Lie Groups and Lie Algebras

Aryaman Maithani

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Contents

Introduction

Notes I'm making for my Lie groups course.

§1. Prerequisite definitions, notations, and conventions

The words "smooth" and "differentiable" will be used interchangeably, referring to the object being $\mathcal{C}^\infty$. We assume that one has the notion of what this means when talking about maps between open subsets of $\mathbb{R}^n$. We recall some notions of manifolds, mainly for the purpose of setting up the notation.
The definition of a submanifold may be different than usual (see Remark 1.4).

Let M be a topological space. A chart c on M is a triple (U, φ, n) where

- $n \in \{0, 1, 2, \ldots\}$,
- $U \subseteq M$ is open,
- $\varphi \colon U \to \mathbb{R}^n$ is a continuous function that is a homeomorphism onto its image.

U is the domain of the chart, and n the dimension.

The charts (U, φ, n) and (V, ψ, m) are compatible if either

- $U \cap V = \varnothing$, or
- $U \cap V \neq \varnothing$ and $\psi \circ \left(\varphi^{-1}|_{\varphi(U \cap V)}\right)$ is a diffeomorphism from $\varphi(U \cap V)$ onto $\psi(U \cap V)$.

The last condition would imply $m = n$.

An atlas on $\mathcal{M}$ is a collection of compatible of charts such that the domains cover M. This leads to the notion of a maximal (or saturated) atlas, which always exists.

Definition 1.1. A differentiable (or smooth) manifold is a tuple $(M, \mathcal{A})$, where M is a Hausdorff topological space and $\mathcal{A}$ is a saturated atlas on M.

By our definition, $\mathbb{R}$ with the discrete topology is a manifold which is zero-dimensional.

As noted earlier, if $x \in M$, then any chart containing x has the same dimension. We denote this dimension by $\dim_x(M)$. One checks that the map $x \mapsto \dim_x(M)$ is locally constant.

If M is a smooth manifold, then every $x \in M$ has a connected neighbourhood. Thus, the connected components of M are open. On the other hand, the connected components of any topological space are closed. Thus, we get:

Theorem 1.2. Let M be a smooth manifold. Then, the connected components of M are both open and closed.

We only needed a topological manifold for the above.

Let M, N be smooth manifolds, and $f\colon M \to N$ a function. Suppose that for all charts (V, ψ, m) on N and (U, φ, n) on M such that $f(U) \subseteq V$ we have that the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f|_U} V \xrightarrow{\psi} \psi(V)$$

is a smooth map. Then, f is said to be smooth.

This gives us the category of (smooth) manifolds.

Definition 1.3. Suppose M is a manifold, and $N \subseteq M$ a subset. N is a submanifold of M if N has the following property: for every $x \in N$, there exists a chart (U, φ, n) with $x \in U$ such that $\varphi(U \cap N) = \varphi(U) \cap \mathbb{R}^\ell$.

Remark 1.4. In the above, we assume $\mathbb{R}^\ell \subseteq R^n$ in the natural canonical way.
For a chart as above, we get a chart $(U \cap N, \varphi|_{U\cap N}, \ell)$ on N. The collection of all such charts give us an atlas on N.

This is a strong definition: this implies that N is locally closed in M.

The above definition implies that the natural inclusion $\iota\colon N \to M$ is smooth since this locally looks like the natural inclusion $\mathbb{R}^\ell \hookrightarrow \mathbb{R}^n$. We then have the following criterion then to check smoothness of maps to submanifolds:

Theorem 1.5. Let $\iota\colon N \hookrightarrow M$ be as above, P a smooth manifold, and $f\colon P \to N$ an arbitrary function.
Let $\widetilde{f}\colon P \to M$ be the composition $P \to N \hookrightarrow M$. Then,

$$f \text{ is smooth} \Leftrightarrow \widetilde{f} \text{ is smooth.}$$

§2. Lie groups: Definition and observations

We start with the basic definition.

§§2.1. Definition

Definition 2.1. A Lie group is a smooth manifold G equipped with smooth functions $m\colon G \times G \to G$ and $\iota\colon G \to G$ such that (G, m, ι) is a group.

A morphism of Lie groups is a smooth group homomorphism.

This gives us the category **LieGrp**.

Recall that (G, m, ι) being a group means the obvious thing: m is associative, unital, and every element has an inverse; the map ι is the inversion map $g \mapsto g^{-1}$.

We will often denote the identity of G by 1 or e. We may include a subscript if necessary.

Example 2.2. $\mathrm{GL}(n, \mathbb{R})$ is a Lie group: this has a natural smooth structure since this is an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The usual multiplication map is a polynomial in the coordinates and hence smooth. Inversion is smooth as well, being a rational function.

Remark 2.3. We will see (Corollary 2.14) that one need not require ι to be smooth.

We also have the following full subcategories of **LieGrp**:

- $\mathcal{C}$**LieGrp**, the category of *connected* Lie groups;
- $\mathcal{SC}$**LieGrp**, the category of *simply-connected* Lie groups.

