# Option Pricing Methods

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April 6, 2024

## Contents

# 1 Introduction

Before we embark on the maths behind pricing options, let's get the financial jargon out the way.

A financial derivative is a financial instrument whose value depends on the performance of an underlying asset. The underlying asset (or just 'underlying') could range from commodities like oil to indices like the S&P 500. An *option* is a type of financial derivative and it has the following definition:

> An option gives the option holder the right but not the obligation to buy or sell the underlying asset within an agreed timeframe in the future at a pre-determined price.

A **call option** gives the holder the right but not the obligation to **buy** the underlying asset within an agreed timeframe in the future at a pre-determined price.

A **put option** gives the holder the right but not the obligation to **sell** the underlying asset within an agreed timeframe in the future at a pre-determined price.

We call the pre-determined price the **strike price**, and we define the **expiration date** to be the date at which the option expires (the option becomes worthless after the expiry date and the holder no longer has the right that the option gave them).

The focus of this essay will be on European options. These are options where the holder can only exercise their right AT expiry.

An important use of options is using them as a means of reducing the risk of a portfolio. Say I have purchased 5 shares of stock A at £20 each. This gives the value of my portfolio exposure to stock A; whose value can fluctuate due to a wide variety of factors. To reduce this risk, I can purchase a Put Option with a strike price of £15 and maturity date 1 year from now. Let's say the price of the Option was £1 per share (so £5 overall). If the value of stock A decreased to £12 per share after 1 year, I can exercise my right on the Put Option and sell the underlying at £15 per share. My net-profit will be $15 \times 5 - 20 \times 5 - 5 =$ -£30 which is a loss. However, without the option, my loss would have been $20 \times 5 - 12 \times 5 =$ £40.

Using options to reduce the riskiness of a portfolio will be key to working out its value. This process is a type of **Hedging**.

Pricing both Put and Call options at expiry is relatively straightforward. Let T be the expiry time of the option with P(S,T), C(S,T) being the value of a

Put and Call Option respectively at time T with S as the underlying.

We can write $P(S,T)$ and $C(S,T)$ as piecewise functions:

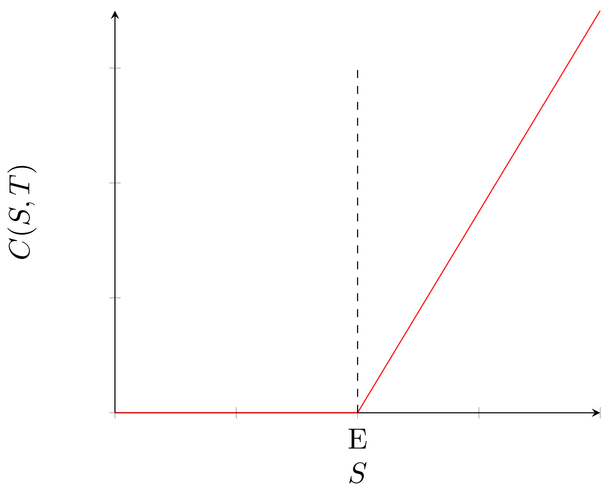
$$C(S,T) = \text{Max}(0, S-E) = \begin{cases} 0 & S \leq E \\ S-E & S > E \end{cases} \tag{1.1}$$

$$P(S,T) = \text{Max}(0, E-S) = \begin{cases} S-E & S \leq E \\ 0 & S > E \end{cases} \tag{1.2}$$

For a Call Option, when the underlying has a price $\leq E$, then it would be best to buy the underlying instead of the option since the underlying is cheaper. Hence, the Call Option is worthless. If the underlying has price $>$ E, then we will buy the Call Option. The benefit of doing this can be measured by the function S-E since it is small when S is close to E and large when S is much larger than E.

The Put Option formula can be explained similarly.The payoff diagram for C(S,T) can be shown below. P(S,T) can be sketched similarly.

Call option payoff at expiry



# 2 Modelling the underlying

The price of an asset is almost impossible to deterministically predict. This is explained by the **Efficient Market Hypothesis**

**Definition 2.1.** *The Efficient Market Hypothesis states:*

1. *The past history of an asset's price is fully reflected in the present price.*
2. *The market responds immediately to new information*

Hence, an asset's price history has no influence on the asset's future price. This is the **Markov Property** and this allows us to use Brownian Motion to model the asset's price.

## 2.1 Brownian Motion

The following definition comes from [3].

**Definition 2.2.** *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space. A standard Brownian Motion or Wiener Process denoted by* $\mathcal{B} = \{\mathcal{B}(t) : t \geq 0\}$ *is a stochastic process on* $(\Omega, \mathfrak{F}, \mathbb{P})$ *that satisfies the following properties:*

1. $\mathcal{B}(0) = 0$
2. *Sample paths of* $\mathcal{B}$ *are continuous with probability 1.*
3. $\forall\ t_1, t_2, ..., t_n$ *st* $0 \leq t_1 < t_2 < ... < t_n < \infty$*:* $\mathcal{B}(t_2) - \mathcal{B}(t_1), \mathcal{B}(t_3) - \mathcal{B}(t_2), ..., \mathcal{B}(t_n) - \mathcal{B}(t_{n-1})$ *are all mutually independent random variables.*
4. *For each* $0 \leq s < t < \infty$*, the increment* $\mathcal{B}(t) - \mathcal{B}(s) \sim N(0, t-s)$

To summarise in words, a Wiener Process starts at 0, always has a continuous sample path, has independent and normally distributed increments. It turns out that a Wiener Process can help us significantly in modelling asset prices.

