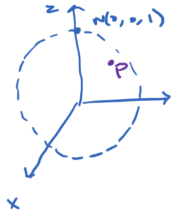
Lecture 1 (03-01-2022)

03 January 2022 13:58

The stereographic projection is a function $\theta : S^2 \longrightarrow \hat{\mathbb{C}}$. ($S^2 \subseteq \mathbb{R}^3$)



Stereographic projection

$P \in S^3$, $P \neq N$.

Define the stereographic projection of $P(x,y,z)$ $\neq N$ as follows:

Join N to P. Extend it. It hits the (equitorial) plane $z=0$ at some point $(x, y, 0)$.

$P \longmapsto x + iy$ is the map.

Analytically, the line is :

$$t(X, Y, Z) + (1-t)(0, 0, 1).$$

We need $tZ + 1 - t = 0$ or $t = \dfrac{1}{1-Z}$.

$$\therefore \quad x = \frac{X}{1-Z} \quad \text{and} \quad y = \frac{Y}{1-Z}. \qquad (\text{note} : Z \neq 1.)$$

Finally, $N \longmapsto \infty$.

(Eg.: Under the above map, $(0, 0, -1) \mapsto (0, 0)$ or $0 + 0i$.)

To sum it up: Define $\theta : S^2 \longrightarrow \hat{\mathbb{C}}$ by

$$\theta(X, Y, Z) = \begin{cases} \dfrac{X}{1-Z} + \dfrac{iY}{1-Z} & ; \quad Z \neq 1, \\ \infty & ; \quad \text{else}. \end{cases}$$

Check: θ is a bijection.

To see that it is onto, let $z = x + iy \in \mathbb{C}$ be arbit.

Check that

$$P(X, Y, Z) := \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right)$$

maps to z. (As usual, $|z| = x^2 + y^2$.)

Q. What happens to P above as $|z| \longrightarrow \infty$?

Evidently $P \longrightarrow N(0, 0, 1)$.

→ Using the above, we can define a topology on $\hat{\mathbb{C}}$.

In fact, we now define a metric on $\hat{\mathbb{C}}$ as follows:

For $w, z \in \hat{\mathbb{C}}$, define the distance between w and z

to be the length of the straight line segment joining $\theta^{-1}(\omega)$ and $\theta^{-1}(z)$, i.e.,

$$d(\omega, z) := \|\theta^{-1}(\omega) - \theta^{-1}(z)\|_{\mathbb{R}^3}$$

after calculations (both $z, \omega \neq \mathbb{C}$)

$$= \frac{\sqrt{2}\,|\omega - z|}{\sqrt{1+|z|^2}\sqrt{1+|\omega|^2}}$$

If $\omega = \infty$ and $z \neq \infty$, we get $d(z, \infty) = \dfrac{\sqrt{2}}{\sqrt{1+|z|^2}}$.

Fix $z \in \hat{\mathbb{C}}$, $r > 0$.

$$B_d(z, r) := \{\omega \in \hat{\mathbb{C}} : d(z, \omega) < r\}.$$

Describe the above set when $z = \infty$

Describe the open nbds in $\hat{\mathbb{C}}$.

Def^n. Define the operators

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Lecture 2 (06-01-2022)

06 January 2022 14:01

Integration:

Integration

Let Ω be a domain in $\mathbb{C}$, and $\gamma : [a,b] \longrightarrow \Omega$ is piecewise - $\mathcal{C}^1$. For any $f \in \mathcal{C}^0(\Omega)$, $(f : \Omega \to \mathbb{C})$

$$\int_\gamma f := \int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\, \gamma'(t)\, dt.$$

Index of a point wrt. a path:

Fix $\gamma : [a,b] \longrightarrow \mathbb{C}$ is piecewise $\mathcal{C}^1$. Assume γ is closed, i.e., $\gamma(a) = \gamma(b)$. Let $\Omega := \mathbb{C} \setminus \operatorname{im}(\gamma)$.

Then, Ω has possibly many connected components, out of which exactly one is unbounded.

Let $z_0 \in \Omega$. We define

$$\operatorname{Ind}_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z_0}\, d\xi$$

well-defined since $z \notin \operatorname{im}(\gamma)$.

$$= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt.$$

Properties:

(1) $\operatorname{Ind}_\gamma$ is an integer-valued function on Ω.

(2) Thus, $\operatorname{Ind}_\gamma$ is constant on the connected components of Ω.

(3) $\operatorname{Ind}_\gamma \equiv 0$ on the unbounded component.

Rem (Cauchy's Theorem)

Cauchy's theorem

Prop^n. (Cauchy's Theorem)   Cauchy's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $f: \Omega \to \mathbb{C}$ be continuous.
TFAE:

(i) $\int_\gamma f = 0$ for every closed γ in Ω.

(ii) $\exists F \in \mathcal{O}(\Omega)$ such that $F' \equiv f$ on Ω.

Consequently, $f \in \mathcal{O}(\Omega)$ (since once differentiable ⇒ always differentiable).

Def^n. Path homotopy

Given $\gamma_0, \gamma_1 : [0,1] \to \Omega$ two closed paths in Ω based at x_0.
A path homotopy between γ_0 and γ_1 is a function

$$H: [0,1] \times [0,1] \to \Omega$$

s.t. ① H is continuous,

② $H(s,0) = \gamma_0(s) \quad \forall s \in [0,1]$,

③ $H(s,1) = \gamma_1(s) \quad \forall s \in [0,1]$

④ $H(0,t) = x_0 = H(1,t) \quad \forall t \in [0,1]$.

Recall: $\gamma_0 \sim \gamma_1$, path-homotopic, null-homotopic ($\gamma \sim 0$).
(equiv. rel'n)

Theorem. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let γ_0, γ_1 be loops based at the same point with $\gamma_0 \sim \gamma_1$. Then,

$$\int_{\gamma_0} f = \int_{\gamma_1} f \quad \text{for all } f \in \mathcal{O}(\Omega).$$

Corollary. Let Ω be a domain and γ be a loop in Ω with $\gamma \sim 0$. Then,

$$\int_\gamma f = 0 \quad \forall f \in \mathcal{O}(\Omega).$$

Def^n. An open set $\Omega \subseteq \mathbb{C}$ is said to be simply-connected if Ω is connected and $\gamma \sim 0$ for every loop γ in Ω.

Corollary. Let Ω be a s-c domain in $\mathbb{C}$ and let γ be a loop in Ω. Then,

$$\int_\gamma f = 0 \qquad \forall\, f \in \mathcal{O}(\Omega).$$

Cor. Let Ω be a s.c. domain in $\mathbb{C}$. Let $f \in \mathcal{O}(\Omega)$. Then, $\exists F \in \mathcal{O}(\Omega)$ s.t. $F' = f$ on Ω.

Cor. Let Ω be a s.c domain in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$ be st. $f(z) \neq 0 \ \forall\, z \in \Omega$. Then, $\exists g \in \mathcal{O}(\Omega)$ s.t.

$$f = \exp \circ g.$$

(g is an analytic branch of logarithm of f.)

