

Calculus

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1 Limits

1.1 Definition

1.1.1 Provisional Definition

Definition 1. $\lim_{x \rightarrow c} f(x) = L$ if we can make $f(x)$ as close as we like to L by requiring that x be sufficiently close to, but not necessarily equal to c .

1.1.2 Rigorous Definition

Definition 2. Let $f(x)$ be a function with domain D .

Then $\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 : \forall x \in D, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$

1.2 Standard Limits

- $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\sin^{-1}(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\tan(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\tan^{-1}(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$
- $\lim_{x \rightarrow 0} \frac{\log_e(1 + x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$
- $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$

1.3 Other Ideas

1.3.1 Uniqueness

The limit of a function at a given point is **unique**. If two or more different values of the limit can be found, the limit is said to not exist. Consequently, the left hand limit of a function at a point must equal the right hand limit at that point for the overall limit to exist.

1.3.2 Sequential Criterion for Existence of Limits

For all sequences that approach c , the answer to $\lim_{x \rightarrow c} f(x)$ should evaluate to the same quantity if the overall limit exists.

$\lim_{x \rightarrow c} f(x) = L$ iff for all sequences x_n (with $x_n \neq c$ for all n) converging to c , $f(x_n)$ converges to L .

Application: Proof that $\lim_{x \rightarrow \infty} \sin(x)$ does not exist.

The idea: We will take two sequences that both blow up to infinity (i.e. get larger and larger as we increase the index variable) and show that when we evaluate the limits using both the sequences, we get different values. By the sequential criterion for existence of limits, we will show that the limit does not exist.

Proof. (kind of) (Using sequential criterion of limits)

Consider the sequence $a_n = n\pi$ where $n \in \mathbb{Z}^+$.

n can be increased to get arbitrarily close to infinity.

One way to evaluate the limit $\lim_{x \rightarrow \infty} \sin(x)$ would be to evaluate $\lim_{n \rightarrow \infty} \sin(a_n)$ where $n \in \mathbb{Z}^+$.

Then,

$$\lim_{n \rightarrow \infty} \sin(a_n) = \lim_{n \rightarrow \infty} \sin(n\pi)$$

But $\sin(n\pi) = 0 \forall n \in \mathbb{Z}$.

Therefore, this would give us the limiting value as 0.

However, now consider the sequence $b_n = (2n+1)\frac{\pi}{2}$.

Again, b_n can be made arbitrarily large by increasing n .

Similarly,

Another way to evaluate the limit $\lim_{x \rightarrow \infty} \sin(x)$ would be to evaluate $\lim_{n \rightarrow \infty} \sin(b_n)$ where $n \in \mathbb{Z}^+$.

Then,

$$\lim_{n \rightarrow \infty} \sin(b_n) = \lim_{n \rightarrow \infty} \sin\left((2n+1)\frac{\pi}{2}\right)$$

But $\sin\left((2n+1)\frac{\pi}{2}\right) = 1 \forall n \in \mathbb{Z}$.

That would give us the limiting value as 1.

But by the property of uniqueness of limits, the limit of a function at a point must be unique.

Therefore the limit does not exist. \square

1.4 L'Hôpital's Rule

If a function f is differentiable in any neighbourhood of a , except maybe at a ,

And $\left[\left(\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0 \right) \text{ or } \left(\lim_{x \rightarrow a} \frac{1}{f(x)} = 0 \right) \text{ and } \lim_{x \rightarrow a} \frac{1}{g(x)} = 0 \right]$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

1.5 Guidelines for Evaluating Limits

Do not apply limits non-uniformly.

If you apply a standard limit formula to one part of an expression, this should be done to the entire expression at once.

Never approximate.

For example, this is incorrect:

$$\lim_{x \rightarrow 0} \frac{e^x - \frac{\sin(x)}{x}}{x} = \lim_{x \rightarrow 0} \frac{e^x - \frac{x}{x}}{x} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

Do not employ L'Hôpital's Rule recklessly.

