Group Theory

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Note the "along with." We don't talk about a group by just talking about a set. It is necessary to have an operation on it as well.

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First, a definition.

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Now, we define what a group is.

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Proof?

Review

The discussion we had is what led to us agreeing upon the above axioms. So, let us only discuss what went wrong with the non-examples.

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Recall vector spaces?

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Recall vector spaces? Verify that any vector space along with its $+$ forms a group.

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- "G is a group under $\cdot$," or
- "G is a group" when $\cdot$ is clear from context.

Notations

Let G be a group and $x \in G$. We define x^n for $n \in \mathbb{Z}$ as follows:

$x^0 := e$.
For $n > 0$, we write x^n to mean

$$\underbrace{x \cdot x \cdots x}_{n \text{ times}}.$$

For $n < 0$, we have $x^n := (x^{-1})^{-n}$, which is the same as $(x^{-n})^{-1}$. (Prove!)

It is convention to write $+$ as the group operation for abelian groups.

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Definition 5 (Order (element))

The order of an element $x \in G$ the smallest positiver integer n such that

$$x^n = e.$$

(Where e is the identity of G.)
If no such n exists, then we say the the element has infinite order.
It is denoted by $|x|$.

Finite groups

Proposition 1

Every element of a finite group has finite order.

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Proof.

Let G be a finite group and let $x \in G$.
It suffices to show that $x^n = e$ for *some* $n \in \mathbb{N}$.
Note that $x^0, x^1, \ldots, x^{|G|}$ are $|G| + 1$ elements of G. By PHP, two of them must be equal. Thus,

$$x^n = x^m$$

for some $0 \leq n < m \leq |G|$.
The above equation gives us

$$e = x^{m-n}.$$

Since $m - n \in \mathbb{N}$, we are done. $\square$

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- Subgrings of rings,
- Subfields of fields,
- Subspaces of (metric/topological) spaces, et cetera.

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One may note that the identity element of $(G, \cdot)$ is always present in H and moreover, it is also the identity of $(H, \cdot|_H)$.

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- $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$.
- The set of $n \times n$ upper invertible diagonal (real) matrices is a subgroup of the group of all invertible $n \times n$ (real) matrices.

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A *(left) coset* of H is a set of the form $g \cdot H$.

Define G/H be the set of cosets, that is,

$$G/H := \{gH : g \in G\}.$$

Note that different elements could correspond to the same coset. In fact, we now see precisely when that is possible.

Proposition 2 (Equality of cosets)

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$$aH = bH \text{ iff } b^{-1}a \in H.$$

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Note that H itself is a coset since it equals $e \cdot H$. (Or $h \cdot H$ for any $h \in H$.)

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Remark. This shows that any two cosets have the same cardinality.

Lagrange's Theorem

With the concept of cosets, we can prove a (quite fundamental) result of group theory.

Theorem 1 (Lagrange's Theorem)

Let G be a finite group and $H \leq G$.
Then, $|H|$ divides $|G|$.

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That completes our proof. □

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The case of groups is particularly simple since there's pretty one much thing that gives the group its structure, the operation $\cdot$.

This leads to the following definition.

Definition 8 (Homomorphism)

Let $(G, \cdot)$ and $(H, \star)$ be groups. A function

$$\varphi : G \to H$$

is said to be a *group homomorphism* if

$$\varphi(a \cdot b) = \varphi(a) \star \varphi(b)$$

for all $a, b \in G$.

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$$G \xrightarrow{\varphi} H \xrightarrow{\psi} K,$$

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Go look up what a Category is. (In the context of Category Theory.)

Examples

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- The map $\exp : \mathbb{R} \to \mathbb{R}^\times$ defined by

$$\exp(x) = e^x$$

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- Given any $n \in \mathbb{Z}$, the map $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined as
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is a homomorphism.
- In general, if G is an abelian group and $n \in \mathbb{Z}$, the map $x \mapsto x^n$ is a homomorphism.

Isomorphisms

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While the groups are not equal (since they don't have the same element), they pretty much are same in terms of the group properties.
This idea can formalised as follows.

Definition 9 (Isomorphism)

Let G and H be groups. A group homomorphism $\varphi : G \to H$ is said to be *isomorphism* if φ is bijective.

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One can note that $\cong$ is an "equivalence relation".

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- In general, the groups $G = \{0, \ldots, n-1\}$ and $H = \{z \in \mathbb{C}^\times : z^n = 1\}$ are isomorphic.
- The map $\exp : \mathbb{R} \to \mathbb{R}^+$ is an isomorphism. (Note that $\mathbb{R}$ is a group under $+$ whereas $\mathbb{R}^+$ is a group under $\cdot$.)

Kernels

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Proposition 5

With the same notations as above, we have

$$\ker \varphi \leq G.$$

Proposition 6

Let $\varphi : G \to H$ and $K = \ker \varphi$.
Then, given any $a \in G$ and $k \in K$, we have

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A curious property about kernels

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This can be written in yet another way as $aK = Ka$.

A curious calculation

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Now, suppose that $a, a', b, b' \in G$ are elements such that $aK = a'K$ and $bK = b'K$.
Then, we see that

$$\begin{aligned}(ab)K &= a(bK) \\ &= a(Kb) \\ &= a(Kb') \\ &= (aK)b' \\ &= (a'K)b' \\ &= a'(Kb') \\ &= (a'b')K.\end{aligned}$$

Let us keep this in mind for now. We shall come back to it later.

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Note that the only property we used was that $gK = Kg$ and not really that K was a kernel.