Lecture 3 (10-01-2022)

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Maximum Principle

① Let $\Omega \subseteq \mathbb{C}$ be a domain, and $f \in \mathcal{O}(\Omega)$.
Let $a \in \Omega$ such that $\exists r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$.
Then,

$$|f(a)| \leq \max_{0 \leq \theta \leq 2\pi} |f(a + re^{i\theta})|.$$

Moreover, equality holds iff f is constant.

② Let Ω be a bounded open set in $\mathbb{C}$.
Let $f \in C^0(\overline{\Omega}) \cap \mathcal{O}(\Omega)$. Then,

$$|f(z)| \leq \max_{\partial\Omega} |f| \qquad \forall z \in \Omega.$$

In words, $|f|$ attains its maximum on the boundary.

Equivalently:

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

EXAMPLE: $\mathbb{H} := \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$.
Define $f(z) = \exp(-z^2)$ on $\overline{\mathbb{H}}$.
$f \in \mathcal{O}(\mathbb{H}) \cap C^0(\overline{\mathbb{H}})$.

Note that $|f(z)| \leq 1$ for $z \in \mathbb{R} = \partial\mathbb{H}$.
But

$$|f(iy)| = e^{y^2} \text{ grows rapidly on } i\mathbb{R}.$$

Thus, MMT need not hold if Ω is unbounded.
Now, we wish to formulate a similar theorem for unbounded.

- Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$.
For $a \in \overline{\Omega}$, define

$$\limsup_{\Omega \ni z \to a} f(z) := \lim_{r \to 0^+} \sup\{|f(z)| : z \in \Omega \cap D(a,r)\}.$$

this limit exists in $[0, \infty]$.

If a is the point at infinity, $D(a,r)$ is the neighbourhood of a in the metric d on the extended complex plane.

The extended boundary of Ω in $\mathbb{C} \cup \{\infty\}$ is denoted by $\partial_\infty \Omega$.
Note:

$$\partial_\infty \Omega = \begin{cases} \partial \Omega & ; \ \Omega \text{ is bounded,} \\ \partial\Omega \cup \{\infty\} & ; \ \text{else.} \end{cases}$$

③ **MMT**: Let Ω be a domain in $\mathbb{C}$, $f \in \mathcal{O}(\Omega)$.
(Not necessarily bounded!)

Suppose that $\limsup_{z \to a} |f(z)| \leq M$ for all $a \in \partial_\infty \Omega$.

Then, $|f| \leq M$ on Ω.

Generalisations of MMT to unbounded domains.

Phragmén – Lindelöf Theorems.

\# Liouville's Theorem: Bounded + entire $\Rightarrow$ constant
Also, recall the following exercise (using Cauchy's estimate, for example):

If $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)| \leq 1 + |z|^{1/3}$, then f is constant.

↳ "Generalisation" of Liouville.
Similarly, we generalise MMT.

(Phragmén - Lindelöf)

Theorem A. Let $\Omega \subseteq \mathbb{C}$ be simply-connected, and $f \in \mathcal{O}(\Omega)$. Fix $M > 0$.
Let $\partial_\infty \Omega = I \cup II$ be such that

(1) $\limsup\limits_{\Omega \ni z \to a} |f(z)| \leq M$ for all $a \in I$, and

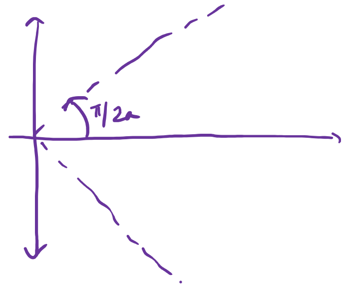
(2) $\exists\, \phi \in \mathcal{O}(\Omega)$, nonvanishing and bounded on Ω such that

$$\limsup_{\Omega \ni z \to a} \left| f(z) \left(\phi(z)\right)^{\eta} \right| \leq M$$

for all $a \in II$ and for all $\eta > 0$.

Then, $|f| \leq M$ on Ω.

__Example.__ Fix $a \geq 1/2$. Let $\Omega = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{2a} \right\}$.



Let $f \in \mathcal{O}(\Omega)$ be s.t.

(a) $\overline{\lim\limits_{z \to \xi \in \partial\Omega}} |f(z)| \leq M$, and

(b) $|f(z)| \leq A \exp(|z|^b)$ for $|z| \gg 1$,

where A and b are positive constants such that $b < a$.

using above theorem

Then, $|f(z)| \leq M \quad \forall\, z \in \Omega$.

Clearly Ω is s-c.

Now, we find $\phi \in \mathcal{O}(\Omega)$ as in P-L.

Consider $\phi(z) = \exp(-z^c)$, where $c > 0$ is chosen later.

Note that this is holo on Ω.

Also, $\phi(z) \neq 0 \quad \forall\, z \in \Omega$

$$|\phi(z)| = |\exp(-z^c)|$$ $z = re^{i\theta}$, $|\theta| < \pi/2a$.

$$= |\exp(-r^c e^{i\theta c})|$$

$$= \exp(-r^c \cos(c\theta)) \leq 1.$$

if $c < a$, then $\cos(c\theta) \geq 0$.

Thus, ϕ is bdd.

Take $I = \partial\Omega$ and $II = \{\infty\}$.

Now, fix $\eta > 0$ and for $z = re^{i\theta} \in \Omega$.

For large z, we have

$$|f(z)\ \phi(z)^\eta| \leq A \exp(|z|^b)\ |\exp(-z^c)|^\eta$$
$$= A \exp(r^b - \eta r^c \cos c\theta)$$
$$\leq A \exp(r^b - \eta r^c \delta).$$

$\delta := \inf_{0 \leq \theta < \pi/2a} \cos(c\theta)$.

The above goes to 0 if $c > b$.

Thus, we can choose any $c \in (b, a)$.

We are now done. ∎

Consider $g_\eta(z) := g(z) := \dfrac{f(z)\ \phi(z)^\eta}{K^\eta}$. $g \in \mathcal{O}(\Omega)$.

Lecture 4 (13-01-2022)

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(Phragmén - Lindëlof)

Theorem B. Fix reals $a < b$, and $B > 0$.

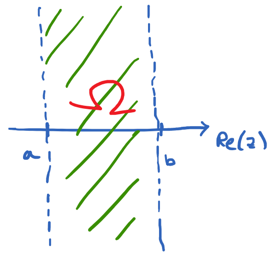
Let $\Omega = \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$, and $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

Assume that:

$$|f| \leq B \quad \text{on} \quad \Omega,$$
$$|f| \leq 1 \quad \text{on} \quad \partial\Omega.$$

Then,

$$|f| \leq 1 \quad \text{on} \quad \Omega.$$



Remark: Note that the above is a type of MMT.

Idea: Introduce a typical multiplicative factor g_ε with $\lim_{\varepsilon \to 0} g_\varepsilon = 1$, such that $|fg_\varepsilon| < M$ on the boundary of a BOUNDED subdomain Ω_ε of Ω. Then, apply usual MMT on Ω_ε. Moreover, pick the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ nicely enough to cover all of Ω. Then take $\varepsilon \to 0$.

