The Binomial Pricing Model

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1 Introduction

The objective of this short exposition is to explain the mathematical foundations behind the attached Excel sheet that I created to price various derivatives under the binomial pricing model.

This project is an extension of the Introduction to Financial Engineering and Term Structure and Credit Derivatives courses taught by M. Haugh and G. Iyengar from the Department of Industrial Engineering and Operations Research at Columbia University.

2 The Binomial Pricing Model

The binomial pricing model is a discrete-time framework used to trace the evolution of an asset's price with the primary assumption that the asset price can move in one of two directions in each period, up or down.

Essentially, the price of a stock S_t can either move up by a factor of u or down by a factor of $d = \frac{1}{u}$ from period to period, i.e.

$$S_{t+1} = \begin{cases} u \cdot S_t & \text{with probability } p \\ d \cdot S_t & \text{with probability } 1-p \end{cases}$$

Following this iterative procedure further into the future, we have

$$S_{t+2} = \begin{cases} u^2 \cdot S_t & \text{with prob. } p^2 \\ S_t & \text{with prob. } p(1-p) \\ d^2 \cdot S_t & \text{with prob. } (1-p)^2 \end{cases}$$

$$S_{t+3} = \begin{cases} u^3 \cdot S_t & \text{with prob. } p^3 \\ u \cdot S_t & \text{with prob. } p^2(1-p) \\ d \cdot S_t & \text{with prob. } p(1-p)^2 \\ d^3 \cdot S_t & \text{with prob. } (1-p)^3 \end{cases}$$

In line with the binomial nature of the model, the probability of a certain payoff occurring which required j up-moves to achieve is given by

$$\binom{n}{j} p^j (1-p)^{n-j}$$

where n denotes the number of periods of the binomial model. Let R denote the risk-free return on a cash account. We posit that no-arbitrage is satisfied if $d < R < u$. Let S_0 denote the price of the asset at time $t = 0$. Consider the following scenarios:

- Suppose $R < d < u$. We borrow S_0 dollars and invest it into the asset at time $t = 0$. Our initial cash flow is 0. At time $t = 1$, we owe $R \cdot S_0$ and our investment is worth either

$$S_1 = \begin{cases} u \cdot S_0 \\ d \cdot S_0 \end{cases}$$

  The net value of our portfolio at time $t = 1$ is then

$$V_1 = \begin{cases} u \cdot S_0 - R \cdot S_0 = (u - R) \cdot S_0 > 0 \\ d \cdot S_0 - R \cdot S_0 = (d - R) \cdot S_0 > 0 \end{cases}$$

  We thus are guaranteed a risk-free profit, violating no-arbitrage.

- Similarly, suppose $d < u < R$. We short sell the asset at time $t = 0$ and invest the proceeds into a cash account. Once again, our initial cash flow is net zero. Our net portfolio value in the following period is then

$$V_1 = \begin{cases} R \cdot S_0 - u \cdot S_0 = (R - u) \cdot S_0 > 0 \\ R \cdot S_0 - d \cdot S_0 = (R - d) \cdot S_0 > 0 \end{cases}$$

  This, too, violates no-arbitrage.

In the binomial model, the price of a derivative on the underlying asset does not depend on the true probability of an up-move or down-move occurring. Rather, up and down moves are determined by a **risk-neutral** probability distribution. The first fundamental theorem of asset pricing states that if there is no arbitrage, then a risk-neutral distribution exists, and vice-versa.

We thus define the risk-neutral probability distribution

$$\mathbb{Q} = \{q, 1 - q\}$$

where

$$q = \frac{R - d}{u - d}$$

Since both $q, 1 - q > 0$ and the probability of up and down moves rely solely on R, d, and u, we designate $\mathbb{Q}$ as a risk-neutral distribution.

3 European Options

Consider an n-period binomial model. Generally, we determine the current price of a derivative security, X_0, in the binomial model by calculating the present discounted value of the expected value of the payoff of the security under the risk-neutral distribution, conditioned on time $t = 0$ information:

$$X_0 = \frac{1}{R^n} \mathbb{E}_0^{\mathbb{Q}} \left[f_X(S_T) \right]$$

Here, T is the expiration date of the derivative, and $f_X(S_T)$ denotes the payoff function of that derivative.

