World's Simplest Poker

A Complete Analysis of the Discrete Analogue

Aryan Gupta, Mihir Tandon

June 1, 2025

Abstract

We investigate the discrete analogue of the World's Simplest Poker model, an abstraction of Poker introduced by David McAdams of Duke University. By defining strategy types and their corresponding payoffs, we analyze the strategic interactions between players under different strategy profiles. Using the Minimax algorithm, we demonstrate that cutoff and betting set strategies are not dominant. Furthermore, we establish the existence of a Nash equilibrium for the case $n = 3$ through a rigorous proof. This result serves as a foundation for future work on generalizing Nash equilibria to arbitrary values of n.

1 Introduction

In this paper, we delve into the workings of a discretized abstraction of the 'World's Simplest Poker', a simplified model of the traditional Texas Hold 'Em developed by Duke Economist David McAdams.

The game consists of two players, Player A and Player B, and implements a symmetric blinds system. Both players pay an 'ante' of \$1 to the pot, and then receive a card drawn from a set of values. In the original model, card value would be modeled by a random variable in the range [0, 1]. However, in the discretized abstraction, players receive a random card with a value from the set $\mathcal{N} = \{1, 2, 3...n\}$, where n is a natural number dictating the number of cards in a given game of our Poker model.

Definition 1. *The* ***playing set***[1]*,* $\mathcal{N} = \{1, 2, 3...n\}$ *is the discrete set of all cards upon which a game is played.*

In general, we will only consider games of $n \geq 3$. For simplicity, we will view the values in the set as synonymous to the cards themselves. The cards are drawn independently and without replacement, so that neither player will receive the same card.

Once both players receive a card, they can then decide to either *bet*, where they contribute another \$1 to the pot in hopes of making a larger profit, or *fold*, cutting their losses. In the case one player decides to fold and the other to bet, the betting player automatically wins

[1] More generally referred to as the deck.

the entirety of the pot (at this stage, \$2), regardless of the value of their card (generally referred to as their hand). However, in the event that both players decide to bet or fold, the game then enters a *showdown*.

In a showdown, the players reveal their hand, and the player with the higher card (i.e. stronger hand) wins the entire pot (\$4 in the case both players bet, \$2 in the case both players fold). The loser of the showdown receives nothing. Formally, we define a winning scenario for Player A as $c_A > c_B$, where c_A, c_B, are Player A's and Player B's card, respectively. As the cards are drawn without replacement, the case $c_A = c_B$ does not exist.

Following the round, the cards are then returned to the deck and can be redrawn in subsequent rounds. It is important to note that this model of Poker has no concept of bankruptcy, and each round is independent of previous rounds. The game is zero-sum.

2 An Analysis of the Three-Card Game

To begin our analysis of the Discrete McAdams Poker model, we shall first start by analyzing the case $n = 3$. By definition, the playing set for the Three-Card game is $\mathcal{N} = \{1, 2, 3\}$.

To begin, our analysis will only consider the situation that both players employ **cutoff strategies**. That is, should one's card be greater than or equal to some cutoff k_x, then Player x should bet, or otherwise fold.

Definition 2. *The* ***cutoff strategy*** *employs an arbitrary cutoff* $k_x \in \mathcal{N}$ *such that Player* x *bets iff* $c_x \geq k_x$.

For example, if Player A's cutoff $k_A = 1$, then Player A will bet on cards 1, 2, and 3. If Player B's cutoff $k_B = 2$, then Player B will bet on cards 2 and 3. With these cutoffs, Player A will win when:

- $c_B = 1$, by forfeit
- $c_A = 3, c_B = 2$, through a double showdown

And will lose when:

- $c_A = 1$ or $2, c_B = 3$, through a double showdown
- $c_A = 1, c_B = 2$, through a double showdown

To organize our results, we can display the expected payoff tables for Player A given Player B's hand, with each entry representing Player A's net earnings based on each player's card.

