When are the natural embeddings of classical invariant rings pure?

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§1. Introduction

These are notes of three lectures given by Prof. Anurag K. Singh at IIT Bombay in June 2024 for the *Recent Trends in Commutative Algebra* workshop. To quote from the abstract,

Consider a reductive linear algebraic group G acting linearly on a polynomial ring S over an infinite field; key examples are the general linear group, the symplectic group, the orthogonal group, and the special linear group, with the classical representations as in Weyl's book: for the general linear group, consider a direct sum of copies of the regular representation and copies of the dual; in the other cases take copies of the regular representation. The invariant rings in the respective cases are determinantal rings, rings defined by Pfaffians of alternating matrices, symmetric determinantal rings, and the Plücker coordinate rings of Grassmannians; these are the classical invariant rings of the title, with S^G in S being the natural embedding.

A subring R of a ring S is pure if the inclusion remains injective upon tensoring with an arbitrary R-module. Over a field of characteristic zero, a reductive group is linearly reductive; it follows that the invariant ring S^G is a direct summand of S as an S^G-module, equivalently that S^G is a pure subring of S. Over fields of positive characteristic, reductive groups are typically no longer linearly reductive. We determine, in the positive characteristic case, precisely when the inclusion of S^G in S is pure. This is joint work with Melvin Hochster, Jack Jeffries, and Vaibhav Pandey.

Notations.

(a) Let R be a noetherian ring, M an R-module, $\mathfrak{a} \subseteq R$ an ideal, and k an integer. Then, $H^k_{\mathfrak{a}}(M)$ is the k-th local cohomology module of M supported at $\mathfrak{a}$.

(b) If M is a graded module, and n an integer, then M_n is the n-th graded component of M. In particular, we may talk about $\left[H^k_{\mathfrak{a}}(M)\right]_n$ if $\mathfrak{a}$ and M are homogeneous.

§2. Pure extensions

Definition 2.1. A ring homomorphism $R \to S$ is pure if $(R \to S) \otimes_R M$ is injective for all R-modules M.

Example 2.2. Recall that a ring homomorphism $f : R \to S$ is said to be (R-)split if there exists an R-linear map $g : S \to R$ such that $g \circ f = \mathrm{id}_R$.

Since tensoring preserves split exact sequences, we see that a split map is pure.

The setup we will be interested in the following.

> Let K be a field, $S := K[x_1, \ldots, x_n]$ a polynomial ring, and G a group acing on S by degree preserving K-algebra homomorphisms. The invariant subring is
>
> $$S^G := \{s \in S : g(s) = s \text{ for all } g \in G\}.$$

Question. Is $S^G \hookrightarrow S$ pure?

Remark 2.3. Having a positive answer to the above question will have the following consequences for $R = S^G$:

(a) R has Finite Frobenius Representation Type.

(b) The F-signature of R is a rational number.

(c) The Hilbert-Kunz multiplicity of R is a rational number.

(d) The ring of K-linear differential operators on R is a simple ring.

(e) The local cohomology modules $H_\mathfrak{a}^k(R)$ have finitely many associated primes for all homogeneous ideals $\mathfrak{a} \subseteq R$ and all integers k.

We examine the purity of $S^G \hookrightarrow S$ in some examples below.

(a) If G is a finite group and $\frac{1}{|G|} \in K$, then there exists a splitting $\frac{1}{|G|} \sum_{g \in G} g : S \to S^G$. Thus, the inclusion is pure. The splitting is often called the Reynolds operator.

(b) If G is the symmetric group Σ_n, then $R := S^G$ is $K[e_1, \ldots, e_n]$, where the e_i are the elementary symmetric polynomials.
The ring S is R-free with basis $\{x_1^{<1} x_2^{<2} \cdots x_n^{<n}\}$, so $R \hookrightarrow S$ splits – even if $\mathrm{char}(K)$ divides $n!$.

(c) If G is the alternating group A_n, then $S^G \hookrightarrow S$ is pure if and only if $\mathrm{char}(K) \nmid |G|$.
The nontrivial direction ($\Rightarrow$) is shown below (Theorem 2.6).

In this case, we have $S^G = K[e_1, \ldots, e_n, \Delta]$, where

$$\Delta := \begin{cases} \displaystyle\prod_{i<j}(x_i - x_j) & \text{if } \mathrm{char}(K) \neq 2, \\[2em] \dfrac{1}{2}\left\{ \displaystyle\prod_{i<j}(x_i - x_j) + \prod_{i<j}(x_i + x_j) \right\} & \text{if } \mathrm{char}(K) = 2. \end{cases}$$

The second definition is to be interpreted as first computing the element in $\mathbb{Q}[\mathbf{x}]$, noting that it is an element of $\mathbb{Z}[\mathbf{x}]$, and then view it in $K[\mathbf{x}]$.