A call option with strike K maturing at time T has the payoff function

$$f(S_T) = \max(S_T - K, 0)$$

In order to find the price of this call option at time $t = 0$, C_0, we must find the expected value of its payoff at time T. We can do this by modelling the evolution of the underlying stock price in the binomial framework, calculating all possible payoffs by evaluating the payoff function f at all possible values of S_T at time T, multiplying each payoff by the probability that it occurs, and finally summing across all states.

$$C_0 = \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} \max(S_0 \cdot u^j d^{n-j} - K, 0)$$

4 Fixed-Rate Derivatives

When pricing fixed-income derivatives, we apply a risk-neutral probability framework to model the short rate r_t. In the model, $r_{i,j}$ denotes the short rate at period i and state (i.e. up-moves required) j.

4.1 Zero Coupon Bonds

Let $Z_{i,j}^k$ denote the time i, state j price of a zero coupon bond that matures at time k. We define the cash account B_t as a security that earns interest at the short rate. We set $B_0 = 1$. While the cash account is not risk-free since B_{t+s} for $s > 1$ is not known at time t, it is locally risk-free since B_{t+1} is known at time t. The cumulative interest of the cash account satisfies

$$B_t = (1 + r_{0,0}) \cdot (1 + r_1) \cdot \ldots \cdot (1 + r_{t-1})$$

which in turn implies that

$$\frac{B_t}{B_{t+1}} = \frac{1}{1 + r_t}$$

We have that the price of a zero coupon bond at time i and state j must equal the present discounted value of the expected value of the security in the following period, i.e.

$$\begin{aligned}
Z_{i,j} &= \frac{1}{1+r_{i,j}} \left[q_u \cdot Z_{i+1,j+1} + q_d \cdot Z_{i+1}j\right] \\
Z_t &= \mathbb{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1}}{1+r_{t,j}}\right] \\
Z_t &= \mathbb{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_{t+1}} Z_{t+1}\right] \\
\frac{Z_t}{B_t} &= \mathbb{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1}}{B_{t+1}}\right]
\end{aligned}$$

Applying the law of iterated expectations, we have

$$\begin{aligned}
\frac{Z_t}{B_t} &= \mathbb{E}_t^{\mathbb{Q}} \left[\mathbb{E}_{t+1}^{\mathbb{Q}} \left[\frac{Z_{t+2}}{B_{t+2}}\right]\right] \\
\implies \frac{Z_t}{B_t} &= \mathbb{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+s}}{B_{t+s}}\right]
\end{aligned}$$

In the lattice framework, we work backward from the time of expiry, where we know the zero coupon bond will pay a face value of, say, 100. We use the corresponding node in the short rate lattice to determine

$$Z_t = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1}}{1+r_t}\right]$$

If the security pays a coupon C_t at time t, we simply add a term that represents the present discounted expected value of the coupon payments.

$$\frac{Z_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1} + C_{t+1}}{1+r_{t,j}}\right]$$

Iterating this equation yields

$$\frac{Z_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{j=t+1}^{t+s} \frac{Z_j}{B_j} + \frac{Z_{t+s}}{B_{t+s}}\right]$$

Computing the price of an option on a zero coupon bond is straightforward once we have the lattice of zero coupon bond prices determined by the specified iterative procedure.

Consider a European call option which gives the buyer the right to purchase a zero coupon bond with expiry $t = k$ with strike K at time $t = s$. The payoff of such an option is given by

$$f(Z_t) = \max(Z_s^k - K, 0)$$

Once we have computed the k-period binomial lattice for the zero coupon bond prices, we select the range of prices corresponding to $t = s$. We then subtract the strike K and work backwards in the lattice to arrive at the present risk-neutral price of the option.

4.2 Caplets

A caplet is similar to a call option on the interest rate r_t. The payoff of a caplet settled in arrears with expiry τ and strike c is given by

$$f(r_t) = \max(r_{\tau-1} - c, 0)$$

This caplet is thus a call option on the short rate prevailing at time $\tau - 1$ settled at time τ. A floorlet is the same but with the opposite payoff, i.e. $\max(c - r_{\tau-1}, 0)$. Caps and floors are simply a series of caplets or floorlets all with the same strikes and different maturities.

To price a caplet, we begin with the short rate lattice. Suppose the caplet has expiry τ and strike c. We select the range of short rates corresponding to time $\tau - 1$ in the short rate lattice and discount them by one period to reflect the settlement at time τ. Namely, the corresponding node in the caplet lattice will be given by the appropriately-discounted payoff of the caplet at time $\tau - 1$

$$\frac{\max(r_{\tau-1} - c, 0)}{1 + r_{\tau-1}}$$

We then work backwards in the caplet lattice in the same way we worked backward in the option lattice, using the risk-neutral probabilities $\mathbb{Q}$.