§§2.2. Natural maps

Given $g \in G$, we get the following natural maps on G:

- The *left translation* $\gamma(g)\colon G \to G$ defined by $h \mapsto gh$.
- The *conjugation* $\mathrm{Int}(g)\colon G \to G$ defined by $h \mapsto ghg^{-1}$.

Both of these maps are diffeomorphisms; the second is also a (iso)morphism of Lie groups.

§§2.3. Basic properties of Lie groups

Theorem 2.4. If G is a group, then the function $g \mapsto \dim_g(G)$ is constant. In other words, a Lie group has the same dimension at every point.

Definition 2.5. The dimension of a Lie group is defined as $\dim_{1_G}(G)$.

Definition 2.6. The connected component of 1 is denoted by G_0; this is a Lie group.

The smooth structure on G_0 follows from Theorem 1.2. That this is a subgroup follows from the fact that left multiplication by an element is continuous, as is inversion. Thus, these maps preserve components. By the same reasoning, we get the following:

Theorem 2.7. G_0 is an open, normal subgroup of G.

If $\varphi \colon G \to H$ is a morphism of Lie groups, then $\varphi(G_0) \subseteq H_0$.

Thus, we get a functor

$$\mathbf{LieGrp} \xrightarrow{(-)_0} \mathcal{C}\mathbf{LieGrp}.$$

We also have the inclusion/forgetful functor

$$\mathcal{C}\mathbf{LieGrp} \xrightarrow{F} \mathbf{LieGrp}.$$

If G is a connected Lie group, and H an arbitrary Lie group, then we have a natural equality of sets

$$\mathrm{Hom}_{\mathbf{LieGrp}}(F(G), H) = \mathrm{Hom}_{\mathcal{C}\mathbf{LieGrp}}(G, H_0)$$

since $G = G_0$.

The above tells us that

Theorem 2.8. The forgetful functor and the connected-component functors are adjoints.

Definition 2.9. A neighbourhood U of 1 is said to be symmetric if $\iota(U) \subseteq U$.

Every neighbourhood U contains a symmetric neighbourhood: for example, $U \cap U^{-1}$.

§§2.4. Differentials of some maps

We can consider the differential of $m \colon G \times G \to G$. Under the natural identification, this gives us a map

$$T_{(1,1)}(m) \colon T_1(G) \oplus T_1(G) \to T_1(G).$$

One might think that this is a useful map giving information about the Lie group. Alas,

Theorem 2.10 (The differential of the multiplication map)**.** For $X, Y \in T_1(G)$, one has

$$T_{(1,1)}(m)(X, Y) = X + Y.$$

In particular, the above is the "same" map for any Lie group.

Sketch. Consider the maps $\iota_1, \iota_2 \colon M \rightrightarrows M \times M$ defined by $m \mapsto (m, 1)$ and $m \mapsto (1, m)$.

Then, $T_1(\iota_*)$ is the corresponding inclusion map $T_1(M) \to T_1(M) \oplus T_1(M)$. We also have $m \circ i_* = \mathrm{id}_M$.
Using the chain rule gives us

$$T_{(1,1)}(X, 0) = X \quad \text{and} \quad T_{(1,1)}(0, Y) = Y. \qquad \square$$

Next, let us compute some explicit differentials of matrix maps. Note that $GL(n, F)$ is an open subset of the vector space $M_n(F)$ and thus, its tangent space at identity can be identified with $M_n(F)$.
In particular, $T_1(F^\times) = F$.

Theorem 2.11. The differential of $\det\colon GL(n, \mathbb{C}) \to \mathbb{C}^\times$ at identity is the trace map

$$\mathrm{trace}\colon M_n(\mathbb{C}) \to \mathbb{C}.$$

This has rank two over $\mathbb{R}$.

Sketch. Consider a tangent vector $T \in T_I(GL(n, \mathbb{C})) = M_n(\mathbb{C})$. Look at the line $s \mapsto I + sT$. We have

$$\begin{aligned}\det(I + sT) &= \det \begin{bmatrix} 1 + sT_{11} & sT_{12} & \cdots & sT_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ sT_{n1} & sT_{n2} & \cdots & 1 + sT_{nn} \end{bmatrix} \\ &= 1 + s(T_{11} + \cdots T_{nn}) + s^2(\cdots).\end{aligned} \qquad \square$$

Theorem 2.12 (Lie bracket on GL)**.** Consider $G := GL(n, \mathbb{R})$. For $A \in G$, we have the conjugation map $\mathrm{int}(A)\colon G \to G$. Let $\mathrm{Ad}(A) := T_I(\mathrm{int}(A))$ denote its differential at identity. Thus, $\mathrm{Ad}(A)$ is a linear map $M_n(\mathbb{R}) \to M_n(\mathbb{R})$.

For $X \in M_n(\mathbb{R})$, we have $\mathrm{Ad}(A)(X) = AXA^{-1}$.

