rank F_1 and rank G_1 compare?

Is there a relation between $\ker \varphi_2$ and $\ker \psi_2$?

Note that we have defined "minimal" last time, least cardinality of generating set at each step. Want to know if it is truly minimal.

The following technical Lemma takes care of it:

Lemma (Splitting Lemma) Let M and N be f.g. R-modules, where $(R, \mathfrak{m}, k)$ is a local Noetherian ring. Let F and G

be free modules mapping onto M and $M \oplus N$, respectively. Further assume that rank $F = \mu(M)$. Then, F split off G (i.e., $\exists$ an R-module P s.t. $G \simeq F \oplus P$) in a "natural way."

Moreover, if $\varphi : F \to M$ and $\psi : G \to M \oplus N$ are the given maps, then $\ker \varphi$ splits off $\ker \psi$.

rank $(F) = \mu(M)$ ensures the minimality.

Proof. Consider the s.e.s. es

$$0 \to L \hookrightarrow G \xrightarrow{\psi} M \oplus N \to 0,$$

$$0 \to K \hookrightarrow F \xrightarrow{\varphi} M \to 0,$$

where $L = \ker \psi$ and $K = \ker \varphi$.

Note that rank $(F) \le$ rank (G) since F maps minimally onto M. (This trivially gives a splitting of F off G, by the way.)

but we want a "natural" one!

Let $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ be bases of F and G (over R), respectively.

Let $x_i = \varphi(e_i)$ and $y_j = \psi(f_j)$. Then, $\{x_1, \dots, x_n\}$ is a minimal gen set of M over R.

(That is, $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a k-basis of $M/\mathfrak{m}M$.)

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] & M \oplus N \arrow[r] \arrow[d, shift left, "\pi"] & 0 \\
0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"] & M \arrow[r] \arrow[u, shift left, "i"] & 0
\end{tikzcd}$$

Under the above inclusions and projection, we can "write y_js in terms of the x_is" and vice-versa.

More precisely:

$$\pi(y_1) = a_{11} x_1 + \cdots + a_{1n} x_n$$
$$\vdots$$
$$\pi(y_m) = a_{m1} x_1 + \cdots + a_{mn} x_n$$

$$\begin{bmatrix} \pi(y_1) \\ \vdots \\ \pi(y_m) \end{bmatrix} = A_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Keep in mind that the

Keep in mind that the elements of the column are "vectors" themselves

$$i(x_1) = b_{11} y_1 + \cdots + b_{1m} y_m$$
$$\vdots$$
$$i(x_n) = b_{n1} y_1 + \cdots + b_{nm} y_m$$

$$\begin{bmatrix} i(x_1) \\ \vdots \\ i(x_n) \end{bmatrix} = B_{n\times m} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

So far, we have

$$\begin{tikzcd} f_j \arrow[rr, mapsto] & & y_j \arrow[ddd, mapsto] \\ G \arrow[r, "\psi"] \arrow[d, dashed] & M \oplus N \arrow[d, shift left, "\pi"] & \\ F \arrow[r, "\varphi"] & M \arrow[u, shift left, "i"] & \\ & & \pi(y_j) = \sum_{k=1}^{n} a_{jk} x_k \end{tikzcd}$$

Want a blue map to make square commute.

Define $G \to F$ by $f_i \mapsto \sum_{k=1}^{n} a_{jk} e_k$.

Since G and F are free modules, we can represent it as a matrix. It is A^T.

Similarly, we have $F \to G$ given as B^T.

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, shift right, "A^T"'] & M \oplus N \arrow[d, shift left, "\pi"] \\ F \arrow[r, "\varphi"'] \arrow[u, shift right, "B^T"'] & M \arrow[u, shift left, "i"] \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, two heads, "\psi"] & M \oplus N \\ F \arrow[u, dashed] \arrow[ur, dashed] \arrow[r, two heads, "\varphi"'] & M \arrow[u] \end{tikzcd}$$

$$BA \left\{ \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & & \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right.$$
$$= \begin{bmatrix} c_{11} x_1 + \cdots + c_{1n} x_n \\ \vdots \end{bmatrix}_{n\times 1} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(The "inner" and "outer" squares commute.)