Some problems will result in an infinite loop of differentiation if you try to solve them with L'Hôpital's Rule. Others will become tedious because of long derivatives. Anyway, evaluating limits should be analytical, not mechanical.

1.6 One-Sided Limits

1.6.1 Left Hand Limit

Definition 3. Let $f(x)$ be a function and let I be an interval within its domain.

Then $\lim_{x \rightarrow a^-} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 : \forall x \in I, 0 < x - a < \delta \Rightarrow 0 < |f(x) - L| < \epsilon$.

1.6.2 Right Hand Limit

Definition 4. Let $f(x)$ be a function and let I be an interval within its domain.

Then $\lim_{x \rightarrow a^+} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 : \forall x \in I, 0 < a - x < \delta \Rightarrow 0 < |f(x) - L| < \epsilon$.

1.7 Cluster Point

Definition 5. Say a function f is defined over set S , then $x = a$ is said to be a cluster point of the function f if $\forall \delta > 0, \exists x \in (a - \delta, a + \delta) : x \neq a$ but $x \in S$.

We can check the limit and continuity of a function only at cluster points. If a point is not a cluster point, it is called an isolated point.

1.8 Polynomial Approximation of Functions to Arbitrary Precision

1.8.1 Taylor Series

If a function f is continuous and differentiable infinitely many times and its successive derivatives are continuous infinitely many times, then for some a in the domain of f ,

$$f(x) = \sum_{n=0}^{\infty} f^n(a) \cdot \frac{(x-a)^n}{n!}$$

Where $f^n(x)$ is the n^{th} derivative of $f(x)$.

1.8.2 Maclaurin Series

Maclaurin Series is just the Taylor series for $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} f^n(0) \cdot \frac{x^n}{n!}$$

Where $f^n(x)$ is the n^{th} derivative of $f(x)$

2 Continuity

2.1 Definition

Definition 6. A function f is said to be continuous at a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.

But we know that the limit only exists if both its right hand limit and left hand limit are equal. So we can expand the condition to:

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$$

$$\text{or } \lim_{h \rightarrow 0^+} f(a+h) = \lim_{h \rightarrow 0^-} f(a+h) = f(a)$$

All polynomial functions, logarithmic function, sine, cosine and exponential function are examples of continuous functions.

2.2 Continuity of Combinations of Continuous Functions

Let f and g be two functions which are continuous at a , then:

1. $h(x) = f(x) + g(x)$ is continuous at $x = a$
2. $h(x) = f(x) - g(x)$ is continuous at $x = a$
3. $h(x) = f(x) \cdot g(x)$ is continuous at $x = a$
4. $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$.

In fact, if f and g are continuous functions, then $\frac{f(x)}{g(x)}$ is continuous for all x except those values for which $g(x) = 0$

2.3 Continuity of Composite Functions

If the function g is continuous at a and the function f is continuous at $g(a)$, then the function $f(g(x))$ is continuous at a .

Example. Given $f(x) = \frac{1}{1-x}$, find the points of discontinuity of $f(f(x))$.

Solution: $f(x)$ is the ratio of two continuous (polynomial) functions. Therefore, it will have a discontinuity where $1-x=0$, i.e. at $x=1$.

So $x=1$ is one of the points where $f(f(x))$ is discontinuous.

Now, $f(f(x))$ will be discontinuous when $f(x)=1$.

$$\begin{aligned} f(x) &= 1 \\ \Rightarrow \frac{1}{1-x} &= 1 \\ \Rightarrow x &= 0 \end{aligned}$$

Therefore, there two points of discontinuity: at $x=0$ and at $x=1$.

