Algebraic Topology

Aryaman Maithani
https://aryamanmaithani.github.io/

August 6, 2020

In what follows, I will denote the closed interval $[0, 1] \subset \mathbb{R}$.
Whenever we talk about a map $f : X \to Y$ between topological spaces X and Y, we will always mean a *continuous function* f.
A path σ in a space X is a map $\sigma : I \to X$. If $x_0 = \sigma(0)$ and $x_1 = \sigma(1)$, we write this as

$$x_0 \xrightarrow{\sigma} x_1.$$

Moreover, x_0 and x_1 are called the *end points* of σ. In particular, x_0 is the initial point and x_1 is the final point.
All the topological spaces are assumed to be nonempty.

§1. Homotopy of Paths

§§1.1. The Fundamental Group

Definition 1.1 (Homotopy)**.** Let σ and τ be paths in a space X with the same end points, i.e., $\sigma(0) = \tau(0)$ and $\sigma(1) = \tau(1)$.
We say that σ and τ are *homotopic with ends points held fixed* written

$$\sigma \simeq \tau \quad \text{rel } \{0, 1\}$$

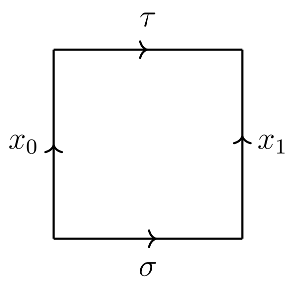
if there is a map $F : I \times I \to X$ such that

1. $F(s, 0) = \sigma(s)$ for all $s \in I$,
2. $F(s, 1) = \tau(s)$ for all $s \in I$,
3. $F(0, t) = x_0$ for all $t \in I$,
4. $F(1, t) = x_1$ for all $t \in I$.

F is called a *homotopy* from σ to τ. We write

$$F : \sigma \simeq \tau \quad \text{rel } \{0, 1\}.$$

The above can be pictorially depicted as



The above picture is interpreted as follows:
Along the (bottom) line $t = 0$, F agrees with σ and along the (top) line $t = 1$, F agrees with $t = 1$.
Similarly, along the (left) line $s = 0$, F is identically equal to x_0 and along the (right) line $s = 1$, it is x_1.

In particular, if σ is a *loop*, i.e., $x_0 = x_1$ and e_{x_0} is the constant loop $s \mapsto x_0$ for $s \in I$, and if $\sigma \simeq e_{x_0} \quad \text{rel } \{0, 1\}$, we say that "$\sigma$ can be shrunk to a point," or is *homotopically trivial*.

Proposition 1.2 ($\simeq$ is an equivalence relation)**.**

1. $\sigma \simeq \sigma \quad \text{rel } \{0, 1\}$,
2. $\sigma \simeq \tau \quad \text{rel } \{0, 1\} \implies \tau \simeq \sigma \quad \text{rel } \{0, 1\}$,
3. $\sigma \simeq \tau \quad \text{rel } \{0, 1\}$ and $\tau \simeq \rho \quad \text{rel } \{0, 1\} \implies \sigma \simeq \rho \quad \text{rel } \{0, 1\}$.

Proof.
1. Define $F(s, t) := \sigma(s)$.
2. Define $F(s, t) := F(s, 1 - t)$.
3. Given $F : \sigma \simeq \tau \quad \text{rel } \{0, 1\}$ and $G : \tau \simeq \rho \quad \text{rel } \{0, 1\}$, define $H : I \times I \to X$ as

$$H(s, t) := \begin{cases} F(s, 2t) & 0 \leq 2t \leq 1, \\ G(s, 2t - 1) & 1 \leq 2t \leq 2. \end{cases}$$

   Note that F and G do agree for $2t = 1$ since we have $F(s, 1) = \tau(s) = G(s, 0)$ for all $s \in I$. It is easy to see that H is well-defined.
   Note that H is continuous (by the pasting lemma) and it satisfies all the four properties of a homotopy (from σ to ρ), since F and G do so. □

Thus, we can consider the homotopy classes $[\sigma]$ of paths σ from x_0 to x_1 under the equivalence relation $\simeq$. (Note very carefully that all paths in an equivalence class have the same end points.)

Definition 1.3 (Multiplication of paths)**.** Let σ be a path from x_0 to x_1 and τ from x_1 to x_2.
The product $\sigma * \tau$ is a path from x_0 to x_2 defined as

$$\sigma * \tau(s) := \begin{cases} \sigma(2s) & 0 \leq 2s \leq 1, \\ \tau(2s - 1) & 1 \leq 2s \leq 2. \end{cases}$$

Once again, it's an easy check that $\sigma\tau$ is well-defined and continuous (using the pasting lemma).

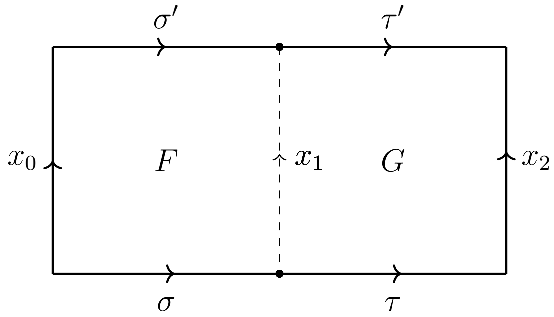
The above $\sigma * \tau$ is essentially the path from x_0 to x_1 obtained by first travelling from x_0 to x_1 via σ and then from x_1 to x_2 via τ.

We will now be lenient with notation and simply denote $\sigma * \tau$ as $\sigma\tau$ unless necessary. The next proposition shows how this product behaves with the equivalence relation.

Proposition 1.4.

$$\sigma \simeq \sigma' \quad \text{rel } \{0, 1\} \text{ and } \tau \simeq \tau' \quad \text{rel } \{0, 1\} \implies \sigma\tau \simeq \sigma'\tau' \quad \text{rel } \{0, 1\}.$$

Proof. The proof is motivated by the following diagram.



