

$$\int (\overset{\circ}{\cup} \overset{\circ}{\cup}) dx$$

MA 406

General Topology

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Spring 2020-21

Lecture 1 (07-01-2021)

07 January 2021 15:04

Def. A topology on a set X is a collection \mathcal{T} of subsets of X having the following properties: (Topology)

(1) \emptyset and X are in \mathcal{T} .

(2) The union of the elements of any subcollection of \mathcal{T} is in \mathcal{T} .

(3) The intersection of the elements of any finite subcollection of \mathcal{T} is in \mathcal{T} .

(Open set)

Any $U \in \mathcal{T}$ is called an open set of X w.r.t. \mathcal{T} .

The pair (X, \mathcal{T}) or just the set X is called a topological space. (abuse of notation)

Can reconcile the above with open sets in \mathbb{R} , or in general, any metric space X . That can be seen as a motivation for the definition.

Examples

(1) $X = \{a, b, c\}$

$\mathcal{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \rightarrow$ Can be seen (fairly easily)

$\mathcal{T}_2 = \{\emptyset, X\}$ that this is a topology

trivial (pun intended, cf. next example)

(2) If X is any set, the collection of all subsets of X is a topology on X , it is called the discrete topology.
 $(\mathcal{T} = P(X), \text{ that is})$ (Discrete topology)

The collection $\{\emptyset, X\}$ is also a topology on X called the indiscrete topology or trivial topology. (Indiscrete topology)
(Trivial topology)

(3) Let X be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\emptyset\}.$$

(Finite complement topology)

Then, \mathcal{T}_f is a topology on X , called the finite complement topology on X .

- $\emptyset \in \mathcal{T}_f$ is clear. $X \in \mathcal{T}_f$ since $|X \setminus X| = 0 < \infty$.
- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be sets in \mathcal{T}_f . Now, $U_\alpha \neq \emptyset \ \forall \alpha$.

$$\begin{aligned} \text{Note } X \setminus \left(\bigcup_{\alpha} U_\alpha \right) &= X \cap \left(\bigcup_{\alpha} U_\alpha^c \right)^c \\ &= \bigcap_{\alpha} (U_\alpha^c)^c \end{aligned}$$

Note that each U_α^c is finite. ($U_\alpha \neq \emptyset$)

Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to $\bigcup_{i=1}^n (U_i^c)$

and conclude as earlier.

(Here, if some U_i were \emptyset , then so would be the intersection.)

(If X is finite, the $\mathcal{T}_f = P(X)$. Thus, we get discrete.)

- (4) Let X be a set.

Let \mathcal{T}_c be the collection of subsets such that $X \setminus U$ is either countable or all of X .

(Generalising the previous.)

(Cocountable topology
Co-countable topology)

Defn

Suppose that \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T}' \supset \mathcal{T}$, we say that \mathcal{T}' is finer than \mathcal{T} and that \mathcal{T} is coarser than \mathcal{T}' .

If $\mathcal{T}' \supsetneq \mathcal{T}$, then the above is strictly finer and strictly

coarser, respectively.

(Finer, coarser, strictly finer, strictly coarser)

(The above gives us a way to compare two topologies)

Example We have the usual topology on \mathbb{R} . ← strictly weaker than this
We also have the discrete topology on \mathbb{R} . ←

Def.ⁿ If X is a set, a basis for a topology on X is a collection B of subsets of X (called basis elements) such that (Basis)

- (1) for each $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.

(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then
 $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Note that in the above, \mathcal{B} is just some collection of subsets of X satisfying (1) & (2). No topology is mentioned so far.

EXAMPLES

- (1) $X = \mathbb{R}^2$, \mathcal{B} is the collection of all discs w/o boundary.
 - (2) \square - rectangles \square
 - (3) Any X . The singletons form a basis.

We now get a topology out of a basis:

Defn: If B is a basis for a topology on X , the topology \mathcal{T} generated by B is described as follows:

A subset U of X is said to be open if for every $x \in U$, there exists $B \in \mathcal{B}$ s.t. $x \in B \subset U$. (Topology generated)

$$x \in B \subset U.$$

(By "open" in above, we mean element of \mathcal{T} . Same thing for what we see in the proof below.)

EXAMPLES (1) & (2) \rightarrow gives standard topology on \mathbb{R}^2
 (3) \rightarrow gives discrete topology on X .

We still have to show that it is topology.

