

Single-Variable Calculus

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1 Limits

1.1 Definition

1.1.1 Provisional Definition

Definition 1. $\lim_{x \rightarrow c} f(x) = L$ if we can make $f(x)$ as close as we like to L by requiring that x be sufficiently close to, but not necessarily equal to c .

1.1.2 Rigorous Definition

Definition 2. Let $f(x)$ be a function with domain D .

Then $\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in D, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

1.2 Standard Limits

- $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\sin^{-1}(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\tan(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\tan^{-1}(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$
- $\lim_{x \rightarrow 0} \frac{\log_e(1 + x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$
- If $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \infty$, then $\lim_{x \rightarrow a} f(x)^{g(x)} = \lim_{x \rightarrow a} e^{(g(x) \cdot (f(x) - 1))}$

1.3 Other Ideas

1.3.1 Uniqueness

The limit of a function at a given point is **unique**. If two or more different values of the limit can be found, the limit is said to not exist. Consequently, the left hand limit of a function at a point must equal the right hand limit at that point for the overall limit to exist.

1.3.2 Sequential Criterion for Existence of Limits

For all sequences that approach c , the answer to $\lim_{x \rightarrow c} f(x)$ should evaluate to the same quantity if the overall limit exists.

$\lim_{x \rightarrow c} f(x) = L$ iff for all sequences x_n (with $x_n \neq a$ for all n) converging to a , $f(x_n)$ converges to L .

Application: Proof that $\lim_{x \rightarrow \infty} \sin(x)$ does not exist.

The idea: We will take two sequences that both blow up to infinity (i.e. get larger and larger as we increase the index variable) and show that when we evaluate the limits using both the sequences, we get different values. By the sequential criterion for existence of limits, we will show that the limit does not exist.

Proof. (kind of) (Using sequential criterion of limits)

Consider the sequence $a_n = n\pi$ where $n \in \mathbb{Z}^+$.

n can be increased to make a_n arbitrarily large.

One way to evaluate the limit $\lim_{x \rightarrow \infty} \sin(x)$ would be to evaluate $\lim_{n \rightarrow \infty} \sin(a_n)$ where $n \in \mathbb{Z}^+$.

Then,

$$\lim_{n \rightarrow \infty} \sin(a_n) = \lim_{n \rightarrow \infty} \sin(n\pi)$$

But $\sin(n\pi) = 0 \forall n \in \mathbb{Z}$.

Therefore, this would give us the limiting value as 0.

However, now consider the sequence $b_n = (2n+1)\frac{\pi}{2}$.

Again, b_n can be made arbitrarily large by increasing n .

Similarly,

Another way to evaluate the limit $\lim_{x \rightarrow \infty} \sin(x)$ would be to evaluate $\lim_{n \rightarrow \infty} \sin(b_n)$ where $n \in \mathbb{Z}^+$.

Then,

$$\lim_{n \rightarrow \infty} \sin(b_n) = \lim_{n \rightarrow \infty} \sin\left((2n+1)\frac{\pi}{2}\right)$$

But $\sin\left((2n+1)\frac{\pi}{2}\right) = 1 \forall n \in \mathbb{Z}$.

That would give us the limiting value as 1.

But by the property of uniqueness of limits, the limit of a function at a point must be unique.

Therefore the limit does not exist. \square

1.4 L'Hôpital's Rule

If a function f is differentiable in any neighbourhood of a , except maybe at a ,

And $\left[\left(\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0 \right) \text{ or } \left(\lim_{x \rightarrow a} \frac{1}{f(x)} = 0 \text{ and } \lim_{x \rightarrow a} \frac{1}{g(x)} = 0 \right) \right]$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

1.5 Guidelines for Evaluating Limits

Do not apply limits non-uniformly.

If you apply a standard limit formula to one part of an expression, this should be done to the entire expression at once.

Never approximate.

For example, this is incorrect:

$$\lim_{x \rightarrow 0} \frac{e^x - \frac{\sin(x)}{x}}{x} = \lim_{x \rightarrow 0} \frac{e^x - \frac{x}{x}}{x} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

Do not employ L'Hôpital's Rule recklessly.

Some problems will result in an infinite loop of differentiation if you try to solve them with L'Hôpital's Rule. Others will become tedious because of long derivatives. Anyway, evaluating limits should be analytical, not mechanical.

1.6 One-Sided Limits

1.6.1 Left Hand Limit

Definition 3. Let $f(x)$ be a function and let I be an interval within its domain.

Then $\lim_{x \rightarrow a^-} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in I, 0 < x - a < \delta \implies 0 < |f(x) - L| < \varepsilon$.

1.6.2 Right Hand Limit

Definition 4. Let $f(x)$ be a function and let I be an interval within its domain.

Then $\lim_{x \rightarrow a^+} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in I, 0 < a - x < \delta \implies 0 < |f(x) - L| < \varepsilon$.

1.7 Limits to Infinity

1.7.1 Intuition

Definition 5. $\lim_{x \rightarrow \infty} f(x) = L$ iff we can make $f(x)$ as close as we like to L by making x sufficiently large.

1.7.2 Formal Definition

Definition 6. Let $f(x)$ be a function.

$\lim_{x \rightarrow \infty} f(x) = L \iff \forall \varepsilon > 0, \exists N > 0$ such that $x > N \implies |f(x) - L| < \varepsilon$

1.8 Cluster Point

Definition 7. Say a function f is defined over set S , then $x = a$ is said to be a cluster point of the function f if $\forall \delta > 0, \exists x \in (a - \delta, a + \delta) : x \neq a$ but $x \in S$.

We can check the limit and continuity of a function only at cluster points. If a point is not a cluster point, it is called an isolated point.

1.9 Polynomial Approximation of Functions to Arbitrary Precision

1.9.1 Taylor Series

If a function f is continuous and differentiable infinitely many times and its successive derivatives are continuous infinitely many times, then for some a in the domain of f ,

$$f(x) = \sum_{n=0}^{\infty} f^n(a) \cdot \frac{(x-a)^n}{n!}$$

Where $f^n(x)$ is the n^{th} derivative of $f(x)$.

1.9.2 Maclaurin Series

Maclaurin Series is just the Taylor series for $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} f^n(0) \cdot \frac{x^n}{n!}$$

Where $f^n(x)$ is the n^{th} derivative of $f(x)$

2 Continuity

2.1 Definition

Definition 8. A function f is said to be continuous at a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.

But we know that the limit only exists if both its right hand limit and left hand limit are equal. So we can expand the condition to:

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$$

$$\text{or } \lim_{h \rightarrow 0^+} f(a+h) = \lim_{h \rightarrow 0^-} f(a+h) = f(a)$$

All polynomial functions, logarithmic function, sine, cosine and exponential function are examples of continuous functions.

2.2 Continuity of Combinations of Continuous Functions

Let f and g be two functions which are continuous at a , then:

1. $h(x) = f(x) + g(x)$ is continuous at $x = a$
2. $h(x) = f(x) - g(x)$ is continuous at $x = a$
3. $h(x) = f(x) \cdot g(x)$ is continuous at $x = a$
4. $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$.

In fact, if f and g are continuous functions, then $\frac{f(x)}{g(x)}$ is continuous for all x except those values for which $g(x) = 0$

2.3 Continuity of Composite Functions

If the function g is continuous at a and the function f is continuous at $g(a)$, then the function $f(g(x))$ is continuous at a .

Example. Given $f(x) = \frac{1}{1-x}$, find the points of discontinuity of $f(f(x))$.

Solution: $f(x)$ is the ratio of two continuous (polynomial) functions. Therefore, it will have a discontinuity where $1-x=0$, i.e. at $x=1$.

So $x=1$ is one of the points where $f(f(x))$ is discontinuous.

Now, $f(f(x))$ will be discontinuous when $f(x)=1$.

$$\begin{aligned} f(x) &= 1 \\ \Rightarrow \frac{1}{1-x} &= 1 \\ \Rightarrow x &= 0 \end{aligned}$$

Therefore, there two points of discontinuity: at $x=0$ and at $x=1$.

2.4 Product of Continuous and Discontinuous Function

Let f be a continuous function at a and g be a discontinuous function at a and assume that all the limits are real and finite.

$$\text{Say } K_1 = \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

$$\text{Say } K_2 = \lim_{x \rightarrow a^-} g(x) \neq \lim_{x \rightarrow a^+} g(x) = g(x) = K_3$$

And let $h(x) = f(x) + g(x)$. We have to comment on the nature of $h(x)$.

$$\lim_{x \rightarrow a^-} h(x) = \lim_{x \rightarrow a^-} (f(x) \cdot g(x)) = K_1 K_2$$

$$\lim_{x \rightarrow a^+} h(x) = \lim_{x \rightarrow a^+} (f(x) \cdot g(x)) = K_1 K_3$$

$$h(x) = f(x) \cdot g(x) = K_1 K_3$$

Clearly, for $h(x)$ to be continuous,

$$\begin{aligned} K_1 K_2 &= K_1 K_3 \\ \Rightarrow K_1 \cdot (K_2 - K_3) &= 0 \end{aligned}$$

(But $K_2 \neq K_3$)

$$\Rightarrow K_1 = 0$$

Conclusion: The product of a continuous and a discontinuous function will be continuous at a point iff the value of the continuous function at that point is zero.

Example. Where will the function $f(x) = [x] \cdot \cos\left(\left(\frac{2x-1}{2}\right)\pi\right)$ be discontinuous?

Solution: Note that $\cos\left(\left(\frac{2x-1}{2}\right)\pi\right)$ is continuous and $[x]$ is discontinuous. The vulnerable points are integer values of x , because $[x]$ is discontinuous at integer values.

But if $x \in \mathbb{Z}$, then $\cos\left(\left(\frac{2x-1}{2}\right)\pi\right) = \cos\left((2x-1)\frac{\pi}{2}\right) = 0$.

Using the above result, since the continuous function in the product is zero at the points of discontinuity of the discontinuous function, this function is continuous everywhere.

Example. Where will the function $f(x) = [x] \cdot \sin(x\pi)$ be discontinuous?

Solution: By the same reasoning as the previous example, if x is an integer (vulnerable point of the floor function), the continuous function $\sin(x\pi) = 0$, so the function product is continuous everywhere.

2.5 Calculus and Zeros

2.5.1 Warm-Up Problem

Problem 1. Let $p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial in x with real coefficients. Assume that $p(0) = 1$, $p(1) = 2$, $p(2) = 3$ and $p(3) = 4$. Then find the value of $p(4)$.

Solution: Let us define a function $g(x) = p(x) - (x+1)$.

Then $g(0) = p(0) - (0+1) = 1 - 1 = 0$.

Similarly $g(1) = g(2) = g(3) = 0$.

Now, $g(x) = p(x) - (x+1) = (x)(x-1)(x-2)(x-3)$ (Because $g(x)$ is a polynomial of degree 4 with roots 0, 1, 2, 3)

So $p(x) = g(x) + (x+1) = x(x-1)(x-2)(x-3) + (x+1)$.

$\therefore p(4) = 4(3)(2)(1) + 5 = 29$

In general, if a problem looks like $f(x) = g(x)$ for some values of x , define $h(x) = f(x) - g(x)$. Then those values of x will be the zeros of $h(x)$.

2.5.2 Bolzano's Theorem

Theorem 1. Bolzano's Theorem: Let a function f be continuous in the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs. Then there exists at least one $c \in (a, b)$ such that $f(c) = 0$. In other words, there exists at least 1 zero of $f(x)$ in (a, b) .

2.5.3 Important Results Derived From Bolzano's Theorem

Theorem 2. Every polynomial of odd degree will have at least one zero.

Proof. (Using Bolzano's Theorem)

Let $f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_2x^2 + a_1x + a_0$ where $a_{2n+1} \neq 0$.

(Note that $f(x)$ is continuous $\forall x \in \mathbb{R}$)

$$\Rightarrow f(x) = x^{2n+1} \left(a_{2n+1} + \frac{a_{2n}}{x} + \frac{a_{2n-1}}{x^2} + \dots + \frac{a_1}{x^{2n}} + \frac{a_0}{x^{2n+1}} \right)$$

Case 1: ($a_{2n+1} > 0$)

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

Using Bolzano's Theorem, there exists at least one root of $f(x)$ in the interval $(-\infty, \infty)$.

Case 2: ($a_{2n+1} < 0$)

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

Again, Using Bolzano's Theorem, there exists at least one root of $f(x)$ in the interval $(-\infty, \infty)$. □

Theorem 3. For a polynomial of even degree whose leading coefficient and constant term are of opposite signs, there exist at least two real zeros.

Proof. (Using Bolzano's Theorem)

Let $f(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \dots + a_2x^2 + a_1x + a_0$.

$$\Rightarrow f(x) = x^{2n} \left(a_{2n} + \frac{a_{2n-1}}{x} + \frac{a_{2n-2}}{x^2} + \dots + \frac{a_3}{x^{2n-2}} + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}} \right)$$

WLOG, let $a_{2n} > 0$ and $a_0 < 0$.

Then $f(0) = a_0 < 0$. (1)

And $\lim_{x \rightarrow \infty} f(x) = \infty > 0$. (2)

Also, $\lim_{x \rightarrow -\infty} f(x) = \infty > 0$. (3)

Applying Bolzano's Theorem on (1) and (2), there exists at least one root of $f(x)$ in $(0, \infty)$

Applying Bolzano's Theorem on (1) and (3), there exists at least one root of $f(x)$ in $(-\infty, 0)$

Therefore, there exist at least two real zeros of $f(x)$. □

2.5.4 Problems based on Bolzano's Theorem

Problem 2. Suppose $f : [a, b] \rightarrow [a, b]$ is a continuous function. Prove that there exists some $x_0 \in [a, b]$ such that $f(x_0) = x_0$. What does it say about the graph of $y = f(x)$?

Proof. Let $g(x) = f(x) - x$.

g is continuous in $[a, b]$ since the difference of two continuous functions is continuous.

$$g(a) = f(a) - a,$$

$$g(b) = f(b) - b.$$

Case 1: $f(a) = a$

In this case, the given statement is trivially true, where $x_0 = a$.

Case 2: $f(b) = b$

In this case also, the given statement is trivially true, where $x_0 = b$.

Case 3: $f(a) \neq a$ and $f(b) \neq b$

Since $f(a) \neq a$ and $f(b) \neq b$ and the co-domain of f is $[a, b]$

$$\Rightarrow f(a) > a \text{ and } f(b) < b$$

$$\Rightarrow f(a) - a > 0 \text{ and } f(b) - b < 0$$

$$\Rightarrow g(a) > 0 \text{ and } g(b) < 0$$

Using Bolzano's Theorem, $g(a) = f(a) - a$ has a root in (a, b) .

$$\Rightarrow \exists x_0 : f(x_0) = x_0.$$

This means, the graph of $y = f(x)$ intersects the graph of $y = x$ at least once.

□

Problem 3. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function and $f(0) = f(1)$. Then prove that there exists some $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

Proof. (Using Bolzano's Theorem)

$$\text{Let } g(x) = f(x) - f\left(x + \frac{1}{2}\right)$$

g is continuous in $\left[0, \frac{1}{2}\right]$ since f is continuous in $[0, 1]$.

$$g(0) = f(0) - f\left(\frac{1}{2}\right) \tag{1}$$

$$g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) \tag{2}$$

$$g(0) + g\left(\frac{1}{2}\right) = f(0) - f(1) = 0 \tag{3}$$

$$g(0) + g\left(\frac{1}{2}\right) = 0$$

$$\text{Case 1: } g(0) = g\left(\frac{1}{2}\right) = 0$$

In this case, the given statement will be trivially true where c can be 0 or $\frac{1}{2}$.

$$\text{Case 2: } g(0) > 0 \text{ and } g\left(\frac{1}{2}\right) < 0$$

$$\text{Using Bolzano's Theorem, } \exists c \in \left(0, \frac{1}{2}\right) : f(c) = f\left(c + \frac{1}{2}\right) \Rightarrow c \in \left[0, \frac{1}{2}\right] \quad \because \left(0, \frac{1}{2}\right) \subset \left[0, \frac{1}{2}\right]$$

$$\text{Case 3: } g(0) < 0 \text{ and } g\left(\frac{1}{2}\right) > 0$$

$$\text{Using Bolzano's Theorem, } \exists c \in \left(0, \frac{1}{2}\right) : f(c) = f\left(c + \frac{1}{2}\right) \Rightarrow c \in \left[0, \frac{1}{2}\right] \quad \because \left(0, \frac{1}{2}\right) \subset \left[0, \frac{1}{2}\right]$$

□

Problem 4. Suppose f is a continuous function and $f(0) = f(1)$. Let n be any natural number, then prove that there exists some c such that $f(c) = f\left(c + \frac{1}{n}\right)$.

Proof. (Note: this is just the generalized version of the previous problem)

$$\text{Let } g(x) = f(x) - f\left(x + \frac{1}{n}\right)$$

g is continuous in \mathbb{R} since f is continuous in \mathbb{R} .

We get the following telescoping series.

$$\begin{aligned} g(0) &= f(0) - f\left(\frac{1}{n}\right) \\ g\left(\frac{1}{n}\right) &= f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right) \\ g\left(\frac{2}{n}\right) &= f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right) \\ &\vdots = \vdots \\ g\left(\frac{n-2}{n}\right) &= f\left(\frac{n-2}{n}\right) - f\left(\frac{n-1}{n}\right) \\ g\left(\frac{n-1}{n}\right) &= f\left(\frac{n-1}{n}\right) - f(1) \end{aligned}$$

$$g(0) + g\left(\frac{1}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) = f(0) - f(1) = 0 \quad [\because f(0) = f(1)]$$

$$\Rightarrow g(0) + g\left(\frac{1}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) = 0$$

$$\text{Case 1: } g(0) = g\left(\frac{1}{n}\right) = \cdots = g\left(\frac{n-1}{n}\right) = 0$$

In this case, the given statement is trivially true for $c = 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$.

Case 2: At least one term is positive and at least one term is negative.

WLOG, Let $g(a)$ be one of the positive term(s) and $g(b)$ be one of the negative term(s) and let $a < b$.

Since g is continuous and $g(a) > 0; g(b) < 0$,

Using Bolzano's Theorem, $\exists c : g(c) = 0$

$$\Rightarrow \exists c : f(c) - f\left(c + \frac{1}{n}\right) = 0$$

$$\Rightarrow \exists c : f(c) = f\left(c + \frac{1}{n}\right)$$

□

2.6 Checking Continuity of Some Weird Functions

2.6.1 Dirichlet Function

The Dirichlet Function $f : \mathbb{R} \rightarrow \{0, 1\}$ is defined in the following way:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

In other words, the Dirichlet Function maps to 1 if x is rational and 0 if x is irrational.

Claim: The Dirichlet Function is discontinuous everywhere.

If the input at some point is rational, we can approach it using a sequence consisting entirely of rational numbers and a sequence consisting of entirely irrational numbers; the former will give us the limiting value as 1 and the latter will give us the limiting value as 0.

Similarly, if the input at some point is irrational, we can again approach it with sequences consisting entirely of rational or irrational terms, resulting in two limiting values: 0 and 1.

Therefore, the limit of this function doesn't exist at any point, which means the Dirichlet function discontinuous everywhere.

2.6.2 Thomae's Function

Thomae's Function (commonly called the Popcorn Function) is defined in the following way:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} : m, n \in \mathbb{Z}; n \neq 0; \gcd(m, n) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Claim: This function is discontinuous at all rational values of x .

Proof. (analogous to that of Dirichlet Function)

Let $x = \frac{m}{n}$ be a rational number where $m, n \in \mathbb{Z}, \gcd(m, n) = 1$

Then $f(x) = \frac{1}{n}$

If we approach x from a sequence of irrationals, we will get the limiting value as 0. Even if the limit does exist, $\frac{1}{n}$ (the value of the function) cannot be equal to 0 (The limiting value).

So it has to be discontinuous for all rational inputs. □

Claim: This function is continuous at all irrational values of x .

Proof. Let x_0 be an irrational number.

$\Rightarrow f(x_0) = 0$

Consider a sequence $x_n (n \geq 1)$ such that $\lim_{n \rightarrow \infty} x_n = x_0$.

Case 1: If x_n is irrational $\forall n$, $\lim_{n \rightarrow \infty} f(x_n) = 0$.

Case 2: Otherwise, Let x_{n_k} be a subsequence of x_n , where $x_{n_k} \in \mathbb{Q} \forall k$. Again, $\lim_{k \rightarrow \infty} x_{n_k} = x_0$

Let $x_{n_k} = \frac{p_{n_k}}{q_{n_k}}$

Then $q_{n_k} \rightarrow \infty$ as $k \rightarrow \infty \Rightarrow \lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} \left(\frac{1}{q_{n_k}} \right) = 0$

$\lim_{x \rightarrow x_0} f(x) = f(x_0) = 0$

Therefore, $f(x)$ is continuous at all irrational points. □

2.7 The Intermediate Value Theorem

2.7.1 Statement and Proof

Theorem 4. If f is a function that is continuous in the closed interval $[a, b]$, then $f(x)$ takes all the values lying between $f(a)$ and $f(b)$ (both inclusive).

Note that although technically Bolzano's Theorem is a corollary of the Intermediate Value Theorem, we will prove the Intermediate Value Theorem using Bolzano's Theorem here. Alternatively, Bolzano's theorem can be proved by Bisection.

Proof. WLOG, let $f(a) < f(b)$.

Let k be any real number between $f(a)$ and $f(b)$.

$\Rightarrow f(a) < k < f(b)$

Then we have to prove that there exists at least one solution to $f(x) = k$.

Let us define $g(x) = f(x) - k$.

Then $g(a) = f(a) - k < 0$

And $g(b) = f(b) - k > 0$

\Rightarrow There exists at least one c such that $g(c) = 0$ or $f(c) - k = 0$ or $f(c) = k$ □

Important: The converse of this is not necessarily true.

2.7.2 Example Problems

Problem 5. Does the function $f(x) = \frac{x^3}{4} - \sin(\pi x) + 3$ take the value $\frac{7}{3}$ in the interval $[-2, 2]$?

Solution. $f(x)$ is continuous everywhere because $\frac{x^3}{4}$, $\sin(\pi x)$ and 3 are continuous everywhere.

$$f(-2) = \frac{(-2)^3}{4} - \sin(-2\pi) + 3 = 1 + \sin(2\pi) = 1$$

$$f(2) = \frac{(2)^3}{4} - \sin(2\pi) + 3 = 5 - \sin(2\pi) = 5$$

$\Rightarrow f(x)$ takes all values in $[1, 5]$ in the interval $[-2, 2]$ (By Intermediate Value Theorem)

And $\frac{7}{3} = 2.333\ldots \in [1, 5]$.

Therefore, yes, $f(x)$ will take the value $\frac{7}{3}$ in the interval $[-2, 2]$.

2.8 Boundedness

2.8.1 Definition

Definition 9. A function f defined on a set S (domain) is said to be bounded if $\exists M > 0$ such that $|f(x)| \leq M \forall x \in S$

Geometrically call a function bounded if the function can be contained between two horizontal lines $y = M$ and $y = -M$

2.9 Absolute Maxima and Minima

2.9.1 Absolute/Global Maximum

Definition 10. Let f be a function defined on set S . The point $c \in S$ is said to be a point of absolute or global maximum if $f(c) \geq f(x) \forall x \in S$

2.9.2 Absolute/Global Minimum

Definition 11. Let f be a function defined on set S . The point $c \in S$ is said to be a point of absolute or global minimum if $f(c) \leq f(x) \forall x \in S$

2.10 Properties of a Continuous Function

If a function f is continuous on the closed interval $[a, b]$, then:

1. f is bounded on the given interval.
2. There is a point of absolute minimum and maximum in this interval.
3. Say m is the least value of f and M is the maximum value of f , then f will take all values lying in the interval $[m, M]$, i.e. $m \leq f(x) \leq M$ (consequence of the Intermediate Value Theorem).
4. A continuous function f defined on $[a, b]$ maps to a closed interval $[c, d]$

2.11 Problems Based on Continuity

Problem 6. Let f be continuous on $[a, b]$ and $x_1, x_2, x_3, \dots, x_{n-1}, x_n \in [a, b]$, then prove that there exists some $c \in [a, b]$ such that $f(c) = \frac{1}{n} [f(x_1) + f(x_2) + f(x_3) + \dots + f(x_{n-1}) + f(x_n)]$

Proof. Since f is continuous on $[a, b]$, f will attain a maximum and minimum. Let m be the minimum and M be the maximum. Then,

$$\begin{aligned} m &\leq f(x_1) \leq M \\ m &\leq f(x_2) \leq M \\ &\vdots \\ m &\leq f(x_n) \leq M \end{aligned}$$

$$n \cdot m \leq f(x_1) + f(x_2) + f(x_3) + \dots + f(x_{n-1}) + f(x_n) \leq n \cdot M$$

Divide throughout by n .

$$m \leq \frac{f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n)}{n} \leq M$$

Therefore, by the Intermediate Value Theorem,

$$\exists c \in [a, b] : f(c) = \frac{f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n)}{n}$$

□

3 Derivatives and Differentiability

3.1 Definition

Definition 12. Let f be a function. Then the derivative of f at $x = a$ is defined as the limit:

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \end{aligned}$$

The function is said to be differentiable at $x = a$ if this limit exists. In other words, the function will be called differentiable if the left hand derivative equals the right hand derivative (which are just the one-sided equivalents of the above).

3.1.1 Left-Hand Derivative

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0^-} \frac{f(a + h) - f(a)}{h}$$

3.1.2 Right-Hand Derivative

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h}$$

3.1.3 Geometric Intuition

Graphically, the derivative of a function at a point represents the slope of the tangent at that point. However, this may not always hold true, such as in cases where drawing the graph isn't possible. In general, the derivative can be thought of as a measure of sensitivity of a function to small changes in its input.

3.2 Differentiability Implies Continuity

If a function is differentiable at a point, then it must be continuous at that point.

However, the converse is not true. If a function is continuous at a point, that doesn't mean it must be differentiable.

The contrapositive is always true, i.e. if a function is discontinuous at a point, it must be non-differentiable at that point.

3.3 Differentiability of Combinations of Differentiable Functions

Let f and g be functions differentiable at $x = a$.

Then,

1. $h(x) = f(x) + g(x)$ is differentiable at $x = a$.
2. $h(x) = f(x) - g(x)$ is differentiable at $x = a$.
3. $h(x) = f(x) \cdot g(x)$ is differentiable at $x = a$.
4. $h(x) = \frac{f(x)}{g(x)}$ is differentiable at $x = a$ if $g(a) \neq 0$.

In fact, if f and g are differentiable functions, then $\frac{f(x)}{g(x)}$ is differentiable for all x except those values for which $g(x) = 0$

3.4 Combination of Differentiable and Non-Differentiable Functions

Let f be a function differentiable at $x = a$ and let g be a function non-differentiable but continuous at $x = a$. Then:

1. $(f + g)$ is non-differentiable at $x = a$.
2. $(f - g)$ is non-differentiable at $x = a$.
3. $(f \cdot g)$ is differentiable at $x = a$ if $f(a) = 0$ and non-differentiable otherwise.
4. $\left(\frac{f}{g}\right)$ is differentiable at $x = a$ if $f(a) = 0$ and $g(a) \neq 0$ and non-differentiable otherwise.

3.5 Differentiability of the Modulus Function

3.5.1 Differentiability of $|x|$ and $|x - a|$ where a is Constant

Let $f(x) = |x|$

The vulnerable point is at $x = 0$. Differentiable everywhere else.

$$\begin{aligned} LHD &= \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \rightarrow 0^-} \left(\frac{|h|}{h} \right) = \lim_{h \rightarrow 0^-} \left(-\frac{h}{h} \right) = -1 \end{aligned}$$

$$\begin{aligned} RHD &= \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \rightarrow 0^+} \left(\frac{|h|}{h} \right) = \lim_{h \rightarrow 0^+} \left(\frac{h}{h} \right) = 1 \end{aligned}$$

Since $LHD \neq RHD$, $|x|$ is non-differentiable only at $x = 0$.

Similarly, $|x - a|$ will be non-differentiable only at $x = a$.

In fact, if g is a differentiable function, then the only points of vulnerability for differentiability of $|g(x)|$ are those points where $g(x) = 0$.

3.5.2 Differentiability of $|x|^a$ where a is Constant

$y = |x|^a$ will be non-differentiable at $x = 0$ for all $a \leq 1$ and differentiable at $x = 0$ for all $a > 1$

Example: Consider $f(x) = |x|^{1.01}$

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{|0+h|^{1.01} - |0|^{1.01}}{h} = \lim_{h \rightarrow 0} \frac{|h|^{1.01}}{h} = \lim_{h \rightarrow 0} \frac{|h| \cdot |h|^{0.01}}{h} \\ LHD &= \lim_{h \rightarrow 0^-} \frac{|h| \cdot |h|^{0.01}}{h} = \lim_{h \rightarrow 0^-} \frac{(-h) \cdot (-h)^{0.01}}{-h} = \lim_{h \rightarrow 0^-} (-h)^{1.01} = 0 \\ RHD &= \lim_{h \rightarrow 0^+} \frac{|h| \cdot |h|^{0.01}}{h} = \lim_{h \rightarrow 0^+} \frac{(h) \cdot (h)^{0.01}}{h} = \lim_{h \rightarrow 0^+} (h)^{1.01} = 0 \end{aligned}$$

Therefore, $|x|^{1.01}$ is differentiable at $x = 0$.

3.6 Important Result on Differentiability

If a function f is differentiable at x , then $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$.

Proof.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} \quad (2)$$

This will only swap the left-hand limit and the right-hand limit, but they're equal since the overall limit exists because function is differentiable.

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - \cancel{f(x)} + \cancel{f(x)} - f(x-h)}{2h} \quad (3)$$

Subtracted eq (2) from eq (1), then divided both sides by 2.

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} \quad (4)$$

□

Important Note: The converse of this statement is not necessarily true. Counterexample: $y = |x|$. Also note that this can be used as an alternative definition of the derivative but should only be used when we know that the function is differentiable at that point.

3.7 Leibniz Formula for Derivatives

Let u and v be functions which are differentiable n times. Then,

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^n \binom{n}{r} u^{(r)} \cdot v^{(n-r)}$$

Where $u^{(r)}$ denotes the r^{th} derivative of u and $v^{(n-r)}$ represents the $(n-r)^{th}$ derivative of v .

3.8 Differentiability, Existence of Derivative and Continuity of Derivative

Differentiability is the same as the existence of derivative.

However, if a function is differentiable at a point (i.e. has a real and finite derivative at that point), that does **NOT** mean derivative of the function is continuous at that point.

More formally, the existence of $f'(a)$ is **NOT** the same as the continuity of $f'(x)$ at $x = a$.

Example. Consider the following function:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = 0$$

$$\text{But } f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) \text{ does not exist.}$$

$$\Rightarrow f'(x) \text{ is discontinuous at } x = 0.$$

3.9 Standard Derivatives

- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$
- $\frac{d^n}{dx^n} x^n = n!$
- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \tan(x) = \sec^2(x)$
- $\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$
- $\frac{d}{dx} \csc(x) = -\csc(x) \cot(x)$
- $\frac{d}{dx} \cot(x) = -\csc^2(x)$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \log_e(x) = \frac{1}{x}$
- $\frac{d}{dx} a^x = a^x \log_e(a)$
- $\frac{d}{dx} \log_a(x) = \frac{1}{x \log_e(a)}$
- $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x| \sqrt{x^2-1}}$
- $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \csc^{-1}(x) = \frac{-1}{|x| \sqrt{x^2-1}}$
- $\frac{d}{dx} \cot^{-1}(x) = \frac{1}{1+x^2}$

4 Miscellaneous Problems on Differential Calculus

Problem 7. Suppose f and g are differentiable functions. Find the derivative of $f \cdot g$.

Solution: Let $p(x) = f(x) \cdot g(x)$

$$\begin{aligned}
 p'(x) &= \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} && \text{Add and subtract } f(x) \cdot g(x+h) \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{g(x+h) \cdot (f(x+h) - f(x)) + f(x) \cdot (g(x+h) - g(x))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{g(x+h) \cdot (f(x+h) - f(x))}{h} + \lim_{h \rightarrow 0} \frac{f(x) \cdot (g(x+h) - g(x))}{h} \\
 &= g(x)f'(x) + f(x)g'(x)
 \end{aligned}$$

This effectively proves the product rule for derivatives.

Problem 8. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is non-differentiable at:

Solution: Note that $\cos(|x|) = \cos(x) \forall x \in \mathbb{R}$ which is differentiable everywhere. So the points of vulnerability will be those where $x^2 - 3x + 2 = 0$.

$$\begin{aligned}
 x^2 - 3x + 2 &= 0 \\
 x^2 - 2x - x + 2 &= 0 \\
 x(x+2) - 1(x-2) &= 0 \\
 (x-1)(x-2) &= 0 \\
 x &= 1 \text{ or } x = 2
 \end{aligned}$$

At $x = 1$, the function will be differentiable because $x^2 - 1 = 1 - 1 = 0$. This comes from the result that for a product of a differentiable function with a non-differentiable function, the resulting function will be differentiable at a point if the differentiable function becomes equal to zero at that point.

That is not the case at $x = 2$

Therefore, the only point where $f(x)$ is non-differentiable is at $x = 2$.

Problem 9. Let f be a function such that $|f(x)| \leq x^2 \forall x$. Prove that f is differentiable at 0.

Proof.

$$\begin{aligned} |f(x)| &\leq x^2 && \text{Substitute } x = 0 \\ |f(0)| &\leq 0 && \text{But } |f(0)| \text{ can't be negative} \\ \Rightarrow f(0) &= 0 \end{aligned}$$

Now,

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{f(x)}{x} \end{aligned}$$

Also,

$$\begin{aligned} |f(x)| &\leq x^2 \quad \forall x \in \mathbb{R} \\ \Rightarrow |f(x)| &\leq |x|^2 \quad \forall x \in \mathbb{R} && \text{Divide by } |x| \\ \Rightarrow \left| \frac{f(x)}{x} \right| &\leq |x| \quad \forall x \in \mathbb{R} - \{0\} \\ \Rightarrow \left| \lim_{x \rightarrow 0} \frac{f(x)}{x} \right| &\leq \lim_{x \rightarrow 0} |x| \\ \Rightarrow \left| \lim_{x \rightarrow 0} \frac{f(x)}{x} \right| &\leq 0 \\ \Rightarrow |f'(0)| &\leq 0 && \text{But } |f'(0)| \text{ can't be negative} \\ \Rightarrow f'(0) &= 0 \end{aligned}$$

Therefore, f is differentiable at 0. □

Problem 10. Let a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation $f(x + y) = f(x) + f(y) \forall x, y \in \mathbb{R}$. If the function $f(x)$ is continuous at $x = 0$, discuss the continuity of $f(x)$ over the set of reals.

Solution:

$$\begin{aligned} f(x + y) &= f(x) + f(y) \quad \forall x, y \in \mathbb{R} && \text{Substitute } x = 0, y = 0 \\ f(0 + 0) &= f(0) + f(0) \\ f(0) &= 2f(0) \\ f(0) &= 0 \end{aligned}$$

Since f is continuous at $x = 0$,

$$\lim_{h \rightarrow 0} f(h) = f(0) = 0$$

Now, let a be any arbitrary real number.

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{h \rightarrow 0} f(a + h) \\ &= \lim_{h \rightarrow 0} (f(a) + f(h)) \\ &= \lim_{h \rightarrow 0} f(a) + \lim_{h \rightarrow 0} f(h) \\ &= f(a) + 0 = f(a) \end{aligned}$$

Therefore, f is continuous $\forall x \in \mathbb{R}$.

Problem 11. If $f(x + y) = f(x) \cdot f(y) \forall x, y$ and $f(x) = 1 + g(x) \cdot G(x)$ where $\lim_{x \rightarrow 0} g(x) = 0$ and $\lim_{x \rightarrow 0} G(x)$ exists, prove that $f(x)$ is continuous for all x .

Proof. If we can prove that f is continuous for some arbitrary a , then that will prove that f is continuous for all real values of x .

Now,

$$\begin{aligned}
 f(x) &= 1 + g(x) \cdot G(x) \forall x && \text{(Given)} \\
 f(h) &= 1 + g(h) \cdot G(h) \forall h \\
 \lim_{h \rightarrow 0} f(h) &= \lim_{h \rightarrow 0} (1 + g(h) \cdot G(h)) \\
 &= 1 + \lim_{h \rightarrow 0} (g(h) \cdot G(h)) && \begin{aligned} &\text{Also } \lim_{x \rightarrow 0} g(x) = 0 \\ &\text{And } \lim_{h \rightarrow 0} G(h) \text{ exists} \end{aligned} \\
 &= 1 + 0 \\
 &= 1 \\
 f(a + h) &= f(a) \cdot f(h) && \text{(Given)} \\
 \Rightarrow \lim_{h \rightarrow 0} f(a + h) &= \lim_{h \rightarrow 0} (f(a) \cdot f(h)) \\
 &= f(a) \cdot \lim_{h \rightarrow 0} f(h) \\
 &= f(a) \cdot 1 \\
 &= f(a) \\
 \Rightarrow \lim_{h \rightarrow a} f(h) &= f(a)
 \end{aligned}$$

$\Rightarrow f$ is continuous at a .

$\Rightarrow f$ is continuous $\forall x \in \mathbb{R}$

□

Problem 12. Find the number of points of discontinuity of $\lfloor 2x^3 - 5 \rfloor$ in the interval $[1, 2]$.

Solution:

$$\lfloor 2x^3 - 5 \rfloor = \lfloor 2x^3 \rfloor - 5$$

This function will be discontinuous at points where $2x^3$ is an integer.

We are given the interval $[1, 2]$.

$$2(1)^3 = 2$$

$$2(2)^3 = 16$$

Between 2 and 16 (both inclusive), we have $16 - 2 + 1 = 15$ integers.

But at $x = 1$, although $2x^3$ is an integer, our domain doesn't go below $x = 1$, so the function will be continuous at $x = 1$.

This means, the total number of points of discontinuity will be 14.

Problem 13. Let $f(x)$ be a continuous function defined for $1 \leq x \leq 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then find the value of $f(1.5)$.

Solution: It is given that $f(x)$ is continuous in $[1, 3]$ and takes only rational values.

Claim: If a continuous function takes only rational values, then it must be constant.

In any neighbourhood of $x = 2$, if f is either strictly increasing or strictly decreasing, then f will take all values between its minimum and maximum, which will include irrational values because between any two rational numbers, there exist infinitely many irrational numbers. So f can't be strictly increasing or strictly decreasing in any neighbourhood of 2. Therefore, f must be constant.

This means, $f(1.5) = 10$.

Problem 14. Evaluate $\lim_{x \rightarrow 0^+} x \left(\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor + \cdots + \left\lfloor \frac{15}{x} \right\rfloor \right)$

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} x \left(\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{15}{x} \right\rfloor \right) &= \lim_{x \rightarrow 0^+} x \left(\frac{1}{x} - \left\{ \frac{1}{x} \right\} + \frac{2}{x} - \left\{ \frac{2}{x} \right\} + \cdots + \frac{15}{x} - \left\{ \frac{15}{x} \right\} \right) \\
 &= \lim_{x \rightarrow 0^+} x \left(\frac{1}{x} + \frac{2}{x} + \cdots + \frac{15}{x} - \left(\left\{ \frac{1}{x} \right\} + \left\{ \frac{2}{x} \right\} + \cdots + \left\{ \frac{15}{x} \right\} \right) \right) \\
 &= \lim_{x \rightarrow 0^+} x \left(\frac{1+2+\cdots+15}{1} - \left(\left\{ \frac{1}{x} \right\} + \left\{ \frac{2}{x} \right\} + \cdots + \left\{ \frac{15}{x} \right\} \right) \right) \\
 &= 120 - \lim_{x \rightarrow 0^+} x \left(\left\{ \frac{1}{x} \right\} + \left\{ \frac{2}{x} \right\} + \cdots + \left\{ \frac{15}{x} \right\} \right)
 \end{aligned}$$

Note that the value of fractional part lies between 0 and 1.

$$\therefore 0 \leq \left(\left\{ \frac{1}{x} \right\} + \left\{ \frac{2}{x} \right\} + \cdots + \left\{ \frac{15}{x} \right\} \right) \leq 15$$

\Rightarrow A number approaching 0 times a number between 0 and 15 will approach 0.

$$\therefore \lim_{x \rightarrow 0^+} x \left(\left\{ \frac{1}{x} \right\} + \left\{ \frac{2}{x} \right\} + \cdots + \left\{ \frac{15}{x} \right\} \right) = 0$$

$$\begin{aligned}
 \Rightarrow \lim_{x \rightarrow 0^+} x \left(\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{15}{x} \right\rfloor \right) &= 120 - 0 \\
 &= 120
 \end{aligned}$$

Problem 15. Prove that $\lim_{x \rightarrow 5} 3x = 15$

Proof. According to the ε - δ definition of a limit, we have to effectively prove that $\forall \varepsilon > 0, \exists \delta > 0 \forall x$ such that $0 < |x - 5| < \delta \Rightarrow |3x - 15| < \varepsilon$.

We are given any $\varepsilon > 0$,

Let us choose $\delta = \frac{\varepsilon}{3}$.

Then,

$$\begin{aligned}
 0 &< |x - 5| < \delta \\
 \Rightarrow 0 &< |x - 5| < \frac{\varepsilon}{3} && \text{Multiply by 3} \\
 \Rightarrow 0 &< 3 \cdot |x - 5| < \varepsilon \\
 \Rightarrow 0 &< |3x - 15| < \varepsilon
 \end{aligned}$$

i.e. $0 < |x - 5| < \delta \Rightarrow 0 < |3x - 15| < \varepsilon$.

Therefore, $\lim_{x \rightarrow 5} 3x = 15$ for $\delta = \frac{\varepsilon}{3}$.

□

Problem 16. Let $g(x) = \log_e(f(x))$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. Then for $N = 1, 2, 3, \dots$, $g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$ is equal to:

- (A) $-4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right)$ (B) $4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right)$
 (C) $-4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N+1)^2}\right)$ (D) $4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N+1)^2}\right)$

Solution: The correct option is **(A)**.

$$f(x+1) = x \cdot f(x) \quad (\text{Given})$$

$$\Rightarrow x = \frac{f(x+1)}{f(x)}$$

$$g(x) = \log_e(f(x)) \quad (\text{Given})$$

$$\Rightarrow g(x+1) - g(x) = \log_e(f(x+1)) - \log_e(f(x))$$

$$= \log_e\left(\frac{f(x+1)}{f(x)}\right)$$

$$= \log_e(x) \quad (\text{Derived above})$$

Differentiating both sides,

$$\Rightarrow g'(x+1) - g'(x) = \frac{1}{x}$$

Again differentiating both sides,

$$\Rightarrow g''(x+1) - g''(x) = \frac{-1}{x^2}$$

Now, replacing x by $\left(x - \frac{1}{2}\right)$,

$$\Rightarrow g''\left(x + \frac{1}{2}\right) - g''\left(x - \frac{1}{2}\right) = \frac{-4}{(2x-1)^2}$$

$$\Rightarrow g''\left(\frac{2x+1}{2}\right) - g''\left(\frac{2x-1}{2}\right) = \frac{-4}{(2x-1)^2}$$

This forms the following telescopic sum:

$$\begin{array}{rcl} g''\left(\frac{2+1}{2}\right) - g''\left(\frac{2-1}{2}\right) & = & \frac{-4}{(2-1)^2} \\ g''\left(\frac{4+1}{2}\right) - g''\left(\frac{4-1}{2}\right) & = & \frac{-4}{(4-1)^2} \\ g''\left(\frac{6+1}{2}\right) - g''\left(\frac{6-1}{2}\right) & = & \frac{-4}{(6-1)^2} \\ \vdots & & \vdots \\ g''\left(\frac{2N+1}{2}\right) - g''\left(\frac{2N-1}{2}\right) & = & \frac{-4}{(2N-1)^2} \\ \hline g''\left(\frac{2N+1}{2}\right) - g''\left(\frac{1}{2}\right) & = & -4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right) \end{array}$$

$$\Rightarrow g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right) \text{ which corresponds to option (A).}$$

Problem 17. Let a function $f : (0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = \lim_{n \rightarrow \infty} \cos^n \left(\frac{1}{n^x} \right)$. Show that f has exactly one point of discontinuity.

Proof.

$$\begin{aligned}
 f(x) &= \lim_{n \rightarrow \infty} \cos^n \left(\frac{1}{n^x} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\cos \left(\frac{1}{n^x} \right) \right)^n \\
 &= e^{\left[\lim_{n \rightarrow \infty} n \left(\cos \left(\frac{1}{n^x} \right) - 1 \right) \right]} \\
 &= e^{\left[\lim_{t \rightarrow 0^+} \left(\frac{\cos(t^x) - 1}{t} \right) \right]} \\
 &= e^{\left[\lim_{t \rightarrow 0^+} (-x t^{x-1} \sin(t^x)) \right]} \\
 &= e^{\left[\lim_{t \rightarrow 0^+} \left(\frac{-x \sin(t^x)}{t} \cdot t^x \right) \right]} \\
 &= e^{\left[\lim_{t \rightarrow 0^+} \left(\frac{-x \sin(t^x)}{t^x} \cdot t^{2x-1} \right) \right]}
 \end{aligned}$$

Note that this is the indeterminate form of 1^∞

Let $n = \frac{1}{t}$,
 $n \rightarrow \infty \Rightarrow t \rightarrow 0^+$

Apply
 L'Hôpital's Rule

Now, if $x > \frac{1}{2}$, then $\lim_{t \rightarrow 0^+} t^{2x-1} = 0$, so the whole limit will evaluate to 0, giving us $f(x) = e^0 = 1$

If $x = \frac{1}{2}$, then the limit will evaluate to $-\frac{1}{2}$, giving us $f(x) = e^{\left(-\frac{1}{2}\right)}$

If $x < \frac{1}{2}$, then the limit becomes $-\infty$, giving us $f(x) = 0$. So we get the following definition of $f(x)$:

$$f(x) = \begin{cases} 1 & \text{if } x > \frac{1}{2} \\ e^{\left(-\frac{1}{2}\right)} & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x < \frac{1}{2} \end{cases}$$

Therefore f is discontinuous at exactly one point, i.e. at $x = \frac{1}{2}$. □

Problem 18. Let $a_1 = 1$ and $a_n = n(a_{n-1} + 1)$ for $n \geq 2$.

Define $P_n = \left(1 + \frac{1}{a_1}\right) \left(1 + \frac{1}{a_2}\right) \left(1 + \frac{1}{a_3}\right) \cdots \left(1 + \frac{1}{a_n}\right)$. Compute $\lim_{n \rightarrow \infty} P_n$.

