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MA 839

Advanced Commutative Algebra

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Lecture 0 (07-01-2021)

07 January 2021 08:08

A Quick Intro.

Setup : A ring is commutative with 1.

Let M be an R-module.

Observation : ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$), we get an R-linear map $R \twoheadrightarrow M$, $a \mapsto ax$ which is onto.

Then, $M \cong R/I$ where I is the kernel.

In this case, $I = \operatorname{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R.

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$.

$= \{ax + by \mid (a, b) \in R^{\oplus 2}\}$

Then, we get an onto R-linear map $\underbrace{Re_1 \oplus Re_2}_{\{e_1, e_2\} \text{ is a basis}} \xrightarrow{\varphi} M$

$e_1 \mapsto x$
$e_2 \mapsto y$ } extend this

this lets us extend the map

In particular, $M \cong R^2/\ker \varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle\ \rangle} \oplus \frac{R}{\langle\ \rangle} ?$$

This has a positive answer: ① R is a field
② R is a PID

CAUTION: We won't include fields as PID.
That is, when we say "PID", we exclude fields !!

③ Suppose M is a finitely generated (f.g.) R-module.

(That is, suppose $M = \langle x_1, \dots, x_n \rangle$.)

Then, M is a quotient of $R^{\oplus n}$.

way to get

way to get this

Then, M is a quotient of $R^{\oplus n}$.

Define $R^{\oplus n} \overset{\varphi}{\twoheadrightarrow} M$ by $e_i \longmapsto x_i$.

$M \simeq R/\ker\varphi$.

④ In general, consider a free module with "M as basis", call it $F(M)$. Then $F(M)$ maps onto M.

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$, then $F(A)$ maps onto M.

Thus, M can be written as a quotient of a free-module.

To Summarise: If M is an R-module, then M can be written as a quotient of a free R-module. Moreover, if M is f.g., then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R-module mapping onto M with kernel K. That is, $\varphi : F \twoheadrightarrow M$ is onto R-linear and $\ker \varphi = K$.

Now, $\exists$ a free R-module G and an onto map $\psi : G \to K$

We capture this in the following diagram

compose ψ with $K \hookrightarrow F$

$$\begin{tikzcd}
G \arrow[rr, dashed, "\psi"] \arrow[dr, "\psi"'] & & F \arrow[r, "\varphi"] & M \arrow[r] & 0 \\
& K \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

means φ is onto

means the map $K \to F$ is 1-1

Note that $\operatorname{im}\psi = K = \ker\varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \longrightarrow 0$.

① φ is onto and $\ker\varphi = \operatorname{im}\psi$.
② G and F are free R-modules.

Note that we can repeat the above process with K instead of F.

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \varphi$, $\varphi_1 := \psi$.

$$\begin{tikzcd}
 & K_1 \arrow[dr] & & & & \\
F_2 \arrow[ur, "\varphi_2"] \arrow[r, "\varphi_2"'] & F_1 \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_1"'] & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
 & & K_0 \arrow[u] \arrow[dr] & & \\
 & & & 0 &
\end{tikzcd}$$

Thus, we get free modules $\{F_n, \varphi_n : F_n \to F_{n-1}\}$ such that $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$ written as

$$\cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

F_is are free, φ_0 is onto & $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to \underbrace{F_1 \xrightarrow{\varphi_1} F_0 \to 0}$$ as an

R free resolution of M.

$\operatorname{im} \varphi_1 = K$, this is not exact here. φ_1 not onto (nec.)

Q: ① If M is f.g.:
Can we get F_is so that $\operatorname{rank}(F_i) < \infty$ $\forall i$.

② If yes, are $\operatorname{rank}(F_i)$s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_is s.t. the maps are "described nicely"?