Proof:

- $\emptyset \in \mathcal{T}$ vacuously
 $X \in \mathcal{T}$ since given any $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
 $B \subset X$ is by definition
- Let $\{U_\alpha\}_{\alpha \in A}$ be open. Let $U := \bigcup_{\alpha} U_\alpha$.
 Fix $\alpha_0 \in A$.
 Let $x \in U$ be arbitrary. Then, $x \in U_{\alpha_0} \leftarrow$ open
 $\therefore \exists B \in \mathcal{B}$ s.t. $x \in B \subset U_{\alpha_0} \subset U$.
 $\therefore U \in \mathcal{T}$.
- Let U_1 and U_2 be open. Put $U := U_1 \cap U_2$.
 Let $x \in U$.
 Then $x \in U_1$ and $x \in U_2$
 \downarrow
 $\exists B_1 \in \mathcal{B}$ \uparrow
 $\exists B_2 \in \mathcal{B}$
 s.t. $x \in B_1 \subset U_1$ s.t. $x \in B_2 \subset U_2$
 $\therefore x \in B_1 \cap B_2 \subset U_1 \cap U_2$
 \downarrow
 $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U$.
 $\Rightarrow U \in \mathcal{T}$

By induction, any finite intersection is in \mathcal{T} . \square

$$\bigcap_{i=1}^n U_i = U_n \cap \left(\bigcap_{i=1}^{n-1} U_i \right).$$

Lecture 2 (11-01-2021)

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Lemma 1. Let B be a basis and \mathcal{T} the topology generated by B . Then, \mathcal{T} is the collection of all unions of elements of B .

Note that \emptyset is empty union.

Proof. Given $\{U_n\} \subset B$, it is clear that $\bigcup U_n \in \mathcal{T}$ since \mathcal{T} is a topology and U_n are open. (By defn.)

Conversely, let $V \in \mathcal{T}$. Given any $x \in V$, $\exists B_x \in B$ st. $x \in B_x \subset V$. (By defn of \mathcal{T} .)

Thus,

$$\bigcup_{x \in V} B_x = V.$$

(\subseteq) since $B_x \subset V$

(\supseteq) Each $x \in V$ is in B_x .

(Note that if $V = \emptyset$, the last union is the empty union!)

[The above gives us a way of extracting a basis B if we are already given a topology \mathcal{T} . Namely, pick any subcollection $B \subset \mathcal{T}$ such that \mathcal{T} is precisely the collection of all unions of elements of B .]

Lemma 2. Let B and B' be bases for the topologies \mathcal{T} and \mathcal{T}' , respectively, on X . TFAE:

- \mathcal{T}' is finer than \mathcal{T} . (recall this means $\mathcal{T} \subset \mathcal{T}'$)
- for each $x \in X$ and each basis element $B \in B$ containing x , $\exists B' \in B'$ s.t. $x \in B' \subset B$.

Proof. (i) \Rightarrow (ii) Let $x \in X$ and $B \in B$ be arbitrary.

Note that B is open in (X, \mathcal{T}) , i.e., $B \in \mathcal{T}$.
Then $B \subset \mathcal{T}'$. (by (i))

Note that B is open in (X, \mathcal{T}) , i.e., $B \in \mathcal{T}$.
 Thus, $B \in \mathcal{T}'$. (by (i))
 Since B is open in \mathcal{T}' , $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.
(Defⁿ of top. generated.)

(ii) \Rightarrow (i) Suppose $U \in \mathcal{T}$. We show that $U \in \mathcal{T}'$.
 Let $x \in U$. By defⁿ of \mathcal{T} , $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
 By (ii), $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B \subset U$.

Since x was arbit., we see that $U \in \mathcal{T}'$. (By defⁿ of \mathcal{T}')
 Thus, $\mathcal{T} \subset \mathcal{T}'$. □

Lemma 3. Let X be a topological space. Suppose \mathcal{C} is a collection of **open sets** of X s.t. for each open set $U \subset X$ and each $x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.

Then \mathcal{C} is a basis for the topology.

Prof. • Showing \mathcal{C} is a basis.

(i) Given any $x \in X$, X is an open set containing x .
 Thus, by hypothesis, $\exists C \in \mathcal{C}$ s.t. $x \in C$.

(ii) Let $C_1, C_2 \in \mathcal{C}$ s.t. $x \in C_1 \cap C_2$.

Note that C_1, C_2 are open and hence, $C_1 \cap C_2$ is open.

By hypothesis, $\exists C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$.

Thus, \mathcal{C} satisfies both properties of a topology.