		c_B		
		1	2	3
c_A	1	X	-2	-2
	2	1	X	-2
	3	1	2	X

Table 1: The payoff table for $k_A = 1, k_B = 2$

Case 1: $k_A = 1$**,** $k_B = 2$ The expected payout for Player A is the sum of all payoffs for Player A divided by the number of hand pairings for both players. For $k_A = 1, k_B = 2$, the expected payout for Player A is

$$\frac{(-2-2+1-2+1+2)}{6} = -\frac{1}{3}$$

More generally, we can define the Expected Payout for Player A, $\mathbb{E}_A$, as

$$\mathbb{E}_A = \frac{\sum \mathbf{E_A}(c_A, c_B)}{n(n-1)}$$

where $\mathbf{E_A}(c_A, c_B)$ is the cutoff strategy payoff function for Player A given Player A's and Player B's hands.

		c_B		
		1	2	3
c_A	1	X	1	-2
	2	1	X	-2
	3	1	1	X

Table 2: The payoff table for $k_A = 1, k_B = 3$

Case 2: $k_A = 1$**,** $k_B = 3$

$$\mathbb{E}_A = \frac{\sum \mathbf{E_A}(c_A, c_B)}{n(n-1)} = \frac{(1-2+1-2+1+1)}{6} = 0$$

		c_B		
		1	2	3
	1	X	-1	-1
c_A	2	1	X	-2
	3	1	1	X

Table 3: The payoff table for $k_A = 2, k_B = 3$

Case 3: $k_A = 2$**,** $k_B = 3$

$$\mathbb{E}_A = \frac{\sum \mathbf{E_A}(c_A, c_B)}{n(n-1)} = \frac{(-1-1+1-2+1+1)}{6} = -\frac{1}{6}$$

Without loss of generalization, we can see the opposite holds for Player B, due to the symmetric nature of the game. One player's loss is the other player's gain.

2.1 Summary of Expected Payouts

The following table summarizes the expected payouts of Player A for varying k_A, k_B.

		k_B		
		1	2	3
	1	0	$-\frac{1}{3}$	0
k_A	2	$\frac{1}{3}$	0	$-\frac{1}{6}$
	3	0	$\frac{1}{6}$	0

Table 4: The expected payouts table for Player A given $n = 3$

Note the cases where $k_A = k_B$, which have an expected payout of 0. This feature is a product of the zero-sum nature of the game.

3 The Analysis of Pure Cutoff Strategies

Now that we have an understanding of the expected payout of the Poker model, we can begin to extend our understanding to arbitrary values of n.

As such, our game breaks down into 6 sub-cases, which we will denote in the form $X_1X_2X_3$. X_1 represents the action of Player A ('B' for bet, 'F' for fold), X_2 represents the action for Player B, and X_3 represents the outcome of the round with respect to Player A.

Sub-case 1, FFW: In this case, both Players A & B fold their hand. In doing so, the total pot has a balance of \$2, and the game proceeds to a showdown. $c_A > c_B$, so Player A wins the showdown.

To calculate the expected payout of this scenario, we must first find the probability of this case occurring. Since both players folded and Player A won the showdown, we have the following constraints:

- $c_A > c_B$
- $1 \leq c_A < k_A$
- $1 \leq c_B < k_B$
- Therefore, $1 \leq c_B < c_A < k_A$
- Therefore, $k_B \leq c_A < k_A$

By these constraints, we can see there are $k_B - 1$ possible values for c_B $(1, 2, 3...k_B - 1)$, and, independently, $k_A - k_B$ possible values for c_A $(k_B, k_B + 1, k_B + 2...k_A)$. As there are $n(n-1)$ possible hand pairings, of which half satisfy the requirement $c_A > c_B$, we have a total probability of

$$P_1 = \frac{1}{2} \cdot \frac{(k_B - 1)(k_B - 2)}{n(n-1)} + \frac{(k_B - 1)(k_A - k_B)}{n(n-1)} = \frac{(k_b - 1)(2k_A - k_B - 2)}{2n(n-1)}$$

As this is a showdown where both players fold, $\mathbf{E_{A}}_1(c_A, c_B) = 1$.

Sub-case 2, FFL: $x < k_B$, $y < k_B$, and $x < y$.

