Morphisms of Schemes: Chevalley's Theorem

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7. Given $f \in A$, A_f will denote the localisation of A at the multiplicative set $\{1, f, f^2, \ldots\}$.

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$$\begin{tikzcd} \mathscr{F}(W) \arrow[r, "\operatorname{res}_{W,V}"] \arrow[dr, "\operatorname{res}_{W,U}"'] & \mathscr{F}(V) \arrow[d, "\operatorname{res}_{V,U}"] \\ & \mathscr{F}(U) \end{tikzcd}.$$

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Slogan 3

Given elements on patches, we can glue them uniquely.

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Moreover, the "obvious diagrams" must commute.

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The collection $\{D(f) : f \in A\}$ forms a basis for the above topology.

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To conclude, the only closed singleton subset of $\mathbb{A}_k^1$ is $\{\langle 0 \rangle\}$.

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In fact, (it follows that) the affine opens form a basis for X.

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The above is a morphism of affine schemes. That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

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Some definitions

Definition 14 (Compact morphism)

Definition 15 (Finite type morphism)

Definition 16 (Noetherian schemes)

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A morphism $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of schemes is compact

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A scheme $(X, \mathscr{O}_X)$ is said to be Noetherian if X can be covered by finitely many affine opens $\operatorname{Spec} A_i$ such that each A_i is a Noetherian ring.

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However, for Noetherian topological spaces (whatever those are), the two are equivalent.

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Chevalley's Theorem

Theorem 21 (Chevalley)

If $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a finite type morphism of Noetherian schemes, then the image of any constructible set is constructible. In particular, the image of π is constructible.

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