## 2.2 Deriving the Stochastic Differential Equation

**Definition 2.3.** *Let* $S$ *be the price of the asset. The* ***return*** *of an asset is the quantity* $\frac{dS}{S}$ *where* $dS$ *is a small change in the asset price.*

We will model the asset price by modelling the asset's return since the asset's return measures asset performance far better than the change in the asset's price.

We can calculate the asset's return by working out what the return of the asset would be if there were no risk-factors (e.g. change in interest rates, conflicts, earnings reports). We can represent this part deterministically. Then, we add these risk-factors afterwards using a stochastic process.

Now, let S be the price of the asset at time t and consider the asset's return at t+dt and let dS be the corresponding change in the asset's price.

The deterministic part relates the average rate of the growth of the asset's price $\mu$. If no risk-factors existed, then the asset would grow at a rate of $\mu$. So over a small increase in time $dt$, the deterministic change in the asset's return is $\mu dt$.

The stochastic part reflects the risk factors. This is where the Wiener Process comes in handy. Let $\sigma$ be the volatility of the asset's return. Then, we model the impact of the risk-factors on the asset's return by $\sigma dX$ where X(t) is the Wiener Process. Also, $dX = X(t + dt) - X(t)$ and $dX \sim N(0, dt)$.

Combining these results leads to the following Stochastic Differential Equation:

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{2.1}$$

# 3 Tools needed to derive Black–Scholes Equation

## 3.1 Itô's Lemma

Ito's Lemma is for random variables what Taylor's Theorem is for deterministic variables. It relates the small change of a function of a random variable to a change in the random variable itself.

We will go through an informal proof of Ito's Lemma in this essay. First, we will state 2 theorems without proof.

**Theorem 3.1. Second-Order Taylor's Formula**

*Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of third order.*

$$\Rightarrow f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(x_0, h) \tag{3.1}$$

*where $\frac{R_2(x_0,h)}{\|h\|^2} \to 0$ as $h \to 0$ where the second sum is over all i's and j's between 1 and n inclusive.*

Second-Order Taylor's Formula approximates at function f at $x_0$ in terms of the partial derivatives up to second order along with a remainder term $R(a, h)$. This theorem was taken from [2, Chapter 3]

**Theorem 3.2.** *Let X denote a Wiener Process defined above.*

*Then $dX^2 \to dt$ as $dt \to 0$ with a probability of 1.*

With these two facts in our arsenal, we can give an informal proof of Itô's lemma.

**Lemma 3.1. Itô's Lemma** *Let $f(S,t) \in C^2(S,t)$ be such that $f : \mathbb{R}^2 \to \mathbb{R}$ where $S$ satisfies $dS = \mu dt + \sigma dX$ where $X$ is a Wiener Process.*

*Then, $df = \sigma \frac{\partial f}{\partial S} dS + \left( \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} dt + \mu \frac{\partial f}{\partial S} \right) dt$*

*Proof.* *The Stochastic Differential equation S satisfies is $dS = \mu dt + \sigma dX$*

$\Rightarrow (dS)^2 = \mu^2 dt^2 + 2\mu\sigma dX dt + \sigma^2 dX^2$

*Note that Theorem 3.2 implies that as $dt \to 0$, $dS^2 \to \sigma^2 dt$*

*Now, consider the function f at (S+dS,t+dt) and apply Taylor's Second Order Formula.*

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 + \ldots$$

*All other terms of Taylor's Second Order Formula can be ignored since terms of order higher than dt converge to 0 faster than dt does. Now substitute the expression for dS into the above equation.*

$$df = \frac{\partial f}{\partial S}(\mu dt + \sigma dX) + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 dt$$

$$\Rightarrow df = \sigma \frac{\partial f}{\partial S} dX + \left( \frac{\partial f}{\partial S}\mu + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 \right) dt$$

□

## 3.2 Heat Equation

The material in this section comes from [4, Chapter 2].

**Definition 3.1.** *The initial boundary problem for the heat equation in 1D consists of finding $u(x,t) : (0,L) \times (0,\infty) \to \mathbb{R}$ such that:*

$$\begin{gathered} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \\ u(x,0) = \phi(x) \end{gathered} \tag{3.2}$$

We will state the next theorem without proof.

**Theorem 3.3.** *The fundamental solution to 3.2 is given by*

$$\int_{\mathbb{R}} S(x-y,t)\phi(y)dy,$$
$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}$$

The fundamental solution to the Heat Equation will be key to solving Black-Scholes Formula. For a full derivation, see [4].