Furthermore, Ad gives us a map $\mathrm{Ad}\colon G \to \mathrm{End}(M_n(\mathbb{R}))$. We may differentiate this at identity to get a linear map

$$T_I(\mathrm{Ad})\colon T_1(G) \to T_1(\mathrm{End}(M_n(\mathbb{R}))).$$

Under the usual identifications, this can be rewritten as a linear map

$$T_I(\mathrm{Ad})\colon M_n(\mathbb{R}) \to \mathrm{Hom}(M_n(\mathbb{R}), M_n(\mathbb{R})).$$

Currying/uncurrying this gives us a *bilinear* map

$$T_I(\mathrm{Ad})\colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}).$$

We have

$$T_I(\mathrm{Ad})(X, Y) = XY - YX$$

for $X, Y \in M_n(\mathbb{R})$.

Note that the above steps could be carried out for a general Lie group G to get a *bilinear* map $T_1(G) \times T_1(G) \to T_1(G)$. This will give us the Lie algebra structure on $T_1(G)$.

Sketch. First, let $A \in G$ and $X \in M_n(\mathbb{R})$ be arbitrary. For $s \in \mathbb{R}$, we note

$$\mathrm{int}(A)(I + sX) = A(I + sX)A^{-1} = I + s(AXA^{-1}).$$

This shows $T_I(\mathrm{int}(A))(X) = AXA^{-1}$.

Now, let $X, Y \in M_n(\mathbb{R})$ be arbitrary. For small s, the element $I + sX$ is invertible, and the above tells us

$$\mathrm{Ad}(I + sX)(Y) = (I + sX)Y(I + sX)^{-1}.$$

For small s, we may expand the above as

$$\begin{aligned}\mathrm{Ad}(I + sX)(Y) &= (I + sX)Y\left(I - sX + (sX)^2 - \cdots\right) \\ &= (Y + sXY)\left(I - sX + (sX)^2 - \cdots\right) \\ &= Y + sXY - sYX - s^2XYX + \text{higher order} \\ &= Y + s(XY - YX) + \text{higher}.\end{aligned}$$

□

§§2.5. Creating Lie groups

We see a criteria here that gives us that many natural groups are Lie groups. In particular, we will see that smoothness of the multiplication map implies that of the inversion map.

For this subsection, we have the following setup:

Let M be a "Lie monoid", i.e., a smooth manifold equipped with a smooth binary operator $m \colon M \times M \to M$ that is associative and unital.
Let G be the subset of regular elements, i.e.,

$$G := \{m \in M : m \text{ is invertible}\}.$$

Then, one has the following.

Theorem 2.13. G is an open subset of M. (Thus, G has a natural manifold structure.)

G is closed under multiplication. The inverse of an element of G is again an element of G.

The induced inversion map on G is smooth.

Thus, G is a Lie group.

We sketch a proof at the end of this subsection.

Corollary 2.14. Suppose G is a smooth manifold, and $m\colon G \times G \to G$ is a smooth binary operator that makes G into a group.
Then, G is a Lie group, i.e., the inversion map $g \to g^{-1}$ is smooth.

The following is another setup where the above can be applied:

Suppose A is a finite-dimensional associative $\mathbb{R}$-algebra with a unit 1. Using a basis, we can identify $A \cong \mathbb{R}^n$; this gives a (basis-independent) smooth structure on A. The multiplication on A is $\mathbb{R}$-bilinear, hence smooth. Thus, the regular elements form a Lie group G.

Note that since G is an open subset of A, we get that $\dim(G) = \dim_{\mathbb{R}}(A)$.

Example 2.15. For A, we may take $M_n(F)$ where F is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.
This gives us the corresponding Lie group $GL(n, \mathbb{F})$ with corresponding dimension.

Sketch of Theorem 2.13. Consider $\Phi\colon M \times M \to M$ defined by $(a, b) \mapsto (a, m(a, b))$.
Thus,

$$T_{(1,1)}(\Phi)(X, Y) = (X, X + Y).$$

(The calculation of $T_{(1,1)}(m)$ done earlier is valid even for Lie monoids.)

Note that the above is an automorphism of tangent spaces, using the inverse function theorem tells us that Φ is a diffeomorphism when restricted to open sets. By restrictions, we may assume that

$$\Phi|_U = \varphi\colon U \to V \times V$$

is a diffeomorphism where $U \subseteq M \times M$ is open, and $V \subseteq M$ is open.
Let ψ be its inverse. Check that ψ looks like

$$\psi(a, b) = (a, m(a, \beta(a, b)))$$

for some smooth β. But then, $m(a, \beta(a, 1)) = 1$ for all $a \in V$. This gives a neighbourhood of 1 where every element has a right inverse. By symmetry, there is one with left inverses. By intersection, there is a neighbourhood of 1 on which every element has an inverse and that this inversion map is smooth. By translation, G is open and inversion is smooth. □

§§2.6. Lie subgroups

Definition 2.16. A Lie subgroup of G is a subgroup of G that is a submanifold (Definition 1.3).

Remark 2.17. Note that by Theorem 1.5, the restricted multiplication map $H \times H \to H$ is smooth. In turn, Corollary 2.14 tells us that H is a Lie group with the induced structure.

As remarked in Remark 1.4, H is locally closed in G. The following tells us more.

Theorem 2.18. Lie subgroups are closed.