4.3 Forward Equations

To simplify the calculation of fixed-rate derivative prices in the binomial model, we define the elementary security P^e. Specifically, let $P^e_{i,j}$ denote the time 0 price of a security that pays 1 at time i, state j and 0 at every other time and state. We call $P^e_{i,j}$ the state price of the elementary security.

Fix the risk-neutral probabilities $q = (1 - q) = \frac{1}{2}$. We know that $P^e_{0,0} = 1$. We can iteratively calculate $P^e_{1,0}$ and $P^e_{1,1}$, the state prices of the elementary securities corresponding to time $t = 1$. Namely,

$$P^e_{1,0} = \frac{1}{1 + r_{1,0}} \left[\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 \right]$$

$$P^e_{1,1} = \frac{1}{1 + r_{1,1}} \left[\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 \right]$$

This follows from the fact that each elementary security at time 1 has a one-half probability of paying off. Iterating this logic into further periods, we see that the price of the elementary securities satisfy the **forward equations**:

$$\begin{aligned} P^e_{k+1,s} &= \frac{P^e_{k,s-1}}{2(1+r_{k,s-1})} + \frac{P^e_{k,s}}{2(1+r_{k,s})} \qquad \text{for } 0 < s < k+1 \\ P^e_{k+1,0} &= \frac{1}{2}\frac{P^e_{k,0}}{(1+r_{k,0})} \\ P^e_{k+1,k+1} &= \frac{1}{2}\frac{P^e_{k,k}}{(1+r_{k,k})} \end{aligned}$$

with $P^e_{0,0} = 1$.

5 The Black-Derman-Toy Model

We must calibrate the parameters of our pricing model to ensure that the calculated prices, at the very least, agree with the market prices of liquid securities such as caps, floors, and swaptions. To simplify the calibration of this model, we fix $q_{i,j} = q = \frac{1}{2}$ for all times and states i, j and assume a parametric form for $r_{i,j}$. This is where the Black-Derman-Toy (BDT) model comes in.

The BDT model assumes that the interest rate at node (i, j) is given by

$$r_{i,j} = a_i e^{b_i j}$$

Here, $\log a_i$ is a drift parameter for $\log r$ and b_i is a volatility parameter for $\log r$. The calibration to the observed market term structure is achieved by choosing the values of a_i and b_i to match market prices.

In the Excel model, we have further simplified the process by assuming that the volatility parameter $b_i = b$ is known for all i. Let $(s_1, \ldots, s_n)$ denote the observed term structure of interest rates in the market. Without loss of generality, we assume that these spot rates are compounded per period. We then have

$$\frac{1}{1+s_i)^i} = \sum_{j=0}^{i} P^e_{i,j} = Z_{0,i}$$

This is because summing the elementary prices across all states at time i will directly give the price of a zero coupon bond that pays 1 at maturity i. Expanding the summation with the forward equations and the BDT parameterization of r, we have

$$\begin{aligned}
\frac{1}{1+s_i)^i} &= \sum_{j=0}^{i} P^e_{i,j} \\
&= P^e_{i,0} + \sum_{j=1}^{i-1} P^e_{i,j} + P^e_{i,i} \\
&= \frac{P^e_{i-1,0}}{2(1+a_{i-1})} + \sum_{j=1}^{i-1} \left(\frac{P^e_{i-1,j}}{2(1+a_{i-1}e^{bj}} + \frac{P^e_{i-1,j-1}}{2(1+a_{i-1}e^{b(j-1)})} \right) \\
&\quad + \frac{P^e_{i-1,i-1}}{2(1+a_{i-1}e^{b(i-1)}}
\end{aligned}$$

In the Excel model, the Solver add-in is used to iteratively solve for the a_i's. To calibrate the model, we specify a model under $\mathbb{Q}(\theta)$-dynamics where θ is a vector of parameters, for instance the a_i's and b_i's of the BDT model. We will price all securities following

$$\frac{Z_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}(\theta)} \left[\sum_{j=t+1}^{t+s} \frac{C_j}{B_j} + \frac{Z_{t+s}}{B_{t+s}} \right]$$