Note that
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = BA \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

(why? $\underbrace{\pi i(x_i) = x_i}, \ldots$)

Go modulo m:
$$\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \overline{BA} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$
i.e.,

$$(\bar{I} - \bar{B}\bar{A}) \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = 0.$$

Since $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a basis, we have $\bar{I} = \bar{B}\bar{A}$.

Thus, $\det(\bar{B}\bar{A}) = \bar{1}$ and hence, $\det(BA) \in 1 + \mathfrak{m}$.

Thus, $\det(BA) \in \mathcal{U}(R)$.

$\left.\begin{array}{l} \text{Hence } BA \text{ is invertible.} \\ \text{Thus, so is } (BA)^T = A^T B^T. \end{array}\right\}$ in $\mathcal{M}_n(R)$

Let $x : F \to F$ be s.t. $x A^T B^T : F \to F$ is id_F.

Thus, $B^T : F \to G$ and $xA^T : G \to F$ are s.t.

$$(xA^T) B^T = \mathrm{id}_F.$$

$B^T x : F \to G$

$A^T : G \to F$

$G - B^T x (F) \oplus \ker A^T$

$$0 \longrightarrow \ker(xA^T) \longrightarrow G \underset{B^T}{\overset{xA^T}{\rightleftarrows}} F \longrightarrow 0$$

Thus, the above s.e.s. splits and hence

$$G = B^T(F) \oplus \ker(xA^T).$$

} shows the naturality!

Next, we show: $L = B^T(K) \oplus (\ker(xA^T) \cap L)$

(Try it!)

- $B^T(K) \subset L$

$$\begin{array}{ccccccccc} 0 & \to & L & \to & G & \xrightarrow{\psi} & M \oplus N & \to & 0 \\ & & & & B^T \uparrow \downarrow A^T & & & & \\ 0 & \to & K & \to & F & \xrightarrow{\varphi} & M & \to & 0 \\ & & & & x \circlearrowleft & & & & \end{array}$$

<u>Proof.</u> Recall: $L = \ker \psi$ and $K = \ker \varphi$.

Let $x \in K = \ker \varphi$.

Then, $\varphi(x) = 0$ and thus, $i(\varphi(x)) = 0$.

But $i(\varphi(x)) = \psi(B^T(x))$.

$\therefore B^T(x) \in \ker \psi = L$. ∎

Lecture 10 (01-02-2021)

01 February 2021 10:31

Recall:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] \arrow[d, shift left, "\alpha"] & M \oplus N \arrow[r] \arrow[d, shift right, "\pi"'] & 0 \\ 0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"] \arrow[u, shift left, "\beta"] & M \arrow[r] \arrow[u, shift right, "i"'] & 0 \end{tikzcd}$$

$(i\varphi = \psi\beta)$

Where β is multiplication by B^T and α by A^T.
(After we fixed an appropriate basis.)

We showed: $G = \beta(F) \oplus \ker(\pi\alpha)$

Claim. $\beta(K)$ is a direct summand of L.

Proof. We show that: $L \cap \beta(F) = \beta(K)$

[This proves the claim since we get $L = \beta(K) \oplus (\ker \pi\alpha \cap L)$.]

($\supseteq$) Let $x \in K$. Then, $\varphi(x) = 0$ and $i\varphi(x) = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \psi\beta(x)$

Thus, $\beta(x) \in \ker\psi = L$.

Thus, $\beta(K) \subseteq L \cap \beta(F)$.

($\subseteq$) Suppose $y \in L \cap \beta(F)$. Then, $y = \beta(x)$ for some $x \in F$.

We show $x \in K$, i.e., $\varphi(x) = 0$.