Given $F : \sigma \simeq \sigma' \quad \text{rel } \{0,1\}$ and $G : \tau \simeq \tau' \quad \text{rel } \{0,1\}$, define $H : I \times I \to X$ as

$$H(s,t) := \begin{cases} F(2s,t) & 0 \le 2s \le 1, \\ G(2s-1,t) & 1 \le 2s \le 2. \end{cases}$$

As earlier, H is well-defined (since $F(1,t) = x_1 = G(0,t)$ for all $t \in I$) and continuous. Moreover, we have

$$H(0,t) = F(0,t) = x_0, \quad H(1,t) = G(1,t) = x_2,$$

$$H(s,0) = \begin{cases} F(2s,0) & 0 \le 2s \le 1, \\ G(2s-1,0) & 1 \le 2s \le 2 \end{cases} = \begin{cases} \sigma(2s) & 0 \le 2s \le 1, \\ \tau(2s-1) & 1 \le 2s \le 2 \end{cases} = \sigma\tau(s),$$

and similarly,

$$H(s,1) = \sigma'\tau'(s) \text{ for all } s \in I.$$

This shows that

$$H : \sigma\tau \simeq \sigma'\tau' \quad \text{rel } \{0,1\}.$$

□

Definition 1.5 (Product of equivalence classes)**.** In view of the above proposition, we define

$$[\sigma] * [\tau] := [\sigma * \tau].$$

The above, of course, is defined only when the final point of σ (and thus, any other representative of $[\sigma]$) equals the initial point of τ (and thus, any other representative of $[\tau]$).

As before, we shall drop the $*$ and simply write $[\sigma][\tau]$.

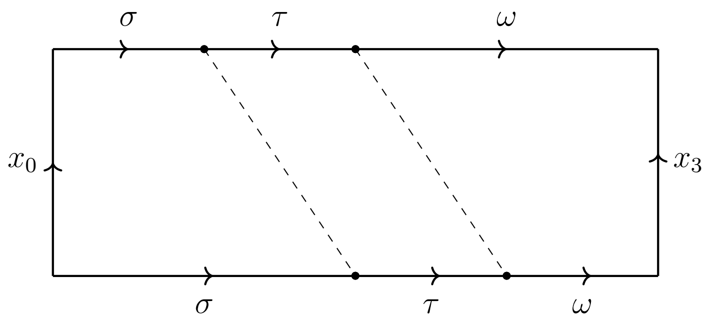
Lemma 1.6. Let σ, τ, ω be paths such that the products $\sigma(\tau\omega)$ and $(\sigma\tau)\omega$ are defined. Then,

$$\sigma(\tau\omega) \simeq (\sigma\tau)\omega \quad \text{rel } \{0,1\}.$$

Proof. Let x_0, x_1, x_2, x_3 be points such that

$$x_0 \xrightarrow{\sigma} x_1 \xrightarrow{\tau} x_2 \xrightarrow{\omega} x_3.$$

We define a homotopy F from $\sigma(\tau\omega)$ to $(\sigma\tau)\omega$. To motivate the definition of F, we may first visualise the homotopy as follows.



One can note that the top line depicts the path $(\sigma\tau)\omega$ and the bottom $\sigma(\tau\omega)$.

We define $F : I \times I \to X$ piece-wise on the three regions (from left to right) as follows:

$$F(s,t) := \begin{cases} \sigma\left(\dfrac{4s}{2-t}\right) & 0 \le s \le \dfrac{1}{4}(2-t), \\ \tau\left(4s+2-t\right) & \dfrac{1}{4}(2-t) \le s \le \dfrac{1}{4}(3-t), \\ \omega\left(\dfrac{4s+t-3}{t+1}\right) & \dfrac{1}{4}(3-t) \le s \le 1. \end{cases}$$

It is clear that F is continuous on each piece. By the pasting lemma, it is continuous everywhere.
The four properties of being a homotopy are also clear, by construction. (The diagram makes it clear why.) $\square$

Definition 1.7 (Inverse path)**.** Given a path σ from x_0 to x_1, its *inverse path* σ^{-1} is a path from x_1 to x_0 given by

$$\sigma^{-1}(s) := \sigma(1-s), \qquad s \in I.$$

The above is simply "travelling backwards σ."

Lemma 1.8. Let $\sigma, \sigma' : I \to X$ be paths such that $\sigma \simeq \sigma' \quad \text{rel } \{0,1\}$. Then,

$$\sigma^{-1} \simeq \sigma'^{-1} \quad \text{rel } \{0,1\}.$$

Proof. Let $F : \sigma \simeq \sigma' \quad \text{rel } \{0,1\}$ be a homotopy. Then, $F'(s,t) := F(1-s,t)$ is a homotopy between the inverses. $\square$

Definition 1.9 (Inverse class)**.** Let $\sigma : I \to X$ be a path. We define the inverse of the class $[\sigma]$ as

$$[\sigma]^{-1} := [\sigma^{-1}].$$

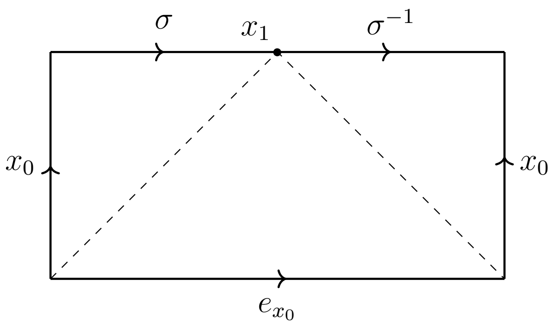
In view of the above lemma, the above definition is indeed well-defined.

Lemma 1.10. Given any path σ from x_0 to x_1, we have

$$e_{x_0} \simeq \sigma\sigma^{-1} \quad \text{rel } \{0,1\},$$

where e_{x_0} denotes the constant loop at x_0.

Proof. As usual, we motivate the proof with a diagram. In this case, it is the following:



The homotopy $F : I \times I \to X$ in this case, is defined as

$$F(s,t) := \begin{cases} \sigma(2s) & 0 \le 2s \le t, \\ \sigma(t) & t \le 2s \le 2-t, \\ \sigma^{-1}(2s-1) & 2-t \le 2s \le 2. \end{cases}$$

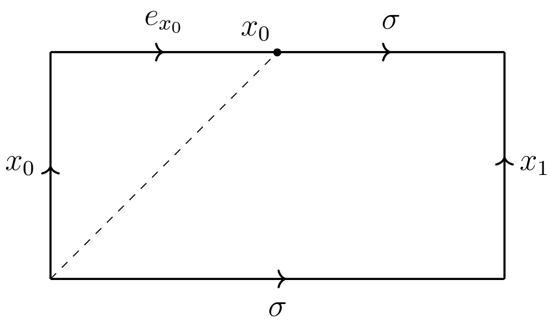
It is clear that the piecewise definitions agree on the dashed line $2s = t$. Observe that $\sigma^{-1}(2s-1) = \sigma(2-2s)$ and thus, the functions do agree on the dashed line $2s = 2-t$ as well.
One can easily check that the four properties of the homotopy are satisfied. To see the bottom line property, note that $F(s,0) = \sigma(0)$ (using the second piece definition) and $\sigma(0) = x_0 = e_{x_0}(s)$ for all $s \in I$. $\square$

Note that since $(\sigma^{-1})^{-1} = \sigma$, the above also shows that $\sigma^{-1}\sigma = e_{x_1}$.