To calibrate to the observed market term structure, we seek to minimize the sum of squared differences between model prices and market prices. The problem is then

$$\min_\theta \sum_i \omega_i (P_i(\text{model}) - P_i(\text{market}))^2 + \lambda \|\theta - \theta_{\text{prev}}\|^2$$

where

- $(P_i(\text{model})$ is the model price of the i^{th} calibration security.
- $(P_i(\text{market})$ is the market price of the i^{th} calibration security.
- ω_i is a positive weight reflecting the relative importance of the i^{th} security or the confidence we have in its market price.
- θ_{prev} is the vector of previously calibrated model parameters.
- λ is a parameter reflecting the relative importance of remaining close to the previous calibration.

Once we have solved the above optimization problem to our satisfaction, we can use the model to more reliably price illiquid and exotic securities. The non-convex optimization problem, however, is difficult to solve in practice and requires several recalibrations daily.

6 Defaultable Bonds

Modelling defaultable bonds requires adjusting the risk-neutral probabilities of up and down moves to include the possibility of default. We will assume that the defaultable bonds pay a face value F and have a recovery value of R, the fraction of face value recovered upon default. We will model the term structure by using the risk-neutral probabilities $\mathbb{Q}$ and with a 1-step default probability

$$h(t) = \mathbb{Q}\left(\text{bond defaults in } [t, t+1) | \mathcal{F}_t\right)$$

where $\mathcal{F}_t$ represents information known at time t. We utilize the binomial lattice for the short rate where each node (i, j) has the prevailing short rate $r_{i,j}$ at time i and state j. We have

$$\mathbb{Q}\left((i+1, s) \mid (i, j)\right) = \begin{cases} q_u & s = j+1 \\ q_d & s = j \\ 0 & \text{otherwise} \end{cases}$$

Each node in the defaultable bond lattice is now given by the split node (i, j, η) where

$$\eta = \begin{cases} 1 & \text{if the bond has defaulted at time } \tau \leq i \\ 0 & \text{if the bond has not yet defaulted} \end{cases}$$

We are now ready to adjust $\mathbb{Q}$ to include the probability of default, also known as the **hazard rate**, h_{ij}. The risk-neutral probabilities during the transition from a non-default state (i.e. $\eta_i = 0$) are

$$\mathbb{Q}\left((i+1, s, \eta) \mid (i, j, 0)\right) = \begin{cases} q_u h_{ij} & s = j+1,\ \eta = 1 \\ q_u(1 - h_{ij}) & s = j+1,\ \eta = 0 \\ q_d h_{ij} & s = j,\ \eta = 1 \\ q_d(1 - h_{ij}) & s = j,\ \eta = 0 \\ 0 & \text{otherwise} \end{cases}$$

These probabilities correspond to the probability of an up-move and default, up-move and no default, down-move and default, and down-move and no default, respectively, at time i. The transition from a state where the bond has already defaulted (i.e. $\eta_i = 0$) is given by

$$\mathbb{Q}\left((i+1, s, \eta) \mid (i, j, 0)\right) = \begin{cases} q_u & s = j+1,\ \eta = 1 \\ q_d & s = j+1,\ \eta = 1 \\ 0 & \text{otherwise} \end{cases}$$

which essentially implies that a previously-defaulted bond cannot "undefault". To price a zero coupon bond with recovery, assume that the random recovery $\tilde{R}$ is independent of both default and interest rate dynamics and let $R = \mathbb{E}[\tilde{R}]$. Let $\bar{Z}^T_{i,j,\eta}$ denote the price of defaultable zero coupon bond maturing on date T in

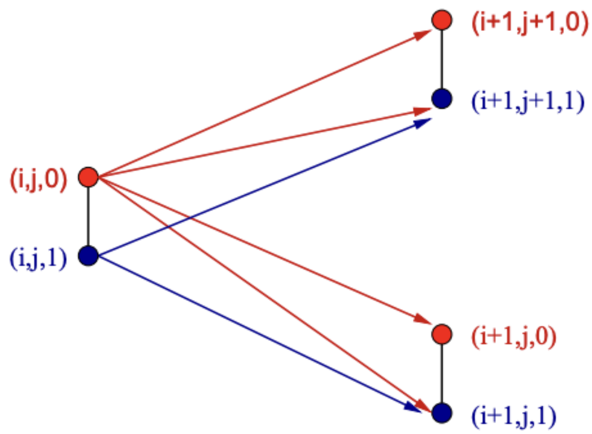

Figure 1: Transition dynamics in the binomial lattice with defaults

node $(i, j, \eta$ after recovery. The defaultable bond pays 1 dollar at every state at time T provided that default has not occurred at any time $t \leq T$. Risk-neutral pricing implies that