This case arises in the scenario that neither Player Bets and Player A loses the showdown. First, we compute the probability that this case occurs: there are $n \cdot (n-1)$ ways to choose x and y, and $(k_B - 1) \cdot (k_B - 2)$ ways to pick $x, y < k_B$. Exactly $1/2$ of them satisfy $x < y$. Thus, we get

$$P_2 = \frac{(k_B - 1)(k_B - 2)}{2n(n-1)}.$$

Note that the expected payout $\mathbb{E}_2 = 1$ as $x > y$ and neither bet.

Sub-case 3, BFW: $x \geq k_A$ and $y < k_B$.

In this case, Player B always folds, and Player A always bets. Thus, Player A always wins, and wins exactly \$1, so

$$\mathbb{E}_4 = 1.$$

To compute the probability for this case, note that there are $n - k_A + 1$ ways to choose x, and then independently $k_B - 1$ ways to choose y. Then

$$P_4 = \frac{n - k_A + 1}{n} \cdot \frac{k_B - 1}{n - 1} = \frac{(n - k_A + 1)(k_B - 1)}{n(n-1)}.$$

Sub-case 4, FBL: $x < k_A$ and $y \geq k_B$.

In this case, Player A always folds, and Player B always bets. Thus, Player B always wins, and Player A loses exactly \$1, so

$$\mathbb{E}_5 = -1.$$

To compute the probability for this case, if we have $k_B \leq x < k_A$, in which we have $k_A - k_B$ choices for x, there are $n - k_B$ choices for y. This comes from the fact that there are $n - k_B + 1$ values in the range $[k_B, n]$, however one of them is already taken by x. Otherwise, if we have $x < k_B$, in which we have $k_B - 1$ choices for x, there are $n - k_B + 1$ choices for y. This comes from the fact that there are $n - k_B + 1$ values in the range $[k_B, n]$, and none of them are taken by x. The total probability then is

$$P_5 = \frac{(k_A - k_B)(n - k_B) + (k_B - 1)(n - k_B + 1)}{n(n-1)}.$$

Sub-case 5, BBW: $x \geq k_A$ and $y < k_A$: or $y \geq k_A$ and $y < x$.

First, we compute the probability for this case. If $x \geq k_A$ There are $n + 1 - k_A$ ways to choose x, and $k_A - k_B$ ways to choose y. If $y \geq k_A$ and $y < x$ there are $n - k_A$ ways to pick y. This gives

$$P_6 = \frac{(n+1-k_A)(k_A - k_B)}{n(n-1)} + \frac{(n+1-k_A)(n-k_A)}{2n(n-1)} = \frac{(n+1-k_A)(n+k_A-2k_B)}{2n(n-1)}.$$

In this case, both players bet and we go to a showdown. Thus

$$\mathbb{E}_6 = 4p - 2 = 4 \cdot 1 - 2 = 2.$$

Sub-case 6, BBL: $x \geq k_A$, $y \geq k_A$, and $x < y$.

First we find the probability for this case. We have $(n - k_A + 1)$ ways to pick x, then $(n - k_A)$ ways to pick y, of which half satisfy $x < y$. This gives

$$P_7 = \frac{(n+1-k_A)(n-k_A)}{2n(n-1)}.$$

Now, player A bets and loses, so

$$\mathbb{E}_7 = -2.$$

Developing the Expected Payout Function To develop a closed-form solution to our expected payout function, we can combine the prior 6 sub-cases. Formally, this equates to

$$\mathbb{E}_{\mathbb{A}}(k_A, k_B) = \sum_{i=1}^{6} \mathbf{E}_{\mathbf{A}i} \cdot P_i$$

Which expands to:

$$\mathbb{E}_{\mathbb{A}}(k_A, k_B) =$$

$$\mathbf{E}_{\mathbf{A}1} \cdot P_1 + \mathbf{E}_{\mathbf{A}2} \cdot P_2 + \mathbf{E}_{\mathbf{A}3} \cdot P_3$$
$$+ \mathbf{E}_{\mathbf{A}4} \cdot P_4 + \mathbf{E}_{\mathbf{A}5} \cdot P_5 + \mathbf{E}_{\mathbf{A}6} \cdot P_6$$