Note that $i\varphi(x) = \psi\beta(x)$
$\qquad\qquad\qquad = \psi(y)$ $\quad \supset y \in L = \ker\psi$
$\qquad\qquad\qquad = 0$

$\therefore$ $i\varphi(x) = 0$. Since i is 1-1, we get $\varphi(x) = 0$.

This finishes the proof. ∎

———— x ————

Restating the lemma:

Lemma: Let M and N be R-modules, $\varphi: F \to M$, $\psi: G \to M \oplus N$ be onto, $\ker \varphi = K$, $\ker \psi = L$, $F = R^{\oplus n}$, $G = R^{\oplus m}$, and $n = \mu(M)$. Consider the s.e.s.es

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] & M \oplus N \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"'] \arrow[u, "\beta"] & M \arrow[r] \arrow[u, "i"'] & 0 \end{tikzcd}$$

Then, $\exists \beta: F \to G$ s.t.

① $\psi\beta = i\varphi$.

② $\exists \gamma: G \to F$ onto s.t. β is a splitting.

③ $\beta|_K : K \to L$ is a splitting.

(Part of ③ that $\beta(K) \subset L$.)

Notation: $K \mid L$ that K is (isomorphic to) a direct summand of L.

Note:

$$0 \to \ker \overset{\overset{x\alpha}{\shortparallel}}{\gamma} \longrightarrow G \underset{\beta}{\overset{\gamma}{\rightleftarrows}} F \to 0$$

$$0 \to \ker \gamma \cap L \longrightarrow L \underset{\beta}{\overset{\gamma}{\rightleftarrows}} K \to 0$$

Some observations:

① Free modules have a lifting property:

$$\begin{tikzcd} G \arrow[r, "\psi"] & M \oplus N \arrow[r] & 0 \\ F \arrow[u, dashed, "\beta"] \arrow[ur, "i\varphi"'] & & \end{tikzcd}$$

constructed this $\to \beta$

$\{e_1, \dots, e_n\}$

Suppose $e_1 \overset{i\varphi}{\longmapsto} z_1 \in M \oplus N$.

Let $z_1' \in G$ be s.t. $\psi(z_1') = z_1$.

Then, we define $\beta(e_1) := z_1'$.

Can do this for every e_i.
Then, this defines a map $\beta : F \to G$ since F is free with basis $\{e_i\}_i$.

All we really used is: ① ψ is onto.
② F is free.

More generally, we have: (Lifting property)

Let φ be onto and F free.
Then, $\exists\, \bar{x} : F \to M$ s.t.
$\varphi \bar{x} = x$.
(rank $F < \infty$ not necessary.)

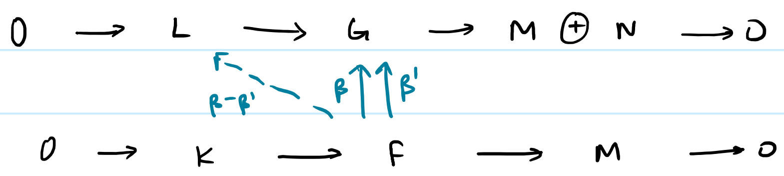
$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \arrow[r] & 0 \\ F \arrow[u, dashed, "\bar{x}"] \arrow[ur, "x"'] & & \end{tikzcd}$$

Def^n. This defines an R-module being projective.
(One that has the lifting property as above.)

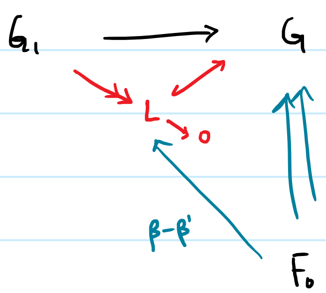
(Projective modules)

Eg. Free modules are projective.