$$\begin{aligned}\bar{Z}^T_{i,j,\eta} &= \frac{1}{1+r_{ij}}\left[q_u(1-h_{ij})\bar{Z}^T_{i+1,j+1,0} + q_d(1-h_{ij})\bar{Z}^T_{i+1,j,0}\right] \\ &\quad + \frac{1}{1+r_{ij}}\left[q_u h_{ij} R + q_d h_{ij} R\right]\end{aligned}$$

We use this relation to work backwards in the binomial lattice for the defaultable zero coupon bonds.

7 Convergence to the Black-Scholes Model

The Black-Scholes model for option pricing is a continuous representation of the discrete-time binomial model. The model assumes that stock prices follow a Geometric Brownian Motion (GBM) given by

$$S_t = S_0 \cdot e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

where W_t follows a Standard Brownian Motion. Further, we assume the continuously-compounded interest rate is r and the stock has an annualized dividend yield and volatility of c and σ, respectively. The Black-Scholes price of a call option is given by

$$C_0 = S_0 \cdot e^{-cT} N(d_1) - K e^{-rt} N(d_2)$$

$$d_1 = \frac{\ln \frac{S_0}{K} + (r - c + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where $N(d) = P(N(0,1) \leq d)$ where $N(0,1)$ is a standard normally-distributed random variable. The formula calculates the price as the expected value of the following replicating strategy

$$C_0 = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT}\max(S_T - K, 0)\right]$$

where under the risk-neutral probability distribution $\mathbb{Q}$, the stock price evolves following

$$S_t = S_0 \cdot e^{(r-c-\frac{\sigma^2}{2})t+\sigma W_t}$$

To demonstrate the convergence of the binomial model to the Black-Scholes model, we must first calibrate the parameters of our binomial model by representing them as functions of the Black-Scholes parameters and the number of periods of our binomial model, n. These calibrations are given by

$$\begin{aligned}
R_n &= \exp(r^{\frac{T}{n}}) \\
R_n - c_n &= \exp((r-c)\frac{T}{N}) \approx 1 + r\frac{T}{n} - c\frac{T}{n} \\
u_n &= \exp(\sigma\sqrt{\frac{T}{n}}) \\
d_n &= \frac{1}{u_n}
\end{aligned}$$

Our corresponding calibrated risk-neutral probability of an up-move, q_n, is then

$$q_n = \frac{e^{(r-c)\frac{T}{n}} - d_n}{u_n - d_n}$$

An n-period binomial model calculates the price of a call option with expiry T and strike K as

$$C_0 = \frac{1}{R^n}\mathbb{E}_0^{\mathbb{Q}}\left[\max(S_T - K, 0)\right]$$

We can expand this into a summation of the products of the payoffs and their respective probabilities, as we did earlier, but this time in terms of our calibrated parameters.

$$C_0 = \frac{1}{R_n^n}\sum_{j=0}^{n}\binom{n}{j}q_n^j(1-q_n)^{n-j}\max(S_0 \cdot u_n^j d_n^{n-j} - K, 0)$$

Define $\eta := \min\{j : S_0 \cdot u_n^j d_n^{n-j} \geq K\}$. Then, η represents the minimum number of up-moves j, such that the terminal value of the stock price S_T is greater than or equal to the strike price, in which case the option will be exercised. This allows us to rewrite the payoff term $\max(S_T - K, 0)$ as simply $S_T - K$ as we are only considering states in which the option is exercised.

We can then split the summation into one part related to the evolution of the stock price and one related to the strike K.

$$\frac{S_0}{R_n^n}\sum_{j=\eta}^{n}\binom{n}{j}q_n^j(1-q_n)^{n-j}u_n^j d_n^{n-j} - \frac{K}{R_n^n}\sum_{j=\eta}^{n}\binom{n}{j}q_n^j(1-q_n)^{n-j}$$

The discrete time model divides the total time to expiry T into n periods of length Δt. Thus, as we take the limit of this above equation as $n \to \infty$, or equivalently as $\Delta t \to 0$, we converge to the continuous-time model, i.e. the Black-Scholes formula.