A deeper theorem of Cartan tells us that the purely topological converse is true:

Theorem 2.19 (Cartan)**.** Any closed subgroup of a Lie group is a Lie subgroup.

§3. Group actions

We now study properties of group actions. The interesting takeaway is how questions about group homomorphisms can be reduced to the general framework of group actions.

Definition 3.1. Let G be a Lie group, and M a smooth manifold. A (smooth) action of G on M is a smooth map $\mu\colon G \to M$ inducing a usual action on M.

"Usual action" means that $\mu(1, m) = m$ for all m and that the diagram

$$\begin{tikzcd} G \times G \times M \arrow[r, "\mathrm{id}_G \times \mu"] \arrow[d, "m \times \mathrm{id}_M"'] & G \times M \arrow[d, "\mu"] \\ G \times M \arrow[r, "\mu"] & M \end{tikzcd}$$

commutes.

§§3.1. Orbits

Given $m \in M$ and $g \in G$, we get the following maps

$$\begin{aligned} \omega_m\colon G &\to M \\ g &\mapsto g \cdot m, \end{aligned}$$

and

$$\begin{aligned} \tau(g)\colon M &\to M \\ m &\mapsto g \cdot m. \end{aligned}$$

Note that $\tau(g)$ is a diffeomorphism and we have

$$\omega_m \circ \gamma(g) = \tau(g) \circ \omega_m.$$

Taking differentials at 1 and noting diffeomorphisms, we get

$$\operatorname{rank}(T_g(\omega_m)) = \operatorname{rank}(T_1(\omega_m)).$$

Corollary 3.2. For a fixed m, the orbit map $\omega_m \colon G \to M$ has constant rank. Thus, ω_m is a subimmersion.

The rank may very well depend on m. E.g., consider the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. The orbit map of $\mathbf{0}$ is different from any other orbit map.

A subimmersion of rank r locally looks like

$$\mathbb{R}^d \ni (x_1, \ldots, x_r, x_{r+1} \ldots, x_d) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0) \in \mathbb{R}^n.$$

§§3.2. Stabilisers

Given $m \in M$, the stabiliser of m is the subgroup

$$\mathrm{Stab}_m \coloneqq \{g \in G : g \cdot m = m\}.$$

We may denote the above as S_m if the context is clear.

Theorem 3.3. The stabiliser Stab_m is a Lie subgroup of G with

$$T_1(\mathrm{Stab}_m) = \ker(T_1(\omega_m)).$$

Corollary 3.4. If $\varphi \colon G \to H$ is a Lie group morphism, then φ is of constant rank and $\ker(\varphi)$ is a Lie subgroup of G with

$$T_1(\ker(\varphi)) = \ker(T_1(\varphi)).$$

Sketch. G acts on M via $(g, h) \mapsto \varphi(g)h$. Then, $\varphi = \omega_{1_H}$ and $\ker(\varphi) = \mathrm{Stab}_{1_H}$. □

Corollary 3.5. If $\varphi \colon G \to H$ is a morphism of Lie groups, then the following are equivalent:

(a) φ is a local diffeomorphism.

(b) $T_1(\varphi)$ is an isomorphism.

(c) $\dim(G) = \dim(H)$ and $\ker(\varphi)$ is zero-dimensional.

Sketch. (a) $\Rightarrow$ (b) is OK.
(b) $\Rightarrow$ (a): constant rank implies $T_g(\varphi)$ is an isomorphism for all g.
(b) $\Leftrightarrow$ (c) follows by Corollary 3.4 and rank-nullity. □

Example 3.6. $SL(n, F)$ is a subgroup of $GL(n, F)$ for $F = \mathbb{R}, \mathbb{C}$ since this is the kernel of $\det\colon GL(n, F) \to F^\times$.

§4. Involutions on algebras

Let A be a finite-dimensional associative unital $\mathbb{R}$-algebra. As noted earlier, A is a canonically a smooth manifold with the multiplication being smooth, and G the subset of invertible elements is a open subset which is a Lie group.

Definition 4.1. An involution on A is a function $\tau\colon A \to A$ satisfying

- τ is $\mathbb{R}$-linear,
- $\tau(\tau(a)) = a$ for all $a \in A$,
- $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in A$.

We use the notation $a^\tau \coloneqq \tau(a)$.

Example 4.2. $A \coloneqq M_n(\mathbb{R})$ is an example of such an $\mathbb{R}$-algebra with the usual matrix multiplication.
The transpose operation is an involution.

Similarly, conjugate-transpose is an involution on $A \coloneqq M_n(\mathbb{C})$. Note that this *is* $\mathbb{R}$-linear (but not $\mathbb{C}$-linear).

Lemma 4.3. $\tau(1) = 1$.

Define

$$H \coloneqq \{g \in A : g \cdot g^\tau = 1 = g^\tau \cdot g\}.$$

Note that $H \subseteq G$.

Lemma 4.4. H is a subgroup of G.

Remark 4.5. Since A is a finite-dimensional $\mathbb{R}$-algebra, the equation $ab = 1$ implies $ba = 1$.