Proof. For each $\varepsilon > 0$, define $g_\varepsilon : \bar{\Omega} \to \mathbb{C}$ by

$$g_\varepsilon(z) := \frac{1}{1 + \varepsilon(z-a)}.$$

↳ denominator is 0 iff $z = a - \frac{1}{\varepsilon} \notin \bar{\Omega}$.

Theorem C. Fix reals $a < b$, and $B > 0$.

Let $\Omega = \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$, and $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar{\Omega}) \setminus \{0\}$.

Assume that $|f| < B$. Define $M : [a,b] \to [0, \infty)$ by

$$M(x) := \sup_{y \in \mathbb{R}} |f(x+iy)|.$$

Then $\log \circ M$ is a convex function on (a, b).

Remarks: (i) For $a \leq x < v < y \leq b$:

$$M(v)^{(y-x)} \leq M(x)^{(y-v)} \cdot M(y)^{(v-x)}.$$

(ii) Since $\log \circ M$ is convex on (a, b), we get

$$M(x) \leq \max\{M(a), M(b)\} \quad \forall x \in [a, b].$$

Proof: Suffices to show that

$$M(v)^{b-a} \leq M(a)^{b-v} \cdot M(b)^{v-a}$$

for $v \in (a, b)$.

What if $M(a)$ or $M(b) = 0$?
Take care of this separately.

Consider the entire function g defined as

$$g(z) := M(a)^{\frac{b-z}{b-a}} \cdot M(b)^{\frac{z-a}{b-a}}.$$

$(\lambda^z := \exp(z \log \lambda))$

Also note that g is nonvanishing.

$$|g(z)| = M(a)^{\frac{b-x}{b-a}} \cdot M(b)^{\frac{x-a}{b-a}}.$$

The above is continuous as a function of x and is non-vanishing. Thus, $\exists C > 0$ s.t. $\frac{1}{|g(z)|} \leq C$ on $\bar{\Omega}$.

Now, consider $\frac{f}{g} \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

$$\left|\frac{f(a+iy)}{g(a+iy)}\right| = \left|\frac{f(a+iy)}{M(a)}\right| \leq 1 \qquad \forall y \in \mathbb{R}.$$

$$\text{ll}^{\text{ly}} \quad \left|\frac{f}{g}\right| \leq 1 \quad \text{on} \quad \partial\Omega.$$

Moreover, $\left|\frac{f}{g}\right| \leq CB$ on Ω

Then, by Theorem B, we have $\left|\frac{f}{g}\right| \leq 1$ on Ω or

$|f| \leq |g|$ on Ω.

Expanding out, we get

$$|f(x+iy)|^{b-a} \leq M(a)^{b-x} \, M(b)^{x-a}$$

$\forall x \in (a,b)$
$\forall y \in \mathbb{R}$.

Take sup over $y \in \mathbb{R}$ and we are done. ∎

Consequences of MMT.

Schwarz Lemma: Let $f \in \mathcal{O}(D(0,1))$ such that $f(0) = 0$ and $|f| \leq 1$.

Then,

(a) $|f'(0)| \leq 1$ and

(b) $|f(z)| \leq |z|$ $\forall z \in D(0,1)$.

Moreover if equality holds either in (a) or for some $z \neq 0$ in (b), then $\exists \lambda \in S^1$ s.t. $f(z) = \lambda z$.

Lecture 5 (17-01-2022)

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- $\mathbb{D} := D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.
- $\mathrm{Aut}(\mathbb{D}) := \{ f : \mathbb{D} \to \mathbb{D} \mid f \text{ is bijective, } f, f^{-1} \in \mathcal{O}(\mathbb{D})\}$.

  ↳ group under composition

Aut(D)

Automorphisms of $\mathbb{D}$ fixing the origin: Automorphisms of the disc

Theorem. If $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = 0$, then f is a rotation, i.e., $\exists \lambda \in \partial\mathbb{D}$ s.t. $f(z) = \lambda z \;\; \forall z \in \mathbb{D}$.

Möbius transforms: Möbius, Mobius

Let $\alpha \in \mathbb{D}$, and consider $z \overset{\varphi_\alpha}{\longmapsto} \dfrac{z - \alpha}{1 - \bar{\alpha} z}$, $z \in \mathbb{D}$.

- Note that φ_α makes sense on $\mathbb{C} \setminus \{1/\bar{\alpha}\} \supseteq \mathbb{D}$.

Moreover, φ_α is holomorphic on $\mathbb{D}$, i.e., $\varphi_\alpha \in \mathcal{O}(\mathbb{D})$.

$\varphi_\alpha(\alpha) = 0$.

- $\varphi_\alpha(\mathbb{D}) = ?$

  Check: $|\varphi_\alpha(e^{it})| \leq 1$ for $t \in \mathbb{R}$.

  Thus, by MMT $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$.

  Also, $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ has inverse as $\varphi_{-\alpha}$.

  Thus, $\varphi_\alpha \in \mathrm{Aut}(\mathbb{D}) \;\; \forall \alpha \in \mathbb{D}$.

Theorem. $\mathrm{Aut}(\mathbb{D}) = \{ \lambda \varphi_\alpha : \lambda \in \partial\mathbb{D}, \; \alpha \in \mathbb{D}\}$.

Let $\alpha, \beta \in \mathbb{D}$. Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be holomorphic and $f(\alpha) = \beta$. Among all such f, what is the maximum possible value of $|f'(\alpha)|$?

$$\begin{array}{ccc} \alpha \;\; \mathbb{D} & \xrightarrow{\;f\;} & \overline{\mathbb{D}} \;\; \beta \\ {\scriptstyle \varphi_{-\alpha}} \uparrow & \curvearrowright & \downarrow {\scriptstyle \varphi_\beta} \end{array}$$

$g := \varphi_\beta \circ f \circ \varphi_{-\alpha}$.

$g(0) = 0$.

$$\begin{tikzcd} \alpha \quad \mathbb{D} \arrow[r] \arrow[d, phantom, ""] & \mathbb{D} \quad \beta \arrow[d, "\psi_\beta"] \\ 0 \quad \mathbb{D} \arrow[u, "\psi_{-\alpha}"] \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{D} \quad 0 \end{tikzcd}$$

$g := \psi_\beta \circ f \circ \psi_{-\alpha}$.

$g(0) = 0$.

By Schwarz, we have $|g'(0)| \leq 1$.