Solution: We have the following recurrence relation:

$$\begin{aligned}
 a_n &= n(a_{n-1} + 1) && \text{Divide by } n \\
 \Rightarrow \frac{a_n}{n} &= a_{n-1} + 1 && \text{Divide by } (n-1)! \\
 \Rightarrow \frac{a_n}{n!} &= \frac{a_{n-1}}{(n-1)!} + \frac{1}{(n-1)!} \\
 \Rightarrow \frac{a_n}{n!} - \frac{a_{n-1}}{(n-1)!} &= \frac{1}{(n-1)!} && \text{Let } V_n = \frac{a_n}{n!} \\
 \Rightarrow V_n - V_{n-1} &= \frac{1}{(n-1)!}
 \end{aligned}$$

This forms the following telescopic sum:

$$\begin{aligned}
 \cancel{V_2} - V_1 &= \frac{1}{1!} \\
 \cancel{V_3} - \cancel{V_2} &= \frac{1}{2!} \\
 \cancel{V_4} - \cancel{V_3} &= \frac{1}{3!} \\
 &\vdots \\
 \cancel{V_{n-1}} - \cancel{V_{n-2}} &= \frac{1}{(n-2)!} \\
 V_n - \cancel{V_{n-1}} &= \frac{1}{(n-1)!} \\
 \hline
 V_n - V_1 &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} && V_1 = \frac{a_1}{1!} = 1 \\
 \hline
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow V_n &= \frac{a_n}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} \\
 \Rightarrow a_n &= n! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} \right)
 \end{aligned}$$

Also, $a_n = n(a_{n-1} + 1)$ (Given)

$$\Rightarrow \frac{a_{n-1} + 1}{a_n} = \frac{1}{n} \tag{1}$$

Now,

$$\begin{aligned}
 P_n &= \left(1 + \frac{1}{a_1}\right) \left(1 + \frac{1}{a_2}\right) \left(1 + \frac{1}{a_3}\right) \cdots \left(1 + \frac{1}{a_n}\right) && \text{(Given)} \\
 &= \left(\frac{a_1 + 1}{a_1}\right) \left(\frac{a_2 + 1}{a_2}\right) \left(\frac{a_3 + 1}{a_3}\right) \cdots \left(\frac{a_{n-1} + 1}{a_{n-1}}\right) \left(\frac{a_n + 1}{a_n}\right) \\
 &= \frac{1}{a_1} \left(\frac{a_1 + 1}{a_2}\right) \left(\frac{a_2 + 1}{a_3}\right) \left(\frac{a_3 + 1}{a_4}\right) \cdots \left(\frac{a_{n-1} + 1}{a_n}\right) (a_n + 1) \\
 &= 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n} \cdot (a_n + 1) = \frac{a_n + 1}{n!} && \text{From (1) and using } a_1 = 1
 \end{aligned}$$

Now, we have to find $\lim_{n \rightarrow \infty} P_n$.

$$\begin{aligned}
\lim_{n \rightarrow \infty} P_n &= \lim_{n \rightarrow \infty} \frac{a_n + 1}{n!} \\
&= \lim_{n \rightarrow \infty} \frac{n! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} \right) + 1}{n!} \\
&= \lim_{n \rightarrow \infty} \frac{n! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} \right) + n! \cdot \left(\frac{1}{n!} \right)}{n!} \\
&= \lim_{n \rightarrow \infty} \frac{n! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} \right)}{n!} \\
&= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} \right) \\
&= \lim_{n \rightarrow \infty} \left(1 + \frac{1^1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots + \frac{1^{n-1}}{(n-1)!} + \frac{1^n}{n!} \right) \\
&= e^1 = e
\end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} P_n = e$

Problem 19. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function satisfying $|f(x+y) - f(x-y) - y| \leq y^2$ for all $x, y \in \mathbb{R}$. Show that $f(x) = \frac{x^2}{2} + c$ where c is constant.

Proof.

$$\begin{aligned}
&|f(x+y) - f(x-y) - y| \leq y^2 \quad \forall x, y \in \mathbb{R} && \text{(Given)} \\
\Rightarrow &|f(x+y) - f(x-y) - y| \leq |y|^2 && \text{Divide by } |y| \\
\Rightarrow &\left| \frac{f(x+y) - f(x-y)}{y} - 1 \right| \leq |y| && \text{Divide by 2} \\
\Rightarrow &\left| \frac{f(x+y) - f(x-y)}{2y} - \frac{1}{2} \right| \leq \frac{1}{2}|y| && \text{Take } \lim_{y \rightarrow 0} \text{ on both sides} \\
\Rightarrow \lim_{y \rightarrow 0} &\left| \frac{f(x+y) - f(x-y)}{2y} - \frac{1}{2} \right| \leq \lim_{y \rightarrow 0} \frac{1}{2}|y|
\end{aligned}$$

Since f is differentiable (Given),

$$\begin{aligned}
\Rightarrow &\left| f'(x) - \frac{1}{2} \right| \leq 0 && \text{But absolute value can't be negative} \\
\Rightarrow &\left| f'(x) - \frac{1}{2} \right| = 0 \\
\Rightarrow &f'(x) = \frac{1}{2} \\
\Rightarrow &f(x) = \int f'(x) dx \\
&= \int \frac{1}{2} dx \\
&= \frac{1}{2}x + c
\end{aligned}$$

□

Problem 20. Let $A_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}}$ (n radical signs). Prove that $A_n = 2 \sin \left(\frac{\pi}{2^{n+1}} \right)$ for $n \geq 2$ and then evaluate $\lim_{n \rightarrow \infty} 2^n A_n$.

Proof. (By induction)

Let $P(n)$ be the predicate that $A_n = 2 \sin \left(\frac{\pi}{2^{n+1}} \right)$.

Base Case: $P(2)$

$$\begin{aligned} A_2 &= \sqrt{2 - \sqrt{2}} \\ 2 \sin \left(\frac{\pi}{2^{2+1}} \right) &= 2 \sin \left(\frac{\pi}{8} \right) \\ &= 2 \cdot \frac{\sqrt{2 - \sqrt{2}}}{2} \\ &= \sqrt{2 - \sqrt{2}} \end{aligned}$$

This proves the base case $P(2)$.

Inductive Step: Assume $P(n)$ is true for some n , prove $P(n+1)$

$$\begin{aligned} A_n &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}} && n \text{ radical signs} \\ \Rightarrow A_n^2 &= 2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}} && (n-1) \text{ radical signs} \\ \Rightarrow 4 - A_n^2 &= 2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}} && (n-1) \text{ radical signs} \\ \Rightarrow \sqrt{4 - A_n^2} &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}} && n \text{ radical signs} \\ \Rightarrow \sqrt{2 - \sqrt{4 - A_n^2}} &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}} && (n+1) \text{ radical signs} \\ &= A_{n+1} \\ \Rightarrow A_{n+1} &= \sqrt{2 - \sqrt{4 - A_n^2}} \\ &= \sqrt{2 - \sqrt{4 - \left(2 \sin \left(\frac{\pi}{2^{n+1}} \right) \right)^2}} && [\because P(n) \text{ was assumed to be true}] \\ &= \sqrt{2 - \sqrt{4 \left(1 - \sin^2 \left(\frac{\pi}{2^{n+1}} \right) \right)}} \\ &= \sqrt{2 - \sqrt{4 \cos^2 \left(\frac{\pi}{2^{n+1}} \right)}} && \left[\because 1 - \sin^2(\theta) = \cos^2(\theta) \forall \theta \in \mathbb{R} \right] \end{aligned}$$

$$\begin{aligned}
\Rightarrow A_{n+1} &= \sqrt{2 - 2 \cos \left(\frac{\pi}{2^{n+1}} \right)} \\
&= \sqrt{2 \cdot \left(1 - \cos \left(\frac{\pi}{2^{n+1}} \right) \right)} \\
&= \sqrt{2 \cdot \left(2 \sin^2 \left(\frac{\pi}{2^{n+2}} \right) \right)} & \left[\because 1 - \cos(2\theta) = 2 \sin^2(\theta) \forall \theta \in \mathbb{R} \right] \\
&= \sqrt{4 \sin^2 \left(\frac{\pi}{2^{n+2}} \right)} \\
&= 2 \sin \left(\frac{\pi}{2^{n+2}} \right)
\end{aligned}$$

This proves the inductive step, i.e. $P(n) \Rightarrow P(n+1)$.

Since $P(2)$ is true (base case) and $P(n) \Rightarrow P(n+1)$ (inductive step), by the induction axiom, $\Rightarrow P(n)$ is true for all $n \geq 2$, $n \in \mathbb{Z}$

□

Now,

$$\begin{aligned}
\lim_{n \rightarrow \infty} (2^n A_n) &= \lim_{n \rightarrow \infty} 2^n \cdot \left(2 \sin \left(\frac{\pi}{2^{n+1}} \right) \right) \\
&= \lim_{n \rightarrow \infty} 2^{n+1} \sin \left(\frac{\pi}{2^{n+1}} \right) \\
&= \lim_{n \rightarrow \infty} \frac{\sin \left(\frac{\pi}{2^{n+1}} \right)}{\left(\frac{1}{2^{n+1}} \right)} & \text{Multiply and divide by } \pi \\
&= \lim_{n \rightarrow \infty} \frac{\sin \left(\frac{\pi}{2^{n+1}} \right)}{\left(\frac{\pi}{2^{n+1}} \right)} \cdot \pi \\
&= \lim_{t \rightarrow 0} \frac{\sin(t)}{t} \cdot \pi & \text{Let } t = \frac{\pi}{2^{n+1}} \\
& & n \rightarrow \infty \Rightarrow t \rightarrow 0 \\
&= 1 \cdot \pi \\
&= \pi & \left[\because \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \right]
\end{aligned}$$

Problem 21. Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $p(x) = x$ does not have any real solution. Prove that $p(p(x)) = x$ does not have any real solution.

Proof. Let $g(x) = p(x) - x$
 $g(x)$ is continuous since $p(x)$ is continuous.
It is given that $p(x) = x$ has no real solution.
 $\Rightarrow g(x)$ has no real zeros.

Claim: $g(x) > 0 \forall x$ or $g(x) < 0 \forall x$

Proof of the claim: (by contradiction) Assume for the purpose of contradiction that there exist some $a, b \in \mathbb{R}$ such that $g(a)g(b) < 0$. Then by Bolzano's Theorem, $g(x)$ will have at least one real root in (a, b) , which contradicts the above statement that $g(x)$ has no real root.

Case 1: $g(x) > 0 \forall x$

$$\begin{aligned}
&\Rightarrow p(x) - x > 0 \quad \forall x \\
&\Rightarrow p(x) > x \quad \forall x && \text{Replace } x \text{ by } p(x) \\
&\Rightarrow p(p(x)) > p(x) \quad \forall x \\
&\Rightarrow p(p(x)) > p(x) > x \quad \forall x \\
&\Rightarrow p(p(x)) > x \quad \forall x \\
&\Rightarrow p(p(x)) \neq x \quad \forall x
\end{aligned}$$

Case 2: $g(x) < 0 \forall x$

$$\begin{aligned}
&\Rightarrow p(x) - x < 0 \quad \forall x \\
&\Rightarrow p(x) < x \quad \forall x && \text{Replace } x \text{ by } p(x) \\
&\Rightarrow p(p(x)) < p(x) \quad \forall x \\
&\Rightarrow p(p(x)) < p(x) < x \quad \forall x \\
&\Rightarrow p(p(x)) < x \quad \forall x \\
&\Rightarrow p(p(x)) \neq x \quad \forall x
\end{aligned}$$

Therefore, $p(p(x)) \neq x \quad \forall x$, i.e. $p(p(x)) = x$ has no real solutions. □

5 Applications of Derivatives

5.1 Derivative as a Rate of Change

Consider a function of time $y = f(t)$.

The average rate of change of f over some time interval Δt is given by $\frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$, which is the same as the slope of the secant to the graph of the function.

Now, the limit of this average rate of change as the time interval Δt approaches 0 is nothing but the derivative of $f(t)$

$$f'(t) = \lim_{t_2 \rightarrow t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

This is called the **instantaneous rate of change** of $f(t)$ with respect to t .

In crude terms, the derivative can be thought of as a measure of how sensitive a function's value is to small changes in its input.

5.2 Tangents and Normals

The derivative of a differentiable and continuous function at a point gives the slope of the tangent to the function's graph at that point.

Consider the function $y = f(x)$ and a point x_0 on the function's graph.

5.2.1 Equation of Tangent at (x_0, y_0)

$$y - y_0 = f'(x_0) [x - x_0]$$

5.2.2 Equation of Normal at (x_0, y_0)

$$y - y_0 = \frac{-1}{f'(x_0)} [x - x_0]$$

5.3 Angle of Intersection of 2 Curves

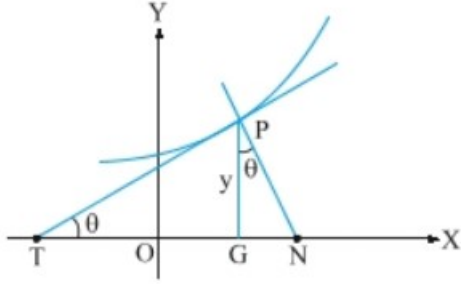
The angle of intersection between two curves is defined as the angle between their tangents at the point of intersection.

Given the slopes of the tangents m_1 and m_2 , we can calculate the angle between them using this result from coordinate geometry:

$$\tan(\theta) = \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right)$$

5.4 Lengths

Let the tangent and normal to $y = f(x)$ at point P meet the x -axis at points T and N respectively. Let G be the foot of the ordinate of P . Then TG and GN are the Cartesian Subtangent and Subnormal respectively, while PT and PN are the lengths of the tangent and normal respectively.



5.4.1 Length of Subtangent

$$\text{Subtangent} = TG = \frac{|y|}{\tan(\theta)} = \frac{|y|}{\left(\frac{dy}{dx}\right)}$$

5.4.2 Length of Subnormal

$$\text{Subnormal} = GN = |y| \tan(\theta) = |y| \frac{dy}{dx}$$

5.4.3 Length of Tangent

$$\text{Tangent} = PT = \frac{|y|}{\sin(\theta)} = |y| \csc(\theta) = |y| \sqrt{1 + \cot^2(\theta)} = \frac{|y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\left|\frac{dy}{dx}\right|}$$

5.4.4 Length of Normal

$$\text{Normal} = PN = |y| \sec(\theta) = |y| \sqrt{1 + \tan^2(\theta)} = |y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

5.4.5 Length of the Curve from a to b

$$\text{Length of curve} = \int_a^b \left(1 + \left(\frac{dy}{dx}\right)^2\right) dx$$

5.4.6 Radius of Curvature

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

5.5 Local Maxima and Minima

5.5.1 Local Maximum

Definition 13. Let a function f be defined on a set S . The point $c \in S$ is called a point of local maximum if there exists an open interval I containing c such that $f(c) \geq f(x) \forall x \in I \cap S$ and I should be a subset of S .

5.5.2 Local Minimum

Definition 14. Let a function f be defined on a set S . The point $c \in S$ is called a point of local minimum if there exists an open interval I containing c such that $f(c) \leq f(x) \forall x \in I \cap S$ and I should be a subset of S .

Note that Local Maxima and Local Minima don't exist at end-points because you can't have an open interval which contains the end-point and is a subset of S .

5.6 Fermat's Theorem

Theorem 5. If a function attains a local maximum or a local minimum at an interior point c , and if the function is differentiable at c , then $f'(c) = 0$.

5.7 Rolle's Theorem

Theorem 6. If a function f is continuous in $[a, b]$ and differentiable in (a, b) and $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

5.7.1 Conclusions

Let f be a function which is infinitely continuously differentiable (i.e. all subsequent derivatives are also continuous), then:

- Between two real roots of $f(x) = 0$ lies at least one real root of $f'(x) = 0$.
- Between n real roots of $f(x) = 0$ lie at least $(n - 1)$ real roots of $f'(x) = 0$
- Between n real roots of $f(x) = 0$ lie at least $(n - 2)$ real roots of $f''(x) = 0$ (Generalizing)
- Between n real roots of $f(x) = 0$ lie at least $(n - k)$ real roots of $f^{(k)}(x) = 0$
- If $f^{(k)}(x) = 0$ has $(n - k)$ roots, then $f(x) = 0$ has at most n roots.

5.8 Linear Differential Equation and Inequality

$$f'(x) + kf(x) = 0$$

$$f'(x) - kf(x) = 0$$

Whenever we see an equation or inequality of this form, we will multiply both sides by the integrating factor. The integrating factor of $f'(x) + kf(x)$ is e^{kx} and that of $f'(x) - kf(x)$ is e^{-kx} . Notice that both are positive and won't change the sign of the inequality.

$$e^{kx} f'(x) + ke^{kx} f(x) = 0$$
$$\frac{d}{dx} (e^{kx} f(x)) = 0$$

5.9 Darboux's Theorem

Theorem 7. *If a function f is differentiable in the closed interval $I = [a, b]$ and $f'(a) < k < f'(b)$, then $\exists c \in (a, b) : f'(c) = k$.*

In other words, $f'(x)$ will have intermediate value property.

5.10 Multiplicity of Roots and Rolle's Theorem

Suppose f is a polynomial of which α is a zero with multiplicity r where $f(x) = (x - \alpha)^r \cdot g(x)$, where g is another polynomial and $g(\alpha) \neq 0$. Then $f'(x)$ has α as one of its zeros with multiplicity $(r - 1)$.

Proof.

$$\begin{aligned} f(x) &= (x - \alpha)^r \cdot g(x) \\ f'(x) &= r(x - \alpha)^{r-1} \cdot g(x) + (x - \alpha)^r \cdot g'(x) \\ &= (x - \alpha)^{r-1} [rg(x) + (x - \alpha)g'(x)] \end{aligned}$$

Now, observing $[rg(x) + (x - \alpha)g'(x)]$. This won't be zero at $x = \alpha$ since $g(\alpha) \neq 0$ [given].

Also, $(x - \alpha)^{r-1}$ has α as a zero with multiplicity $(r - 1)$.

Therefore, $f'(x)$ will have α as one of its zeros with multiplicity $(r - 1)$. □

Similarly, the multiplicity of α in $f''(x)$ will be $(r - 2)$ and so on...
The converse is also true.

5.11 Problems on Rolle's Theorem and Roots

Problem 22. Prove that $xe^x = 1$ has exactly one root in $(0, 1)$.

Proof. Consider a function $f(x) = xe^x - 1$.

We need to prove that $f(x)$ has at least one root in $(0, 1)$.

Now, f is continuous everywhere, so it's also continuous in $[0, 1]$.

Applying Bolzano's Theorem:

$$\begin{aligned}f(0) &= 0 - 1 = -1 < 0 \\f(1) &= e - 1 > 0 \\ \Rightarrow f(0) \cdot f(1) &< 0\end{aligned}$$

Therefore, there exists **at least one** root of $f(x)$ in $(0, 1)$.

But we have to prove that it has exactly one root.

Assume for the purpose of contradiction that $f(x)$ has two roots in $(0, 1)$.

Then by Rolle's theorem, $f'(x)$ will have at least one root in $(0, 1)$.

$$f'(x) = e^x + xe^x$$

But $e^x + xe^x > 0 \forall x \in (0, 1)$

Or, $f'(x)$ has no root in $(0, 1)$.

This contradicts our assumption that $f(x)$ has two roots in $(0, 1)$.

$\therefore f$ has exactly one root in $(0, 1)$. □

Problem 23. Comment on the number of roots of $f(x) = x^4 - 4x - 1$.

Solution: The given polynomial is of even degree and has the leading coefficient and constant term of opposite signs. Therefore it has at least 2 real roots: one in $(-\infty, 0)$ and the other in $(0, \infty)$.

Assume $f(x)$ has 3 or more roots.

Then $f'(x)$ should have at least 2 real roots.

$$\begin{aligned}f'(x) &= 4x^3 - 4 \\ &= 4(x^3 - 1)\end{aligned}$$

But $f'(x)$ has exactly 1 real root (i.e. $x = 1$). This is a contradiction.

Therefore, $f(x)$ has exactly 2 roots.

Problem 24. Let f be a function which is differentiable on \mathbb{R} and α and β be two numbers such that $\alpha < \beta$ and $f(\alpha) = f(\beta) = 0$. Then which of the following is/are correct?

- (A) There exists $c \in (\alpha, \beta)$ such that $f'(c) = f(c)$
- (B) There exists $c \in (\alpha, \beta)$ such that $f'(c) = 2f(c)$
- (C) There exists $c \in (\alpha, \beta)$ such that $f'(c) = 3f(c)$
- (D) There exists $c \in (\alpha, \beta)$ such that $f'(c) = kf(c)$ for any real k

Solution: (A, B, C, D) D is just a generalization of A, B and C. So if we prove D, that will prove all the options.

We have to prove that $\exists c \in (\alpha, \beta) \mid f'(c) - kf(c) = 0 \forall k > 0$

Consider $g(x) = e^{-kx}f(x)$

g is continuous in $[a, b]$ and differentiable in (a, b)

Now, $g(\alpha) = 0$ and $g(\beta) = 0$ because $f(\alpha) = f(\beta) = 0$ [Given]

Then by Rolle's Theorem,

$$\begin{aligned} \exists c \in (\alpha, \beta) : g'(c) &= 0 \\ \Rightarrow e^{-kc} f'(c) - ke^{-kc} f(c) &= 0 && \text{Divide by } e^{-kc} \\ \Rightarrow f'(c) - kf(c) &= 0 \\ \Rightarrow f'(c) &= kf(c) \end{aligned}$$

This completes the proof. Hence all the options **A, B, C and D** are correct.

Problem 25. If $\frac{a}{3} + \frac{b}{2} + c = 0$, then prove that the equation $ax^2 + bx + c = 0$ has at least one root in $(0, 1)$

Proof. We will treat $ax^2 + bx + c$ as the derivative of some function.

Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$.

$$\begin{aligned} f'(x) &= ax^2 + bx + c \\ f(x) &= \frac{ax^3}{3} + \frac{bx^2}{2} + cx \\ f(1) &= \frac{a}{3} + \frac{b}{2} + c = 0 \\ f(0) &= 0 \end{aligned}$$

Since $f(x)$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$, Applying Rolle's Theorem,

$$\begin{aligned} \Rightarrow \exists k \in (0, 1) : f'(k) &= 0 \\ \Rightarrow \exists k \in (0, 1) : ak^2 + bk + c &= 0 \end{aligned}$$

i.e. $ax^2 + bx + c = 0$ has at least one root in $(0, 1)$ □

Problem 26. The Legendre Polynomial is defined by the following formula (Rodrigues Formula).

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Prove that $P_n(x)$ has n different real roots, all of which are found between -1 and 1 .

Proof. Notice that $\frac{1}{2^n n!}$ is just a constant. So the roots of $P_n(x)$ will be the same as the roots of $\frac{d^n}{dx^n} (x^2 - 1)^n$.

Consider $g(x) = (x^2 - 1)^n$. This is a polynomial of degree $2n$. If we differentiate it n times, we will get a polynomial of degree n .

So $\frac{d^n}{dx^n} (x^2 - 1)^n$ is a polynomial of degree n . By the Fundamental Theorem of Algebra, it will have at most n roots.

$$\begin{aligned} g(x) &= (x^2 - 1)^n \\ &= (x + 1)^n (x - 1)^n \end{aligned}$$

$g(x)$ has two roots: -1 and 1 , each having multiplicity n .

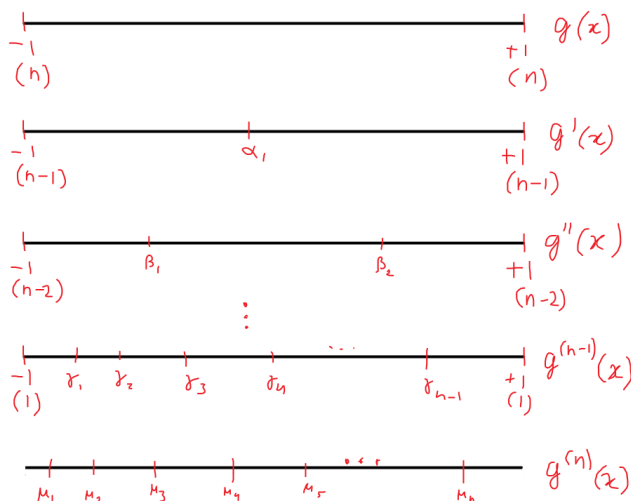
By Rolle's Theorem, $g'(x)$ will have at least one root in $(-1, 1)$. Also $g'(x)$ will have -1 and 1 as roots, each having multiplicity $(n - 1)$. So $g'(x)$ has at least 3 distinct roots.

Again applying Rolle's Theorem, between 3 roots of $g'(x)$ will lie at least 2 real roots of $g''(x)$. And $g''(x)$ will have -1 and 1 as roots of multiplicity $(n - 2)$. So $g''(x)$ will have at least 4 distinct roots.

And so on...

Applying Rolle's Theorem $(n - 1)$ times. Then $g^{(n-1)}(x)$ will have at least $n - 1$ roots in $(0, 1)$ and also -1 and 1 as roots of multiplicity 1 each. So $g^{(n-1)}(x)$ will have at least $(n + 1)$ real roots.

Finally, applying Rolle's Theorem, $g^{(n)}(x)$ will have at least n roots, all of which lie in $(0, 1)$.



Therefore, $P_n(x)$ will have exactly n real roots, all of which lie in $(0, 1)$

□

5.12 Mean Value Theorems

5.12.1 Lagrange's Mean Value Theorem

Theorem 8. *If a function f is continuous in $[a, b]$ and differentiable in (a, b) , then there exists at least one $c \in (a, b)$ such that:*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Geometrically, it means at that at some point in the interval (a, b) , the tangent will be parallel to the chord.

Proof. Consider the following function.

$$g(x) = (f(b) - f(a))x - f(x) \cdot (b - x)$$

Then,

$$g'(x) = f(b) - f(a) - f'(x) \cdot (b - x)$$

Now, g is continuous in $[a, b]$ and differentiable in (a, b) and $g(a) = g(b)$. Applying Rolle's Theorem. $\exists c \in (a, b)$ such that $g'(c) = 0$.

$$\begin{aligned} \Rightarrow f(b) - f(a) - f'(c) \cdot (b - a) &= 0 \\ \Rightarrow f(b) - f(a) &= f'(c) \cdot (b - a) \\ \Rightarrow \frac{f(b) - f(a)}{b - a} &= f'(c) \end{aligned}$$

□

5.12.2 Cauchy's Mean Value Theorem

Theorem 9. *Let f and g be two functions continuous in $[a, b]$ and differentiable in (a, b) , then there exists at least one $c \in (a, b)$ such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$*

Proof. We need to prove that for some $x = c$, where $c \in (a, b)$, $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$.

Or $g'(x) \cdot (f(b) - f(a)) - f'(x) \cdot (g(b) - g(a)) = 0$. Consider the following function:

$$h(x) = g(x) \cdot (f(b) - f(a)) - f(x) \cdot (g(b) - g(a))$$

Then,

$$h'(x) = g'(x) \cdot (f(b) - f(a)) - f'(x) \cdot (g(b) - g(a))$$

Now, $h(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and $h(a) = h(b)$. Applying Rolle's Theorem.

$\exists c \in (a, b)$ such that $h'(c) = 0$

Or $g'(c) \cdot (f(b) - f(a)) - f'(c) \cdot (g(b) - g(a)) = 0$

$$\Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

□

5.12.3 Important Result from Lagrange's Mean Value Theorem

Theorem 10. If $f'(x) > g'(x) \forall x$ and $f(a) = g(a)$, then $f(x) > g(x) \forall x > a$ and $f(x) < g(x) \forall x < a$.

Proof. Consider $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x)$

Also, $h'(x) > 0 \forall x$

Then we have to prove that $h(x) > 0 \forall x > a$ and $h(x) < 0 \forall x < a$.

Now,

$$h(a) = f(a) - g(a) = 0 \qquad \because f(a) = g(a) \text{ [given]}$$

Consider any real number $x \neq a$.

Note that h is continuous and differentiable. Applying Lagrange's Mean Value Theorem.

$$\exists c : \frac{h(x) - h(a)}{x - a} = h'(c) > 0$$

Case 1: $x > a$

Then $x - a > 0$,

Therefore,

$$h(x) - h(a) > 0$$

$$\Rightarrow h(x) > h(a)$$

Case 2: $x < a$

Then $x - a < 0$,

Therefore,

$$h(x) - h(a) < 0$$

$$\Rightarrow h(x) < h(a)$$

□

5.12.4 Example Problems

Problem 27. Suppose $f'(x) \geq M$ in the interval $[a, b]$, then prove that $f(b) \geq f(a) + M(b - a)$

Proof. $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) .
Applying Lagrange's Mean Value Theorem.

$$\exists c \in (a, b) : \frac{f(b) - f(a)}{b - a} = f'(c)$$

But $f'(x) \geq M \forall x \in [a, b]$ (Given), so

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &\geq M \\ f(b) - f(a) &\geq M(b - a) \\ f(b) &\geq f(a) + M(b - a) \end{aligned}$$

□

Problem 28. Suppose $f'(x) \leq M$ in the interval $[a, b]$, then prove that $f(b) \leq f(a) + M(b - a)$

Proof. $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) .
Applying Lagrange's Mean Value Theorem.

$$\exists c \in (a, b) : \frac{f(b) - f(a)}{b - a} = f'(c)$$

But $f'(x) \leq M \forall x \in [a, b]$ (Given), so

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &\leq M \\ f(b) - f(a) &\leq M(b - a) \\ f(b) &\leq f(a) + M(b - a) \end{aligned}$$

□

Problem 29. Let the second derivative f'' of a function be continuous and satisfy $|f''(x)| \leq 1$. If $f(0) = f(1)$, show that $|f'(x)| < 1 \forall x \in (0, 1)$.

Proof.

f is continuous in $[0, 1]$ and differentiable in $(0, 1)$, and $f(0) = f(1)$. Applying Rolle's Theorem.

$\exists c \in (0, 1) : f'(c) = 0$ or $|f'(c)| \leq 1$

But we have to prove it for all x not just some specific value.

General advice: If you see an inequality in the second derivative, using Lagrange's Mean Value Theorem could be helpful.

Applying Lagrange's Mean Value Theorem.

Consider any $x \in (0, 1) - \{c\}$.

The function f' is continuous in $[x, c]$ or $[c, x]$ and differentiable in (x, c) or (c, x)

$$\frac{f'(x) - f'(c)}{x - c} = f''(c_1) \text{ for some } c_1 \in (x, c) \text{ or } (c, x)$$

$$|f'(x) - f'(c)| = |x - c| \cdot |f''(c_1)| \quad [\text{Now, } f'(c) = 0 \text{ and } |f''(c_1)| \leq 1]$$

$$|f'(x)| = |x - c| \cdot |f''(c_1)| \leq |x - c| < 1$$

□

Problem 30. Assume that $|f''(x)| \leq m$ for each x in the interval $[0, a]$ and assume that f takes its largest value at an interior point in this interval. If f'' is continuous in the closed interval $[0, a]$, show that $|f'(0)| + |f'(a)| \leq am$ where $f'(0)$ denotes the right-hand derivative of f at 0 and $f'(a)$ denotes the left-hand derivative of f at a .

Proof. f takes its largest value at an interior point of $[0, a]$.

Or, f has a point of maximum in $(0, a)$.

By Fermat's Theorem,

$$\exists c \in (0, a) : f'(c) = 0$$

Applying Lagrange's Mean Value Theorem on f' in $[0, c]$.

$$\frac{f'(0) - f'(c)}{0 - c} = f''(c_1) \quad \text{for some } c_1 \in (0, c)$$

But $f'(c) = 0$

Taking modulus on both sides and multiplying by $|c|$

$$|f'(0)| = |c| \cdot |f''(c_1)|$$

But $|f''(c_1)| \leq m$

$$|f'(0)| = |c| \cdot |f''(c_1)| \leq |c| \cdot m \tag{1}$$

Applying Lagrange's Mean Value Theorem on f' in $[c, a]$.

$$\frac{f'(a) - f'(c)}{a - c} = f''(c_2) \quad \text{for some } c_2 \in (c, a)$$

But $f'(c) = 0$ Taking modulus on both sides and multiplying by $|a - c|$

$$|f'(a)| = |a - c| \cdot |f''(c_2)|$$

But $|f''(c_2)| \leq m$

$$|f'(a)| = |a - c| \cdot |f''(c_2)| \leq |a - c| \cdot m \tag{2}$$

Adding (1) and (2),

$$\begin{aligned} |f'(0)| + |f'(a)| &\leq |c| \cdot m + |a - c| \cdot m \\ |f'(0)| + |f'(a)| &\leq m \cdot (|c| + |a - c|) \\ |f'(0)| + |f'(a)| &\leq m \cdot |a| \quad \text{But } a \geq 0 \text{ so } |a| = a \\ \Rightarrow |f'(0)| + |f'(a)| &\leq am \end{aligned}$$

□

Problem 31. Suppose f is continuous on $[a, b]$ and differentiable on (a, b) . Assume further that $f(b) - f(a) = b - a$, then prove that for all positive integer n , there exist distinct points $c_1, c_2, c_3, \dots, c_n$ such that $f'(c_1) + f'(c_2) + f'(c_3) + \dots + f'(c_n) = n$.

Proof. We will try to prove it for $n = 1$ and $n = 2$, and then we'll try to draw inspiration from it to prove for any general n .

For $n = 1$,

f is continuous in $[a, b]$ and differentiable in (a, b) .

Applying Lagrange's Mean Value Theorem.

$$\exists c_1 \in (a, b) : \frac{f(b) - f(a)}{b - a} = f'(c_1)$$

But $f(b) - f(a) = b - a$ (Given),

$$\begin{aligned} \Rightarrow \exists c_1 \in (a, b) : \frac{b-a}{b-a} &= f'(c_1) \\ \Rightarrow \exists c_1 \in (a, b) : f'(c_1) &= 1 \end{aligned}$$

And this proves the result for $n = 1$.

For $n = 2$,

We need to prove that there exist c_1 and c_2 such that $f'(c_1) + f'(c_2) = 2$. We will divide the interval $[a, b]$ into 2 equal parts using the point $\frac{a+b}{2}$.

Applying Lagrange's Mean Value Theorem and Adding the equations.

For some c_1 and c_2 ,

$$\begin{array}{rcl} f\left(\frac{a+b}{2}\right) - f(a) &= & \frac{b-a}{2} f'(c_1) \\ f(b) - f\left(\frac{a+b}{2}\right) &= & \frac{b-a}{2} f'(c_2) \\ \hline f(b) - f(a) &= & \frac{b-a}{2} (f'(c_1) + f'(c_2)) \\ \hline \end{array}$$

$$\Rightarrow \frac{f(b) - f(a)}{b - a} = \frac{1}{2} (f'(c_1) + f'(c_2))$$

But $f(b) - f(a) = b - a$ (Given),

$$\Rightarrow \frac{b-a}{b-a} = \frac{1}{2} (f'(c_1) + f'(c_2))$$

$$f'(c_1) + f'(c_2) = 2$$

This proves the result for $n = 2$. Now we will prove it for any general n by the same analogy.

For any General n ,

For the case of $n = 2$, we divided the interval (a, b) into two equal parts.

So for the general case, we will try to divide the interval into n equal parts, i.e. $\left(a, a + \frac{b-a}{n}\right), \left(a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}\right), \left(a + \frac{2(b-a)}{n}, a + \frac{3(b-a)}{n}\right), \dots, \left(a + \frac{(n-1)(b-a)}{n}, b\right)$

Applying Lagrange's Mean Value Theorem n times (on each of these intervals) and adding all the resulting equations. This makes a telescopic sum. There exist $c_1, c_2, c_3, \dots, c_n$ such that:

$$\begin{aligned}
 & f\left(a + \frac{b-a}{n}\right) - f(a) = \left(\frac{b-a}{n}\right) \cdot f'(c_1) \\
 & f\left(a + \frac{2(b-a)}{n}\right) - f\left(a + \frac{b-a}{n}\right) = \left(\frac{b-a}{n}\right) \cdot f'(c_2) \\
 & f\left(a + \frac{3(b-a)}{n}\right) - f\left(a + \frac{2(b-a)}{n}\right) = \left(\frac{b-a}{n}\right) \cdot f'(c_3) \\
 & f\left(a + \frac{4(b-a)}{n}\right) - f\left(a + \frac{3(b-a)}{n}\right) = \left(\frac{b-a}{n}\right) \cdot f'(c_4) \\
 & \vdots \\
 & f(b) - f\left(a + \frac{(n-1)(b-a)}{n}\right) = \left(\frac{b-a}{n}\right) \cdot f'(c_n) \\
 \hline
 & f(b) - f(a) = \left(\frac{b-a}{n}\right) \cdot (f'(c_1) + f'(c_2) + \dots + f'(c_n)) \\
 \hline
 \end{aligned}$$

$$\Rightarrow \frac{f(b) - f(a)}{b - a} = \frac{1}{n} (f'(c_1) + f'(c_2) + \dots + f'(c_n))$$

But $f(b) - f(a) = b - a$

$$\begin{aligned}
 & \Rightarrow \frac{b-a}{b-a} = \frac{1}{n} (f'(c_1) + f'(c_2) + \dots + f'(c_n)) \\
 & \Rightarrow f'(c_1) + f'(c_2) + f'(c_3) + \dots + f'(c_n) = n
 \end{aligned}$$

□

Proving Some Inequalities

Problem 32. Prove that $\forall a, b \in \mathbb{R}$,

$$|\sin(a) - \sin(b)| \leq |a - b|$$

Proof. (using Lagrange's Mean Value Theorem)

Case 1: If $a = b$

This inequality is trivially true if $a = b$, because $LHS = \sin(a) - \sin(a) = 0$ and $RHS = a - a = 0$

Case 2: If $a \neq b$

Consider a function $f(x) = \sin(x)$ f is continuous and differentiable everywhere. Applying Lagrange's Mean Value Theorem.

There exists $c \in (a, b)$ or (b, a) such that

$$\begin{aligned} \frac{\sin(a) - \sin(b)}{a - b} &= \cos(c) \\ \Rightarrow \left| \frac{\sin(a) - \sin(b)}{a - b} \right| &= |\cos(c)| \leq 1 && \text{Multiply by } |a - b| \\ \Rightarrow |\sin(a) - \sin(b)| &\leq |a - b| \end{aligned}$$

□

Problem 33. Prove that $\forall x > 0$,

$$\frac{x}{1+x} < \log_e(1+x) < x$$

Proof. Consider the function $f(t) = \log_e(t)$

f is continuous in $[1, 1+x]$ and differentiable in $(1, 1+x)$. Applying Lagrange's Mean Value Theorem.

There exists $c \in (1, 1+x)$ such that

$$\frac{\ln(1+x) - \ln(1)}{1+x-1} = \frac{1}{c}$$

Now,

$$\begin{aligned} 1 &< c < 1+x \\ \Rightarrow \frac{1}{1+x} &< \frac{1}{c} < 1 \\ \Rightarrow \frac{1}{1+x} &< \frac{\ln(1+x) - \ln(1)}{1+x-1} < 1 \end{aligned}$$

But $\ln(1) = 0$

$$\Rightarrow \frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1$$

Now, $x > 0$ (Given), so multiplying by x ,

$$\frac{x}{1+x} < \ln(1+x) < x$$

□

Problem 34. Show that the square roots two successive natural numbers greater than N^2 differ by less than $\frac{1}{2N}$.

Proof. Consider any two consecutive natural numbers m and $m + 1$, both greater than N^2 .

Also consider the function $f(x) = \sqrt{x}$.

Applying Lagrange's Mean Value Theorem,

$f(x)$ is continuous in $[m, m + 1]$ and differentiable in $(m, m + 1)$.

Therefore, there exists $c \in (m, m + 1)$ such that

$$\begin{aligned}\frac{\sqrt{m+1} - \sqrt{m}}{m+1 - m} &= f'(c) \\ \sqrt{m+1} - \sqrt{m} &= \frac{1}{2\sqrt{c}}\end{aligned}$$

Now,

$$\begin{aligned}m &< c < m + 1 \\ \sqrt{m} &< \sqrt{c} < \sqrt{m+1} \\ \frac{1}{\sqrt{m+1}} &< \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{m}}\end{aligned}$$

Also, it is given that

$$\begin{aligned}N^2 &< m < m + 1 \\ \Rightarrow N &< \sqrt{m} < \sqrt{m+1} \\ \Rightarrow \frac{1}{\sqrt{m+1}} &< \frac{1}{\sqrt{m}} < \frac{1}{N}\end{aligned}$$

So $\frac{1}{\sqrt{c}} < \frac{1}{\sqrt{m}}$ and $\frac{1}{\sqrt{m}} < \frac{1}{N}$

$$\Rightarrow \frac{1}{\sqrt{c}} < \frac{1}{N}$$

Therefore,

$$\begin{aligned}\sqrt{m+1} - \sqrt{m} &= \frac{1}{2\sqrt{c}} < \frac{1}{2N} \\ \Rightarrow \sqrt{m+1} - \sqrt{m} &< \frac{1}{2N}\end{aligned}$$

□

Problem 35. If f and g are differentiable functions in $(0, 1)$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in [0, 1]$:

- (A) $2f'(c) = g'(c)$
- (B) $2f'(c) = 3g'(c)$
- (C) $f'(c) = g'(c)$
- (D) $f'(c) = 2g'(c)$

Solution: (D) Simply invoke Cauchy's Mean Value Theorem.

$$\begin{aligned}\exists c \in (0, 1) : \frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{f'(c)}{g'(c)} \\ \Rightarrow \frac{f'(c)}{g'(c)} &= \frac{6 - 2}{2 - 0} = 2 \Rightarrow f'(c) = 2g'(c)\end{aligned}$$

5.12.5 Differentiation of Determinants

To differentiate a determinant, we differentiate each row (or column) one at a time, leaving the rest unchanged, and take the sum.

$$\text{Let } p(x) = \begin{vmatrix} a(x) & b(x) & c(x) \\ d(x) & e(x) & f(x) \\ g(x) & h(x) & i(x) \end{vmatrix}$$

Then

$$p'(x) = \begin{vmatrix} a'(x) & b'(x) & c'(x) \\ d(x) & e(x) & f(x) \\ g(x) & h(x) & i(x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c(x) \\ d'(x) & e'(x) & f'(x) \\ g(x) & h(x) & i(x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c(x) \\ d(x) & e(x) & f(x) \\ g'(x) & h'(x) & i'(x) \end{vmatrix}$$

Or

$$p'(x) = \begin{vmatrix} a'(x) & b(x) & c(x) \\ d'(x) & e(x) & f(x) \\ g'(x) & h(x) & i(x) \end{vmatrix} + \begin{vmatrix} a(x) & b'(x) & c(x) \\ d(x) & e'(x) & f(x) \\ g(x) & h'(x) & i(x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c'(x) \\ d(x) & e(x) & f'(x) \\ g(x) & h(x) & i'(x) \end{vmatrix}$$

5.12.6 The General Mean Value Theorem

Theorem 11. Let f , g and h be functions which are continuous in $[a, b]$ and differentiable in (a, b) , then there exists at least one $c \in (a, b)$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Proof. Consider the function p defined as the following.

$$p(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

Then,

$$\begin{aligned} p'(x) &= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ f(b) & g(b) & h(b) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ 0 & 0 & 0 \end{vmatrix} \\ &= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \end{aligned}$$

Now, p is continuous in $[a, b]$ and differentiable in (a, b) because f , g and h are continuous in $[a, b]$ and differentiable in (a, b) .

Also, $p(a) = 0$ (because that would make the first two rows equal) and similarly $p(b) = 0$.

Applying Rolle's Theorem on p ,

$$\exists c \in (a, b) : p'(c) = 0$$

$$\exists c \in (a, b) : \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

□

5.13 Monotonicity

5.13.1 Definitions

Definition 15. A function f is said to be non-decreasing or monotonically increasing in an interval if for all x_1, x_2 in the interval, $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$

Definition 16. A function f is said to be non-increasing or monotonically decreasing in an interval if for all x_1, x_2 in the interval, $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$

Definition 17. A function f is said to be strictly increasing in an interval if for all x_1, x_2 in the interval, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

Definition 18. A function f is said to be strictly decreasing in an interval if for all x_1, x_2 in the interval, $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

Note that by these definitions, the constant function is both monotonically increasing and monotonically decreasing.

Also note that a function doesn't have to be continuous or differentiable in order for it to be increasing or decreasing.

5.13.2 Properties

1. If a function is increasing in an interval $[a, b]$, then for this interval, the function assumes its greatest value at b and its least value at a .
2. Similarly, if a function is decreasing in an interval $[a, b]$, then for this interval, the function assumes its greatest value at a and its least value at b .
3. Let f be a function which is continuous in $[a, b]$ and differentiable in (a, b) . Then,
 - (a) If $f'(x) \geq 0 \forall x \in (a, b)$, then the function is monotonically increasing (or non-decreasing) in the interval.
 - (b) If $f'(x) \leq 0 \forall x \in (a, b)$, then the function is monotonically decreasing (or non-increasing) in the interval.
 - (c) If $f'(x) > 0 \forall x \in (a, b)$, then the function is strictly increasing in the interval.
 - (d) If $f'(x) < 0 \forall x \in (a, b)$, then the function is strictly decreasing in the interval.
4. A function can have its derivative equal to zero somewhere and still be strictly increasing or decreasing (For example, $y = x^3$ is strictly increasing). Let f be a function continuous in $[a, b]$ and differentiable in (a, b) . Then,
 - (a) If $f'(x) \geq 0 \forall x \in (a, b)$ and the equality holds at discrete points, then the function is strictly increasing.
 - (b) If $f'(x) \leq 0 \forall x \in (a, b)$ and the equality holds at discrete points, then the function is strictly decreasing.
5. Let f be a continuous and differentiable function.
Then, f is strictly increasing or strictly decreasing $\Leftrightarrow f$ is one-one.

6. If f is increasing and g is increasing, then $f \circ g$ is increasing.

Proof. Consider any two real numbers a and b such that $a < b$.

$\Rightarrow g(a) < g(b)$ because g is increasing.

$\Rightarrow f(g(a)) < f(g(b))$ because f is increasing.

$\Rightarrow (f \circ g)$ is increasing

□

7. Similarly if f is decreasing and g is decreasing, then $f \circ g$ is increasing.

5.13.3 Problems

Problem 36. Find the total number of positive integer solutions of $a^b = b^a$ (where $b > a$).

Solution:

$$\begin{aligned} a^b &= b^a && \text{Raise both sides to } \frac{1}{ab} \text{ power} \\ \Rightarrow a^{\left(\frac{1}{a}\right)} &= b^{\left(\frac{1}{b}\right)} && \text{Take natural log on both sides} \\ \Rightarrow \frac{1}{a} \ln(a) &= \frac{1}{b} \ln(b) \\ \Rightarrow \frac{\ln(a)}{a} &= \frac{\ln(b)}{b} \\ \Rightarrow \frac{a}{\ln(a)} &= \frac{b}{\ln(b)} \end{aligned}$$

Consider the function $f(x) = \frac{x}{\ln(x)}$, $x > 0$

Then we're looking for solutions of $f(a) = f(b)$ where $a, b \in \mathbb{Z}^+$, $b > a$.

$$f(x) = \frac{x}{\ln(x)}$$

$$\Rightarrow f'(x) = \frac{\ln(x) - 1}{(\ln(x))^2}$$

Now, $f'(x) = 0$ at $x = e$

So $f(x) > 0$ for $x \in (e, \infty)$ and $f(x) < 0$ for $x \in (0, e)$

Therefore f is strictly decreasing in $(0, e)$ and f is strictly increasing in (e, ∞)

$\Rightarrow f$ is one-one in $(0, e)$ and one-one in (e, ∞)

In other words, $f(a) = f(b)$ has no solution for $a, b \in (0, e)$ and $a, b \in (e, \infty)$

Then the only case that remains is when $a \in (0, e)$ and $b \in (e, \infty)$.

There are only two positive integers in $(0, e)$, namely 1 and 2.

If $a = 1$, we get

$$1^b = b^1$$

where $b > 1$.

LHS = 1 but RHS will always be greater than 1.

So there will be no solution corresponding to $a = 1$ Now, if $a = 2$, we get

$$2^b = b^2$$

There exists exactly one integer solution to this, i.e. $a = 2$ and $b = 4$.

$$2^4 = 4^2$$

Therefore, $a^b = b^a$ will have exactly one positive-integer solution for $b > a$.

Problem 37. Let α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x), B(x)$ and $C(x)$ be polynomials of degree 3, 4 and 5 respectively. Then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$.

Proof. α is a root of $f(x)$.

$$\begin{aligned} f(\alpha) &= 0 \\ \Rightarrow f(x) &= a(x - \alpha)^2 \end{aligned}$$

Also, since α is a repeated root of $f(x)$, this implies α is also a root of $f'(x)$.

$$f'(\alpha) = 0$$

Now

$$\begin{aligned} \text{Let } h(x) &= \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \\ h(\alpha) &= 0 \\ h'(x) &= \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} + 0 + 0 = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \end{aligned}$$

Therefore,

$$h'(\alpha) = 0$$

$\Rightarrow \alpha$ has a multiplicity of 2 in $h(x)$.

$\Rightarrow h(x)$ is divisible by $f(x)$.

□

Problem 38. Let $p(x)$ be a polynomial such that $p(1) = 0$ and $\frac{d}{dx}p(x) > p(x) \forall x > 1$. Show that $p(x) > 0 \forall x > 1$.