Then, Δ^2 is fixed by Σ_n and hence, $\Delta^2 = f(\mathbf{e})$ for some polynomial f. This gives us a presentation $S^G \cong K[E_1, \ldots, E_n, D]/(D^2 - f(\mathbf{E}))$. In particular, S^G is a hypersurface.

(d) As a generalisation of (a), if G is linearly reductive, e.g., $\mathrm{SL}(\mathbb{C}), \mathrm{GL}(\mathbb{C}), \mathrm{Sp}(\mathbb{C}), \mathrm{O}(\mathbb{C})$, then there exists a splitting $S \to S^G$. A linear algebraic group (Zariski closed subgroup of GL) over $\mathbb{C}$ is linearly reductive precisely if it has a Zariski dense subgroup that is a compact real Lie group; average over the compact subgroup with respect to the Haar measure to obtain the splitting. Elements fixed by a (Zariski) dense subgroup are fixed by everything.

Question. What if $\mathbb{C}$ is replaced by $\overline{\mathbb{F}_p}$, say for classical representations?

In the above examples of invariant subrings, we deduced purity from the *a priori* stronger notion of splitting. The following proposition indicates why.

Proposition 2.4. Suppose $R \hookrightarrow S$ is a degree preserving inclusion of $\mathbb{N}$-graded normal rings, that are finitely generated over a field $R_0 = S_0$. Set $d := \dim(R)$, $\mathfrak{m}_R$ the homogeneous maximal ideal of R, and E_R the injective hull of $R/\mathfrak{m}_R$ in the category of graded R-modules.

Consider the following statements:

(1) $R \hookrightarrow S$ is R-split.

(2) $R \hookrightarrow S$ is pure.

(3) $R \otimes_R E_R \to S \otimes_R E_R$ is injective.

(4) $H^d_{\mathfrak{m}_R}(S) \neq 0$.

Then, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4).
If R is a polynomial ring of char $p > 0$, then (4) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1): Suppose $R \otimes_R E_R \to S \otimes_R E_R$ is injective.
Apply the (graded) Hom functor $\mathrm{Hom}_R(-, E_R)$ to this map. This yields the surjection

$$\mathrm{Hom}_R(S \otimes_R E_R, E_R) \twoheadrightarrow \mathrm{Hom}_R(R \otimes_R E_R, E_R).$$

Now, using tensor-Hom adjunction, along with $\mathrm{Hom}_R(E_R, E_R) \cong R$, we get a surjection

$$\mathrm{Hom}_R(S, R) \twoheadrightarrow \mathrm{Hom}_R(R, R).$$

Thus, there is an element $g \in \mathrm{Hom}_R(S, R)$ that maps to id_R. This is precisely what a splitting is.[1]

Thus, we have proven the equivalence of (1)–(3).

(2) $\Rightarrow$ (4): Suppose $R \hookrightarrow S$ is pure. Then, tensoring this map with $H^d_{\mathfrak{m}_R}(R)$ gives us an injection

$$H^d_{\mathfrak{m}_R}(R) \hookrightarrow H^d_{\mathfrak{m}_R}(R) \otimes_R S \cong H^d_{\mathfrak{m}_R}(S),$$

where the last isomorphism follows since we are working with the top local cohomology. Since $d = \dim(R)$, we see that the module on the left is nonzero, implying the same for the desired module on the right.

Lastly, assume that R is a polynomial ring of positive characteristic $p > 0$.
(4) $\Rightarrow$ (3): Suppose that $R = K[x_1, \ldots, x_d]$ and $H^d_{\mathfrak{m}_R}(S) \neq 0$. We wish to show that $R \otimes_R E_R \to S \otimes_R E_R$ is injective. Since R is a polynomial ring, the injective hull is simply the top local cohomology. As noted before, we have $H^d_{\mathfrak{m}_R}(R) \otimes_R S \cong H^d_{\mathfrak{m}_R}(S)$.
Thus, we wish to show

$$\varphi : H^d_{\mathfrak{m}_R}(R) \to H^d_{\mathfrak{m}_R}(S)$$

is injective. We will compute these local cohomology modules using the Čech complex on $\{x_1, \ldots, x_d\}$. Set