Substituting the values we derived for each case:

$$\mathbb{E}_{\mathbb{A}}(k_A, k_B) =$$

$$(1) \cdot \frac{(k_B - 1)(2k_A - k_B - 2)}{2n(n-1)} \tag{FFW}$$
$$+(1) \cdot \frac{(k_B - 1)(k_B - 2)}{2n(n-1)} \tag{FFL}$$
$$+(1) \cdot \frac{(n - k_A + 1)(k_B - 1)}{n(n-1)} \tag{BFW}$$
$$+(-1) \cdot \frac{(k_A - k_B)(n - k_B) + (k_B - 1)(n - k_B + 1)}{n(n-1)} \tag{FBL}$$
$$+(2) \cdot \frac{(n + 1 - k_A)(n + k_A - 2k_B)}{2n(n-1)} \tag{BBW}$$
$$+(-2) \cdot \frac{(n + 1 - k_A)(n - k_A)}{2n(n-1)} \tag{BBL}$$

Combining all sub-cases yields the expected payout:

$$\mathbb{E}_{\mathbb{A}}(k_A, k_B) = \frac{1}{2n(n-1)} \Bigg[(k_B - 1)(2k_A - k_B - 2) + (k_B - 1)(k_B - 2) + 2(n - k_A + 1)(k_B - 1)$$
$$-2\Big[(k_A - k_B)(n - k_B) + (k_B - 1)(n - k_B + 1)\Big] + 2(n + 1 - k_A)(n + k_A - 2k_B) - 2(n + 1 - k_A)(n - k_A)\Bigg]$$

Which simplifies to:

$$\boxed{\mathbb{E}_{\mathbb{A}}(k_A, k_B)_{k_A > k_B} = \frac{3k_A k_B - k_B^2 - 2k_A^2 + 2k_A - 2k_B + nk_A - nk_B}{n(n-1)}}$$

in the case $k_A > k_B$.

We can quickly calculate the reverse case, $k_A < k_B$, by swapping k_A and k_B and evaluating $-\mathbb{E}_{\mathbb{A}}(k_B, k_A)_{k_A > k_B}$.

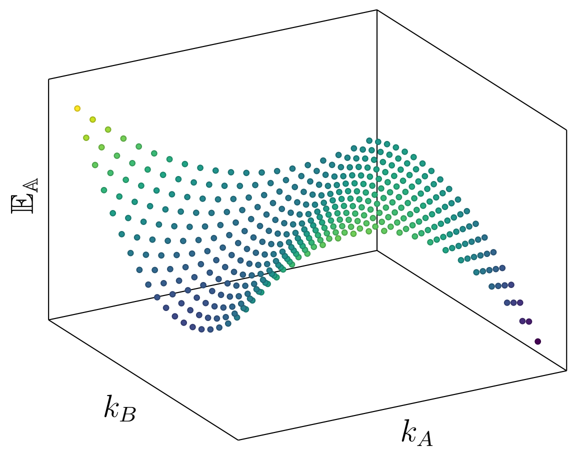
$$\boxed{\mathbb{E}_{\mathbb{A}}(k_A, k_B)_{k_A < k_B} = \frac{-3k_A k_B + k_A^2 + 2k_B^2 - 2k_B + 2k_A - nk_B + nk_A}{n(n-1)}}$$

when $k_A < k_B$.

Thus, the expected payout of Player A as a function of cutoffs k_A, k_B can be expressed as:

$$\boxed{\mathbb{E}_{\mathbb{A}}(k_A,k_B)=\begin{cases}\frac{3k_Ak_B-k_B^2-2k_A^2+2k_A-2k_B+nk_A-nk_B}{n(n-1)}, & k_A>k_B\\ \frac{-3k_Ak_B+k_A^2+2k_B^2-2k_B+2k_A-nk_B+nk_A}{n(n-1)}, & k_A<k_B\\ 0, & else\end{cases}}$$

Analyzing this over the positive integers, we see the following lattice take place:



The lattice features regions where the expected payout of Player A is negative (shaded darkly) and positive (shaded brightly). While the lattice alone fails to give us any further insight on the relationship between cutoffs and payout, we can analyze the relationship between $\mathbb{E}_A(k_A,k_B)$ and the cutoffs in two-dimensions in order to better understand the individual effects of changing strategies.