• \mathcal{C} generates the topology.

Let \mathcal{T} denote the topology of X . Let \mathcal{T}' be the topology generated by \mathcal{C} .

Let $U \in \mathcal{T}'$, then U is some union of elements of \mathcal{C} .

but elements of \mathcal{C} are elements of \mathcal{T} and thus, $U \in \mathcal{T}$.
(\mathcal{T} is topo)

Thus, $\mathcal{T}' \subseteq \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. for each $x \in U$, $\exists C_x \in \mathcal{C}$ s.t.
 $x \in C_x \subset U$.

As earlier,

$$U = \bigcup_{x \in U} C_x \in \mathcal{T}'$$

Thus, $\mathcal{T} \subseteq \mathcal{T}'$. 3

Def. Let \mathcal{B} be the collection of all bounded intervals.

That is,

$$\mathcal{B} = \{(a, b) : -\infty < a < b < \infty\}.$$

\mathcal{B} is a basis and the topology generated by \mathcal{B} is called the standard topology on \mathbb{R} . (Standard topology on \mathbb{R})

If \mathcal{B}' is the collection of all half open intervals of the form $[a, b)$, then \mathcal{B}' is also a basis and the topology generated by \mathcal{B}' is called the lower limit topology on \mathbb{R} . (Lower limit topology on \mathbb{R})

Lemma 4. The lower limit topology is strictly finer than the standard topology.

Proof. Let \mathcal{T} denote the standard topology and \mathcal{T}' the lower limit.

- $\mathcal{T} \subseteq \mathcal{T}'$. Let (a, b) be an arbit. basis element and let $x \in (a, b)$. Then, $[x, b)$ is a basis element for \mathcal{T}' & $x \in [x, b) \subset (a, b)$.

Thus, $T \subseteq T'$, by Lemma 2.

$\cdot T' \neq T$. Note that $[0, 1] \in T'$.

but given $0 \in [0, 1]$, there is no $(a, b) \ni 0$
s.t. $(a, b) \subset [0, 1]$. \blacksquare

Def'n: A subbasis S for a topology is a collection of subsets of X whose union is X .
(Subbasis, sub basis)

(Note that no topology given so far. Similar to what we saw for basis.)

The topology generated by the subbasis S is defined to be the collection of all unions of finite intersections of elements of S .

We need to show that the topology defined above is actually a basis.

Let B be the collection of finite intersections of elements of S . We show B is a basis. (This suffices. Why? Lemma 1.)

(i) Let $x \in X$. Then, $\exists S \in S$ s.t. $x \in S$. ($\because \bigcup_{S \in S} S = X$)
But $S \in B$.

(ii) Let $B_1, B_2 \in B$ and $x \in B_1 \cap B_2$.
But note that $B_1 \cap B_2 \in B$. (Why?)

Thus, both the conditions are satisfied.

Remark: The standard topology of \mathbb{R} is also called the order topology on \mathbb{R} , because of the order relation of \mathbb{R} .

(We will see this in general, later.)

Defn. Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection \mathcal{B} of all sets of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open. (Product topology)

(Open in the respective topologies, i.e.)

Note that \mathcal{B} is a basis because:

- (i) $X \times Y$ is itself a basis element
- (ii) $U \times V, U' \times V' \in \mathcal{B} \Rightarrow (U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V') \in \mathcal{B}$
↓ ↓
intersection of
open sets

Note \mathcal{B} itself won't be the topology. (In general.)

Thm 5. If \mathcal{B} is a basis for a topology \mathcal{T}_x on X , and \mathcal{C} for \mathcal{T}_y on Y , then the collection

$$\mathcal{D} = \{ B \times C : B \in \mathcal{B}, C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We check that the hypotheses of Lemma 3 are satisfied.

Let $W \subseteq X \times Y$ be open and $(x, y) \in W$.

Then, by defⁿ of prod. top., $\exists U \in \mathcal{T}_x, V \in \mathcal{T}_y$ s.t.

$$(x, y) \in U \times V \subset W.$$

Since \mathcal{B} is a basis for \mathcal{T}_x , $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

11) $\exists c \in C$ s.t. $y \in c \subset V$.

$\Rightarrow (x, y) \in B \times C \subset U \times V \subset W$. 



Lecture 3 (14-01-2021)

14 January 2021 15:28

By last lecture's discussion, we know that

$$\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$$

is a basis for the product topology on \mathbb{R}^2 .
This is called the standard topology on \mathbb{R}^2 .