② Let β, β' be two lifts. How are they related?
$\beta - \beta'$ must be in ker:



Suppose we have the free resolution:



By the lifting property, $\exists$ a map $F_0 \to G_1$ as:

$$
\begin{tikzcd}
G_1 \arrow[r] & G_0 \arrow[r] & {} \\
F_1 \arrow[r] & F_0 \arrow[r] \arrow[u, shift left] \arrow[u, shift right] \arrow[ul, dashed] & {}
\end{tikzcd}
$$

This idea will define a "homotopy" between two maps of chain complexes

③ Definition of chain maps :

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & F_0 \arrow[r] \arrow[dd, dashed] \arrow[ddr, phantom, "2"] & M \arrow[r] \arrow[dd] & 0 \\
& F_1 \arrow[u] \arrow[ur] \arrow[dd, dashed] & & & \\
0 \arrow[r] & L \arrow[r] & G_0 \arrow[r] & M \oplus N \arrow[r] & 0 \\
& G_1 \arrow[u] \arrow[ur] & & &
\end{tikzcd}
$$

This idea helps to define maps between chain complexes.

Lecture 11 (02-02-2021)

02 February 2021 11:33

Consequences of the proof of the Splitting Lemma:

1. if $N = 0$, then $L = \beta(K) \oplus \ker(\chi\, \alpha)$.
2. if $m = n$, then ($N = 0$ and) $L = \beta(K)$.

   In particular, $L \cong K$.

Consequence of the Splitting Lemma:

Thm. Let $(R, \mathfrak{m}, \mathbb{k})$ be a Noetherian local ring, M a f.g. R-module and $F_\bullet \to M$ be a minimal R-free resolution of M. If $G_\bullet \to M$ is any R-free resolution of M, then $\forall i \geq 0$, $\exists$ injective maps $\beta_i : F_i \to G_i$ satisfying:

(1)
$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, equal, "id_M"'] \arrow[dr, phantom, "\circlearrowleft"] & F_0 \arrow[l, "\varphi_0"'] \arrow[d, "\beta_0"] \arrow[dr, phantom, "\circlearrowleft"] & F_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta_1"] & \cdots \arrow[l] \\
0 & M \arrow[l] & G_0 \arrow[l, "\psi_0"] & G_1 \arrow[l, "\psi_1"] & \cdots \arrow[l]
\end{tikzcd}$$

$\psi_i \beta_i = \beta_{i-1} \varphi_i$ or $\psi\beta = \beta\varphi$ or each square commutes

(2) $\beta_i(F_i) \mid G_i$, i.e., $G_i = \beta_i(F_i) \oplus \cdots$

In particular, $\operatorname{rank}(F_i) \leq \operatorname{rank}(G_i)$. (Since β_i is 1-1.)

Proof. We use induction on i $(=: n)$ to show that

$\exists \beta_n : F_n \to G_n$ satisfying (1) and (2)

and (3) $\ker \psi_n = \beta(\ker(\varphi_n)) \oplus \cdots$

The base case $n = 0$ is the splitting lemma.

By induction, assume that $\forall i \leq n$, we have

$\beta_i : F_i \to G_i$

satisfying ①, ②, and ③.

$$\begin{array}{ccccccccc} 0 & \to & L_n & \to & G_n & \xrightarrow{\psi_n} & G_{n-1} & \to & \cdots \\ & & & & \uparrow \beta_n & \circlearrowright & \uparrow \beta_{n-1} & & \\ 0 & \to & K_n & \to & F_n & \xrightarrow[\varphi_n]{} & F_{n-1} & \to & \cdots \end{array}$$

We know that $L_n = \beta(K_n) \oplus \cdots$

The splitting lemma applied to

$$\begin{array}{ccccccccc} 0 & \to & \ker \psi_{n+1} & \to & G_{n+1} & \to & L_n & \to & 0 \\ & & & & & & \uparrow \beta_n & & \\ 0 & \to & \ker \varphi_{n+1} & \to & F_{n+1} & \to & K_n & \to & 0 \end{array}$$

gives β_{n+1} satisfying ①, ② and ③.