Lemma 1.11. Let $x_0 \xrightarrow{\sigma} x_1$ and e_{x_0} be the constant path at x_0. Then,

$$\sigma \simeq e_{x_0}\sigma \quad \text{rel } \{0,1\}.$$

Proof. The proof is motivated by this diagram.



The homotopy is $F : I \times I \to X$ defined as

$$F(s,t) := \begin{cases} x_0 & 0 \leq 2s \leq t, \\ \sigma\left(\dfrac{2s-t}{2-t}\right) & t \leq 2s \leq 2. \end{cases}$$ □

As one would expect, we have a lemma in the other direction as well.

Lemma 1.12. Let $x_1 \xrightarrow{\sigma} x_0$ and e_{x_0} be the constant path at x_0. Then,

$$\sigma \simeq \sigma e_{x_0} \quad \text{rel } \{0,1\}.$$

Proof. Similar as in the last case and we omit it. □

The astute reader might have sensed a group sneaking around the corner.
However, note that the product of equivalence classes defined above is not a binary operation unless the endpoints are the same. Due to this, we restrict ourselves to loops in the next theorem.

Theorem 1.13. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X at x_0.
If multiplication in $\pi_1(X, x_0)$ is defined as above, $\pi_1(X, x_0)$ becomes a group, in which the neutral element is the class $[e_{x_0}]$ and the inverse of a class $[\sigma]$ is the class of the inverse $[\sigma^{-1}]$.

Proof. Interpreting Lemmas 1.6 to 1.12 as equalities of the equivalence classes shows that $\pi_1(X, x_0)$ verifies the group axioms. □

The next proposition tells us how $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are related in the case that x_0 and x_1 lie in the same path-connected component. (In the case that they do not, nothing can be said.)

Proposition 1.14. Let α be a path from x_0 to x_1. The mapping $\widehat{\alpha}$ defined by

$$[\sigma] \mapsto [\alpha^{-1}] * [\sigma] * [\alpha] = [\alpha^{-1}\sigma\alpha]$$

is an isomorphism of the group $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$.

Note that the above is well-defined since $*$ is well-defined.

Proof. We first note that if $[\sigma] \in \pi_1(X, x_0)$, then $\alpha^{-1}\sigma\alpha$ is path as follows:

$$x_1 \xrightarrow{\alpha^{-1}} x_0 \xrightarrow{\sigma} x_0 \xrightarrow{\alpha} x_1$$

and thus, $[\alpha^{-1}\sigma\alpha]$ is indeed an element of $\pi_1(X, x_1)$.
Moreover, note that

$$\begin{aligned} \widehat{\alpha}([\sigma\sigma']) &= [\alpha^{-1}\sigma\sigma'\alpha] \\ &= [\alpha^{-1}\sigma][\sigma'\alpha] \\ &= [\alpha^{-1}\sigma][\alpha\alpha^{-1}][\sigma'\alpha] \\ &= [\alpha^{-1}\sigma\alpha][\alpha^{-1}\sigma'\alpha] \\ &= \widehat{\alpha}([\sigma])\widehat{\alpha}([\sigma']). \end{aligned}$$

This shows that $\widehat{\alpha}$ is a homomorphism. That this is an isomorphism follows by noting that it has as inverse $\widehat{\alpha^{-1}}$. □

Corollary 1.15. If X is pathwise connected, the group $\pi_1(X, x_0)$ is independent of the point x_0, up to isomorphism.

Note that if C is a connected component of X containing x_0, then $\pi_1(X, x_0) = \pi_1(C, x_0)$ since any loop at x_0 must necessarily lie in C. For this reason, we might as well only work with pathwise connected spaces.

Definition 1.16. If X is pathwise connected, we write $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it *the fundamental group* of X.

Note that this group depends on x_0 in the sense that the elements of the group depend on the base point x_0 but the isomorphism class does not.

Definition 1.17 (Simply connected)**.** A space X is called simply connected if it is pathwise connected and its fundamental group is trivial.

§§1.2. Functoriality

We now wish to turn π_1 into a functor. Since we need to take care of the base points, we look at the category of *Pointed Topological spaces*.

Definition 1.18 (Pointed Topological Spaces)**.** The category $\mathsf{Top}_\bullet$ of *pointed topological spaces* is the category whose objects and morphisms are given as follows:

- Objects: Pairs (X, x_0) where X is a topological space and $x_0 \in X$,
- Morphisms: $f : (X, x_0) \to (Y, y_0)$ such that $f : X \to Y$ is a continuous function and $f(x_0) = y_0$.

That the above is a category can be easily verified.

Definition 1.19. Let $h : (X, x_0) \to (Y, y_0)$ be a morphism. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([\sigma]) = [h \circ \sigma].$$

The map h_* is called the *homomorphism induced by* h, relative to the base point x_0.

To see that h_* is well-defined, we note that if

$$F : \sigma \simeq \sigma' \quad \text{rel } \{0, 1\}$$

for loops σ, σ' in X at x_0, then

$$h \circ F : h \circ \sigma \simeq h \circ \sigma' \quad \text{rel } \{0, 1\}.$$

That is to say, if two loops at x_0 are homotopic, then so are the loops obtained by pre-composing h.

To see that h_* is a homomorphism, first note that

$$(h \circ \sigma)(h \circ \sigma') = h \circ (\sigma\sigma').$$

(This follows from the definition of the product of paths.)
Then, we see that

$$h_*([\sigma\sigma']) = [h \circ (\sigma\sigma')] = [h \circ \sigma][h \circ \sigma'] = h_*([\sigma])h_*([\sigma']).$$

Theorem 1.20 (Functoriality)**.** If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are morphisms, then

$$(k \circ h)_* = k_* \circ h_*.$$

If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof. By definition, we have

$$\begin{aligned}(k \circ h)_*([\sigma]) &= [(k \circ h) \circ \sigma] \\ &= [k \circ (h \circ \sigma)] \\ &= k_*([h \circ \sigma]) \\ &= k_*(h_*([\sigma])) \\ &= (k_* \circ h_*)([\sigma]).\end{aligned}$$

Thus, $(k \circ h)_* = k_* \circ h_*$.

