Topic- Application of Functional Analysis with respect to Brownian Motion

NAME: Aryan Raj
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Introduction

The concept of **Brownian motion**, which is a fundamental concept in probability theory and stochastic calculus, is not traditionally included in Functional Analysis. Below is offered a presentation which relies on some of the big theorems from functional analysis, and especially on $L2$-Hilbert spaces, built from probability measures on function spaces.

Here is also included a brief discussion of Brownian motion in order to illustrate infinite Cartesian products and unitary one-parameter group $\{U(t)\}t \in \mathrm{R}$ acting in Hilbert space.

NOTE: Hilbert Space: An L^2 space is a Hilbert space, which means it is a complete inner product space. Completeness implies that every Cauchy sequence in the space converges to a limit within the space. The completeness property is crucial in functional analysis.

Definitions

Definition . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e.,

- $\Omega =$ sample space
- $\mathcal{F} =$ sigma algebra of events
- $\mathbb{P} =$ probability measure defined on $\mathcal{F}$.

A function $X : \Omega \to \mathbb{R}$ is called a *random variable if it is a measurable function, i.e., if for all intervals $(a,b) \subset \mathbb{R}$ the inverse image*

$$X^{-1}((a,b)) = \{\omega \in \Omega \,|\, X(\omega) \in (a,b)\}$$

is in $\mathcal{F}$; and we write $\{a < X(\omega) < b\} \in \mathcal{F}$ in short-hand notation.

We say that X is Gaussian (written $N(m, \sigma^2)$) if

$$\mathbb{P}(\{a < X(\omega) < b\}) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} dx.$$

(The function under the integral in is called the Gaussian distribution.)

Note: Snapshot taken from the book Functional Analysis with Applications

Measurable Functions in Functional Analysis

Other Definitions

Another Definition can be when, Events A, $B \in$ F are said to be *independent* if,

P($A \cap B$) = P(A)P(B).

Random variables X and Y are said to be independent iff (Def.) X−1 (I) and Y −1 (J) are independent for all intervals I and J.

Definition A family $\{X_t\}_{t\in\mathbb{R}}$ of random variables for $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion iff (Def.) for every $n \in \mathbb{N}$,

1. the random variables $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are jointly Gaussian with

$$\mathbb{E}(X_t) = \int_\Omega X_t(\omega)\, d\mathbb{P}(\omega) = 0, \; \forall t \in \mathbb{R};$$

2. if $t_1 < t_2 < \cdots < t_n$ then $X_{t_{i+1}} - X_{t_i}$ and $X_{t_i} - X_{t_{i-1}}$ are independent;
3. for all $s, t \in \mathbb{R}$,

$$\mathbb{E}\left(|X_t - X_s|^2\right) = |t - s|.$$

The Path Space

Theorem: Set Ω = C (R) = (all continuous real valued function on R), F = the sigma algebra generated by cylinder-sets, i.e., determined by

finite systems t1,...,tn, and intervals J1,...,Jn;

Cyl(t1,...,tn,J1,...,Jn)=ω ∈C(R) ω(ti)∈Ji, i=1,2...,n . The measure P *is determined by its value on cylinder sets, and and an integral over*

Gaussians; it is called the Wiener-measure. Set
Xt (ω) = ω (*t*) , ∀*t* ∈ R, ω ∈ Ω (= *C* (R)) .

If 0<*t*1 <*t*2 <···<*tn,and the cylinder set is as in,then* P(*Cyl*(*t*1,...,*tn*,*J*1,...,*Jn*))

$$= \int_{J_1} \cdots \int_{J_n} g_{t_1}(x_1)\, g_{t_2-t_1}(x_2-x_1) \cdots g_{t_n-t_{n-1}}(x_n-x_{n-1})\, dx_1 \cdots dx_n$$

where

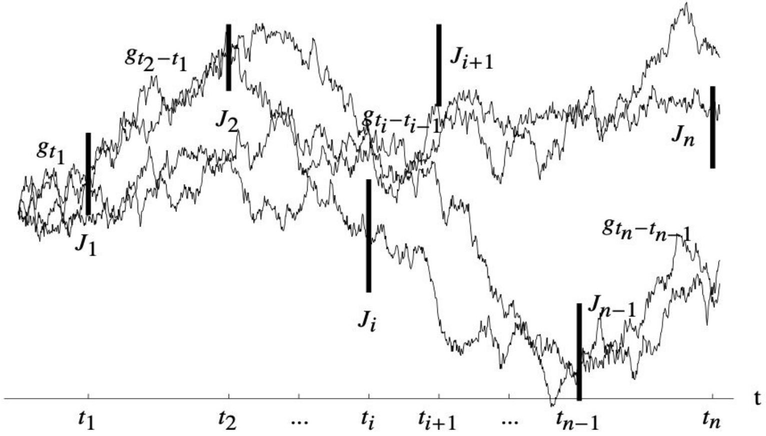
$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \ \forall t > 0,$$

i.e., the $N(0,t)$-Gaussian.