Proof.

$$\begin{aligned} \frac{d}{dx}(p(x)) &> p(x) && \forall x > 1 \\ \Rightarrow p'(x) &> p(x) && \forall x > 1 \\ \Rightarrow p'(x) - p(x) &> 0 && \forall x > 1 \end{aligned}$$

Multiply by the integrating factor e^{-x}

$$\begin{aligned} \Rightarrow e^{-x}p'(x) - e^{-x}p(x) &> 0 && \forall x > 1 \\ \Rightarrow \frac{d}{dx}(e^{-x} \cdot p(x)) &> 0 && \forall x > 1 \end{aligned}$$

Let $g(x) = e^{-x} \cdot p(x)$.

Then, $g'(x) > 0 \forall x > 1 \Rightarrow g$ is strictly increasing $\forall x > 1$

And $g(1) = e^{-1}p(1) = 0$

$\Rightarrow g(x) > 0 \forall x > 1$

$\Rightarrow e^{-x} \cdot p(x) > 0$

$\Rightarrow p(x) > 0 \forall x > 1$

□

Actually, this problem is fundamentally flawed. We can prove that no such polynomial exists by contradiction. However, such a function can exist. Had $p(x)$ been given as a differentiable function, this problem and solution would have been valid.

Proof that such a polynomial doesn't exist:

Proof. (By contradiction)

Assume for the purpose of contradiction that such a polynomial exists. Let us call it $p(x)$.

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \\ \Rightarrow p'(x) &= n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + (n-2) a_{n-2} x^{n-3} + \cdots + a_1 \end{aligned}$$

Now, it is given that $p'(x) > p(x) \forall x > 1$

i.e. $p'(x) - p(x) > 0 \forall x > 1$

$$\begin{aligned} p'(x) - p(x) &= -a_n x^n + x^{n-1} (n a_n - a_{n-1}) + x^{n-2} ((n-1) a_{n-1} - a_{n-2}) + \cdots \\ &= -a_n x^n - x^{n-1} (a_{n-1} - n a_n) - x^{n-2} (a_{n-2} - (n-1) a_{n-1}) - \cdots \end{aligned}$$

If $p'(x) - p(x) > 0 \forall x > 1$

$$-a_n x^n - x^{n-1} (a_{n-1} - n a_n) - x^{n-2} (a_{n-2} - (n-1) a_{n-1}) - \cdots > 0 \forall x > 1$$

Then this should also be true for values of x approaching infinity.

$$\Rightarrow a_n < 0$$

But we proved that $p(x) > 0 \forall x > 1$.

$$\Rightarrow a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 > 0 \forall x > 1$$

This should also be true for infinitely great values of x .

$$\Rightarrow a_n > 0$$

And we have derived a contradiction.

Therefore the assumption that such a polynomial exists is incorrect.

Hence, the given problem is fundamentally flawed. □

Problem 39. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x) \forall x \in \mathbb{R}$ and $f(0) = 1$.
Then:

- (A) $f(x) > e^{2x}$ in $(0, \infty)$
- (B) $f(x)$ is decreasing in $(0, \infty)$
- (C) $f(x)$ is increasing in $(0, \infty)$
- (D) $f'(x) < e^{2x}$ in $(0, \infty)$

Solution: (AC)

$$\begin{aligned}
 f'(x) &> 2f(x) && \text{(Given)} \\
 \Rightarrow f'(x) - 2f(x) &> 0 && \text{(Multiply by the integrating factor)} \\
 \Rightarrow e^{-2x} f'(x) - 2e^{-2x} f(x) &> 0 \\
 \Rightarrow \frac{d}{dx} (e^{-2x} f(x)) &> 0 && \text{Let } g(x) = e^{-2x} f(x) \\
 \Rightarrow g'(x) &> 0 && \forall x \in \mathbb{R} \\
 \Rightarrow g(x) \text{ is strictly increasing.} \\
 \Rightarrow g(x) &> g(0) && \forall x > 0 \\
 \Rightarrow g(x) &> e^0 f(0) && \forall x > 0 \\
 \Rightarrow e^{-2x} f(x) &> 1 && \forall x > 0 \\
 \Rightarrow f(x) &> e^{2x} && \forall x \in (0, \infty)
 \end{aligned}$$

Therefore, option A is correct

Now, since $f'(x) > 2f(x)$ and $f(x) > e^{2x}$,

$$\begin{aligned}
 \Rightarrow f'(x) &> 2f(x) > 2e^{2x} && \forall x \in (0, \infty) \\
 \Rightarrow f'(x) &> 2e^{2x} && \forall x \in (0, \infty) \\
 \Rightarrow f'(x) &> 0 && \forall x \in (0, \infty) \\
 \Rightarrow f(x) \text{ is strictly increasing in } (0, \infty)
 \end{aligned}$$

Therefore, option C is correct

5.14 Testing for Local Maxima and Minima

Definition 19. The critical point of a function is defined as a point where the derivative of the function is either zero or not defined.

Algorithm 1. Consider a function $f(x)$. We have to find its local maxima and minima.

1. Differentiate f to get $f'(x)$.
2. Find all the points where $f'(x)$ changes sign.
3. If $f'(x)$ changes sign from positive to negative, then the point is a local maximum. If $f'(x)$ changes sign from negative to positive, then the point is a local minimum.

This algorithm is preferred as it gives the answer definitively and in exactly 3 steps.

Problem 40. The function $f(x) = x(x^2 - 4)^n(x^2 - x + 1)$, $n \in \mathbb{N}$ assumes a local minimum at $x = 2$, then:

- (A) n can be any odd number
- (B) n can only be an odd prime number
- (C) n can be any even integer
- (D) n can only be a multiple of four

Solution: (C)

$$f(x) = x(x^2 - 4)^n(x^2 - x + 1)$$

$$\begin{aligned} f'(x) &= (x^2 - 4)^n \cdot (x^2 - x + 1) + n(x^2 - 4)^{n-1} \cdot (2x)(x)(x^2 - x + 1) + x(x^2 - 4)^n \cdot (2x - 1) \\ &= (x^2 - 4)^{n-1} \left((x^2 - 4)(x^2 - x + 1) + n(2x)(x)(x^2 - x + 1) + x(x^2 - 4)(2x - 1) \right) \end{aligned}$$

By Fermat's Theorem, since f attains a local maximum at $x = 2$,

$$f'(2) = 0$$

$$\Rightarrow n - 1 \geq 1$$

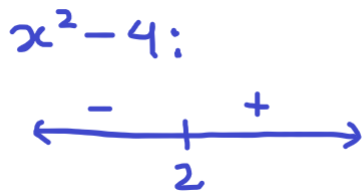
$$\Rightarrow n \geq 2$$

Now,

$((x^2 - 4)(x^2 - x + 1) + n(2x)(x)(x^2 - x + 1) + x(x^2 - 4)(2x - 1))$ is positive in the neighbourhood of $x = 2$.

So $(x^2 - 4)^{n-1}$ must change sign at $x = 2$ from negative to positive.

This is only possible when $n - 1$ is odd.



$\Rightarrow n$ is any even number.

Problem 41. The least value of $a \in \mathbb{R}$ for which $4ax^2 + \frac{1}{x} \geq 1$ for all $x > 0$ is:

- (A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Solution (C):

Method 1: Using Calculus

Consider the function $f(x) = 4ax^2 + \frac{1}{x}$ in the interval $(0, \infty)$.

The problem stipulates that $f(x) \geq 1 \forall x \in (0, \infty)$

Proposition: If the minimum value of the function is greater than or equal to 1, then it is necessary and sufficient for the condition to be satisfied.

$$f'(x) = 8ax - \frac{1}{x^2}$$

Finding Critical points:

$$8ax^3 - 1 = 0$$

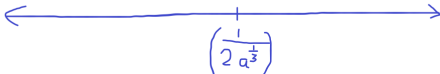
$$\Rightarrow x^3 = \frac{1}{8a}$$

$$\Rightarrow x = \frac{1}{2(a)^{\frac{1}{3}}}$$

Therefore $f(x)$ attains its minimum value at $x = \frac{1}{2(a)^{\frac{1}{3}}}$, which is $f\left(\frac{1}{2(a)^{\frac{1}{3}}}\right)$.

$f(x)$: Decreasing Increasing

$f'(x)$: $-$ $+$



$$\Rightarrow f\left(\frac{1}{2(a)^{\frac{1}{3}}}\right) \geq 1$$

$$\Rightarrow 4a \cdot \left(\frac{1}{2a^{\frac{1}{3}}}\right)^2 + 2a^{\frac{1}{3}} \geq 1$$

$$\Rightarrow 3a^{\frac{1}{3}} \geq 1$$

$$\Rightarrow a^{\frac{1}{3}} \geq \frac{1}{3}$$

$$\Rightarrow a \geq \left(\frac{1}{3}\right)^3$$

$$\Rightarrow a \geq \frac{1}{27}$$

Therefore, option **C** is correct.

Method 2: Using AM-GM Inequality

$$f(x) = 4ax^2 + \frac{1}{x} \geq 1, \quad x \in (0, \infty)$$

In order to apply the AM-GM inequality, we must ensure that the terms are all non-negative, which will be the case if and only if a is non-negative.

Claim: $a \geq 0$

Proof. Assume for the purpose of contradiction that a is negative.
Then,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} 4ax^2 + \frac{1}{x} < 0$$

This contradicts the fact that $f(x) \geq 1 \quad \forall x \in (0, \infty)$

Therefore, $a \geq 0$. □

Now, applying AM-GM inequality on $4ax^2$, $\frac{1}{2x}$ and $\frac{1}{2x}$

$$\begin{aligned} 4ax^2 + \frac{1}{2x} + \frac{1}{2x} &\geq 3 \left((4ax^2) \left(\frac{1}{2x} \right) \left(\frac{1}{2x} \right) \right)^{\frac{1}{3}} \\ &\Rightarrow 4ax^2 + \frac{1}{x} \geq 3a^{\frac{1}{3}} \end{aligned}$$

The equality holds when $4ax^2 = \frac{1}{2x}$ or at $x^3 = \frac{1}{8a} \Rightarrow x = \frac{1}{2a^{\frac{1}{3}}}$, which is permissible.

This implies that the minimum value of $f(x)$ is $3a^{\frac{1}{3}}$, but $f(x) \geq 1 \quad \forall x \in (0, \infty)$.

$$\begin{aligned} &\Rightarrow 3a^{\frac{1}{3}} \geq 1 \\ &\Rightarrow a^{\frac{1}{3}} \geq \frac{1}{3} \\ &\Rightarrow a \geq \frac{1}{3^3} \\ &\Rightarrow a \geq \frac{1}{27} \end{aligned}$$

Therefore, option **C** is correct.

Problem 42. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x equal to:

- (A) -2 (B) $-\frac{2}{3}$ (C) 2 (D) $\frac{2}{3}$

Solution (AB):

We know that,

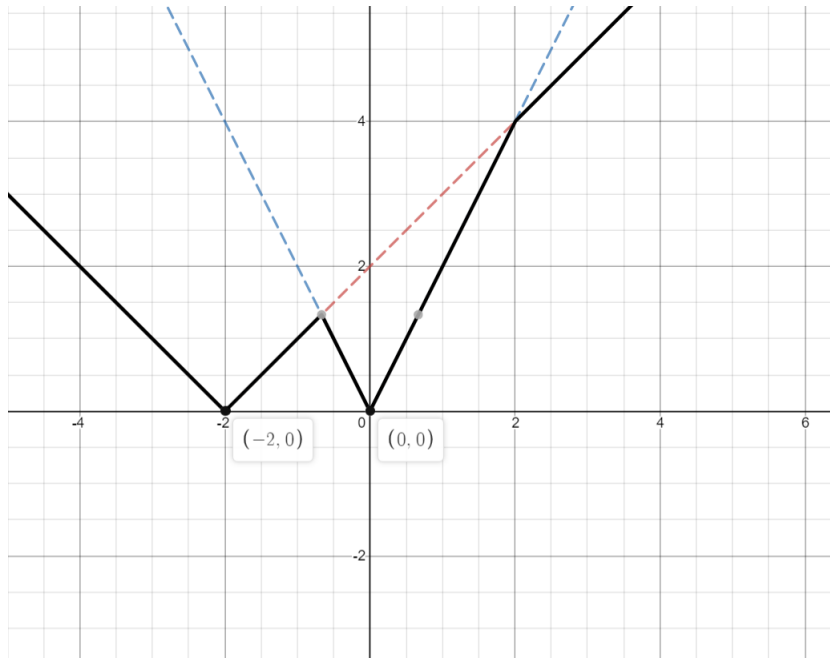
$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

$$\min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}$$

Then,

$$f(x) = 2 \cdot \min\{|x - 2|, 2|x|\}$$

We can plot the graph here.



The critical points are at $x = -2, -\frac{2}{3}, 0, 2$

The point at $x = 2$ is neither that of a minimum, nor of a maximum, because the derivative is positive both to its immediate left and right.

The points of minima are at $x = -2$ and $x = 0$, and the point of maximum is at $x = \frac{-2}{3}$.

Therefore, options **(A)** and **(B)** are correct.

Problem 43. If the roots of the equation $x^4 - 4x^3 + ax^2 + bx + 1 = 0$ are all positive, find the value of $(a - b)$.

Solution:

$$x^4 - 4x^3 + ax^2 + bx + 1 = 0$$

Say the roots are x_1, x_2, x_3 and x_4 .

Then,

$$\text{Sum of roots} = x_1 + x_2 + x_3 + x_4 = 4$$

$$\Rightarrow \frac{x_1 + x_2 + x_3 + x_4}{4} = 1$$

And

$$\text{Product of roots} = x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 1$$

$$\Rightarrow (x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{\frac{1}{4}} = 1^{\frac{1}{4}} = 1$$

That means, AM of the roots = GM of the roots.

The equality holds for the AM-GM inequality if and only if all the terms are equal.

Therefore,

$$x_1 = x_2 = x_3 = x_4 = 1$$

Then,

$$a = \text{Sum of roots taken two at a time} = {}^4C_2 = 6$$

$$-b = \text{Sum of roots taken three at a time} = {}^4C_3 = 4$$

$$\Rightarrow b = -4$$

$$a - b = 6 - (-4)$$

$$= 6 + 4$$

$$= 10$$

Therefore, we get the value of $(a - b)$ as 10.

Problem 44. Find the total number of points of global maxima of $f(x) = \cos(x) + \cos(\sqrt{2}x)$

Solution: We know that the maximum value of $\cos(A) + \cos(B)$ can be at most 1 because the range of $\cos(x)$ is $[-1, 1]$.

It is clear that $\cos(x) + \cos(\sqrt{2}x)$ will have the value 2 at $x = 0$. Therefore, $x = 0$ is a point of global maximum.

To check if other such points exist, we can solve $\cos(x) = 1$ and $\cos(\sqrt{2}x) = 1$ simultaneously, which will give us $x = 0$ as the only solution.

5.15 Concavity

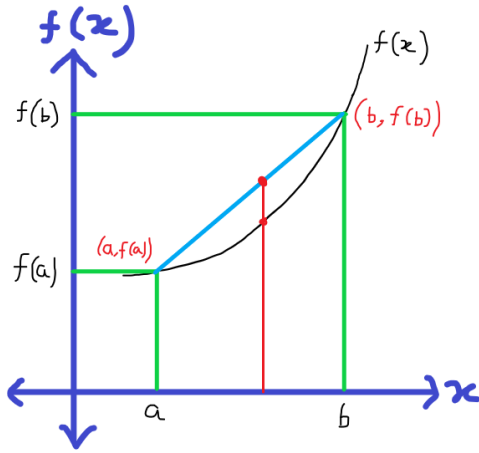
Definition 20. A function is said to be (weakly) concave up if a line joining any two arbitrary points of the curve lies on or above the curve.

Definition 21. A function is said to be strictly concave up if a line joining any two arbitrary points always lies above the curve.

Definition 22. A function is said to be (weakly) concave down if a line joining any two arbitrary points of the curve lies on or below the curve.

Definition 23. A function is said to be strictly concave down if a line joining any two arbitrary points always lies below the curve.

5.15.1 Mathematical Condition for a Function to be Concave Up



Equation of line joining the points:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\Rightarrow y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Condition for concave up:

$$\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \geq f(x) \quad \forall x \in [a, b]$$

$$\Rightarrow \frac{f(b) - f(a)}{b - a} \geq \frac{f(x) - f(a)}{x - a} \quad \forall x \in [a, b]$$

Similarly,

For strictly concave up:

$$\frac{f(b) - f(a)}{b - a} > \frac{f(x) - f(a)}{x - a} \quad \forall x \in (a, b)$$

For concave down:

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(x) - f(a)}{x - a} \quad \forall x \in [a, b]$$

For strictly concave down:

$$\frac{f(b) - f(a)}{b - a} < \frac{f(x) - f(a)}{x - a} \quad \forall x \in (a, b)$$

5.15.2 Properties

Let f be a differentiable function, then:

1. If f is concave up in (a, b) then $f'(a) \leq f'(b)$ and f' is non-decreasing.
2. If f is strictly concave up in (a, b) then $f'(a) < f'(b)$ and f' is strictly increasing.
3. If f is concave down in (a, b) then $f'(a) \geq f'(b)$ and f' is non-increasing.
4. If f is strictly concave down in (a, b) , then $f'(a) > f'(b)$ and f' is strictly decreasing.

Let f be a twice-differentiable function, then:

1. f is concave up when $f''(x) \geq 0$.
2. f is strictly concave up when $f''(x) > 0$.
3. f is concave down when $f''(x) \leq 0$.
4. f is strictly concave down when $f''(x) < 0$.

Suppose f is a twice-differentiable function in (a, b) and strictly concave up. If $f(a) = f(b)$, then $f(x) < f(a) \quad \forall x \in (a, b)$

5.16 Inflection Points

Definition 24. A point c is called a point of inflection for a function f if:

1. There exists a tangent at c , and
2. f changes its concavity about c .

Definition 25. If f is a doubly-differentiable function, then c is said to be a point of inflection when:

1. There exists a tangent at c ,
2. $f''(c) = 0$ and
3. $f''(x)$ must change its sign about the point c .

Problem 45. For what values of a will the curve $y = x^4 + ax^3 + 3x^2 + 1$ be concave about the entire real number line?

Solution:

$$\begin{aligned}y &= x^4 + ax^3 + 3x^2 + 1 \\ \Rightarrow y' &= 4x^3 + 3ax^2 + 6x \\ \Rightarrow y'' &= 12x^2 + 6ax + 6 \\ \Rightarrow y'' &= 6 \cdot (2x^2 + ax + 1)\end{aligned}$$

For the curve to be concave up, $y'' \geq 0 \quad \forall x$.

This is only possible when the discriminant of the quadratic is non-positive.

$$\begin{aligned}\Rightarrow a^2 - 8 &\leq 0 \\ \Rightarrow (a + 2\sqrt{2})(a - 2\sqrt{2}) &\leq 0 \\ \Rightarrow a &\in [-2\sqrt{2}, 2\sqrt{2}]\end{aligned}$$

Problem 46. What conditions must the coefficients a, b and c satisfy for the curve $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ to have point(s) of inflection?

Solution:

$$\begin{aligned}f(x) &= ax^4 + bx^3 + cx^2 + dx + e \\ f'(x) &= 4ax^3 + 3bx^2 + 2cx + d \\ f''(x) &= 12ax^2 + 6bx + 2c\end{aligned}$$

For f to have a point of inflection, f must be differentiable (which it is, because f is a polynomial function), and should have a point c where $f''(c) = 0$ and $f''(x)$ changes sign at c .

This means the discriminant of the quadratic must be positive.

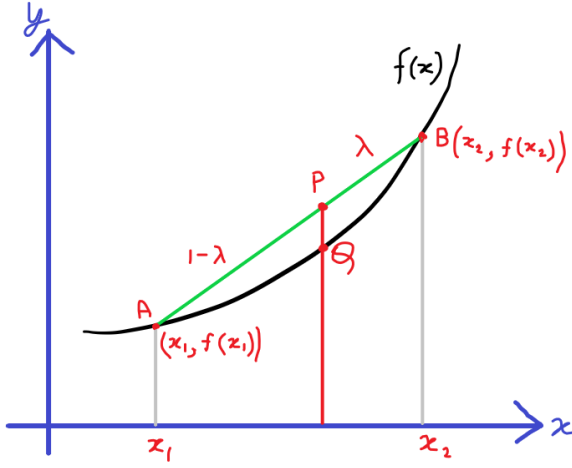
$$\begin{aligned}\Rightarrow (6b)^2 - 4(12a)(2c) &> 0 \\ \Rightarrow 36b^2 - 96ac &> 0 \\ \Rightarrow 6b^2 - 16ac &> 0 \\ \Rightarrow 3b^2 - 8ac &> 0 \\ \Rightarrow 3b^2 &> 8ac\end{aligned}$$

Therefore, for f to have a point of inflection, the coefficients a, b and c must satisfy the condition $3b^2 > 8ac$.

5.17 Jensen's Inequality

Say a function is concave up in an interval (x_1, x_2)

Say the point P divides the line segment AB in the ratio $(1 - \lambda) : \lambda$



Then, using section formula,

P will have coordinates $(\lambda x_1 + (1 - \lambda)x_2, \lambda f(x_1) + (1 - \lambda)f(x_2))$.

Then, Q will have coordinates $(\lambda x_1 + (1 - \lambda)x_2, f(\lambda x_1 + (1 - \lambda)x_2))$

As λ varies from 0 to 1, the point P takes all positions between A and B .

Since the curve is concave up,

y -coordinate of $P > y$ -coordinate of Q

$$\Rightarrow \lambda f(x_1) + (1 - \lambda)f(x_2) > f(\lambda x_1 + (1 - \lambda)x_2)$$

Say $\lambda = \lambda_1$ and $(1 - \lambda) = \lambda_2$

$\lambda_1 > 0, \lambda_2 > 0$

$\lambda_1 + \lambda_2 = 1$

$$\Rightarrow \lambda_1 f(x_1) + \lambda_2 f(x_2) \geq f(\lambda_1 x_1 + \lambda_2 x_2)$$

Generalizing,

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$$

where $0 < \lambda_i < 1, i \in \{1, 2, 3, \dots, n\}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$

This is the general Jensen's inequality for a function that is concave up. The generalization can be proved using Mathematical Induction. However, we are more interested in the special case when

$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$.

$$\frac{f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right)$$

The sign reverses for a function that is concave down.

So Jensen's inequality becomes:

$$\frac{f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right)$$

if the function is concave up, and

$$\frac{f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right)$$

if the function is concave down.

5.17.1 Proof of the AM-GM Inequality

Proof. Consider $f(x) = \ln(x)$. Note that $x > 0$ (from the domain of f).

$$f'(x) = \frac{1}{x}$$

$$f''(x) = \frac{-1}{x^2}$$

i.e. $f''(x) < 0 \forall x > 0$

$\Rightarrow f(x) = \ln(x)$ is concave down. Applying Jensen's Inequality:

$$\begin{aligned} \frac{f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)}{n} &\leq f\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right) \\ \frac{1}{n} [\ln(x_1) + \ln(x_2) + \ln(x_3) + \cdots + \ln(x_n)] &\leq \ln\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right) \\ \Rightarrow \frac{1}{n} \ln(x_1 x_2 x_3 \cdots x_n) &\leq \ln\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right) \\ \Rightarrow \ln\left((x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}}\right) &\leq \ln\left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right) \end{aligned}$$

Now, since $\ln(x)$ is an increasing function, this means $\ln(x_1) \leq \ln(x_2) \Leftrightarrow x_1 \leq x_2$.

$$\begin{aligned} \Rightarrow (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} &\leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \\ \Rightarrow \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} &\geq (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} \\ \Rightarrow AM &\geq GM \end{aligned}$$

□

5.17.2 Problems

Problem 47. Prove that in a triangle ABC , $\sin(A) + \sin(B) + \sin(C) \leq \frac{3\sqrt{3}}{2}$.

Proof. Since A, B, C are angles in a triangle,

$0 < A, B, C < \pi$ and $A + B + C = \pi$.

Consider the function $f(x) = \sin(x)$ in the interval $[0, \pi]$.

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f''(x) \leq 0 \quad \forall x \in [0, \pi]$$

$\sin(x)$ is concave down in $[0, \pi]$. Applying Jensen's Inequality:

$$\begin{aligned} \frac{\sin(A) + \sin(B) + \sin(C)}{3} &\leq \sin\left(\frac{A + B + C}{3}\right) \\ \Rightarrow \frac{\sin(A) + \sin(B) + \sin(C)}{3} &\leq \sin\left(\frac{\pi}{3}\right) \\ \Rightarrow \frac{\sin(A) + \sin(B) + \sin(C)}{3} &\leq \frac{\sqrt{3}}{2} \\ \Rightarrow \sin(A) + \sin(B) + \sin(C) &\leq \frac{3\sqrt{3}}{2} \end{aligned}$$

□

Problem 48. Prove that for an acute-angled triangle ABC , $\tan(A) + \tan(B) + \tan(C) \geq 3\sqrt{3}$

Proof.

$$A + B + C = \pi$$

$$0 < A, B, C < \frac{\pi}{2}$$

Consider the function $f(x) = \tan(x)$ in the interval $\left[0, \frac{\pi}{2}\right)$.

$$f(x) = \tan(x)$$

$$f'(x) = \sec^2(x)$$

$$f''(x) = 2 \sec(x) \tan(x) \geq 0 \quad \forall x \in \left[0, \frac{\pi}{2}\right)$$

i.e. $\tan(x)$ is concave up in $\left[0, \frac{\pi}{2}\right)$. Applying Jensen's Inequality:

$$\begin{aligned} \frac{\tan(A) + \tan(B) + \tan(C)}{3} &\geq \tan\left(\frac{A + B + C}{3}\right) \\ \Rightarrow \tan(A) + \tan(B) + \tan(C) &\geq 3 \tan\left(\frac{\pi}{3}\right) \\ \Rightarrow \tan(A) + \tan(B) + \tan(C) &\geq 3\sqrt{3} \end{aligned}$$

□

5.18 Problems on Functional Equations

Problem 49. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation $f(x + y) = f(x) \cdot f(y) \forall x, y \in \mathbb{R}$ and $f(x) \neq 0$ for any x in \mathbb{R} . Let the function be differentiable at 0 and let $f'(0) = 2$. Find the value of $f(\ln(10))$.

Solution:

$$\begin{aligned} f(x + y) &= f(x) \cdot f(y) \forall x, y \in \mathbb{R} \\ \Rightarrow f(0 + 0) &= f(0) \cdot f(0) \\ \Rightarrow f(0) &= (f(0))^2 \\ \Rightarrow f(0) &= 1 \vee f(0) = 0 \end{aligned}$$

But $f(x) \neq 0$ for any $x \in \mathbb{R}$ (Given).

Therefore,

$$f(0) = 1$$

But why did the problem say that $f(x)$ is never zero?

Suppose $f(0) = 0$

Then $f(x + 0) = f(x) \cdot f(0) \forall x$

Or $f(x) = 0 \forall x \in \mathbb{R}$

i.e. f will become constant and equal to 0.

In fact, if a function satisfying such a functional equation is zero anywhere, then it's actually zero everywhere.

If $f(c) = 0$,

$$f(x + c) = f(x) \cdot f(c) = 0 \forall x \in \mathbb{R}$$

$$\Rightarrow f(x + c) = 0 \forall x \in \mathbb{R}$$

$$\Rightarrow f(x) = 0 \forall x \in \mathbb{R}$$

Claim 1: If the function is continuous at 0, then it is continuous everywhere.

Proof. f is continuous at 0

$$\Rightarrow \lim_{h \rightarrow 0} f(h) = f(0) = 1$$

If we can prove that $\lim_{h \rightarrow 0} f(x + h) = f(x)$, then we're done.

$$\Rightarrow \lim_{h \rightarrow 0} f(x + h) = \lim_{h \rightarrow 0} (f(x) \cdot f(h)) = f(x) \cdot 1 = f(x)$$

□

Claim 2: If the function is differentiable at 0, then it is differentiable everywhere.

Proof.

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} = 2$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h} = \lim_{h \rightarrow 0} f(x) \cdot \left(\frac{f(h) - 1}{h} \right) = 2f(x)$$

Therefore, it is differentiable everywhere.

□

Now,

$$f'(x) = 2f(x)$$

Since we know that $f(x)$ is not zero anywhere, we can divide by $f(x)$.

$$\frac{f'(x)}{f(x)} = 2$$

Integrating both sides

$$\int \frac{f'(x)}{f(x)} dx = \int 2 dx$$

Let $f(x) = t$. This means $f'(x) dx = dt$.

$$\Rightarrow \int \frac{dt}{t} = \int 2 dx$$

$$\Rightarrow \ln(f(x)) = 2x + c$$

$$\Rightarrow f(x) = e^{2x+c}$$

$$\Rightarrow f(x) = e^{2x} \cdot e^c$$

But e^c is just a constant. Call it k .

$$\Rightarrow f(x) = ke^{2x}$$

$$f(0) = ke^0$$

$$\Rightarrow f(0) = k$$

$$\Rightarrow k = 1$$

$$\Rightarrow f(x) = e^{2x}$$

But $f(0) = 1$ (Given)

Now,

$$\begin{aligned} f(\ln(10)) &= e^{2\ln(10)} \\ &= e^{\ln(10^2)} \\ &= e^{\ln(100)} \\ &= 100 \end{aligned}$$

Therefore, $f(\ln(10)) = 100$.

5.19 Approximations

5.19.1 First Order Approximations

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Say $h = \Delta x$ is sufficiently small.

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f(x + \Delta x) \approx f(x) + \Delta x f'(x)$$

Higher order approximations can be made using the Taylor Series of the function we're interested in approximating.

6 The Antiderivative

6.1 The First Fundamental Theorem of Calculus

Theorem 12. Let f be an integrable function in $[a, b]$ and let $F(x) = \int_a^x f(x) dx$. Let f be continuous at a point $c \in [a, b]$. Then $F'(c) = f(c)$.

In crude terms, it means that the indefinite integral (a.k.a the antiderivative) is the inverse operator of differentiation and does the reverse of that.

6.2 Definition of the Antiderivative

Definition 26. A function P is called the *primitive* (or the *antiderivative*) of a function f in an open interval I if the derivative of P is f , i.e. $P'(x) = f(x) \forall x \in I$.

$$F'(x) = f(x) \Rightarrow \int f(x) dx = F(x) + c$$

6.3 Properties of the Antiderivative

1. $\int k \cdot f(x) dx = k \cdot \int f(x) dx$
2. $\int [f(x_1) \pm f(x_2) \pm f(x_3) \pm \cdots \pm f(x_n)] dx = \int f(x_1) dx \pm \int f(x_2) dx \pm \cdots \pm \int f(x_n) dx$
3. $\int \frac{f'(x)}{f(x)} dx = \log_e(|f(x)|) + c$
4. $\int (f(x))^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$
5. If $\int f(x) dx = F(x)$, then $\int f(ax+b) dx = \frac{F(ax+b)}{a} + c$

6.4 Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
2. $\int \frac{1}{x} dx = \log_e(|x|) + c$
3. $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$
4. $\int \sqrt{x} dx = \frac{2}{3}x\sqrt{x} + c$
5. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

6. $\int \cos(x) \, dx = \sin(x) + c$
7. $\int \sin(x) \, dx = -\cos(x) + c$
8. $\int \sec^2(x) \, dx = \tan(x) + c$
9. $\int \sec(x) \tan(x) \, dx = \sec(x) + c$
10. $\int \csc(x) \cot(x) \, dx = -\csc(x) + c$
11. $\int \csc^2(x) \, dx = -\cot(x) + c$
12. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$
13. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left(\left| \frac{x-a}{x+a} \right| \right) + c$
14. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left(\left| \frac{a+x}{a-x} \right| \right) + c$
15. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log_e \left(\left| x + \sqrt{x^2 + a^2} \right| \right) + c$
16. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e \left(\left| x + \sqrt{x^2 - a^2} \right| \right) + c$
17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c$
18. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left(\left| \frac{x}{a} \right| \right) + c$
19. $\int \left(\sqrt{x^2 + a^2} \right) dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log_e \left(\left| x + \sqrt{x^2 + a^2} \right| \right) + c$
20. $\int \left(\sqrt{x^2 - a^2} \right) dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e \left(\left| x + \sqrt{x^2 - a^2} \right| \right) + c$
21. $\int \left(\sqrt{a^2 - x^2} \right) dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c$

6.5 Integration by Transformation and Adjustment

Problem 50. $\int \frac{2 \, dx}{1 + \cos(2x)}$

$$\begin{aligned} I &= \int \frac{2 \, dx}{1 + \cos(2x)} \\ &= \int \frac{\cancel{2} \, dx}{\cancel{2} \cos^2(x)} \\ &= \int \frac{dx}{\cos^2(x)} \\ &= \int \sec^2(x) \, dx \\ &= \tan(x) + c \end{aligned}$$

Problem 51. $\int \frac{1}{\sin^2(x) \cos^2(x)} \, dx$

$$\begin{aligned} I &= \int \frac{1}{\sin^2(x) \cos^2(x)} \, dx \\ &= \int \frac{\sin^2(x) + \cos^2(x)}{\sin^2(x) \cos^2(x)} \, dx \\ &= \int \frac{\cancel{\sin^2(x)}}{\cancel{\sin^2(x)} \cos^2(x)} \, dx + \int \frac{\cancel{\cos^2(x)}}{\sin^2(x) \cancel{\cos^2(x)}} \, dx \\ &= \int \sec^2(x) \, dx + \int \csc^2(x) \, dx \\ &= \tan(x) - \cot(x) + c \end{aligned}$$

Problem 52. $\int \frac{x^3 - x^2 + x - 1}{x - 1} \, dx$

$$\begin{aligned} I &= \int \frac{x^3 - x^2 + x - 1}{x - 1} \, dx \\ &= \int \frac{x^2(x - 1) + 1(x - 1)}{x - 1} \, dx \\ &= \int \frac{(x^2 + 1)\cancel{(x - 1)}}{\cancel{x - 1}} \, dx \\ &= \frac{x^3}{3} + x + c \end{aligned}$$

Problem 53. $\int \cos(x) \cos(2x) \cos(3x) \, dx$

$$\begin{aligned}
 I &= \int \cos(x) \cos(2x) \cos(3x) \, dx \\
 &= \int \cos(x) \cos(3x) \cos(2x) \, dx \\
 &= \frac{1}{2} \int (\cos(4x) + \cos(2x)) \cos(2x) \, dx \\
 &= \frac{1}{2} \left[\int \left(\frac{\cos(6x) + \cos(2x)}{2} \right) \, dx + \int \cos^2(2x) \, dx \right] \\
 &= \frac{1}{2} \left[\frac{\sin(6x)}{12} + \frac{\sin(2x)}{4} + \int \left(\frac{1 + \cos(4x)}{2} \right) \, dx \right] \\
 &= \frac{1}{2} \left[\frac{\sin(6x)}{12} + \frac{\sin(2x)}{4} + \frac{x}{2} + \frac{\sin(4x)}{8} \right] + c \\
 &= \frac{1}{4} \left(\frac{\sin(6x)}{6} + \frac{\sin(2x)}{2} + x + \frac{\sin(4x)}{4} \right) + c \\
 &= \frac{1}{4} \left(\frac{\sin(6x)}{6} + \frac{\sin(2x)}{2} + \frac{\sin(4x)}{4} + x \right) + c
 \end{aligned}$$

Problem 54. $\int \frac{1}{1 - \sin(x)} \, dx$

$$\begin{aligned}
 I &= \int \frac{1}{1 - \sin(x)} \, dx \\
 &= \int \left(\frac{1}{1 - \sin(x)} \cdot \frac{1 + \sin(x)}{1 + \sin(x)} \right) \, dx \\
 &= \int \frac{1 + \sin(x)}{1 - \sin^2(x)} \, dx \\
 &= \int \frac{1 + \sin(x)}{\cos^2(x)} \, dx \\
 &= \int \sec^2(x) \, dx + \int \sec(x) \tan(x) \, dx \\
 &= \tan(x) + \sec(x) + c
 \end{aligned}$$

Multiply and divide by the conjugate

Problem 55. $\int \frac{x-1}{x+1} dx$

$$\begin{aligned}
 I &= \int \frac{x-1}{x+1} dx \\
 &= \int \frac{x+1-1-1}{x+1} dx \\
 &= \int \frac{x+1-2}{x+1} dx \\
 &= \int \frac{\cancel{x+1}}{\cancel{x+1}} dx - 2 \int \frac{dx}{x+1} \\
 &= x - 2 \log_e (|x+1|) + c
 \end{aligned}$$

Problem 56. $\int \frac{x^2-1}{x^2+1} dx$

$$\begin{aligned}
 I &= \int \frac{x^2-1}{x^2+1} dx \\
 &= \int \frac{x^2+1-1-1}{x^2+1} dx \\
 &= \int \frac{\cancel{x^2+1}}{\cancel{x^2+1}} dx - 2 \int \frac{dx}{1+x^2} \\
 &= x - 2 \tan^{-1}(x) + c
 \end{aligned}$$

6.6 Integration By Substitution

This method of integration is sort-of like reverse Chain Rule.

Generic Example:

Let the antiderivative of f be F .

$$\int f(g(x)) g'(x)$$

Let $g(x) = t$.

$\Rightarrow g'(x) dx = dt$

So the integral becomes

$$\int f(t) dt = F(t) + c = F(g(x)) + c$$

If we want to assume $g(x) = t$, first ensure that $g'(x)$ is present along with dx in the numerator.

Problem 57. $\int \sin^2(x) \cos(x) dx$

$$I = \int \sin^2(x) \cos(x) dx$$

Let $\sin(x) = t$

$\Rightarrow \cos(x) dx = dt$

$$\begin{aligned}\Rightarrow I &= \int t^2 dt \\ &= \frac{t^3}{3} + c \\ &= \frac{\sin^3(x)}{3} + c\end{aligned}$$

Problem 58. $\int x^3 \sin(x^4)$

$$I = \int x^3 \sin(x^4)$$

Let $x^4 = t$

$\Rightarrow 4x^3 dx = dt$

$$\begin{aligned}I &= \int \frac{\sin(t) dt}{4} \\ &= \frac{1}{4} \int \sin(t) dt \\ &= \frac{-1}{4} \cos(t) + c \\ &= \frac{-1}{4} \cos(x^4) + c\end{aligned}$$

Problem 59. $\int \frac{\sin(x) + \cos(x)}{(\sin(x) - \cos(x))^3} dx$

$$I = \int \frac{\sin(x) + \cos(x)}{(\sin(x) - \cos(x))^3} dx$$

Let $\sin(x) - \cos(x) = t$
 $\Rightarrow (\cos(x) + \sin(x)) dx = dt$
 $\Rightarrow (\sin(x) + \cos(x)) dx = dt$

$$\begin{aligned} I &= \int \frac{1}{t^3} dt \\ &= \int t^{-3} dt \\ &= \frac{t^{-3+1}}{-3+1} + c \\ &= \frac{t^{-2}}{-2} + c \\ &= \frac{(\sin(x) - \cos(x))^{-2}}{-2} + c \\ &= \frac{-1}{2(\sin(x) - \cos(x))^2} + c \end{aligned}$$

Problem 60. $\int \frac{x dx}{(ax^2 + b)^p}$

$$I = \int \frac{x dx}{(ax^2 + b)^p}$$

Let $ax^2 + b = t$
 $\Rightarrow 2ax dx = dt$
 $\Rightarrow x dx = \frac{dt}{2a}$

$$\begin{aligned} I &= \frac{1}{2a} \int \frac{1}{t^p} dt \\ &= \frac{1}{2a} \int t^{-p} dt \\ &= \frac{1}{2a} \cdot \frac{t^{(-p+1)}}{-p+1} + c \\ &= \frac{t^{(1-p)}}{2a(1-p)} + c \end{aligned}$$

Problem 61. $\int \frac{dx}{e^x + e^{-x}}$

$$\begin{aligned} I &= \int \frac{dx}{e^x + e^{-x}} \\ &= \int \frac{dx}{e^{-x}(1 + e^{2x})} \\ &= \int \frac{e^x dx}{1 + e^{2x}} \end{aligned}$$

Let $e^x = t$
 $\Rightarrow e^x dx = dt$

$$\begin{aligned} \Rightarrow I &= \int \frac{dt}{1 + t^2} \\ &= \tan^{-1}(t) + c \\ &= \tan^{-1}(e^x) + c \end{aligned}$$

Problem 62. $\int \frac{x^3}{\sqrt{x^8 + 1}} dx$

$$I = \int \frac{x^3}{\sqrt{x^8 + 1}} dx$$

Let $x^4 = t$
 $\Rightarrow 4x^3 dx = dt$
 $\Rightarrow x^3 dx = \frac{dt}{4}$

$$\begin{aligned} I &= \frac{1}{4} \int \frac{1}{\sqrt{t^2 + 1}} dt \\ &= \frac{1}{4} \ln \left(\left| t + \sqrt{t^2 + 1} \right| \right) + c \\ &= \frac{1}{4} \ln \left(\left| x^4 + \sqrt{x^8 + 1} \right| \right) + c \end{aligned}$$

$$\text{Using } \int \frac{dx}{\sqrt{x^2 + a^2}} = \log_e \left(\left| x + \sqrt{x^2 + a^2} \right| \right) + c$$

Problem 63. $\int \frac{\sec^2(x)}{(\sec(x) + \tan(x))^{\frac{3}{2}}} dx$

$$I = \int \frac{\sec^2(x)}{(\sec(x) + \tan(x))^{\frac{3}{2}}} dx$$

Let $\sec(x) + \tan(x) = t$

Then $\sec^2(x) - \tan^2(x) = 1$

$\Rightarrow (\sec(x) + \tan(x)) (\sec(x) - \tan(x)) = 1$

$\Rightarrow t (\sec(x) - \tan(x)) = 1$

$\Rightarrow \sec(x) - \tan(x) = \frac{1}{t}$

$$\sec(x) + \tan(x) = t$$

$$\sec(x) - \tan(x) = \frac{1}{t}$$

$$2 \sec(x) = t + \frac{1}{t}$$

$\Rightarrow \sec(x) = \frac{1}{2} \left(t + \frac{1}{t} \right)$

$\sec(x) + \tan(x) = t$

$\Rightarrow \sec(x) (\sec(x) + \tan(x)) dx = dt$

$\Rightarrow \sec(x) dx = \frac{dt}{t}$

$$\begin{aligned} I &= \int \frac{\frac{1}{2} \left(t + \frac{1}{t} \right) \frac{dt}{t}}{t^{\frac{9}{2}}} \\ &= \frac{1}{2} \int \frac{\left(t + \frac{1}{t} \right) \frac{1}{t}}{t^{\frac{9}{2}}} dt = \frac{1}{2} \int \frac{\left(1 + \frac{1}{t^2} \right)}{t^{\frac{9}{2}}} dt \\ &= \frac{1}{2} \int t^{-\frac{9}{2}} dt + \frac{1}{2} \int t^{-\frac{13}{2}} dt \\ &= \frac{1}{2} \left(\frac{t^{-\frac{7}{2}}}{-\frac{7}{2}} \right) + \frac{1}{2} \left(\frac{t^{-\frac{11}{2}}}{-\frac{11}{2}} \right) + c = \frac{1}{t^{\frac{11}{2}}} \left[\frac{1}{11} + \frac{1}{7} t^2 \right] + c \\ &= \frac{-1}{(\sec(x) + \tan(x))^{\frac{11}{2}}} \left[\frac{1}{11} + \frac{1}{7} (\sec(x) + \tan(x))^2 \right] + c \end{aligned}$$

Problem 64. $\int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx$

$$\begin{aligned} I &= \int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx \\ &= \int \left(\frac{x^2 - 1}{x^5 \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} \right) dx \end{aligned}$$

Let $2 - \frac{2}{x^2} + \frac{1}{x^4} = t$

$$\Rightarrow 2 - 2x^{-2} + x^{-4} = t$$

$$\Rightarrow (4x^{-3} - 4x^{-5}) dx = dt$$

$$\Rightarrow \left(\frac{1}{x^3} - \frac{1}{x^5} \right) dx = \frac{dt}{4}$$

$$\Rightarrow \left(\frac{x^2 - 1}{x^5} \right) dx = \frac{dt}{4}$$

$$\begin{aligned} \Rightarrow I &= \frac{1}{4} \int \frac{dt}{\sqrt{t}} \\ &= \frac{1}{4} (2\sqrt{t}) + c \\ &= \frac{\sqrt{t}}{2} + c \\ &= \frac{\sqrt{2x^4 - 2x^2 + 1}}{2} + c \end{aligned}$$

Problem 65. $\int \frac{x^{n-1}}{x^{2n} + a^2} dx$

$$I = \int \frac{x^{n-1}}{x^{2n} + a^2} dx$$

Let $x^n = t$

$$\Rightarrow nx^{n-1} dx = dt$$

$$\Rightarrow x^{n-1} dx = \frac{dt}{n}$$

$$\begin{aligned} \Rightarrow I &= \frac{1}{n} \int \frac{dt}{t^2 + a^2} \\ &= \frac{1}{a} \tan^{-1} \left(\frac{t}{a} \right) + c \\ &= \frac{1}{a} \tan^{-1} \left(\frac{x^n}{a} \right) + c \end{aligned}$$

Problem 66. $\int \frac{dx}{x^2(x^4 + 1)^{3/4}}$

$$\begin{aligned} I &= \int \frac{dx}{x^2(x^4 + 1)^{3/4}} \\ &= \int \frac{dx}{x^2 \left[x^4 \left(1 + \frac{1}{x^4} \right) \right]^{3/4}} \\ &= \int \frac{dx}{x^5 \left(1 + \frac{1}{x^4} \right)^{3/4}} \end{aligned}$$

Let $1 + \frac{1}{x^4} = t$

$$\Rightarrow \frac{-4}{x^5} dx = dt$$

$$\Rightarrow \frac{dx}{x^5} = \frac{-dt}{4}$$

$$\begin{aligned} \Rightarrow I &= \frac{-1}{4} \cdot \int \frac{1}{t^{3/4}} dt \\ &= \frac{-1}{4} \cdot \frac{t^{(-\frac{3}{4}+1)}}{\left(-\frac{3}{4}+1\right)} + c \\ &= \frac{-1}{4} \cdot \frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c \\ &= -t^{1/4} + c \\ &= -\left(1 + \frac{1}{x^4}\right)^{1/4} + c \\ &= \frac{-(x^4 + 1)^{1/4}}{x} + c \end{aligned}$$

Problem 67. $\int \frac{x^2 - 1}{(x^4 + 3x^2 + 1) \tan^{-1} \left(\frac{x^2 + 1}{x} \right)} dx$

$$\begin{aligned} I &= \int \frac{x^2 - 1}{(x^4 + 3x^2 + 1) \tan^{-1} \left(\frac{x^2 + 1}{x} \right)} dx \\ &= \int \frac{1 - \frac{1}{x^2}}{\left[\left(x + \frac{1}{x} \right)^2 + 1 \right] \tan^{-1} \left(x + \frac{1}{x} \right)} dx \end{aligned}$$

Let $x + \frac{1}{x} = t$
 $\left(1 - \frac{1}{x^2} \right) dx = dt$

$$\Rightarrow I = \int \frac{dt}{(t^2 + 1) \tan^{-1}(t)}$$

Let $\tan^{-1}(t) = u$
 $\Rightarrow \frac{dt}{1 + t^2} = du$

$$\begin{aligned} \Rightarrow I &= \int \frac{du}{u} \\ &= \ln(|u|) + c \\ &= \ln \left(\left| \tan^{-1}(t) \right| \right) + c \\ &= \ln \left(\left| \tan^{-1} \left(x + \frac{1}{x} \right) \right| \right) + c \end{aligned}$$

Problem 68. $\int \frac{dx}{a^2 \sin^2(x) + b^2 \cos^2(x)}$

$$I = \int \frac{dx}{a^2 \sin^2(x) + b^2 \cos^2(x)} \quad \text{Divide numerator and denominator by } \cos^2(x)$$

$$= \int \frac{\sec^2(x)}{a^2 \tan^2(x) + b^2} dx$$

Let $\tan(x) = t$
 $\Rightarrow \sec^2(x) dx = dt$

$$\begin{aligned} \Rightarrow I &= \int \frac{dt}{a^2 t^2 + b^2} \\ &= \frac{1}{a^2} \int \frac{dt}{t^2 + \frac{b^2}{a^2}} \\ &= \frac{1}{a^2} \cdot \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1} \left[\frac{t}{\left(\frac{b}{a}\right)} \right] + c \\ &= \frac{1}{ab} \tan^{-1} \left(\frac{at}{b} \right) + c \\ &= \frac{1}{ab} \tan^{-1} \left(\frac{a \tan(x)}{b} \right) + c \end{aligned}$$

Problem 69. $\int \frac{1}{x \ln(x)} dx$

$$I = \int \frac{1}{x \ln(x)} dx$$

Let $\ln(x) = t$
 $\Rightarrow \frac{1}{x} dx = dt$

$$\begin{aligned} \Rightarrow I &= \int \frac{1}{t} dt \\ &= \ln(|t|) + c \\ &= \ln(|\ln(x)|) + c \end{aligned}$$

6.7 Integration by Parts

We know from the product rule that

$$\begin{aligned}\frac{d}{dx}(uv) &= v \frac{du}{dx} + u \frac{dv}{dx} \\ \Rightarrow d(uv) &= v du + u dv \\ \Rightarrow u dv &= d(uv) - v du\end{aligned}$$

Integrate both sides

$$\Rightarrow \int u dv = uv - \int v du$$

We use this result to split the integrand into two “parts” u and dv .
We differentiate the first part u and integrate the second part dv .