Q. If two modules have "isomorphic" free resolutions, are they isomorphic?

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_3"] & F_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2", "\wr"'] & F_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1", "\wr"'] & F_0 \arrow[r] \arrow[d, "\psi_0", "\wr"'] & 0 \\
\cdots \arrow[r, "\varphi_3'"] & F_2' \arrow[r, "\varphi_2'"] & F_1' \arrow[r, "\varphi_1'"] & F_0' \arrow[r] & 0
\end{tikzcd}$$

$M \simeq F_0 / \operatorname{im} \varphi_1$

$M' \simeq F_0' / \operatorname{im} \varphi_1'$

$$\varphi_1' \psi_1 = \psi_0 \varphi_1 \quad (*)$$

Claim. $\gamma_0(\mathrm{im}\ \varphi_1) = \mathrm{im}\ \varphi_1'$

($\subseteq$) clear by (*)

($\supseteq$) clear again since $\varphi_1' = \gamma_0 \varphi_1 \gamma_1^{-1}$

Thus, $M_0 \simeq F_0 / \mathrm{im}\varphi \simeq \dfrac{\psi_0(F_0)}{\psi_0(\mathrm{im}\varphi)} = \dfrac{F_0'}{\mathrm{im}\varphi_1'} \simeq M_0'$

Lecture 1 (11-01-2021)

11 January 2021 11:07

Free modules:

(Free modules)

As usual : R is a (commutative) ring (with 1)
M is an R-module.

Def^n. ① Let $A \subset M$. A is said to be a generating set of M (as an R-module) if
$\forall x \in M$, $\exists x_1, \dots, x_n \in A$ and $(a_1, \dots, a_n) \in R^n$ s.t.

$$x = a_1 x_1 + \dots + a_n x_n.$$

(Note that A need not be finite.)

Notation : $M = \langle A \rangle$

If $A = \{x_1, \dots, x_n\}$ is finite, then $M = \langle x_1, \dots, x_n \rangle$ and M is said to be finitely generated.

② ⓐ Let $x_1, \dots, x_n \in M$. We say $\{x_1, \dots, x_n\}$ is linearly independent (over R) if for $(a_1, \dots, a_n) \in R^n$,

$$a_1 x_1 + \dots + a_n x_n = 0 \Rightarrow (a_1, \dots, a_n) = 0 \text{ in } R^n.$$

ⓑ A subset $A \subset M$ is linearly independent (over R) if every finite subset of A is linearly independent (over R).

③ A subset $A \subset M$ is a basis of M (over R) if $M = \langle A \rangle$ (over R) and A is linearly independent (over R).

④ M is free (over R) if M has a basis (over R).

REMARKS. ① Not every R-module has a basis.
② A minimal generating set need not be lin. indep.
③ A maximal lin. indep. set need not be a gen. set.

Q. If every R-module has a basis, is R a field?
(Yes. Take a non-field ring R and any non-trivial ideal $I \subsetneq R$.
Then, R/I has no lin. indep. set over R.)

Q. If an R-module M has a basis, does every basis have the same cardinality?
Ans. Yes. This is called the Invariant Basis Number (IBN) property of R.
Remark. This is not true if R is non-commutative. (That is, we can find a counterexample of a non-commutative ring.)
If R is a division ring, then again we have IBN.

Defn If M has a finite basis, say B, then we define
$$\mathrm{rank}(M) := |B|.$$
If M is free with an infinite basis, $\mathrm{rank}(M) := \infty$.
} well-defined, by IBN

(Rank)
(When we do say "rank", we will usually mean "finite rank.")

EXAMPLES. ① $R^{\oplus n}$ is a free R-module of rank n
$M_{m\times n}(R)$ of rank mn
$R[x]$ of rank ∞

② Let A be a non-empty set and

$$\mathcal{F}_0(A, R) = \{f : A \to R \mid f(a) = 0 \text{ for all but fin. many } a \in A\}.$$

Then, $\mathcal{F}_0(A, R)$ is an R-module under pointwise operations.
In fact, $\mathcal{F}_0(A, R)$ is a free R-module with basis $\{\chi_a\}_{a\in A}$, where

$$\chi_a(b) = \begin{cases} 0 & ; \ b \neq a \\ 1 & ; \ b = a \end{cases}$$

To see where the above set is generating, given any $f \in \mathcal{F}_0(A, R)$, we can write

$$f = \sum_{a \in A} f(a)\, \chi_a .$$

the sum is actually finite since $f(a) = 0$ for all but finitely many.
(it is to be understood that 0s are ignored.)