Defn. Given any two sets X and Y , we have the two projection maps $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ given as

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \\ \forall (x, y) \in X \times Y.$$

(Projections)

Note that $\pi_1^{-1}(U) = U \times Y$ for any $U \subseteq X$ and similarly $\pi_2^{-1}(V) = X \times V$ for any $V \subseteq Y$.

Thm. The collection

$$S = \{\pi_1^{-1}(U) \mid U \subseteq X \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \subseteq Y \text{ open}\} \subseteq P(X \times Y)$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let T_p denote the product topology on $X \times Y$.
Let T_S ——— topology generated by S .

Note that any element of S is of the form $U \times Y$ or $X \times V$.

$$\begin{matrix} U \subseteq X \\ \text{open} \end{matrix} \quad \begin{matrix} V \subseteq Y \\ \text{open} \end{matrix}$$

Thus, $S \subseteq T_p$ since both the above are actually basis elements. Since T_p is a topology, it is closed under arbitrary unions of finite intersections. Thus, $T_S \subseteq T_p$.

On the other hand, consider any arbitrary basis elt. of \mathcal{T}_p .
 It is of the form $U \times V$. $U \subseteq X, V \subseteq Y$ open.
 Note now

$$U \times V = \underbrace{\pi_1^{-1}(U)}_{\in \mathcal{T}_s} \cap \underbrace{\pi_2^{-1}(V)}_{\in \mathcal{T}_s} \in \mathcal{T}_s$$

Thus, $U \times V \in \mathcal{T}_s$. Since \mathcal{T}_s is a topology,
 arbitrary union of basis elements of \mathcal{T}_p is in \mathcal{T}_s .
 Thus, $\mathcal{T}_p \subseteq \mathcal{T}_s$. \square

Defn. Let (X, \mathcal{T}) be a topological space and $Y \subseteq X$. Then,
 the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y , called the **subspace topology**.

With this topology, (Y, \mathcal{T}_Y) is called a **subspace** of (X, \mathcal{T}) .

(Subspace topology)

(We will often just say "Y is a subspace of X" if it is clear.)

We now check that \mathcal{T}_Y is actually a topology.

Check. (i) Since $\emptyset \in \mathcal{T}$, we get $\emptyset = \emptyset \cap Y \in \mathcal{T}_Y$.

$$\text{If } Y = Y \cap X \in \mathcal{T}_Y.$$

(ii,iii) Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_Y$. Then, we have $\{U'_i\}_{i \in I} \subseteq \mathcal{T}$ s.t.

$$U'_i \cap Y = U_i \quad \forall i \in I.$$

Then,

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} (U'_i \cap Y) = \left(\bigcup_{i \in I} U'_i \right) \cap Y$$

and similarly,

$$\bigcap_{i \in I} U_i = \left(\bigcap_{i \in I} U'_i \right) \cap Y. \quad \square$$

and similarly,

$$\bigcup_{i \in I} U_i = \left(\bigcup_{i \in I} U_i \right) \cap Y.$$

□
if $|I| < \infty$

Lemma 2. If \mathcal{B} is a basis for (X, \mathcal{T}) , then the collection

$$\mathcal{B}_Y = \{ B \cap Y : B \in \mathcal{B} \}$$

is a basis for (Y, \mathcal{T}_Y) , i.e., the subspace topology on Y .

Proof We use Lemma 3 from Lecture 2.

Using that, it suffices to show that for any $U \in \mathcal{T}_Y$ and any $x \in U$, $\exists B \in \mathcal{B}_Y$ s.t. $x \in B \subset U$.

To this end, let u, U be as given. Then,
 $U = U' \cap Y$ for some $U' \in \mathcal{T}$.

Clearly, $x \in U'$.

Then, $\exists B' \in \mathcal{B}$ s.t. $x \in B' \subset U'$. (\mathcal{B} is a basis for \mathcal{T} .)

Then, $B = B' \cap Y \in \mathcal{B}_Y$ and
 $x \in B \subset U$. □

Lemma 3. Let Y be a subspace of X . If U is open in Y and Y is open in X , then U is open in X .

Proof. $U = U' \cap Y$ for some $U' \subset X$ open.

Since U' and Y are open in X , so is $U = U' \cap Y$. (Finite intersection of open sets.)

EXAMPLES. (i) If $Y = [0, 1] \subset X = \mathbb{R}$ in subspace topology,
it has a basis given as
 $\{ (a, b) \cap Y : a < b \in \mathbb{R} \}$.