Note: A Wiener measure set refers to a particular set of paths of a Brownian motion.

Note: Cylinder sets are a fundamental concept in probability theory and stochastic processes, allowing for the precise description of events or constraints on the behavior of random processes over finite time intervals. They provide a structured way to understand the dependence of a stochastic process on specific time points.

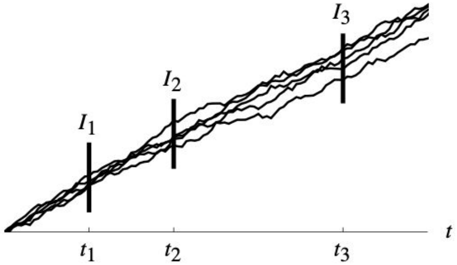
Stochastic Processes indexed by time



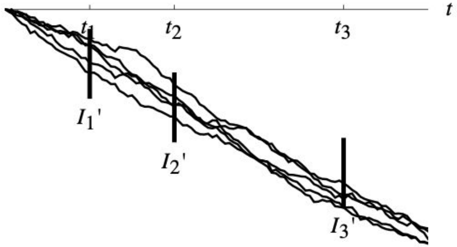
Stochastic processes indexed by time: A cylinder set C is a special subset of the space of all paths, i.e., functions of a time variable. A fixed cylinder set C is specified by a finite set of sample point on the time-axis (horizontal), and a corresponding set of "windows" (intervals on the vertical axis). When sample points and intervals are given, we define the corresponding cylinder set C to be the set of all paths that pass through the respective windows at the sampled times. In the figure we illustrate sample points (say future relative to $t = 0$). Imagine the set C of all outcomes with specification at the points $t_1, t_2, \ldots$ etc.

Note: Usage of Stochastic Processes can be found in many industries, be it Finance and Economics, Healthcare and Epidemiology, Quality Control and Supply Chain Analysis and many more. Functional Analysis enables us with these applications.

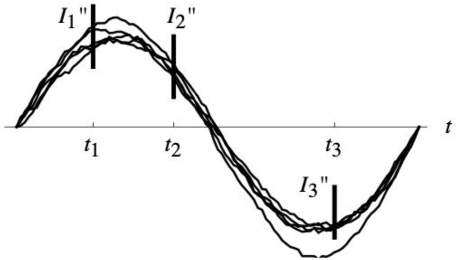
The cylinder sets of C, C` and C``



(a) C optimistic



(b) C' pessimistic



(c) C'' mixed

NOTE: Optimistic Scenario (e.g., Success, High Value): In this scenario, the stochastic process takes values that correspond to favorable or optimistic outcomes.

Pessimistic Scenario (e.g., Failure, Low Value): In this scenario, the stochastic process takes values that represent unfavorable or pessimistic outcomes.

Mixed Scenario (e.g., Uncertainty): The mixed scenario accounts for cases where outcomes are not entirely optimistic or pessimistic but fall somewhere in between.

Let $\Omega = \prod_{k=1}^{\infty}\{1,-1\}$ be the infinite Cartesian product of $\{1,-1\}$ with the product topology. Ω is compact and Hausdorff by Tychnoff's theorem.

For each $k \in \mathbb{N}$, let $X_k : \Omega \to \{1,-1\}$ be the k^{th} coordinate projection, and assign probability measures μ_k on Ω so that $\mu_k \circ X_k^{-1}\{1\} = a$ and $\mu_k \circ X_k^{-1}\{-1\} = 1-a$, where $a \in (0,1)$. The collection of measures $\{\mu_k\}$ satisfies the consistency condition, i.e., μ_k is the restriction of μ_{k+1} onto the k^{th} coordinate space. By Kolomogorov's extension theorem, there exists a unique probability measure P on Ω so that the restriction of P to the k^{th} coordinate is equal to μ_k.