Q. What if we take $\mathcal{F}(A, R)$? (All functions.)

Universal Property of free modules: (Free R-module on A)

Def^n: Given a non-empty set A, a **free R-module on A** is a pair $(F(A), e)$ where (i) $F(A)$ is an R-module,
(ii) $e : A \to F(A)$ is a (set) function

satisfying :

Given an R-module M and a function $f : A \to M$, there exists a unique R-linear $\tilde{f} : F(A) \to M$ making the following diagram commute.

$$\begin{tikzcd} & F(A) \arrow[d, dashed, "\tilde{f}"] \\ A \arrow[ur, "e"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & M \end{tikzcd}$$

(That is, $\tilde{f} e = f$.)

Remarks. ① Given $A = \emptyset$, a free R-module on A exists, and is unique up to isomorphism.
Moreover, $e : A \to F(A)$ is one-one and $F(A)$ is free with basis $\{e_a\}_{a \in A}$, where $e_a := e(a)$.

② If M is a free R-module, then $M \cong F(B)$, where B is a(ny) basis of M.

Thus, an R-module M is free iff $M \cong F(A)$ for some A.

What the universal property is really saying is that: given a free R-module M with basis A, every R-linear

$$M \to N$$

R-module

is completely determined by its action on A.

(The above is in the sense that given any assignment of values on A, we do get an R-linear map.)

EXAMPLE: Given an R-module M, such that $M = \langle A \rangle$, we can write M as a quotient of $F(A)$.

(What we did last lec.)

Lecture 2 (12-01-2021)

12 January 2021 08:35

Weyl Algebra

Ex. $\mathbb{k}$ is a field, $\mathbb{k}[x_1, \dots, x_d]$

$\partial_1, \dots, \partial_d \to$ partial diff op.

$A_d(k) = k[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$ D-modules

non-comm. How would you define product?

Tensor Product

(Tensor product)

Tensor product (of two modules) essentially converts the study of bilinear maps to linear maps.

Defn. Given R-modules M and N, the tensor product of M and N (over R) is a pair (T, θ), where T is an R-module, $\theta : M \times N \to T$ is R-bilinear satisfying:

Given (K, φ) where K is an R module, $\varphi : M \times N \to K$ is R-bilinear, there exists a unique R-linear map $\tilde{\varphi} : T \to K$ making the following diagram commute

$$\begin{tikzcd} & T \arrow[d, dashed, "\tilde{\varphi}"] \\ M \times N \arrow[ur, "\theta"] \arrow[r, "\varphi"'] & K \end{tikzcd}$$

, i.e., $\tilde{\varphi}\theta = \varphi$.

(We are using "with", but can use "and" once we prove $M \otimes_R N \cong N \otimes_R M$.)

Thm. A tensor of M with N exists and is unique, up to isomorphism.

Uniqueness follows by universal property.

Notation: $M \otimes_R N$

Construction:

Want
$$\begin{tikzcd} M \times N \arrow[rr, dashed, "\theta"] \arrow[dr, "\varphi"'] & & T \arrow[dl, dashed, "\tilde{\varphi}"] \\ & K & \end{tikzcd}$$
$\theta(x_1 + x_2, y) = \theta(x_1, y) + \theta_2(x, y)$

Step 1: Let $F = F(M \times N)$, the free module on the set $M \times N$.
We get a map
$$e : M \times N \to F(M, N)$$
$$(x, y) \mapsto e_{(x,y)}$$

$\{e_{(x,y)} : x \in M, y \in M\}$ is a basis for F.