Using chain rule, we have:

$$g'(0) = \psi_\beta'(f(\psi_{-\alpha}(0))) \cdot f'(\psi_{-\alpha}(0)) \cdot \psi_{-\alpha}'(0)$$

$$= \psi_\beta'(f(\alpha)) \cdot f'(\alpha) \cdot \psi_{-\alpha}'(0)$$

$$= \psi_\beta'(\beta) \cdot f'(\alpha) \cdot \psi_{-\alpha}'(0)$$

$$= \frac{1-\bar{\beta}\beta}{(1-\bar{\beta}\beta)^2} \cdot f'(\alpha) \cdot \frac{1-(\alpha\bar{\alpha})}{1^2}$$

$\psi_\beta(z) := \dfrac{z-\beta}{1-\bar{\beta}z}$

$\Rightarrow \psi_\beta'(z) = \dfrac{1-\bar{\beta}z-(z-\beta)(-\bar{\beta})}{(1-\bar{\beta}z)^2}.$

$$= \frac{1-|\alpha|^2}{1-|\beta|^2} \cdot f'(\alpha)$$

$$\therefore \ |f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}.$$

Note that equality is possible. For example, $f = \varphi_{-\beta} \circ \varphi_\alpha$. In fact, it happens iff $\exists \lambda \in S^1$ s.t.

$$f = \varphi_{-\beta} \circ r_\lambda \circ \varphi_\alpha.$$

Towards the Riemann-Mapping Theorem

$$\mathcal{O}(\Omega) \subseteq \mathcal{C}^0(\Omega; \mathbb{C}).$$

Given any open $\Omega \subseteq \mathbb{C}$, $\exists$ a sequence $\{K_n\}_n$ of compact subset of $\mathbb{C}$ s.t.

(1) $\Omega = \bigcup_{n=1}^{\infty} K_n^\circ$,

(2) $K_n \subseteq K_{n+1}$ $\forall n \in \mathbb{N}$,

(3) for any compact $K \subseteq \Omega$, $\exists n \in \mathbb{N}$ s.t. $K \subseteq K_n$.

Proof For each $n \in \mathbb{N}$, let

$$K_n := \overline{D(0,n)} \cap \{z \in \Omega : \operatorname{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/n\}.$$

Using the above, we define a metric on $C^0(\Omega; \mathbb{C})$.

Fix some $\{K_n\}_n$ as given by compact exhaustion.
Let $f, g \in C^0(\Omega; \mathbb{C})$.
Define

$$\rho_n(f, g) := \sup_{z \in K_n} |f(z) - g(z)|.$$

Finally, define

$$\rho(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f,g)}{1 + \rho_n(f,g)}.$$

Ex ① $(C^0(\Omega; \mathbb{C}), \rho)$ is a metric space.

② A sequence $\{f_k\}_{k \geq 1}$ converges to f in $(C^0(\Omega; \mathbb{C}), \rho)$ iff $f_k \to f$ uniformly on compact subsets of Ω.
What are open sets in $(C^0(\Omega; \mathbb{C}), \rho)$?

This ex. shows that the topology does not depend on $\{K_n\}_{n \geq 1}$.

Lecture 6 (20-01-2022)

20 January 2022 14:19

$\mathcal{O}(\Omega) \subseteq \mathcal{C}^0(\Omega; \mathbb{C})$.

metric space

↳ subspace topology

Prop^n. $\mathcal{O}(\Omega)$ is closed in $\mathcal{C}^0(\Omega; \mathbb{C})$.
That is, if $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$ and $f_n \longrightarrow f$ in $\mathcal{C}^0(\Omega; \mathbb{C})$, then $f \in \mathcal{O}(\Omega)$.
Moreover, $f_n^{(k)} \longrightarrow f_n^{(k)}$ in $\mathcal{O}(\Omega)$ for all $k \geq 1$.

Normal Families

Def^n. Let $\Omega \subseteq \mathbb{C}$ be a domain, and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$.
$\mathcal{F}$ is said to be *normal* if for every sequence $(f_n)_n \in \mathcal{F}^{\mathbb{N}}$, it is possible to extract a subsequence $(f_{n_k})_k$ such that either

(a) $(f_{n_k})_k$ converges uniformly on compact subsets of Ω, or

(b) given any pair of compact sets $K \subset \Omega$, $L \subset \mathbb{C}$, $\exists\, k_0 = k_0(K, L) \in \mathbb{N}$ s.t.

$$f_{n_k}(K) \cap L = \emptyset \qquad \forall\, k \geq k_0.$$

$(f_{n_k} \longrightarrow \infty$ uniformly on compact subsets of Ω.)

REMARKS. (i) If (a) is true and $f_{n_k} \longrightarrow f$, the $f \in \mathcal{O}(\Omega)$.
(ii) However, f above need not be in $\mathcal{F}$.

Theorem (Montel's Theorem)
Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be *locally uniformly bounded* on Ω, i.e., for all compact $K \subseteq \Omega$, $\exists\, M = M(K) > 0$ such that

$$|f(z)| \leq M \qquad \forall f \in \mathcal{F}, \ \forall z \in K.$$

Then, $\mathcal{F}$ is a normal family.
In fact, $\mathcal{F}$ is normal and satisfying (a) of the def^n.

Theorem (Arzelà - Ascoli Theorem)
Let $\mathcal{F} \subset \mathcal{C}^0(\Omega; \mathbb{C})$.
Every sequence in $\mathcal{F}$ admits a convergent subsequence in $(\mathcal{C}(\Omega; \mathbb{C}), \rho)$ iff:
(i) $\mathcal{F}$ is pointwise bounded, i.e., $\exists M : \Omega \longrightarrow [0, \infty)$ s.t.
$$|f(z)| \leq M(z) \qquad \forall z \in \Omega, \quad \text{and}$$
(ii) $\mathcal{F}$ is equicontinuous at each point of Ω.

Lecture 7 (24-01-2022)

24 January 2022 14:02

EXAMPLE. Monte l's Theorem fails on $\mathbb{R}$.

Indeed, consider the family $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$, where $f: \mathbb{R} \to \mathbb{R}$ is defined as $f_n(x) := \sin(nx)$.

Clearly, $\mathcal{F}$ is locally uniformly bounded as $|f_n(x)| \leq 1 \quad \forall x \in \mathbb{R} \ \forall n \in \mathbb{N}$.

However, given any $\delta > 0$, pick n s.t. $x = \frac{\pi}{2n} < \delta$.

Then, $|f_n(x) - f_n(0)| = \left|\sin\left(\frac{\pi}{2}\right)\right| = 1$.

Thus, no δ exists for $\varepsilon = 1$.

Thus, $\mathcal{F}$ is not equicontinuous.

Theorem (Hurwitz's Theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that $\exists a \in \Omega, r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$ such that f has no zeroes on $\partial D(a,r)$.
Then, $\exists N \in \mathbb{N}$ such that f and f_n have the same number of zeroes counting multiplicities in $D(a,r)$ for all $n \geq N$.

Corollary 1. Let Ω be a domain in $\mathbb{C}$, $f_n \in \mathcal{O}(\Omega) \ \forall n$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that each f_n is non-vanishing on Ω.
Then, either $f \equiv 0$ or f is also nonvanishing.

Corollary 2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that each f_n is injective on Ω, then f is injective on Ω.

Theorem (RMT). Let $\Omega \subsetneq \mathbb{C}$ be simply-connected.

Theorem (RMT). Let $\Omega \subsetneq \mathbb{C}$ be simply-connected.
Then, Ω is biholomorphic to $D(0,1)$.