$$\eta := \left[\frac{1}{x_1 \cdots x_d} \right].$$

We will write η_R or η_S according to whether this is considered as an element in either $H^d_{\mathfrak{m}_R}(R)$ or $H^d_{\mathfrak{m}_R}(S)$. Since R is a polynomial ring, we know that η_R is a generator for the socle of $H^d_{\mathfrak{m}_R}(R)$. Thus, if φ is not injective, then η_R is in the kernel, i.e., $\eta_S = 0$. Applying the Frobenius map, we get that

$$\left[\frac{1}{(x_1 \cdots x_d)^q} \right]_S = 0 \text{ for all } q = p^e.$$

But these elements generate $H^d_{\mathfrak{m}_R}(S)$ as an S-module. This shows that $H^d_{\mathfrak{m}_R}(S) = 0$, a contradiction. $\qquad\square$

Porism 2.5. Suppose $R \hookrightarrow S$ is a degree preserving inclusion of $\mathbb{N}$-graded normal rings, that are finitely generated over a field $R_0 = S_0$. Then,

$$R \hookrightarrow S \text{ is pure} \quad \Rightarrow \quad H^{\dim(R)}_{\mathfrak{m}_R}(R) \to H^{\dim(R)}_{\mathfrak{m}_R}(S) \text{ is injective.}$$

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[1] One needs to check that the map $\mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(R, R)$ is the natural map, i.e., the one obtained by applying $\mathrm{Hom}_R(-, R)$ to $R \to S$. This follows from the naturality of the identifications made earlier.

We now look at examples to see how various hypotheses in the above theorem cannot be dropped.

(a) "Graded" cannot be replaced with "local": [CGM16]

The inclusion

$$\mathbb{C}[x]_{(x)} \hookrightarrow \frac{\mathbb{C}[x,y]_{(x,y)}}{(xy^2 - y + x)}$$

is pure but not split.
The above map does split if we pass to the completion.

(b) (4) $\Rightarrow$ (2) fails if R is only assumed to be Gorenstein instead of polynomial in positive characteristic: The inclusion

$$R := \frac{\mathbb{F}_p[x,y,z]}{(x^3 + y^3 + z^3)} \hookrightarrow R^{1/p}$$

is pure $\Leftrightarrow p \equiv 1 \mod 3$.

However, $H^2_{\mathfrak{m}_R}(R^{1/p}) \neq 0$ for all primes.

(c) (4) $\Rightarrow$ (2) fails in characteristic zero: the inclusion

$$R := \mathbb{C}[x_1, x_2, x_3] \hookrightarrow \frac{\mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]}{((x_1 x_2 x_3)^2 - \sum_i y_i x_i^3)}$$

is not pure since $(R \to S) \otimes_R R/(x_1^3, x_2^3, x_3^3)$ is not injective: the nonzero element $(x_1 x_2 x_3)^2$ is in the kernel.

However, $H^3_{(x_1, x_2, x_3)}(S) \neq 0$, as shown in [Rob94].

Note that the above is still a graded inclusion of normal domains if we consider the grading $\deg(x_i) = 1$ and $\deg(y_i) = 3$.

Note that in the last example, the target was not polynomial:

Question. Suppose $R \hookrightarrow S$ is a graded inclusion of polynomial rings of characteristic 0 and $H^{\dim(R)}_{\mathfrak{m}_R}(S) \neq 0$. Is $R \hookrightarrow S$ pure?

We now return to the alternating group.

Theorem 2.6. Let $G := A_n$ act on $S := \mathbb{F}_p[x_1, \ldots, x_n]$ by permuting the variables. If p divides $|G|$, then $S^G \hookrightarrow S$ is not pure.

In [Gla92] the above was shown for when p divides $n(n-1)$. It was strengthened to p dividing $|G| = \frac{n!}{2}$ in [Sin98]. The proof we give is adapted from [GJS24].

Remark 2.7. In our proof, we will use the following facts about the degree $(-n)$ component of local cohomology:

- $\left[H^n_{\mathfrak{m}_{S^G}}(S)\right]_{-n}$ is one-dimensional.

- $\left[H^n_{\mathfrak{m}_{S^G}}(S^G)\right]_{-n} \neq 0$.

The first is true since S is a standard graded polynomial ring in n variables and $\mathfrak{m}_{S^G}S$ is $\mathfrak{m}_S$-primary.

The second is true for any subgroup $G \leqslant A_n$, by [GJS24, Theorem 4.4].

For $G = A_n$ specifically, this can be explicitly worked out using the description of S^G from (c) on page 2. Following that notation, one checks that $\left[\frac{\Delta}{e_1\cdots e_n}\right] \in H^n_{\mathfrak{m}_{S^G}}(S^G)_{-n}$ is nonzero.