2.4 Product of Continuous and Discontinuous Function

Let f be a continuous function and g be a discontinuous function and assume that all the limits are real and finite.

$$\text{Say } K_1 = \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

$$\text{Say } K_2 = \lim_{x \rightarrow a^-} g(x) \neq \lim_{x \rightarrow a^+} g(x) = g(a) = K_3$$

And let $h(x) = f(x) + g(x)$. We have to comment on the nature of $h(x)$.

$$\lim_{x \rightarrow a^-} h(x) = \lim_{x \rightarrow a^-} (f(x) \cdot g(x)) = K_1 K_2$$

$$\lim_{x \rightarrow a^+} h(x) = \lim_{x \rightarrow a^+} (f(x) \cdot g(x)) = K_1 K_3$$

$$h(x) = f(x) \cdot g(x) = K_1 K_3$$

Clearly, for $h(x)$ to be continuous,

$$\begin{aligned} K_1 K_2 &= K_1 K_3 \\ \Rightarrow K_1 \cdot (K_2 - K_3) &= 0 \end{aligned}$$

(But $K_2 \neq K_3$)

$$\Rightarrow K_1 = 0$$

Conclusion: The product of a continuous and a discontinuous function will be continuous at a point iff the value of the continuous function at that point is zero.

Example. Where will the function $f(x) = [x] \cdot \cos\left(\left(\frac{2x-1}{2}\right)\pi\right)$ be discontinuous?

Solution: Note that $\cos\left(\left(\frac{2x-1}{2}\right)\pi\right)$ is continuous and $[x]$ is discontinuous. The vulnerable points are integer values of x , because $[x]$ is discontinuous at integer values.

But if $x \in \mathbb{Z}$, then $\cos\left(\left(\frac{2x-1}{2}\right)\pi\right) = \cos\left((2x-1)\frac{\pi}{2}\right) = 0$.

Using the above result, since the continuous function in the product is zero at the points of discontinuity of the discontinuous function, this function is continuous everywhere.

Example. Where will the function $f(x) = [x] \cdot \sin(x\pi)$ be discontinuous?

Solution: By the same reasoning as the previous example, if x is an integer (vulnerable point of the floor function), the continuous function $\sin(x\pi) = 0$, so the function product is continuous everywhere.

2.5 Calculus and Zeros

2.5.1 Warm-Up Problem

Example. Let $p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial in x with real coefficients. Assume that $p(0) = 1$, $p(1) = 2$, $p(2) = 3$ and $p(3) = 4$. Then find the value of $p(4)$.

Solution: Let us define a function $g(x) = p(x) - (x+1)$.

Then $g(0) = p(0) - (0+1) = 1 - 1 = 0$.

Similarly $g(1) = g(2) = g(3) = 0$.

Now, $g(x) = p(x) - (x+1) = (x)(x-1)(x-2)(x-3)$ (Because $g(x)$ is a polynomial of degree 4 with roots 0, 1, 2, 3)

So $p(x) = g(x) + (x+1) = x(x-1)(x-2)(x-3) + (x+1)$.

$\therefore p(4) = 4(3)(2)(1) + 5 = 29$

In general, if a problem looks like $f(x) = g(x)$ for some values of x , define $h(x) = f(x) - g(x)$. Then those values of x will be the zeros of $h(x)$.

2.5.2 Bolzano's Theorem

Theorem 1. Bolzano's Theorem: Let a function f be continuous in the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs. Then there exists at least one $c \in (a, b)$ such that $f(c) = 0$. In other words, there exists at least 1 zero of $f(x)$ in (a, b) .

2.5.3 Important Results Derived From Bolzano's Theorem

Theorem 2. Every polynomial of odd degree will have at least one zero.

Proof. (Using Bolzano's Theorem)

Let $f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_2x^2 + a_1x + a_0$ where $a_{2n+1} \neq 0$.

(Note that $f(x)$ is continuous $\forall x \in \mathbb{R}$)

$$\Rightarrow f(x) = x^{2n+1} \left(a_{2n+1} + \frac{a_{2n}}{x} + \frac{a_{2n-1}}{x^2} + \dots + \frac{a_1}{x^{2n}} + \frac{a_0}{x^{2n+1}} \right)$$

Case 1: ($a_{2n+1} > 0$)

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

Using Bolzano's Theorem, there exists at least one root of $f(x)$ in the interval $(-\infty, \infty)$.