Proof of RMT Let $\Omega \subsetneq \mathbb{C}$ be as specified.
Fix $p \in \Omega$.
Let

$$\mathcal{F} = \{ f \in \mathcal{O}(\Omega) : f(p) = 0,\ f \text{ is injective},\ f(\Omega) \subseteq D(0,1) \}.$$

If we can find $f_0 \in \mathcal{F}$ such that $f(\Omega) = D(0,1)$, then we are done since f_0^{-1} is also holomorphic.

Steps: (I) $\mathcal{F} \neq \emptyset$.

(II) $\displaystyle\sup_{f \in \mathcal{F}} |f'(p)| = |f_0'(p)|$ for some $f_0 \in \mathcal{F}$.

(III) f_0 (as above) is onto.

Lecture 8 (27-01-2022)

27 January 2022 14:00

Infinite Products.

Def^n. Suppose that $(u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Define the sequence $(p_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ by

$$p_n := (1+u_1)\cdots(1+u_n).$$

If $\lim_{n\to\infty} p_n =: p$ exists (in $\mathbb{C}$), then we write

$$p = \prod_{n=1}^{\infty} (1+u_n).$$

p_n are called the partial products of the infinite product $\prod_{n=1}^{\infty}(1+u_n)$.

In this case, we say that $\prod_{n=1}^{\infty}(1+u_n)$ converges (to p).

- Suppose that $z_n \neq 0$ $\forall n$. Assume $z := \prod_{n\geq 1} z_n$ exists and $z \neq 0$.

Let $p_n := z_1 \cdots z_n$. Then, $\lim_n (z_n) = \lim_n \left(\frac{p_{n+1}}{p_n}\right) = \frac{\lim_n p_{n+1}}{\lim_n p_n} = \frac{z}{z} = 1.$

(Each p_n is nonzero and $p_n \to z \neq 0$.)

Lemma. Let $u_1, \ldots, u_N \in \mathbb{C}$. Define

$$p_N := \prod_{n=1}^{N}(1+u_n), \qquad p_N^* := \prod_{n=1}^{N}(1+|u_n|).$$

Then,

(i) $p_N^* \leq \exp(|u_1| + \cdots + |u_N|)$,

(ii) $|p_N - 1| \leq p_N^* - 1$.

Lecture 9 (31-01-2022)

31 January 2022 14:03

Theorem Let X be a metric space. Let $u_n : X \to \mathbb{C}$ be a sequence of functions such that $\sum_{n=1}^{\infty} |u_n|$ converges uniformly to a bounded function.
(Say, bounded by $M > 0$.)

Then, (1) $\prod_{n=1}^{\infty} (1 + u_n)$ converges uniformly on X.

Define $f(x) := \prod_{n=1}^{\infty} (1 + u_n(x))$ for $x \in X$.

(2) For $x_0 \in X$: $f(x_0) = 0 \iff u_M(x_0) = -1$ for some $M \in \mathbb{N}$.

(3) For every permutation $\sigma \in S_{\mathbb{N}}$, the infinite product

(Rearrangement)

$$\prod_{k=1}^{\infty} \left(1 + u_{\sigma(k)}(x)\right)$$ converges to $f(x)$, for all $x \in X$.

Theorem. Let Ω be a domain in $\mathbb{C}$. Let $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$ be such that no f_n is identically zero.
Suppose that $\sum_{n=1}^{\infty} |1 - f_n|$ converges uniformly on compact subsets of Ω.

(1) Then, $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of Ω.

Consequently $f := \prod_{n=1}^{\infty} f_n$ is holomorphic.

(2) Let $a \in \Omega$. If $f(a) = 0$, then $f_n(a) = 0$ for some n.
Moreover, this is true for only finitely many n.
Lastly,

$$\mathrm{ord}_f(a) = \sum_{n=1}^{\infty} \mathrm{ord}_{f_n}(a).$$

Lecture 10 (03-02-2022)

03 February 2022 14:00

Def^n. $E_0(z) = 1 - z$ for $z \in \mathbb{C}$.
For $p \in \mathbb{N}$, define

$$E_p(z) := (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

These functions are called (Weierstrass) Elementary factors.

Below, we have $p \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$.

- Each E_p vanishes precisely at 1.

1 is a simple zero (order $= 1$) for each E_p.

- $E_p(0) = 1$

Lemma. For every $p \geq 0$,

$$|1 - E_p(z)| \leq |z|^{p+1} \qquad \text{if} \quad |z| \leq 1.$$

Theorem. Let $(a_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$ be such that $a_n \neq 0$ $\forall n \geq 1$ and
$|a_n| \to \infty$ as $n \to \infty$.

(Note: the sequence need not consist of distinct points.
However, $|a_n| \to \infty$ forces that no point appears inf. often.)

IF $(p_n)_n \in \mathbb{N}_0^{\mathbb{N}}$ is such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$

for every $r > 0$, THEN:

(i) $\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ converges in $\mathcal{O}(\mathbb{C})$.

Write f for the above function.

(ii) $f \in \mathcal{O}(\mathbb{C})$ and $Z(f) = \{a_n : n \in \mathbb{N}\}$.

(ii) The multiplicity of any zero is precisely the number of times that it appears in the sequence.

Remarks : (1) Since $|a_n| \longrightarrow \infty$ as $n \longrightarrow \infty$, for every $r > 0$, $\exists N_0 = N_0(r) \in \mathbb{N}$ s.t.

$$|a_n| > 2r \quad \text{for all } n \geq N_0.$$

Thus,

$$\left(\frac{r}{|a_n|}\right) < \frac{1}{2} \qquad \forall \; n \geq N_0.$$

In turn,

$$\left(\frac{r}{|a_n|}\right)^{p_n+1} < \left(\frac{1}{2}\right)^{p_n+1} \qquad \forall \; n \geq N_0.$$

Thus, $p_n = n-1$ ALWAYS works for any $(a_n)_n$ with $|a_n| \longrightarrow \infty$.

(2) Suppose that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$.

Then, $p_n \equiv 0$ works!

In this case,

$$f(z) = E_0\left(\frac{z}{a_n}\right)$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \quad \text{works.}$$

(3) If $\displaystyle\sum \frac{1}{|a_n|} = \infty$ but $\displaystyle\sum \frac{1}{|a_n|^2} < \infty$, then $p_n \equiv 1$ works.

$$\therefore \; f(z) = \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right)$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}\right).$$

Lecture 11 (07-02-2022)

07 February 2022 14:03

Theorem. Let $\Omega \subsetneq \mathbb{C} \cup \{\infty\}$ be an open set.
Suppose $A \subset \Omega$ has no limit points in Ω.
Let $m : A \to \mathbb{N}$ be any function.
Then, $\exists f \in \mathcal{O}(\Omega)$ such that $Z(f) = A$, and f has a zero of multiplicity $m(\alpha)$ for every $\alpha \in A$.

Proof. It suffices to prove the theorem in the special case where:
Ω is a deleted neighbourhood of ∞ and $\infty \notin \overline{A}$.
$\Omega = \mathbb{C} \setminus K$ for some compact $K \subseteq \mathbb{C}$.

Justification.