It follows that $\{X_k\}$ is a sequence of independent identically distributed (i.i.d.) random variables in $L^2(\Omega,P)$ with $\mathbb{E}(X_k) = 0$ and $Var[X_k^2] = 1$; and $L^2(\Omega,P) = \overline{span}\{X_k\}$.

Remark 6.1. Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\{u_k\}$. The map $\varphi : u_k \mapsto X_k$ extends linearly to an isometric embedding of $\mathscr{H}$ into $L^2(\Omega,P)$. Moreover, let $\mathcal{F}_+(\mathscr{H})$ be the symmetric Fock space. $\mathcal{F}_+(\mathscr{H})$ is the closed span of the the algebraic tensors $u_{k_1} \otimes \cdots \otimes u_{k_n}$, thus φ extends to an isomorphism from $\mathcal{F}_+(\mathscr{H})$ to $L^2(\Omega,P)$.

NOTE: In essence, this passage is establishing the mathematical framework for constructing a sequence of independent and identically distributed. random variables on a specific probability space (Ω, P) and linking it to a Hilbert space and Fock space, highlighting the connection between probability theory and functional analysis.

Decomposition of Brownian Motion

NOTE: Compact Operator, The integral kernel K is also seen as the covariance function of a Brownian motion process. This means that K describes the statistical relationship between different points in a Brownian motion. A compact operator in this context helps in analyzing and understanding the properties of the process.

The integral kernel $K : [0,1] \times [0,1] \to \mathbb{R}$

$$K(s,t) = s \wedge t$$ (gives the minimum of the 2 values)

is a compact operator on $L^2[0,1]$, where

$$Kf(x) = \int (x \wedge y) f(y) dy.$$

Kf is a solution to the differential equation

$$-\frac{d^2}{dx^2} u = f$$

with zero boundary conditions.

K is also seen as the covariance functions of Brownian motion process. A stochastic process is a family of measurable functions $\{X_t\}$ defined on some sample probability space $(\Omega, \mathfrak{B}, P)$, where the parameter t usually represents time. $\{X_t\}$ is a Brownian motion process if it is a mean zero Gaussian process such that

$$E[X_s X_t] = \int_\Omega X_s X_t dP = s \wedge t.$$

It follows that the corresponding increment process $\{X_t - X_s\} \sim N(0, t-s)$. P is called the Wiener measure.

Building $(\Omega, \mathfrak{B}, P)$ is a fancy version of Riesz's representation theorem as in Theorem 2.14 of Rudin's book. It turns out that

$$\Omega = \prod_t \bar{\mathbb{R}}$$

which is a compact Hausdorff space.

$$X_t : \Omega \to \mathbb{R}$$

is defined as

$$X_t(\omega) = \omega(t)$$

i.e. X_t is the continuous linear functional of evaluation at t on Ω.

For Brownian motion, the increment of the process $\triangle X_t$, in some statistical sense, is proportional to $\sqrt{\triangle t}$. i.e.

$$\triangle X_t \sim \sqrt{\triangle t}.$$

It it this property that makes the set of differentiable functions have measure zero. In this sense, the trajectory of Brownian motion is nowhere differentiable.

An very important application of the spectral theorem of compact operators in to decompose the Brownian motion process.

$$B_t(\omega) = \sum_{n=1}^{\infty} \frac{\sin(n\pi t)}{n\pi} Z_n(\omega)$$

where

$$s \wedge t = \sum_{n=1}^{\infty} \frac{\sin(n\pi s)\sin(n\pi t)}{(n\pi)^2}$$

and $Z_n \sim N(0,1)$.

Conclusion

In conclusion, the application of functional analysis in understanding and modeling Brownian motion is a powerful and insightful approach that sheds light on the intricate behavior of this fundamental stochastic process. Through the lens of functional analysis, we have explored several key aspects of Brownian motion:

Path Space and Function Spaces: Functional analysis provides the mathematical framework to rigorously define and analyze the path space of Brownian motion. The path space consists of all possible trajectories the process can take. By embedding these paths in suitable function spaces, we gain a deeper understanding of the regularity and continuity properties of Brownian motion.

Brownian Motion as a Stochastic Process: Functional analysis allows us to treat Brownian motion as a stochastic process indexed by time. We can study its properties, such as sample path continuity and differentiability, and analyze its behavior over various time intervals.

References:

1. Jorgensen, Palle & Tian, Feng. (2014). Functional Analysis with Applications.
2. Bobrowski, A. (2005). Brownian motion and Hilbert spaces. In *Functional Analysis for Probability and Stochastic Processes: An Introduction* (pp. 121-146). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511614583.005