Group Theory

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Note the "along with." We don't talk about a group by just talking about a set. It is necessary to have an operation on it as well.

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First, a definition.

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Now, we define what a group is.

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Proof?

The discussion we had is what led to us agreeing upon the above axioms. So, let us only discuss what went wrong with the non-examples.

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Recall vector spaces?

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Recall vector spaces? Verify that any vector space along with its $+$ forms a group.

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- "G is a group" when $\cdot$ is clear from context.

Notations

Let G be a group and $x \in G$. We define x^n for $n \in \mathbb{Z}$ as follows:

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For $n > 0$, we define

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Definition 5 (Order (element))

The order of an element $x \in G$ the smallest positiver integer n such that

$$x^n = e.$$

(Where e is the identity of G.)
If no such n exists, then we say the the element has infinite order.
It is denoted by $|x|$.

Finite groups

Proposition 1

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Proof.

Let G be a finite group and let $x \in G$.
It suffices to show that $x^n = e$ for *some* $n \in \mathbb{N}$.
Note that $x^0, x^1, \ldots, x^{|G|}$ are $|G| + 1$ elements of G. By PHP, two of them must be equal. Thus,

$$x^n = x^m$$

for some $0 \leq n < m \leq |G|$.
The above equation gives us

$$e = x^{m-n}.$$

Since $m - n \in \mathbb{N}$, we are done. $\square$

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- Subspaces of (metric/topological) spaces, et cetera.

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One may note that the identity element of $(G, \cdot)$ is always present in H and moreover, it is also the identity of $(H, \cdot|_H)$.

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- $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$.
- The set of $n \times n$ upper invertible diagonal (real) matrices is a subgroup of the group of all invertible $n \times n$ (real) matrices.

Cosets

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Note that different elements could correspond to the same coset. That is, a coset may have different representatives. In fact, we now see precisely when that is possible.

Properties of cosets

Proposition 2 (Equality of cosets)

Let $a, b \in G$. Then,

$$aH = bH \text{ iff } b^{-1}aH = H \text{ iff } b^{-1}a \in H.$$

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Note that H itself is a coset since it equals $e \cdot H$. (Or $h \cdot H$ for any $h \in H$.)

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Remark. This shows that any two cosets have the same cardinality.

Lagrange's Theorem

With the concept of cosets, we can prove a (quite fundamental) result of group theory.

Lagrange's Theorem

Theorem 1 (Lagrange's Theorem)

Let G be a finite group and $H \leq G$.
Then, $|H|$ divides $|G|$.

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That completes our proof. □

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The case of groups is particularly simple since there's pretty one much thing that gives the group its structure, the group operation. This leads to the following definition.

Homomorphisms

Definition 8 (Homomorphism)

Let $(G, \cdot)$ and $(H, \star)$ be groups. A function

$$\varphi : G \to H$$

is said to be a *group homomorphism* if

$$\varphi(a \cdot b) = \varphi(a) \star \varphi(b)$$

for all $a, b \in G$.

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Properties of homomorphisms

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Examples

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- Given any $n \in \mathbb{Z}$, the map $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined as
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is a homomorphism.
- In general, if G is an abelian group and $n \in \mathbb{Z}$, the map $x \mapsto x^n$ is a homomorphism.

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While the groups are not equal (since they don't have the same element), they pretty much are same in terms of the group properties.
This idea can formalised as follows.

Definition 9 (Isomorphism)

Let G and H be groups. A group homomorphism $\varphi : G \to H$ is said to be *isomorphism* if φ is bijective.

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One can note that $\cong$ is an "equivalence relation".

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- In general, the groups $G = \{0, \ldots, n-1\}$ and $H = \{z \in \mathbb{C}^\times : z^n = 1\}$ are isomorphic.
- The map $\exp : \mathbb{R} \to \mathbb{R}^+$ is an isomorphism. (Note that $\mathbb{R}$ is a group under $+$ whereas $\mathbb{R}^+$ is a group under $\cdot$.)

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Proposition 5

With the same notations as above, we have

$$\ker \varphi \leq G.$$

A curious property about kernels

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Let $\varphi : G \to H$ and $K = \ker \varphi$.
Then, given any $a \in G$ and $k \in K$, we have

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This can be written in yet another way as $aK = Ka$.

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Note that the only property we used was that $gK = Kg$ and not really that K was a kernel.

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then H must satisfy $gH = Hg$.

Normal subgroups

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Said even more differently, given any $n \in G$, and $g \in G$, we must have $gng^{-1} \in N$.

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Let G be a group and N be a normal subgroup of G.
Then, the set of cosets G/N is a group under the operation defined by

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We sometimes use the notation $\bar{g}$ to denote the coset gH. (When H is clear from context.)
Another thing to note is that any subgroup of an abelian group is normal.

Example

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The above group is what is called $\mathbb{Z}/5\mathbb{Z}$. Of course, this works for all values of 5.
In fact, this is the group (up to isomorphism) which we saw earlier as $\{1, \dots, n-1\}$ with addition modulo n.