Theorem 4.6. H is a Lie subgroup of G with

$$T_1(H) = \{a \in A : a = -a^\tau\}.$$

The above gives us the dimension of H as well.

Example 4.7. The orthogonal matrix group

$$O(n) \coloneqq \{A \in M_n(\mathbb{R}) : AA^\mathsf{T} = I\}$$

is a Lie group of dimension $\frac{n(n-1)}{2}$.

The unitary matrix group

$$U(n) \coloneqq \{A \in M_n(\mathbb{C}) : AA^* = I\}$$

is a Lie group of dimension n^2.

Example 4.8. Considering the kernels of the det map on the above groups, we get $SO(n)$ and $SU(n)$ as Lie subgroups of $O(n)$ and $U(n)$ respectively. These are the subgroups of matrices of determinant 1, called the special orthogonal group and special unitary group respectively. (These are subgroups by Corollary 3.4.)

Note that $\det(O(n)) = \{\pm 1\}$. Thus, $SO(n)$ is an open subset of $O(n)$. This gives us $\dim(SO(n)) = \dim(O(n))$.

Similarly, we have $\det\colon U(n) \to S^1$. By Corollary 3.4, we know that

$$\dim(SU(n)) = \dim(\ker(\det)) = \dim_\mathbb{R}(\ker(T_1(\det))) = \operatorname{nullity}(T_1(\det)).$$

Recall from Theorem 2.11 that $T_1(\det) = \operatorname{trace}$. Restricting this to $T_1(U(n)) = \{A \in M_n(\mathbb{C}) : A = -A^*\}$ is still a nonzero map. Thus, $\det\colon U(n) \to S^1$ is a rank-one map. This gives us

$$\dim(SU(n)) = \operatorname{nullity}(T_1(\det)) = \dim(U(n)) - 1.$$

§5. Quaternions and symplectic group

Recall that the algebra of quaternions $\mathbb{H}$ is a four-dimensional associative unital $\mathbb{R}$-algebra. The usual representation is $\mathbb{H} = \mathbb{R}(1 \oplus \imath \oplus \jmath \oplus k)$ with the "cross-product" relations between $\imath, \jmath, k$. The conjugation map on $\mathbb{H}$ is the linear map fixing 1 and negating the other three basis elements.

$\mathbb{H}$ can also be identified with the subalgebra A of $M_2(\mathbb{C})$ given by

$$A \coloneqq \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

The determinant of the matrix above is $|\alpha|^2 + |\beta|^2 = 1$. Moreover, the usual conjugate-transpose involution on $M_2(\mathbb{C})$ restricts to an involution on A. This corresponds to the conjugation on $\mathbb{H}$.

In turn, this leads to an involution on $M_n(\mathbb{H})$ given by conjugate-transpose. As in Example 4.7, we see that the symplectic group defined as

$$\mathrm{Sp}(n) \coloneqq \{A \in M_n(\mathbb{H}) : AA^* = I\}$$

is a compact Lie group of dimension $n(2n+1)$.

Note that $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

§6. Connectedness of some groups

Theorem 6.1. $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, and $\mathrm{U}(n)$ are connected Lie groups.

Note that $\mathrm{O}(n)$ is not connected.

Sketch for $\mathrm{SO}(n)$. By induction on n. The case $n = 1$ is trivial. For $n = 2$, we explicitly have

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

is connected. Assume $n \geqslant 3$.

The action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ restricts to a smooth action of $\mathrm{SO}(n)$ on S^{n-1}. Consider the north pole $p = (1, 0, \ldots, 0)^T \in S^{n-1}$. Note that Mp is the first column of M. This gives us that the differential of the orbit map ω_p is

$$\begin{aligned} T_1(\mathrm{SO}(n)) &\to T_1(S^{n-1}) \\ M &\mapsto \text{first column of } M. \end{aligned}$$

We know $T_1(\mathrm{SO}(n)) = \{\text{skew-symmetric matrices}\}$ and $T_1(S^{n-1}) \leqslant T_1(\mathbb{R}^n)$ consists of column vectors with first coordinate zero. This description tells us that $T_1(\omega_p)$ is surjective, hence a submersion, hence an open map (using the local description of a submersion).

Thus, $\omega_p(\mathrm{SO}(n))$ is an open subset of S^{n-1}. But this is also compact. Connectedness of S^{n-1} tells us that $\omega_p(\mathrm{SO}(n)) = S^{n-1}$.

Suppose $\mathrm{SO}(n)$ is not connected. Let G_0 be the component of I. The same arguments applied to G_0 tell us that $\omega_p(G_0) = S^{n-1}$ as well.

Check that

$$\mathrm{Stab}_p = \left\{ \begin{bmatrix} 1 & \\ & U \end{bmatrix} : U \in \mathrm{SO}(n-1) \right\}.$$

By induction, the above is connected and hence, $\mathrm{Stab}_p \leqslant G_0$.