Now, if i is the identity map, then we have

$$i_*([\sigma]) = [i \circ \sigma] = [\sigma],$$

showing that i_* is the identity map of $\pi_1(X, x_0)$. $\square$

The above then shows that π_1 defines a functor from the category Top_* to Grp.
Since functors preserve isomorphisms in general, we get the following corollary.

Corollary 1.21. If $h : (X, x_0) \to (Y, y_0)$ is a morphism such that $h : X \to Y$ is a homeomorphism, then

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Since we aren't discussing Category Theory, we give a proof for this special example of functors.

Proof. Let $h^{-1} : Y \to X$ be the inverse, which is continuous since h is a homeomorphism. Moreover, $h^{-1}(y_0) = x_0$ and thus, $h^{-1} : (Y, y_0) \to (X, x_0)$ is a morphism and the inverse of h.
Now, note that,

$$(h_*) \circ ((h^{-1})^*) = (h \circ h^{-1})^* = (\mathrm{id}_{(Y,y_0)})^* = \mathrm{id}_{\pi_1(Y,y_0)},$$

by functoriality. Similarly, we have

$$((h^{-1})^*) \circ (h_*) = \mathrm{id}_{(X,x_0)},$$

proving the corollary. $\square$

§2. Homotopy of Maps

In the previous section, we talked about homotopy of special types of maps. More precisely, we only considered maps $I \to X$. However, we can replace I by an arbitrary topological space Y. In the place of endpoints, we just consider a subspace $A \subset Y$.

Definition 2.1 (Relative homotopy)**.** Given maps $f, g : Y \to X$ such that $f|_A = g|_A$, we say f and g are homotopic relative to A written

$$f \simeq g \quad \text{rel} \ \ A$$

if there is a map $F : Y \times I \to X$ satisfying

1. $F(y, 0) = f(y)$ for all $y \in Y$,
2. $F(y, 1) = g(y)$ for all $y \in Y$,
3. $F(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$.

This map F is called a homotopy from f to g relative to A and we write

$$F : f \simeq g \quad \text{rel} \ \ A.$$

Note that the "second coordinate" above is still I.
Note that (3) is satisfied vacuously if $A = \varnothing$ and we have $f|_A = g|_A$ for all maps $f, g : Y \to X$. Keeping this in mind, we have the following definition.

Definition 2.2 (Homotopy)**.** Maps $f, g : Y \to X$ are said to be *homotopic* if f and g are homotopic relative to $\varnothing$.
We write this more simply as

$$f \simeq g.$$

Moreover, any F as before is simply called a homotopy from f to g.
As before, we write

$$F : f \simeq g.$$

Once again, we obtain an equivalence. The homotopies defined as in the proof of Proposition 1.2 work again.

Definition 2.3 (Contractible space)**.** If X is a topological space such that the identity map on X is homotopic to a constant map on some point in X, we say that X is *contractible*.

Proposition 2.4. X is contractible if and only if for any space Y, any two maps of Y into X are homotopic. A contractible space is pathwise connected.

Proof. ($\implies$) Let X be contractible and Y be any space. Fix any $x_0 \in X$ such that id_X is homotopic to the constant map $e_{x_0} : X \to X$.
Let $f_{x_0} : Y \to X$ denote the constant map $y \mapsto x_0$.
Now, given any map $f : Y \to X$, we show that it is homotopic to f_{x_0}.
This will prove that any two maps of Y into X are homotopic since $\simeq$ is an equivalence relation.
Let $H : \mathrm{id}_X \simeq e_{x_0}$ be any homotopy. Then, we have

$$H(x, 0) = x, \quad H(x, 1) = x_0; \qquad \text{for all } x \in X.$$

(Note that H is continuous.)
Now, we define $F : Y \times I \to X$ as

$$F(y, t) = H(f(y), t).$$

It is clear that F is a map. (That is, F is continuous.)
Moreover, note that

$$F(y, 0) = H(f(y), 0) = f(y), \quad F(y, 1) = H(f(y), 1) = x_0 = f_{x_0}(y); \qquad \text{for all } y \in Y.$$

This shows that $F : f \simeq f_{x_0}$, as desired.

($\impliedby$) To show that X is contractible, simply consider $Y = X$ and consider the maps id_X and e_{x_0}. (Both of these are indeed continuous.)
By hypothesis, these maps are homotopic and by definition, X is contractible.

Now, we show that X is pathwise connected assuming that it is contractible.
Let x_0 and x_1 be any two points in X. As X is contractible, ($\implies$) tells us that the maps e_{x_0} and e_{x_1} are homotopic.
Let F be any homotopy from e_{x_0} and e_{x_1}. Define $\sigma : I \to X$ as

$$\sigma(t) := F(x_0, t).$$

σ is clearly continuous. Moreover, we have

$$\begin{aligned} \sigma(0) &= F(x_0, 0) = e_{x_0}(x_0) = x_0, \\ \sigma(1) &= F(x_0, 1) = e_{x_1}(x_0) = x_1. \end{aligned}$$

Thus, σ is path from x_0 to x_1 in X, proving the proposition. □

Example 1. Every convex subset X of Euclidean space is contractible.
Given maps $f_1, f_2 : Y \to X$, we have a homotopy $F : f_1 \simeq f_2$ given by

$$F(y, t) = tf_2(y) + (1 - t)f_1(y), \quad y \in Y, t \in I.$$

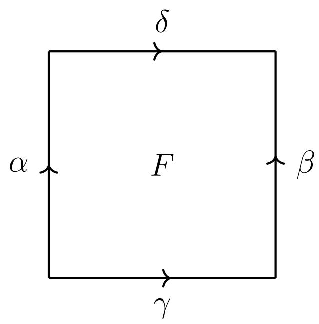
By the convexity assumption, the above F is indeed a map into X.
By the previous proposition, this shows that X is contractible.