Definition 3. *A* ***risky player*** *is a player with a cutoff strategy such that k_x is low.*

Definition 4. *A* ***safe player*** *is a player with a cutoff strategy such that k_x is high.*

Here, we fix k_B and allow k_A to vary. We see that for small values of k_B (shaded darkly), which we label as a 'risky' cutoff strategy for Player B, $\mathbb{E}_A(k_A,k_B)$ decreases at an increasing rate as k_A approaches n. However, there seems to be an inflection (roughly between $0.2n < k_A < 0.4n$).

For larger values of k_B (shaded lightly), the curves decrease at a decreasing rate (that is, they approach zero) as k_A approaches n. Thus, we can begin to build some intuition regarding the relationship between cutoff strategies and expected payout:

Conjecture 1. *When playing against a risky player, it is in one's favor to play slightly less risky (i.e. with a slightly higher cutoff).*

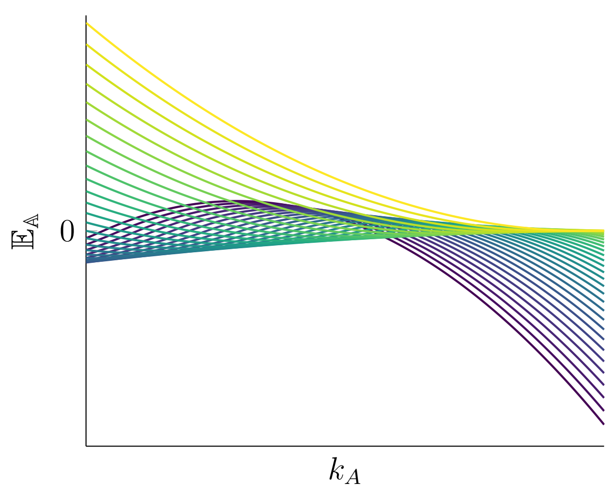


Figure 1: $\mathbb{E}_A(k_A,k_B)$ when only k_B is fixed

In contrast, when playing against a safe player, ($k_B \approx n$), we see that $\mathbb{E}_A(k_A,k_B)$ is highest when $k_A=0$ and approaches 0 as k_A grows. Knowing this, we can establish that:

Conjecture 2. *When playing against a safe player, it is in one's favor to play risky (i.e. $k_A = 0$).*

With these basic intuitions, we can begin to develop patterns to search for strategies that maximize one's payout.

3.1 Maximizing a Player's Expectation

Let us now begin to develop a concrete understanding of how one can develop a counter-strategy to maximize their expected payout, given their opponent's cutoff strategy. To do this, we make use of the Minimax Algorithm. Mathematically, this means to find a solution to the following optimization question:

$$\max_{k_A} \mathbb{E}_A(k_A, (\min_{k_B}(\mathbb{E}_A(k_A, k_B)))),$$

Where the inner minimization function returns an optimal cutoff counter-strategy, k_B^*, such that Player B's expected payout is maximized against Player A's initial strategy, k_A, and the outer maximization function returns an optimal cutoff [counter-]counter-strategy, k_A^*, such that Player A's expected payout is maximized against the optimal counter-strategy of Player B.

To maximize the expected value of Player A, let us first work under the assumption that $k_A > k_B$. That is, Player A is playing more safely than Player B. Then,

$$\mathbb{E}_{\mathbb{A}}(k_A, k_B) = \frac{3k_Ak_B - k_B^2 - 2k_A^2 + 2k_A - 2k_B + nk_A - nk_B}{n(n-1)}$$

.

Fixing k_B, we see that

$$\frac{\partial \mathbb{E}_{\mathbb{A}}}{\partial k_A} = \frac{-4k_A + (2 + n + 3k_B)}{n(n-1)}$$

has a critical point at

$$\frac{\partial \mathbb{E}_{\mathbb{A}}}{\partial k_A} = 0 \Longrightarrow \frac{-4k_A + (2 + n + 3k_B)}{n(n-1)} = 0 \Longrightarrow k_A = \frac{2 + n + 3k_B}{4}$$

Note that k_A must satisfy $k_A > k_B$ for the case to be valid, or

$$\frac{2 + n + 3k_B}{4} > k_B \Longleftrightarrow 2 + n + 3k_B > 4k_B \Longleftrightarrow k_B < n + 2,$$

which is always true. We also require that $k_A \leq n$, or

$$\frac{2 + n + 3k_B}{4} \leq n \Longleftrightarrow 2 + n + 3k_B \leq 4n \Longleftrightarrow k_B \leq n - \frac{2}{3}.$$

This holds unless $k_B = n$, in which there exists no k_A such that $k_A > k_B$. Thus, we have shown the optimal value of k_A for this sub-case is

$$\boxed{k_A^* = \left\lfloor \frac{2+n+3k_B}{4} \right\rceil}$$

where $\lfloor x \rceil$ denotes the closest integer to a real number x.