Proof. Suppose $p \mid |G|$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of S^G. Consider the trace map

$$\mathrm{Tr} := \sum_{g \in G} g : S \to S^G.$$

It is straightforward to check that the map does take values in S^G and is S^G-linear. Moreover, by Dedekind's independence of characters, the above map is nonzero.

Note however that Tr is far from a splitting: The composition

$$S^G \hookrightarrow S \xrightarrow{\mathrm{Tr}} S^G \tag{2.1}$$

is the zero map. Indeed, if $s \in S^G$, then $\mathrm{Tr}(s) = \sum_{g \in G} g(s) = \sum_{g \in G} s = |G|s = 0$.

Nevertheless, we have a short exact sequence of S^G-modules given as

$$S \xrightarrow{\mathrm{Tr}} S^G \to S^G/\mathrm{im}(\mathrm{Tr}) \to 0.$$

Since S^G is a domain, the cokernel has strictly smaller dimension. Thus, the right exact functor $H^n_{\mathfrak{m}}(-)$ vanishes on $S^G/\mathrm{im}(\mathrm{Tr})$ giving us a surjection

$$H^n_{\mathfrak{m}}(S) \xrightarrow{\mathrm{Tr}} H^n_{\mathfrak{m}}(S^G).$$

The above surjection is degree-preserving; this gives us a sequence of K-vector spaces:

$$H^n_{\mathfrak{m}}(S^G)_{-n} \to H^n_{\mathfrak{m}}(S)_{-n} \xrightarrow{\mathrm{Tr}} H^n_{\mathfrak{m}}(S^G)_{-n}.$$

As noted after (2.1), the above composition is zero. Since the middle vector space is one-dimensional and the outer vector spaces are nonzero, we conclude that the left map is not injective. By Porism 2.5, $S^G \hookrightarrow S$ is not pure. $\qquad\square$

§3. A Helpful Vanishing Theorem

We recall the following vanishing theorem from [PS73, Proposition III.4.1].

Theorem 3.1 (Peskine-Szpiro)**.** Suppose S is a polynomial ring of characteristic $p > 0$, and $\mathfrak{a}$ a homogeneous ideal such that $S/\mathfrak{a}$ is Cohen-Macaulay. Then,

$$H_{\mathfrak{a}}^k(S) = 0 \quad \text{for all } k \neq \operatorname{ht}(\mathfrak{a}).$$

Proof. Let $h \coloneqq \operatorname{ht}(\mathfrak{a})$. Since S is Cohen-Macaulay, we know that $\operatorname{depth}_{\mathfrak{a}}(S) = \operatorname{ht}(\mathfrak{a})$. Thus, the first nonvanishing of local cohomology occurs at the h-th index. We show that $H_{\mathfrak{a}}^k(S) = 0$ for all $k > h$.

Since $S/\mathfrak{a}$ is Cohen-Macaulay, the Auslander-Buchsbaum formula gives us that

$$\operatorname{pdim}_S(S/\mathfrak{a}) = \operatorname{depth}(S) - \operatorname{depth}(S/\mathfrak{a}) = \dim(S) - \dim(S/\mathfrak{a}) = \operatorname{ht}(\mathfrak{a}) = h.$$

Thus, the minimal free resolution $F_\bullet$ of $S/\mathfrak{a}$ has length h:

$$F_\bullet : 0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_h \leftarrow 0.$$

Since S is a polynomial ring, the Frobenius endomorphism Frob is flat. Thus, $\operatorname{Frob}^e(F_\bullet)$ is acyclic, and resolves $S/\mathfrak{a}^{[p^e]}$ for all $e \geqslant 1$. In particular, these quotients have projective dimension $\leqslant h$. This give us

$$H_{\mathfrak{a}}^k(S) = \operatorname*{colim}_{e \to \infty} \operatorname{Ext}_S^k(S/\mathfrak{a}^{[p^e]}, S) = 0 \quad \text{for all } k > h. \qquad \square$$

Remark 3.2. Lyubeznik [Lyu06] showed that if S is a polynomial ring of positive characteristic, and $\mathfrak{a}$ is a homogeneous ideal, then

$$H_{\mathfrak{a}}^k(S) = 0 \quad \Leftrightarrow \quad H_{\mathfrak{m}_S}^{\dim(S)-k}(S/\mathfrak{a}) \text{ is annihilated by some } \operatorname{Frob}^e.$$

§4. Classical Group Actions

The next sections will discuss the following *classical group actions*, in the following order.