Example 2. $\mathbb{R}^n$ is contractible for any n.
To see this, we could either appeal to the previous example or do it directly by defining a homotopy $F : e_0 \simeq \mathrm{id}_{\mathbb{R}^n}$ as

$$F(x,t) = tx.$$

We would now like to show that any contractible space is simply connected. What we do know is that any loop would be homotopic to a point. However, we do not know if this homotopy is relative to $\{0,1\}$. Indeed, to show that we do have a homotopy relative to $\{0,1\}$, we would need to use the fact that X is contractible once again.
Before proving that, we first look at a lemma.

Lemma 2.5. Let $F : I \times I \to X$ be a map. Set $\alpha(t) = F(0,t)$, $\beta(t) = F(1,t)$, $\gamma(s) = F(s,0)$, and $\delta(s) = F(s,1)$, as in the diagram



Then, $\delta = \alpha^{-1}\gamma\beta$.

Proof. The proof is quite intuitive. First, we define the paths

$$\sigma : I \to I \times I, \quad \tau : I \to I \times I$$

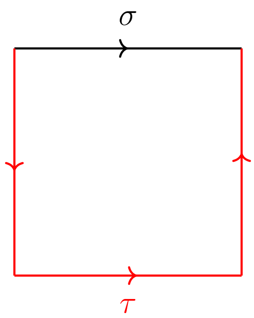
as

$$\sigma(s) := (t,1)$$

and

$$\tau(s) := \begin{cases} (0, 1-4s) & 0 \leq 4s \leq 1, \\ (4s-1, 0) & 1 \leq 4s \leq 2, \\ (1, 2s-1) & 1 \leq 2s \leq 2. \end{cases}$$

These paths are the following ones in I^2 :



As it should be clear from the diagram (and one can easily check), we have

$$\delta = F \circ \sigma, \quad (\alpha^{-1}\gamma)\beta = F \circ \tau.$$

(Note that the bracketing in $(\alpha^{-1}\gamma)\beta$ is necessary.)
Also, since I^2 is convex, we see that σ and τ are homotopic relative to $\{0, 1\}$ with $H : I \times I \to I \times I$ being a required homotopy defined as

$$H(s, t) := (1 - t)\sigma(s) + t\tau(s).$$

Thus,

$$\begin{aligned} &F \circ H : F \circ \sigma \simeq F \circ \tau \quad \text{rel } \{0, 1\} \\ \implies &F \circ H : \delta \simeq (\alpha^{-1}\gamma)\beta \quad \text{rel } \{0, 1\}, \end{aligned}$$

as desired. □

Theorem 2.6. Let X be a contractible space. Then, X is simply connected.

Proof. Note that by Proposition 2.4, we know that X is pathwise connected. Now we show that that $\pi_1(X)$ is trivial.
Let $x_0 \in X$ be arbitrary and $\alpha : I \to X$ be a loop at x_0 in X.
If we show that $\alpha \simeq e_{x_0} \quad \text{rel } \{0, 1\}$, then we are done.

To do this, we will use the earlier lemma after constructing an appropriate F.
Using that X is contractible, we fix a homotopy $H : \text{id}_X \simeq f_{x_0}$ where $f_{x_0} : X \to X$ is the constant function $x \mapsto x_0$.
(This is different from e_{x_0} since the domains are different in general.)
To recall, H has the following properties:

$$H(x, 0) = x, \ H(x, 1) = x_0 \quad \text{for all } x \in X.$$

Now, we define $F : I \times I \to X$ as

$$F(s, t) := H(\sigma(s), t).$$

Now, note that if we set $\alpha, \beta, \gamma, \delta$ as in the previous lemma, we have

$$\begin{aligned} \alpha(t) = F(0,t) = H(\sigma(0),t) &= H(x_0,t) \\ &= H(\sigma(1),t) = F(1,t) = \beta(t), \end{aligned}$$

$$\gamma(s) = F(s,0) = H(\sigma(s),0) = \sigma(s),$$
$$\delta(s) = F(s,1) = H(\sigma(s),1) = x_0.$$

In other words, we have

$$\alpha = \beta, \gamma = \sigma, \delta = e_{x_0}.$$

By the previous lemma, we know that $[\delta] = [\alpha^{-1}\gamma\beta]$, where $[.]$ is the homotopy class of a path relative to $\{0,1\}$. Thus, we have

$$\begin{aligned} & [e_{x_0}] = [\alpha^{-1}\sigma\alpha] \\ \implies & [\alpha][e_{x_0}][\alpha^{-1}] = [\sigma] \\ \implies & [e_{x_0}] = [\sigma] \\ \implies & e_{x_0} \simeq \sigma \quad \text{rel } \{0,1\}, \end{aligned}$$

finishing the proof. $\square$

Proposition 2.7. Let $f, g : Y \to X$ be maps which are homotopic by means of a homotopy $F : Y \times I \to X$.
Let $y_0 \in Y$, $x_0 := f(y_0) = F(y_0, 1)$, and $x_1 := g(y_0) = F(y_0, 1)$.
Let $\alpha : I \to X$ be a path from x_0 to x_1 given by

$$\alpha(t) = F(y_0, t) \quad t \in I.$$

Then, the following diagram commutes.

$$\begin{tikzcd} \pi_1(Y, y_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(X, x_0) \arrow[d, "\widehat{\alpha}"] \\ & \pi_1(X, x_1) \end{tikzcd}$$

Proof. The diagram commuting is just saying that

$$\widehat{\alpha} \circ f_* = g_*.$$

Let $[\sigma] \in \pi_1(Y, y_0)$ be arbitrary. Showing that the above is true is equivalent to showing that

$$(\widehat{\alpha} \circ f_*)([\sigma]) = g_*([\sigma]).$$

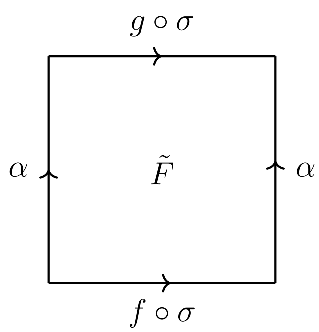
Using the definitions of $\widehat{\alpha}$ and f_*, we note that

$$\begin{aligned} & (\widehat{\alpha} \circ f_*)([\sigma]) = g_*([\sigma]) \\ \iff & \widehat{\alpha}(f_*([\sigma])) = g_*([\sigma]) \\ \iff & \widehat{\alpha}([f \circ \sigma]) = [g \circ \sigma] \\ \iff & [\alpha^{-1}(f \circ \sigma)\alpha] = [g \circ \sigma]. \end{aligned}$$

Now, defining $\tilde{F} : I \times I \to X$ as

$$\tilde{F}(s, t) = F(\sigma(s), t).$$

Then, we have the following diagram as in Lemma 2.5 which proves the proposition.