Now, given $T \in \mathrm{SO}(n)$, the calculation earlier gives us $Tp = Sp$ for some $S \in G_0$. Thus, $S^{-1}T \in \mathrm{Stab}_p \subseteq G_0$. □

§7. A map from SU(2) to SO(3)

By earlier, we have $\dim(\mathrm{SU}(2)) = 2^2 - 1 = 3$ and $\dim(\mathrm{SO}(3)) = 3(3-1)/2 = 3$. Thus,

$$\dim(\mathrm{SU}(2)) = \dim(\mathrm{SO}(3)).$$

We will show that we have a two-fold covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

First, note that $\mathrm{SU}(2)$ is simply-connected since topologically $\mathrm{SU}(2) \cong S^3 \subseteq \mathbb{C}^2$ since

$$\mathrm{SU}(2) = \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \begin{array}{c} \alpha, \beta \in \mathbb{C}, \\ |\alpha|^2 + |\beta|^2 = 1 \end{array} \right\}.$$

Consider the linear map $T : \mathbb{R}^3 \to M_2(\mathbb{C})$ given by

$$(x, y, z) \mapsto \begin{bmatrix} x & y + \iota z \\ y - \iota z & -x \end{bmatrix}.$$

Let $\mathcal{H} := \mathrm{im}(T)$. Check that

- $\dim(\mathcal{H}) = 3$.
- $\mathcal{H}$ is precisely the space of traceless self-adjoint matrices.
- $\det(T(x, y, z)) = -(x^2 + y^2 + z^2) = -\|(x, y, z)\|^2$.

Thus, $(\mathcal{H}, -\det)$ is isomorphic to $(\mathbb{R}^3, \|\cdot\|)$ as normed spaces.

By the description of $\mathcal{H}$, we see that $\mathrm{SU}(2)$ acts on $\mathcal{H}$ by

$$(T, H) \mapsto THT^*.$$

Moreover, $-\det(THT^*) = -\det(H)$. Thus, $\mathrm{SU}(2)$ acts via norm-preserving isomorphisms, giving us a homomorphism $\mathrm{SU}(2) \to \mathrm{O}(\mathcal{H}) \cong \mathrm{O}(\mathbb{R}^3)$. Since $\mathrm{SU}(2)$ is connected, this map has image inside $\mathrm{SO}(3)$.

One can calculate the map to get that the homomorphism $\rho : SU(2) \to SO(3)$ is

$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \mapsto \begin{bmatrix} |\alpha|^2 - |\beta|^2 & 2\mathfrak{R}(\alpha\overline{\beta}) & -2\mathfrak{I}(\alpha\overline{\beta}) \\ -2\mathfrak{R}(\alpha\beta) & \mathfrak{R}(\alpha^2 - \beta^2) & -\mathfrak{I}(\alpha^2 + \beta^2) \\ -2\mathfrak{I}(\alpha\beta) & \mathfrak{I}(\alpha^2 - \beta^2) & \mathfrak{R}(\alpha^2 + \beta^2) \end{bmatrix}.$$

The above gives us that $\ker(\rho) = \{\pm I\}$. By Corollary 3.5, ρ is a local diffeomorphism. We will see later Lemma 9.1 that this implies that ρ is a covering map. We already checked that $SU(2)$ is simply-connected. Thus, we have constructed the universal cover of $SO(3)$. It would also follow by Theorem 9.5 that $\pi_1(SO(3), I) \cong \mathbb{Z}/2$.

§8. An open mapping theorem

Question. Suppose G acts transitively on M, and $m \in M$. Is the orbit map $\omega_m : G \to M$ a submersion?
Answer. **No!** Consider $G \coloneqq \mathbb{R}_{disc}$ and $M \coloneqq \mathbb{R}$. By our definitions, G is indeed a (zero-dimensional) Lie group.
We have the transitive action of G on M given by $(x, y) \mapsto x + y$. The orbit map of 0 is the "identity" map $\mathbb{R}_{disc} \to \mathbb{R}$. However, this map cannot be a submersion since the dimensions are incorrect.
We will see that under mild assumptions Theorem 8.6 such a thing cannot happen.

For this section, just a topological action (i.e., a continuous action) is sufficient to prove the theorems.

Definition 8.1. A topological space is said to be locally compact if it is Hausdorff and every point has a compact neighbourhood.

Definition 8.2. A locally compact space is countable at ∞ if it is the union of countably many compact sets.

$\mathbb{R}_{disc}$ is locally compact but not countable at ∞.
The Euclidean space $\mathbb{R}^n$ is countable at ∞.

Lemma 8.3. Let G be a connected topological group and U a neighbourhood of 1. Then,

$$G = \bigcup_{n \geqslant 1} U^n,$$

where U^n is the set of all products of n elements of U.

In particular, any neighbourhood of the identity of a connected Lie group generates the group.

Sketch. We may assume that U is open and symmetric, in which case the union is an open subgroup H. But G is then the disjoint union of the cosets of H. Since each coset is also open, we must have $H = G$. □

Corollary 8.4. Let G be a connected topological group. If G is locally compact, then G is countable at ∞.

Lemma 8.5. Let G be a Lie group. The following are equivalent:

(a) G is countable at ∞.

(b) G has countably many components.

In the above, we only need that G is a *topological* manifold (with continuous multiplication and inversion).