Remark. The compatibility of β with φ and ψ shows that every free resolution of M contains a minimal free resolution.
(Can think of β_i's as inclusions $F_i \hookrightarrow G_i$ and ψ_i's restrict to maps $F_i \to F_{i-1}$ where it becomes φ_i.)

Next consequence:

Thm If F and G are two minimal resolutions of M over R, then $\operatorname{rank}(F_i) = \operatorname{rank}(G_i)$.

Remark. In fact, the two resolutions are "isomorphic".
(We have chain maps which are isomorphisms. ← definition pending)

Defn Let $F_\bullet$ be a minimal free resolution (m.f.r.) of a f.g. module M over a Noetherian local ring R.
Then,

① the i^{th} Betti number of M over R, denoted $\beta_i^R(M) = \operatorname{rank}_R(F_i)$.

② $\ker \varphi_i = \operatorname{im} \varphi_{i+1}$ is called the $(i+1)^{st}$ syzygy module

of M over R, denoted $\Omega^R_{i+1}(M)$.
(Note the shift of index, $K_0 = \Omega^R_1(M)$.)

The above is well-defined, by the discussion above.

Note

① Thus, $F_\bullet$ breaks into short exact sequences (with $\Omega_0(M)=M$):

$$0 \longrightarrow \Omega_{i+1}(M) \longrightarrow R^{\oplus \beta_i} \longrightarrow \Omega_i(M) \to 0.$$

$\Omega_{i+1}(M)$ is the first syzygy of $\Omega_i(M)$ since β_i chosen minimaly

② $\beta_i = \mu(\Omega_i(M))$.

Remark The above also makes sense in the category of graded modules over graded rings.
(There's a notion of graded local, graded NAK, graded free resolutions, graded free resolution)

Testing minimality:

Q. What does it mean for $\{x_1, \ldots, x_n\}$ to be a minimal generating set of M?
(Notation: Let $\varphi: F = \bigoplus_{i=1}^{n} Re_i \twoheadrightarrow M$ with $e_i \mapsto x_i$. Assume it actually is a gen. set.)

$\{x_1, \ldots, x_n\}$ is a minimal gen. set of M over R

$\Leftrightarrow$ $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is a K-basis of $M/\mathfrak{m}M$

Lecture 12 (04-02-2021)

04 February 2021 09:31

Q. What does it mean for $\{x_1, \dots, x_n\}$ to be a minimal generating set of M?

(Notation: Let $\varphi : F = \bigoplus_{i=1}^{n} Re_i \twoheadrightarrow M$ with $e_i \mapsto x_i$. Assume it actually is a gen. set.)

$\{x_1, \dots, x_n\}$ is a minimal gen. set of M over R

$\Leftrightarrow$ $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a K-basis of $M/\mathfrak{m}M$ $\Leftrightarrow$ rank $F = \mu(M)$

Observe:

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ F/\mathfrak{m}F \arrow[r, dashed, "\bar{\varphi}"'] & M/\mathfrak{m}F \end{tikzcd}$$

$\varphi : F \twoheadrightarrow M$ induces an onto map $\bar{\varphi} : F/\mathfrak{m}F \to M/\mathfrak{m}M$ of k-vector spaces. (Can verify manually or observe this as tensor $\otimes_R k$.)

Now: $\{x_1, \dots, x_n\}$ is a minimal gen set of M $\Leftrightarrow$ $\bar{\varphi}$ is an iso.

($\Rightarrow$) Can use right-exactness to show $\bar{\varphi}$ is onto.

However, $\dim_k (F/\mathfrak{m}F) = \dim_k (M/\mathfrak{m}F) = n < \infty$. $\therefore \bar{\varphi}$ iso. $\blacksquare$

($\Leftarrow$) If $S = \{x_1, \dots, x_n\}$ is not minimal, then $\exists\, A \subsetneq S$ minimal.

But then $\bar{A}$ is a basis with $< n$ elements.