Let Ω_1 and A_1 be as in the hypothesis of the theorem.
Fix $\infty \neq a \in \Omega_1 \setminus A_1$. Define

$$T(z) = \frac{1}{z - a}.$$

T is a linear fractional transformation from $\hat{\mathbb{C}}$ onto itself.
T is a homeomorphism of Ω_1 onto $T(\Omega_1) =: \Omega$.
Define $A := T(A_1)$. Then, A is BOUNDED as well, has no limit points in Ω.
Now, Ω and A satisfy the requirements of the special case.
Now, if theorem holds for special case, we can translate it back.

Now, we prove the theorem for the special case.
If $A = \{a_1, \ldots, a_n\}$, take

$$f(z) := \frac{(z - a_1)^{m_1} \cdots (z - a_{n})^{m_n}}{(z - b)^{m_1 + \cdots + m_n}}$$

for some $b \in \mathbb{C} \setminus \Omega$.

Lecture 12 (10-02-2022)

10 February 2022 13:52

Recall: Had reduced theorem to special case.

We now prove it for the special case:

$\Omega = \mathbb{C} \setminus K'$ for $K' \neq \emptyset$ compact, (If $\Omega = \mathbb{C}$, we already know.)

$\infty \notin \bar{A}$.

Had done it for finite A.

$(z_n)_{n \geq 1}$: enumeration of A, with multiplicities.

$(w_n)_{n \geq 1}$: satisfy $\text{dist}(z_n, \mathbb{C} \setminus \Omega) = |z_n - w_n|$.

↳ lie in $\mathbb{C} \setminus \Omega$

If $|z_n - w_n| \not\to 0$, then $\exists$ subsequence s.t. $|z_{n_k} - w_{n_k}| \geq \delta > 0$.

But A is bounded. $\exists (z_{n_{k_m}})$ s.t. $z_{n_{k_m}} \to z_0 \in \mathbb{C} \setminus \Omega$.

But then $|z_{n_{k_m}} - w_{n_{k_m}}| \to 0$. $\rightarrow\leftarrow$

Thus, $|z_n - w_n| \longrightarrow 0$.

Note that if $a \in \Omega$, $b \notin \Omega$, then $z \mapsto E_p\left(\frac{a-b}{z-b}\right)$ is holo on Ω and has a simple zero at a.

Claim: $z \mapsto \prod_{n=1}^{\infty} E_n\left(\frac{z_n - w_n}{z - w_n}\right)$ converges in $\mathcal{O}(\Omega)$.

From the claim, everything follows.

Proof. Suffices to show that

$$z \mapsto \sum_{n=1}^{\infty} \left| 1 - E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| \quad \text{converges in } \mathcal{O}(\Omega).$$

Fix $K \subseteq \Omega$. Then, $\text{dist}(K, \mathbb{C} \setminus \Omega) =: \delta > 0$.

For $z \in K$:

$$\left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{|z_n - w_n|}{\delta} \longrightarrow 0.$$

$$\therefore \left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{1}{2} \quad \forall n >> 0.$$

$$\therefore \left| 1 - \varepsilon_n \left(\frac{z_n - w_n}{z - w_n} \right) \right| \leq \left(\frac{1}{2}\right)^{n+1} \qquad \forall\, n >> 0. \qquad \blacksquare$$

LAPLACIAN IN POLAR: Define $u(z) = \log(|z|)$.

Write $z = re^{i\theta}$, $|z| = r$.

$x = r\cos\theta$, $y = r\sin\theta$.

(Exercise) $$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Lecture 15 (28-02-2022)

28 February 2022 13:49

Recall: Dirichlet problem on $D(0,1) = \mathbb{D}$.

Given $f : \partial D(0,1) \longrightarrow \mathbb{R}$ continuous, we wish to find

$$u : \overline{\mathbb{D}} \to \mathbb{R}$$

such that (i) $u \in C^0(\overline{\mathbb{D}})$,

(ii) u is harmonic on $\mathbb{D}$,

(iii) $u|_{\partial \mathbb{D}} = f$.

We defined u as follows:

$$u(z) := \begin{cases} f(z) & ; z \in \partial D, \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{Re}\left(\dfrac{e^{it}+z}{e^{it}-z}\right) f(e^{it})\, dt & ; z \in \mathbb{D}. \end{cases}$$

u was seen to be harmonic as it was the real part of the holomorphic $F : \mathbb{D} \longrightarrow \mathbb{C}$ defined by

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} f(e^{it})\, dt.$$

Prop^n. Let $u : \overline{\mathbb{D}} \to \mathbb{R}$, $u \in \mathcal{C}^0(\overline{\mathbb{D}})$, u is harmonic on $\mathbb{D}$.
Then, $u|_{\mathbb{D}}$ is the real part of $F : \mathbb{D} \to \mathbb{C}$ is defined as

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, u(e^{it})\, dt.$$

\# Poisson kernel on $\mathbb{D} = D(0,1)$.

$$P : D(0,1) \times \partial D(0,1) \longrightarrow \mathbb{R}$$
$$(z, s) \longmapsto \frac{1-|z|^2}{|z-s|^2}.$$

\# Poisson kernel on $D(a,R)$. $\to$ s' again!

Poisson kernel on $D(a,R)$.

s^1 again!

$$\tilde{P} : D(a,R) \times \partial \mathbb{D} \longrightarrow \mathbb{R}$$
$$(z, \zeta) \longmapsto P\left(\frac{z-a}{R}, \zeta\right).$$

Generalised Poisson Integral Formula:

Prop^n Let u be harmonic on $D(a, R)$ and continuous on $\overline{D(a,R)}$. Then, for any $z \in D(a,r)$, we have

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{P}(z, e^{it})\, u(a + Re^{it})\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - |z-a|^2}{|z-a-Re^{it}|^2}\, u(a+Re^{it})\, dt.$$

Obs1. Suppose further that $u \geq 0$ (Cont. on $\overline{D}$, harmonic on D).

$$\frac{R^2 - |z-a|^2}{|z-a-Re^{it}|^2} \leq \frac{R^2 - |z-a|^2}{(R-|z-a|)^2} = \frac{R+|z-a|}{R-|z-a|}$$

$$\frac{R^2 - |z-a|^2}{|z-a-Re^{it}|^2} \geq \frac{R^2 - |z-a|^2}{|z-a|+R} = \frac{R-|z-a|}{R+|z-a|}$$

That is: $$\frac{R-|z-a|}{R+|z-a|} \leq \frac{R^2-|z-a|^2}{|z-a-Re^{it}|^2} \leq \frac{R+|z-a|}{R-|z-a|}.$$

Can multiply with $u(e^{it}) \geq 0$ to integrate and get:

$$u(a)\left(\frac{R-|z-a|}{R+|z-a|}\right) \leq u(z) \leq u(a)\left(\frac{R+|z-a|}{R-|z-a|}\right).$$

Harnack's Inequality

(We can relax u to not extend continuously on ∂D)

Obs 2. Let $(u_n)_n$ be a seq. of nonnegative harmonic functions on $D(a,R)$.