# 4 Deriving the Black-Scholes Equation

Now, we have all the tools we need to derive the Black-Scholes Equation. The Black-Scholes Equation helps us determine the price of a European Call Option. Note that if we know the value of the Call Option, we can work out the value of a Put Option with same underlying, strike and expiry via Put-Call Parity. See the Appendix for an explanation.

We make some fundamental assumptions about the nature of financial markets. These assumptions come from [1].

1. The underlying asset S obeys 2.1.
2. The risk-free interest rate $r$ and asset volatility $\sigma$ are constant. The risk-free interest rate is the rate of return on an asset that has no risk like a bank deposit.
3. No transaction costs associated with hedging a portfolio.
4. The underlying asset pays no dividends during the life of the option.
5. There are no arbitrage opportunities. Arbitrage is a trading strategy which guarantees a profit for a trader. The trader does this by taking advantage of price differences of the same product in different markets. If no arbitrage opportunities exist, then each risk-free asset generates the same rate of return.
6. The underlying asset can be traded continuously.
7. Short selling is allowed and the assets are divisible. Short selling occurs when a trader borrows an asset from a broker, sells the asset, rebuys the asset at a lower price and returns it to the broker with interest. We can buy or sell as many units of the underlying asset as we wish.

Now, let C(S,t) be the value of a call option with S being the underlying asset and t being time. Here, we can use Ito's Lemma to find an expression for dC.

$$dC = \sigma S\frac{\partial C}{\partial S}dX + \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt \tag{4.1}$$

Note that the presence of S comes from the fact that we are modelling $\frac{dS}{S}$ using Brownian Motion instead of just dS.

Now, let's construct a simple portfolio where we buy a call option C and sell a certain number of the underlying asset. Call this $\triangle$. Let's call this portfolio $\Pi = C - \triangle S$. Let the rate of change of the portfolio's value be $d\Pi$.

$\Rightarrow d\Pi = dC - \triangle dS$. Using 4.1 and 2.1, we can simplify this to give:

$$d\Pi = \sigma S\left(\frac{\partial C}{\partial S} - \triangle\right)dX + \left(\mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu \triangle S\right)dt$$

This is where the hedging part comes into play. We can eliminate the risk of this portfolio by setting $\triangle = \dfrac{\partial C}{\partial S}$. The stochastic element has been eliminated meaning the portfolio is now completely deterministic. $\dfrac{\partial C}{\partial S}$ is known as the **delta** of an option and hence, this process is called **Delta Hedging**.

$$\Rightarrow d\Pi = \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right)dt$$

Now, we can use the fact that no arbitrage opportunities exist,zero transaction costs and the permission to engage in short selling. Let r be our risk-free rate. Investing $\Pi$ into a bank gives a return of $\Pi rdt$ over time dt. There are 3 cases: $d\Pi > \Pi rdt$, $d\Pi < \Pi rdt$ and $d\Pi = \Pi rdt$.

*Case 1*: $\mathbf{d\Pi > \Pi rdt}$
There is an arbitrage opportunity in this case. A trader could borrow $\Pi$ from a bank at interest rate $r$ and invest in the portfolio. This is a guaranteed profit as the gain of portfolio exceeds the interest rate.

*Case 2*: $\mathbf{d\Pi < \Pi rdt}$
There is an arbitrage opportunity here as well. The trader can short the portfolio and invest in a savings account in a bank. This guarantees a profit.

Since the first two cases cannot happen, we must have that $d\Pi = \Pi rdt$. I.e. the return of the portfolio must be the same as the return given by a risk-free asset over time.

$$\Rightarrow \Pi rdt = \left(\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right)dt$$

We know $\Pi = C - \triangle S$ and $\triangle = \dfrac{\partial C}{\partial S}$. Substituting these into the above result and re-arrange gives the **Black- Scholes Partial-Differential Equa-**

**tion**.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{4.2}$$

## 4.1 Boundary and Final Conditions

Determining the price of an option as a function of S and t is a Final Value Problem. We know the price of the call option at expiry T. This is 1.1.

Also, when S=0 we know using 2.1 that dS = 0 meaning S is constant. Hence, the value of the call option is also 0. Another key point is that as $S \to \infty, C(S,T) \to S$. As the asset tends to infinity, the option becomes more and more likely to be exercised making the exercise price less and less important. Hence, the call option becomes more and more like the underlying asset.

## 4.2 Solving Black-Scholes Partial Differential Equation

Let us solve for a call option, C(S,t) first. The problem reads

$$\begin{aligned}
&\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \\
&C(0,t) = 0 \\
&C(S,t) \to S \text{ as } t \to \infty \\
&C(S,T) = \text{Max}(S - E, 0)
\end{aligned} \tag{4.3}$$

Before we solve, we need scale our partial differential equation so it achieves two things:

1. Make the equation dimensionless.
2. Remove the $S$ and $S^2$ terms so it looks more like the heat equation.

The scaling below achieves these objectives:

$$S = Ee^x, \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad C = Ev(x,\tau)$$