Problem 70. $\int x \sin(x) dx$

$$I = \int x \sin(x) dx$$

Let us choose $u = x$ and $dv = \sin(x) dx$.
 $\Rightarrow du = dx$ and $v = -\cos(x)$

$$\begin{aligned}\Rightarrow I &= \int u dv = uv - \int v du \\ &= -x \cos(x) - \int (-\cos(x)) dx \\ &= -x \cos(x) + \int \cos(x) dx \\ &= \sin(x) - x \cos(x) + c\end{aligned}$$

Problem 71. $\int \sin^{-1}(x) dx$

$$I = \int \sin^{-1}(x) dx$$

Let us choose $u = \sin^{-1}(x)$ and $dv = dx$.

Then $\frac{1}{\sqrt{1-x^2}} dx = du$ and $v = x$

$$\begin{aligned}\Rightarrow I &= \int u dv = uv - \int v du \\ &= x \sin^{-1}(x) - \int \frac{x dx}{\sqrt{1-x^2}} \\ &= x \sin^{-1}(x) - \frac{1}{2} \int \frac{dt}{\sqrt{1-t}} \\ &= x \sin^{-1}(x) + \frac{1}{2} \cdot 2\sqrt{1-t} + c \\ &= x \sin^{-1}(x) + \sqrt{1-t} + c \\ &= x \sin^{-1}(x) + \sqrt{1-x^2} + c\end{aligned}$$

$$\text{Let } x^2 = t \Rightarrow 2x dx = dt$$

Problem 72. $\int \left(x - \frac{1}{x^3}\right) e^{\left(x + \frac{1}{x}\right)} dx$

$$\begin{aligned}
 I &= \int \left(x - \frac{1}{x^3}\right) e^{\left(x + \frac{1}{x}\right)} dx \\
 &= \int \left(\frac{x^4 - 1}{x^3}\right) e^{\left(x + \frac{1}{x}\right)} dx \\
 &= \int \left(\frac{(x^2 + 1)(x^2 - 1)}{x^3}\right) e^{\left(x + \frac{1}{x}\right)} dx \\
 &= \int \left(\left(\frac{x^2 + 1}{x}\right) \left(\frac{x^2 - 1}{x^2}\right)\right) e^{\left(x + \frac{1}{x}\right)} dx \\
 &= \int \left(x + \frac{1}{x}\right) \left(1 - \frac{1}{x^2}\right) e^{\left(x + \frac{1}{x}\right)} dx
 \end{aligned}$$

Let $x + \frac{1}{x} = t$, then $\left(1 - \frac{1}{x^2}\right) dx = dt$

$$\Rightarrow I = \int t \cdot e^t dt$$

This essentially reduces to the next problem but let's just do it twice.

Integrating by parts.

Let us choose $u = t$ and $dv = e^t dt$

Then $du = dt$ and $v = e^t$

$$I = \int u dv = uv - \int v du$$

$$= t \cdot e^t - \int e^t dt$$

$$= t \cdot e^t - e^t + c$$

Substitute back

$$= \left(x + \frac{1}{x}\right) \cdot e^{\left(x + \frac{1}{x}\right)} - e^{\left(x + \frac{1}{x}\right)} + c$$

Problem 73. $\int x \cdot e^x dx$

$$I = \int x \cdot e^x dx$$

Integrating by parts.

Let us choose $u = x$ and $dv = e^x dx$

$\Rightarrow du = dx$ and $v = e^x$

$$\begin{aligned}\Rightarrow I &= \int u dv = uv - \int v du \\ &= x \cdot e^x - \int e^x dx \\ &= x \cdot e^x - e^x + c\end{aligned}$$

Problem 74. $\int e^x \cdot x^2 dx$

$$I = \int e^x \cdot x^2 dx$$

Integrating by parts.

Let us choose $u = x^2$ and $dv = e^x dx$

Then $du = 2x dx$ and $v = e^x$

$$\begin{aligned}\Rightarrow I &= \int u dv = uv - \int v du \\ &= x^2 \cdot e^x - \int e^x \cdot (2x dx) \\ &= x^2 \cdot e^x - 2 \int x \cdot e^x dx\end{aligned}$$

Integrating by parts again.

Let us choose $p = x$ and $dq = e^x dx$

Then $dp = dx$ and $q = e^x$

$$\begin{aligned}\Rightarrow I &= x^2 \cdot e^x - 2 \int p dq = x^2 \cdot e^x - 2 \left(pq - \int q dp \right) \\ &= x^2 \cdot e^x - 2 \left(xe^x - \int e^x dx \right) \\ &= x^2 \cdot e^x - 2(xe^x - e^x) + c\end{aligned}$$

Note: We can also solve this problem using the method of indefinite coefficients.

Problem 75. $\int e^x \cdot \cos(x) \, dx$

Let $I = \int e^x \cos(x) \, dx$

Integrating by parts.

Let $u = \cos(x)$ and $dv = e^x \, dx$

Then $du = -\sin(x) \, dx$ and $v = e^x$

$$\begin{aligned}\Rightarrow I &= uv - \int v \, du \\ &= e^x \cos(x) + \int e^x \sin(x) \, dx\end{aligned}$$

Integrating by parts again.

Let $p = \sin(x)$ and $dq = e^x \, dx$

Then $dp = \cos(x) \, dx$ and $q = e^x$

$$\begin{aligned}\Rightarrow I &= e^x \cos(x) + pq - \int q \, dp \\ &= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) \, dx \\ &= e^x \cos(x) + e^x \sin(x) - I + c_1 \\ I &= e^x \cos(x) + e^x \sin(x) - I + c_1 \\ 2I &= e^x \cos(x) + e^x \sin(x) + c_1 \\ I &= \frac{e^x}{2} [\cos(x) + \sin(x)] + c\end{aligned}$$

6.8 The Method of Indefinite Coefficients

6.8.1 For Polynomial Functions

Claim: Let $P_n(x)$ be a polynomial in x of degree n . Then,

$$\int e^{ax} P_n(x) dx = e^{ax} Q_n(x) + c$$

where $Q_n(x)$ is some polynomial in x of degree n .

Problem 76. $\int e^x \cdot x^2 dx$

$$I = \int e^x \cdot x^2 dx$$

Using the method of indefinite coefficients,

$$\int e^x \cdot x^2 dx = e^x \cdot (Ax^2 + Bx + C) + c$$

for some A , B and C .

Differentiate both sides

$$e^x \cdot x^2 = e^x (Ax^2 + Bx + C + 2Ax + B)$$

$$\Rightarrow x^2 = Ax^2 + (2A + B)x + (B + C)$$

Comparing coefficients

$$A = 1$$

$$2A + B = 0$$

$$B + C = 0$$

$$\Rightarrow B = -2$$

$$\Rightarrow C = 2$$

Therefore,

$$I = e^x (x^2 - 2x + 2) + c$$

Problem 77. $\int e^x \cdot (x^3 + 3x^2 + 3x + 9) dx$

$$I = \int e^x \cdot (x^3 + 3x^2 + 3x + 9) dx$$

Using the method of indefinite coefficients,

$$I = e^x \cdot (Ax^3 + Bx^2 + Cx + D) + c$$

for some A, B, C and D .

Differentiate both sides

$$\begin{aligned} \cancel{e^x} (x^3 + 3x^2 + 3x + 9) &= \cancel{e^x} (Ax^3 + Bx^2 + Cx + D + 3Ax^2 + 2Bx + C) \\ \Rightarrow (x^3 + 3x^2 + 3x + 9) &= Ax^3 + (B + 3A)x^2 + (C + 2B)x + (C + D) \end{aligned}$$

Comparing coefficients

$$A = 1$$

$$B + 3A = 3 \Rightarrow B = 0$$

$$C + 2B = 3 \Rightarrow C = 3$$

$$C + D = 9 \Rightarrow D = 6$$

Therefore,

$$I = e^x (x^3 + 3x + 6) + c$$

This idea can be extended to trigonometric functions as well.

6.8.2 For Trigonometric Functions

Extending the idea to Trigonometric functions,

$$\int e^{ax} \sin(px + q) dx = e^{ax} (A \sin(px + q) + B \cos(px + q)) + c$$

for some A and B .

Also,

$$\int e^{ax} \cos(px + q) dx = e^{ax} [A \cos(px + q) + B \sin(px + q)] + c$$

for some A and B

6.9 Integrals of the form $\int e^x (f(x) + f'(x)) dx$

$$\int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

Problem 78. $\int e^x \left[\frac{x^2 + 1}{(x + 1)^2} \right] dx$

$$\begin{aligned}
 I &= \int e^x \left[\frac{x^2 + 1}{(x + 1)^2} \right] dx \\
 &= \int e^x \left[\frac{x^2 - 1 + 2}{(x + 1)^2} \right] dx \\
 &= \int e^x \left[\frac{x^2 - 1}{(x + 1)^2} + \frac{2}{(x + 1)^2} \right] dx \\
 &= \int e^x \left[\frac{(x+1)(x-1)}{(x+1)^2} + \frac{2}{(x+1)^2} \right] dx \\
 &= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx
 \end{aligned}$$

Now, we can see that $\frac{d}{dx} \left(\frac{x-1}{x+1} \right) = \frac{x-1-(x+1)}{(x+1)^2} = \frac{-2}{(x+1)^2}$

$$\Rightarrow I = e^x \left(\frac{x-1}{x+1} \right) + c$$

Problem 79. $\int e^x \left(\frac{2 + \sin(2x)}{1 + \cos(2x)} \right) dx$

$$\begin{aligned}
 I &= \int e^x \left(\frac{2 + \sin(2x)}{1 + \cos(2x)} \right) dx \\
 &= \int e^x \left(\frac{2 + 2 \sin(x) \cos(x)}{1 + \cos(2x)} \right) dx \\
 &= \int e^x \left(\frac{2(1 + \sin(x) \cos(x))}{2 \cos^2(x)} \right) dx \\
 &= \int e^x \left(\sec^2(x) + \tan(x) \right) dx \\
 &= e^x \tan(x) + c
 \end{aligned}$$

Notice that $\frac{d}{dx} \tan(x) = \sec^2(x)$

Problem 80. $\int \left[\ln(\ln(x)) + \frac{1}{\ln(x)} \right] dx$

$$I = \int \left[\ln(\ln(x)) + \frac{1}{\ln(x)} \right] dx$$

Let $\ln(x) = t$
 $\Rightarrow x = e^t$
 $\Rightarrow dx = e^t dt$

$$\begin{aligned}
 \Rightarrow I &= \int \left[\ln(t) + \frac{1}{\ln(t)} \right] e^t dt \\
 &= e^t \ln(t) + c \\
 &= e^{\ln(x)} \cdot \ln(\ln(x)) + c
 \end{aligned}$$

6.10 General Formula for Integration by Parts

$$\int u(x) \cdot v(x) \, dx = u(x) \cdot v_1(x) - u'(x) \cdot v_2(x) + u''(x) \cdot v_3(x) - \cdots + (-1)^{(n-1)} u^{(n-1)}(x) \cdot v_n(x) + (-1)^n \int u^{(n)}(x) \cdot v_n(x) \, dx$$

Where $u^{(i)}(x) = \frac{d^i}{dx^i} u(x)$ for $i \in \{1, 2, 3, \dots, n\}$

And $v_1(x) = \int v(x) \, dx$,

$v_2(x) = \int v_1(x) \, dx$,

\vdots

$v_i(x) = \int v_{i-1}(x) \, dx$ for $i \in \{2, 3, 4, \dots, n\}$

Problem 81. Find a formula for $\int P_n(x) \cdot e^{kx} \, dx$ where $P_n(x)$ is a polynomial in x of degree n .

$$I = \int P_n(x) \cdot e^{kx} \, dx$$

Using the general formula for integration by parts.

Let us choose $u(x) = P_n(x)$ and $v(x) = e^x$

Then $u'(x) = P'_n(x)$, $u''(x) = P''_n(x)$, \dots , $u^{(n)}(x) = P_n^{(n)}(x)$

And,

$$v_1(x) = \frac{e^{kx}}{k}$$

$$v_2(x) = \frac{e^{kx}}{k^2}$$

$$v_3(x) = \frac{e^{kx}}{k^3}$$

\vdots

$$v_i(x) = \frac{e^{kx}}{k^i} \quad \forall i$$

\vdots

$$v_n(x) = \frac{e^{kx}}{k^n}$$

Applying the formula,

$$I = \int P_n(x) \cdot e^{kx} \, dx = P_n(x) \frac{e^{kx}}{k} - P'_n(x) \frac{e^{kx}}{k^2} + P''_n(x) \frac{e^{kx}}{k^3} - \cdots + (-1)^{n-1} P_n^{(n-1)}(x) \frac{e^{kx}}{k^n} + (-1)^n P_n^{(n)}(x) \frac{e^{kx}}{k^{n+1}}$$

The advantage of using this formula is that this process will end in exactly n (i.e. finitely many) steps.

6.11 Reduction Formulas

Reduction formulas are basically just recurrence relations in integrals. Here, we try to express an integral for n in terms of its successor or predecessor.

Problem 82. $I_n = \int \frac{dx}{(x^2 + a^2)^n}$

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

Integrating by parts,

Let $u = \frac{1}{(x^2 + a^2)^n}$ and $dv = dx$

$\Rightarrow du = \frac{-n(2x) dx}{(x^2 + a^2)^{n+1}}$ and $v = x$

$$\begin{aligned} \Rightarrow I_n &= \int u dv = uv - \int v du \\ &= \frac{x}{(x^2 + a^2)^n} + \int \frac{2n(x^2) dx}{(x^2 + a^2)^{n+1}} \\ &= \frac{x}{(x^2 + a^2)^n} + \int \frac{2n(x^2 + a^2 - a^2)}{(x^2 + a^2)^{n+1}} dx \\ &= \frac{x}{(x^2 + a^2)^n} + \int \frac{2n(x^2 + a^2)}{(x^2 + a^2)^{n+1}} dx - 2na^2 \int \frac{dx}{(x^2 + a^2)^{n+1}} \\ &= \frac{x}{(x^2 + a^2)^n} + \int \frac{2n}{(x^2 + a^2)^n} dx - 2na^2 \int \frac{dx}{(x^2 + a^2)^{n+1}} \\ &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{1}{(x^2 + a^2)^n} dx - 2na^2 \int \frac{dx}{(x^2 + a^2)^{n+1}} \\ I_n &= \frac{x}{(x^2 + a^2)^n} + 2n \cdot I_n - 2na^2 \cdot I_{n+1} \\ \Rightarrow 2na^2 \cdot I_{n+1} &= \frac{x}{(x^2 + a^2)^n} + 2n \cdot I_n - I_n \\ &= \frac{x}{(x^2 + a^2)^n} + (2n - 1) \cdot I_n \\ \Rightarrow I_{n+1} &= \frac{1}{2na^2} \cdot \left(\frac{x}{(x^2 + a^2)^n} \right) + \frac{2n - 1}{2na^2} \cdot I_n \end{aligned}$$

Problem 83. Find a reduction formula for $I_n = \int (\ln(x))^n dx$

$$I_n = \int (\ln(x))^n dx$$

Integrating by parts,

Let us choose $u = (\ln(x))^n$ and $dv = dx$

Then, $du = \frac{n (\ln(x))^{n-1}}{x} dx$ and $v = x$.

$$\begin{aligned} I_n &= \int u dv = uv - \int v du \\ &= x (\ln(x))^n - \int x \cdot \frac{n (\ln(x))^{n-1}}{x} dx \\ &= x (\ln(x))^n - n \int (\ln(x))^{n-1} dx \\ &= x (\ln(x))^n - n \cdot I_{n-1} \end{aligned}$$

So our reduction formula becomes

$$I_n = x (\ln(x))^n - n \cdot I_{n-1}$$

Problem 84. $I_n = \int x^\alpha (\ln(x))^n$

Integrating by parts.

Choosing $u = (\ln(x))^n$ and $dv = x^\alpha dx$

Then, $du = \frac{n (\ln(x))^{n-1}}{x} dx$ and $v = \frac{x^{(\alpha+1)}}{(\alpha+1)}$.

$$\begin{aligned} \Rightarrow I_n &= \int u dv = uv - \int v du \\ &= (\ln(x))^n \cdot \frac{x^{(\alpha+1)}}{(\alpha+1)} - \int \frac{x^{(\alpha+1)}}{(\alpha+1)} \cdot \frac{n (\ln(x))^{n-1}}{x} dx \\ &= (\ln(x))^n \cdot \frac{x^{(\alpha+1)}}{(\alpha+1)} - \frac{n}{\alpha+1} \int \frac{(x^{(\alpha+1)}) (\ln(x))^{n-1}}{x} dx \\ &= (\ln(x))^n \cdot \frac{x^{(\alpha+1)}}{(\alpha+1)} - \frac{n}{\alpha+1} \int \frac{(x^{(\alpha+1)}) (\ln(x))^{n-1}}{x} dx \\ &= (\ln(x))^n \cdot \frac{x^{(\alpha+1)}}{(\alpha+1)} - \frac{n}{\alpha+1} \int x^\alpha (\ln(x))^{n-1} dx \\ &= (\ln(x))^n \cdot \frac{x^{(\alpha+1)}}{(\alpha+1)} - \frac{n}{\alpha+1} (I_{n-1}) \\ &= \frac{1}{\alpha+1} \left[x^{\alpha+1} (\ln(x))^n - n \cdot I_{n-1} \right] \end{aligned}$$

This gives us the following reduction formula:

$$I_n = \frac{1}{\alpha+1} \left[x^{\alpha+1} (\ln(x))^n - n \cdot I_{n-1} \right]$$

The Beta Function

More about it in Definite Integration. The Beta function actually has the limits of integration from 0 to 1. For now, we're concerned with just the antiderivative and a reduction formula for it.

Problem 85. $\beta(m, n) = \int x^m(1-x)^{n-1} dx$

$$\beta(m, n) = \int x^m(1-x)^{n-1} dx$$

Integrating by parts

Let us choose $u = (1-x)^{n-1}$ and $dv = x^m dx$

$\Rightarrow du = -(n-1)(1-x)^{n-2} dx$ and $v = \frac{x^{m+1}}{m+1}$

$$\begin{aligned}\Rightarrow \beta(m, n) &= \int u dv = uv - \int v du \\ &= (1-x)^{n-1} \cdot \frac{x^{m+1}}{m+1} + \int \frac{x^{m+1}}{m+1} (n-1)(1-x)^{n-2} dx \\ &= (1-x)^{n-1} \cdot \frac{x^{m+1}}{m+1} + \frac{n-1}{m+1} \int x^{m+1}(1-x)^{n-2} dx \\ &= (1-x)^{n-1} \cdot \frac{x^{m+1}}{m+1} + \frac{n-1}{m+1} \beta(m+1, n-2)\end{aligned}$$

The Gamma Function

Again, more details in Definite Integration. The Gamma Function actually has limits from 0 to ∞ but for now, we're just concerned with its reduction formula. It is used to extend the idea of factorial to the set of real numbers.

Problem 86. $\Gamma(n) = \int x^n e^{-x} dx$

$$\Gamma(n) = \int x^n e^{-x} dx$$

Integrating by parts

(Note that we are violating the "ILATE" principle. It is not really a rule and just a matter of what's easy to integrate and differentiate.)

Let us choose $u = e^{-x}$ and $dv = x^n dx$

$\Rightarrow du = -e^{-x} dx$ and $v = \frac{x^{n+1}}{n+1}$

$$\begin{aligned}\Gamma(n) &= \int u dv = uv - \int v du \\ &= e^{-x} \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot (-e^{-x}) dx \\ &= e^{-x} \cdot \frac{x^{n+1}}{n+1} + \frac{1}{n+1} \int x^{n+1} \cdot e^{-x} dx \\ &= e^{-x} \cdot \frac{x^{n+1}}{n+1} + \frac{1}{n+1} \cdot \Gamma(n+1)\end{aligned}$$

So we get the following reduction formula:

$$\Gamma(n) = \frac{x^{n+1}}{n+1} (e^{-x}) + \frac{1}{n+1} \Gamma(n+1)$$

Problem 87. $I_n = \int (a^2 - x^2)^n dx$

$$I_n = \int (a^2 - x^2)^n dx$$

Integrating by parts

Let us choose $u = (a^2 - x^2)^n$ and $dv = dx$

$\Rightarrow du = -2nx(a^2 - x^2)^{n-1} dx$ and $v = x$

$$\begin{aligned} I_n &= \int u dv = uv - \int v du \\ &= x(a^2 - x^2)^n - \int x \cdot (-2nx(a^2 - x^2)^{n-1}) dx \\ &= x(a^2 - x^2)^n - \int 2n(a^2 - x^2 + x^2) \cdot (a^2 - x^2)^{n-1} dx \\ &= x(a^2 - x^2)^n - 2n \int (a^2 - x^2)^n dx + 2na^2 \int (a^2 - x^2)^{n-1} dx \\ &= x(a^2 - x^2)^n - 2nI_n + 2na^2I_{n-1} \end{aligned}$$

$$I_n = x(a^2 - x^2)^n - 2nI_n + 2na^2I_{n-1}$$

Problem 88. $I_{(n,-m)} = \int \frac{\sin^n(x)}{\cos^m(x)} dx$

$$I_{(n,-m)} = \int \frac{\sin^n(x)}{\cos^m(x)} dx$$

$$I_{(n,-m)} = \int \sin^{n-1}(x) \frac{\sin(x)}{\cos^m(x)} dx$$

Integrating by parts

Let us choose $u = \sin^{n-1}(x)$ and $dv = \frac{\sin(x)}{\cos^m(x)} dx$

$\Rightarrow du = (n-1)\sin^n(x)\cos(x) dx$ and $v = \frac{(\cos(x))^{-m+1}}{-m+1}$

$$\begin{aligned} I_{(n,-m)} &= \int u dv = uv - \int v du \\ &= \frac{\sin^{n-1}(x) \cdot (\cos(x))^{-m+1}}{m-1} - \frac{n-1}{m-1} \int \frac{\sin^{n-2}(x)}{\cos^{m-2}(x)} dx \\ &= \frac{\sin^{n-1}(x) \cdot (\cos(x))^{-m+1}}{m-1} - \frac{n-1}{m-1} I_{(n-2,-(m-2))} \end{aligned}$$

This gives us the reduction formula as:

$$I_{(n,-m)} = \frac{\sin^{n-1}(x) \cdot (\cos(x))^{-m+1}}{m-1} - \frac{n-1}{m-1} (I_{(n-2,-(m-2))})$$

6.12 Integration of Rational Functions and Integration by Partial Fractions

An expression is said to be rational if it is of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.

The rational function is said to be **proper** when the degree of $Q(x)$ is greater than the degree of $P(x)$; otherwise it is called **improper**.

Given an improper rational function as the integrand, our first job is to convert it into a proper or mixed fraction, possibly by long division.

Step 1: If $\deg(P(x)) > \deg(Q(x))$, then use long division to convert to mixed fraction, and try to integrate. Proceed to step 2

Say from long division, $P(x) = q(x) \cdot Q(x) + r(x)$. Then,

$$\int \frac{P(x)}{Q(x)} dx = \int q(x) dx + \int \frac{r(x)}{Q(x)} dx$$

Step 2: Here, we have two cases:

Case 1: If $Q(x)$ is a polynomial of odd degree, then $Q(x)$ will have at least one root. Even if all other roots are non-real conjugates, they can be made into quadratics. Try to factor $Q(x)$ in such a form to get only linear or quadratic factors.

$$Q(x) = (x - \alpha)(x^2 + \beta x + \gamma) \cdots$$

Case 2: If $Q(x)$ is a polynomial of even degree, try to factor $Q(x)$. If all roots of $Q(x)$ are non-real, it can be expressed as a product of quadratics.

In general, we'll try to factor $Q(x)$ as a product of linear and quadratic expressions.

Step 3: Once that is done, we'll try to integrate, possibly by Partial Fractions.

6.12.1 Integration by Partial Fractions

The idea is to decompose a given rational function into a sum of simpler fractions (called partial fractions) that can be integrated by the techniques discussed earlier.

We will separate the discussion into cases depending on the way in which the denominator $Q(x)$ can be factored.

Case 1: If $Q(x)$ can be expressed as a product of distinct linear factors, such as

$$Q(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n)$$

then we can find constants A_i for all i such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \frac{A_3}{x - \alpha_3} + \cdots + \frac{A_n}{x - \alpha_n}$$

Case 2: If $Q(x)$ can be represented as a product of linear or quadratic factors, some of which are repeated, such as,

$$Q(x) = (x - \alpha)^m (x^2 + \beta x + \gamma)^n$$

Then,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - \alpha} + \frac{A_2}{(x - \alpha)^2} + \cdots + \frac{A_m}{(x - \alpha)^m} + \frac{B_1 x + C_1}{(x^2 + \beta x + \gamma)} + \frac{B_2 x + C_2}{(x^2 + \beta x + \gamma)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + \beta x + \gamma)^n}$$

Then, we can make a common denominator (LCM) on the right and compare coefficients, giving us a system of equations to solve. Alternatively, we can substitute values that make the variable vanish.

Problem 89. $\int \frac{2x-3}{(x-1)(2x-1)} dx$

$$I = \int \frac{2x-3}{(x-1)(2x-1)} dx$$

$$\begin{aligned}\text{Let } \frac{2x-3}{(x-1)(2x-1)} &= \frac{A}{x-1} + \frac{B}{2x-1} \\ &= \frac{A(2x-1) + B(x-1)}{(x-1)(2x-1)} \\ \Rightarrow 2x-3 &= A(2x-1) + B(x-1)\end{aligned}$$

Substitute $x = 1$

$$\Rightarrow 2 - 3 = A(2 - 1) \Rightarrow A = -1$$

Substitute $x = \frac{1}{2}$

$$\Rightarrow B = 4$$

$$\begin{aligned}\Rightarrow I &= \int \left(\frac{-1}{x-1} + \frac{4}{2x-1} \right) dx \\ &= - \int \frac{dx}{x-1} + 4 \int \frac{dx}{2x-1} \\ &= -\ln(|x-1|) + \frac{4 \ln(|2x-1|)}{2} + c \\ &= -\ln(|x-1|) + 2 \ln(|2x-1|) + c\end{aligned}$$

6.13 Integration of Irrational Functions

6.13.1 Type 1:

$$\int \frac{dx}{\text{linear}\sqrt{\text{linear}}}$$

If you see something of this form, substitute $\sqrt{\text{linear}} = t$.

Why? It will give us $\frac{dx}{2\sqrt{\text{linear}}} = dt$. Then, we can express the linear part outside the square root in terms of t and integrate.

Problem 90. $\int \frac{dx}{(2x+3)\sqrt{x+1}}$

$$I = \int \frac{dx}{(2x+3)\sqrt{x+1}}$$

$$\text{Let } \sqrt{x+1} = t$$

$$\Rightarrow \frac{dx}{2\sqrt{x+1}} = dt$$

$$\Rightarrow \frac{dx}{\sqrt{x+1}} = 2 dt$$

$$x+1 = t^2$$

$$\Rightarrow 2x+2 = 2t^2$$

$$\Rightarrow 2x+3 = 2t^2 + 1$$

$$\begin{aligned}\Rightarrow I &= \int \frac{2 dt}{2t^2 + 1} \\ &= \int \frac{2 dt}{2\left(t^2 + \frac{1}{2}\right)} \\ &= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \tan^{-1} \left(\frac{t}{\left(\frac{1}{\sqrt{2}}\right)} \right) + c \\ &= \sqrt{2} \tan^{-1} (\sqrt{2}t) + c \\ &= \sqrt{2} \tan^{-1} (\sqrt{2} \cdot \sqrt{x+1}) + c \\ &= \sqrt{2} \tan^{-1} (\sqrt{2(x+1)}) + c\end{aligned}$$

6.13.2 Type 2:

$$\int \frac{dx}{(ax^2 + b)\sqrt{cx^2 + d}}$$

For integrals of this type, substitute $x = \frac{1}{t}$.

This will give us $dx = -\frac{1}{t^2} dt$.

Problem 91. $\int \frac{dx}{(x^2 + 1)\sqrt{x^2 + 2}}$

$$I = \int \frac{dx}{(x^2 + 1)\sqrt{x^2 + 2}}$$

Let $x = \frac{1}{t}$.

$$\Rightarrow dx = -\frac{1}{t^2} dt$$

$$\begin{aligned} \Rightarrow I &= - \int \frac{\frac{dt}{t^2}}{\left(\frac{1}{t^2} + 1\right) \sqrt{\frac{1}{t^2} + 2}} \\ &= - \int \frac{\frac{dt}{t^2}}{\left(\frac{t^2 + 1}{t^2}\right) \sqrt{\frac{1 + 2t^2}{t^2}}} \\ &= - \int \frac{t dt}{(t^2 + 1)\sqrt{1 + 2t^2}} \end{aligned}$$

$$\text{Let } 1 + 2t^2 = u^2$$

$$\Rightarrow 4t dt = 2u du$$

$$\Rightarrow t dt = \frac{u du}{2}$$

$$\text{Now } 1 + 2t^2 = u^2$$

$$\Rightarrow 2t^2 = u^2 - 1$$

$$\Rightarrow t^2 = \frac{u^2 - 1}{2}$$

$$\Rightarrow t^2 + 1 = \frac{u^2 - 1}{2} + 1 = \frac{u^2 + 1}{2}$$

$$\begin{aligned} \Rightarrow I &= -\frac{1}{2} \int \frac{\cancel{t} du}{\left(\frac{u^2 + 1}{\cancel{2}}\right) \cancel{t}} = - \int \frac{du}{u^2 + 1} \\ &= -\tan^{-1}(u) + c \\ &= -\tan^{-1}(1 + 2t^2) + c \\ &= -\tan^{-1}\left(1 + \frac{2}{x^2}\right) + c \end{aligned}$$

6.13.3 Type 3:

$$\int \frac{dx}{\text{linear}\sqrt{\text{quadratic}}}$$

For this type, we substitute $\text{linear} = \frac{1}{t}$.

Problem 92.
$$\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$$

$$I = \int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$$

$$\text{Let } x+2 = \frac{1}{t}$$

$$\Rightarrow dx = -\frac{1}{t^2} dt$$

$$\text{And } x = \left(\frac{1}{t} - 2\right)$$

$$\Rightarrow x^2 + 6x + 7 = \left(\frac{1}{t} - 2\right)^2 + 6\left(\frac{1}{t} - 2\right) + 7 = \frac{1 + 2t - t^2}{t^2}$$

$$\begin{aligned}\Rightarrow I &= \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \sqrt{\frac{1 + 2t - t^2}{t^2}}} \\ &= - \int \frac{dt}{\sqrt{1 + 2t - t^2}} \\ &= - \int \frac{dt}{\sqrt{2 - (t-1)^2}} \\ &= -\sin^{-1} \left(\frac{\sqrt{2}}{(t-1)} \right) + c \\ &= \sin^{-1} \left(\frac{\sqrt{2}}{(1-t)} \right) + c\end{aligned}$$

6.13.4 Type 4:

If there is a polynomial function of the form

$$f\left(x, x^{p_1/q_1}, x^{p_2/q_2} + x^{p_3/q_3} + \dots + x^{p_r/q_r}\right),$$

Then substitute $x = t^s$ where $s = LCM(q_1, q_2, q_3, \dots, q_r)$.

Similarly, if we have

$$f\left(x, (ax+b)^{p_1/q_1}, (ax+b)^{p_2/q_2} + (ax+b)^{p_3/q_3} + \dots + (ax+b)^{p_r/q_r}\right)$$

Then substitute $(ax+b) = t^s$ where $s = LCM(q_1, q_2, q_3, \dots, q_r)$.

This will make it a rational function which can be evaluated.

Problem 93. $\int \frac{x + x^{2/3} + x^{1/6}}{x(1 + x^{1/3})} dx$

$$I = \int \frac{x + x^{2/3} + x^{1/6}}{x(1 + x^{1/3})} dx$$

Let $x = t^6$

$$\Rightarrow dx = 6t^5 dt$$

$$\Rightarrow I = \int \frac{6t^5 (t^6 + t^4 + t)}{t^6 (1 + t^2)} dt$$

$$= 6 \int \frac{t^6 + t^4 + t}{t(1 + t^2)} dt$$

Long division

$$= 6 \int \left[t^3 + \frac{1}{t^2 + 1} \right] dt$$

$$= 6 \left[\frac{t^4}{4} + \tan^{-1}(t) \right] + c$$

$$= 6 \left[\frac{x^{4/6}}{4} + \tan^{-1}(x^{1/6}) \right] + c$$

$$= \frac{3}{2}x^{2/3} + 6 \tan^{-1}(x^{1/6}) + c$$

6.13.5 Type 5:

Integrals of the type $\int x^m (a + bx^n)^p dx$

1. If m and p are positive integers, simply use Binomial Theorem and evaluate.
2. If p is an integer but m and n are not integers (but rational), then substitute $x = t^s$ where s is the LCM of the denominators of m and n .
3. If m and p are not integers but $\frac{m+1}{n}$ is an integer, then substitute $a + bx^n = t^s$ where s is the denominator of the fraction p .
4. If m and p are not integers but $\frac{m+1}{n} + p$ is an integer, then put $ax^{-n} + b = t^s$ where s is the denominator of the fraction p .

6.14 Other “Special” Integrals

6.14.1 When the Denominator is a Quadratic in x^2

Divide the numerator and denominator by x^2 and try to make some substitution from here.

Problem 94. $\int \frac{x^2 + 1}{x^4 + 1} dx$

$$\begin{aligned} I &= \int \frac{x^2 + 1}{x^4 + 1} dx \\ &= \int \left(\frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} \right) dx \end{aligned}$$

Let $x - \frac{1}{x} = t$.

Then $\left(1 + \frac{1}{x^2}\right) dx = dt$

$$\left(x - \frac{1}{x}\right)^2 = t^2$$

$$\Rightarrow x^2 - 2 + \frac{1}{x^2} = t^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = t^2 + 2$$

$$\begin{aligned} \Rightarrow I &= \int \frac{dt}{t^2 + 2} \\ &= \int \frac{dt}{(\sqrt{2})^2 + t^2} \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + c \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\left(x - \frac{1}{x}\right)}{\sqrt{2}} \right) + c \end{aligned}$$

Problem 95. $\int \frac{x^2 - 1}{x^4 + 1} dx$

$$\begin{aligned} I &= \int \frac{x^2 - 1}{x^4 + 1} dx \\ &= \int \left(\frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} \right) dx \end{aligned}$$

Let $x + \frac{1}{x} = t$.

Then $\left(1 - \frac{1}{x^2}\right) dx = dt$

$$\left(x + \frac{1}{x}\right)^2 = t^2$$

$$\Rightarrow x^2 + 2 + \frac{1}{x^2} = t^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$$

$$\Rightarrow I = \int \frac{dt}{t^2 - 2}$$

$$\Rightarrow I = \int \frac{dt}{t^2 - (\sqrt{2})^2}$$

Using $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left(\left| \frac{x - a}{x + a} \right| \right) + c$

$$\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left(\left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| \right) + c$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left(\left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| \right) + c$$

6.14.2 Special Integrals Involving Sine and Cosine

Integrals of the form $\int \sin^m(x) \, dx$ or $\int \cos^m(x) \, dx$, where $m \in \mathbb{Z}$

If m is even:

$$\text{Use } \sin^2(x) = \frac{1 - \cos(2x)}{2} \text{ or } \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Otherwise, if m is odd:

Break it into power 1 and the nearest even power and use one of the following:

$$\sin^2(x) = 1 - \cos^2(x) \text{ and substitute } \cos(x) = t$$

$$\text{Or } \cos^2(x) = 1 - \sin^2(x) \text{ and substitute } \sin(x) = t$$

Problem 96. $\int \sin^4(x) \, dx$

$$\begin{aligned} I &= \int \sin^4(x) \, dx \\ &= \int \left(\sin^2(x) \right)^2 \, dx \\ &= \int \left(\frac{1 - \cos(2x)}{2} \right)^2 \, dx \\ &= \frac{1}{4} \int \left(1 - 2\cos(2x) + \cos^2(2x) \right) \, dx \\ &= \frac{1}{4} \int \, dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx \\ &= \frac{x}{4} - \frac{1}{2} \cdot \frac{\sin(2x)}{2} + \frac{1}{4} \int \frac{1 + \cos(4x)}{2} \, dx \\ &= \frac{x}{4} - \frac{1}{2} \cdot \frac{\sin(2x)}{2} + \frac{1}{8} \int (1 + \cos(4x)) \, dx \\ &= \frac{x}{4} - \frac{1}{2} \cdot \frac{\sin(2x)}{2} + \frac{1}{8} \left(x + \frac{\sin(4x)}{4} \right) + c \\ &= \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + c \\ &= \frac{12x}{32} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + c \\ &= \frac{12x - 8\sin(2x) + \sin(4x)}{32} + c \end{aligned}$$

Problem 97. $\int \cos^5(x) \, dx$

$$\begin{aligned}
 I &= \int \cos^5(x) \, dx \\
 &= \int \cos(x) \cdot \cos^4(x) \, dx \\
 &= \int \cos(x) \cdot \left(\cos^2(x)\right)^2 \, dx \\
 &= \int \cos(x) \cdot \left(1 - \sin^2(x)\right)^2 \, dx
 \end{aligned}$$

Let $\sin(x) = t$
 $\Rightarrow \cos(x) \, dx = dt$

$$\begin{aligned}
 \Rightarrow I &= \int \left(1 - t^2\right)^2 \, dt \\
 &= \int \left(1 - 2t^2 + t^4\right) \, dt \\
 &= t - \frac{2t^3}{3} + \frac{t^5}{5} + c \\
 &= \sin(x) - \frac{2\sin^3(x)}{3} + \frac{\sin^5(x)}{5} + c
 \end{aligned}$$

Integrals of the form $\int \sin^m(x) \cdot \cos^n(x) \, dx$, **where** $m, n \in \mathbb{Z}^+$

If any of m and n is/are odd:

Break the (least) odd power to nearest even like before and use one of the following:

$\sin^2(x) = 1 - \cos^2(x)$ and substitute $\cos(x) = t$

Or $\cos^2(x) = 1 - \sin^2(x)$ and substitute $\sin(x) = t$

Else if $m = n$, i.e. $\int \sin^m(x) \cos^m(x) \, dx$:

Multiply and divide by 2^m and try to solve like before.

Integrals of the form $\int \sin(mx) \cdot \sin(nx) \, dx$, $\int \cos(mx) \cdot \cos(nx) \, dx$ **or** $\int \sin(mx) \cdot \cos(nx) \, dx$,
where $m, n \in \mathbb{R}^+$

Multiply and divide by 2; use the transformation formulae from trigonometry to split them into sum/difference of sine and cosine.

7 Definite Integrals

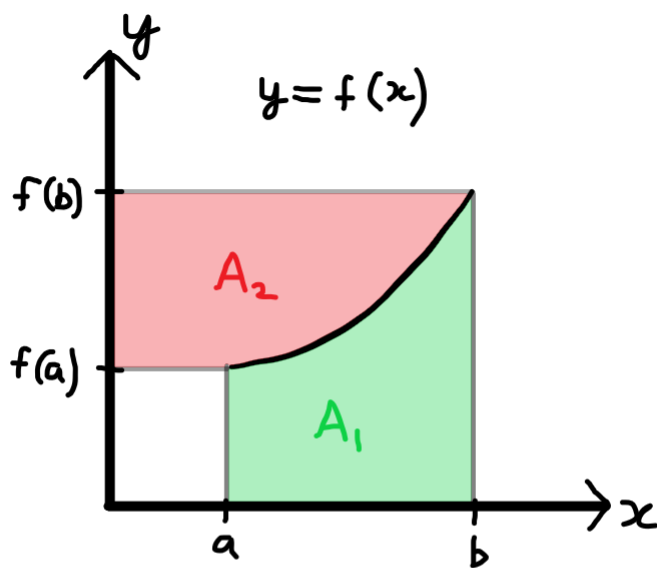
7.1 The Second Fundamental Theorem of Calculus

Theorem 13. Let f be a continuous function on $[a, b]$ and $f(x) = g'(x) \quad \forall x \in [a, b]$. Then,

$$\int_a^b f(x) dx = \int_a^b f = g(b) - g(a)$$

7.2 Geometrical Interpretation

Let f be a non-negative function on the closed interval $[a, b]$. Then the integral $\int_a^b f(x) dx = R(f, a, b)$ represents the area bounded by the function f , the lines $x = a$, $x = b$ and the x -axis.



$$A_1 = \int_a^b y dx$$

$$A_2 = \int_{f(a)}^{f(b)} x dy$$

7.3 Examples

Problem 98. $\int_0^{\frac{\pi}{2}} \sin(2x) \, dx$

$$\int_0^{\pi/2} \sin(2x) \, dx = \left[\frac{-\cos(2x)}{2} \right]_0^{\pi/2} = -\frac{\cos(\pi)}{2} + \frac{\cos(0)}{2} = \frac{-(-1)}{2} + \frac{1}{2} = 1$$

Problem 99. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos(x)}{\sin^3(x)} \, dx$

$$I = \int \frac{\cos(x)}{\sin^3(x)} \, dx$$

Let $\sin(x) = t$
 $\Rightarrow \cos(x) \, dx = dt$

$$I = \int \frac{dt}{t^3} = \frac{t^{-2}}{-2} + c = \frac{(\sin(x))^{-2}}{-2} + c$$

So,

$$\int_{\pi/6}^{\pi/2} \frac{\cos(x)}{\sin^3(x)} \, dx = \left[\frac{(\sin(x))^{-2}}{-2} \right]_{\pi/6}^{\pi/2} = \left[\frac{-1}{2 \sin^2(x)} \right]_{\pi/6}^{\pi/2} = \frac{-1}{2 \sin^2\left(\frac{\pi}{6}\right)} + \frac{1}{2 \sin^2\left(\frac{\pi}{2}\right)} = -2 + \frac{1}{2} = -\frac{3}{2}$$

Problem 100. $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$

$$\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \int_0^2 \frac{dx}{\sqrt{4^2-x^2}} = \left[\sin^{-1}\left(\frac{x}{4}\right) \right]_0^2 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6}$$

7.4 Properties of Definite Integrals

$$1. \int_a^a f(x) \, dx = 0$$

$$2. \int_a^b f(x) \, dx = \int_a^b f(t) \, dt$$

$$3. \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$4. \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

$$5. \int_a^b f(x) \, dx = \int_a^{c_1} f(x) \, dx + \int_{c_1}^{c_2} f(x) \, dx + \int_{c_2}^{c_3} f(x) \, dx + \cdots + \int_{c_n}^b f(x) \, dx$$

$$6. \int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx$$

$$7. \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$$

$$8. \int_0^{2a} f(x) \, dx = \int_0^a f(x) \, dx + \int_0^a f(2a-x) \, dx$$

$$9. \int_0^{2a} f(x) \, dx = \begin{cases} 2 \int_0^a f(x) \, dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$10. \int_{-a}^a f(x) \, dx = \int_0^a [f(x) + f(-x)] \, dx$$

$$11. \int_{-a}^a f(x) \, dx = \begin{cases} 2 \int_0^a f(x) \, dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } f(-x) = -f(x) \text{ (odd function)} \end{cases}$$

12. If $f(x)$ is a periodic function with period T , then

$$\int_0^{nT} f(x) \, dx = n \int_0^T f(x) \, dx$$

7.5 Substitution in Definite Integrals

When we use the method of Integration by Substitution in Definite Integrals, we must ensure that the substituted function satisfies some criteria.

Say we have $\int_a^b f(x) dx$. Then if we want to make the substitution $x = g(t)$, we must ensure that g is:

1. Defined on the interval $[a, b]$
2. Continuously differentiable on $[a, b]$
3. One-one

Also, upon making a substitution, we must update the limits of the integral accordingly. So the following is “obviously” incorrect:

$$I = \int_{-1}^1 x^4 dx$$

Let $x^2 = t$

Then, $t = 1$ when $x = -1$ and $t = 1$ when $x = 1$

So the integral becomes

$$I = \int_1^1 t^2 dt = 0$$

7.6 Derivative of an Integral and Leibniz' Formula

$$\text{If } h(x) = \int_{q(x)}^{p(x)} f(t) dt$$

$$\text{Then } h'(x) = f(p(x)) \cdot p'(x) - f(q(x)) \cdot q'(x)$$

Problem 101. Consider the function f defined as:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ \sqrt{x} & \text{if } 1 < x \leq 2 \end{cases}$$

Find $\int_0^2 f(x) \, dx$.

Solution: Note that the function is continuous at $x = 1$.