Case-by-Case Analysis of k_A^* and $\mathbb{E}_A(k_A, k_B)$

Substituting out optimal cutoff $k_A^* = \lfloor \frac{2+n+3k_B}{4} \rceil$ into $\mathbb{E}_A(k_A, k_B)$, we get:

$$\mathbb{E}_A(k_A, k_B) = \frac{\frac{(2+n+3k_B)^2}{8} - (k_B^2 + nk_B + 2k_B)}{n(n-1)}.$$

We can now analyze $\mathbb{E}_A(k_A, k_B)$ based on the value of $2+n+3k_B \pmod 4$.

Note: Previously, we only had two conditions for the expected value, $\mathbb{E}_A(k_A, k_B)$, which were when $k_A > k_B$ or $k_A < k_B$. However, we are now considering sub-cases of $k_A > k_B$, where we evaluate ${k_A}^*$ based on k_B, given $k_A > k_B$. By symmetry, the reverse is true as well.

Case 1: $2+n+3k_B \equiv 0 \pmod 4$ Here, $2+n+3k_B$ is divisible by 4, so k_A^* is an integer:

$$k_A^* = \frac{2+n+3k_B}{4}$$

The expected payout is:

$$\mathbb{E}_A(k_A, k_B) = \frac{\frac{(2+n+3k_B)^2}{8} - (k_B^2 + nk_B + 2k_B)}{n(n-1)} = \boxed{\frac{4 - 4k_B + 4n + n^2 + k_B^2 - 2nk_B}{8n(n-1)}}$$

To ensure $\mathbb{E}_A(k_A, k_B) > 1$, we require:

$$4 - 4k_B + 4n + n^2 + k_B^2 - 2nk_B > n(n-1)$$

Case 2: $2+n+3k_B \equiv 1 \pmod 4$ Here, $2+n+3k_B = 4k+1$ for some integer k, so:

$$k_A^* = \left\lceil k + \frac{1}{4} \right\rfloor = k$$

The expected payout is:

$$\mathbb{E}_A(k_A, k_B) = \boxed{\frac{-2k^2 + (2+n+3k_B)k - (k_B^2 + nk_B + 2k_B)}{n(n-1)}}$$

To ensure $\mathbb{E}_A(k_A, k_B) > 1$, we require:

$$-2k^2 + (2+n+3k_B)k - (k_B^2 + nk_B + 2k_B) > n(n-1)$$

Case 3: $2 + n + 3k_B \equiv 2 \pmod 4$ Here, $2 + n + 3k_B = 4k + 2$ for some integer k, so:

$$k_A^* = \left\lceil k + \frac{1}{2} \right\rceil = k + 1$$

The expected payout is:

$$\mathbb{E}_A(k_A, k_B) = \boxed{\frac{-2(k+1)^2 + (2 + n + 3k_B)(k+1) - (k_B^2 + nk_B + 2k_B)}{n(n-1)}}$$

To ensure $\mathbb{E}_A(k_A, k_B) > 1$, we require:

$$-2(k+1)^2 + (2 + n + 3k_B)(k+1) - (k_B^2 + nk_B + 2k_B) > n(n-1)$$

Which is the same as Case 2, substituting k for $k+1$.