This results in the following equation and initial condition:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_1 - 1)\frac{\partial v}{\partial x} - k_1 v \tag{4.4}$$

$$v(x,0) = \text{Max}(e^x - 1, 0), \text{ where } k_1 = \frac{2r}{\sigma^2} \tag{4.5}$$

Now, we use a clever change of variable. Let $v = e^{\alpha x + \beta t}u(x,\tau)$ for some $\alpha$ and $\beta$ to be found. Substituting this into 4.4 gives:

$$\beta u + \frac{\partial u}{\partial r} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k_1 - 1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - k_1 u \tag{4.6}$$

We can choose $\alpha$ and $\beta$ so that there is no u term and no $\frac{\partial u}{\partial x}$ term. These choices are:

$$\begin{aligned}\beta &= \alpha^2 + (k_1-1)\alpha - k_1\\ \alpha &= \frac{1-k_1}{2}\\ \Rightarrow \beta &= -\frac{1}{4}(k_1+1)^2\end{aligned}$$

This changes 4.4 to:

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } \tau > 0 \text{ and } -\infty < x < \infty \tag{4.7}$$

$$u(x,0) = \phi(x) = \text{Max}\left(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0\right) \tag{4.8}$$

The fundamental solution to 4.7 reads:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \phi(y) e^{\frac{-(x-y)^2}{4\tau}} dy \tag{4.9}$$

We won't evaluate the integral above here for conciseness but note the presence of $e^{\frac{-(x-y)^2}{4\tau}}$ in the integral which is over $\mathbb{R}$. This suggests to us that the Normal Distribution will play a part in the solution and indeed it does.

$$\begin{aligned}u(x,\tau) &= e^{\frac{1}{2}(k_1+1)x+\frac{1}{4}(k_1+1)^2\tau}N(d_1) - e^{\frac{1}{2}(k_1-1)x+\frac{1}{4}(k_1-1)^2\tau}N(d_2),\\ d_1 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k_1+1)\sqrt{2\tau},\\ d_2 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k_1-1)\sqrt{2\tau},\\ N(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}x^2}dx \text{ for z} \in \mathbb{R}.\end{aligned}$$

Now, we work out v(x,t) from u(x,$\tau$) and undo the scaling we did.

$v(x,t) = e^{-\frac{1}{2}(k_1-1)x-\frac{1}{4}(k_1+1)^2\tau}u(x,\tau)$

Now, letting $x = log\left(\frac{S}{E}\right)$, $\tau = \frac{1}{2}\sigma^2(T-t)$, $C = Ev(x,\tau)$ gives us the solution for the Call Option:

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{4.10}$$

where $d_1 = \frac{log\left(\frac{S}{E}\right)+\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$ and $d_2 = \frac{log\left(\frac{S}{E}\right)+\left(r-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$ after unwinding our scaling.

# 5 Option Pricing via Risk-Neutrality

There are numerous ways to price options. Here, we will explore how to use the Forward and Backward Kolmogorov Equations along with the Risk-Neutral Random Walk to price options.

## 5.1 Transition Probability Density Function

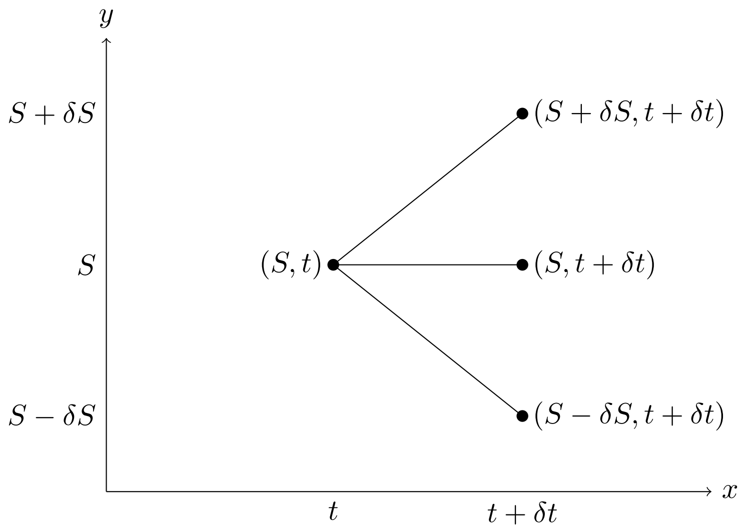
**Definition 5.1. Transition Probability Density Function**

Let $dy = A(y,t)dt + B(y,t)dX$. Then the transition probability density function p(y,t;y',t') satisfies

$$\mathbb{P}(a < y < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y,t;y',t')dy'$$

The transition probability density function answers the question 'Given the random walk is at $y$ at time $t$, what is the probability at time $t'$ in the future, $y$ will be between $a$ and $b$?'.

We can use the trinomial model to approximate the random walk for $y$.




This is a discrete approximation to the random walk and at each time step, $y$ has 3 choices: Increase, decrease or stay the same.

The size of the increase and decrease will be the same. The probabilities of each choice will be such that the mean and variance of the trinomial model will be the same as the continuous random walk over the same time step.

There are two stochastic differential equations that the transition probability density function satisfies. These are the **Forward Kolmogorov Equation** and **Backward Kolmogorov Equation**.