Theorem 8.6. Let G be a locally compact topological group with countably many components. Let X be a locally compact topological space. Suppose G acts on X continuously and that this is action is transitive. Let $x \in X$. The orbit map

$$\begin{aligned} \omega_x : G &\to X \\ g &\mapsto gx \end{aligned}$$

is open.

In the case that everything is smooth, the map ω_x is an open submersion.

Note that in the smooth case, we already knew that ω_x is a subimmersion. Thus, being open gives us that it is an immersion. (Recalling the local description of a subimmersion.)

Sketch. Let $U \subseteq G$ be an open neighbourhood of 1. Pick a compact symmetric neighbourhood $1 \in V \subseteq U$ with $V^2 \subseteq U$. Since G is countable at ∞, there is a countable set $\{g_n \in G : n\}$ such that $(g_n \cdot \mathrm{int}(V))_n$ is an open cover of G. Hence, $(g_n \cdot V)_n$ is a compact cover of G. We have that $U_n := X \setminus \omega_x(g_n \cdot V)$ is open in X with $\bigcap_n U_n = \varnothing$. By the Baire category theorem, some U_n is not dense in X. Thus, $\omega_x(g_n \cdot V) = g_n \cdot V \cdot x$ has nonempty interior. Thus, $V \cdot x$ is a neighbourhood of $g \cdot x$ for some g. But then $U \cdot x$ is a neighbourhood of x. □

§9. Covering maps

For a smooth map $p\colon X \to Y$ of smooth connected manifolds, we say that p is a covering map if p is surjective and given any $y \in Y$, there exists a connected open neighbourhood U of y such that $p^{-1}(U)$ is a disjoint neighbourhood of components such that p restricted

to any component is a diffeomorphism onto U.
In this case, U is said to be an evenly covered neighbourhood.

Lemma 9.1. Assume G and H are *connected* Lie groups, and $p\colon G \to H$ is a Lie group morphism. The following are equivalent:

(a) p is a covering map.

(b) $T_1(p)$ is a linear isomorphism.

(c) p is a local diffeomorphism.

In such a case, $D \coloneqq \ker(p)$ is a discrete subgroup contained in the center of G.

Sketch. (b) $\Leftrightarrow$ (c) was Corollary 3.5.

(c) $\Rightarrow$ (a): $p(G)$ is open and hence, generates H. But $p(G)$ is already a subgroup. Thus, $p(G) = H$ and hence, p is surjective. Let V be a neighbourhood of 1_G on which p is bijective. We must have that $D \coloneqq \ker(p)$ does not intersect V. In turn, D is discrete.
Given $g \in G$, the map $\operatorname{int}(g)$ lands inside D by normality. A continuous map $G \to D$ must be constant and hence, D is central.
Now, let $W \subseteq V$ be a small enough open neighbourhood of 1_G such that $p|_W$ is a diffeomorphism, W is connected, $W = W^{-1}$, $W^2 \subseteq V$. Then $p(W)$ is evenly covered since

$$p^{-1}(p(W)) = \bigsqcup_{d \in D} dW. \qquad \square$$

§§9.1. Universal cover

Recall that given any connected smooth manifold with basepoint (X, x_0), there exists a connected manifold $(\widetilde{X}, \widetilde{x}_0)$ with a covering map $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that any other covering map $q\colon (Y, y_0) \to (X, x_0)$ factors uniquely as

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{x}_0) \arrow[rr, dashed, "\exists! r"] \arrow[dr, "p"'] & & (Y, y_0) \arrow[dl, "q"] \\
& (X, x_0). &
\end{tikzcd}$$

This is called the universal cover of (X, x_0) and is unique up to diffeomorphism.

Now, if G is a connected Lie group, then there exists a universal covering space $p\colon (\widetilde{G}, \widetilde{1}) \to (G, 1)$ using the fact that G has a smooth structure.
To emphasise, $\widetilde{G}$ is only a smooth manifold with a distinguished basepoint $\widetilde{1}$, and p is only a smooth map.

Theorem 9.2. There exists a unique smooth binary operator $\widetilde{m}$ on $\widetilde{G}$ that makes p a multiplicative map. This map $\widetilde{m}$ then also makes $\widetilde{G}$ a group with $\widetilde{1}$ as the identity.

Thus, $\widetilde{G}$ is then called the universal covering group.

The above will follows from the LIFTING THEOREM: Suppose we have a covering $(Y, y_0) \xrightarrow{p} (X, x_0)$, a simply-connected space Z, and a smooth map $(Z, z_0) \xrightarrow{F} (X, x_0)$. Then, there is a unique map F' making the following commute:

$$\begin{tikzcd}
(Z, z_0) \arrow[r, dashed, "F'"] \arrow[dr, "F"'] & (Y, y_0) \arrow[d, "p"] \\
& (X, x_0)
\end{tikzcd}$$

Sketch of the theorem. To make p multiplicative is precisely asking for a dashed arrow in the following diagram to exist (such that the diagram commutes):

$$\begin{tikzcd}
(\widetilde{G} \times \widetilde{G}, (\widetilde{1}, \widetilde{1})) \arrow[r, dashed] \arrow[d, "p \times p"'] \arrow[dr, "m \circ (p \times p)"] & (\widetilde{G}, \widetilde{1}) \arrow[d, "p"] \\
(G \times G, (1, 1)) \arrow[r, "m"'] & (G, 1).
\end{tikzcd}$$

The LIFTING THEOREM gives the existence and uniqueness of such a map, which we call $\widetilde{m}$. Note that this maps satisfies $\widetilde{m}(\widetilde{1}, \widetilde{1}) = \widetilde{1}$.