Case 4: $2 + n + 3k_B \equiv 3 \pmod 4$ Here, $2 + n + 3k_B = 4k + 3$ for some integer k, so:

$$k_A^* = \left\lceil k + \frac{3}{4} \right\rceil = k + 1$$

The expected payout is:

$$\mathbb{E}_A(k_A, k_B) = \boxed{\frac{-2(k+1)^2 + (2 + n + 3k_B)(k+1) - (k_B^2 + nk_B + 2k_B)}{n(n-1)}}$$

To ensure $\mathbb{E}_A(k_A, k_B) > 1$, we require:

$$-2(k+1)^2 + (2 + n + 3k_B)(k+1) - (k_B^2 + nk_B + 2k_B) > n(n-1)$$

Which is the same as Case 3.

The optimal expected payout $\mathbb{E}_A(k_A^*, k_B)$ is defined as a piecewise function based on the value of $2 + n + 3k_B \pmod 4$. In any case, if $\mathbb{E}_A(k_A^*, k_B) \leq 1$, the default value is 1.

$$\mathbb{E}_A(k_A^*, k_B) = \begin{cases} 1, & \mathbb{E}_A(k_A^*, k_B) \leq 1, \\ \frac{4 - 4k_B + 4n + n^2 + k_B^2 - 2k_B n}{8n(n-1)}, & 2 + n + 3k_B \equiv 0 \pmod 4 \\ \frac{-2k^2 + (2+n+3k_B)k - (k_B^2 + nk_B + 2k_B)}{n(n-1)}, & 2 + n + 3k_B \equiv 1 \pmod 4 \\ \frac{-2(k+1)^2 + (2+n+3k_B)(k+1) - (k_B^2 + nk_B + 2k_B)}{n(n-1)}, & 2 + n + 3k_B \equiv 2 \pmod 4 \\ \frac{-2(k+1)^2 + (2+n+3k_B)(k+1) - (k_B^2 + nk_B + 2k_B)}{n(n-1)}, & 2 + n + 3k_B \equiv 3 \pmod 4 \end{cases}$$

Now, suppose that Player A adopts a strategy with $k_A < k_B$. Then

$$\mathbb{E}_A(k_A^*, k_B) = \frac{-3k_Ak_B + k_A^2 + 2k_B^2 - 2k_B + 2k_A - nk_B + nk_A}{n(n-1)}.$$
$$= \frac{k_A^2 + (n+2-3k_B)k_A - (nk_B + 2k_B - 2k_B^2)}{n(n-1)}.$$

Since this quadratic as a positive leading coefficient, it does not have a global maximum, but rather a minimum over the reals. Then by the Extreme Value Theorem, the maximum must occur at one of the endpoints. In this discrete setting, the endpoints are $k_A = 1$ and $k_A = k_B - 1$. We examine each case individually.

- $k_A = 1$: We have

$$\mathbb{E}_A(k_A^*, k_B) = \frac{1^2 + (n+2-3k_B)\cdot 1 - (nk_B + 2k_B - 2k_B^2)}{n(n-1)}$$
$$= \frac{1 + n + 2 - 3k_B - (nk_B + 2k_B - 2k_B^2)}{n(n-1)}$$
$$= \frac{3 + n - 5k_B - nk_B + 2k_B^2}{n(n-1)}.$$

- $k_A = k_B - 1$: We have

$$\mathbb{E}_A(k_A^*, k_B) = \frac{(k_B-1)^2 + (n+2-3k_B)(k_B-1) - (nk_B + 2k_B - 2k_B^2)}{n(n-1)}$$
$$= \frac{k_B^2 - 2k_B + 1 + nk_B - n + 2k_B - 2 - 3k_B^2 + 3k_B - nk_B - 2k_B + 2k_B^2}{n(n-1)}.$$

Combining like terms, we have

$$\mathbb{E}_A(k_A^*, k_B) = \frac{(k_B^2 - 3k_B^2 + 2k_B^2) + (-2k_B + 2k_B + 3k_B - 2k_B) + (nk_B - nk_B) - n + (1-2)}{n(n-1)}$$
$$\implies \mathbb{E}_A(k_A^*, k_B) = \frac{(0) + (k_B) + (0) - n - 1}{n(n-1)} = \frac{k_B - n - 1}{n(n-1)}.$$

However, note that since $k_B \le n$, $\mathbb{E}_A(k_A^*, k_B) < 0$. That means that this strategy is never optimal. Thus, given k_B, the optimal value of k_A is either 1 or $\left\lfloor \frac{2+n+3k_B}{4} \right\rceil$ (whichever maximizes $\mathbb{E}_A(k_A^*, k_B)$).