To see that the sides are indeed as labeled, recall that σ is a loop at y_0 and note that

$$\begin{aligned} \tilde{F}(0, t) &= F(\sigma(0), t) = F(y_0, t) = \alpha(t), \\ \tilde{F}(1, t) &= F(\sigma(1), t) = F(y_0, t) = \alpha(t), \\ \tilde{F}(s, 0) &= F(\sigma(s), 0) = g(\sigma(s)) = (g \circ \sigma)(s), \\ \tilde{F}(s, 1) &= F(\sigma(s), 1) = f(\sigma(s)) = (f \circ \sigma)(s). \end{aligned}$$

By the conclusion of Lemma 2.5, we are done. □

Recall that $\widehat{\alpha}$ is an isomorphism and thus, we get the following corollary.

Corollary 2.8. With the same setup as above, f_* is an isomorphism if and only if g_*.

What the above corollary says is that if f and g are homotopic, then f_* is an isomorphism iff g_* is.

Definition 2.9 (Homotopy equivalence)**.** A map $f : Y \to X$ is said to be a *homotopy equivalence* if there exists a map $f' : X \to Y$ such that

$$\begin{aligned} ff' &\simeq \mathrm{id}_X, \\ f'f &\simeq \mathrm{id}_Y\,. \end{aligned}$$

If such a map exists, we say that X and Y are *homotopically equivalent spaces*.

It can be checked that being homotopically equivalent is an "equivalence relation."

Corollary 2.10. If $f : Y \to X$ is a homotopy equivalence, then f_* is an isomorphism

$$\pi_1(Y, y_0) \to \pi_1(X, f(y_0))$$

for any $y_0 \in Y$.

Proof. Let $f' : X \to Y$ be as in the definition.
Then, $ff' \simeq \mathrm{id}_X$. By the previous corollary, we have that $(ff')_*$ is an isomorphism. (Since $(\mathrm{id}_X)_*$ is.)
Similarly, $(f'f)_*$ is an isomorphism. Since $(ff')_* = f_* \circ f'_*$ and $(f'f)_* = f'_* \circ f_*$, we see that f_* is a bijection and hence, an isomorphism. □

The above shows that the fundamental group of a path-connected space is a *homotopy invariant*. We had shown earlier that this was a topological invariant.
Note that being homotopically equivalent is a weaker concept than being topologically invariant (i.e., homeomorphic). Clearly, if $f : X \to Y$ is a homeomorphism, it also a homotopy equivalence with $f' = f^{-1}$.
However, the closed interval I is homotopically equivalent to the point space but clearly not homeomorphic. In fact, one can note that X is contractible if and only if it is homeomorphic to a point.

§3. Fundamental Group of the Circle

In this section, we prove a more general result. S^1 will turn out to be a special case of that. First, we need a lemma.

Lemma 3.1. Let K be a compact metric space and G a topological group. Let $V \subset G$ be open such that $1 \in V$.
If $f : K \to G$ is continuous, then there exists $\delta > 0$ such that

$$d(k, k') < \delta \implies f(k)(f(k'))^{-1} \in V.$$

The above is essentially mimicking something like "uniform continuity."

Proof.

Claim 1. There exists an open set $U \subset G$ such that

1. $1 \in U \subset V$,
2. $g, g' \in U \implies gg^{-1} \in V$.

Proof. The function $\varphi : G \times G \to G$ defined as

$$\varphi(g, g') := g(g')^{-1}$$

is continuous. Thus, $\varphi^{-1}(V)$ is open.
Note that $(1, 1) \in \varphi^{-1}(V)$. Thus, there exists a basis element of the form $U_1 \times U_2$ satisfying

$$(1, 1) \in U_1 \times U_2 \subset \varphi^{-1}(V).$$

Let $U := U_1 \cap U_2 \cap V$. Clearly, U is open and $1 \in U \subset V$.
Moreover,

$$g, g' \in U \implies (g, g') \in U_1 \times U_2 \subset \varphi^{-1}(V) \implies \varphi(g, g') \in V \implies g(g')^{-1} \in V,$$

as desired. □

With this, we can mimic the proof of continuous functions being uniformly continuous on compact sets. (The above U will help us use "triangle inequality" in the codomain.) Let U be as in the above claim.

Claim 2. Given any $k \in K$, there exists $\delta_k > 0$ such that

$$d(k, k') < \delta_k \implies f(k)(f(k'))^{-1} \in U,$$
$$d(k, k') < \delta_k \implies f(k')(f(k))^{-1} \in U.$$

Proof. The function $f_k : K \to G$ defined by $f_k(k') = f(k)(f(k'))^{-1}$ is continuous with $f_k(k) = 1 \in U$.
Consider the open set $f_k^{-1}(U)$. Since it contains k, there exists $\delta > 0$ such that $B_\delta(k) \subset f_k^{-1}(U)$. Thus, if $k' \in B_\delta(k)$, then $f_k(k') \in U$, as desired for the first condition.

Note that we can find a suitable δ'_k for the other condition as well. Taking the minimum of the two proves the claim. □

Let $V_k = B_{\delta_k/2}(k)$. Clearly, $\{V_k\}_{k\in K}$ is an open cover of K. Since K is compact, we may extract a finite subcover.
Let $k_1, \dots, k_n$ be the indices of one such. Set

$$\delta := \min_{1\leq i\leq n} \frac{1}{2}\delta_{k_i}.$$

Clearly, $\delta > 0$. Moreover, it satisfies the condition of the lemma. To see this, let $k, k' \in K$ be such that $d(k, k') < \delta$.
Since $\{V_{k_i}\}_{1\leq i\leq n}$, is an open cover, k lies in V_{k_i} for some $1 \leq i \leq n$. That is, $2d(k, k_i) < \delta_i$. Now, using triangle inequality, note that

$$d(k', k_i) \leq d(k', k) + d(k, k_i) < \delta + \frac{1}{2}\delta_i \leq \frac{1}{2}\delta_i + \frac{1}{2}\delta_i = \delta_i.$$

Thus, both k and k' are at most δ_i from k_i. By the definition of δ_i (from Claim 2), we see that $f(k)(f(k_i))^{-1} \in U$ and $f(k_i)(f(k'))^{-1} \in U$.
By the property of U, we have

$$(f(k)(f(k_i))^{-1})((k_i)(f(k'))^{-1}) = f(k)(f(k'))^{-1} \in V,$$

as desired. □

Now, for the remainder of this section, we shall fix G as any simply connected topological group and $H \leq G$ is a *discrete* normal subgroup of G. We will show that $\pi_1(G/H, 1) \cong H$.
(In the special case that $G = \mathbb{R}$ and $H = \mathbb{Z}$, we see that $\pi_1(S^1, 1) \cong \mathbb{Z}$ or simply, $\pi_1(S^1) \cong \mathbb{Z}$.)