Using $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$

$$\begin{aligned} I &= \int_0^2 f(x) \, dx = \int_0^1 f(x) \, dx + \int_1^2 f(x) \, dx \\ &= \int_0^1 x^2 \, dx + \int_1^2 \sqrt{x} \, dx \\ &= \left[\frac{x^3}{3} \right]_0^1 + \left[\frac{2}{3} x\sqrt{x} \right]_1^2 \\ &= \frac{1}{3} + \frac{2}{3} (2\sqrt{2} - 1) \\ &= \frac{1}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} \\ &= \frac{4\sqrt{2} - 1}{3} \end{aligned}$$

Problem 102. $\int_0^2 |1 - x| \, dx$

Using $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$.

$$\begin{aligned} I &= \int_0^2 |1 - x| \, dx = \int_0^1 |1 - x| \, dx + \int_1^2 |1 - x| \, dx \\ &= \int_0^1 (1 - x) \, dx + \int_1^2 (x - 1) \, dx \\ &= \left[x - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^2 \\ &= 1 - \frac{1}{2} + 0 - \left(\frac{-1}{2} \right) \\ &= 1 \end{aligned}$$

7.7 Newton's Proposition

Suppose f is an increasing, integrable and invertible function. Then,

$$\int_a^b f^{-1} = b \cdot f^{-1}(b) - a \cdot f^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f$$

Note that the above notation is perfectly acceptable and is equivalent to:

$$\int_a^b f^{-1}(x) \, dx = b \cdot f^{-1}(b) - a \cdot f^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(x) \, dx$$

7.7.1 Formal Proof

Proof. Let $I = \int_a^b f^{-1}(x) \, dx$

Let us make the substitution $x = f(t)$. Note that this is a valid substitution since f is one-one because it is invertible.

$$\Rightarrow dx = f'(t) \, dt$$

$$x = a \Rightarrow t = f^{-1}(a)$$

$$\text{And } x = b \Rightarrow t = f^{-1}(b)$$

So,

$$\begin{aligned} I &= \int_a^b f^{-1}(x) \, dx \\ &= \int_{f^{-1}(a)}^{f^{-1}(b)} f^{-1}(f(t)) \cdot f'(t) \, dt \\ &= \int_{f^{-1}(a)}^{f^{-1}(b)} t \cdot f'(t) \, dt \end{aligned}$$

Integrating by parts,

$$\text{Let } u = t \text{ and } dv = f'(t) \, dt$$

$$\Rightarrow du = dt \text{ and } v = f(t)$$

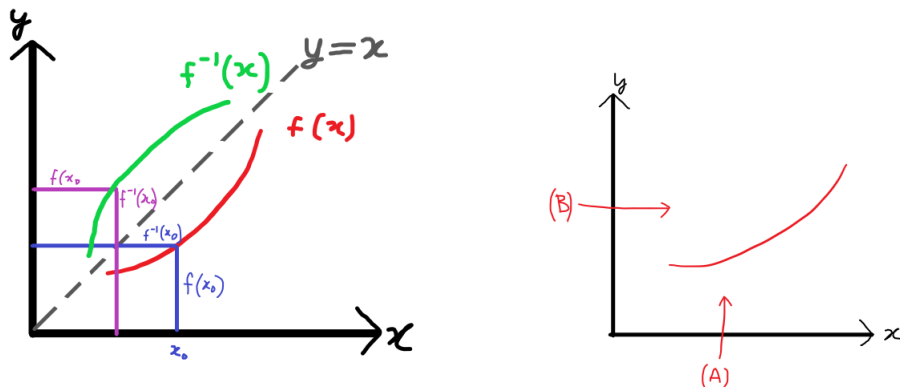
$$\int u \, dv = uv - \int v \, du$$

$$\begin{aligned} \Rightarrow I &= \left[t \cdot f(t) \right]_{f^{-1}(a)}^{f^{-1}(b)} - \int_{f^{-1}(a)}^{f^{-1}(b)} f(t) \, dt \\ &= \left[f^{-1}(b) \cdot f(f^{-1}(b)) - f^{-1}(a) \cdot f(f^{-1}(a)) \right] - \int_{f^{-1}(a)}^{f^{-1}(b)} f(t) \, dt \\ &= b \cdot f^{-1}(b) - a \cdot f^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(x) \, dx \end{aligned}$$

□

7.7.2 Geometric Intuition

We know that the graph of $y = f^{-1}(x)$ is simply the mirror image of the graph of $y = f(x)$ along the line $y = x$ since the mirror image of any point (x_0, y_0) along $y = x$ is (y_0, x_0) and f^{-1} also does the same thing that is changes (x, y) to (y, x) .

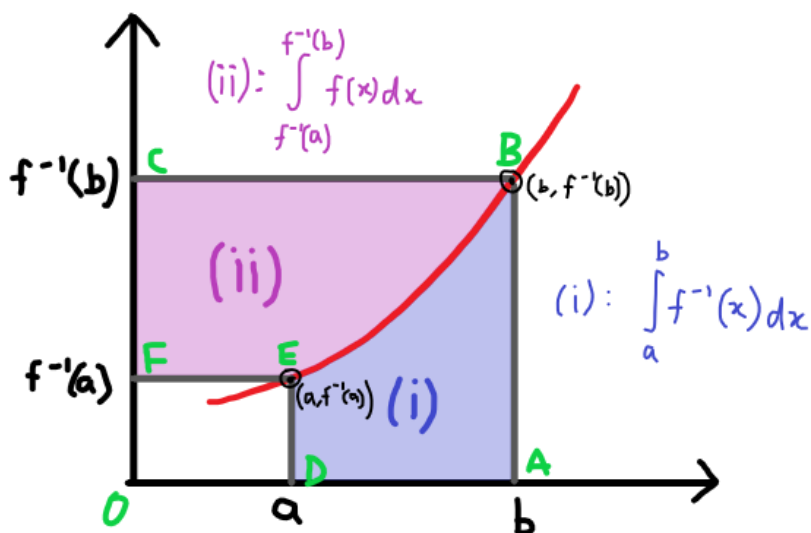


So, if we look from (A), it's $f(x)$ and if we look from (B), it's $f^{-1}(x)$.
Now, looking at Newton's Proposition.

$$\int_a^b f^{-1} = b \cdot f^{-1}(b) - a \cdot f^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f$$

$$\Leftrightarrow \int_a^b f^{-1} + \int_{f^{-1}(a)}^{f^{-1}(b)} f = b \cdot f^{-1}(b) - a \cdot f^{-1}(a)$$

\Leftrightarrow Area of region (i) + Area of region (ii) = Area of rectangle OABC – Area of rectangle ODEF



7.8 Young's Inequality

Suppose that f is a continuous increasing function with $f(0) = 0$. Then for $a, b > 0$,

$$\int_0^a f(x) \, dx + \int_0^b f^{-1}(x) \, dx \geq ab$$

and the equality holds iff $a = f^{-1}(b)$

7.8.1 Formal Proof

Note: In such theoretical proofs, one nice way of substitution can be such that the limits of integration can be made equal to those of the integral on the RHS, just like we did for proving Newton's Proposition.

Proof.

$$\begin{aligned} I &= \int_0^a f(x) \, dx + \int_0^b f^{-1}(x) \, dx && \text{Using Newton's Proposition} \\ &= \int_0^a f(x) \, dx + b \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x) \, dx && \text{Using } \int_a^b f = - \int_b^a f \\ &= \int_{f^{-1}(b)}^0 f(x) \, dx + \int_0^a f(x) \, dx + b \cdot f^{-1}(b) && \text{Using } \int_a^c f + \int_c^b f = \int_a^b f \\ &= \int_{f^{-1}(b)}^a f(x) \, dx + b \cdot f^{-1}(b) && \text{Add and subtract } ab \\ &= \int_{f^{-1}(b)}^a f(x) \, dx + b \cdot f^{-1}(b) - ab + ab && \text{Factor out } b \end{aligned}$$

$$\left[\text{Wishful thinking: if we can prove } \int_{f^{-1}(b)}^a f(x) \, dx + b \cdot f^{-1}(b) - ab \text{ is nonnegative, we are done.} \right]$$

$$\begin{aligned} \Rightarrow I &= \int_{f^{-1}(b)}^a f(x) \, dx + b \cdot (f^{-1}(b) - a) + ab && \text{Express } f^{-1}(b) - a = \int_a^{f^{-1}(b)} 1 \, dx \\ &= \int_{f^{-1}(b)}^a f(x) \, dx + b \int_a^{f^{-1}(b)} dx + ab && \text{Using } \int_a^b f = - \int_b^a f \text{ and } k \int_a^b f = \int_a^b kf \\ &= \int_{f^{-1}(b)}^a f(x) \, dx - \int_{f^{-1}(b)}^a b \, dx + ab && \text{Using } \int_a^b f + \int_a^b g = \int_a^b (f + g) \\ &= \int_{f^{-1}(b)}^a (f(x) - b) \, dx + ab \end{aligned}$$

We have $I = \int_{f^{-1}(b)}^a (f(x) - b) dx + ab$.

Three cases arise.

Case 1: If $a = f^{-1}(b)$

$$\begin{aligned} I &= \int_a^a (f(x) - b) dx + ab && \text{Using } \int_a^a f = 0 \\ &= 0 + ab \\ &= ab \geq ab \end{aligned}$$

This proves that equality holds if $a = f^{-1}(b)$

Case 2: If $a > f^{-1}(b)$

$$\begin{aligned} I &= \int_{f^{-1}(b)}^a (f(x) - b) dx + ab \\ &\Rightarrow f^{-1}(b) \leq x \leq a \end{aligned}$$

Since f is a strictly increasing function, we can compose this with f .

$$\begin{aligned} &\Rightarrow f(f^{-1}(b)) \leq f(x) \leq f(a) \\ &\Rightarrow b \leq f(x) \leq f(a) \end{aligned}$$

So $f(x) \geq b$

Therefore $f(x) - b \geq 0$

And since $a > f^{-1}(b)$,

$$\begin{aligned} &\int_{f^{-1}(b)}^a (f(x) - b) > 0 \\ &\Rightarrow \int_{f^{-1}(b)}^a (f(x) - b) + ab > ab \end{aligned}$$

Case 3: If $a < f^{-1}(b)$

$$\begin{aligned} I &= \int_{f^{-1}(b)}^a (f(x) - b) dx + ab = \int_a^{f^{-1}(b)} (b - f(x)) dx + ab \\ &\Rightarrow a \leq x \leq f^{-1}(b) \Rightarrow f(a) \leq f(x) \leq b \end{aligned}$$

Therefore $b - f(x) \geq 0$

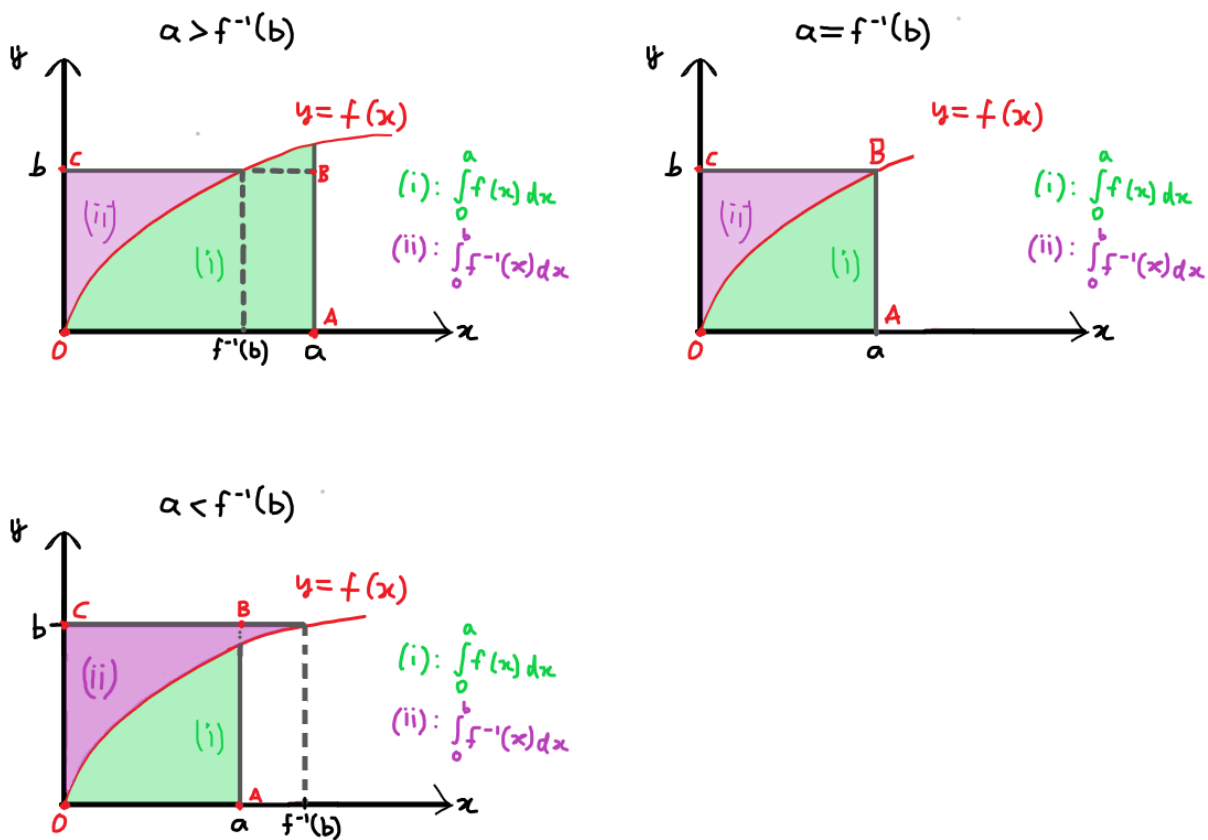
And since $f^{-1}(b) > a$,

$$\begin{aligned} &\int_a^{f^{-1}(b)} (b - f(x)) dx > 0 \\ &\Rightarrow \int_a^{f^{-1}(b)} (b - f(x)) dx + ab > ab \end{aligned}$$

□

7.8.2 Graphical Intuition

f is a strictly increasing function and $f(0) = 0$.



In all of the cases,

Area of region (i) + Area of region (ii) \geq Area of rectangle $OABC$

$$\Rightarrow \int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \geq ab$$

$$\int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \geq ab$$

□

7.9 Problems

Problem 103. $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx$

$$I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$= \int_1^5 (6-x-1)(6-x-2)(6-x-3)(6-x-4)(6-x-5) dx$$

$$= \int_1^5 (5-x)(4-x)(3-x)(2-x)(1-x) dx$$

$$= - \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx$$

$$= -I$$

$$I = -I$$

$$\Rightarrow I = 0$$

Problem 104. $\int_{\sqrt{\ln(2)}}^{\sqrt{\ln(3)}} \frac{x \sin(x^2)}{\sin(x^2) + \sin(\ln(6) - x^2)} dx$

$$I = \int_{\sqrt{\ln(2)}}^{\sqrt{\ln(3)}} \frac{x \sin(x^2)}{\sin(x^2) + \sin(\ln(6) - x^2)} dx$$

Substitute $x^2 = t \Rightarrow 2x dx = dt$

$x = \sqrt{\ln(2)} \Rightarrow t = \ln(2)$ and $x = \sqrt{\ln(3)} \Rightarrow t = \ln(3)$

$$\Rightarrow I = \frac{1}{2} \int_{\ln(2)}^{\ln(3)} \frac{\sin(t)}{\sin(t) + \sin(\ln(6) - t)} dt \quad (1)$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\Rightarrow I = \frac{1}{2} \int_{\ln(2)}^{\ln(3)} \frac{\sin(\ln(6) - t)}{\sin(\ln(6) - t) + \sin(t)} dt \quad (2)$$

Add (1) and (2),

$$2I = \frac{1}{2} \int_{\ln(2)}^{\ln(3)} \frac{\cancel{\sin(t)} + \cancel{\sin(\ln(6) - t)}}{\cancel{\sin(t)} + \cancel{\sin(\ln(6) - t)}} dt = \frac{1}{2} (\ln(3) - \ln(2))$$

$$\Rightarrow I = \frac{1}{4} (\ln(3) - \ln(2)) = \frac{1}{4} \ln\left(\frac{3}{2}\right)$$

Problem 105. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)} + \sqrt{\cos(x)}} dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)} + \sqrt{\cos(x)}} dx \quad (1)$$

Using property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \\ &\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos(x)}}{\sqrt{\cos(x)} + \sqrt{\sin(x)}} dx \end{aligned} \quad (2)$$

Add equations (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(x)}}{\sqrt{\cos(x)} + \sqrt{\sin(x)}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos(x)}}{\sqrt{\cos(x)} + \sqrt{\sin(x)}} dx$$

We can combine the integrals here.

$$\begin{aligned} \Rightarrow 2I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos(x)} + \sqrt{\sin(x)}}{\sqrt{\cos(x)} + \sqrt{\sin(x)}} dx \\ &\Rightarrow 2I = \frac{\pi}{2} \\ &\Rightarrow I = \frac{\pi}{4} \end{aligned}$$

Problem 106. Let $A = \int_0^1 \frac{e^t}{t+1} dt$

Then, express the following in terms of A.

(i) $I_1 = \int_0^1 \frac{t \cdot e^{t^2}}{t^2+1} dt$

Let $x = t^2 \Rightarrow dx = 2t dt$

$$\Rightarrow \int_0^1 \frac{t \cdot e^{t^2}}{t^2+1} dt = \frac{1}{2} \int_0^1 \frac{e^x}{x+1} dx = \frac{1}{2} \int_0^1 \frac{e^t}{t+1} dt = \frac{1}{2}(A) = \frac{A}{2}$$

(ii) $I_2 = \int_{a-1}^a \frac{e^{-t}}{t-a-1} dt$

Let $u = t - a + 1 \Rightarrow du = dt$

$\Rightarrow t = u + a - 1$

$$\Rightarrow \int_{a-1}^a \frac{e^{-t}}{t-a-1} dt = \int_0^1 \frac{e^{-u-a+1}}{u+a-1-a-1} du = \int_0^1 \frac{e^{-u-a+1}}{u-2} du$$

Using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\Rightarrow I_2 = \int_0^1 \frac{e^{u-a+1}}{1-u-2} du = \int_0^1 \frac{e^{u-a}}{-u-1} du = -e^{-a} \int_0^1 \frac{e^u}{u+1} du = -e^{-a} \int_0^1 \frac{e^t}{t+1} dt = -e^{-a} \cdot (A)$$

(iii) $I_3 = \int_0^1 \frac{e^t}{(t+1)^2} dt$

Integrating by parts.

Let $u = e^t$ and $dv = \frac{dt}{(t+1)^2}$

$\Rightarrow du = e^t dt$ and $v = \frac{-1}{t+1}$

$\int u dv = uv - \int v du$

$$I_3 = \left[\frac{-e^t}{t+1} \right]_0^1 - \int_0^1 \frac{-e^t}{t+1} dt = \frac{-e}{2} - (-1) + \int_0^1 \frac{e^t}{t+1} dt = 1 - \frac{e}{2} + A$$

Problem 107. Let $f(x) = \int_1^x \frac{\ln(t)}{t+1} dt$. If $x > 0$, compute $f(x) + f\left(\frac{1}{x}\right)$. Hence or otherwise, show that $f(2) + f\left(\frac{1}{2}\right) = \frac{1}{2} (\ln(2))^2$.

Solution:

$$f(x) = \int_1^x \frac{\ln(t)}{t+1} dt = \int_1^x \frac{\ln(u)}{u+1} du$$

$$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln(t)}{t+1} dt$$

$$\text{Let } t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2} du$$

$$t = 1 \Rightarrow u = 1$$

$$t = \frac{1}{x} \Rightarrow u = x$$

$$= \int_1^x \frac{-\ln\left(\frac{1}{u}\right)}{\frac{1}{u}+1} \left(\frac{du}{u^2}\right)$$

$$= \int_1^x \frac{\ln(u)}{u(u+1)} du$$

$$\text{Now, } f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left[\frac{\ln(u)}{u+1} + \frac{\ln(u)}{u(u+1)} \right] du$$

$$= \int_1^x \frac{u \ln(u) + \ln(u)}{u(u+1)} du$$

$$= \int_1^x \frac{\ln(u) \cdot \cancel{(u+1)}}{u \cancel{(u+1)}} du$$

$$\text{Let } v = \ln(u) \Rightarrow \frac{1}{u} du = dv$$

$$= \int_0^{\ln(x)} v dv = \left[\frac{v^2}{2} \right]_0^{\ln(x)}$$

$$= \frac{1}{2} (\ln(x))^2$$

$$\therefore f(2) + f\left(\frac{1}{2}\right) = \frac{1}{2} (\ln(2))^2$$

□

Problem 108. A function F is defined by the following definite integral.

$$F(x) = \int_1^x \frac{e^t}{t} dt$$

Prove that $\int_1^x \frac{e^t}{t+a} dt = e^{-a} [F(x+a) - F(1+a)]$.

Proof.

$$I = \int_1^x \frac{e^t}{t+a} dt$$

$$= \int_{1+a}^{x+a} \frac{e^{u-a}}{u} du$$

$$= e^{-a} \int_{1+a}^{x+a} \frac{e^u}{u} du$$

$$= e^{-a} \left[\int_{1+a}^1 \frac{e^u}{u} du + \int_1^{x+a} \frac{e^u}{u} du \right]$$

$$= e^{-a} \left[- \int_1^{1+a} \frac{e^u}{u} du + \int_1^{x+a} \frac{e^u}{u} du \right]$$

$$= e^{-a} \left[\int_1^{x+a} \frac{e^u}{u} du - \int_1^{1+a} \frac{e^u}{u} du \right]$$

$$= e^{-a} [F(x+a) - F(1+a)]$$

$$\text{Let } t+a = u$$

$$\Rightarrow dt = du$$

$$t=1 \Rightarrow u=1+a$$

$$t=x \Rightarrow u=x+a$$

$$\text{Using } \int_a^c f + \int_c^b f = \int_a^b f$$

$$\text{Using } \int_a^b f = - \int_b^a f$$

□

Problem 109. Find a continuous function f such that

$$[f(x)]^2 = \int_0^x \frac{f(t) \cdot \sin(t)}{2 + \cos(t)} dt$$

Solution:

$$[f(x)]^2 = \int_0^x \frac{f(t) \cdot \sin(t)}{2 + \cos(t)} dt$$

Differentiate both sides

$$\Rightarrow 2\cancel{f(x)} \cdot f'(x) = \frac{\cancel{f(x)} \cdot \sin(x)}{2 + \cos(x)}$$

Divide by 2

$$\Rightarrow f'(x) = \frac{1}{2} \cdot \frac{\sin(x)}{2 + \cos(x)}$$

$$\begin{aligned} f(x) &= \int f'(x) dx \\ &= \frac{1}{2} \int \frac{\sin(x)}{2 + \cos(x)} dx \end{aligned}$$

Let $\cos(x) = t$
 $\Rightarrow -\sin(x) dx = dt$

$$\begin{aligned} &= -\frac{1}{2} \int \frac{dt}{2 + t} \\ &= -\frac{1}{2} \ln(|2 + t|) + c \\ &= -\frac{1}{2} \ln(|2 + \cos(x)|) + c \end{aligned}$$

c is an arbitrary constant

One such function is $f(x) = -\frac{1}{2} \ln(|2 + \cos(x)|)$

Problem 110. Prove that for any integral with finite limits a and b , one can always choose a linear substitution $x = pt + q$, (where p and q are constants) so as to transform the integral into a new one with limits from 0 to 1.

Proof.

Let us consider any such integral $I = \int_a^b f(x) \, dx$

Let $x = pt + q$

$\Rightarrow dx = p \, dt$

Now,

$$x = a \text{ at } t = 0$$

$$\Rightarrow pt + q = a \text{ at } t = 0$$

$$\therefore q = a$$

And,

$$x = b \text{ at } t = 1$$

$$\Rightarrow pt + q = b \text{ at } t = 1$$

$$\Rightarrow p + q = b$$

$$\Rightarrow p = b - a$$

So,

$$\int_a^b f(x) \, dx = p \int_0^1 f(pt + q) \, dt$$

where $p = b - a$ and $q = a$.

So for any integral with finite limits a to b , we can substitute $x = (b - a)t + a$ to change the limits to 0 to 1.

$$\int_a^b f(x) \, dx = (b - a) \cdot \int_0^1 f((b - a)t + a) \, dt$$

□

Problem-Solving Tip: In a general problem involving addition of two integrals, try to convert both of them into an integral of the form \int_0^1 and then add.

Problem 111. $\int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{1/3}^{2/3} e^{9(x-\frac{2}{3})^2} dx$

Let $p(x) = e^{(x+5)^2}$ and $q(x) = e^{9(x-\frac{2}{3})^2}$

$$I = \int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{1/3}^{2/3} e^{9(x-\frac{2}{3})^2} dx$$

$$= \int_{-4}^{-5} p(x) dx + 3 \int_{1/3}^{2/3} q(x) dx$$

Using $\int_a^b f(x) dx = (b-a) \int_0^1 f((b-a)t+a) dt$

$$= \int_0^1 p(-t-4) dt + \int_0^1 q\left(\frac{1}{3}t + \frac{1}{3}\right) dt$$

$$= -1 \int_0^1 e^{((-t-4)+5)^2} dt + 3 \left(\frac{1}{3}\right) \int_0^1 e^{9(\frac{1}{3}t + \frac{1}{3} - \frac{2}{3})^2} dt$$

$$= - \int_0^1 e^{(1-t)^2} dt + \int_0^1 e^{9(\frac{t-1}{3})^2} dt$$

$$= - \int_0^1 e^{(1-t)^2} dt + \int_0^1 e^{(t-1)^2} dt$$

$$= - \int_0^1 e^{(t-1)^2} dt + \int_0^1 e^{(t-1)^2} dt$$

$$= 0$$

Problem 112. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Then find the value of

$$I = \int_{-\pi/2}^{\pi/2} [f(x) + f(-x)] \cdot [g(x) - g(-x)] dx$$

Let $h(x) = [f(x) + f(-x)] \cdot [g(x) - g(-x)]$

$$\begin{aligned} h(x) &= [f(x) + f(-x)] \cdot [g(x) - g(-x)] \\ \Rightarrow h(-x) &= [f(-x) + f(x)] \cdot [g(-x) - g(x)] \\ &= [f(-x) + f(x)] \cdot (-1) [g(x) - g(-x)] \\ &= -h(x) \end{aligned}$$

$\Rightarrow h(x)$ is an odd function.

We know that $\int_{-a}^a f = 0$ if $f(-x) = -f(x)$

$$\therefore I = 0$$

And we're done.

Remark: It is often a good idea to shift the origin to the midpoint of the limits of integration, thereby making them of the form \int_{-a}^a .

Problem 113. $\int_{-2}^0 \left(x^3 + 3x^2 + 3x + 3 + (x+1) \cos(x+1) \right) dx$

$$\begin{aligned} I &= \int_{-2}^0 \left(x^3 + 3x^2 + 3x + 3 + (x+1) \cdot \cos(x+1) \right) dx \\ &= \int_{-2}^0 \left((x+1)^3 + 2 + (x+1) \cdot \cos(x+1) \right) dx \end{aligned}$$

Let us shift the origin this time.

Let $x+1 = t$

$\Rightarrow dx = dt$

$$\begin{aligned} \Rightarrow I &= \int_{-1}^1 \left(t^3 + 2 + t \cos(t) \right) dt && \text{Note that } t^3 + t \cos(t) \text{ is an odd function} \\ &= \int_{-1}^1 2 dt \\ &= [2t]_{-1}^1 \\ &= 4 \end{aligned}$$

Problem 114. $\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx$ [Putnam]

$$I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx$$

Shifting origin again,

Let $x - \frac{1}{2} = t$

$\Rightarrow dx = dt$

$$\begin{aligned} \Rightarrow I &= \int_{-1/2}^{1/2} \sqrt[3]{2 \left(t - \frac{1}{2} \right)^3 - 3 \left(t - \frac{1}{2} \right)^2 - \left(t - \frac{1}{2} \right) + 1} dt \\ &= \int_{-1/2}^{1/2} \sqrt[3]{2t^3 - \frac{5t}{2}} dt && \text{This is an odd function} \\ &= 0 \end{aligned}$$

Problem 115. $\int_{1/2014}^{2014} \frac{\tan^{-1}(x)}{x} dx$

$$I = \int_{1/2014}^{2014} \frac{\tan^{-1}(x)}{x} dx \quad (1)$$

Let $x = \frac{1}{t}$
 $\Rightarrow dx = \frac{-1}{t^2} dt$

$$\begin{aligned} \Rightarrow I &= \int_{2014}^{1/2014} \frac{\tan^{-1}\left(\frac{1}{t}\right)}{\frac{1}{t}} \cdot \left(\frac{-dt}{t^2}\right) \\ &= - \int_{2014}^{1/2014} \frac{\tan^{-1}\left(\frac{1}{t}\right)}{t} dt \\ \Rightarrow I &= \int_{1/2014}^{2014} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{x} dx \quad (2) \end{aligned}$$

Adding (1) and (2),

$$\begin{aligned} 2I &= \int_{1/2014}^{2014} \frac{\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)}{x} dx && \text{Using } \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} \\ &= \frac{\pi}{2} [\ln(x)]_{1/2014}^{2014} \\ &= \pi \ln(2014) \\ \Rightarrow I &= \frac{\pi}{2} \ln(2014) \end{aligned}$$

Problem 116. Let f be a differentiable function such that $f(f(x)) = x \forall x \in [0, 1]$. Suppose $f(0) = 1$. Determine the value of the following integral:

$$\int_0^1 (x - f(x))^{2016} dx$$

Solution:

$$\begin{aligned} f(f(x)) &= x \forall x \in [0, 1] \\ \Rightarrow f(f(0)) &= 0 \\ \Rightarrow f(1) &= 0 \end{aligned}$$

Since f is given as a differentiable function, we are allowed to differentiate it.

$$\begin{aligned} f(f(x)) &= x \\ \Rightarrow f'(f(x)) \cdot f'(x) &= 1 \end{aligned}$$

So, $f'(x)$ can never be zero.

By Darboux's Theorem, f' has intermediate value property.

$\Rightarrow f'$ must preserve its sign.

$\Rightarrow f'(x) > 0 \forall x$ or $f'(x) < 0 \forall x$

But $f(0) = 1$ and $f(1) = 0$, is inconsistent with the condition that $f'(x) > 0 \forall x$

$\Rightarrow f'(x) < 0 \forall x$

$\Rightarrow f$ is strictly decreasing.

$\Rightarrow f : [0, 1] \rightarrow [0, 1]$ is one-one and onto, implying f is invertible.

$$I = \int_0^1 (x - f(x))^{2016} dx = \int_0^1 (t - f(t))^{2016} dt \quad (1)$$

Substitute $x = f(t)$. This is allowed because f is one-one.

$\Rightarrow dx = f'(t) dt$

$x = 0 \Rightarrow t = 1$ and $x = 1 \Rightarrow t = 0$

$$\begin{aligned} \Rightarrow I &= \int_1^0 (f(t) - f(f(t)))^{2016} \cdot f'(t) dt \\ &= - \int_0^1 (f(t) - t)^{2016} \cdot f'(t) dt \end{aligned}$$

$$\Rightarrow I = - \int_0^1 (t - f(t))^{2016} \cdot f'(t) dt \quad (2)$$

Add (1) and (2),

$$2I = \int_0^1 (t - f(t))^{2016} \cdot (1 - f'(t)) dt$$

$$2I = \int_0^1 (t - f(t))^{2016} \cdot (1 - f'(t)) \, dt$$

$$\begin{aligned} \text{Let } t - f(t) &= u \\ \Rightarrow (1 - f'(t)) \, dt &= du \end{aligned}$$

$$\begin{aligned} \text{At } t &= 0, \\ u &= t - f(t) \\ &= 0 - f(0) \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{At } t &= 1, \\ u &= t - f(t) \\ &= 1 - f(1) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow 2I &= \int_{-1}^1 u^{2016} \, du \\ &= \left[\frac{u^{2017}}{2017} \right]_{-1}^1 \\ &= \frac{1}{2017} - \frac{(-1)}{2017} \\ &= \frac{2}{2017} \\ \Rightarrow I &= \frac{1}{2017} \end{aligned}$$

Problem 117. $\int_{-1}^1 \frac{\sqrt[3]{x}}{\sqrt[3]{1-x} + \sqrt[3]{1+x}} \, dx$

$$I = \int_{-1}^1 \frac{\sqrt[3]{x}}{\sqrt[3]{1-x} + \sqrt[3]{1+x}} \, dx$$

$$\text{Let } f(x) = \frac{\sqrt[3]{x}}{\sqrt[3]{1-x} + \sqrt[3]{1+x}}$$

$$\Rightarrow f(-x) = \frac{\sqrt[3]{-x}}{\sqrt[3]{1+x} + \sqrt[3]{1-x}} = \frac{-\sqrt[3]{x}}{\sqrt[3]{1-x} + \sqrt[3]{1+x}} = -f(x)$$

$\Rightarrow f$ is an odd function.

$$\Rightarrow I = \int_{-1}^1 f(x) \, dx = 0$$

Problem 118. Let a and b be positive real numbers. Evaluate:

$$\int_a^b \frac{e^{x/a} - e^{b/x}}{x} dx$$

Solution:

$$I = \int_a^b \frac{e^{x/a} - e^{b/x}}{x} dx$$

$$\begin{aligned} \text{Let } x &= \frac{ab}{t} \\ \Rightarrow dx &= \frac{-ab}{t^2} dt \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{x}{a} &= \frac{\left(\frac{ab}{t}\right)}{a} = \frac{b}{t} \\ \frac{b}{x} &= \frac{b'}{\left(\frac{ab}{t}\right)} = \frac{t}{a} \end{aligned}$$

$$\begin{aligned} I &= \int_b^a \frac{e^{b/t} - e^{t/a}}{\left(\frac{ab}{t}\right)} \cdot \frac{(-ab)}{t^2} dt \\ &= \int_b^a \frac{e^{t/a} - e^{b/t}}{t} dt \\ &= - \int_a^b \frac{e^{t/a} - e^{b/t}}{t} dt \\ &= - \int_a^b \frac{e^{x/a} - e^{b/x}}{t} dx \\ &= -I \end{aligned}$$

$$\Rightarrow I = 0$$

7.10 Comparison Theorem

7.10.1 Statement

Theorem 14. Say f and g are two functions such that $f(x) \geq g(x) \quad \forall x \in (a, b)$, then $\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$ and the equality holds iff $f(x) = g(x) \quad \forall x \in (a, b)$.

7.10.2 Corollaries

Corollary 1. If $f(x) \geq 0 \quad \forall x \in (a, b)$, then $\int_a^b f(x) \, dx \geq 0$.

Corollary 2. If $f(x) \geq 0 \quad \forall x \in (a, b)$ and $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$.

7.10.3 Problems

Problem 119. Arrange I_1, I_2, I_3 and I_4 in decreasing order.

$$I_1 = \int_0^{\pi/4} e^{x^2} \, dx$$

$$I_2 = \int_0^{\pi/4} e^x \, dx$$

$$I_3 = \int_0^{\pi/4} e^{x^2} \cdot \cos(x) \, dx$$

$$I_4 = \int_0^{\pi/4} e^{x^2} \cdot \sin(x) \, dx$$

Solution: For $x \in \left(0, \frac{\pi}{4}\right)$, which is a subset of $(0, 1)$,

$$e^x > e^{x^2} > e^{x^2} \cdot \cos(x) > e^{x^2} \cdot \sin(x)$$

Using Comparison Theorem,

$$\begin{aligned} \Rightarrow \int_0^{\pi/4} e^x &> \int_0^{\pi/4} e^{x^2} > \int_0^{\pi/4} e^{x^2} \cdot \cos(x) > \int_0^{\pi/4} e^{x^2} \cdot \sin(x) \\ &\Rightarrow I_2 > I_1 > I_3 > I_4 \end{aligned}$$

Problem 120. Find the total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$.

Solution:

$$\text{Let } f(x) = \int_0^x \frac{t^2}{1+t^4} dt - 2x + 1$$

We are concerned with the number of roots of $f(x)$ in $[0, 1]$.

Note that f is continuous in $[0, 1]$ and differentiable in $(0, 1)$. Applying Bolzano's Theorem:

$$f(0) = \int_0^0 \frac{t^2}{1+t^4} dt - 0 + 1 = 1$$

$$\Rightarrow f(0) > 0$$

$$f(1) = \int_0^1 \frac{t^2}{1+t^4} dt - 1$$

Now, for $t \in (0, 1)$, $t^2 < 1$ and $1 + t^4 > 1$

$$\Rightarrow \frac{t^2}{1+t^4} < 1$$

$$\Rightarrow \frac{t^2}{1+t^4} < 1$$

Applying Comparison Theorem,

$$\Rightarrow \int_0^1 \frac{t^2}{1+t^4} dt < 1$$

$$\Rightarrow f(1) < 0$$

By Bolzano's Theorem, since f is continuous, $f(0) > 0$ and $f(1) < 0$, there exists **at least one** root of $f(x)$ in $(0, 1)$.

Now, assume for the purpose of contradiction that f has two (or more) roots in $(0, 1)$.

Then, by Rolle's Theorem, $f'(x)$ has at least one root in $(0, 1)$.

$$f'(x) = \frac{x^2}{1+x^4} - 2$$

Since $\frac{x^2}{1+x^4} < 1 \forall x \in (0, 1)$, this implies $f'(x)$ can't be zero for any $x \in (0, 1)$

$\Rightarrow f'(x)$ has no root in $(0, 1)$. This is a contradiction.

$\therefore f(x)$ has exactly one root in $(0, 1)$.

Therefore, there exists exactly one value of $x \in [0, 1]$ such that $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$.

Problem 121. Find all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ satisfying

$$\int_0^1 f(x) \, dx = \frac{1}{3} + \int_0^1 \left(f(x^2) \right)^2 \, dx$$

Solution:

$$\int_0^1 f(x) \, dx = \frac{1}{3} + \int_0^1 \left(f(x^2) \right)^2 \, dx$$

Let us try to express $\frac{1}{3}$ as an integral, it may help us simplify stuff.

$$\frac{1}{3} = \int_0^1 x^2 \, dx$$

$$\Rightarrow \int_0^1 f(x) \, dx = \int_0^1 x^2 \, dx + \int_0^1 \left(f(x^2) \right)^2 \, dx$$

Now let us try to express the whole thing in terms of x^2 .

Substitute $x = t^2$ in LHS. This is allowed because t^2 is one-one for $t \in (0, 1)$

$\Rightarrow dx = 2t \, dt$

$$\Rightarrow \int_0^1 f(t^2) \cdot 2t \, dt = \int_0^1 x^2 \, dx + \int_0^1 \left(f(x^2) \right)^2 \, dx$$

$$\Rightarrow \int_0^1 f(x^2) \cdot 2x \, dx = \int_0^1 x^2 \, dx + \int_0^1 \left(f(x^2) \right)^2 \, dx$$

Combine the integrals, bring to one side.

$$\Rightarrow \int_0^1 \left[\left(f(x^2) \right)^2 + x^2 - 2x \cdot f(x^2) \right] \, dx = 0$$

$$\Rightarrow \int_0^1 \left(f(x^2) - x \right)^2 \, dx = 0$$

Now, $\left(f(x^2) - x \right)^2 \geq 0 \, \forall x$ (\because the square of a real number is always non-negative)

$$\Rightarrow \int_0^1 \left(f(x^2) - x \right)^2 \, dx \geq 0$$

By Comparison Theorem, $\left(f(x^2) - x \right)^2 = 0$

$$\Rightarrow f(x^2) = x$$

$$\Rightarrow f(x) = \sqrt{x}$$

Therefore, $f(x) = \sqrt{x}$ is the only continuous function that satisfies the property.

Problem 122. Determine the continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ satisfying

$$\int_0^1 f(x) \cdot (x - f(x)) \, dx = \frac{1}{12}$$

Solution:

$$\int_0^1 f(x) \cdot (x - f(x)) \, dx = \frac{1}{12}$$

Let us try to express $\frac{1}{12}$ as an integral with limits 0 to 1.

$$\Rightarrow \int_0^1 (x \cdot f(x) - (f(x))^2) \, dx = \int_0^1 \frac{x^2}{4} \, dx$$

$$\Rightarrow \int_0^1 \left((f(x))^2 - x \cdot f(x) + \frac{x^2}{4} \right) \, dx = 0$$

$$\Rightarrow \int_0^1 \left(f(x) - \frac{x}{2} \right)^2 \, dx = 0$$

$$\text{Now, } \left(f(x) - \frac{x}{2} \right)^2 \geq 0 \, \forall x$$

$$\Rightarrow \int_0^1 \left(f(x) - \frac{x}{2} \right)^2 \, dx \geq 0$$

Using the comparison theorem (Corollary 2),

$$\Rightarrow f(x) - \frac{x}{2} = 0$$

$$\Rightarrow f(x) = \frac{x}{2}$$

7.11 Average Value of a Function

The average value of a function f on an interval $[a, b]$ is defined as

$$\frac{\int_a^b f(x) \, dx}{b - a}$$

The average value of a function f over a function g on the interval $[a, b]$ is defined as

$$\frac{\int_a^b f(x) \cdot g(x) \, dx}{\int_a^b g(x) \, dx}$$

7.12 Mean Value Theorems for Integrals

Theorem 15. *Say a function f is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that:*

$$\int_a^b f(x) \, dx = f(c) \cdot (b - a)$$

Proof. Since f is continuous on a closed interval, f will have at least one absolute minimum and at least one absolute maximum value.

Let m and M be the minimum and maximum values attained by f respectively.

$$m \leq f(x) \leq M$$

Using Comparison Theorem, integrating throughout.

$$\begin{aligned} \int_a^b m \, dx &\leq \int_a^b f(x) \, dx \leq \int_a^b M \, dx \\ \Rightarrow m(b - a) &\leq \int_a^b f(x) \, dx \leq M(b - a) \end{aligned}$$

Since $b \geq a$, we can divide by $b - a$ without changing the signs of the inequality.

$$\Rightarrow m \leq \frac{\int_a^b f(x) \, dx}{b - a} \leq M$$

By Intermediate Value Theorem, $\frac{\int_a^b f(x) \, dx}{b - a}$ will take all the values between its maximum M and minimum m .

$$\begin{aligned} \exists c \in [a, b] : \frac{\int_a^b f(x) \, dx}{b - a} &= f(c) \\ \Rightarrow \exists c \in [a, b] : \int_a^b f(x) \, dx &= f(c) \cdot (b - a) \end{aligned}$$

□

7.12.1 The General Mean Value Theorems

Theorem 16. *Let f and g be two continuous functions in $[a, b]$ such that on the given interval, g does not change sign. Then there exists some $c \in [a, b]$ such that:*

$$\int_a^b f(x) \cdot g(x) \, dx = f(c) \cdot \int_a^b g(x) \, dx$$

Proof. It is given that g doesn't change sign.

WLOG, consider the case where $g(x) > 0$.

Since f is continuous on a closed interval, f will have at least one absolute minimum and at least one absolute maximum value.

Let m and M be the minimum and maximum values attained by f respectively.

$$m \leq f(x) \leq M$$

Multiply by $g(x)$

$$m \cdot g(x) \leq f(x) \cdot g(x) \leq M \cdot g(x)$$

Again, using comparison theorem, we can integrate.

$$\begin{aligned} \int_a^b m \cdot g(x) \, dx &\leq \int_a^b f(x) \cdot g(x) \, dx \leq \int_a^b M \cdot g(x) \, dx \\ \Rightarrow m \cdot \int_a^b g(x) \, dx &\leq \int_a^b f(x) \cdot g(x) \, dx \leq M \cdot \int_a^b g(x) \, dx \end{aligned}$$

Since $g(x) > 0$, by comparison theorem, $\int_a^b g(x) \, dx > 0$.

Divide by $\int_a^b g(x) \, dx$.

$$m \leq \frac{\int_a^b f(x) \cdot g(x) \, dx}{\int_a^b g(x) \, dx} \leq M$$

By Intermediate Value Theorem $\frac{\int_a^b f(x) \cdot g(x) \, dx}{\int_a^b g(x) \, dx}$ will take all values between its minimum m and maximum M .

$\therefore \exists c \in [a, b]$ such that $\int_a^b f(x) \cdot g(x) \, dx = f(c) \cdot \int_a^b g(x) \, dx$

□

Theorem 17. Say f and g are continuous functions and say the derivative of f is continuous and does not change sign. Then, there exists some $c \in [a, b]$ such that:

$$\int_a^b f(x) \cdot g(x) \, dx = f(a) \int_a^c g(x) \, dx + f(b) \int_c^b g(x) \, dx$$

Proof.

$$\text{Let } G(x) = \int_a^x g(t) \, dt$$

$$\Rightarrow G'(x) = g(x) - g(a) \cdot \frac{d}{dx}(a) = g(x)$$

$$\text{Let } I = \int_a^b f(x) \cdot g(x) \, dx$$

$$\Rightarrow I = \int_a^b f(x) \cdot G'(x) \, dx$$

Integrating by parts.

Let $u = f(x)$ and $dv = G'(x) \, dx$

Then, $du = f'(x) \, dx$ and $v = G(x)$

$\int u \, dv = uv - \int v \, du$

$$\begin{aligned} \Rightarrow I &= [f(x) \cdot G(x)]_a^b - \int_a^b f'(x) \cdot G(x) \, dx \\ &= f(b) \int_a^b g(x) \, dx - f(a) \int_a^a g(x) \, dx - \int_a^b f'(x) \cdot G(x) \, dx \\ &= f(b) \int_a^b g(x) \, dx - G(c) \int_a^b f'(x) \, dx \\ &= f(b) \int_a^b g(x) \, dx - \left(\int_a^c g(x) \, dx \right) \cdot (f(b) - f(a)) \\ &= f(b) \int_a^b g(x) \, dx - f(b) \int_a^c g(x) \, dx + f(a) \int_a^c g(x) \, dx \\ &= -f(b) \int_b^c g(x) \, dx + f(a) \int_a^c g(x) \, dx \\ &= f(a) \int_a^c g(x) \, dx + f(b) \int_c^b g(x) \, dx \end{aligned}$$

Using the previous general mean value theorem.

For some c

□

Problem 123. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a non-decreasing continuous function. If $0 \leq x < y < z$, show that:

$$(z-x) \int_y^z f(u) \, du \geq (z-y) \int_x^z f(u) \, du$$

Proof. Effectively, we have to prove that:

$$\frac{\int_y^z f(u) \, du}{(z-y)} \geq \frac{\int_x^z f(u) \, du}{(z-x)}$$

i.e. Fundamentally, we have to prove that the average of the function is a non-decreasing function.

$$\text{Let } g(t) = \frac{\int_t^z f(u) \, du}{(z-t)}$$

$$\begin{aligned} g'(t) &= \frac{-(z-t) \cdot f(t) - \left(\int_t^z f(u) \, du \right) (-1)}{(z-t)^2} \\ &= \frac{\int_t^z f(u) \, du - (z-t) \cdot f(t)}{(z-t)^2} \end{aligned}$$

Wishful thinking:
Denominator is non-negative.
If we can prove the numerator
is non-negative, we are done

Now, $\int_t^z f(u) \, du = (z-t)f(c)$ for some $c \in (t, z)$

Using Mean Value Theorem
for integrals

$$\begin{aligned} \Rightarrow g'(t) &= \frac{(z-t)f(c) - (z-t)f(t)}{(z-t)^2} \\ &= \frac{(z-t)(f(c) - f(t))}{(z-t)^2} \end{aligned}$$

Now, $c \in [t, z] \Rightarrow c > t$
 $\Rightarrow f(c) \geq f(t)$, since f is non-decreasing.
Also, $t \leq z$.

$$\Rightarrow g'(t) \geq 0$$

$\Rightarrow g(t)$ is non-decreasing.

$$\begin{aligned} &\Rightarrow \frac{\int_y^z f(u) \, du}{(z-y)} \geq \frac{\int_x^z f(u) \, du}{(z-x)} \\ &\Rightarrow (z-x) \int_y^z f(u) \, du \geq (z-y) \int_x^z f(u) \, du \end{aligned}$$

□

Problem 124. If ϕ'' is continuous and non-zero on the closed interval $[a, b]$ and if there is a constant $m > 0$ such that $\phi'(t) \geq m \forall t \in [a, b]$, then prove that $\left| \int_a^b \sin(\phi(t)) dt \right| \leq \frac{4}{m}$

Proof. Let $I = \int_a^b \sin(\phi(t)) dt$

$$\Rightarrow I = \int_a^b \frac{\sin(\phi(t))}{\phi'(t)} \cdot \phi'(t) dt$$

Let $f(t) = \frac{1}{\phi'(t)}$ and $g(t) = \sin(\phi(t)) \cdot \phi'(t)$

Then,

$$I = \int_a^b f(t) \cdot g(t) dt$$

$$f'(t) = \frac{-\phi'(t)}{[\phi''(t)]^2}$$

Since ϕ'' is continuous and non-zero, this implies ϕ'' preserves its sign.

Using the Second General Mean Value Theorem, $\exists c \in [a, b]$ such that

$$\begin{aligned} I &= f(a) \int_a^c g(t) dt + f(b) \int_c^b g(t) dt \\ &\Rightarrow \int_a^b \frac{\sin(\phi(t))}{\phi'(t)} \cdot \phi'(t) dt = \frac{1}{\phi'(a)} \int_a^c \sin(\phi(t)) \cdot \phi'(t) dt + \frac{1}{\phi'(b)} \int_c^b \sin(\phi(t)) \cdot \phi'(t) dt \end{aligned}$$

Using the Triangle Inequality $|x + y| \leq |x| + |y| \forall x, y \in \mathbb{C}$,

$$\begin{aligned} \Rightarrow |I| &\leq \left| \frac{1}{\phi'(a)} \int_a^c \sin(\phi(t)) \phi'(t) dt \right| + \left| \frac{1}{\phi'(b)} \int_c^b \sin(\phi(t)) \phi'(t) dt \right| \\ &\Rightarrow |I| \leq \left| \frac{[-\cos(\phi(t))]_a^c}{\phi'(a)} \right| + \left| \frac{[-\cos(\phi(t))]_c^b}{\phi'(b)} \right| \end{aligned}$$

Now,

$$\phi'(t) \geq m \Rightarrow \frac{1}{\phi'(t)} \leq \frac{1}{m}$$

And the maximum difference of $\cos(x)$ is 2.

$$\begin{aligned} \Rightarrow |I| &\leq \frac{2}{m} + \frac{2}{m} \\ &\Rightarrow \left| \int_a^b \sin(\phi(t)) dt \right| \leq \frac{4}{m} \end{aligned}$$

□

7.13 Estimation of Integrals

Note: Estimates may be good or bad, but are seldom incorrect.