Say we are currently at time t with the underlying asset price S. We can create two differential equations to solve for the transition probability density function. The Forward Kolmogorov Equation models transitioning from $(S, t)$ to $(S', t')$ and hence, it solves $p(S, t; S', t')$. The Backward Kolmogorov Equation models how to transition from a past time t' to the present time t. Hence, it solves $p(S', t'; S, t)$. Note that in this context, we have fixed $(S, t)$ as our starting point and have moved forwards in the Forward Kolmogorov Equation and backwards in the Backward Kolmogorov Equation.

**Definition 5.2.** *The Forward-Kolmogorov Equation is:*

$$\frac{\partial p}{\partial t'} - \frac{1}{2}\frac{\partial^2}{\partial y'^2}(\mathrm{B(y',t')}^2 p) + \frac{\partial}{\partial y'}(\mathrm{A(y',t')p}) = 0$$

*The Backward-Kolmogorov Equation is:*

$$\frac{\partial p}{\partial t} + \frac{1}{2}B(y, t)^2\frac{\partial^2 p}{\partial y^2} + A(y, t)\frac{\partial p}{\partial y} = 0$$

For a derivation of these equations, see the Appendix.

**Example 5.1.** Let's find the probability density function for the random walk 2.1 which reads:

$$dS = \mu S dt + \sigma S dt$$

The Forward Kolmogorov Equation for this random walk is

$$\frac{\partial p}{\partial t'} - \frac{1}{2}\frac{\partial^2}{\partial y'^2}(\sigma^2 S^2 p) + \frac{\partial}{\partial y'}(\mu S p) = 0$$

The Backward Kolmogorov Equation for this random walk is

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 p}{\partial S^2} + \mu S\frac{\partial p}{\partial S} = 0$$

There are two extra boundary conditions that $p(S, t; S', t')$ satisfy:

1. $p(S, t; 0, t') = 0$. If the asset is non-zero at some time, then it cannot be 0 at a later time. We can understand this by considering the transformation $f(S) = logS$. In this random walk, $logS$ can never reach $-\infty$ which corresponds to S=0.
2. $p(S, t; S', t') \to 0$ as $S' \to \infty$. The probability of a stock increasing by a very large amount in a small timestep is extremely small.

For conciseness, I will just state the solution satisfying both stochastic differential equations above:

$$p(S,t;S',t') = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}} e^{-\frac{\left(log\left(\frac{S'}{S}\right)-(\mu-\frac{1}{2}\sigma^2)(t'-t)\right)^2}{2\sigma^2(t'-t)}} \tag{5.1}$$

## 5.2 Risk Neutrality

**Definition 5.3. Risk-Neutral Random Walk** *Let S be the price of a stock. S follows a risk-neutral random walk if dS satisfies:*

$$dS = rSdt + \sigma Sdt$$

*I.e. The average return of the stock is the same as the return on a bank deposit ($\mu = r$)*

The Backward Kolmogorov Equation for this random walk reads:

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} = 0 \tag{5.2}$$

Let $C(S,t)$ be the price of a Call Option with S as the underlying, maturity T and strike E. An interesting point about the Black-Scholes Formula is that it doesn't depend on $\mu$; the rate of growth of the underlying. This means that individuals who have different estimations for $\mu$ can agree upon the price of an Option; assuming no other parameters change. Hence, there must be a pricing method which doesn't involve $\mu$ at all. One such method is to consider the present value of the expected return at expiry.

Let U(S,t) be the expected payoff $\Rightarrow C(S,t) = e^{-r(T-t)}U(S,t)$ where U(S,t) is the expected return at expiry.

We call $e^{-r(T-t)}$ the discounting factor. It is a solution to the $\frac{dy}{dt} = ry$ with initial condition $y(T) = 1$. This equation can be reframed to answer the question 'If at time T we had 1 unit in a bank deposit which gave risk-free rate of r, how much did we have in the bank at some time t before T?'. The answer is the discounting factor and it is the present value of my bank deposit if it is 1 at time T. Hence, to get the present value of expected return at expiry, we multiply U(S,t) by the discount factor.

If we substitute this into the Black-Scholes Partial Differential Equation, we get:

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0 \tag{5.3}$$

This is the Backward Kolmogorov Equation above meaning that U is the probability density function for the risk-neutral random walk.

Since $\mu = r$, we have the same solution to the asset price random walk 5.1. We will write this with r instead of $\mu$

$$p(S,t;S',T) = \frac{1}{\sigma S'\sqrt{2\pi(T-t)}} e^{-\frac{\left(log\left(\frac{S'}{S}\right)-(r-\frac{1}{2}\sigma^2)(T-t)\right)^2}{2\sigma^2(T-t)}} \tag{5.4}$$

$\mathbb{E}_t[\mathrm{Max}(S-E,0)] = \int_0^\infty \mathrm{Max}(S-E,0)p(S^*,t^*,S,T)dS$ is the value of the option where the integral represents the expected return of the option at expiry. It can be shown that this integral can be evaluated to give the Black-Scholes Formula derived in previous sections.