Let G be the submodule of F generated by

- $e_{(x_1 + x_2, y)} - e_{(x_1, y)} - e_{(x_2, y)}$
- $e_{(x, y_1 + y_2)} - e_{(x, y_1)} - e_{(x, y_2)}$
- $e_{(ax, y)} - a\, e_{(x,y)}$
- $e_{(x, ay)} - a\, e_{(x,y)}$

$\forall\, x, x_1, x_2 \in M$, $\forall\, y, y_1, y_2 \in M$, $\forall a \in R$

Step 2: Define $T = F/G$. Let $\pi : F \twoheadrightarrow T$ be the natural map.
Set $\pi(e_{(x,y)}) =: x \otimes y$.

Note that
$$\left.\begin{aligned} (x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\ (ax) \otimes y &= a(x \otimes y) = x \otimes (ay) \end{aligned}\right\} \begin{aligned} &\forall x, \ldots \in M \\ &\forall y, \ldots \in N \\ &\forall a \in R \end{aligned}$$

Consider $\theta = \pi e : M \times N \to T$
$$(x, y) \mapsto x \otimes y$$

$$\begin{tikzcd} & F(M \times N) \arrow[dr, "\pi"] & \\ M \times N \arrow[ur, "e"] \arrow[rr, dashed, "\theta"] & & T \end{tikzcd}$$
$e_{(x,y)}$, (x, y), $x \otimes y$

Step 3. Now, suppose we are given a bilinear
$$\varphi : M \times N \to K. \qquad (K \text{ is some } R\text{-module})$$

$$\begin{tikzcd} M\times N \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists!\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

Q1: Is there even a func. $\tilde{\varphi}$?

Q2: Is it R-linear?

Q3: Is it unique?

Note also

$$\begin{tikzcd} & F \arrow[d, dashed] & \\ M\times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists!\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

Note we have a set map $M\times N \xrightarrow{\varphi} K$ which induces an R linear map $\bar{\varphi} : F \to K$. (UMP of free modules)

$$\begin{tikzcd} & F \arrow[d, dashed, "\bar{\varphi}"] & \\ M\times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists!\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

We now want to show that $\bar{\varphi}$ factors through T. It would suffice to show that $G \subseteq \ker \bar{\varphi}$.

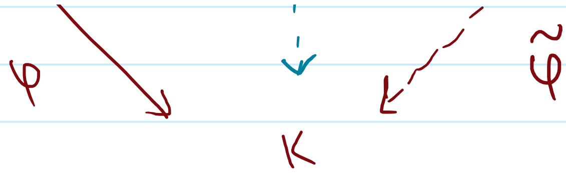
Using bilinearity of φ, it follows that all our (four types of) generators of G are in $\ker \bar{\varphi}$.

Thus, $\bar{\varphi}$ factors through quotient. That is, $\exists!$ R-linear $\tilde{\varphi} : T \to K$ s.t.

$$\begin{tikzcd} F \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi"'] & K \\ T \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes. That is, $\bar{\varphi} = \tilde{\varphi}\pi$.

$$\begin{tikzcd} & F \arrow[d, dashed, "\bar{\varphi}"] \arrow[dr, dashed, "\pi"] & \\ M\times N \arrow[ur, dashed, "e"] \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\tilde{\varphi}"] \\ & {} & \end{tikzcd}$$



Can now verify $\tilde{\varphi}\,\theta = \varphi$. (Use commutation of diff. triangles.)
Can also verify that $\tilde{\varphi}$ is unique R-linear such.

Basic Properties:

(1) ["Identity"] $\quad R \otimes_R M \simeq M$

(2) [Commutativity] $\quad M \otimes_R N \simeq N \otimes_R M$

(3) [Associativity] $\quad (M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$

(4) [Distributivity] $\quad M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$

Q. How does get an R-linear map $M \otimes_R N \to L$?
A. Give an R-bilinear map $M \times N \to L$.
Pretty much the only way. $x \otimes y$ would be 0 even if $x, y \neq 0$.
Thus, checking "well-defined" ness would become quite difficult.