7.13.1 Method - 1: Comparison Theorem

Suppose we need to estimate $\int_a^b f(x) dx$.

We will estimate $f(x)$ between two functions whose integrals can be evaluated.

$$g(x) < f(x) < h(x)$$

Then, we'll use comparison theorem and evaluate.

Also, we can use the fact that $\left| \int_a^b f(x) dx \right| \geq \int_a^b |f(x)| dx$.

For example, if we need to estimate $\int_0^1 e^{-x^2} dx$, then e^{-x^2} can be estimated in two ways:

$$1 - x^2 \leq e^{-x^2} \leq \frac{1}{1 - x^2} \quad \text{Better, More precise}$$

$$e^{-x} \leq e^{-x^2} \leq 1 \quad \text{Not as precise}$$

7.13.2 Method - 2: Using a Minimum and Maximum Value

Say f is a continuous function.

To estimate $\int_a^b f(x) dx$, we can find the minimum and maximum values of f . Say the minimum is m and the maximum is M .

$$m \leq f(x) \leq M$$

Apply comparison theorem,

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

7.13.3 Method - 3: Mean Value Theorem for Integrals

$$\int_a^b f(x) dx = f(c) \cdot (b-a), \quad \text{for some } c \in [a, b]$$

If f is a non-decreasing function,
then $a \leq c \leq b \Rightarrow f(a) \leq f(c) \leq f(b)$

$$\Rightarrow f(a) \cdot (b-a) \leq \int_a^b f(x) dx \leq f(b) \cdot (b-a)$$

If f is a non-increasing function,
then $a \leq c \leq b \Rightarrow f(a) \geq f(c) \geq f(b)$

$$\Rightarrow f(a) \cdot (b-a) \geq \int_a^b f(x) dx \geq f(b) \cdot (b-a)$$

7.13.4 Method - 4: General Mean value Theorem for Integrals

$$\int_a^b f(x) \cdot g(x) \, dx = f(c) \cdot \int_a^b g(x) \, dx, \quad \text{for some } c \in [a, b]$$

If f is a non-decreasing function,

then $a \leq c \leq b \Rightarrow f(a) \leq f(c) \leq f(b)$

$$\Rightarrow f(a) \cdot \int_a^b g(x) \, dx \leq \int_a^b f(x) \cdot g(x) \, dx \leq f(b) \cdot \int_a^b g(x) \, dx$$

If f is a non-increasing function,

then $a \leq c \leq b \Rightarrow f(a) \geq f(c) \geq f(b)$

$$\Rightarrow f(a) \cdot \int_a^b g(x) \, dx \geq \int_a^b f(x) \cdot g(x) \, dx \geq f(b) \cdot \int_a^b g(x) \, dx$$

7.13.5 Method - 5: Cauchy-Schwarz Inequality

If f and g are two functions integrable on $[a, b]$. Then,

$$\left| \int_a^b f(x) \cdot g(x) \, dx \right| \leq \sqrt{\int_a^b (f(x))^2 \, dx \cdot \int_a^b (g(x))^2 \, dx}$$

and equality holds iff $f(x) = \lambda \cdot g(x)$ for some constant λ .

Proof.

$$\text{Let } F(x) = [f(x) - \lambda \cdot g(x)]^2$$

Now, $F(x) \geq 0 \, \forall x \in \mathbb{R}$, because the square of a real number is always non-negative.

Using comparison theorem,

$$\begin{aligned} & \int_a^b (f(x) - \lambda \cdot g(x))^2 \, dx \geq 0 \\ \Rightarrow & \int_a^b (f(x))^2 \, dx - 2\lambda \int_a^b f(x) \cdot g(x) \, dx + \lambda^2 \int_a^b (g(x))^2 \, dx \geq 0 \\ \Rightarrow & \lambda^2 \left(\int_a^b (g(x))^2 \, dx \right) - 2\lambda \left(\int_a^b f(x) \cdot g(x) \, dx \right) + \int_a^b (f(x))^2 \, dx \geq 0 \end{aligned}$$

This is a quadratic in λ . For the quadratic to be non-negative for all real λ , $D \leq 0$.

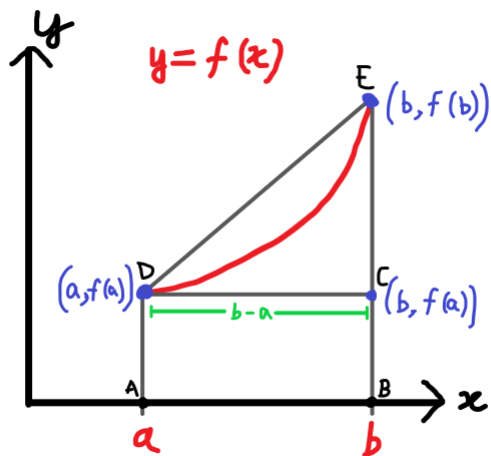
$$\begin{aligned} \Rightarrow & 4\lambda^2 \left(\int_a^b f(x) \cdot g(x) \, dx \right)^2 - 4 \int_a^b (g(x))^2 \cdot \int_a^b (f(x))^2 \, dx \leq 0 \\ \Rightarrow & \left| \int_a^b f(x) \cdot g(x) \, dx \right| \leq \sqrt{\int_a^b (f(x))^2 \, dx \cdot \int_a^b (g(x))^2 \, dx} \end{aligned}$$

□

7.13.6 Method - 6: Estimation Based on Shapes

Say f is a strictly increasing function.

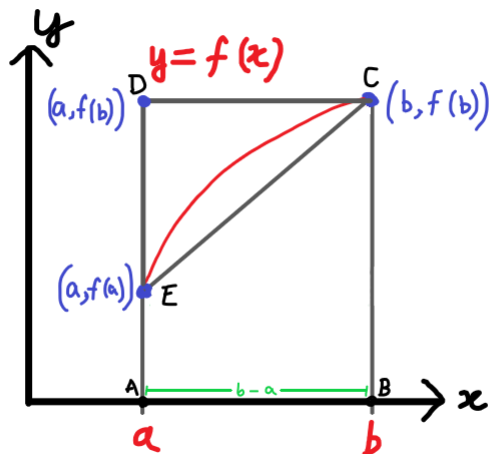
If f is concave up,



Area of rectangle ABCD \leq Area under curve $f \leq$ Area of trapezium ABED

$$\Rightarrow (b-a) \cdot f(a) \leq \int_a^b f(x) dx \leq \frac{1}{2} (f(a) + f(b)) (b-a)$$

Similarly, if f is concave down,



Area of trapezium ABCE \leq Area under curve $f \leq$ Area of rectangle ABCD

$$\frac{1}{2} (f(a) + f(b)) (b-a) \leq \int_a^b f(x) dx \leq (b-a) \cdot f(b)$$

Problem 125. Show that $0 < \int_0^1 \frac{x^7}{\sqrt[3]{1+x^9}} dx < \frac{1}{8}$

For $x \in (0, 1)$,

$$\frac{x^7}{\sqrt[3]{1+x^9}} > 0$$

By comparison theorem,

$$\int_0^1 \frac{x^7}{\sqrt[3]{1+x^9}} dx > 0$$

Also, for $x \in (0, 1)$,

$$\Rightarrow \frac{x^7}{\sqrt[3]{1+x^9}} < \frac{x^7}{1}$$

By comparison theorem,

$$\Rightarrow \int_0^1 \frac{x^7}{\sqrt[3]{1+x^9}} dx < \int_0^1 x^7 dx$$

$$\Rightarrow \int_0^1 \frac{x^7}{\sqrt[3]{1+x^9}} dx < \left[\frac{x^8}{8} \right]_0^1$$

$$\Rightarrow \int_0^1 \frac{x^7}{\sqrt[3]{1+x^9}} dx < \frac{1}{8}$$

Therefore,

$$0 < \int_0^1 \frac{x^7}{\sqrt[3]{1+x^9}} dx < \frac{1}{8}$$

Problem 126. Show that $0 < \int_0^1 e^{x^2} dx < e$

For $x \in (0, 1)$,

$$x^2 < x$$

Since $\exp(x)$ is an increasing function,

$$0 < e^{x^2} < e^x$$

Apply comparison theorem,

$$\int_0^1 0 dx < \int_0^1 e^{x^2} dx < \int_0^1 e^x dx$$

$$\Rightarrow 0 < \int_0^1 e^{x^2} dx < [e^x]_0^1$$

$$\Rightarrow 0 < \int_0^1 e^{x^2} dx < e - 1 < e$$

$$\Rightarrow 0 < \int_0^1 e^{x^2} dx < e$$

Problem 127. Estimate $\int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx$

We'll approach this with multiple methods.

Method 1

$$\text{Let } f(x) = \frac{\sin(x)}{x}$$

$$\begin{aligned} \Rightarrow f'(x) &= \frac{x \cos(x) - \sin(x)}{x^2} \\ &= \frac{\cos(x) [x - \tan(x)]}{x^2} \end{aligned}$$

Now, for $x \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$,

$$\begin{aligned} \cos(x) &> 0, \quad x^2 > 0 \quad \text{and} \quad \tan(x) > x \\ \Rightarrow f'(x) &< 0 \end{aligned}$$

$\Rightarrow f(x)$ is strictly decreasing for $x \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$.

Also, since $f(x)$ is continuous in $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$,

$$f\left(\frac{\pi}{4}\right) \geq f(x) \geq f\left(\frac{\pi}{3}\right)$$

So for $x \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$,

$$\begin{aligned} \Rightarrow \frac{\sin\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)} &< \frac{\sin(x)}{x} < \frac{\sin\left(\frac{\pi}{4}\right)}{\left(\frac{\pi}{4}\right)} \\ \Rightarrow \frac{3}{\pi} \cdot \frac{\sqrt{3}}{2} &< \frac{\sin(x)}{x} < \frac{4}{\pi} \cdot \frac{1}{\sqrt{2}} \end{aligned}$$

Using comparison theorem,

$$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{3\sqrt{3}}{2\pi} dx &< \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx < \int_{\pi/4}^{\pi/3} \frac{4}{\pi\sqrt{2}} dx \\ \Rightarrow \frac{3\sqrt{3}}{2\pi} \left[\frac{\pi}{3} - \frac{\pi}{4}\right] &< \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx < \frac{4}{\pi\sqrt{2}} \left[\frac{\pi}{3} - \frac{\pi}{4}\right] \\ \Rightarrow \frac{\sqrt{3}}{8} &< \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx < \frac{\sqrt{2}}{6} \end{aligned}$$

Method 2 For $x \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$,

$$\begin{aligned}\sin(x) &< x \\ \Rightarrow \frac{\sin(x)}{x} &< 1\end{aligned}$$

By Comparison Theorem,

$$\begin{aligned}\int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx &< \int_{\pi/4}^{\pi/3} 1 dx \\ \Rightarrow \int_{\pi/3}^{\pi/4} \frac{\sin(x)}{x} dx &< \frac{\pi}{4} - \frac{\pi}{3} \\ \Rightarrow \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx &< \frac{\pi}{12}\end{aligned}$$

Method 3

$$\begin{aligned}\text{Let } I &= \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx \\ \Rightarrow I &= \int_{\pi/4}^{\pi/3} \left(\frac{1}{x}\right) (\sin(x)) dx\end{aligned}$$

Using the General Mean Value Theorem for integrals, $\exists c \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ such that

$$\begin{aligned}I &= \frac{1}{c} \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx \\ &= \frac{1}{c} [-\cos(x)]_{\pi/4}^{\pi/3} \\ &= \frac{1}{c} \left[-\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right) \right] \\ &= \frac{1}{c} \left[\frac{1}{\sqrt{2}} - \frac{1}{2} \right]\end{aligned}$$

$$\text{Now, } \frac{\pi}{4} < c < \frac{\pi}{3} \Rightarrow \frac{3}{\pi} < \frac{1}{c} < \frac{4}{\pi}$$

$$\Rightarrow \frac{3}{\pi} \left[\frac{1}{\sqrt{2}} - \frac{1}{2} \right] < \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{x} dx < \frac{4}{\pi} \left[\frac{1}{\sqrt{2}} - \frac{1}{2} \right]$$

Problem 128. Show that $\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \frac{1}{10}$

For $x \in (0, 1)$,

$$\frac{x^9}{\sqrt{1+1}} \leq \frac{x^9}{\sqrt{1+x}} \leq \frac{x^9}{1}$$

Using Comparison Theorem,

$$\begin{aligned} \int_0^1 \frac{x^9}{\sqrt{2}} dx &\leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \int_0^1 \frac{x^9}{1} dx \\ \Rightarrow \frac{1}{\sqrt{2}} \left[\frac{x^{10}}{10} \right]_0^1 &\leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \left[\frac{x^{10}}{10} \right]_0^1 \\ &\Rightarrow \frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \frac{1}{10} \end{aligned}$$

Problem 129. Show that $0.692 \leq \int_0^1 x^x dx \leq 1$

$$x^x \leq 1 \quad \forall x \in (0, 1)$$

Applying Comparison Theorem

$$\int_0^1 x^x dx \leq 1$$

Now, let $f(x) = x^x$

$$\Rightarrow f'(x) = x^x [1 + \ln(x)]$$

$$f'(x) = 0 \Leftrightarrow x = \frac{1}{e}$$

$f(x)$ attains its minimum value at $x = \frac{1}{e}$

So the minimum value of f becomes $f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}$

$$\Rightarrow \left(\frac{1}{e}\right)^{\frac{1}{e}} \leq x^x \quad \forall x \in (0, 1)$$

Using Comparison Theorem,

$$\begin{aligned} \Rightarrow \left(\frac{1}{e}\right)^{1/e} &\leq \int_0^1 x^x dx \\ \Rightarrow 0.692 &\leq \int_0^1 x^x dx \\ \Rightarrow 0.692 &\leq \int_0^1 x^x dx \leq \frac{1}{10} \end{aligned}$$

Problem 130. Show that $0.41 < \int_0^1 x^{\sin(x)+\cos(x)} dx < 0.5$

We know that

$$\begin{aligned} 1 &< \sin(x) + \cos(x) < \sqrt{2} \quad \forall x \in (0, 1) \\ \Rightarrow x^{\sqrt{2}} &< x^{\sin(x)+\cos(x)} < x \quad [\because x \in (0, 1)] \end{aligned}$$

Using Comparison Theorem,

$$\begin{aligned} \Rightarrow \int_0^1 x^{\sqrt{2}} dx &< \int_0^1 x^{\sin(x)+\cos(x)} dx < \int_0^1 x dx \\ \Rightarrow \frac{1}{\sqrt{2}+1} &< \int_0^1 x^{\sin(x)+\cos(x)} dx < \frac{1}{2} \\ \Rightarrow \sqrt{2}-1 &< \int_0^1 x^{\sin(x)+\cos(x)} dx < \frac{1}{2} \end{aligned}$$

Since $0.41 < \sqrt{2}-1$,

$$\Rightarrow 0.41 < \int_0^1 x^{\sin(x)+\cos(x)} dx < 0.5$$

Problem 131. Show that $4 \leq \int_1^3 \sqrt{3+x^3} dx \leq 2\sqrt{30}$

$$\text{Let } f(x) = \sqrt{3+x^3}$$

$$f'(x) = \frac{1}{2\sqrt{3+x^3}} \cdot 3x^2$$

Since $f'(x) > 0 \forall x \in (1, 3)$, f is strictly increasing.

$$\begin{aligned} \Rightarrow f(1) &\leq f(x) \leq f(3) \quad \forall x \in (1, 3) \\ \Rightarrow \sqrt{3+1^3} &\leq \sqrt{3+x^3} \leq \sqrt{3+3^3} \\ \Rightarrow 2 &\leq \sqrt{3+x^3} \leq 2\sqrt{3} \end{aligned}$$

Using Comparison Theorem,

$$\begin{aligned} 2 \cdot (3-1) &\leq \int_1^3 \sqrt{3+x^3} dx \leq 2\sqrt{3} \cdot (3-1) \\ \Rightarrow 4 &\leq \int_1^3 \sqrt{3+x^3} dx \leq 4\sqrt{3} < 2\sqrt{30} \\ \Rightarrow 4 &\leq \int_1^3 \sqrt{3+x^2} dx \leq 2\sqrt{30} \end{aligned}$$

Problem 132. Show that $\int_0^{\pi/2} e^{-R \sin(x)} dx < \frac{\pi}{2R} (1 - e^{-R})$ for $R > 0$.

The right-hand side can be written as

$$\frac{\pi}{2R} (1 - e^{-R}) = \int_0^{\pi/2} e^{\frac{-2Rx}{\pi}} dx$$

So we have to prove:

$$\int_0^{\pi/2} e^{-R \sin(x)} dx < \int_0^{\pi/2} e^{\frac{-2Rx}{\pi}} dx$$

So, by Comparison Theorem, if we can prove $e^{-R \sin(x)} < e^{\frac{-2Rx}{\pi}} \quad \forall x \in \left(0, \frac{\pi}{2}\right)$, we're done.

Proof. Consider $f(x) = \frac{\sin(x)}{x}$

We know by differentiation that $f(x)$ is continuous and decreasing in $\left(0, \frac{\pi}{2}\right]$,

So it takes its least value at $\frac{\pi}{2}$.

$$\Rightarrow \frac{\sin(x)}{x} < \frac{2}{\pi} \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

Since $x > 0$,

$$\Rightarrow \sin(x) < \frac{2x}{\pi} \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

Since $R > 0$,

$$\Rightarrow -R \sin(x) > \frac{-2Rx}{\pi}$$

Now, since e^x is strictly increasing,

$$\Rightarrow e^{-R \sin(x)} < e^{\frac{-2Rx}{\pi}} \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

Using Comparison Theorem,

$$\Rightarrow \int_0^{\pi/2} e^{-R \sin(x)} dx < \int_0^{\pi/2} e^{\frac{-2Rx}{\pi}} dx$$

□

Problem 133. Let $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$.

Then which of the following is/are correct?

- (A) $I < \frac{49}{50}$ (B) $I < \ln(99)$ (C) $I > \frac{49}{50}$ (D) $I > \ln(99)$

Solution: (B, C)

For $x \in (k, k+1)$,

$$\begin{aligned} \frac{k+1}{x(x+1)} &< \frac{\cancel{k+1}}{x(\cancel{k+1})} \\ \Rightarrow \frac{k+1}{x(x+1)} &< \frac{1}{x} \end{aligned}$$

Using Comparison Theorem,

$$\begin{aligned} \int_k^{k+1} \frac{k+1}{x(x+1)} &< \left[\ln(|x|) \right]_k^{k+1} \\ \int_k^{k+1} \frac{k+1}{x(x+1)} &< \ln(k+1) - \ln(k) \end{aligned}$$

Therefore,

$$\sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} < \sum_{k=1}^{98} (\ln(k+1) - \ln(k))$$

Now the right hand side becomes a telescopic sum. Let $v_k = \ln(k+1) - \ln(k)$

$$\begin{aligned} v_1 &= \ln(\cancel{2}) - \ln(1) \\ v_2 &= \ln(\cancel{3}) - \ln(\cancel{2}) \\ &\vdots \quad \quad \quad \vdots \\ v_{97} &= \ln(\cancel{98}) - \ln(\cancel{97}) \\ v_{98} &= \ln(99) - \ln(\cancel{98}) \\ \hline \sum_{k=1}^{98} v_k &= \ln(99) - \ln(1) = \ln(99) \\ \hline \end{aligned}$$

$$\therefore I < \ln(99)$$

Similarly, for $x \in (k, k+1)$,

$$\frac{\cancel{k+1}}{(\cancel{k+1})(x+1)} < \frac{k+1}{x(x+1)}$$

By Comparison Theorem,

$$I > \sum_{k=1}^{98} [\ln(k+2) - \ln(k+1)] = \ln(100) - \ln(2) = \ln(50)$$

$$\Rightarrow I > \ln(50) > \frac{49}{50}$$

Therefore, options B and C are correct.

Problem 134. Let $f(x)$ (where $x \geq 0$) be a continuous and non-negative function and let

$F(x) = \int_0^x f(t) dt$ ($x \geq 0$). If for some $c > 0$, $f(x) \leq cF(x) \forall x \geq 0$, show that $f(x) = 0 \forall x \geq 0$.

Proof.

$$F(x) = \int_0^x f(t) dt$$

Using the Fundamental Theorem of Calculus,

$$F'(x) = f(x)$$

Now, it is given that there exists $c > 0$ such that

$$\begin{aligned} f(x) &\leq cF(x) \quad \forall x \geq 0 \\ \Rightarrow F'(x) - cF(x) &\leq 0 \end{aligned}$$

Multiply by integrating factor e^{-cx}

$$\begin{aligned} \Rightarrow e^{-cx} F'(x) - c \cdot e^{-cx} F(x) &\leq 0 \\ \Rightarrow \frac{d}{dx} [e^{-cx} F(x)] &\leq 0 \end{aligned}$$

Let $g(x) = e^{-cx} F(x)$. Then $g(x)$ is non-increasing $\forall x \geq 0$

$$\Rightarrow g(x) \leq g(0) \quad \forall x \geq 0$$

$$\text{Now, } g(0) = e^0 F(0) = F(0) = \int_0^0 f(t) dt = 0$$

$$\begin{aligned} \Rightarrow g(x) &\leq 0 \quad \forall x \geq 0 \\ \Rightarrow e^{-cx} F(x) &\leq 0 \quad \forall x \geq 0 \end{aligned}$$

But $e^{-cx} > 0 \quad \forall x \in \mathbb{R}$,

$$\Rightarrow F(x) \leq 0 \quad \forall x \geq 0$$

Now, it is given that $f(x) \geq 0$. By Comparison Theorem, for all non-negative x ,

$$\begin{aligned} F(x) &= \int_0^x f(x) \geq 0 \quad \forall x \geq 0 \\ \Rightarrow F(x) &\leq 0 \quad \forall x \geq 0 \text{ and } F(x) \geq 0 \quad \forall x \geq 0 \\ \Rightarrow F(x) &= 0 \quad \forall x \geq 0 \end{aligned}$$

□

Problem 135. Let f be a function on $[a, b]$ such that $f(x) \geq 0$.

Prove that the expression $\int_a^b f(x) \, dx \cdot \int_a^b \frac{1}{f(x)} \, dx$ takes its least value when f is a constant function.

Proof. Using the Cauchy-Schwarz Inequality

$$\begin{aligned} \left(\int_a^b \sqrt{f(x)} \cdot \frac{1}{\sqrt{f(x)}} \, dx \right)^2 &\leq \int_a^b \left(\sqrt{f(x)} \right)^2 \, dx \cdot \int_a^b \left(\frac{1}{\sqrt{f(x)}} \right)^2 \, dx \\ &\Rightarrow \int_a^b f(x) \, dx \cdot \int_a^b \frac{1}{f(x)} \, dx \geq (b-a)^2 \end{aligned}$$

We know that for the Cauchy-Schwarz Inequality, the equality holds when $f(x) = \lambda g(x)$. So in this case, the expression will take its minimum value when the equality holds, i.e. when

$$\begin{aligned} f(x) &= \lambda \cdot \frac{1}{f(x)} \\ \Rightarrow (f(x))^2 &= \lambda \\ \Rightarrow |f(x)| &= \sqrt{\lambda} \end{aligned}$$

i.e. $f(x)$ is a constant function.

□

7.14 Reduction Formulae

7.14.1 The Beta Function

The Beta Function is defined as:

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Claim: $\beta(m, n) = \beta(n, m)$

Proof.

Using $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\begin{aligned}\beta(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ &= \int_0^1 (1-x)^{m-1} (1-1+x)^{n-1} dx \\ &= \int_0^1 (1-x)^{m-1} (x)^{n-1} dx \\ &= \beta(n, m)\end{aligned}$$

□

Finding a Reduction Formula

$$\beta(m+1, n+1) = \int_0^1 x^m (1-x)^n dx$$

Integrating by parts.

Let us choose $u = (1-x)^n$ and $dv = x^m dx$

Then $du = -n(1-x)^{n-1}$ and $v = \frac{x^{m+1}}{m+1}$

$$\int u dv = uv - \int v du$$

$$\begin{aligned}\Rightarrow \beta(m+1, n+1) &= \left[(1-x)^n \cdot \frac{x^{m+1}}{m+1} \right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1} \cdot (-n(1-x)^{n-1}) dx \\ &= 0 + \frac{n}{m+1} \int_0^1 x^{m+1} (1-x)^{n-1} dx \\ \Rightarrow \beta(m+1, n+1) &= \frac{n}{m+1} \beta(m+2, n)\end{aligned}$$

And we have our reduction formula.

Finding an explicit formula

$$\begin{aligned}
\beta(m+1, n+1) &= \frac{n}{m+1} \cdot \beta(m+2, n) \\
&= \frac{n}{m+1} \cdot \frac{(n-1)}{(m+2)} \cdot \beta(m+3, n-1) \\
&= \frac{n}{m+1} \cdot \frac{(n-1)}{(m+2)} \cdot \frac{(n-2)}{(m+3)} \cdot \beta(m+4, n-2) \\
&= \frac{n}{m+1} \cdot \frac{(n-1)}{(m+2)} \cdot \frac{(n-2)}{(m+3)} \cdot \dots \cdot \frac{2}{(m+n-1)} \cdot \frac{1}{(m+n)} \cdot \beta(m+n+1, 1) \\
&= \frac{n!}{(m+1)(m+2)(m+3) \cdots (m+n-1)(m+n)} \cdot \beta(m+n+1, 1) \\
&= \frac{n!}{(m+1)(m+2)(m+3) \cdots (m+n-1)(m+n)} \cdot \int_0^1 x^{m+n} (1-x)^0 dx \\
&= \frac{n!}{(m+1)(m+2)(m+3) \cdots (m+n-1)(m+n)} \cdot \int_0^1 x^{m+n} dx
\end{aligned}$$

Multiply and divide by $m!$

$$\begin{aligned}
&= \frac{n! \cdot m!}{m!(m+1)(m+2)(m+3) \cdots (m+n-1)(m+n)} \cdot \int_0^1 x^{m+n} dx \\
&= \frac{n! \cdot m!}{(m+n)!} \cdot \frac{1}{(m+n+1)} \\
&= \frac{1}{{}^{m+n}C_n} \cdot \frac{1}{(m+n+1)} \\
\therefore \beta(m+1, n+1) &= \int_0^1 x^m (1-x)^n dx = \frac{1}{{}^{m+n}C_n} \cdot \frac{1}{(m+n+1)}
\end{aligned}$$

This idea connects discrete and continuous mathematics – combinatorics and calculus. Such intersections of two seemingly divergent branches of mathematics don't appear very often. But when they do, they can produce some really elegant results.

There is another method to derive the same result. That involves introducing a parameter (say t) and invoking the Binomial Theorem. While we can surely use Reduction Formulae to solve problems, the other method has its own elegance, and can offer new insight into problem-solving.

Another Method (Variation of Parameter, Binomial Theorem)

$$\beta(m+1, n+1) = \int_0^1 x^m (1-x)^n dx$$

Wishful thinking: $x^m(1-x)^n$ is highly reminiscent of a term in a binomial expansion.

Assume that t is a parameter independent of x and consider the expansion of $(tx + (1-x))^{m+n}$.

$$\begin{aligned} \int_0^1 (tx + (1-x))^{m+n} dx &= \int_0^1 (x(t-1) + 1)^{m+n} dx \\ &= \left[\frac{(x(t-1) + 1)^{m+n+1}}{(m+n+1)(t-1)} \right]_0^1 \\ &= \frac{t^{m+n+1} - 1}{(m+n+1)(t-1)} \\ &= \frac{t^{m+n+1} - 1}{t-1} \cdot \frac{1}{m+n+1} \end{aligned}$$

Now, $\frac{t^{m+n+1} - 1}{t-1}$ can be thought of as the sum of a geometric progression.

$$\Rightarrow \int_0^1 (tx + (1-x))^{m+n} dx = \left(1 + t + t^2 + \dots + t^{m+n}\right) \cdot \frac{1}{m+n+1}$$

Now, using Binomial Theorem,

$$\begin{aligned} \int_0^1 (tx + (1-x))^{m+n} dx &= \int_0^1 \sum_{r=0}^{m+n} {}^{m+n}C_r \cdot (tx)^r \cdot (1-x)^{m+n-r} dx \\ &= \sum_{r=0}^{m+n} \int_0^1 {}^{m+n}C_r \cdot t^r x^r \cdot (1-x)^{m+n-r} dx \\ &= \sum_{r=0}^{m+n} {}^{m+n}C_r \cdot t^r \int_0^1 x^r (1-x)^{m+n-r} dx \end{aligned}$$

This should give us a polynomial in t . However, we have already evaluated the integral as a polynomial in t . Equating them now.

$$\Rightarrow \sum_{r=0}^{m+n} {}^{m+n}C_r \cdot t^r \int_0^1 x^r (1-x)^{m+n-r} dx = \left(1 + t + t^2 + \dots + t^{m+n}\right) \cdot \frac{1}{m+n+1}$$

Equating the coefficient of t^m from both sides.

$${}^{m+n}C_m \cdot \int_0^1 x^m (1-x)^n dx = \frac{1}{m+n+1}$$

Isolate the integral, and we have our result:

$$\beta(m+1, n+1) = \int_0^1 x^m (1-x)^n dx = \frac{1}{{}^{m+n}C_m} \cdot \frac{1}{m+n+1}$$

$$\beta(m+1, n+1) = \int_0^1 x^m (1-x)^n dx = \frac{1}{m+n+1} \cdot \frac{1}{m+n+1}$$

A Special Case

Note that the following is just a special case of the above result:

$$\begin{aligned}\beta(r+1, n-r+1) &= \int_0^1 x^r (1-x)^{n-r} dx = \frac{1}{n+1} \cdot \frac{1}{n+1} \\ \Rightarrow \int_0^1 x^r (1-x)^{n-r} dx &= \frac{1}{(n+1) \cdot {}^nC_r} \\ \Rightarrow \frac{1}{n+1} &= (n+1) \int_0^1 x^r (1-x)^{n-r} dx\end{aligned}$$

This can serve as a nice result for problem-solving.

Problem 136. $\int_0^1 x^{70} (1-x)^{30} dx$

Solution: (Directly using Beta Function)

$$\begin{aligned}\int_0^1 x^{70} (1-x)^{30} dx &= \beta(71, 31) = \frac{1}{70+30} \cdot \frac{1}{70+30+1} \\ &= \frac{1}{101 \cdot {}^{100}C_{30}}\end{aligned}$$

Problem 137. Evaluate $\sum_{r=0}^n \frac{(-1)^r}{{}^nC_r}$

Solution: Note that this problem could be easily solved with prior knowledge on an ad-hoc basis if n were odd.

For the case where n is odd,

Using ${}^nC_r = {}^nC_{n-r}$

$$\begin{aligned}\sum_{r=0}^n \frac{(-1)^r}{{}^nC_r} &= \frac{1}{{}^nC_0} - \frac{1}{{}^nC_1} + \cdots - \frac{1}{{}^nC_{(n-1)/2}} + \frac{1}{{}^nC_{(n+1)/2}} - \cdots + \frac{1}{{}^nC_{n-1}} - \frac{1}{{}^nC_n} \\ &= \frac{1}{\cancel{{}^nC_0}} - \frac{1}{\cancel{{}^nC_1}} + \cdots - \frac{1}{\cancel{{}^nC_{(n-1)/2}}} + \frac{1}{\cancel{{}^nC_{(n-1)/2}}} - \cdots + \frac{1}{\cancel{{}^nC_1}} - \frac{1}{\cancel{{}^nC_0}} \\ &= 0\end{aligned}$$

If n were odd, everything would beautifully cancel, and give us 0.

However, we are interested in the general case, where n is any non-negative integer.

For that we'll make the substitution for $\frac{1}{{}^nC_r}$ that we derived above:

$$\frac{1}{{}^nC_r} = (n+1) \int_0^1 x^r (1-x)^{n-r} dx$$

$$\begin{aligned}
\sum_{r=0}^n \frac{(-1)^r}{n C_r} &= \sum_{r=0}^n (-1)^r \cdot (n+1) \int_0^1 x^r (1-x)^{n-r} dx \\
&= (n+1) \sum_{r=0}^n \int_0^1 (-1)^r \cdot x^r (1-x)^{n-r} dx \\
&= (n+1) \int_0^1 \sum_{r=0}^n (-1)^r \cdot x^r (1-x)^{n-r} dx \\
&= (n+1) \int_0^1 \left[(1-x)^n - x(1-x)^{n-1} + x^2(1-x)^{n-2} - \dots + (-1)^n x^n \right] dx
\end{aligned}$$

Now, notice that $[(1-x)^n - x(1-x)^{n-1} + x^2(1-x)^{n-2} - \dots + (-1)^n x^n]$ is just the sum of terms of a geometric progression of $n+1$ terms with common ratio $\frac{-x}{1-x}$ and first term $(1-x)^n$. So we can use the GP sum formula.

$$\begin{aligned}
\Rightarrow \sum_{r=0}^n \frac{(-1)^r}{n C_r} &= (n+1) \int_0^1 (1-x)^n \cdot \left(\frac{1 - \left(\frac{-x}{1-x} \right)^{n+1}}{1 - \left(\frac{-x}{1-x} \right)} \right) dx \\
&= (n+1) \int_0^1 (1-x)^{n+1} \cdot \left(1 - \left(\frac{-x}{1-x} \right)^{n+1} \right) dx \\
&= (n+1) \int_0^1 \left((1-x)^{n+1} - (1-x)^{n+1} \cdot \left(\frac{-x}{1-x} \right)^{n+1} \right) dx \\
&= (n+1) \int_0^1 \left((1-x)^{n+1} - (-x)^{n+1} \right) dx \\
&= (n+1) \int_0^1 \left((1-x)^{n+1} - (-1)^{n+1} (x)^{n+1} \right) dx \\
&= (n+1) \int_0^1 \left((1-x)^{n+1} + (-1)^{n+2} (x)^{n+1} \right) dx \\
&= (n+1) \left[\int_0^1 (1-x)^{n+1} dx + (-1)^{n+2} \int_0^1 x^{n+1} dx \right] \\
&= (n+1) \left\{ \left[\frac{-(1-x)^{n+2}}{n+2} \right]_0^1 + (-1)^{n+2} \left[\frac{x^{n+2}}{n+2} \right]_0^1 \right\} \\
\Rightarrow \sum_{r=0}^n \frac{(-1)^r}{n C_r} &= (n+1) \left[\frac{1}{n+2} + \frac{(-1)^{n+2}}{n+2} \right] = \frac{(n+1)}{(n+2)} [1 + (-1)^n]
\end{aligned}$$

And we have our answer for the general case. Also we can see from the general formula that the sum will equal 0 if n is odd.

7.14.2 Wallis Product and $\int_0^{\pi/2} \sin^n(x) dx$

Finding a Reduction Formula

Let $I_n = \int_0^{\pi/2} \sin^n(x) dx$

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin(x) dx \\ &= \int_0^{\pi/2} \sin^{n-1}(x) \cdot \sin(x) dx \end{aligned} \quad \text{Integrating by parts}$$

Let $u = \sin^{n-1}(x)$ and $dv = \sin(x) dx$
 Then $du = (n-1) \sin^{n-2}(x) \cos(x) dx$ and $v = -\cos(x)$
 $\int u dv = uv - \int v du$

$$\begin{aligned} \Rightarrow I_n &= \left[-\sin^{n-1}(x) \cos(x) \right]_0^{\pi/2} - \int_0^{\pi/2} -\cos^2(x) \cdot (n-1) \sin^{n-2}(x) dx \\ &= -\left[\sin^{n-1}(x) \cos(x) \right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^2(x) \sin^{n-2}(x) dx \\ &= 0 + (n-1) \int_0^{\pi/2} [1 - \sin^2(x)] \sin^{n-2}(x) dx \\ &= (n-1) \int_0^{\pi/2} (\sin^{n-2}(x) - \sin^n(x)) dx \\ &= (n-1) \left[\int_0^{\pi/2} \sin^{n-2}(x) dx - \int_0^{\pi/2} \sin^n(x) dx \right] \\ &= (n-1) I_{n-2} - (n-1) I_n \\ \Rightarrow I_n + (n-1) I_n &= (n-1) I_{n-2} \\ \Rightarrow n \cdot I_n &= (n-1) I_{n-2} \\ \Rightarrow I_n &= \left(\frac{n-1}{n} \right) I_{n-2} \end{aligned}$$

And we have our reduction formula

$$I_n = \frac{n-1}{n} \cdot I_{n-2}$$

Finding an Explicit Formula

Case 1: If n is odd, i.e. $n = 2m + 1$ for some m

$$\begin{aligned}
 I_n &= \frac{n-1}{n} \cdot I_{n-2} \\
 \Rightarrow I_{2m+1} &= \frac{2m}{2m+1} \cdot I_{2m-1} \\
 &= \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot I_{2m-3} \\
 &= \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdot \dots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \\
 &= \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdot \dots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \int_0^{\pi/2} \sin(x) \, dx \\
 &= \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdot \dots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot [-\cos(x)]_0^{\pi/2} \\
 &= \frac{(2m)(2m-2)(2m-4) \cdot \dots \cdot 4 \cdot 2}{(2m+1)(2m-1)(2m-3) \cdot \dots \cdot 5 \cdot 3} \quad \begin{array}{l} \text{[Multiply and} \\ \text{divide by} \\ \text{numerator]} \end{array} \\
 &= \frac{(2m)(2m-2)(2m-4) \cdot \dots \cdot 4 \cdot 2}{(2m+1)(2m-1)(2m-3) \cdot \dots \cdot 5 \cdot 3} \cdot \frac{(2m)(2m-2)(2m-4) \cdot \dots \cdot 4 \cdot 2}{(2m)(2m-2)(2m-4) \cdot \dots \cdot 4 \cdot 2} \\
 &= \frac{(2^m \cdot m!)^2}{(2m+1)!} \\
 &\Rightarrow \int_0^{\pi/2} \sin^{2m+1}(x) \, dx = \frac{(2^m \cdot m!)^2}{(2m+1)!}
 \end{aligned}$$

Case - 2: If n is even, i.e. $n = 2m$ for some m

$$\begin{aligned}
 I_n &= \frac{n-1}{n} \cdot I_{n-2} \\
 \Rightarrow I_{2m} &= \frac{2m-1}{2m} \cdot I_{2m-2} \\
 &= \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot I_{2m-4} \\
 &= \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \dots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 \\
 &= \frac{(2m-1)(2m-3) \cdot \dots \cdot 3 \cdot 1}{(2m)(2m-2) \cdot \dots \cdot 4 \cdot 2} \cdot \int_0^{\pi/2} \sin^0(x) \overset{1}{\rightarrow} dx \\
 &= \frac{(2m-1)(2m-3) \cdot \dots \cdot 3 \cdot 1}{(2m)(2m-2) \cdot \dots \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \quad \begin{array}{l} \text{[Multiply and} \\ \text{divide by} \\ (2m) \cdot \dots \cdot 4 \cdot 2] \end{array} \\
 &= \frac{(2m-1)(2m-3) \cdot \dots \cdot 3 \cdot 1}{(2m)(2m-2) \cdot \dots \cdot 4 \cdot 2} \cdot \frac{(2m)(2m-2) \cdot \dots \cdot 4 \cdot 2}{(2m)(2m-2) \cdot \dots \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \\
 &= \frac{(2m)!}{(2^m \cdot m!)^2} \\
 &\Rightarrow \int_0^{\pi/2} \sin^{2m}(x) \, dx = \frac{(2m)!}{(2^m \cdot m!)^2}
 \end{aligned}$$

Now,

$$\sin^{2m+1}(x) \leq \sin^{2m}(x) \leq \sin^{2m-1}(x) \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

Using Comparison Theorem,

$$\begin{aligned} \Rightarrow \int_0^{\pi/2} \sin^{2m+1}(x) dx &\leq \int_0^{\pi/2} \sin^{2m}(x) dx \leq \int_0^{\pi/2} \sin^{2m-1}(x) dx \\ \Rightarrow I_{2m+1} &\leq I_{2m} \leq I_{2m-1} \end{aligned}$$

Divide by I_{2m+1} . This won't change the sign as $I_{2m+1} > 0$ (By Comparison Theorem).

$$\Rightarrow 1 \leq \frac{I_{2m}}{I_{2m+1}} \leq \frac{I_{2m-1}}{I_{2m+1}} \quad (1)$$

Now,

$$\begin{aligned} I_{2m+1} &= \frac{(2^m \cdot m!)^2}{(2m+1)!} \\ I_{2m-1} &= I_{2 \cdot (m-1) + 1} = \frac{(2^{(m-1)} \cdot (m-1)!)^2}{(2m-1)!} \\ \Rightarrow \frac{I_{2m-1}}{I_{2m+1}} &= \frac{\left(\frac{(2^{m-1} \cdot (m-1)!)^2}{(2m-1)!}\right)}{\left(\frac{(2^m \cdot m!)^2}{(2m+1)!}\right)} = \frac{2m+1}{2m} = 1 + \frac{1}{2m} \end{aligned}$$

Substitute in (1),

$$\Rightarrow 1 \leq \frac{I_{2m}}{I_{2m+1}} \leq 1 + \frac{1}{2m}$$

Now,

$$\lim_{m \rightarrow \infty} \left(1 + \frac{1}{2m}\right) = 1 + 0 = 1$$

Therefore, using Squeeze Theorem,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{I_{2m}}{I_{2m+1}} &= 1 \\ \Rightarrow \lim_{m \rightarrow \infty} I_{2m} \cdot \frac{1}{I_{2m+1}} &= 1 \\ \Rightarrow \lim_{m \rightarrow \infty} \left(\frac{(2m-1)(2m-3) \cdots 3 \cdot 1}{(2m)(2m-2) \cdots 4 \cdot 2} \cdot \frac{\pi}{2} \cdot \frac{(2m+1)(2m-1) \cdots 5 \cdot 3}{(2m)(2m-2) \cdots 4 \cdot 2} \right) &= 1 \\ \Rightarrow \lim_{m \rightarrow \infty} \left(\frac{(2m)(2m-2) \cdots 4 \cdot 2}{(2m-1)(2m-3) \cdots 3 \cdot 1} \cdot \frac{2}{\pi} \cdot \frac{(2m)(2m-2) \cdots 4 \cdot 2}{(2m+1)(2m-1) \cdots 5 \cdot 3} \right) &= 1 \\ \Rightarrow \lim_{m \rightarrow \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2m) \cdot (2m)}{1 \cdot 1 \cdot 3 \cdot 3 \cdots (2m-1)(2m-1)(2m+1)} &= \frac{\pi}{2} \\ \Rightarrow \lim_{m \rightarrow \infty} \frac{2 \cdot 4 \cdots (2m)}{1 \cdot 3 \cdots (2m-1)} \cdot \frac{1}{\sqrt{2m+1}} &= \sqrt{\frac{\pi}{2}} \end{aligned}$$

This is known as the Wallis Product.

7.14.3 Another Wallis Integral

Let us define

$$J_{p,q} = \int_0^{\pi/2} \sin^p(\theta) \cdot \cos^q(\theta) \, d\theta$$

Finding a Reduction Formula

Claim: $J_{p,q} = \frac{p-1}{p+q} \cdot J_{p,q-2}$

Proof.

$$J_{p,q} = \int_0^{\pi/2} \sin^p(\theta) \cdot \cos^{q-1}(\theta) \cdot \cos(\theta) \, d\theta$$

Integrating by parts

Let $u = \sin^p(\theta) \cdot \cos^{q-1}(\theta)$ and $dv = \cos(\theta) \, d\theta$

Then,

$$du = \left[p \cdot \sin^{p-1}(\theta) \cos^q(\theta) - (q-1) \cos^{q-2}(\theta) \cdot \sin^{p+1}(\theta) \right] d\theta$$

And

$$v = \sin(\theta)$$

$$\int u \, dv = uv - \int v \, du$$

$$\Rightarrow J_{p,q} = \left[\sin^{p+1}(\theta) \cdot \cos^{q-1}(\theta) \right]_0^{\pi/2} - \int_0^{\pi/2} \left[p \cdot \sin^p(\theta) \cos^q(\theta) - (q-1) \cos^{q-2}(\theta) \cdot \sin^{p+2}(\theta) \right] d\theta$$

$$= 0 - p \int_0^{\pi/2} \sin^p(\theta) \cdot \cos^q(\theta) \, d\theta + (q-1) \int_0^{\pi/2} \cos^{q-2}(\theta) \cdot \sin^{p+2}(\theta) \, d\theta$$

$$= -p \cdot J_{p,q} + (q-1) \int_0^{\pi/2} \cos^{q-2}(\theta) \sin^p(\theta) \cdot \sin^2(\theta) \, d\theta$$

$$\Rightarrow J_{p,q} \cdot (1+p) = (q-1) \int_0^{\pi/2} \cos^{q-2}(\theta) \cdot \sin^p(\theta) \cdot (1 - \cos^2(\theta)) \, d\theta$$

$$\Rightarrow J_{p,q} = \frac{(q-1)}{(p+1)} \cdot \left[\int_0^{\pi/2} \cos^{q-2}(\theta) \cdot \sin^p(\theta) \, d\theta - \int_0^{\pi/2} \sin^p(\theta) \cdot \cos^q(\theta) \, d\theta \right]$$

$$= \frac{(q-1)}{(p+1)} \cdot [J_{p,q-2} - J_{p,q}]$$

$$\Rightarrow J_{p,q} \cdot \left[1 + \frac{q-1}{p+1} \right] = \frac{q-1}{p+1} \cdot J_{p,q-2}$$

$$\Rightarrow J_{p,q} \cdot \left[\frac{p+q}{p+1} \right] = \frac{q-1}{p+1} \cdot J_{p,q-2}$$

$$\Rightarrow J_{p,q} = \frac{q-1}{p+q} \cdot J_{p,q-2}$$

□

Relation Between the Beta Function and $J_{p,q}$

We have

$$\beta(m+1, n+1) = \int_0^1 x^m (1-x)^n dx$$

Let us substitute $x = \sin^2(\theta)$. Note that this substitution is valid because $\sin^2(\theta)$ is one-one in $\left(0, \frac{\pi}{2}\right)$.

$$dx = 2 \sin(\theta) \cdot \cos(\theta) d\theta$$

$$\Rightarrow \beta(m+1, n+1) = \int_0^{\pi/2} \sin^{2m}(\theta) \cdot [1 - \sin^2(\theta)]^n \cdot 2 \sin(\theta) \cos(\theta) d\theta$$

$$= \int_0^{\pi/2} \sin^{2m}(\theta) \cdot \cos^{2n}(\theta) \cdot 2 \sin(\theta) \cos(\theta) d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m+1}(\theta) \cos^{2n+1}(\theta) d\theta$$

$$= 2 \cdot J_{2m+1, 2n+1}$$

$$\text{Using } J_{p,q} = \frac{q-1}{p+q} \cdot J_{p,q-2}$$

$$= 2 \cdot \frac{2n}{2m+2n+2} \cdot J_{2m+1, 2n-1}$$

$$= 2 \cdot \frac{2n}{2m+2n+2} \cdot \frac{2n-2}{2m+2n} J_{2m+1, 2n-3}$$

$$= 2 \cdot \frac{2n}{2m+2n+2} \cdot \frac{2n-2}{2m+2n} \cdot \dots \cdot \frac{2}{2m+4} J_{2m+1, 1}$$

$$= 2 \cdot \frac{2n}{2m+2n+2} \cdot \frac{2n-2}{2m+2n} \cdot \dots \cdot \frac{2}{2m+4} \int_0^{\pi/2} \sin^{2m+1}(\theta) \cos(\theta) d\theta$$

$$\begin{aligned} \text{Let } \sin(\theta) &= t \\ \Rightarrow \cos(\theta) d\theta &= dt \end{aligned}$$

$$= 2 \cdot \frac{2n}{2m+2n+2} \cdot \frac{2n-2}{2m+2n} \cdot \dots \cdot \frac{2}{2m+4} \cdot \int_0^1 t^{2m+1} dt$$

$$= 2 \cdot \frac{2n}{2(m+n+1)} \cdot \frac{2(n-1)}{2(m+n)} \cdot \dots \cdot \frac{2}{2(m+2)} \cdot \left(\frac{1}{2m+2} \right)$$

$$= \frac{n}{(m+n+1)} \cdot \frac{(n-1)}{(m+n)} \cdot \dots \cdot \frac{1}{(m+2)} \cdot \left(\frac{2}{2(m+1)} \right)$$

Multiply and divide by $m!$

$$= \frac{1}{m+n+1} \cdot \frac{m! \cdot n!}{(m+n) \cdots (m+2)(m+1)m!}$$

$$= \frac{1}{m+n+1} \cdot \frac{m! \cdot n!}{(m+n)!}$$

$$= \frac{1}{m+n+1} \cdot \frac{1}{m+n C_m}$$

Problem 138. Let $I_n = \int_0^{\pi/2} \theta \cdot \sin^n(\theta) d\theta$. Then prove that $I_n = \frac{n-1}{n} \cdot I_{n-2} + \frac{1}{n^2}$.