(I) (Special Linear Group) $SL_t(K)$ acts on $K[Y_{t \times n}]$, where the action is given by

$$M : Y \mapsto MY.$$

To recall: $SL_t(K) = \{M : \det(M) = 1\} \leqslant GL_t(K)$.

(II) (General Linear Group) $GL_t(K)$ acts on $K[Y_{m \times t}, Z_{t \times n}]$, where the action is given by

$$M : \begin{cases} Y \mapsto YM^{-1}, \\ Z \mapsto MZ. \end{cases}$$

(III) (Symplectic Group) $Sp_{2t}(K)$ acts on $K[Y_{2t \times n}]$, where the action is given by

$$M : Y \mapsto MY.$$

To recall: Define

$$\Omega := \begin{bmatrix} 0 & id_t \\ -id_t & 0 \end{bmatrix} \in GL_{2t}(K).$$

Then, $Sp_{2t}(K) = \{M : M^\mathsf{T}\Omega M = \Omega\} \leqslant GL_{2t}(K)$.

(IV) (Orthogonal Group) $O_t(K)$ acts on $K[Y_{t \times n}]$, where the action is given by

$$M : Y \mapsto MY.$$

To recall: $O_t(K) = \{M : M^\mathsf{T}M = 1\} \leqslant GL_t(K)$.

We use the example of $SL_t(K)$ to explain our notation: $K[Y_{t \times n}]$ is the polynomial ring in tn variables over K. We imagine the variables set as $\{y_{ij} : 1 \leqslant t \leqslant i, 1 \leqslant j \leqslant n\}$. We shall use Y to refer to the corresponding obvious $t \times n$ matrix. The action of $M \in SL_t(K)$ is given by mapping the entries of Y to the corresponding entries of MY.

The next four sections will elaborate on the purity of invariant subrings of each of these subgroups in positive characteristic. As noted earlier (cf. (d) on page 3), if $char(K) = 0$, then the subrings are pure.

The common underlying template in the following sections is the following:

STEP 1. Identify the invariant subring $R := S^G$. We will be assuming that the field K is infinite, see however Remark 5.3. In each case, there will be some obvious invariants, and these will end up generating the invariant subring.

STEP 2. "Understand" the nullcone: $S/\mathfrak{m}_R S$. For example, is this Cohen-Macaulay? If not, can we "understand" the minimal primes of $\mathfrak{m}_R S$?

STEP 3. Apply Peskine-Szpiro 3.1 and Proposition 2.4 appropriately.

STEP 4. Deduce that purity implies restrictive numerical conditions on t, n. For such conditions, it is (comparatively) easy to check purity. Thus, obtain a complete characterisation.

> **Remark 4.1.** Each of these cases have the corresponding analogous question:
>
> Is S^G a (graded) pure subring of *some* polynomial ring?
>
> We will see in each section that the invariant subrings are interesting objects in their own right. By Remark 2.3, having a positive answer to the above will shed light on various properties of S^G.

Remark 4.2. In the process of proving these results, it was proven for each of the four classical group actions (in any characteristic), the irreducible components of the nullcone are normal and Cohen-Macaulay.

This leads to the following question: Is there some linearly reductive group (say over $\mathbb{C}$) and some action for which the above is not true?

§5. SL

The easiest and most illustrative case is the special linear group.

> **Setup.**
>
> $G := SL_t(K)$ is acting on $S := K[Y_{t \times n}]$, where $M : Y \mapsto MY$.

One checks that the $t \times t$ minors of Y are fixed by G, and so $K[t \times t \text{ minors}] \subseteq S^G$.

Theorem 5.1 ([Igu54], [DP76])**.** If K is infinite, then $K[t \times t \text{ minors}] = S^G$.

The above ring is of independent interest since $\text{Proj}(S^G) = \text{Grass}(t, n)$, the Grassmannian variety.

Theorem 5.2 ([HJPS23, Theorem 3.1])**.** If $\text{char}(K) > 0$ and $t \leqslant n$, then

$$K[t \times t \text{ minors}] \hookrightarrow K[Y_{t \times n}] \text{ is pure} \quad \Leftrightarrow \quad t = 1 \text{ or } t = n.$$

Remark 5.3. The above statement, as written, is true even if K is a finite field. However, $K[t \times t \text{ minors}]$ is not the complete $SL_t(K)$-invariant subring in that case.

Proof. Let $R := K[t \times t \text{ minors}]$ and $S := K[Y_{t \times n}]$. In this case, the nullcone[2] $S/\mathfrak{m}_R S$ is precisely $S/I_t(Y)$. By [EN62] or [HE71], we have that the nullcone is Cohen-Macaulay and $\text{ht}(I_t(Y)) = n - t + 1$.