Q. Let M be an R-module. $R \subset S$ subring.
Can you identify a natural S-module on $S \otimes_R M$?
(Base change)

Lecture 3 (14-01-2021)

14 January 2021 09:31

Distributivity: Given R-modules L, M, N

$$L \otimes_R (M \oplus N) \cong (L \otimes_R M) \oplus (L \otimes_R N)$$

How do we show? Want something like:

$$x \otimes (y, z) \longmapsto (x \otimes y, x \otimes z)$$

Note that elements of this form only GENERATE the tensor.

We now need to show the above map is well-defined. (As a function)
To do so, we go back to $L \times (M \oplus N)$ and use the universal property.

This ϕ is well defined. Every elt. here is uniquely written in the given form.

$$\begin{tikzcd} L \times (M \oplus N) \arrow[r, "\phi"] \arrow[d, "\theta"'] & (L \otimes_R M) \oplus L \otimes_R N \\ L \otimes (M \oplus N) \arrow[ur, dashed, "\tilde{\phi}"'] & \end{tikzcd}$$

$$(x, (y, z)) \longmapsto (x \otimes y, x \otimes z)$$

Note that ϕ is R-bilinear, thus an R-linear map $\tilde{\phi}$ (as indicated) which makes the diagram commute does exist.

To now show that is an isomorphism, we construct an inverse

$$\psi : (L \otimes_R M) \oplus (L \otimes_R N) \longrightarrow L \otimes_R (M \oplus N)$$
$$(x \otimes y, 0) \longmapsto x \otimes (y, 0)$$
$$(0, x \otimes z) \longmapsto x \otimes (0, z)$$

verify these are well defined (again universal property)

Can verify now that ψ is the two-sided inverse of $\tilde{\phi}$.

REMARK. Suppose M and N are R-modules.

1. If $x = 0 \in M$, then $x \otimes y = 0$ $\forall y \in N$.
   However if $x \otimes y = 0$ for some $x \in M, y \in N$, we cannot conclude $x = 0$ or $y = 0$.

   Example: Take $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$.
   In fact, look at $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ ← this is the zero module.
   Thus $x \otimes y = 0$ $\forall y \in N \not\Rightarrow x = 0$.

2. $M \otimes_R N$ is generated by $\{x \otimes y : x \in M, y \in N\}$ as R-module.
   In particular, if M and N are f.g., then so is $M \otimes_R N$.

   Take fin. gen. sets S_M and S_N. Then
   $$M \otimes_R N = \langle x \otimes y : x \in S_M, y \in S_N \rangle$$

3. If M and N are free, then so is $M \otimes_R N$. Identify a basis.

   For finite rank: write
   $$M = \overbrace{R \oplus \cdots \oplus R}^{m}$$
   $$N = R \oplus \cdots \oplus R$$

For finite rank: write $M = \overbrace{R \oplus \cdots \oplus R}^{m}$

$N = \underbrace{R \oplus \cdots \oplus R}_{n}$

Then, $M \otimes_R N = (R \oplus \cdots \oplus R) \otimes_R (R \oplus \cdots \oplus R)$

} distribute and use $R \otimes_R R \cong R$.

④ It is possible that $M \neq 0 \neq N$ but $M \otimes_R N = 0$.
(See 1.)

Q. Given a simple tensor $x \otimes y$, how can we determine if it's 0?
Concrete ex: Is $2 \otimes 3 \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ non-zero?