Proof.

$$\begin{aligned} I_n &= \int_0^{\pi/2} \theta \cdot \sin^n(\theta) d\theta \\ &= \int_0^{\pi/2} \sin^{n-1}(\theta) \cdot \theta \cdot \sin^{n-1}(\theta) d\theta \end{aligned}$$

Integrating by parts,

Let $u = \sin^{n-1}(\theta)$ and $dv = \theta \cdot \sin(\theta) d\theta$

$\Rightarrow du = (n-1) \sin^{n-2}(\theta) \cdot \cos(\theta) d\theta$

And $v = \int \theta \cdot \sin(\theta) d\theta$.

Let $p = \theta$ and $dq = \sin(\theta) d\theta$

$\Rightarrow v = \int p dq = pq - \int q dp = \sin(\theta) - \theta \cdot \cos(\theta)$

$\int u dv = uv - \int v du$

$$\Rightarrow I_n = \left[\sin^n(\theta) - \theta \cdot \sin^{n-1}(\theta) \cdot \cos(\theta) \right]_0^{\pi/2} - \int_0^{\pi/2} [\sin(\theta) - \theta \cdot \cos(\theta)] \cdot (n-1) \sin^{n-2}(\theta) \cdot \cos(\theta) d\theta$$

$$= 1 - (n-1) \int_0^{\pi/2} [\sin(\theta) - \theta \cdot \cos(\theta)] \sin^{n-2}(\theta) \cdot \cos(\theta) d\theta$$

$$= 1 - (n-1) \left[\int_0^{\pi/2} \sin^{n-1}(\theta) \cdot \cos(\theta) d\theta - \int_0^{\pi/2} \theta \cdot \sin^{n-2}(\theta) \cdot \cos^2(\theta) d\theta \right]$$

Let $\sin(\theta) = t \Rightarrow \cos(\theta) d\theta = dt$

$$\Rightarrow I_n = 1 - (n-1) \left[\int_0^1 t^{n-1} dt - \int_0^{\pi/2} \theta \cdot \sin^{n-2}(\theta) \cdot (1 - \sin^2(\theta)) d\theta \right]$$

$$\Rightarrow I_n = 1 - (n-1) \left[\frac{1}{n} - \int_0^{\pi/2} \theta \cdot \sin^{n-2}(\theta) d\theta + \int_0^{\pi/2} \theta \sin^n(\theta) d\theta \right]$$

$$= 1 - \frac{n-1}{n} + (n-1) \cdot I_{n-2} - (n-1) \cdot I_n$$

$$\Rightarrow I_n \cdot (n) = \cancel{1} - \cancel{1} + \frac{1}{n} + (n-1) \cdot I_{n-2}$$

$$\Rightarrow I_n = \frac{1}{n^2} + \frac{n-1}{n} \cdot I_{n-2}$$

□

Problem 139. Show that $\int_0^1 (1-x^2)^n dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2n}{2n+1}$

Proof.

$$\text{Let } I_n = \int_0^1 (1-x^2)^n dx$$

Substitute $x = \sin(\theta)$
 $\Rightarrow dx = \cos(\theta) d\theta$

$$\begin{aligned} \Rightarrow I_n &= \int_0^{\pi/2} (1 - \sin^2(\theta)) \cdot \cos(\theta) d\theta \\ &= \int_0^{\pi/2} (\cos^2(\theta))^n \cdot \cos(\theta) d\theta \\ &= \int_0^{\pi/2} (\cos^{2n+1}(\theta)) d\theta \\ &= \int_0^{\pi/2} \sin^{2n+1}(\theta) d\theta \\ &= \frac{(2^n \cdot n!)^2}{(2n+1)!} \\ &= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2n}{2n+1} \end{aligned}$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

This is just the Wallis integral

□

Problem 140. Show that $\int_0^\infty \frac{1}{(1+x^2)^n} dx = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-3}{2n-2} \cdot \frac{\pi}{2}$

Proof. (Similar to the previous problem)

$$\text{Let } I_n = \int_0^\infty \frac{1}{(1+x^2)^n} dx$$

Substitute $x = \tan(\theta)$
 $\Rightarrow dx = \sec^2(\theta) d\theta$

$$\begin{aligned} \Rightarrow I_n &= \int_0^{\pi/2} \frac{1}{(1 + \tan^2(\theta))^n} \cdot \sec^2(\theta) d\theta \\ &= \int_0^{\pi/2} \frac{1}{\sec^{2n-2}(\theta)} d\theta = \int_0^{\pi/2} \cos^{2n-1}(\theta) d\theta \\ &= \int_0^{\pi/2} \sin^{2n-2}(\theta) d\theta = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-3}{2n-2} \cdot \frac{\pi}{2} \end{aligned}$$

□

7.14.4 Finding $\int_0^{\infty} e^{-x^2} dx$

This is going to be interesting because the antiderivative of e^{-x^2} cannot be expressed in terms of elementary functions, meaning we can't find it using the tools we have. However, we can find out the value of the definite integral $\int_0^{\infty} e^{-x^2} dx$, which we will attempt now.

Claim 1: $1 - x^2 \leq e^{-x^2}$ for $0 \leq x \leq 1$

Proof. Let $x^2 = t$

$$0 \leq x \leq 1 \implies 0 \leq x^2 \leq 1 \implies 0 \leq t \leq 1$$

Let $f(t) = e^{-t} + t - 1$

Then, for $0 \leq t \leq 1$,

$$f(t) = e^{-t} + t - 1$$

$$\Rightarrow f'(t) = -e^{-t} + 1 = 1 - e^{-t} = 1 - \frac{1}{e^t} \geq 0$$

$\Rightarrow f$ is strictly increasing.

$$\Rightarrow f(t) > f(0) \quad \forall t > 0$$

Also,

$$f(0) = 0$$

$$\Rightarrow f(t) \geq 0 \text{ for } 0 \leq t \leq 1$$

$$\Rightarrow f(x^2) \geq 0 \text{ for } 0 \leq x \leq 1$$

$$\Rightarrow e^{-x^2} \geq 1 - x^2 \quad \forall x \in [0, 1]$$

□

Claim 2: $e^{-x^2} \leq \frac{1}{1+x^2}$ for $x \geq 0$

Proof. Again, let $x^2 = t$

$$x \geq 0 \implies x^2 \geq 0 \implies t \geq 0$$

Let $f(t) = (1+t) \cdot e^{-t} - 1$

Then, for $t \geq 0$,

$$f(t) = (t+1) \cdot e^{-t} - 1$$

$$\Rightarrow f'(t) = -(t+1) \cdot e^{-t} + e^{-t} = -t \cdot e^{-t} \leq 0$$

$\Rightarrow f$ is strictly decreasing

$$\Rightarrow f(t) < f(0) \text{ for } t > 0$$

Also,

$$f(0) = -1$$

$$\Rightarrow f(t) \leq 0 \text{ for } t \geq 0$$

$$\Rightarrow f(x^2) \leq 0 \text{ for } x^2 \geq 0$$

$$\Rightarrow (1+x^2) \cdot e^{-x^2} - 1 \leq 0 \quad \forall x \geq 0$$

$$\Rightarrow e^{-x^2} \leq \frac{1}{1+x^2} \quad \forall x \geq 0$$

□

Then, we have the following result for $x \in [0, 1]$:

$$1 - x^2 \leq e^{-x^2} \leq \frac{1}{1+x^2} \quad \forall x \in [0, 1]$$

[This is a nice inequality to remember.]

$$1 - x^2 \leq e^{-x^2} \leq \frac{1}{1+x^2} \quad \forall x \in [0, 1]$$

Raise to the n^{th} power (where $n \geq 0$). Note that we don't need to worry about the sign since all terms are non-negative.

$$(1 - x^2)^n \leq e^{-nx^2} \leq \frac{1}{(1+x^2)^n}$$

Using comparison theorem,

$$\begin{aligned} \int_0^1 (1 - x^2)^n dx &\leq \int_0^1 e^{-nx^2} dx \leq \int_0^1 \frac{1}{(1+x^2)^n} dx \\ \Rightarrow \int_0^1 (1 - x^2)^n dx &\leq \int_0^1 e^{-nx^2} dx \leq \int_0^1 \frac{1}{(1+x^2)^n} dx \leq \int_0^\infty \frac{1}{(1+x^2)^n} dx \\ &\Rightarrow \int_0^1 (1 - x^2)^n dx \leq \int_0^1 e^{-nx^2} dx \leq \int_0^\infty \frac{1}{(1+x^2)^n} dx \end{aligned}$$

Let $y = \sqrt{n} \cdot x$
 $\Rightarrow dy = \sqrt{n} \cdot dx$

$$\Rightarrow \int_0^1 (1 - x^2)^n dx \leq \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} dy \leq \int_0^\infty \frac{1}{(1+x^2)^n} dx$$

Multiply by \sqrt{n}

$$\Rightarrow \sqrt{n} \int_0^1 (1 - x^2)^n dx \leq \int_0^{\sqrt{n}} e^{-y^2} dy \leq \sqrt{n} \int_0^\infty \frac{1}{(1+x^2)^n} dx$$

Take the limit as $n \rightarrow \infty$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \sqrt{n} \int_0^1 (1 - x^2)^n dx &\leq \lim_{n \rightarrow \infty} \int_0^{\sqrt{n}} e^{-y^2} dy \leq \lim_{n \rightarrow \infty} \sqrt{n} \int_0^\infty \frac{1}{(1+x^2)^n} dx \\ \Rightarrow \lim_{n \rightarrow \infty} \sqrt{n} \int_0^1 (1 - x^2)^n dx &\leq \int_0^\infty e^{-y^2} dy \leq \lim_{n \rightarrow \infty} \sqrt{n} \int_0^\infty \frac{1}{(1+x^2)^n} dx \\ \Rightarrow \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} (1 - \sin^2(\theta))^n \cdot \cos(\theta) d\theta &\leq \int_0^\infty e^{-y^2} dy \leq \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \frac{1}{(1 + \tan^2(\theta))^n} \cdot \sec^2(\theta) d\theta \\ \Rightarrow \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \cos^{2n+1}(\theta) d\theta &\leq \int_0^\infty e^{-y^2} dy \leq \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \cos^{2n-2}(\theta) d\theta \\ \Rightarrow \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \sin^{2n+1}(\theta) d\theta &\leq \int_0^\infty e^{-y^2} dy \leq \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \sin^{2n-2}(\theta) d\theta \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \sin^{2n+1}(\theta) d\theta \leq \int_0^{\infty} e^{-y^2} dy \leq \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \sin^{2n-2}(\theta) d\theta$$

Now, we will use

$$\int_0^{\pi/2} \sin^{2n+1}(\theta) d\theta = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2n}{2n+1} \quad \text{and} \quad \int_0^{\pi/2} \sin^{2n-2}(\theta) d\theta = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-3}{2n-2} \cdot \frac{\pi}{2}$$

Also, Wallis Product:

$$\lim_{m \rightarrow \infty} \frac{2 \cdot 4 \cdot \dots \cdot (2m)}{1 \cdot 3 \cdot \dots \cdot (2m-1)} \cdot \frac{1}{\sqrt{2m+1}} = \sqrt{\frac{\pi}{2}}$$

So,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \sin^{2n+1}(\theta) d\theta &= \lim_{n \rightarrow \infty} \sqrt{n} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n+1} \\ &= \lim_{n \rightarrow \infty} \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{\sqrt{2n+1}} \cdot \frac{\sqrt{n}}{\sqrt{2n+1}} \quad \text{Using Wallis Product} \\ &= \sqrt{\frac{\pi}{2}} \cdot \lim_{n \rightarrow \infty} \sqrt{\frac{n}{2n+1}} \\ &= \sqrt{\frac{\pi}{2}} \cdot \lim_{n \rightarrow \infty} \sqrt{\frac{\mathcal{N}}{\mathcal{N}\left(2 + \frac{1}{n}\right)}} = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{2} \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{n} \int_0^{\pi/2} \sin^{2n-2}(\theta) d\theta &= \lim_{n \rightarrow \infty} \sqrt{n} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-3}{2n-2} \cdot \frac{\pi}{2} \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-1}{2n} \cdot \frac{\pi}{2} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{2n+1}} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-1}{2n} \cdot \sqrt{2n+1} \cdot \frac{\pi}{2} \quad \text{Using Wallis Product} \\ &= \sqrt{\frac{2}{\pi}} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{2n+1}} \cdot \frac{\pi}{2} \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2\sqrt{2}} = \frac{\cancel{\sqrt{2}}}{\sqrt{\pi}} \cdot \frac{\pi}{2\cancel{\sqrt{2}}} = \frac{\sqrt{\pi}}{2} \\ &\Rightarrow \frac{\sqrt{\pi}}{2} \leq \int_0^{\infty} e^{-y^2} dy \leq \frac{\sqrt{\pi}}{2} \\ &\Rightarrow \int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \\ &\Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \end{aligned}$$

7.14.5 The Gamma Function

The Gamma Function is defined as:

$$\Gamma(x) = \int_0^{\infty} e^{-t} \cdot t^{x-1} dt$$

Finding a Reduction Formula

$$\Gamma(x) = \int_0^{\infty} e^{-t} \cdot t^{x-1} dt$$

Integrating by parts,

Let $u = t^{x-1}$ and $dv = e^{-t} dt$

$\Rightarrow du = (x-1)t^{x-2} dt$ and $v = -e^{-t}$

$$\int u dv = uv - \int v du$$

$$\begin{aligned}\Rightarrow \Gamma(x) &= \left[t^{x-1} \cdot (-e^{-t}) \right]_0^{\infty} - \int_0^{\infty} (-e^{-t}) \cdot (x-1)t^{x-2} dt \\ &= \left[\frac{t^{x-1}}{e^t} \right]_{\infty}^0 + (x-1) \int_0^{\infty} e^{-t} \cdot t^{x-2} dt \\ &= 0 + (x-1) \cdot \Gamma(x-1) \\ \Rightarrow \Gamma(x) &= (x-1) \cdot \Gamma(x-1)\end{aligned}$$

Base Case

$$\Gamma(1) = \int_0^{\infty} e^{-t} \cdot t^{1-1} dt = \int_0^{\infty} e^{-t} dt = [-e^{-t}]_0^{\infty} = 1$$

Finding an Explicit Formula

$$\begin{aligned}\Gamma(n) &= (n-1) \cdot \Gamma(n-1) \\ &= (n-1) \cdot (n-2) \cdot \Gamma(n-3) \\ &= (n-1) \cdot (n-2) \cdot (n-3) \cdot \Gamma(n-4) \\ &= (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 3 \cdot 2 \cdot \Gamma(1) \\ &= (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 3 \cdot 2 \cdot 1 \\ &= (n-1)!\end{aligned}$$

$$\Rightarrow \Gamma(n+1) = n!$$

$$\Rightarrow \Gamma(n+1) = \int_0^{\infty} e^{-t} \cdot t^n dt = n!$$

This is another beautiful connection between Discrete and Continuous mathematics.

The Gamma function can be used to extend the domain of Factorial from integers to all real numbers. So we could say something like the following:

$$\text{Consider } I = \int_0^{\infty} e^{-x^2} dx$$

$$\begin{aligned} \text{Let } x^2 &= u \\ \Rightarrow dx &= \frac{1}{2\sqrt{u}} du \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \int_0^{\infty} e^u \cdot \frac{1}{2\sqrt{u}} du \\ &= \frac{1}{2} \int_0^{\infty} e^u \cdot u^{-1/2} du \\ &= \frac{1}{2} \int_0^{\infty} e^u \cdot u^{(1/2)-1} du \\ &= \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \end{aligned}$$

$$\text{But we know that } I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Now,

$$\begin{aligned} \Gamma\left(\frac{3}{2}\right) &= \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \\ \Rightarrow \left(\frac{1}{2}\right)! &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

7.15 Feynman's Technique for Integration

Let f be a function whose first derivative is continuous in a given region with respect to the variables involved, and

$$\Phi(y) = \int_{q(y)}^{p(y)} f(x, y) \, dx$$

Then,

$$\Phi'(y) = \int_{q(y)}^{p(y)} f_y(x, y) \, dx + f(p(y), y) \cdot p'(y) - f(q(y), y) \cdot q'(y)$$

Where $f_y(x, y) = \frac{\partial}{\partial y} f(x, y)$,

i.e. $f_y(x, y)$ is the partial derivative of $f(x, y)$ with respect to y .

Problem 141. Evaluate $\int_0^a \frac{\log_e(1+ax)}{1+x^2} \, dx$ and hence find $\int_0^1 \frac{\log_e(1+x)}{1+x^2} \, dx$.

Solution: The value of the integral depends on a .

Let us define

$$\Phi(a) = \int_0^a \frac{\log_e(1+ax)}{1+x^2} \, dx$$

Now,

$$\frac{\partial}{\partial a} \left(\frac{\log_e(1+ax)}{1+x^2} \right) = \frac{x}{(1+x^2)(1+ax)}$$

Using Feynman's Technique,

$$\begin{aligned} \Phi'(a) &= \int_0^a \frac{x}{(1+x^2)(1+ax)} \, dx + \frac{\log_e(1+a^2)}{1+a^2} - \frac{\log_e(1+0)}{1+0} \\ &= \int_0^a \frac{x}{(1+x^2)(1+ax)} \, dx + \frac{\log_e(1+a^2)}{1+a^2} \end{aligned}$$

$$\text{Let } I = \int_0^a \frac{x}{(1+x^2)(1+ax)} \, dx$$

Evaluating integral I by partial fractions.

$$\begin{aligned} \text{Let } \frac{x}{(1+x^2)(1+ax)} &= \frac{Ax+B}{1+x^2} + \frac{C}{1+ax} \\ &= \frac{(Ax+B)(1+ax) + C(1+x^2)}{(1+x^2)(1+ax)} \\ \Rightarrow x &= (Ax+B)(1+ax) + C(1+x^2) \\ &= Ax + Aax^2 + B + Bax + C + Cx^2 \\ &= (Aa+C)x^2 + (A+Ba)x + B+C \end{aligned}$$

$$\text{Solve to get } A = \frac{1}{1+a^2}, \quad B = \frac{a}{1+a^2}, \quad C = \frac{-a}{1+a^2}$$

$$\begin{aligned}\Rightarrow \frac{x}{(1+x^2)(1+ax)} &= \frac{\left(\frac{1}{1+a^2}\right)x + \left(\frac{a}{1+a^2}\right)}{1+x^2} + \frac{\left(\frac{-a}{1+a^2}\right)}{1+ax} \\ &= \frac{x+a}{(1+a^2)(1+x^2)} - \frac{a}{(1+a^2)(1+ax)}\end{aligned}$$

Then,

$$\begin{aligned}I &= \int_0^a \frac{x}{(1+x^2)(1+ax)} dx = \int_0^a \frac{x+a}{(1+a^2)(1+x^2)} dx - \int_0^a \frac{a}{(1+a^2)(1+ax)} dx \\ &= \int_0^a \frac{x}{(1+a^2)(1+x^2)} dx + \int_0^a \frac{a}{(1+a^2)(1+x^2)} dx - \int_0^a \frac{a}{(1+a^2)(1+ax)} dx \\ &= \frac{1}{2(1+a^2)} \int_0^a \frac{2x}{1+x^2} dx + \frac{a}{1+a^2} \int_0^a \frac{1}{1+x^2} dx - \frac{a}{1+a^2} \int_0^a \frac{1}{1+ax} dx \\ &\quad \left[\text{Let } 1+x^2 = t \Rightarrow 2x dx = dt \right] \\ &= \frac{1}{2(1+a^2)} \left[\log_e(1+x^2) \right]_0^a + \frac{a}{1+a^2} \left[\tan^{-1}(x) \right]_0^a - \frac{a}{1+a^2} \cdot \frac{1}{a} \left[\log_e(|1+ax|) \right]_0^a \\ &= \frac{\log_e(1+a^2)}{2(1+a^2)} + \frac{a \cdot \tan^{-1}(a)}{1+a^2} - \frac{\log_e(1+a^2)}{1+a^2} \\ &= \frac{a \cdot \tan^{-1}(a)}{1+a^2} - \frac{1}{2} \cdot \frac{\log_e(1+a^2)}{(1+a^2)}\end{aligned}$$

$$\begin{aligned}\Rightarrow \Phi'(a) &= I + \frac{\log_e(1+a^2)}{1+a^2} \\ &= \frac{a \cdot \tan^{-1}(a)}{1+a^2} - \frac{1}{2} \cdot \frac{\log_e(1+a^2)}{(1+a^2)} + \frac{\log_e(1+a^2)}{1+a^2} \\ &= \frac{1}{2} \cdot \frac{\log_e(1+a^2)}{(1+a^2)} + \frac{a \cdot \tan^{-1}(a)}{1+a^2}\end{aligned}$$

Integrate with respect to a to get $\Phi(a)$:

$$\Phi(a) = \int \frac{\log_e(1+a^2)}{2(1+a^2)} da + \int \frac{a \cdot \tan^{-1}(a)}{1+a^2} da$$

Integrating by parts,

$$\text{Let } u = \frac{1}{2} \log_e(1+a^2) \Rightarrow du = \frac{1}{2(1+a^2)} \cdot 2a da = \frac{a}{(1+a^2)} da$$

$$\text{Let } dv = \frac{1}{1+a^2} da \Rightarrow v = \tan^{-1}(a)$$

$$\begin{aligned}\Phi(a) &= \int u dv + \int \frac{a \cdot \tan^{-1}(a)}{1+a^2} da \\ &= uv - \int v du + \int \frac{a \cdot \tan^{-1}(a)}{1+a^2} da \\ &= \frac{1}{2} \log_e(1+a^2) \cdot \tan^{-1}(a) - \int \frac{a \cdot \tan^{-1}(a)}{(1+a^2)} da + \int \frac{a \cdot \tan^{-1}(a)}{1+a^2} da + c \\ &= \frac{1}{2} \log_e(1+a^2) \cdot \tan^{-1}(a) + c\end{aligned}$$

$$\Rightarrow \Phi(a) = \frac{1}{2} \log_e(1 + a^2) \cdot \tan^{-1}(a) + c$$

$$\begin{aligned} \Rightarrow \Phi(0) &= \frac{1}{2} \log_e(1) \cdot \tan^{-1}(1) + c \\ &= 0 + c \end{aligned}$$

$$\Rightarrow \Phi(0) = c \tag{1}$$

But we defined

$$\begin{aligned} \Phi(a) &= \int_0^a \frac{\log_e(1 + ax)}{1 + x^2} dx \\ \Rightarrow \Phi(0) &= \int_0^0 \frac{\log_e(1 + 0)}{1 + x^2} dx = 0 \\ \Phi(0) &= 0 \end{aligned} \tag{2}$$

Now, from equations (1) and (2),

$$\begin{aligned} c &= 0 \\ \Rightarrow \Phi(a) &= \frac{1}{2} \log_e(1 + a^2) \cdot \tan^{-1}(a) \end{aligned}$$

Now,

$$\begin{aligned} \int_0^1 \frac{\log_e(1 + x)}{1 + x^2} dx &= \Phi(1) \\ &= \frac{1}{2} \log_e(1 + 1^2) \cdot \tan^{-1}(1) \\ &= \frac{1}{2} \log_e(2) \cdot \tan^{-1}(1) \\ &= \log_e\left(2^{1/2}\right) \cdot \frac{\pi}{4} \\ &= \log_e\left(\sqrt{2}\right) \cdot \frac{\pi}{4} \\ &= \frac{\pi}{4} \log_e\left(\sqrt{2}\right) \end{aligned}$$

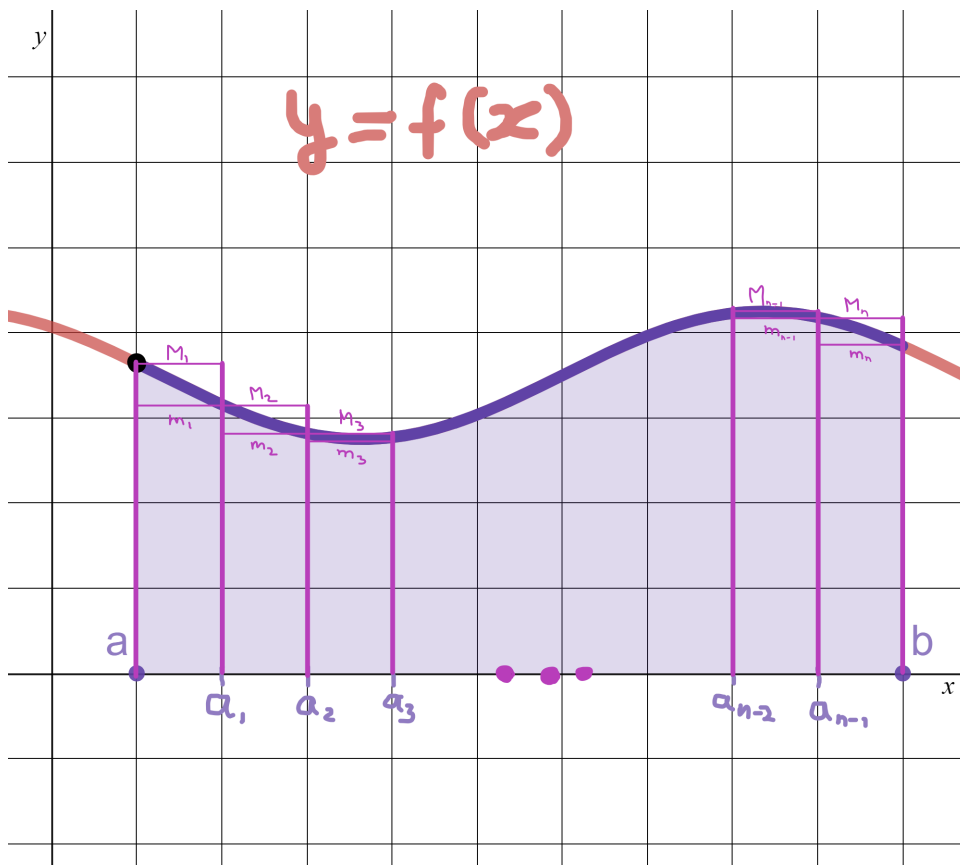
7.16 Integral as the Limit of a Sum

Let f be a continuous function in a closed interval $[a, b]$.

Let us partition the interval into n parts $[a, a_1], [a_1, a_2], [a_2, a_3], \dots, [a_{n-2}, a_{n-1}], [a_{n-1}, b]$.

Now, let m_i denote the minimum value of f in the i^{th} interval.

And let M_i denote the maximum value of f in the i^{th} interval.



Say the set P is the partition set:

$$P = \{a_1, a_2, a_3, \dots, a_{n-1}, a_n\}$$

$$m_1(a_1 - a) + m_2(a_2 - a_1) + \dots + m_n(b - a_{n-1}) \leq \int_a^b f(x) dx \leq M_1(a_1 - a) + M_2(a_2 - a_1) + \dots + M_n(b - a_{n-1})$$

Then the Riemann Lower Sum with respect to partition P is defined as:

$$L(f, P) = m_1(a_1 - a) + m_2(a_2 - a_1) + \dots + m_n(b - a_{n-1}) = \sum_{i=1}^n m_i \Delta x_i$$

And the Riemann Upper Sum with respect to partition P is defined as:

$$U(f, P) = M_1(a_1 - a) + M_2(a_2 - a_1) + \dots + M_n(b - a_{n-1}) = \sum_{i=1}^n M_i \Delta x_i$$

7.16.1 Condition for Integrability

A function is said to be (Riemann-Darboux) Integrable iff $\forall \varepsilon > 0, \exists p$ such that

$$|U(f, P) - L(f, P)| < \varepsilon$$

(This is known as the Cauchy Criterion for Riemann Integrability)

The upper integral is defined as

$$\overline{\int_a^b f(x) dx} = \inf (U(f, P))$$

And the lower integral is defined as

$$\underline{\int_a^b f(x) dx} = \sup (L(f, P))$$

Where sup stands for the supremum, a.k.a. the lowest upper bound and inf stands for the infimum, a.k.a. the greatest lower bound.

The Cauchy Criterion is equivalent to the statement that a function f is integrable in $[a, b]$ iff

$$\begin{aligned} \overline{\int_a^b f(x) dx} &= \underline{\int_a^b f(x) dx} \\ \Leftrightarrow \inf (U(f, P)) &= \sup (L(f, P)) \end{aligned}$$

7.16.2 Defining the Integral as the Limit of a Sum

Say f is an increasing function. Let us partition the interval $[a, b]$ into n parts of equal width. Then the minima in the intervals will be at their left endpoints. If we take these minima and multiply them with the width of the intervals, and take their sum, we'll get the Riemann Lower Sum.

$$\begin{aligned} \left(\frac{b-a}{n}\right) \left[f(a) + f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right) \right] &= L(f, P) \\ \Rightarrow \left(\frac{b-a}{n}\right) \sum_{r=0}^{n-1} f\left(a + \frac{r(b-a)}{n}\right) &= L(f, P) \end{aligned}$$

Similarly for the Riemann Upper Sum:

$$\begin{aligned} \left(\frac{b-a}{n}\right) \left[f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{n(b-a)}{n}\right) \right] &= U(f, P) \\ \Rightarrow \left(\frac{b-a}{n}\right) \sum_{r=1}^n f\left(a + \frac{r(b-a)}{n}\right) &= U(f, P) \end{aligned}$$

So we have the following if f is increasing.

$$\Rightarrow \left(\frac{b-a}{n}\right) \sum_{r=0}^{n-1} f\left(a + \frac{r(b-a)}{n}\right) \leq \int_a^b f(x) dx \leq \left(\frac{b-a}{n}\right) \sum_{r=1}^n f\left(a + \frac{r(b-a)}{n}\right)$$

Similarly, if f is decreasing, the upper sum and lower sum would get interchanged, giving us:

$$\Rightarrow \left(\frac{b-a}{n}\right) \sum_{r=1}^n f\left(a + \frac{r(b-a)}{n}\right) \leq \int_a^b f(x) dx \leq \left(\frac{b-a}{n}\right) \sum_{r=0}^{n-1} f\left(a + \frac{r(b-a)}{n}\right)$$

For an integrable function, the limit of the upper and lower sums as the number of partitions $n \rightarrow \infty$ equals the value of the integral.

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{b-a}{n}\right) \sum_{r=0}^{n-1} f\left(a + \frac{r(b-a)}{n}\right) = \lim_{n \rightarrow \infty} \left(\frac{b-a}{n}\right) \sum_{r=1}^n f\left(a + \frac{r(b-a)}{n}\right) = \int_a^b f(x) dx$$

This is an important result.

It can be thought of like the $\frac{b-a}{n}$ serves as the dx and the $a + \frac{r(b-a)}{n}$ serves as the x in the integral.

For problems involving infinite series, to find the lower and upper limits of the integral, plug in extreme values of r and evaluate the limit.

Problem 142. Evaluate $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} \right)$

Solution:

Observation: Note that this limit doesn't simply evaluate to zero, because the number of terms is n , which is a variable, so we can't really split it into n different limits, each evaluating to zero, since the limit itself is on $n \rightarrow \infty$.

$$\text{Let } S = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} \right)$$

$$\begin{aligned} S &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{n+r} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{1}{1 + \left(\frac{r}{n} \right)} \right) \end{aligned}$$

Comparing it with the definition of an integral as the limit of a sum.

$$\begin{aligned} \text{Now, } b-a=1 &\Rightarrow a + \frac{r(b-a)}{n} = a + \frac{r}{n} = 1 + \frac{r}{n} \\ &\Rightarrow a=1 \text{ and } b=2 \end{aligned}$$

$$\begin{aligned} \therefore S &= \int_1^2 \frac{1}{x} dx \\ &= \ln(2) - \ln(1) \\ &= \ln(2) \\ \therefore S &= \ln(2) \end{aligned}$$

Problem 143. Evaluate $\lim_{n \rightarrow \infty} \frac{\pi}{n} \left(\sin \left(\frac{\pi}{n} \right) + \sin \left(\frac{2\pi}{n} \right) + \sin \left(\frac{3\pi}{n} \right) + \cdots + \sin \left(\frac{n\pi}{n} \right) \right)$

Solution:

$$\text{Let } S = \lim_{n \rightarrow \infty} \frac{\pi}{n} \left(\sin \left(\frac{\pi}{n} \right) + \sin \left(\frac{2\pi}{n} \right) + \sin \left(\frac{3\pi}{n} \right) + \cdots + \sin \left(\frac{n\pi}{n} \right) \right)$$

$$\Rightarrow S = \lim_{n \rightarrow \infty} \frac{\pi}{n} \cdot \sum_{r=1}^n \sin \left(\frac{r\pi}{n} \right)$$

Comparing with the limit of a sum definition.

$$b - a = \pi \Rightarrow a + \frac{r(b-a)}{n} = a + \frac{r\pi}{n} = \frac{r\pi}{n}$$

$$\Rightarrow a = 0 \text{ and } b = \pi$$

$$\begin{aligned} \Rightarrow S &= \int_0^{\pi} \sin(x) \, dx \\ &= -[\cos(x)]_0^{\pi} \\ &= -[(-1) - (+1)] \\ &= 2 \end{aligned}$$

Problem 144. Evaluate $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{4n^2 - 1}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \frac{1}{\sqrt{4n^2 - 3^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right)$

Solution:

$$\text{Let } S = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{4n^2 - 1}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \frac{1}{\sqrt{4n^2 - 3^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right)$$

$$\begin{aligned} \Rightarrow S &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{\sqrt{4n^2 - r^2}} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{\sqrt{n^2 \cdot \left(4 - \frac{r^2}{n^2} \right)}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^n \left(\frac{1}{\sqrt{4 - \left(\frac{r}{n} \right)^2}} \right) \end{aligned}$$

Comparing with the limit of a sum definition.

$$b - a = 1 \Rightarrow a + \frac{r}{n} = \frac{r}{n}$$

$$\Rightarrow a = 0 \text{ and } b = 1$$

$$\Rightarrow S = \int_0^1 \frac{1}{\sqrt{4 - x^2}} \, dx = \left[\sin^{-1} \left(\frac{x}{2} \right) \right]_0^1 = \frac{\pi}{6}$$

Problem 145. Evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{2}{n}} + \sqrt{1 + \frac{3}{n}} + \cdots + \sqrt{1 + \frac{n}{n}} \right)$

Solution:

$$\begin{aligned} \text{Let } S &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{2}{n}} + \sqrt{1 + \frac{3}{n}} + \cdots + \sqrt{1 + \frac{n}{n}} \right) \\ \Rightarrow S &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \sqrt{1 + \frac{r}{n}} \end{aligned}$$

Comparing with the limit of a sum definition.

$$\begin{aligned} \Rightarrow b - a = 1 &\Rightarrow a + \frac{r}{n} = 1 + \frac{r}{n} \\ \Rightarrow a &= 1 \text{ and } b = 2 \end{aligned}$$

$$\begin{aligned} \Rightarrow S &= \int_1^2 \sqrt{x} \, dx \\ &= \frac{2}{3} \left[x^{3/2} \right]_1^2 \\ &= \frac{2}{3} [2\sqrt{2} - 1] \end{aligned}$$

Problem 146. Evaluate $\lim_{n \rightarrow \infty} \frac{1 + \sqrt[3]{2} + \sqrt[3]{3} + \sqrt[3]{4} + \cdots + \sqrt[3]{n}}{\sqrt[3]{n^4}}$

$$\begin{aligned} \text{Let } S &= \lim_{n \rightarrow \infty} \frac{1 + \sqrt[3]{2} + \sqrt[3]{3} + \sqrt[3]{4} + \cdots + \sqrt[3]{n}}{\sqrt[3]{n^4}} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \sqrt[3]{2} + \sqrt[3]{3} + \sqrt[3]{4} + \cdots + \sqrt[3]{n}}{\sqrt[3]{n^3 \cdot n}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \sqrt[3]{\frac{3}{n}} + \cdots + \sqrt[3]{\frac{n}{n}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \sqrt[3]{\frac{r}{n}} \end{aligned}$$

Comparing with the limit of a sum definition.

$$\begin{aligned} b - a = 1 &\Rightarrow a + \frac{r}{n} = \frac{r}{n} \\ \Rightarrow a &= 0 \text{ and } b = 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow S &= \int_0^1 \sqrt[3]{x} \, dx \\ &= \int_0^1 x^{1/3} \, dx \\ &= \frac{3}{4} \left[x^{4/3} \right]_0^1 \\ &= \frac{3}{4} \end{aligned}$$

Problem 147. Evaluate $\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n}$

Solution:

$$\begin{aligned}
 \text{Let } A &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n} && \text{Take natural log on both sides} \\
 \Rightarrow \ln(A) &= \ln \left(\lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \ln \left(\left(\frac{n!}{n^n} \right)^{1/n} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{n!}{n^n} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \dots \cdot \frac{n}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\ln \left(\frac{1}{n} \right) + \ln \left(\frac{2}{n} \right) + \ln \left(\frac{3}{n} \right) + \dots + \ln \left(\frac{n}{n} \right) \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \ln \left(\frac{r}{n} \right)
 \end{aligned}$$

Comparing with the limit of a sum definition.

$$\begin{aligned}
 b - a &= 1 \Rightarrow a + \frac{r}{n} = \frac{r}{n} \\
 \Rightarrow a &= 0 \text{ and } b = 1
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \ln(A) &= \int_0^1 \ln(x) \, dx && \text{Integrate by parts} \\
 &= [x \cdot \ln(x) - x]_0^1 \\
 &= -1 \\
 \Rightarrow A &= e^{-1} \\
 &= \frac{1}{e}
 \end{aligned}$$

Generally, if we see a product of n terms in a limit problem, it may be a good idea to take the natural logarithm on both sides to turn it into a problem of an infinite sum, or an integral as the limit of a sum.

Problem 148. Evaluate $\lim_{n \rightarrow \infty} \frac{[(n+1) \cdot (n+2) \cdot (n+3) \cdot \dots \cdot (n+n)]^{1/n}}{n}$

Solution:

$$\text{Let } A = \lim_{n \rightarrow \infty} \frac{[(n+1) \cdot (n+2) \cdot (n+3) \cdot \dots \cdot (n+n)]^{1/n}}{n}$$

$$\begin{aligned} \Rightarrow \ln(A) &= \ln \left(\lim_{n \rightarrow \infty} \frac{[(n+1) \cdot (n+2) \cdot (n+3) \cdot \dots \cdot (n+n)]^{1/n}}{n} \right) \\ &= \lim_{n \rightarrow \infty} \ln \left(\left[\frac{(n+1) \cdot (n+2) \cdot (n+3) \cdot \dots \cdot (n+n)}{n^n} \right]^{1/n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\left[\left(\frac{n+1}{n} \right) \cdot \left(\frac{n+2}{n} \right) \cdot \left(\frac{n+3}{n} \right) \cdot \dots \cdot \left(\frac{n+n}{n} \right) \right] \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\ln \left(\frac{n+1}{n} \right) + \ln \left(\frac{n+2}{n} \right) + \ln \left(\frac{n+3}{n} \right) + \dots + \ln \left(\frac{n+n}{n} \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^n \ln \left(\frac{n+r}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^n \ln \left(1 + \frac{r}{n} \right) \end{aligned}$$

Comparing with the limit of a sum definition,

$$\begin{aligned} b - a = 1 &\Rightarrow a + \frac{r}{n} = 1 + \frac{r}{n} \\ \Rightarrow a = 1 \text{ and } b = 2. \end{aligned}$$

$$\begin{aligned} \Rightarrow \ln(A) &= \int_1^2 \ln(x) \, dx && \text{Integrate by parts} \\ &= [x \cdot \ln(x) - x]_1^2 \\ &= (2 \ln(2) - 2) - (\ln(1) - 1) \\ &= 2 \ln(2) - 1 \\ \Rightarrow A &= e^{2 \ln(2) - 1} \\ &= e^{\ln(4) - \ln(e)} \\ &= e^{\ln(4/e)} \\ &= \frac{4}{e} \end{aligned}$$

Problem 149. Evaluate $\lim_{n \rightarrow \infty} \left[\tan\left(\frac{\pi}{2n}\right) \cdot \tan\left(\frac{2\pi}{2n}\right) \cdot \tan\left(\frac{3\pi}{2n}\right) \cdot \dots \cdot \tan\left(\frac{n\pi}{2n}\right) \right]^{1/n}$

Solution:

$$\text{Let } A = \lim_{n \rightarrow \infty} \left[\tan\left(\frac{\pi}{2n}\right) \cdot \tan\left(\frac{2\pi}{2n}\right) \cdot \tan\left(\frac{3\pi}{2n}\right) \cdot \dots \cdot \tan\left(\frac{n\pi}{2n}\right) \right]^{1/n}$$

Taking the natural logarithm on both sides.

$$\begin{aligned} \ln(A) &= \ln \left(\lim_{n \rightarrow \infty} \left[\tan\left(\frac{\pi}{2n}\right) \cdot \tan\left(\frac{2\pi}{2n}\right) \cdot \tan\left(\frac{3\pi}{2n}\right) \cdot \dots \cdot \tan\left(\frac{n\pi}{2n}\right) \right]^{1/n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\ln \left(\tan\left(\frac{\pi}{2n}\right) \right) + \ln \left(\tan\left(\frac{2\pi}{2n}\right) \right) + \ln \left(\tan\left(\frac{3\pi}{2n}\right) \right) + \dots + \ln \left(\tan\left(\frac{n\pi}{2n}\right) \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \ln \left(\tan\left(\frac{r\pi}{2n}\right) \right) \end{aligned}$$

Comparing with the limit of a sum definition.

$$\begin{aligned} b - a = 1 &\Rightarrow a + \frac{r}{n} = \frac{r}{n} \\ \Rightarrow a = 0 \text{ and } b = 1 \end{aligned}$$

$$\Rightarrow \ln(A) = \int_0^1 \ln \left(\tan\left(\frac{\pi}{2}x\right) \right) dx$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$= \int_0^1 \ln \left(\tan\left(\frac{\pi}{2}(1-x)\right) \right) dx$$

Add

$$\Rightarrow 2\ln(A) = \int_0^1 \left[\ln \left(\tan\left(\frac{\pi}{2}x\right) \right) + \ln \left(\tan\left(\frac{\pi}{2} - \frac{\pi}{2}x\right) \right) \right] dx$$

$$= \int_0^1 \left[\ln \left(\tan\left(\frac{\pi}{2}x\right) \right) + \ln \left(\cot\left(\frac{\pi}{2}x\right) \right) \right] dx$$

$$= \int_0^1 \ln \left(\tan\left(\frac{\pi}{2}x\right) \cdot \cot\left(\frac{\pi}{2}x\right) \right) dx$$

$$= \int_0^1 \ln(1) dx$$

$$\Rightarrow 2\ln(A) = 0$$

$$\Rightarrow \ln(A) = 0$$

$$\Rightarrow A = e^0$$

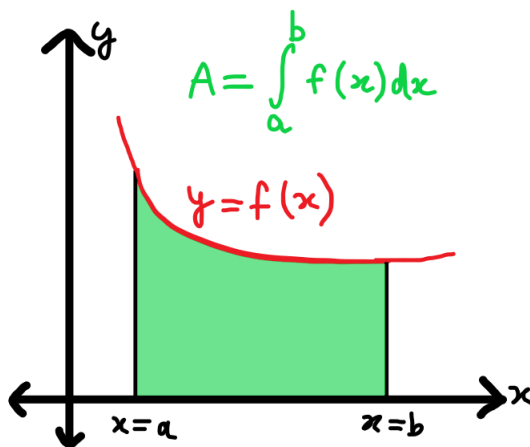
$$= 1$$

$$\Rightarrow A = 1$$

7.17 Areas

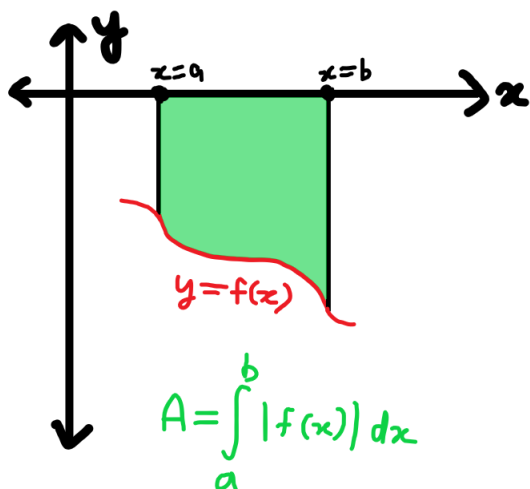
1. Let f be a non-negative function on the interval $[a, b]$. Then the area bounded by the function $y = f(x)$ with the x -axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b y \, dx = \int_a^b f(x) \, dx$$



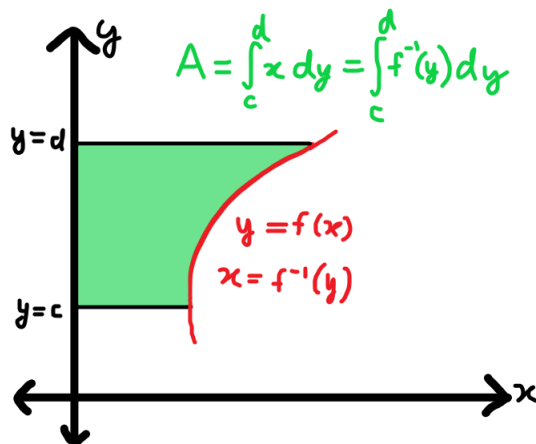
2. In general, if f doesn't change sign in $[a, b]$, then the area bounded by the function with the x -axis and lines $x = a$ and $x = b$ is

$$\int_a^b |f(x)| \, dx$$



3. Say we want to find the area bounded by the curve with the y -axis and the lines $y = c$ and $y = d$.
If x is non-negative,

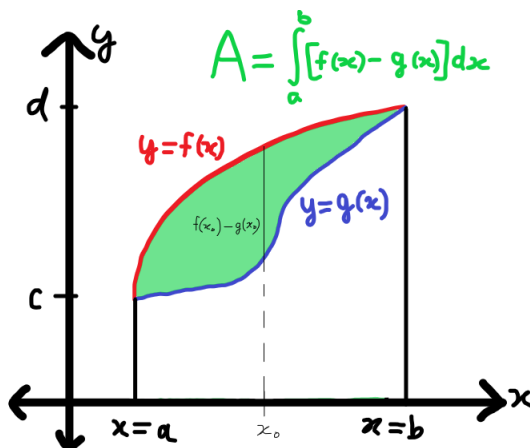
$$\text{Area} = \int_c^d x \, dy = \int_c^d f^{-1}(y) \, dy$$



In general,
$$\text{Area} = \int_c^d |x| \, dy = \int_a^b |f^{-1}(y)| \, dy$$

4. Say in the interval $[a, b]$, $f(x) \geq g(x)$ and the curves intersect at the points a and b . Then the area bounded between $f(x)$ and $g(x)$ is

$$\int_a^b (f(x) - g(x)) \, dx$$

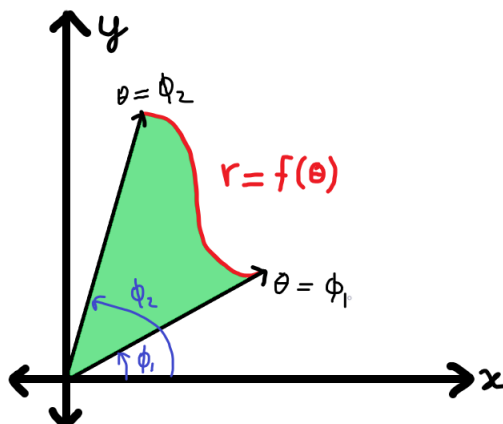


5. The other way of finding the same area could be

$$\int_c^d (x_1 - x_2) dy = \int_c^d (g^{-1}(y) - f^{-1}(y)) dy$$

We can use this if the functions are easier to deal with in terms of y .