More explicitly, here we have

1 2 3 ... n

$$(a, b) \cap Y = \begin{cases} (a, b) & a \in Y \ni b \\ [a, b) & a \notin Y \ni b \\ (a, 1] & a \in Y \ni b \\ \emptyset \text{ or } Y & a \notin Y \ni b \end{cases}$$

(2) Consider $Y = [0, 1] \cup \{2\} \subset \mathbb{R}$.

Note that

$$\{2\} = (1.5, 2.5) \cap Y.$$

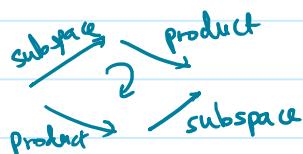
Thus, $\{2\}$ is open in Y . (Was not open in \mathbb{R} !)

Similarly, $[0, 1]$ is open in Y but not \mathbb{R} .

Thm 4. If A is a subspace of X and B of Y , then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Note that the above tells us that the ^{following} two ways of topologising $A \times B$ are the same:

- consider A and B as spaces by themselves and give $A \times B$ the product topology
- consider the topological space $X \times Y$ in product topology.
Note that $A \times B$ is a subset of $X \times Y$ and hence, can be given the subspace topology.



Proof.

Note the following:

typical basis
elt of $X \times Y$



$$\{(U \times V) \cap (A \times B) : U \subseteq X, V \subseteq Y \text{ open}\}$$

basis for subspace topology on $A \times B$
by Lemma 2

$$= \{ (U \cap A) \times (V \cap B) : U \subseteq X, V \subseteq Y \text{ open} \}$$

↓ ↓
 a general open set
 in the subspace
 topology of $A \subseteq X$
 or $B \subseteq Y$

basis for prod top.
 on $A \times B$

Thus, both the topologies have a common basis.

Dfⁿ: A subset of a topological space is said to be **closed** if its complement is open.

(Closed set)

Example: (1) $[a, b] \subseteq \mathbb{R}$ is closed because

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty) \text{ is open.}$$

(2) $[0, \infty) \times [0, \infty) \subseteq \mathbb{R}^2$ is closed because

$$\mathbb{R}^2 \setminus [0, \infty)^2 = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0)) \text{ is open.}$$

(3) In the discrete topology, every set is open and hence, every set is closed.

(4) Consider $Y = [-1, 0] \cup (2, 3) \subseteq \mathbb{R}$.

Both $[-1, 0]$ and $(2, 3)$ are open in Y .

$$\begin{matrix} " \\ (-2, 1) \cap Y \end{matrix}$$

Since they are complements of each other (in Y), we have that both the sets are closed as well, in Y .

Thms: Let X be a topological space. Then,

- (i) \emptyset and X are closed,
- (ii) arbitrary intersection of closed sets is closed,
- (iii) finite union of closed sets is closed.

Proof: $X \setminus \emptyset = X, X \setminus X = \emptyset$.

$$X \setminus \left(\bigcap_{i \in I} C_i \right) = \bigcup_{i \in I} (X \setminus C_i).$$

$$X \setminus \left(\bigcup_{i=1}^n C_i \right) = \bigcap_{i \in I} (X \setminus C_i).$$

Conclude.

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Remark. The above is also a way to define a topology.

Thm 6. Let Y be a subspace of X and $A \subseteq Y$.

Then, A is closed in Y iff A equals the intersection of a closed set (in X) with Y .

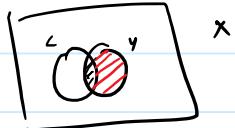
Proof. (\Leftarrow) Suppose $A = C \cap Y$ for some closed set $C \subseteq X$.

Then,

$$Y \setminus A = (X \setminus C) \cap Y$$

open in X

open in Y



$\therefore A$ is closed in Y .

(\Rightarrow) Suppose A is closed in Y .

Then, $Y \setminus A$ is open in Y . Thus, $\exists U \subseteq X$ open s.t.

$$Y \setminus A = U \cap Y$$

$$\begin{aligned} \text{Then, } Y \setminus (Y \setminus A) &= Y \setminus (U \cap Y) \\ &\stackrel{A}{=} Y \cap (U \cap Y)^c \\ &= Y \cap (U^c \cup Y^c) \\ &= (Y \cap U^c) \cap (Y \cup Y^c) \end{aligned}$$

$$\Rightarrow A = Y \cap U^c$$

The $(C)^c$ is
complement
in X .

Since $U^c \subseteq X$ is closed, we are done.