Q. Is it possible that $M \otimes_R M = 0$ even if $M \neq 0$?
Yes. Take $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$.

$$\left(\frac{a}{b} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right) = \left(\frac{a}{db} + \mathbb{Z}\right) \otimes \big(\underbrace{c + \mathbb{Z}}_{0}\big) = 0.$$

Tensor Algebra

(Tensor Algebra)

The tensor algebra of M

$$T(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$$

$= R \oplus M \oplus T_2(M) \oplus T_3(M) \oplus \cdots$

$T(M)$ is clearly an additive group. One can define multiplication by "concatenation."

$$(x_1 \otimes \cdots \otimes x_m) \cdot (y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n.$$

m^{th} piece $\qquad$ n^{th} piece $\qquad$ $(m+n)^{th}$ piece

Elements of $T(M)$ are written as formal sums: $\underset{R}{z_0} + \underset{M}{z_1} + \cdots + \underset{M^{\otimes n} = T_n(M)}{z_n}$

Identify $T(R^{\oplus n})$.

Some quotients of $T(M)$:

① Symmetric algebra

(Symmetric Algebra)

Given $x, y \in M$, $x \otimes y \neq^* y \otimes x$.

($\neq^*$: not necessarily equal)

Define $\mathrm{Sym}(M) = \dfrac{T(M)}{\langle x \otimes y - y \otimes x \mid x, y \in M \rangle}$ $= R \oplus M \oplus \mathrm{Sym}_2(M) \oplus \cdots$

$\mathrm{Sym}(M)$ is now a commutative algebra.

R and M not affected. Only $M^{\otimes 2}$ onwards.

② Exterior algebra. [wedge (M)]

(Exterior algebra)

$$\Lambda(M) = \frac{T(M)}{\langle x \otimes y + y \otimes x \rangle}$$

$= R \oplus M \oplus \Lambda^2(M) \oplus \cdots$

Q. What are $\mathrm{Sym}(R^{\oplus n})$ and $\Lambda(R^{\oplus n})$?

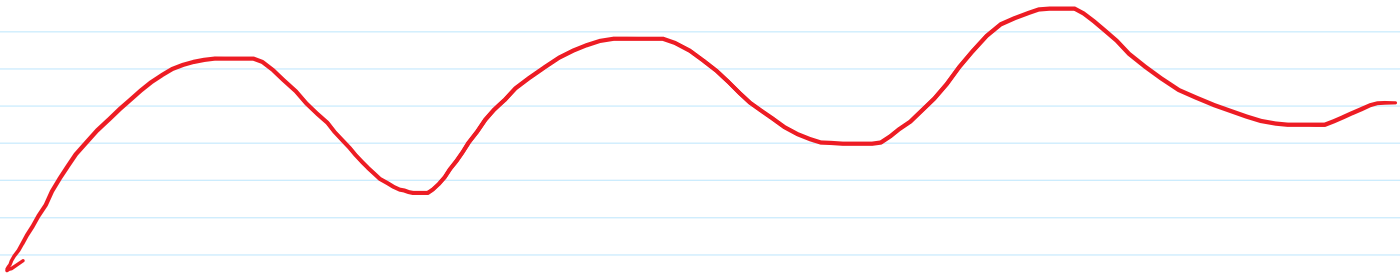
(Trivial extension)

③ $R \ltimes M = \dfrac{T(M)}{\langle x \otimes y \mid x, y \in M \rangle}$ (Trivial extension or)

③ $R \ltimes M = \dfrac{T(M)}{\langle x \otimes y \mid x, y \in M \rangle}$ (Trivial extension or idealisation.)

In this algebra, M is an ideal, with $M^2 = 0$.
This is called an idealisation of M.

Q. What is $R \ltimes R^{\oplus n}$?



Lecture 4 (18-01-2021)

18 January 2021 10:33

Base change:

(Base change or extension of scalars)

Let R and S be rings and $\varphi : R \to S$ a ring homomorphism. Then, we say that S is an R-algebra "via φ", i.e., S has an R-module structure defined by

$$a \cdot x = \varphi(a)x \qquad \forall a \in R \ \forall x \in S.$$

Two key examples: ① R is a subring of S (φ is 1-1)
② S is a quotient of R (φ is onto)
(and their compositions)

Example. If $I \subset R[x_1, \dots, x_d]$ is an ideal, then

$$S = \frac{R[x_1, \dots, x_d]}{I} \text{ is an } R\text{-algebra}$$

(via the natural maps $R \hookrightarrow R[x_1, \dots, x_d] \twoheadrightarrow S$)

A consequence of the Hilbert Basis Theorem:

Thm. Every finitely generated algebra over a Noetherian ring is a Noetherian ring.