# 6 Local Volatility Surface

The biggest drawback of the Black-Scholes formula is that it assumes that the volatility of the underlying is constant. This is never true in the real world. However, the Black-Scholes formula is a good starting point to uncover more about how to model the volatility of the underlying asset.

The aim will be to determine a surface with S,t as parameters for the volatility of the underlying. This is the **Local Volatility Surface** and is written as $\sigma(S,t)$.

## 6.1 Deriving the Local Volatility Surface for a Call Option

We will assume that there is a distribution of call options with varying strike prices and maturities. We will fix the asset price at $S^*$ and time to $t^*$ and assume S obeys the risk-neutral random walk.

$$C(S,T) = e^{-r(T-t^*)}\int_0^\infty \mathrm{Max}(S-E,0)p(S^*,t^*,S,T)dS \tag{6.1}$$

$$= e^{-r(T-t^*)}\int_E^\infty (S-E)p(S^*,t^*,S,T)dS \tag{6.2}$$

We will need $\dfrac{\partial C}{\partial E}$ and $\dfrac{\partial^2 C}{\partial E^2}$ later:

$$\frac{\partial C}{\partial E} = e^{-r(T-t^*)}\int_E^\infty p(S^*,t^*,S,T)dS \tag{6.3}$$

$$\frac{\partial^2 C}{\partial E^2} = e^{-r(T-t^*)}p(S^*,t^*;S,T) \tag{6.4}$$

p(S*,t*,S,T) satisfies the Forward Kolmogorov Equation:

$$\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S^2}(\sigma^2 S^2 p) - \frac{\partial}{\partial S}(rSp) \tag{6.5}$$

Let's differentiate 6.2 with respect to T. This gives:

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t^*)} \int_E^\infty (S-E)\frac{\partial p}{\partial T} dS \tag{6.6}$$

Now substitute 6.5 in place of $\dfrac{\partial p}{\partial T}$ giving

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t^*)} \int_E^\infty (S-E)\left(\frac{1}{2}\frac{\partial^2}{\partial S^2}(\sigma^2 S^2 p) - \frac{\partial}{\partial S}(rSp)\right) dS \tag{6.7}$$

Now, we will integrate by parts. Let I be the integral in the above expression. We will evaluate I separately:

$$\begin{aligned}
I &= \left[(S-E)\left(\frac{1}{2}\frac{\partial}{\partial S}(\sigma^2 S^2 p) - rSp\right)\right]_E^\infty - \int_E^\infty \frac{1}{2}\frac{\partial}{\partial S}(\sigma^2 S^2 p) - rSpdS \\
&= -\int_E^\infty \frac{1}{2}\frac{\partial}{\partial S}(\sigma^2 S^2 p) - rSpdS \\
&= \frac{1}{2}\sigma^2 E^2 p - r\int_E^\infty SpdS
\end{aligned}$$

This simplification is possible since $p(S,t;S',t') \to 0$ as $S' \to \infty$ in Forward and Backward Kolmogorov Equations for the asset price random walk.

Now, we can sub this into 6.7 and after some clever algebraic manipulation we get:

$$\begin{aligned}
\frac{\partial C}{\partial T} &= -rC + \frac{1}{2}e^{-r(T-t^*)}\sigma^2 E^2 p - re^{-r(T-t^*)}\int_E^\infty SpdS \\
&= -rC + \frac{1}{2}e^{-r(T-t^*)}\sigma^2 E^2 p - re^{-r(T-t^*)}\int_E^\infty (S-E)pdS \\
&\quad - re^{-r(T-t^*)}E\int_E^\infty pdS \\
&= \frac{1}{2}e^{-r(T-t^*)}\sigma^2 E^2 \frac{\partial^2 C}{\partial E^2} - rE\frac{\partial C}{\partial E}
\end{aligned}$$

Note how we have used 6.3 and 6.4 to simplify the above. Now, we can rearrange for $\sigma$:

$$\sigma = \sqrt{\frac{\dfrac{\partial C}{\partial t} + rE\dfrac{\partial C}{\partial E}}{\frac{1}{2}E^2\dfrac{\partial^2 C}{\partial E^2}}} \tag{6.8}$$

This gives the local volatility surface in terms of the market price for an option. The formula above is known as **Dupire's Formula**.

Dupire's Formula is an improvement on the constant assumption of volatility we made. However, it's not perfect since we need a numerical method for the partial derivative computations and differentiation is numerically unstable.

For a deeper study into volatility modelling, see [5]

# 7 Appendix

## 7.1 Put-Call Parity

We can use Put-Call Parity formula to work out the price of a Put Option using the price of a Call Option with the same underlying, same strike and same expiry T.

Let $\Pi = S+P-C$ where P is a Put and C is a Call with the same underlying S, same strike price E and same expiry date T. Π is a portfolio.

The value of the portfolio at time T is $S + \text{Max}(E - S, 0) - \text{Max}(S - E, 0)$.