- Assume that $u_n(a) \to 0$.
  Then, Harnack's inequality tells us that $u_n(z) \to 0$ for all $z \in D(a,R)$. Moreover, this is uniform on every CC subdisk.
- OTOH, if $(u_n(a))_n$ is bounded, then $(u_n)_n$ is locally uniformly bounded.

Theorem. Let $\Omega \subseteq \mathbb{C}$ be a domain.
Let $u_n : \Omega \to \mathbb{C}$ be a sequence of nonnegative harmonic functions.

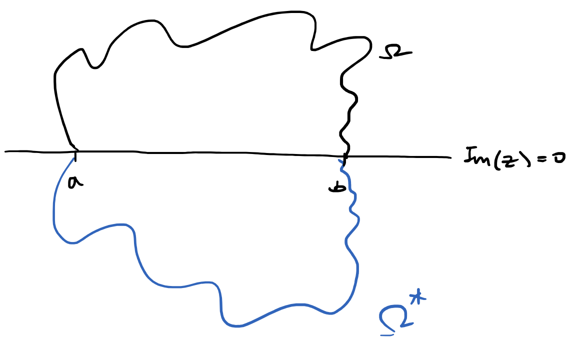
- If $\exists z_0 \in \Omega$ s.t. $u_n(z_0) \to \infty$, then $u_n \to \infty$ uniformly on compact subsets.
- If $\exists z_0 \in \Omega$ s.t. $(u_n(z_0))_n$ is bdd, then $(u_n)_n$ is bdd uniformly on compact subsets.

Prop^n. Let $\Omega \subseteq \mathbb{C}$ be a domain.
Let $u \in \mathcal{C}^0(\Omega)$ and suppose that u has the mean value property.
Then, u is harmonic.

Schwarz Reflection Principle for Harmonic Functions

Schwarz Reflection Principle for Harmonic Functions

Prop^n.



Let u be harmonic on Ω (where Ω is as shown). (open)

Define
$$\Omega^* = \{ z \in \mathbb{C} : \bar{z} \in \Omega \}.$$

Assume that for all $x \in (a,b)$, we have
$$\lim_{\Omega \ni z \to x} u(z) = 0.$$

Then, we can extend u to $u^* : \Omega \cup \Omega^* \cup (a,b) \to \mathbb{R}$ as

$$u^*(z) = \begin{cases} u(z) & ; \quad z \in \Omega \\ 0 & ; \quad z \in (a,b) \end{cases}$$

$$u^*(z) = \begin{cases} u(z) & ; \quad z \in \Omega \\ 0 & ; \quad z \in (a,b) \\ -u(\bar{z}) & ; \quad z \in \Omega^* \end{cases}$$

Then, u^* is harmonic on $\underbrace{\Omega \cup (a,b) \cup \Omega^*}_{=\Omega'}$.

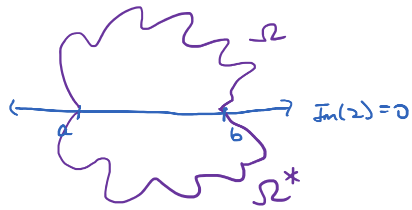
Lecture 16 (03-03-2022)

03 March 2022 13:54

Schwarz Reflection Principle For Holomorphic Functions

Theorem: Let $G \subseteq \mathbb{C}$ be a domain in $\mathbb{C}$ such that $G \cap \mathbb{R} = (a, b)$.
Let $\Omega = \{z \in G : Im(z) > 0\}$.
Suppose $F \in \mathcal{O}(\Omega)$ and

$$\lim_{\Omega \ni z \to x} Im(F(z)) = 0$$



for all $x \in (a, b)$.
Then, $\exists F^* \in \mathcal{O}(\Omega \cup (a, b) \cup \Omega^*)$ s.t. $F^*|_\Omega = F$.
Furthermore, F^* is given as

$$F^*(z) := \begin{cases} F(z) & ; z \in \Omega, \\ \lim_{\Omega \ni \zeta \to z} F(\zeta) & ; z \in (a, b), \\ \overline{F(\bar{z})} & ; z \in \Omega^*. \end{cases}$$

Theorem. (Runge's Theorem)
Let $K \subseteq \mathbb{C}$ be compact.
Let f be holomorphic on a neighbourhood Ω of K.
Suppose $E \subseteq \hat{\mathbb{C}} \setminus K$ containing (at least) one point from each connected component of $\hat{\mathbb{C}} \setminus K$.
Then, for any $\varepsilon > 0$, there is a rational function R such that

$$\sup_{z \in K} |f(z) - R(z)| < \varepsilon$$

and $Poles(f) \subseteq E$.

Corollary: Let $K \subseteq \mathbb{C}$ be compact such that $\hat{\mathbb{C}} \setminus K$ is connected. Let $\varepsilon > 0$.
Then, taking $E = \{\infty\}$ $(\infty \notin K)$ shows that we can find a polynomial P s.t. $\|P - f\|_K < \varepsilon$.

Lemma. Every nonempty open set $\Omega \subseteq \mathbb{C}$ is the union of a sequence $(K_n)_{n\geq 1}$ of compact sets such that:

(i) $\Omega = \bigcup_{n=1}^{\infty} K_n$,

(ii) $K_n \subseteq K_{n+1}^{\circ}$ for all $n \in \mathbb{N}$,

$\Rightarrow$ every compact $K \subseteq \Omega$ is contained in some K_n.

(iii) every connected component of $\hat{\mathbb{C}} \setminus K_n$ contains a component of $\hat{\mathbb{C}} \setminus \Omega$.

"K_n has no other holes than those forced upon it by Ω"

Theorem. (Runge's Theorem ver. 2)

Let $\Omega \subseteq \mathbb{C}$ be an open set. Let A be a set intersecting each component of $\hat{\mathbb{C}} \setminus \Omega$. Let $f \in \mathcal{O}(\Omega)$. Then, there is a sequence of rational functions $(R_n)_{n\geq 1}$ with poles in A s.t.

$$R_n \longrightarrow f$$

uniformly on compact subsets of Ω.

Corollary. If $\hat{\mathbb{C}} \setminus \Omega$ is connected, then R_n can be chosen to be polynomials.

Lecture 17 (07-03-2022)

07 March 2022 14:00

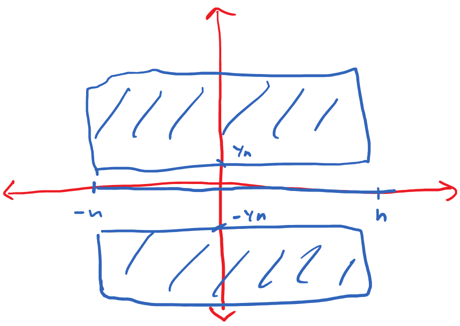
EXAMPLES ① Is there a sequence $(p_n)_{n\geq 1}$ of polynomials such that

$$\lim_{n\to\infty} p_n(z) = \begin{cases} -1 & ; \ \operatorname{Im} z > 0 \\ 0 & ; \ \operatorname{Im} z = 0 \\ 1 & ; \ \operatorname{Im} z < 0 \end{cases}$$

Call this $f(z)$.