It now remains to check that $\widetilde{m}$ as defined above makes $\widetilde{G}$ into a group with $\widetilde{1}$ as the identity. This follows from using the LIFTING THEOREM in different ways appropriately.

The point to note is the following: if Z is simply-connected and $f, g : (Z, z_0) \to (\widetilde{G}, \widetilde{1})$ are two maps such that $f \circ p = g \circ p$, then we must have $f = g$.

Thus, for example, checking associativity means that we want the diagram

$$\begin{tikzcd}
\widetilde{G} \times \widetilde{G} \times \widetilde{G} \arrow[r, "\widetilde{m} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \widetilde{m}"'] & \widetilde{G} \times \widetilde{G} \arrow[d, "\widetilde{m}"] \\
\widetilde{G} \times \widetilde{G} \arrow[r, "\widetilde{m}"'] & \widetilde{G}
\end{tikzcd}$$

to commute. But the diagram does commute after composing with p since we have associativity in G. We check that all the maps above do preserve the obvious basepoints. $\square$

Same techniques as earlier give us the following lifting theorem as well.

Theorem 9.3. Let G and H be connected Lie groups, and $\varphi \colon G \to H$ a morphism of Lie groups. Then, there is a unique smooth map $\widetilde{\varphi}$ that makes

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "p_G"'] & \widetilde{H} \arrow[d, "p_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

commute and satisfies $\widetilde{\varphi}(1_{\widetilde{G}}) = 1_{\widetilde{H}}$.
Moreover, this map is also a morphism of Lie groups.

Thus, we have a new functor

$$\mathcal{C}\mathbf{LieGrp} \xrightarrow{\widetilde{(-)}} \mathcal{S}\mathcal{C}\mathbf{LieGrp}.$$

In all, we have the functors

$$\mathcal{S}\mathcal{C}\mathbf{LieGrp} \underset{\widetilde{(-)}}{\rightleftarrows} \mathcal{C}\mathbf{LieGrp} \underset{(-)_0}{\rightleftarrows} \mathbf{LieGrp},$$

where the top functors are the natural inclusion/forgetful functors.

Theorem 9.4. All sensible possibilities of pairs of functors above are adjoints.

§§9.2. Deck transformations

Let $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering. A deck transformation is a diffeomorphism $T \colon \widetilde{X} \to \widetilde{X}$ such that

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "T"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes.

Note that T will not usually preserve the basepoint $\widetilde{x}_0$. However, it will permute the fiber $p^{-1}(x_0)$.

Usual theory of covering spaces tells us that

$$\pi_1(X, x_0) \cong \{\text{group of deck transformations}\}.$$

Back to our setup of Lie groups, we have:

$$p \colon (\widetilde{G}, \widetilde{1}) \to (G, 1).$$

We showed that $D := \ker(p) = p^{-1}(1)$ is central and discrete (Lemma 9.1).

For $g \in G$, recall the left translation map $\gamma(g) : G \to G$ defined by $h \mapsto gh$.

Theorem 9.5. We have an isomorphism

$$D \cong \{\text{group of deck transformations}\}$$

given by $d \mapsto \gamma(d)$.
Thus, $\pi_1(G, 1) \cong D$. In particular, this is abelian.

§10. Matrix Lie groups and properties

In this section, we compile the list of examples and their properties that we saw across various sections. We give references to the earlier sections where we verified the properties mentioned.

We mention which groups are compact. This is an easy check using Heine-Borel.

(a) $GL(n, \mathbb{R})$ is a Lie group of dimension n^2.

(b) $GL(n, \mathbb{C})$ is a Lie group of dimension $(2n)^2$.

(c) $GL(n, \mathbb{H})$ is a Lie group of dimension $(4n)^2$.
See Example 2.15 for the above three.

(d) $SL(n, \mathbb{R})$ is a Lie group.

(e) $SL(n, \mathbb{C})$ is a Lie group.
See Example 3.6 for the above two.

(f) $O(n)$ is a compact Lie group of dimension $n(n-1)/2$.

(g) $U(n)$ is a connected, compact Lie group of dimension n^2.
See Example 4.7 for the above two.

(h) $SO(n)$ is a connected, compact Lie group of dimension $n(n-1)/2$.

(i) $SU(n)$ is a connected, compact Lie group of dimension $n^2 - 1$.
See Example 4.8 for the above two. For connectedness, see Section 6.

(j) $Sp(n)$ is a connected, compact Lie group of dimension $n(2n+1)$. See Sections 5 and 6.