When $S \leq E$, the value of the portfolio is $S + (E - S) + 0 = E$. When $S > E$, the value of the portfolio is $S + 0 - (E - S) = E$. Hence, the value of the portfolio is E for any value of the underlying.

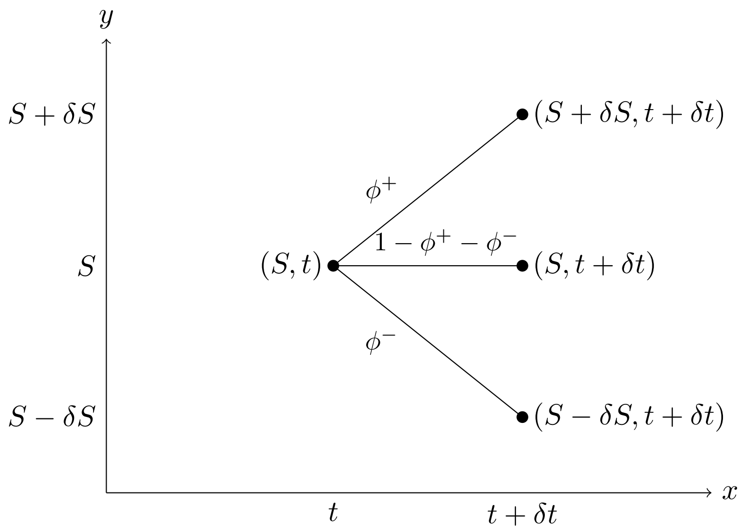
How much should I pay initially for such a portfolio where the value at expiry is E?

Before we answer this, let us answer this question: How much should I place in my bank deposit at time t so at time T, the amount in my deposit is E? Let M be the value put in my bank deposit.

$\Rightarrow \dfrac{\mathrm{d}M}{\mathrm{d}t} = rM$ where r is the risk-free rate. This yields a solution $M = Ee^{r(T-t)}$. If the amount I pay for portfolio Π is different to M at time t, then an arbitrage opportunity would arise. $e^{r(T-t)}$ is known as the **Discounting Factor** and $Ee^{r(T-t)}$ is the **Present Value** for receiving E at time T

Hence, at time T, $\mathbf{\Pi = S + P - C = Ee^{r(T-t)}}$. This is known as **Put-Call Parity**. If we know the price of the Call, we can rearrange this to get the price of the Put.

## 7.2 Forward and Backward Kolmogorov Equation




Let $\triangle\ y$ be the change in y over timestep $\delta t$. Let $\phi^+(y,t)$ be probability $\triangle\ y > 0$ and $\phi^-(y,t)$ be probability of a $\triangle\ y < 0$.

Let's derive a key fact about the transition probability density function under the random walk defined by $dy = A(y,t)dt + B(y,t)dX$.

**Proposition 7.1.**

$$\phi^+ = \frac{1}{2}\frac{\delta t}{\delta y^2}(\mathrm{B(y,t)}^2 + \mathrm{A(y,t)}\delta y) \tag{7.1}$$

$$\phi^- = \frac{1}{2}\frac{\delta t}{\delta y^2}(\mathrm{B(y,t)}^2 - \mathrm{A(y,t)}\delta y) \tag{7.2}$$

*Proof.* $\mathbb{E}[\triangle\ y] = \phi^+\delta y + (1-\phi^+-\phi^-).0 + \phi^-\delta y = (\phi^+ - \phi^-)\delta y$ *where* $\delta y$ *is the size of the increase/decrease. Note that the probability of increase,decrease and staying the same sum to 1.*

$\mathrm{Var}[\triangle\ y] = \mathbb{E}[\triangle\ y^2] - \mathbb{E}[\triangle\ y]^2$. *After simplifying, we get*
$\mathrm{Var}(\triangle\ y) = \delta y^2(\phi^+ + \phi^- - (\phi^+ - \phi^-)^2)$

*The mean of the continuous random walk is A(y,t)*$\delta t$ *and the variance is B(y,t)*$^2\delta t$.

*Hence, we have the following 2 equations:*

$$\mathrm{A(y,t)}\delta t = (\phi^+ - \phi^-)\delta y \tag{7.3}$$

$$\mathrm{B(y,t)}^2\delta t = \delta y^2(\phi^+ + \phi^- - (\phi^+ - \phi^-)^2) \tag{7.4}$$

*Solving these simultaneous equations for $\phi^+$ and $\phi^-$ give:*

$$\phi^+ = \frac{1}{2}\frac{\delta t}{\delta y^2}(\mathrm{B(y,t)}^2 + \mathrm{A(y,t)}\delta y) \tag{7.5}$$

$$\phi^- = \frac{1}{2}\frac{\delta t}{\delta y^2}(\mathrm{B(y,t)}^2 - \mathrm{A(y,t)}\delta y) \tag{7.6}$$

□

We are now ready to sketch a derivation for the Backward Kolmogorov Equation.