Let $\Omega = \mathbb{C}$. $K_n = \{ z \in \mathbb{C} : \frac{1}{n} \leq |\operatorname{Im} z| \leq n, \ |\operatorname{Re} z| \leq n \}$

$\cup \{ x \in \mathbb{R} : |x| \leq n \}$.



Define

$$f_n(z) := \begin{cases} -1 & \operatorname{Im}(z) > 1/2n \\ 0 & |\operatorname{Im}(z)| < 1/4n \\ 1 & \operatorname{Im}(z) < -1/2n \end{cases}$$

Note: f_n is defined on an open nbd Ω_n of K_n and is holomorphic on it.

Also, $\hat{\mathbb{C}} \setminus K_n$ is connected. Also, $\mathbb{C} = \bigcup_{n\geq 1} K_n$.

Thus, by Runge, $\exists$ polynomial P_n s.t.

$$\| p_n - f_n \|_{K_n} < 1/n.$$

Now, given $z \in \mathbb{C}$, $\exists\, N \in \mathbb{N}$ s.t. $z \in K_n \ \forall n \geq N$.

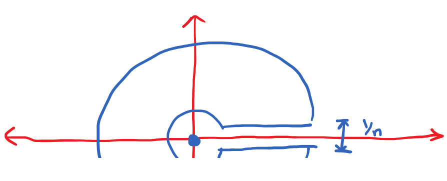
$\therefore \ |p_n(z) - f_n(z)| < 1/n$ for all $n \geq N$.

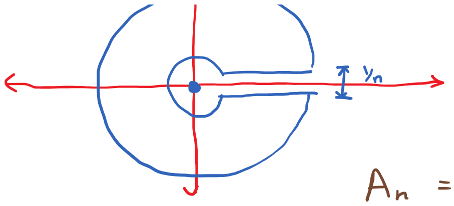
Letting $n \to \infty$ gives the desired result as $f_n(z) = f(z)$ for all $n \geq N$. ⊠

② Is there a sequence of polynomials $(p_n)_{n\geq 1}$ s.t. $p_n(0) \to 1$ as $n \to \infty$ and $p_n \to 0$ in $\mathcal{O}(\mathbb{C} \setminus [0, \infty))$, i.e., on compact subsets of $\mathbb{C} \setminus [0, \infty)$

Consider K_n as in the diagram:

$K_n = \{0\} \cup A_n$, where





$K_n = \{0\} \cup A_n$, where

$$A_n = \left\{ z \in \mathbb{C} : \tfrac{1}{n} \le |z| \le n \right\} \setminus \left\{ z : \operatorname{Re} z > 0, |\operatorname{Im} z| < \tfrac{1}{2n} \right\}.$$

Note: $\hat{\mathbb{C}} \setminus K_n$ is connected $\forall n$.

Define

$$f_n(z) := \begin{cases} 1 & ; \quad z \in D(0, 1/4n), \\ 0 & ; \quad z \in \mathbb{C} \setminus D(0, \tfrac{1}{3n}). \end{cases}$$

As before f_n are defined and holo on an open nbd of K_n $\forall n \in \mathbb{N}$.

Can construct a polynomial p_n s.t.

$$\| f_n - p_n \|_{K_n} < 1/n. \quad \text{Done as before.} \quad \boxtimes$$

Lemma (Pole pushing lemma)

Let $K \subseteq \mathbb{C}$ be compact. Let $P \in \hat{\mathbb{C}} \setminus K$ and U be the connected component of $\hat{\mathbb{C}} \setminus K$ containing P. If $\epsilon > 0$ and $Q \in U \setminus \{\infty\}$, then there is a rational function R with pole only at P s.t.

$$\sup_{z \in K} \left| \frac{1}{z - Q} - R(z) \right| < \epsilon.$$

Theorem (Mittag-Leffler)

Let $\Omega \subseteq \mathbb{C}$ be open, and $A \subseteq \Omega$ be s.t. A has no limit point in Ω. Suppose that for each $\alpha \in A$, we are given:

- $m(\alpha) \in \mathbb{Z}^+$, and
- $P_\alpha(z) = \displaystyle\sum_{j=1}^{m(\alpha)} \frac{A_{j,\alpha}}{(z-\alpha)^j}$ for $A_{j,\alpha} \in \mathbb{C}$.

Then, $\exists f$ meromorphic on Ω s.t. $\operatorname{Poles}(f) = A$ and the principal part of f at α is P_α ($\forall \alpha \in A$).

Lecture 20 (17-03-2022)

17 March 2022 13:52

Introduction To Several Complex Variables

Let $\Omega \subseteq \mathbb{C}^n$ be open. Let $f : \Omega \to \mathbb{C}$.

Some possible definitions:

(A) f is holomorphic on Ω iff $f \in \mathcal{C}^1(\Omega)$ and $\dfrac{\partial f(z)}{\partial \bar{z}_j} = 0$

for all $z \in \Omega$ and $j \in [n]$.

(B) f is holomorphic on Ω iff for each $a \in \Omega$ and any polydisc $D^n(a, \vec{r}) \subset\subset \Omega$, we have

↳ compactly contained

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D(a_n,r_n)} \cdots \int_{\partial D(a_2,r_2)} \int_{\partial D(a_1,r_1)} \frac{f(\omega)}{\prod_{j=1}^{n} (\omega_j - a_j)} \, d\omega_1 \, d\omega_2 \cdots d\omega_n .$$

(Cauchy Integral Formula.)

(C) f is holomorphic on Ω iff for each $a \in \Omega$ and any polydisk $D^n(a, \vec{r}) \subset\subset \Omega$, f admits a power series expansion:

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (z-a)^\alpha \qquad \forall z \in D^n(a, \vec{r}),$$

where the RHS converges absolutely and uniformly on each compact subset of $D^n(a, \vec{r})$.

(D) f is holomorphic on Ω iff f is holomorphic in each variable separately. (Not even assuming f continuous a priori.)

That is: for each $a \in \mathbb{C}$ and $j \in [n]$, consider the subset
$$\{ \xi \in \mathbb{C} : (z_1, \dots, z_{j-1}, \xi, z_{j+1}, \dots, z_n) \in \Omega \}$$
and demand that
$$\xi \mapsto f(z_1, \dots, z_{j-1}, \xi, z_{j+1}, \dots, z_n) \text{ is}$$

$\xi \mapsto f(z_1, \ldots, z_{j-1}, \xi, z_{j+1}, \ldots, z_n)$ is holo. on the above subset.

Theorem Fix $n \geq 2$.
Let $\Omega \subseteq \mathbb{C}^n$ be open. Let $K \subseteq \Omega$ be compact. Suppose that $\Omega \setminus K$ is connected. Then, for any $f \in \mathcal{O}(\Omega \setminus K)$, there exists $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus K} = f$.

Corollary. Let $\Omega \subseteq \mathbb{C}^n$ be open, $n \geq 2$.
Then, there does not exist $f \in \mathcal{O}(\Omega)$ having a compact zero set.