We also fix the map $\varphi : G \to G/H$ to be the projection $g \mapsto gH$.

Lemma 3.2. There exists an open neighbourhood U of 1 in G which is mapped homeomorphically onto an open neighbourhood V of 1 in G/H be φ.

Proof. Since H is discrete, $\{1\}$ is open in H. Thus, there exists an open neighbourhood U_1 of 1 in G such that $U_1 \cap H = \{1\}$.
As in claim 1 of the previous proof, we may find a subset $U \subset U_1$ such that $g, g' \in U \implies gg'^{-1} \in U_1$. Clearly, $U \cap H = \{1\}$ as well.

> **Claim 1.** $\varphi|_U$ is injective.
>
> *Proof.* Let $g_1, g_2 \in U$ with $\varphi(g_1) = \varphi(g_2)$.
> Then, $g_1H = g_2H$ or $Hg_1 = Hg_2$ or $Hg_1g_2^{-1} = H$ or $g_1g_2^{-1} \in H$.
> Since $g_1, g_2 \in U$, we also have $g_1g_2^{-1} \in U_1$. Since $U_1 \cap H = \{1\}$, we see that $g_1g_2^{-1} = 1$ or $g_1 = g_2$. □

Let $V = \varphi(U)$. Clearly, φ maps U bijectively onto V, in view of the previous claim. Moreover, this must be a homeomorphism. To see this, we recall a general result.

> **Claim 2.** The quotient map $\phi : G \to G/H$ is open.
>
> *Proof.* Let W be an open subset of G. The set
>
> $$WH = \{wh : w \in W, h \in H\}$$
>
> is open since $WH = \bigcup_{h \in H} Wh$, which is a union of open subsets of G since right multiplication is a homeomorphism.
> Note that $\varphi^{-1}(\varphi(W)) = WH$. Since φ is a quotient map and WH is open, we see that $\varphi(W)$ is open, as desired. □

Thus, we see that $\varphi|_U : U \to V$ is a bijective open map. In particular, it is a homeomorphism. □

For the remainder of this section, we fix $U \subset G$ and $V \subset G/H$ as above. Moreover, we fix

$$\psi := (\varphi|_U)^{-1}.$$

By our above discussion, $\psi : V \to U$ is a continuous function.
For better clarity, we shall use 1 for the identity of G/H and 1_G for the identity of G.

Now, we prove two key lemmas.

Lemma 3.3 (Lifting Lemma)**.** If σ is a path in G/H with initial point 1, there is a unique path σ' in G with initial point 1_G such that

$$\varphi \circ \sigma' = \sigma.$$

Lemma 3.4 (Covering Homotopy Lemma)**.** If τ is also a path in G/H with the initial point 1 such that

$$F : \sigma \simeq \tau \quad \text{rel } \{0, 1\},$$

then there is a unique $F' : I \times I \to G$ such that

$$\begin{aligned} F' : \sigma' \simeq \tau' \quad \text{rel } \{0, 1\}, \\ \varphi \circ F' = F. \end{aligned}$$

(Note that τ' above is the unique path in G as given by the Lifting Lemma.)

Proof. We prove both results together.
Let $(K, f : Y \to G/H, 0 \in K)$ be either $(I, \sigma, 0 \in I)$ or $(I \times I, F, (0, 0) \in I \times I)$. The first choice corresponds to Lemma 3.3 and the second to Lemma 3.4.

For the sake of less ugly notation, we shall use a/b or $\frac{a}{b}$ to denote ab^{-1} for $a, b \in G/H$. (Note that we are fixing this to mean ab^{-1} without any assumption of abelianity.)

Since K is compact, there exists $\epsilon > 0$ such that

$$|k - k'| < \epsilon \implies f(k)/(f(k')) \in V,$$

by Lemma 3.1.
In particular, for such k and k', $\psi\left(\dfrac{f(k)}{f(k')}\right)$ is defined. Fix $N \in \mathbb{N}$ large enough such that

$$|k| < N\epsilon$$

for all $k \in K$. (This can be done since K is bounded by 2.)
Now, define

$$f' : K \to G$$

by

$$\begin{aligned} f'(k) :=& \psi\left(f(k)/f\left(\frac{N-1}{N}k\right)\right) \\ & \cdot \psi\left(f\left(\frac{N-1}{N}k\right)\Big/ f\left(\frac{N-2}{N}k\right)\right) \\ & \cdots \psi\left(f\left(\frac{1}{N}k\right)/f\left(0\right)\right). \end{aligned}$$

Then, f' is continuous $K \to G$, $f'(0) = (\varphi(1))^N = 1_G$, and $\varphi \circ f' = f$. To see the last point, note that φ is a homomorphism and thus,

$$\begin{aligned}(\varphi \circ f')(k) =& \varphi\left[\psi\left(f(k)/f\left(\frac{N-1}{N}k\right)\right)\right] \\ & \cdot \varphi\left[\psi\left(f\left(\frac{N-1}{N}k\right)\Big/ f\left(\frac{N-2}{N}k\right)\right)\right] \\ & \cdots \varphi\left[\psi\left(f\left(\frac{1}{N}k\right)/f\left(0\right)\right)\right].\end{aligned}$$

Now, using that $\varphi\psi(k) = k$, we see that the fractions cancel and we are left with

$$(\varphi \circ f')(k) = f(k)/f(0) = f(k),$$

since $f(0) = 1_G$, in either case.