Contradiction since $\bar{F}$ has dim. n. $\blacksquare$

Q. What does this say about $\ker \varphi$?

Ans. $\ker \varphi \subset \mathfrak{m}F$.

Proof. Let $y \in \ker \varphi$. Write $y = \sum a_i e_i$.

Claim: $a_i \in \mathfrak{m}$ $\forall i$. (This would prove $\ker \varphi \subset \mathfrak{m}F$.)

Proof. $\sum a_i x_i = 0$ in M over R

$\Rightarrow \sum \bar{a}_i \bar{x}_i = 0$ in $M/\mathfrak{m}M$ over $R/\mathfrak{m}$

$\Rightarrow \bar{a}_i = 0$ $\forall i$ in $R/\mathfrak{m}$

$\Rightarrow \quad a_i \in \mathfrak{m} \quad \forall i \quad$ in R

Alternate proof by diagram:

$$\begin{tikzcd} & y \arrow[r, mapsto] & y \arrow[r, mapsto] & 0 \\ 0 \arrow[r] & \ker\varphi \arrow[r] & F \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ & & F/\mathfrak{m}F \arrow[r, "\bar\varphi"'] & M/\mathfrak{m}M \end{tikzcd}$$

$\therefore \; y \mapsto 0 \in M/\mathfrak{m}M$

$\therefore \; y \mapsto 0 \in F/\mathfrak{m}F$

$\therefore \; y \in \mathfrak{m}F.$

The above is again equivalent. $\rightsquigarrow$ do element wise or use the diagram

Thus, we have : $(R, \mathfrak{m}, \Bbbk)$ local Noetherian, M f.g.,
$M = \langle x_1, \ldots, x_n \rangle$.

$F = Re_1 \oplus \cdots \oplus Re_n$, $\quad \varphi : F \to M$ where $e_i \mapsto x_i$.

Then TFAE

(1) $\{x_1, \ldots, x_n\}$ is a minimal gen. set of M.

(2) $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is a basis of $M/\mathfrak{m}M$ (over $\Bbbk$)

(3) rank $F = \mu(M)$

(4) $\bar\varphi : F/\mathfrak{m}F \to M/\mathfrak{m}M$ is an iso.

(5) $\ker\varphi \subset \mathfrak{m}F$

Consider $F = F_0$ and F_1 a free module mapping onto $\ker\varphi$.
Then, $\operatorname{im} F_1 \subset \mathfrak{m}F_0$. Equivalently, if $\operatorname{im} F_1 \subset \mathfrak{m}F_0$, then $\ker\varphi \subset \mathfrak{m}F_0$ and hence, $\{x_1, \ldots, x_n\}$ is a min. gen. set of M.

Thus, we have the following: With M and R as above, let $F_\bullet \to M$ be a free resolution of M over R.
Then, $F_\bullet$ is minimal $\iff \operatorname{im}\varphi_i \subset \mathfrak{m}F_{i-1} \quad \forall i \geq 1$

$$\begin{tikzcd} & & \ker\varphi_{i-1} \arrow[dr] & & \\ \cdots \arrow[r] & F_{i+1} \arrow[r] & F_i \arrow[u] \arrow[r, "\varphi_i"] & F_{i-1} \arrow[r, "\varphi_{i-1}"] & \cdots \end{tikzcd}$$

$\operatorname{im}\varphi_i = \ker \varphi_{i-1} \subset \mathfrak{m}F_{i-1} \hookleftarrow$ since F_{i-1} maps minimally onto $\ker\varphi_{i-2}$.

$\iff$ (writing φ_i as a matrix) the entries of φ_i are in $\mathfrak{m}$.

Q. Let $G_\cdot \to M$ be a free resolution of M over R. (R local Noetherian, M f.g.). If $\operatorname{rank}(G_i) = n_i$, what can we say about the Betti numbers of M?

Q. If $G_\cdot$ is minimal after one stage, then can we write it as $(F_\cdot) \oplus N$?

Can we drop local-ness?