(Not saying Noetherian as an R-module! Hilbert's Basis Thm does not give that!)

Note. If S is an R-algebra (via φ) and M is an S-module, then M has a natural R-module structure (via φ).

Q. What about the reverse? If M is an R-module, is there a "natural" way to induce an S-module structure on it?

[illegible]
"natural" way to induce an S-module structure on it?
No, in general. Take $\mathbb{Z}$ as a $\mathbb{Z}$ module and $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$.
Is there any "good" $\mathbb{Q}$-module structure on $\mathbb{Z}$?

Example. Let M be an R-module and $I \trianglelefteq R$ an ideal, $A \subset R$ m.c.s.

① When is M an R/I-module? $\Big\}$ induced multiplication
② When is M an R_A-module?

In general, we can create modules over R/I and R_A starting from M: They are M/IM and M_A, respectively.

Obs. $M/IM \simeq R/I \otimes_R M$ and $M_A \simeq R_A \otimes_R M$
(Isomorphic as R-modules.)

Base change (or extension of scalars):

Let S be an R-algebra (via φ) and M be an R-module. The R-module

$$S \otimes_R M$$

has a natural S-module structure defined by

$$a(b \otimes x) := (ab) \otimes x. \qquad \forall a, b \in S \ \forall x \in M$$

(Note that the above is only being defined for simple tensors.)

Note: ① If $M = \langle x_1, \ldots, x_n \rangle$ over R, then

$$S \otimes_R M = S\langle 1 \otimes x_i : 1 \leq i \leq n \rangle.$$

② If $M \simeq R^{\oplus n}$, $S \otimes_R M \cong S^{\oplus n}$ as R-modules.

$\otimes$ distributes over $\oplus$

In fact, $S \otimes_R M \cong S^{\oplus n}$ as S-modules as well.

Q. Given an R-linear map $\psi : M \to N$, will this induce an S-linear map $\bar{\psi} : S \otimes_R M \to S \otimes_R N$?

Does this help in the above?

③ If $M = R[x]$, then $S \otimes_R M \cong S[x]$ as S-modules.

④ If M is a free R-module, then $S \otimes_R M$ is a free S-module.

Example of base change: "Mod p test for irreducibility of a polynomial in $\mathbb{Z}[x]$"

Recall the test: Given $f = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$. If we can find a prime p s.t. $f \bmod p$ is irred in $(\mathbb{Z}/p\mathbb{Z})[x]$, then f is irred.*

(*Need to take care of degree not falling.)

This is an example of base change with $R = \mathbb{Z}$ and $S = \mathbb{Z}/p\mathbb{Z}$.

Complexes and Homology

(Complexes and Homology)

Example: Construct a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

$$\cdots \to 0 \to 0 \to 0 \to 0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

$$x \mapsto 2x \qquad 2\mathbb{Z} \xrightarrow{\ker} \qquad 1 \longmapsto \bar{1}$$

Q1. What is a generating set for $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$?

Ans. $\{\bar{1}\}$ ← singleton.

Thus, we map one copy of $\mathbb{Z}$ onto $\mathbb{Z}/2\mathbb{Z}$. That is,

$$\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

that is,

$$\begin{aligned}\mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z}\\ 1 &\longmapsto \bar{1}\end{aligned}$$

Q2. What is ker ?
Ans. It is $2\mathbb{Z}$.