Thus, by Peskine-Szpiro 3.1, we have that

$$H_{\mathfrak{m}_R S}^{\dim(R)}(S) \neq 0 \quad \Leftrightarrow \quad \dim(R) = \text{ht}(\mathfrak{m}_R S).$$
$$\underset{t(n-t)+1}{\|} \qquad \underset{n-t+1}{\|}$$

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[2]Strictly speaking, the term "nullcone" is used in the invariant theory context. As noted in Remark 5.3, this is slightly iffy in the finite field case.

Rearranging terms gives us

$$H^{\dim(R)}_{m_R S}(S) \neq 0 \Leftrightarrow (t-1)(n-t) = 0 \Leftrightarrow t \in \{1, n\}.$$

Now, if $R \hookrightarrow S$ is pure, then Proposition 2.4 tells us that $H^{\dim(R)}_{m_R S}(S) \neq 0$, i.e., $t \in \{1, n\}$. This proves ($\Rightarrow$).

($\Leftarrow$) If $t = 1$, then $R = S$ and the inclusion is pure. If $t = n$, then $R = K[\det(Y)]$. We may extend $\det(Y)$ to a homogeneous system of parameters for S. This gives us a Noether normalisation $A \subseteq S$. Since S is Cohen-Macaulay, S is free over A, which is in turn free over R. Thus, $R \hookrightarrow S$ splits.   □

Remark 5.4. If $t = n - 1 > 1$, then the above tells us that the natural inclusion

$$K[(n-1) \times (n-1) \text{ minors}] \subseteq K[Y_{(n-1)\times n}]$$

is **not** pure in positive characteristic. However, the subring on the left is known to be a polynomial ring.

Question. If $\text{char}(K) > 0$ and $2 \leqslant t \leqslant n - 2$, is $R := K[t \times t \text{ minors}]$ a (graded) pure subring of some polynomial ring over K?

The first example of the preceding question is $n = 4$ and $t = 2$. In this case, we are asking whether

$$R \cong \frac{K[u, v, w, x, y, z]}{(ux + vy + wz)}$$

is a (graded) pure subring of some polynomial ring over K. This is open at the time of writing these notes.

As remarked earlier, SL was the easiest case out of the four. For the groups in the next sections, we will state the theorem and outline some relevant ideas.

§**6.** GL

Setup.

$G := GL_t(K)$ is acting on $S := K[Y_{m\times t}, Z_{t\times n}]$, where

$$M : \begin{cases} Y \mapsto YM^{-1}, \\ Z \mapsto MZ. \end{cases}$$

One checks that the entries of YZ are fixed by G, and so $K[YZ] \subseteq S^G$.

Theorem 6.1 ([DP76], [Has05]). If K is infinite, then $K[YZ] = S^G$.

The above ring is of independent interest since $K[YZ] \cong K[X_{m \times n}]/I_{t+1}(X)$ is the generic determinantal ring.

Theorem 6.2 ([HJPS23, Theorem 4.2]). If $\mathrm{char}(K) > 0$, then

$$K[YZ] \hookrightarrow K[Y_{m \times t}, Z_{t \times n}] \text{ is pure} \quad \Leftrightarrow \quad \begin{cases} t = 1, & \text{or} \\ m \leqslant t, & \text{or} \\ n \leqslant t. \end{cases}$$

In this case, the nullcone is $K[Y, Z]/(YZ)$, where $(YZ) = I_1(YZ)$ denotes the ideal generated by the entries of YZ. This nullcone is usually not Cohen-Macaulay (or even equidimensional).

We analyse the algebraic set cut out by (YZ). Suppose $(\overline{Y}, \overline{Z}) \in V(YZ)$. This gives us that $\overline{Y} \cdot \overline{Z} = 0$, i.e.,

$$K^m \overset{\overline{Y}}{\leftarrow} K^t \overset{\overline{Z}}{\leftarrow} K^n$$

is a complex, i.e., $\mathrm{im}(\overline{Z}) \subseteq \ker(\overline{Y})$. By rank-nullity, this implies $\mathrm{rank}(\overline{Y}) + \mathrm{rank}(\overline{Z}) \leqslant t$. Thus, the following ideals are natural to consider:

$$\mathfrak{p}_{ij} := I_{i+1}(Y) + I_{j+1}(Z) + (YZ),$$

where i, j are nonnegative integers satisfying $i + j = t$.

Then, the following statements are true:

- Each $\mathfrak{p}_{ij}$ is a prime ideal.