Case 2: ($a_{2n+1} < 0$)

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

Again, Using Bolzano's Theorem, there exists at least one root of $f(x)$ in the interval $(-\infty, \infty)$. □

Theorem 3. For a polynomial of even degree whose leading coefficient and constant term are of opposite signs, there exist at least two real zeros.

Proof. (Using Bolzano's Theorem)

Let $f(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \dots + a_2x^2 + a_1x + a_0$.

$$\Rightarrow f(x) = x^{2n} \left(a_{2n} + \frac{a_{2n-1}}{x} + \frac{a_{2n-2}}{x^2} + \dots + \frac{a_3}{x^{2n-2}} + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}} \right)$$

WLOG, let $a_{2n} > 0$ and $a_0 < 0$.

Then $f(0) = a_0 < 0$. (1)

And $\lim_{x \rightarrow \infty} f(x) = \infty > 0$. (2)

Also, $\lim_{x \rightarrow -\infty} f(x) = \infty > 0$. (3)

Applying Bolzano's Theorem on (1) and (2), there exists at least one root of $f(x)$ in $(0, \infty)$

Applying Bolzano's Theorem on (1) and (3), there exists at least one root of $f(x)$ in $(-\infty, 0)$

Therefore, there exist at least two real zeros of $f(x)$. □

2.5.4 Problems based on Bolzano's Theorem

Example. Suppose $f : [a, b] \rightarrow [a, b]$ is a continuous function. Prove that there exists some $x_0 \in [a, b]$ such that $f(x_0) = x_0$. What does it say about the graph of $y = f(x)$?

Proof. Let $g(x) = f(x) - x$.

g is continuous in $[a, b]$ since the difference of two continuous functions is continuous.

$$g(a) = f(a) - a,$$

$$g(b) = f(b) - b.$$

Case 1: $f(a) = a$

In this case, the given statement is trivially true, where $x_0 = a$.

Case 2: $f(b) = b$

In this case also, the given statement is trivially true, where $x_0 = b$.

Case 3: $f(a) \neq a$ and $f(b) \neq b$

Since $f(a) \neq a$ and $f(b) \neq b$ and the co-domain of f is $[a, b]$

$$\Rightarrow f(a) > a \text{ and } f(b) < b$$

$$\Rightarrow f(a) - a > 0 \text{ and } f(b) - b < 0$$

$$\Rightarrow g(a) > 0 \text{ and } g(b) < 0$$

Using Bolzano's Theorem, $g(a) = f(a) - a$ has a root in (a, b) .

$$\Rightarrow \exists x_0 : f(x_0) = x_0.$$

This means, the graph of $y = f(x)$ intersects the graph of $y = x$ at least once.

□

Example. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function and $f(0) = f(1)$. Then prove that there exists some $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

Proof. (Using Bolzano's Theorem)

$$\text{Let } g(x) = f(x) - f\left(x + \frac{1}{2}\right)$$

g is continuous in \mathbb{R} since f is continuous in \mathbb{R} .

$$g(0) = f(0) - f\left(\frac{1}{2}\right) \tag{1}$$

$$g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) \tag{2}$$

$$g(0) + g\left(\frac{1}{2}\right) = f(0) - f(1) = 0 \tag{3}$$

$$g(0) + g\left(\frac{1}{2}\right) = 0$$

$$\text{Case 1: } g(0) = g\left(\frac{1}{2}\right) = 0$$

In this case, the given statement will be trivially true where c can be 0 or $\frac{1}{2}$.

$$\text{Case 2: } g(0) > 0 \text{ and } g\left(\frac{1}{2}\right) < 0$$

$$\text{Using Bolzano's Theorem, } \exists c \in \left(0, \frac{1}{2}\right) : f(c) = f\left(c + \frac{1}{2}\right) \Rightarrow c \in \left[0, \frac{1}{2}\right] \quad \because \left(0, \frac{1}{2}\right) \subset \left[0, \frac{1}{2}\right]$$

$$\text{Case 3: } g(0) < 0 \text{ and } g\left(\frac{1}{2}\right) > 0$$

$$\text{Using Bolzano's Theorem, } \exists c \in \left(0, \frac{1}{2}\right) : f(c) = f\left(c + \frac{1}{2}\right) \Rightarrow c \in \left[0, \frac{1}{2}\right] \quad \because \left(0, \frac{1}{2}\right) \subset \left[0, \frac{1}{2}\right]$$

□

Example. Suppose f is a continuous function on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number, then prove that there exists some c such that $f(c) = f\left(c + \frac{1}{n}\right)$.