We will ignore the dependence on (y',t') for conciseness. There are 3 ways of getting to y' at time t' from the time step t'+$\delta t$ as shown in the diagram. Now, by the law of total probability:

$$\begin{aligned} p(y,t) =& \phi^+(y,t)p(y+\delta y, t+\delta t) + (1-\phi^+(y,t)-\phi^-(y,t))p(y,t+\delta t) + \\ & \phi^-(y,t)p(y-\delta y, t+\delta t) \end{aligned} \tag{7.7}$$

Let's evaluate the Taylor Series about (y,t) $p(y+\delta y, t+\delta t)$, $p(y, t+\delta t)$ and $p(y-\delta y, t+\delta t)$.

$$\begin{aligned} p(y+\delta y, t+\delta t) &\approx \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 \\ p(y, t+\delta t) &\approx \frac{\partial p}{\partial t}\delta t \\ p(y-\delta y, t+\delta t) &\approx -\frac{\partial p}{\partial y}\delta y + \frac{\partial p}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 \end{aligned}$$

Note that the rest of the Taylor Series is small enough to be ignored for each of the 3 transition probability density functions. We can substitute these 3 results into 7.7 to get the following differential equation:

$$\left(\phi^+(y,t) - \phi^-(y,t)\right)\frac{\partial p}{\partial y}\delta y + \frac{1}{2}\left(\phi^+(y,t) + \phi^-(y,t)\right)\frac{\partial^2 p}{\partial y^2}\delta y^2 + \frac{\partial p}{\partial t} = 0 \tag{7.8}$$

Then, we can use 7.1 and 7.2 to sub for $\phi^+$ and $\phi^-$ to give the Backward Kolmogorov Equation:

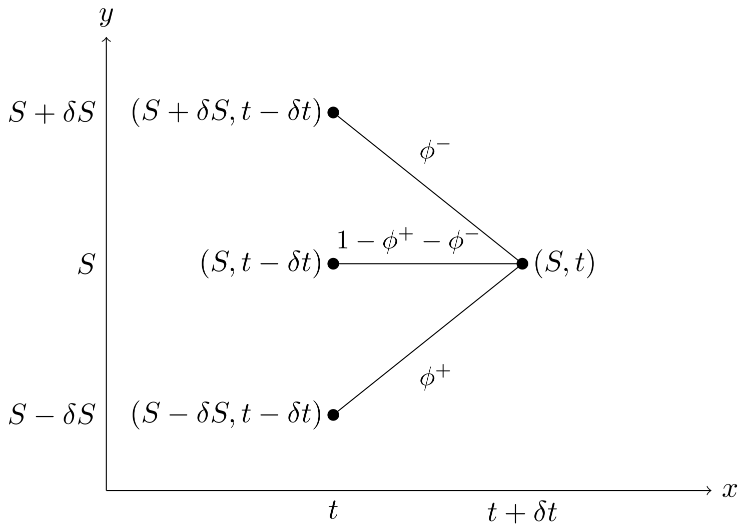
$$\frac{\partial p}{\partial t} + \frac{1}{2}B(y,t)^2\frac{\partial^2 p}{\partial y^2} + A(y,t)\frac{\partial p}{\partial y} = 0 \tag{7.9}$$

Boundary conditions are mostly unique to the random walk under consideration. The final condition is the same for most problems however:

$$p(y,t';y',t') = \delta(y-y') = \begin{cases} 0 & y \neq y' \\ \infty & y = y' \end{cases} \tag{7.10}$$

This basically means that at time t', the value of y can only be y'. This is known as the **Dirac Delta Function**.

Let's sketch a derivation of the Forward Kolmogorov Equation. It is very similar to the Backward Kolmogorov Equation derivation.




From the diagram, we see there are 3 ways to get to y' at time t' and this is reflected in the 3 expressions summed together to get p(y,t;y',t'). From this, we can use the Taylor's Series about (y',t') on $p(y, t; y' + \delta y, t' - \delta t)$,$p(y, t; y', t' - \delta t)$,$p(y, t; y' - \delta y, t' - \delta t)$. I will omit the expansion and simplification since it is similar to the Backward Kolmogorov Equation. We will omit the dependence on (y,t) for conciseness.

$$\begin{aligned}\text{p(y',t')} =& \phi^-(y' + \delta y, t' - \delta t)p(y' + \delta y, t' - \delta t) \\ &+ (1 - \phi^-(y', t' - \delta t) - \phi^+(y', t - \delta t))p(y', t' - \delta t) \\ &+ \phi^+(y' - \delta y, t - \delta t)p(y' - \delta y, t' - \delta t)\end{aligned}$$

The result is

$$\frac{\partial p}{\partial t'} - \frac{1}{2}\frac{\partial^2}{\partial y'^2}(\text{B(y',t')}^2 p) + \frac{\partial}{\partial y'}(\text{A(y',t')p}) = 0 \tag{7.11}$$

The initial condition is:

$$p(y', t') = \delta(y - y') = \begin{cases} 0 & y \neq y' \\ \infty & y = y' \end{cases} \tag{7.12}$$

## References


[1] Sam Howinson. *Option Pricing.* 1st edition, 1993.

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[5] Paul Wilmott. *Paul Wilmott on Quantitative Finance.* John Wiley & Sons Ltd, 2nd edition, 2000.