- The minimal elements of $\{\mathfrak{p}_{ij} : i + j = t\}$ are the minimal primes of $\mathfrak{m}_R S$, i.e., these define the irreducible components of $V(YZ)$.

Kempf showed ([Kem75], [Kem76]) that $\mathbb{C}[Y, Z]/\mathfrak{p}_{ij}$ has rational singularities and is a normal Cohen-Macaulay domain. De Concini-Strickland [DS81, Theorem 2.7] and Huneke [Hun81, Theorem 6.2] independently showed that $K[Y, Z]/\mathfrak{p}_{ij}$ is a normal Cohen-Macaulay domain for any field K.

Using this along with the Mayer-Vietoris sequence and Peskine-Szpiro 3.1 gives the result.

§7. Sp

Setup.

$G := \mathrm{Sp}_{2t}(K)$ is acting on $K[Y_{2t \times n}]$, where $M : Y \mapsto MY$.

As noted earlier, we have $\mathrm{Sp}_{2t}(K) = \{M : M^\mathsf{T}\Omega M = \Omega\}$, where $\Omega := \begin{bmatrix} 0 & \mathrm{id}_t \\ -\mathrm{id}_t & 0 \end{bmatrix}$.

It is straightforward to note that the entries of $Y^\mathsf{T}\Omega Y$ are fixed; this follows from the symbolic computation that

$$Y^\mathsf{T}\Omega Y \mapsto Y^\mathsf{T}M^\mathsf{T}\Omega M Y = Y^\mathsf{T}\Omega Y.$$

Thus, $K[Y^\mathsf{T}\Omega Y] \subseteq S^G$.

Theorem 7.1 ([DP76], [Has05])**.** If K is infinite, then $K[Y^\mathsf{T}\Omega Y] = S^G$.

The above ring is of independent interest since $K[Y^\mathsf{T}\Omega Y] \cong K[X_{n \times n}^{\mathrm{alt}}]/\mathrm{Pf}_{2t+2}(X)$ is the generic Pfaffian ring.

Theorem 7.2 ([HJPS23, Theorem 6.9])**.** If $\mathrm{char}(K) > 0$, then

$$K[Y^\mathsf{T}\Omega Y] \hookrightarrow K[Y_{2t \times n}] \text{ is pure} \quad \Leftrightarrow \quad n \leqslant t+1.$$

While proving this result, it was shown in [HJPS23, Theorem 6.8] that

$$\frac{K[Y_{2t \times n}]}{(Y^\mathsf{T}\Omega Y)}$$

is a normal Cohen-Macaulay domain for any field K; previously in [KS14], it was shown that $\mathbb{C}[Y]/(Y^\mathsf{T}\Omega Y)$ is a normal domain.

Remark 7.3. Suppose $\overline{Y} \in V(Y^\mathsf{T}\Omega Y)$. Then,

$$K^n \xleftarrow{\overline{Y}^\mathsf{T}} K^{2t} \xleftarrow{\Omega} K^{2t} \xleftarrow{\overline{Y}} K^n$$

is a complex. Since Ω is invertible, rank-nullity as before gives us

$$\mathrm{rank}(\overline{Y}^\mathsf{T}) + \mathrm{rank}(\overline{Y}) \leqslant 2t.$$

Since $\mathrm{rank}(\overline{Y}^\mathsf{T}) = \mathrm{rank}(\overline{Y})$, we get $\mathrm{rank}(\overline{Y}) \leqslant t$.

Nullstellensatz tells us that $I_{t+1}(Y) \subseteq \mathrm{rad}(Y^\mathsf{T}\Omega Y)$.

Exercise 7.4. Show that $I_{t+1}(Y) \subseteq (Y^\mathsf{T}\Omega Y)$.

Of course, this follows from some of the results stated above since they imply that $(Y^\mathsf{T}\Omega Y)$ is a prime ideal.

§8. O

> **Setup.**
>
> $G := O_t(K)$ is acting on $K[Y_{t\times n}]$, where $M : Y \mapsto MY$.

It is straightforward to note that the entries of Y^TY are fixed; this follows from the symbolic computation that

$$Y^TY \mapsto Y^TM^TMY = Y^TY.$$

Thus, $K[Y^TY] \subseteq S^G$.

Theorem 8.1 ([DP76], [Ric89]). If K is infinite, then

- $K[Y^TY] = S^G$ if $\mathrm{char}(K) \neq 2$, and
- $K[Y^TY, \sum_i Y_{ij}] = S^G$ if $\mathrm{char}(K) = 2$.