Proof. (Note: this is just the generalized version of the previous problem)

$$\text{Let } g(x) = f(x) - f\left(x + \frac{1}{n}\right)$$

g is continuous in \mathbb{R} since f is continuous in \mathbb{R} .

We get the following telescoping series.

$$\begin{aligned} g(0) &= f(0) - f\left(\frac{1}{n}\right) \\ g\left(\frac{1}{n}\right) &= f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right) \\ g\left(\frac{2}{n}\right) &= f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right) \\ &\vdots = \vdots \\ g\left(\frac{n-2}{n}\right) &= f\left(\frac{n-2}{n}\right) - f\left(\frac{n-1}{n}\right) \\ g\left(\frac{n-1}{n}\right) &= f\left(\frac{n-1}{n}\right) - f(1) \end{aligned}$$

$$g(0) + g\left(\frac{1}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) = f(0) - f(1) = 0 \quad [\because f(0) = f(1)]$$

$$\Rightarrow g(0) + g\left(\frac{1}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) = 0$$

$$\text{Case 1: } g(0) = g\left(\frac{1}{n}\right) = \cdots = g\left(\frac{n-1}{n}\right) = 0$$

In this case, the given statement is trivially true for $c = 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$.

Case 2: At least one term is positive and at least one term is negative.

WLOG, Let $g(a)$ be one of the positive term(s) and $g(b)$ be one of the negative term(s) and let $a < b$.

Since g is continuous and $g(a) > 0; g(b) < 0$,

Using Bolzano's Theorem, $\exists c : g(c) = 0$

$$\Rightarrow \exists c : f(c) - f\left(c + \frac{1}{n}\right) = 0$$

$$\Rightarrow \exists c : f(c) = f\left(c + \frac{1}{n}\right)$$

□

2.6 Checking Continuity of Some Weird Functions

2.6.1 Dirichlet Function

The Dirichlet Function $f : \mathbb{R} \rightarrow \{0, 1\}$ is defined in the following way:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

In other words, the Dirichlet Function maps to 1 if x is rational and 0 if x is irrational.

Claim: The Dirichlet Function is discontinuous everywhere.

If the input at some point is rational, we can approach it using a sequence consisting entirely of rational numbers and a sequence consisting of entirely irrational numbers; the former will give us the limiting value as 1 and the latter will give us the limiting value as 0.

Similarly, if the input at some point is irrational, we can again approach it with sequences consisting entirely of rational or irrational terms, resulting in two limiting values: 0 and 1.

Therefore, the limit of this function doesn't exist at any point, which means the Dirichlet function discontinuous everywhere.

2.6.2 Thomae's Function

Thomae's Function (commonly called the Popcorn Function) is defined in the following way:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} : m, n \in \mathbb{Z}; n \neq 0; \gcd(m, n) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Claim: This function is discontinuous at all rational values of x .

Proof. (analogous to that of Dirichlet Function)

Let $x = \frac{m}{n}$ be a rational number where $m, n \in \mathbb{Z}, \gcd(m, n) = 1$

Then $f(x) = \frac{1}{n}$

If we approach x from a sequence of irrationals, we will get the limiting value as 0. Even if the limit does exist, $\frac{1}{n}$ (the value of the function) cannot be equal to 0 (The limiting value).

So it has to be discontinuous for all rational inputs. □

Claim: This function is continuous at all irrational values of x .

Proof. Let x_0 be an irrational number.

$\Rightarrow f(x_0) = 0$

Consider a sequence $x_n (n \geq 1)$ such that $\lim_{n \rightarrow \infty} x_n = x_0$.