6. Polar Coordinates:



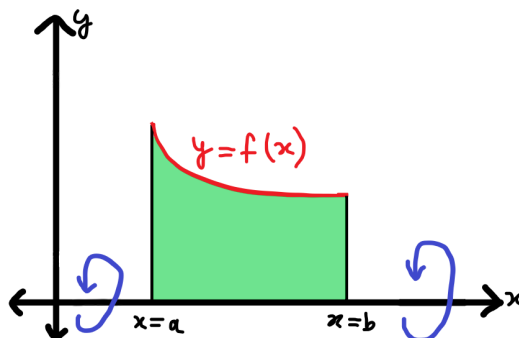
$$\text{Area of small sector} = \frac{1}{2} (d\theta) r^2$$

$$\text{Area bounded by the curve} = \int_{\phi_1}^{\phi_2} \frac{1}{2} r^2 d\theta$$

7. Volume of Revolution:

If we take the curve and rotate it about the x -axis, we'll get a cylinder-like solid whose volume will be given by

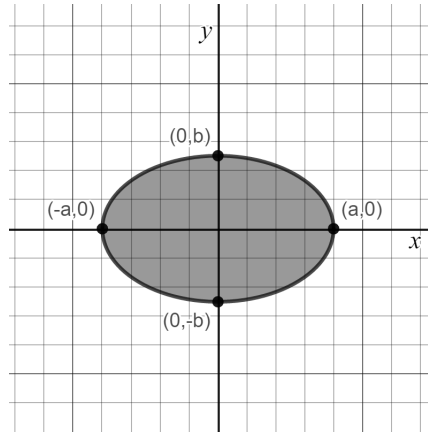
$$\text{Volume} = \int_a^b \pi (f(x))^2 dx$$



Problem 150. Find the area bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are non-zero real numbers.

Solution:

Let the required area be A .



Then, since the ellipse is symmetric, if we find the area bounded in the first quadrant and multiply it by 4, we'll get the total area.

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \Rightarrow \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2}\end{aligned}$$

First quadrant:

$$\begin{aligned}y &= b\sqrt{\frac{a^2 - x^2}{a^2}} \\ \Rightarrow y &= \frac{b}{a}\sqrt{a^2 - x^2}\end{aligned}$$

So the required area becomes:

$$\begin{aligned}A &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx \\ &= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx \\ &= \frac{4b}{a} \cdot \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]_0^a \\ &= \frac{4b}{a} \cdot \left[\frac{a}{2} (0) + \frac{a^2}{2} \sin^{-1} \left(\frac{a}{a} \right) \right] \\ &= \frac{4b}{a} \cdot \left[\frac{a^2}{2} \cdot \frac{\pi}{2} \right] \\ &= \pi ab\end{aligned}$$

Alternatively:

We have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let S be the set of ordered pairs from which x and y take their values.

Then, our objective is to find the value of $\int_S y \, dx$.

Let us substitute $\frac{x}{a} = u$ and $\frac{y}{b} = v$.

Or in other terms,

$$y = bv$$

$$x = au$$

$$\Rightarrow dx = a \, du$$

Let S' be the corresponding set from which u and v take their values.

$$\int_S y \, dx = \int_{S'} (bv)a \, du = ab \int_{S'} v \, du$$

And the equation becomes

$$u^2 + v^2 = 1$$

Now for the relation $u^2 + v^2 = 1$, the integral $\int_{S'} v \, du$ represents the area of the unit circle centred at the origin.

$$\text{i.e. } \int_{S'} v \, du = \pi$$

So the area of the ellipse represented by the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes

$$\begin{aligned} A &= \int_S y \, dx \\ &= ab \int_{S'} v \, du \\ &= ab \cdot \pi \\ &= \pi ab \end{aligned}$$

Problem 151. Find the area bounded by the curve $y = \sqrt{4-x}$ with the x and y axes.

Solution:

$$y = \sqrt{4-x}$$

Finding intercepts:

$$x = 0 \iff y = 2$$

$$y = 0 \iff x = 4$$

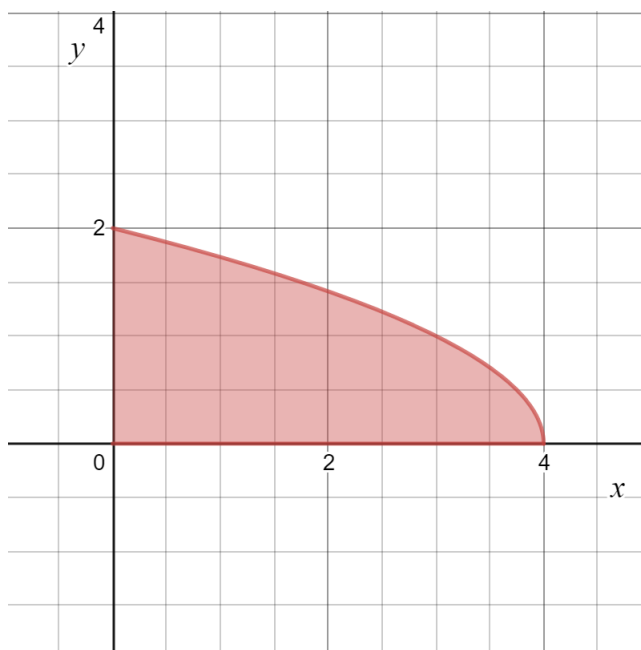
Differentiate:

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{2\sqrt{4-x}} \\ &= \frac{-1}{2} (4-x)^{-1/2} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{-1}{4} (4-x)^{-3/2}\end{aligned}$$

Now,

$\frac{dy}{dx} \leq 0$ and the equality holds only at a discrete point $\implies y$ is strictly decreasing.

Similarly $\frac{d^2y}{dx^2} < 0$ and the equality holds only at a discrete point $\implies y$ is strictly concave down.



$$y^2 = 4 - x$$

$$\implies x = 4 - y^2$$

$$\text{Area} = \int_0^2 x \, dy = \int_0^2 (4 - y^2) \, dy = \left[4y - \frac{y^3}{3} \right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

Problem 152. If the line $x = \alpha$ divides the area of the region

$$R = \left\{ (x, y) : (x, y) \in \mathbb{R}^2, \quad x^3 \leq y \leq x, \quad 0 \leq x \leq 1 \right\}$$

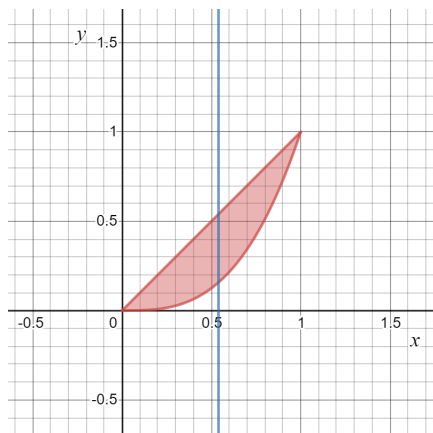
in two parts of equal area, then which of the following is/are true?

- (A) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $0 < \alpha \leq \frac{1}{2}$ (D) $\frac{1}{2} < \alpha < 1$

Solution: (A, D)

For $x \in [0, 1]$, we have

$$\begin{aligned} x^3 &\leq y \leq x \\ \Rightarrow y &\geq x^3 \text{ and } y \leq x \end{aligned}$$



$$\begin{aligned} \int_0^{\alpha} (x - x^3) dx &= \int_{\alpha}^1 (x - x^3) dx \\ \Rightarrow \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^{\alpha} &= \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_{\alpha}^1 \\ \Rightarrow \frac{\alpha^2}{2} - \frac{\alpha^4}{4} &= \frac{1}{2} - \frac{1}{4} - \frac{\alpha^2}{2} + \frac{\alpha^4}{4} \\ \Rightarrow \alpha^2 - \frac{\alpha^4}{2} &= \frac{1}{4} \\ \Rightarrow 4\alpha^2 - 2\alpha^4 &= 1 \\ \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 &= 0 \end{aligned}$$

\therefore Option **(A)** is correct.

Let $\alpha^2 = t$

$$\begin{aligned} \Rightarrow 2t^2 - 4t + 1 &= 0 \\ \Rightarrow t &= \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{1}{\sqrt{2}} \end{aligned}$$

But $0 < \alpha < 1 \implies 0 < t < 1 \implies t = \alpha^2 = 1 - \frac{1}{\sqrt{2}} \approx 0.3$

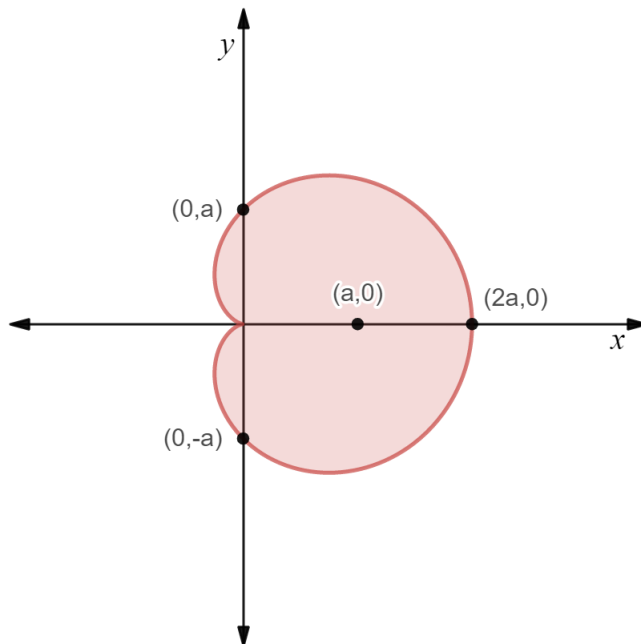
$\Rightarrow \frac{1}{2} < \alpha < 1$, therefore option **(D)** is correct.

Problem 153. Find the area of a cardioid (using polar coordinates).

Solution: A cardioid is represented by the following polar equation and gets its name from its shape resembling a heart.

$$r = a(1 + \cos(\theta))$$

It is defined as the locus of a point on the perimeter of a circle that is rolling around a fixed circle of the same radius (without slipping).



$$\begin{aligned}
 \text{Area} &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} a^2 (1 + \cos(\theta))^3 d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} (1 + \cos(\theta))^2 d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} (1 + 2\cos(\theta) + \cos^2(\theta)) d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} \left(1 + 2\cos(\theta) + \frac{1 + \cos(2\theta)}{2}\right) d\theta \\
 &= \frac{a^2}{2} \left[\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) \right]_0^{2\pi} \\
 &= \frac{a^2}{2} [2\pi + \pi] \\
 &= \frac{3\pi}{2} a^2
 \end{aligned}$$

Problem 154. Find the area bounded by the loop $4y^2 = 4x^2 - x^3$.

Solution:

$$4y^2 = 4x^2 - x^3$$

First, we observe that if we replace y by $-y$ in the above equation, there is no change to the overall equation, implying that the curve is symmetric about the x -axis.

Also, since the square of a real number is always non-negative,

$$4x^2 - x^3 \geq 0$$

$$\Rightarrow x^2(4 - x) \geq 0$$

$$\Rightarrow x \leq 4$$

So the graph of the equation doesn't exist to the right of the line $x = 4$.

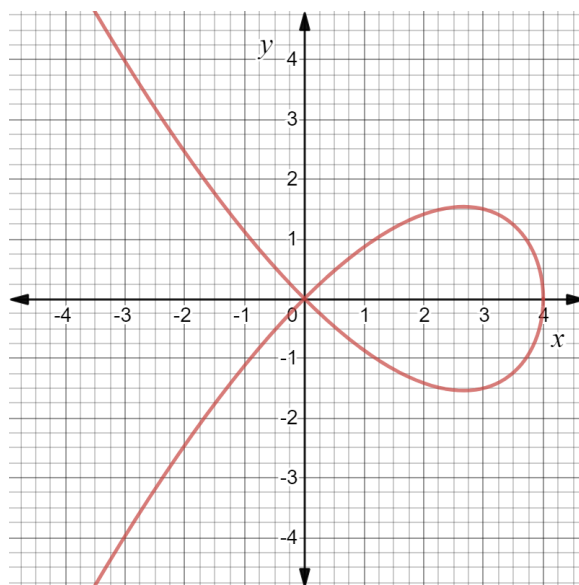
At $x = 4$,

$$4y^2 = 4(4)^2 - 4^3 = 0$$

$$\Rightarrow y = 0$$

Another trivial solution for $y = 0$ comes out to be $x = 0$.

After working with the sign of derivatives, we can figure out which parts of the function are increasing and which parts are decreasing. Then we can plot the curve.



Since the curve is symmetric about the x -axis, we can find the area of the loop above the x -axis and multiply it by two.

$$\text{Area (in sq. units)} = 2 \int_0^4 \sqrt{\frac{4x^2 - x^3}{4}} dx = \int_0^4 \sqrt{4x^2 - x^3} dx = \int_0^4 x\sqrt{4-x} dx$$

$$\text{Let } 4 - x = t^2 \implies dx = -2t dt$$

$$\text{Area (in sq. units)} = \int_0^2 2(4 - t^2)t^2 dt = \int_0^2 (8t^2 - 2t^4) dt = \left[\frac{8t^3}{3} - \frac{2t^5}{5} \right]_0^2 = \frac{64}{3} - \frac{64}{5} = \frac{128}{15}$$

Problem 155. Find the area enclosed by the loop $y^2 = x^2(a^2 - x^2)$

Solution:

$$y^2 = x^2(a^2 - x^2)$$

First, if we replace x by $-x$ or y by $-y$ in the above equation, it remains unchanged. This means the loop is symmetric about the x and y axes. Now,

$$x = 0 \implies y = 0$$

And

$$y = 0 \implies x = 0 \text{ or } x = \pm a$$

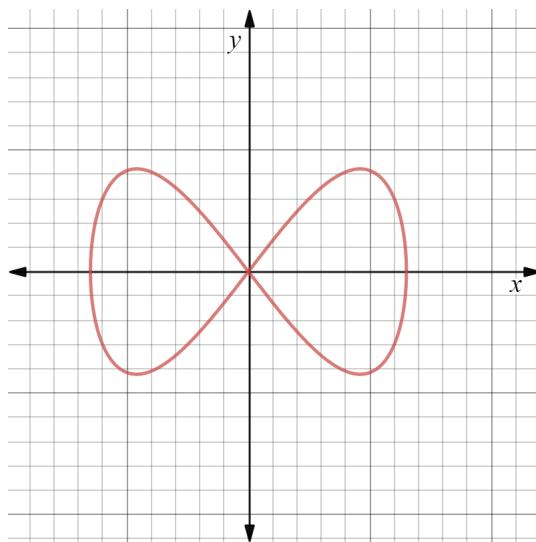
Also, since the square of a real number is always non-negative,

$$x^2(a^2 - x^2) \geq 0$$

$$\implies a^2 - x^2 \geq 0$$

$$\implies -a \leq x \leq a$$

Now, we have enough information to graph the equation.



Since the curve is symmetric, we will find the area enclosed in the first quadrant and then multiply it by 4.

$$\text{Area (in sq. units)} = 4 \int_0^a x \sqrt{a^2 - x^2} \, dx$$

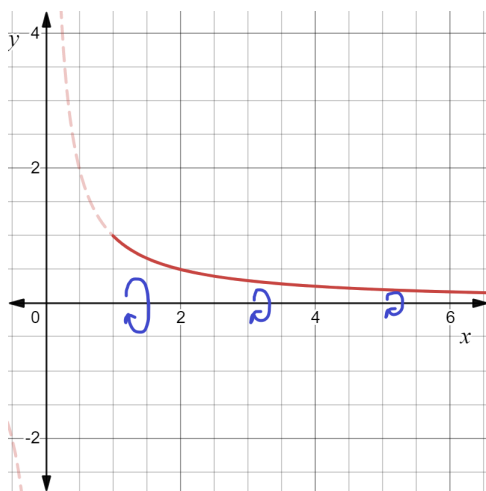
$$\text{Let } a^2 - x^2 = t \implies -2x \, dx = dt$$

$$\begin{aligned} A &= \frac{-4}{2} \int_{a^2}^0 \sqrt{t} \, dt \\ &= \frac{4}{2} \int_0^{a^2} \sqrt{t} \, dt \\ &= \frac{4}{2} \cdot \frac{2}{3} \left[t^{3/2} \right]_0^{a^2} \\ &= \frac{4a^3}{3} \end{aligned}$$

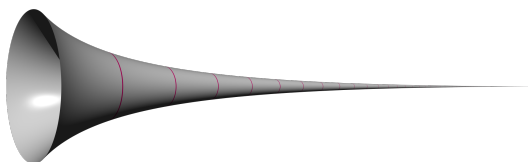
7.17.1 Gabriel's Horn and the Painter's Paradox

Consider the graph of $y = \frac{1}{x}$ for $x \geq 1$.

Now, if we rotate this curve about the x -axis in three dimensions, we'll get its solid of revolution.



The resulting solid of revolution is known as Gabriel's Horn, a.k.a. Torricelli's Trumpet. Notice that this extends indefinitely to the right.



Let us find the volume of this solid.
As discussed before,

$$\text{Volume} = \int_a^b \pi (f(x))^2 dx$$

So the volume of this solid becomes:

$$\begin{aligned} V &= \int_1^{\infty} \pi \left(\frac{1}{x} \right)^2 dx \\ &= \pi \int_1^{\infty} \frac{1}{x^2} dx \\ &= \pi \cdot \lim_{k \rightarrow \infty} \left[\frac{-1}{k} - \frac{-1}{1} \right] \\ &= \pi \end{aligned}$$

So this horn extending infinitely has a finite volume, and therefore we can fill it from inside completely with exactly π cubic units of paint.

But what if we had to paint it completely, could that be done with a finite amount of paint? Our intuition might suggest that, but to truly answer that, let us find the surface area of this horn.

$$\text{Surface Area} = \int_a^b 2\pi \cdot y \, dS$$

where $dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

So for Gabriel's Horn, the surface area becomes:

$$S = \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx$$

Now, for $x \in [1, \infty)$,

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

Since x is positive, we can divide both sides by x without changing the sign of the inequality.

$$\frac{1}{x} \sqrt{1 + \frac{1}{x^4}} > \frac{1}{x}$$

Then, using Comparison Theorem,

$$\int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx > \int_1^{\infty} \frac{1}{x} \, dx$$

$$S > \int_1^{\infty} \frac{1}{x} \, dx$$

$$S > \lim_{k \rightarrow \infty} \ln(k) - \ln(1)$$

$$S > \lim_{k \rightarrow \infty} \ln(k)$$

But $\ln(x)$ can be made arbitrarily large by making x sufficiently large, implying Gabriel's Horn has infinite surface area.

This is the paradox. If we want to paint Gabriel's Horn, we need an infinite amount of paint, but we can fill it completely from inside with a finite amount of paint (exactly π cubic units).

8 Differential Equations

8.1 Introduction

A differential equation is an equation that relates a function to its derivatives. These equations generally pop up when it's easier to talk about change than absolute amounts of something.

Differential equations are generally classified under two headings: Ordinary and Partial.

An example of an ordinary differential equation can be

$$f(x) = f'(x)$$

which is satisfied by the function $f(x) = e^x$.

A partial differential equation will deal with multivariable functions and their partial derivatives, for example the heat equation describes how temperature of a substance changes with position (space) and time:

$$\frac{\partial T}{\partial t}(x, t) = \alpha \cdot \frac{\partial^2 T}{\partial x^2}(x, t)$$

We will however limit our discussion from here to ordinary differential equations.

Definition 27. A differential equation involving derivatives of a dependent variable with respect to only one independent variable is called an ordinary differential equation.

8.2 Order and Degree

Definition 28. The order of a differential equation is the order of the highest order derivative appearing in the equation.

Definition 29. The degree (if defined) of a differential equation is the degree (highest power) of the highest order derivative involved in it, when the differential coefficients are free from radicals and fractions (i.e. the differential equation is written as a polynomial in the derivatives).

Example. $\left(\frac{d^2 y}{dx^2}\right)^3 + \left(\frac{d^2 y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^5 + y^{10} = \sin(x).$

This differential equation has order 2 and degree 3.

8.2.1 Heuristics for Finding Degree

- Express the differential equation as a polynomial in the differential coefficients.
- Ensure that the exponents of the differential coefficients are natural numbers.
- The degree of a differential equation is defined only when expressing the differential equation in such a way is possible.
- Once that is done, look for the terms with the highest order derivative. The highest exponent of these will be the degree.

Problem 156. Find the order and degree of the differential equation: $\frac{d^2 y}{dx^2} = \left(y + \left(\frac{dy}{dx} \right)^6 \right)^{1/4}$

Solution:

$$\frac{d^2 y}{dx^2} = \left(y + \left(\frac{dy}{dx} \right)^6 \right)^{1/4}$$

Clearly, the highest order derivative in the given equation is a second derivative. So the order of this differential equation is 2.

Now, let us try to express the derivatives in polynomial form.

$$\frac{d^2 y}{dx^2} = \left(y + \left(\frac{dy}{dx} \right)^6 \right)^{1/4}$$

Raising both sides to the fourth power:

$$\left(\frac{d^2 y}{dx^2} \right)^4 = y + \left(\frac{dy}{dx} \right)^6$$

Now we can see that the highest order derivative is raised to the fourth power, so the degree is 4.

Problem 157. Find the order and degree: $\frac{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}}{\frac{dy}{dx}} = x + \left(\frac{dy}{dx} \right)^2$

Solution:

$$\frac{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}}{\frac{dy}{dx}} = x + \left(\frac{dy}{dx} \right)^2$$

Since we don't see any second or higher derivatives in the equation, it has order 1.

Now to find the degree, we will first multiply both sides by $\frac{dy}{dx}$.

$$\sqrt{1 + \left(\frac{dy}{dx} \right)^2} = x \frac{dy}{dx} + \left(\frac{dy}{dx} \right)^3$$

Now, let us square both sides,

$$1 + \left(\frac{dy}{dx} \right)^2 = \left(x \frac{dy}{dx} + \left(\frac{dy}{dx} \right)^3 \right)^2$$

At this point, it is clear that upon using the binomial expansion for index 2, we'll get the degree of this differential equation as 6.

Problem 158. Find the order and degree: $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = x$

Solution:

Since there are no higher derivatives, the order of the differential equation is 1.

Because of the sine function, we can't express it as a polynomial in the derivative. Even if we use Taylor/Maclaurin series to expand the sine function, we'll get infinitely many terms, so it won't be a polynomial.

Therefore, the degree of this differential equation is not defined.

Problem 159. Find the order and degree: $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Solution:

By now it is clear that the order is 1.

For finding the degree, we have the equation as

$$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$$

Let us take the sine on both sides.

$$\frac{dy}{dx} = \sin(x + y)$$

So that gives us the degree as 1.

Problem 160. Find the order and degree: $y = 1 + \frac{dy}{dx} + \frac{1}{1 \cdot 2} \left(\frac{dy}{dx}\right)^2 + \frac{1}{1 \cdot 2 \cdot 3} \left(\frac{dy}{dx}\right)^3 + \dots$

Solution:

Clearly, the order is 1.

Finding degree:

$$y = 1 + \frac{dy}{dx} + \frac{1}{1 \cdot 2} \left(\frac{dy}{dx}\right)^2 + \frac{1}{1 \cdot 2 \cdot 3} \left(\frac{dy}{dx}\right)^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{dy}{dx}\right)^4 + \dots$$

$$\Rightarrow y = 1 + \frac{dy}{dx} + \frac{1}{2!} \left(\frac{dy}{dx}\right)^2 + \frac{1}{3!} \left(\frac{dy}{dx}\right)^3 + \frac{1}{4!} \left(\frac{dy}{dx}\right)^4 + \dots$$

$$\Rightarrow y = e^{\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \ln(y) = \frac{dy}{dx}$$

So the degree is 1.

8.3 Solutions of Differential Equations

8.3.1 General Solution

The general solution to an ordinary differential equation will be a class (family) of infinitely many functions/curves that satisfy it. Such a solution will contain arbitrary constant(s), where each value assigned to the arbitrary constant(s) will correspond to a possible solution of the differential equation. The general solution to a differential equation of order n will contain n independent arbitrary constants.

For example, consider the differential equation $\frac{dy}{dx} = \cos(x)$.

The general solution to this equation will be $y = \sin(x) + C$ where C is an arbitrary constant.

8.3.2 Particular Solution

When we substitute a specific value for the arbitrary constant in the general solution of a differential equation, we get a particular solution to the differential equation. If we are asked to solve a differential equation with another condition given to us (such as the value of the function at a point), then we may be expected to find the particular situation.

For example, consider the differential equation $\frac{dy}{dx} = \frac{y}{x}$ with the condition that $y(1) = 1$.

The general solution to the differential will be $y = mx$ where m is an arbitrary constant/parameter, so it represents a family of straight lines passing through the origin.

The particular solution for the given condition will be $y = x$.

8.3.3 Singular Solution

A singular solution is a solution which has no arbitrary constant and isn't derived from the general solution. Instead, it exists separately in the differential equation.

For example, consider the differential equation $(y - 2x^2)(x dy - y dx) = 0$.

Then, for this equation, $y - 2x^2 = 0$ is a singular solution.

8.4 Forming Differential Equations from the General Solution

Given a general solution with n arbitrary constants, we will differentiate it n times and then eliminate all of the arbitrary constants. The resulting equation will be the required differential equation.

Example. Form a differential equation with general solution as $y = a \cos(x) + b \sin(x)$, where a and b are arbitrary constants.

$$\begin{aligned}y &= a \cos(x) + b \sin(x) \\y' &= -a \sin(x) + b \cos(x) \\y'' &= -a \cos(x) - b \sin(x) \\\Rightarrow y'' &= -(a \cos(x) + b \sin(x)) \\\Rightarrow y'' &= -y\end{aligned}$$

Or in Leibniz' Notation,

$$\frac{d^2 y}{dx^2} = -y$$

Problem 161. Form a differential equation for general solution $y = ae^x + be^{-x}$, where a and b are arbitrary constants.

Solution:

$$\begin{aligned}y &= ae^x + be^{-x} \\ \Rightarrow y' &= ae^x - be^{-x} \\ \Rightarrow y'' &= ae^x + be^{-x} \\ &\Rightarrow y'' = y \\ &\Rightarrow y'' - y = 0\end{aligned}$$

Or in Leibniz Notation,

$$\frac{d^2 y}{dx^2} - y = 0$$

Problem 162. Form a differential equation for general solution $y = c(x - c)^2$, where c is an arbitrary constant.

Solution:

$$\begin{aligned}y &= c(x - c)^2 & (1) \\ \Rightarrow y' &= 2c(x - c) & (2)\end{aligned}$$

Divide equation (1) by equation (2),

$$\begin{aligned}\frac{y}{y'} &= \frac{\cancel{c}(x - c)^2}{2\cancel{c}(x - c)} \\ \Rightarrow x - c &= \frac{2y}{y'} \\ \Rightarrow c &= x - \frac{2y}{y'}\end{aligned}$$

Substitute in equation (1),

$$y = \left(x - \frac{2y}{y'}\right) \left(\frac{2y}{y'}\right)^2$$

Problem 163. Form a differential equation for all circles touching the x -axis at the origin and centred at the y -axis.

Solution:

Let such a general circle have its centre at $(0, a)$.

Then, the general solution of the differential equation will be

$$x^2 + (y - a)^2 = a^2$$

where a is an arbitrary constant/parameter.

$$\Rightarrow x^2 + y^2 - 2ay = 0$$

Differentiating it gives us

$$\begin{aligned}2x + 2yy' - 2ay' &= 0 \\ x + yy' - ay' &= 0 \\ \Rightarrow a &= \frac{x + yy'}{y'}\end{aligned}$$

Substitute in the original equation:

$$x^2 + y^2 = 2y \left(\frac{x + yy'}{y'} \right) \Rightarrow y' = \frac{2xy}{x^2 - y^2}$$

Problem 164. Find the differential equation for all non-vertical lines in a plane.

Solution:

Any non-vertical line in a plane will be of the form

$$y = mx + c$$

where m and c are arbitrary constants/parameters. Then

$$y' = m$$

$$y'' = 0$$

And the arbitrary constants got eliminated on their own this time; we didn't have to do anything. So the required differential equation becomes:

$$\frac{d^2 y}{dx^2} = 0$$

Problem 165. Find the differential equation for all lines in the $x - y$ plane passing through the origin.

Solution:

The general form of such a line will be:

$$y = mx$$

Where m is an arbitrary constant.

Differentiate both sides:

$$\begin{aligned} y' &= m \\ \Rightarrow \frac{y}{y'} &= \frac{mx}{m} \\ \Rightarrow y &= \frac{dy}{dx} x \\ \Rightarrow y \, dx &= x \, dy \end{aligned}$$

8.5 Solving First Order and First Degree Differential Equations

8.5.1 Separable Differential Equations

A first order and first degree differential equation will be of the form

$$\frac{dy}{dx} = F(x, y)$$

If $F(x, y)$ can be expressed as a product $p(x)q(y)$, i.e. one function is exclusively in terms of x and the other exclusively in terms of y , then we can solve the equation by separating the variables. This will result in our differential equation becoming:

$$\begin{aligned} \frac{dy}{dx} &= p(x) \cdot q(y) \\ \Rightarrow \frac{1}{q(y)} \, dy &= p(x) \, dx \end{aligned}$$

And now after separating variables, we can take the antiderivative on both sides.

$$\Rightarrow \int \frac{1}{q(y)} \, dy = \int p(x) \, dx$$

If the antiderivatives of $p(x)$ and $\frac{1}{q(y)}$ are $P(x)$ and $Q(y)$ respectively, the general solution of the differential equation will become

$$Q(y) = P(x) + C$$

where C is an arbitrary constant.

Problem 166. Solve: $y \, dx - x \, dy = 0$

Solution:

$$\begin{aligned}y \, dx - x \, dy &= 0 \\ \Rightarrow y \, dx &= x \, dy \\ \Rightarrow \frac{dx}{x} &= \frac{dy}{y} && \text{Integrate both sides} \\ \Rightarrow \int \frac{1}{x} \, dx &= \int \frac{1}{y} \, dy \\ \Rightarrow \ln(|x|) + C &= \ln(|y|) && \text{Let } C = \ln(m) \\ \Rightarrow \ln(|y|) &= \ln(|x|) \cdot \ln(m) \\ \Rightarrow \ln(|y|) &= \ln(m|x|) && \text{Since the log function is one-one} \\ \Rightarrow |y| &= m|x|\end{aligned}$$

Problem 167. Solve: $\frac{dy}{dx} = \frac{x-1}{2-y}$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x-1}{2-y} \\ \Rightarrow (2-y) \, dy &= (x-1) \, dx && \text{Integrating both sides.} \\ \Rightarrow \int (2-y) \, dy &= \int (x-1) \, dx \\ \Rightarrow 2y - \frac{y^2}{2} &= \frac{x^2}{2} - x + C \\ \Rightarrow 4y - y^2 - x^2 + x + 2C &= 0 \\ \Rightarrow x^2 + y^2 - 4y - x - 2C &= 0 && \text{Let } -2C = D \\ \Rightarrow x^2 + y^2 - 4y - x + D &= 0\end{aligned}$$

Problem 168. Solve: $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1+y^2}{1+x^2} && \text{Separating the variables} \\ \Rightarrow \frac{dy}{1+y^2} &= \frac{dx}{1+x^2} && \text{Integrating both sides} \\ \Rightarrow \int \frac{dy}{1+y^2} &= \int \frac{dx}{1+x^2} \\ \Rightarrow \tan^{-1}(y) &= \tan^{-1}(x) + C\end{aligned}$$

Problem 169. Find the particular solution of $\frac{dy}{dx} = -4xy^2$ given that $x = 0$ when $y = 1$.

Solution:

$$\begin{aligned}\frac{dy}{dx} &= -4xy^2 && \text{Separate variables and integrate} \\ \Rightarrow \int \frac{dy}{y^2} &= -4 \int x \, dx \\ \Rightarrow \frac{-1}{y} &= -2x^2 + C \\ \Rightarrow y &= \frac{-1}{C - 2x^2}\end{aligned}$$

This is our general solution.

Now we're given that $x = 0$ when $y = 1$. Substituting $x = 0$ and $y = 1$ in the general solution.

$$\begin{aligned}\Rightarrow 1 &= \frac{-1}{C} \\ \Rightarrow C &= -1\end{aligned}$$

So the particular solution becomes

$$\begin{aligned}y &= \frac{-1}{-1 - 2x^2} \\ \Rightarrow y &= \frac{1}{2x^2 + 1}\end{aligned}$$

Problem 170. Solve the differential equation $(1 + e^{2x}) \, dy + (1 + y^2) \, e^x \, dx = 0$ if $y(0) = 1$.

Solution:

$$\begin{aligned}(1 + e^{2x}) \, dy + (1 + y^2) \, e^x \, dx &= 0 \\ \Rightarrow (1 + e^{2x}) \, dy &= - (1 + y^2) \, e^x \, dx \\ \Rightarrow \frac{dy}{1 + y^2} &= - \frac{e^x}{1 + e^{2x}} \, dx && \text{Integrate both sides} \\ \Rightarrow \int \frac{dy}{1 + y^2} &= - \int \frac{e^x}{1 + e^{2x}} \, dx && \text{Let } e^x = u \implies e^x \, dx = du \\ \Rightarrow \tan^{-1}(y) &= - \int \frac{1}{1 + u^2} \, du \\ \Rightarrow \tan^{-1}(y) &= - \tan^{-1}(u) + C \\ \Rightarrow \tan^{-1}(y) &= - \tan^{-1}(e^x) + C\end{aligned}$$

This is our general solution. Now, we're given $y(0) = 1$.

So substituting $x = 0$ and $y = 1$:

$$\begin{aligned}\Rightarrow \tan^{-1}(1) &= - \tan^{-1}(e^0) + C \\ \Rightarrow C &= 2 \tan^{-1}(1) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}\end{aligned}$$

That gives us the general solution as

$$\tan^{-1}(y) = - \tan^{-1}(e^x) + \frac{\pi}{2}$$

Problem 171. Let the population of rabbits in an area at time t be governed by $\frac{d(P(t))}{dt} = \frac{1}{2}P(t) - 200$ where $P(0) = 100$. Find an expression for $P(t)$.

Solution:

$$\begin{aligned}\frac{d(P(t))}{dt} &= \frac{1}{2}P(t) - 200 \\ \Rightarrow \frac{d(P(t))}{\frac{1}{2}P(t) - 200} &= dt\end{aligned}$$

$$\text{Let } \frac{1}{2}P(t) - 200 = y \implies d(P(t)) = 2 dy$$

$$\Rightarrow 2 \frac{dy}{y} = dt$$

Integrating both sides,

$$\begin{aligned}2 \int \frac{dy}{y} &= dt \\ \Rightarrow 2 \ln(|y|) &= t + C_1 \\ \Rightarrow 2 \ln\left(\left|\frac{1}{2}P(t) - 200\right|\right) &= t + C_1 \\ \Rightarrow \ln\left(\left|\frac{1}{2}P(t) - 200\right|\right) &= \frac{t}{2} + C_2\end{aligned}$$

$$\text{where } C_2 = \frac{C_1}{2}$$

$$\Rightarrow \frac{1}{2}P(t) - 200 = \pm e^{t/2} \cdot e^{C_2}$$

$$\text{Let } \pm e^{C_2} = K$$

$$\begin{aligned}\Rightarrow \frac{1}{2}P(t) - 200 &= K e^{t/2} \\ \Rightarrow P(t) &= 2K e^{t/2} + 400\end{aligned}$$

$$\text{Now, } P(0) = 100,$$

$$\begin{aligned}\Rightarrow 100 &= 2K + 400 \\ \Rightarrow K &= \frac{-300}{2} = -150\end{aligned}$$

So that gives us the population function as

$$P(t) = -300e^{t/2} + 400$$

Problem 172. Solve: $\frac{2 + \sin(x)}{y + 1} \cdot \frac{dy}{dx} = -\cos(x)$

Solution:

$$\frac{2 + \sin(x)}{y + 1} \cdot \frac{dy}{dx} = -\cos(x)$$

Separating variables

$$\frac{1}{y + 1} dy = \frac{-\cos(x)}{2 + \sin(x)} dx$$

Integrating both sides

$$\int \frac{1}{y + 1} dy = - \int \frac{\cos(x)}{2 + \sin(x)} dx$$

$$\Rightarrow \ln(|y + 1|) = -\ln(|2 + \sin(x)|) + C_1$$

Let $C_1 = \ln(C_2)$

$$\Rightarrow \ln(|y + 1|) = +\ln\left(\frac{C_2}{|2 + \sin(x)|}\right)$$

$$\Rightarrow y + 1 = \frac{\pm C_2}{2 + \sin(x)}$$

Let $K = \pm C_2$

$$\Rightarrow (y + 1)(2 + \sin(x)) = K$$

And this is our general solution.

Problem 173. Solve: $\frac{dy}{dx} = \frac{1 - \cos(x)}{1 + \cos(x)}$

Solution:

$$\frac{dy}{dx} = \frac{1 - \cos(x)}{1 + \cos(x)}$$

Separate variables and integrate:

$$\int dy = \int \frac{1 - \cos(x)}{1 + \cos(x)} dx$$

$$\Rightarrow y = \int \frac{1 - \cos(x)}{1 + \cos(x)} dx$$

$$= \int \frac{2 \sin^2\left(\frac{x}{2}\right)}{2 \cos^2\left(\frac{x}{2}\right)} dx$$

$$= \int \tan^2\left(\frac{x}{2}\right) dx$$

$$= \int \left(\sec^2\left(\frac{x}{2}\right) - 1 \right) dx$$

$$= 2 \tan\left(\frac{x}{2}\right) - x + C$$

$$\Rightarrow y = 2 \tan\left(\frac{x}{2}\right) - x + C$$

And we have our general solution.

8.5.2 Reducible to Separable Form

Some differential equations are not directly expressed in variable-separable form but can be reduced to the variable separable form using a suitable substitution.

It's generally a good idea to try such substitutions on differential equations of the form:

$$\frac{dy}{dx} = f(ax + by + c)$$

We can substitute $t = ax + by + c$

$$\text{Implying } \frac{dt}{dx} = a + b \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{\frac{dt}{dx} - a}{b}$$

So the differential equation becomes

$$\frac{\frac{dt}{dx} - a}{b} = f(t)$$

This is now separable:

$$\frac{dt}{b \cdot f(t) + a} = dx$$

From here, we can just integrate and get the general solution.

Problem 174. Solve: $\frac{dy}{dx} = x + y + 1$

Solution:

$$\frac{dy}{dx} = x + y + 1$$

Substitute $x + y + 1 = t$

$$\implies 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\implies \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\implies \frac{dt}{dx} - 1 = t$$

$$\implies \frac{dt}{dx} = t + 1$$

Separate variables and integrate

$$\implies \int dx = \int \frac{1}{t+1} dt$$

$$\implies x = \ln(|t+1|) + C$$

$$\implies x = \ln(|x+y+2|) + C$$

$$\implies x + y + 2 = e^{x-C}$$

$$\implies y = e^{x-C} - x - 2$$

Let $e^{-C} = k$

$$\implies y = ke^x - x - 2$$

Problem 175. Solve: $\frac{dy}{dx} = \cos(x + y)$

Solution:

$$\frac{dy}{dx} = \cos(x + y)$$

Substitute $x + y = t$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\Rightarrow \frac{dt}{dx} - 1 = \cos(t)$$

$$\Rightarrow \frac{dt}{dx} = \cos(t) + 1$$

Separate variables and integrate

$$\Rightarrow \int dx = \int \frac{1}{1 + \cos(t)} dt$$

$$\Rightarrow x = \int \frac{1}{2 \cos^2\left(\frac{t}{2}\right)} dt$$

$$= \frac{1}{2} \int \sec^2\left(\frac{t}{2}\right) dt$$

$$= \tan\left(\frac{t}{2}\right) + C$$

$$\Rightarrow x = \tan\left(\frac{x + y}{2}\right) + C$$

8.5.3 Homogeneous Differential Equation

Let us first generalize the idea of a homogeneous polynomial to define a homogeneous function. Recall that a homogeneous polynomial is a polynomial expression where each term has the same degree.

Definition 30. A homogeneous function is a (possibly multivariate) function that shows multiplicative scaling behaviour. If all of its constants are multiplied by a factor, then its value is multiplied by some integer power of that factor.

i.e. a function $f(x, y)$ is said to be homogeneous iff

$$f(\lambda x, \lambda y) = \lambda^n \cdot f(x, y)$$

for some integer n .

In general, a function $g(x_1, x_2, x_3, \dots, x_k)$ is said to be homogeneous iff

$$g(\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_k) = \lambda^n \cdot f(x_1, x_2, x_3, \dots, x_k)$$

for some integer n .

Here, n is called the **degree of homogeneity**.

Notice that by this definition, all homogeneous polynomials will also be homogeneous functions.

Let us consider some examples of functions.

Example. $f(x, y) = \cos\left(\frac{x}{y}\right)$

$$f(\lambda x, \lambda y) = \cos\left(\frac{\lambda x}{\lambda y}\right) = \lambda^0 \cdot f(x, y)$$

So this is a homogeneous function of degree 0.

Example. $f(x, y) = 2x - 3y$

$$f(\lambda x, \lambda y) = 2\lambda x - 3\lambda y = \lambda(2x - 3y) = \lambda^1 \cdot f(x, y)$$

This is a homogeneous function of degree 1.

Example. $f(x, y) = \sin(x) + \cos(y)$

$$f(\lambda x, \lambda y) = \sin(\lambda x) + \cos(\lambda y) \neq \lambda^n f(x, y)$$

This is a non-homogeneous function.

Claim: Any homogeneous function $f(x, y)$ of degree n can be expressed as

$$f(x, y) = x^n \cdot g\left(\frac{y}{x}\right)$$

or

$$f(x, y) = y^n \cdot h\left(\frac{x}{y}\right)$$

Proof. We are given that $f(x, y)$ is homogeneous, so it satisfies the following property by definition:

$$f(\lambda x, \lambda y) = \lambda^n \cdot f(x, y)$$

Since λ is an arbitrary parameter, we are allowed to substitute it with any real number.

Substitute $\lambda = \frac{1}{x}$.

$$\Rightarrow f\left(1, \frac{y}{x}\right) = \frac{1}{x^n} \cdot f(x, y)$$

$$\Rightarrow f(x, y) = x^n \cdot f\left(1, \frac{y}{x}\right)$$

$$\text{Let } f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right)$$

$$\Rightarrow f(x, y) = x^n \cdot g\left(\frac{y}{x}\right)$$

Similarly, substituting $\lambda = \frac{1}{y}$ will give us

$$f(x, y) = y^n \cdot f\left(\frac{x}{y}, 1\right)$$

$$\Rightarrow f(x, y) = y^n \cdot h\left(\frac{x}{y}\right)$$

□

Homogeneous differential equations:

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

is said to be homogeneous if f and g are homogeneous functions of the same degree.

In other words, a differential equation of the form

$$\frac{dy}{dx} = P(x, y)$$

is homogeneous if P is a homogeneous function of degree zero.

To solve a homogeneous differential equation of the form

$$\frac{dy}{dx} = P(x, y)$$

We first observe that P is a homogeneous function of degree zero. So we can express it as x^0 times a function of $\frac{y}{x}$.

$$\Rightarrow \frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Now, we substitute $y = v \cdot x$

Which means $\frac{y}{x} = v$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = g(v)$$

$$\Rightarrow x \frac{dv}{dx} = g(v) - v$$

Now, this differential equation is separable and can be solved by separating variables and integrating both sides.

$$\Rightarrow \frac{dv}{g(v) - v} = \frac{dx}{x}$$

Problem 176. Solve: $x \cos\left(\frac{y}{x}\right) \cdot \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$

Solution:

$$x \cos\left(\frac{y}{x}\right) \cdot \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)}$$

$$\text{Let } f(x, y) = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)}$$

$$f(\lambda x, \lambda y) = \frac{\lambda y \cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda x}{\lambda x \cos\left(\frac{y}{x}\right)} = \lambda^0 \cdot f(x, y)$$

So this is a homogeneous differential equation.

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

So the differential equation becomes:

$$v + x \frac{dv}{dx} = \frac{vx \cos(v) + x}{x \cos(v)}$$

$$\begin{aligned}
&\Rightarrow v + x \frac{dv}{dx} = \frac{v \cos(v) + 1}{\cos(v)} \\
&\Rightarrow x \frac{dv}{dx} = \frac{v \cos(v) + 1}{\cos(v)} - v \\
&\Rightarrow x \frac{dv}{dx} = \frac{\cancel{v \cos(v)} + 1 - \cancel{v \cos(v)}}{\cos(v)} \\
&\Rightarrow x \frac{dv}{dx} = \frac{1}{\cos(v)} \quad \text{Separate variables and integrate.} \\
&\Rightarrow \int \frac{1}{x} dx = \int \cos(v) dv \\
&\Rightarrow \ln(|x|) + \ln(|c|) = \sin(v) \\
&\Rightarrow \sin\left(\frac{y}{x}\right) = \ln(|cx|)
\end{aligned}$$

Problem 177. Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Proof. We have the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

This is a homogeneous differential equation.

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 \left(1 + \frac{y^2}{x^2}\right)}{x^2} = \frac{1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

Let $y = v \cdot x$
 $\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$
 Substitute:

$$\begin{aligned}
&\Rightarrow v + x \frac{dv}{dx} = \frac{1 + v^2}{2v} \\
&\Rightarrow x \frac{dv}{dx} = \frac{1 + v^2}{2v} - v \\
&\Rightarrow x \frac{dv}{dx} = \frac{1 - v^2}{2v} \quad \text{Separate variables and integrate.} \\
&\Rightarrow \int \frac{1}{x} dx = \int \frac{2v}{1 - v^2} dv \quad \text{Let } v^2 = u \implies 2v dv = du \\
&\Rightarrow \int \frac{1}{1 - u} du = \ln|x| + \ln|k| \\
&\Rightarrow -\ln\left|1 - \frac{y^2}{x^2}\right| = \ln|kx| \\
&\Rightarrow \frac{x^2 - y^2}{x^2} = \pm \frac{1}{kx} \quad \text{Let } c = \pm \frac{1}{k} \\
&\Rightarrow x^2 - y^2 = cx
\end{aligned}$$

□

Problem 178. Solve: $\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right)$

Solution:

$$\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right)$$

This is clearly a homogeneous differential equation.

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \sin(v)$$

$$\Rightarrow x \frac{dv}{dx} = -\sin(v)$$

Separate variables and integrate

$$\Rightarrow \int \frac{1}{x} dx = - \int \csc(v) dv$$

$$\Rightarrow \ln|x| + \ln|c| = \ln|\csc(v) + \cot(v)|$$

$$\Rightarrow \csc(v) + \cot(v) = kx$$