Now, suppose we had $f'' : K \to G$ satisfying $f''(0) = 1_G$, and $\varphi \circ f'' = f$.
Then, we would have $[\varphi \circ (f'/f'')](s) = \varphi(f'(s))/\varphi(f''(s))$, since φ is a homomorphism. However, this equals $f(s)/f(s) = 1$.
Thus, f'/f'' is a continuous map from Y into $\ker \varphi = H$.
Since Y is connected and H is discrete, f'/f'' is a constant. Since $f'(0)/f''(0) = 1_G$, we see that $f' = f''$.
This proves the uniqueness of f'.

Note that in the case of the first lemma (that is $Y = I$), we have $f'(0) = 1_G$ and thus, f' is the required σ'.

For the case of the second lemma, we still have to prove that $F' = f'$ is the desired (relative) homotopy.
First, we show that F' is indeed a (not necessarily relative) homotopy. To see this, set $\alpha(s) := F'(s, 0)$ and $\beta(s) = F'(s, 1)$.
Note that $\varphi \circ \alpha(s) = \varphi \circ F'(s, 0) = F(s, 0) = \sigma(s)$ and $\alpha(0) = F'(0, 0) = 1_G$.
Since σ' is the unique such path, we see that $\alpha = \sigma'$.
Similarly, we can conclude $\beta = \tau$ if we show that $\beta(0) = 1_G$. By definition, we have $\beta(0) = F'(0, 1)$.
Note that F' is continuous and $\varphi \circ F'$ is 1 on $\{0\} \times I$. Thus, $F'|_{\{0\}\times I}$ maps into $\ker \varphi = H$.
As before, we see that F' is constant on $\{0\} \times I$. Thus, $F'(0, 1) = F'(0, 0) = 1_G$ and hence, $\beta = \tau'$.
In fact, we have even proven that F' is constant on $\{0\} \times I$. This shows that F' is a homotopy relative to $\{0\}$. All that remains is to show that it is constant on $\{1\} \times I$ as well.

For that, we once again note that $\varphi \circ F' = F$ is constant on $\{1\} \times I$. Thus, $F'|_{\{1\}\times I}$ maps into a coset of $\ker \varphi = H$. Since the coset is homeomorphic to H, it must be discrete as well. This proves that F' is constant on $\{1\} \times I$ as well, proving that

$$F' : \sigma' \simeq \tau' \quad \text{rel } \{0, 1\}. \qquad \square$$

Corollary 3.5. The end point of σ' only depends on the homotopy class of σ.
In particular, if σ is a loop at 1, then $\sigma'(1) \in H$.

Proof. Let σ, τ be paths in the same homotopy class. Let $F : \sigma \simeq \tau \quad \text{rel } \{0, 1\}$ be a (relative) homotopy.
Then, F' is a homotopy from σ' to τ' relative to $\{0, 1\}$.
In particular, we have $\sigma'(1) = F(1, 0) = F(1, 1) = \tau'(1)$. This proves the first statement.

For the second statement, note that $\varphi \circ \sigma'(1) = \sigma(1) = 1$ and thus, $\sigma'(1) \in \ker \varphi = H$. $\square$

Now, we have the following theorem.

Theorem 3.6. If G is a simply connected topological group, H a discrete normal subgroup, then

$$\pi_1(G/H, 1) \cong H.$$

Proof. Using Corollary 3.5, we define $\chi : \pi_1(G/H, 1) \to H$ by

$$\chi([\sigma]) = \sigma'(1).$$

> **Claim 1.** χ is a homomorphism.
>
> *Proof.* Let $[\sigma], [\tau] \in \pi_1(G/H, 1)$.
> Let $h_1 = \sigma'(1)$ and $h_2 = \tau'(1)$. (Again, we see that these are well-defined and elements of H by Corollary 3.5.)
> Let τ'' be the path from h_1 to $h_1 h_2$ in G given by
>
> $$\tau''(s) = h_1 \tau'(s).$$
>
> (Note that $\tau''(0) = \tau'(0)h_1 = 1_G h_1 = h_1$ and $\tau''(1) = h_1\tau'(1) = h_1 h_2$.)
> Note that
>
> $$(\varphi \circ \tau'')(s) = \varphi(\tau'(s)h_1) = \varphi(\tau'(s))\varphi(h_1) = \tau(s).$$
>
> (Note that $\varphi(h_1) = 1$ since $h_1 \in H = \ker \varphi$.)

Since, $\sigma'(1) = \tau''(0) = h_1$, we can consider the path $\tau''\sigma'$ in G. Note that

$$\varphi \circ (\tau''\sigma')(s) = \begin{cases} \varphi(\sigma'(2s)) & 0 \le 2s \le 1 \\ \varphi(\tau''(2s-1)) & 1 \le 2s \le 2. \end{cases} = (\sigma\tau)(s).$$

Thus, $\tau''\sigma'$ is the unique lift of $\sigma\tau$ as given by the Lifting Lemma.
Thus,

$$\chi([\sigma][\tau]) = \chi[\sigma\tau] = (\tau''\sigma')(1) = h_1h_2 = \chi[\sigma]\chi[\tau].$$

□

Now, we show that χ is bijective.

Claim 2. χ is injective.

Proof. It suffices to show that $\ker\chi$ is trivial.
Let $[\sigma] \in \ker\chi$. Then, $\sigma'(1) = 1_G$.
In other words, σ' is a loop at 1_G in G. Since G is simply connected, σ' is path homotopic to a constant loop. We may choose the constant loop to be e_{1_G}.
Thus, $\sigma' \simeq e_{1_G} \quad \text{rel } \{0,1\}$.
Applying φ, we get that $\sigma \simeq e_1 \quad \text{rel } \{0,1\}$ or $[\sigma] = [e_1]$, the identity of $\pi_1(G/H, 1)$. □

Claim 3. χ is surjective.

Proof. Let $h \in H$ be arbitrary.
Since G is simply-connected, it is pathwise connected. Let σ' be path from 1_G to h in G.
Then, $\varphi \circ \sigma' : I \to G/H$ is a loop at 1 in G/H with

$$\chi[\sigma] = \sigma'(1) = h.$$

□

With that, we are done! □

Corollary 3.7. The fundamental group of S^1 is (isomorphic to) $\mathbb{Z}$.

(Since S^1 is pathwise connected, we need not care about base point.)
In particular, the above corollary shows that S^1 is not simply connected. This is our first example of a non-simply connected space.