Q3. What can map onto $2\mathbb{Z}$?
Ans. $\mathbb{Z}$. $\quad x \mapsto 2x$

Q4. What is ker ?
It is 0. The map is 1-1. This gives the diagram

Note that we could have also written

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

since $2\mathbb{Z}$ itself is a free $\mathbb{Z}$-module. The reason we did not do this is because we are sticking to writing free R-module as copies of R.

Q. Let R be a ring and $a \in R$. Is the following a free resolution of $R/\langle a\rangle$ over R?

$$0 \to R \xrightarrow{a} R \to R/\langle a \rangle \to 0$$
$$x \mapsto ax$$

(finite free resolution) Called a finite free resolution.

Obs. • Note that if some ker is free, we can stop there.
(That is basically saying that $R^{\oplus n} \to$ ker will be 1-1.)

• The above does happen if R is a PID and M is f.g. So, in that case, the resolution stops right away as above. (At Stage 1.)

$$\cdots \to 0 \to 0 \to 0 \to \overset{\text{stage 1}}{R^{\oplus m}} \longrightarrow \underset{\text{stage 0}}{R^{\oplus n}} \longrightarrow M \longrightarrow 0$$

$(m \leq n)$

Thus, every f.g. module has a finite free resolution of "length" 1.
(Recall that a submodule of a free module over a PID is free.)

Thus, every f.g. module has a finite free resolution of "length" 1.
(Recall that a submodule of a f.g. free module over a PID is free.)

- Question of "optimality" of free resolution

We know $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$
is a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

So is
$$0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \xrightarrow{\begin{bmatrix} 2 & 0 \end{bmatrix}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$
$$f \mapsto e_2 \qquad e_1 \mapsto 2, \quad e_2 \mapsto 0$$

Note that we can go on and create an extra copy of $\mathbb{Z}$ and make it longer.

$$0 \to \mathbb{Z}g \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 0 \end{bmatrix}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$
$$e_1 \longmapsto 2$$
$$f_1 \longmapsto e_2 \longmapsto 0$$
$$g \longmapsto f_2 \longmapsto 0$$

Q. If $\operatorname{rank}(F_i) \geq \operatorname{rank}(F_{i-1})$, does that mean non-optimal?
I don't think so. If a column is 0, then yes. But otherwise, don't think so.

Q. What is a free resolution of R over itself

$$0 \to R \xrightarrow{id} R \to 0.$$

(Dropping the module: $0 \to R \to 0$)

In fact, the above is true for any free R-module F.

$$0 \to F \to F \to 0.$$

(Dropping the module $0 \to F \to 0$.)

$\hookrightarrow$ f.g.

Have to conclude I is free. That would imply I is principal.

INCOMPLETE.

Back to example: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is a $\mathbb{Z}$-free resolution of $\mathbb{Z}/2\mathbb{Z}$.

Tensor with $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}$. $\leftarrow$ We get a sequence of $\mathbb{Z}/6\mathbb{Z}$ modules via base change.

① Note that $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}/6\mathbb{Z}$ module. ($\bar{1}_{\mathbb{Z}/6\mathbb{Z}} \mapsto \bar{1}_{\mathbb{Z}/2\mathbb{Z}}$ gives a ring homo)

② $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}/(6\mathbb{Z}+2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

$\hookrightarrow$ Should be generated by $\{\bar{1} \otimes \bar{1}\}$ (recall tensor generated by tensor of gen.)

③ $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$

Thus, tensoring the free resolution with $\mathbb{Z}/6\mathbb{Z}$ gives

$$0 \to \mathbb{Z}/6\mathbb{Z} \xrightarrow[\bar{2}]{?} \mathbb{Z}/6\mathbb{Z} \xrightarrow{?} \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\bar{1} \mapsto \bar{1}$$

$$\bar{1} \mapsto \bar{2}$$

Is this a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/6\mathbb{Z}$?

No! The map $\bar{1} \mapsto \bar{2}$ is not injective.

$3\mathbb{Z}/6\mathbb{Z}$ is the kernel!

$\hookrightarrow$ not free :(

After base change, the free resolution does not remain free.