The above ring is of independent interest since $K[Y^TY] \cong K[X^{\mathrm{sym}}_{n\times n}]/I_{t+1}(X)$ is the symmetric determinantal ring.

Theorem 8.2 ([HJPS23, Theorems 7.14–7.15]). If $\mathrm{char}(K) \geq 3$, then

$$K[Y^TY] \hookrightarrow K[Y_{t\times n}] \text{ is pure} \quad \Leftrightarrow \quad t = 1, \text{ or } t = 2, \text{ or } n \leq (t+2)/2.$$

If $\mathrm{char}(K) = 2$, then the following are equivalent:

- $K[Y^TY] \hookrightarrow K[Y_{t\times n}]$ is pure.
- $K[Y^TY, \sum_i Y_{ij}] \hookrightarrow K[Y_{t\times n}]$ is pure.
- $t = 1$ or $n \leq (t+1)/2$.

In proving the above, the following was shown.

Theorem 8.3 ([HJPS23, §7]). Let K be any field, $S := K[Y_{t\times n}]$, and $\mathfrak{a} := I_1(Y^TY)$.

(a) Suppose $\mathrm{char}(K) \neq 2$ and $\iota := \sqrt{-1} \in K$. Then,

- $\mathfrak{a}$ is radical $\Leftrightarrow 2n \leq d$.
- $\mathfrak{a}$ is prime $\Leftrightarrow 2n < d$.
- If t is odd <u>or</u> $2n < t$, then $S/\mathrm{rad}(\mathfrak{a})$ is a Cohen-Macaulay domain.
- If t is even <u>and</u> $2n \geq t$, then $\mathfrak{a}$ has minimal primes $\mathfrak{P}$ and $\mathfrak{Q}$; $S/\mathfrak{P}$ and $S/\mathfrak{Q}$ are Cohen-Macaulay domains.

(b) Suppose $\operatorname{char}(K) = 2$. Then $S/\operatorname{rad}(\mathfrak{a})$ is a Cohen-Macaulay domain.

Remark 8.4. Let $\overline{Y} \in V(\mathfrak{a})$. Then we have a complex

$$K^n \xleftarrow{\overline{Y}^\mathsf{T}} K^d \xleftarrow{\overline{Y}} K^n$$

Hence, $I_{\lfloor t/2 \rfloor + 1}(\overline{Y}) = 0$.

Thus, as before, Nullstellensatz tells us that $I_{\lfloor t/2 \rfloor + 1}(Y) \subseteq \operatorname{rad}(\mathfrak{a})$.

Unlike before, one cannot drop the "rad": Consider $t = n = 2$. We have

$$Y^\mathsf{T}Y = \begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix} \cdot \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} y_{11}^2 + y_{21}^2 & y_{11}y_{12} + y_{21}y_{22} \\ y_{11}y_{12} + y_{21}y_{22} & y_{12}^2 + y_{22}^2 \end{bmatrix}.$$

The entries of the matrix generate $\mathfrak{a}$. In particular, the degree two elements of $\mathfrak{a}$ are K-linear combinations of these elements. Using this, we see that

$$I_{\lfloor t/2 \rfloor + 1}(Y) = I_2(Y) = (y_{11}y_{22} - y_{12}y_{21}) \not\subseteq \mathfrak{a}.$$

The minimal primes of $\mathfrak{a}$ are

$$(y_{11} - \iota y_{21}, y_{12} - \iota y_{22}),$$
$$(y_{11} + \iota y_{21}, y_{12} + \iota y_{22}).$$

Remark 8.5. Suppose $t = 2k$. Let A be a $k \times n$ matrix, and $Q \in O_k$. Consider the vertically stacked matrix

$$\overline{Y} := \begin{bmatrix} A \\ \iota QA \end{bmatrix}.$$

Then,

$$\overline{Y}^\mathsf{T}\overline{Y} = \begin{bmatrix} A^\mathsf{T} & \iota A^\mathsf{T}Q^\mathsf{T} \end{bmatrix} \begin{bmatrix} A \\ \iota QA \end{bmatrix} = \begin{bmatrix} A^\mathsf{T}A + \iota^2 A^\mathsf{T}Q^\mathsf{T}QA \end{bmatrix} = \begin{bmatrix} A^\mathsf{T}A - A^\mathsf{T}A \end{bmatrix} = 0.$$

This gives us a map

$$O_k \times \mathbb{A}^{k \times n} \to V(\mathfrak{a}).$$

The two components of $V(\mathfrak{a})$ correspond to

$$O_k = SO_k \sqcup (O_k \setminus SO_k).$$

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