Case 1: If x_n is irrational $\forall n$, $\lim_{x \rightarrow \infty} f(x_n) = 0$.

Case 2: Otherwise, Let x_{n_k} be a subsequence of x_n , where $x_{n_k} \in \mathbb{Q} \forall k$. Again, $\lim_{k \rightarrow \infty} x_{n_k} = x_0$

Let $x_{n_k} = \frac{p_{n_k}}{q_{n_k}}$

Then $q_{n_k} \rightarrow \infty$ as $k \rightarrow \infty \Rightarrow \lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} \left(\frac{1}{q_{n_k}} \right) = 0$

$\lim_{x \rightarrow x_0} f(x) = f(x_0) = 0$

Therefore, $f(x)$ is continuous at all irrational points. □

2.7 The Intermediate Value Theorem

2.7.1 Statement and Proof

Theorem 4. If f is a function that is continuous in the closed interval $[a, b]$, then $f(x)$ takes all the values lying between $f(a)$ and $f(b)$ (both inclusive).

Note that although technically Bolzano's Theorem is a corollary of the Intermediate Value Theorem, we will prove the Intermediate Value Theorem using Bolzano's Theorem here. Alternatively, Bolzano's theorem can be proved by Bisection.

Proof. WLOG, let $f(a) < f(b)$.

Let k be any real number between $f(a)$ and $f(b)$.

$\Rightarrow f(a) < k < f(b)$

Then we have to prove that there exists at least one solution to $f(x) = k$.

Let us define $g(x) = f(x) - k$.

Then $g(a) = f(a) - k < 0$

And $g(b) = f(b) - k > 0$

\Rightarrow There exists at least one c such that $g(c) = 0$ or $f(c) - k = 0$ or $f(c) = k$ □

Important: The converse of this is not necessarily true.

2.7.2 Example Problems

Example. Does the function $f(x) = \frac{x^3}{4} - \sin(\pi x) + 3$ take the value $\frac{7}{3}$ in the interval $[-2, 2]$?

Solution. $f(x)$ is continuous everywhere because $\frac{x^3}{4}$, $\sin(\pi x)$ and 3 are continuous everywhere.

$$f(-2) = \frac{(-2)^3}{4} - \sin(-2\pi) + 3 = 1 + \sin(2\pi) = 1$$

$$f(2) = \frac{(2)^3}{4} - \sin(2\pi) + 3 = 5 - \sin(2\pi) = 5$$

$\Rightarrow f(x)$ takes all values in $[1, 5]$ in the interval $[-2, 2]$ (By Intermediate Value Theorem)

And $\frac{7}{3} = 2.333 \dots \in [1, 5]$.

Therefore, yes, $f(x)$ will take the value $\frac{7}{3}$ in the interval $[-2, 2]$.

2.8 Boundedness

2.8.1 Definition

Definition 7. A function f defined on a set S (domain) is said to be bounded if $\exists M > 0$ such that $|f(x)| \leq M \forall x \in S$

Geometrically call a function bounded if the function can be contained between two horizontal lines $y = x$ and $y = -x$

2.9 Absolute Maxima and Minima

2.9.1 Absolute/Global Maximum

Definition 8. Let f be a function defined on set S . The point $c \in S$ is said to be a point of absolute or global maximum if $f(c) \geq f(x) \forall x \in S$

2.9.2 Absolute/Global Minimum

Definition 9. Let f be a function defined on set S . The point $c \in S$ is said to be a point of absolute or global minimum if $f(c) \leq f(x) \forall x \in S$

2.10 Properties of a Continuous Function

If a function f is continuous on the closed interval $[a, b]$, then:

1. f is bounded on the given interval.
2. There is a point of absolute minimum and maximum in this interval.
3. Say m is the least value of f and M is the maximum value of f , then f will take all values lying in the interval $[m, M]$, i.e. $m \leq f(x) \leq M$ (consequence of the Intermediate Value Theorem).
4. A continuous function f defined on $[a, b]$ maps to a closed interval $[c, d]$