4 The Analysis of Betting Set Strategies

Having thoroughly explored the cutoff strategies, let us now analyze the behaviors present in games where both players adopt **betting set strategies**. Formally, let us define

Definition 5. *The **betting set strategy** of a player, $\mathcal{B}_x$ is the set of tuples (c_i, p_i) such that Player x will bet on card c_i with probability p_i.*

For example, should Player A adopt a betting set strategy $\mathcal{B}_A = \{1, 2, 3\}$ and receive a card $c_A = 2$, they would choose to bet.

Set Theory tells us that we can efficiently generate all possible betting set strategies of a Player By calculating the power set of the deck, $\mathbb{P}(\mathcal{N})$. Thus, by analyzing the $2^n \times 2^n$ possible betting set match-ups between Players A and B, we can perform a rigorous analysis of the game. For the case $n = 3$, this looks like:

$\mathcal{B}_B \backslash \mathcal{B}_A$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\emptyset$	0	4	2	0	6	4	2	6
$\{1\}$	-4	0	1	-1	5	3	3	8
$\{2\}$	-2	-1	0	1	1	2	3	4
$\{3\}$	0	1	-1	0	1	-1	-1	0
$\{1,2\}$	-6	-5	-1	0	0	1	5	6
$\{1,3\}$	-4	-3	-2	-1	-1	0	1	2
$\{2,3\}$	-2	-4	-3	1	-5	-1	0	-2
$\{1,2,3\}$	-6	-8	-4	0	-6	-2	2	0

Table 5: $\mathbb{E}_A(\mathcal{B}_A, \mathcal{B}_B)$ for betting sets $\mathcal{B}_A, \mathcal{B}_B$ when $n = 3$.

Notice that the table is skew-symmetric, a feature that arises due to the zero-sum nature of the game. That is, one player's loss is another player's gain, and swapping strategies between players also swaps their expected payouts.

4.1 Betting Set and Cutoff Strategies

Recall that a cutoff strategy is a strategy such that a player bets if and only if their card, c_x, is greater than or equal to their cutoff, k_x. Alternatively, we can define such a strategy in terms of betting sets. That is, $c_x \geq k_x \iff \mathcal{B}_x = \{k_x, k_x + 1, k_x + 2 \ldots n\}$.

Keep this in mind when referring to the following section on **intransitivity**.

4.2 Iteration & Intransitivity

Let us explore the nuances of the game when iterating multiple rounds. To begin, assume that Player B adopts a betting set strategy $\mathcal{B}_B = \{1, 3\}$.

Then, with this in mind, Player A looks to maximize their expected payout. They do this by referring to the aforementioned table, locating the row corresponding to $\mathcal{B}_B = \{1, 3\}$, and finding the column with the highest value for $\mathbb{E}_A(\mathcal{B}_A, \mathcal{B}_B)$. In this case, the resulting strategy is $\mathcal{B}_A = \{1, 2, 3\}$, with an expected payout of 2.

Knowing this, Player B looks to employ a counter strategy to Player A's betting set.

This can be done by locating the column associated with $\mathcal{B}_A = \{1, 2, 3\}$ and finding the row which minimizes Player A's expected value (and, therefore, maximizes Player B's gain). The resulting strategy is $\mathcal{B}_B = \{2, 3\}$.

Player A once again looks to maximize their profit, now given Player B's counter strategy. By the same process, we see that the optimal betting set strategy for Player A is now $\mathcal{B}_A = \{3\}$.

Repetition of this pattern shows that the two players fall into a cyclical loop. That is, no matter what strategy Player A chooses, there exists another strategy for Player B such that Player A's expected payout is minimized (and vice versa). Below is one such cycle:

$$\mathcal{B}_B = \{1, 3\} \to \mathcal{B}_A = \{1, 2, 3\}, \mathcal{B}_B = \{1, 3\} \to \mathcal{B}_A = \{1, 2, 3\}, \mathcal{B}_B = \{2, 3\}$$
$$\to \mathcal{B}_A = \{3\}, \mathcal{B}_B = \{2, 3\} \to \mathcal{B}_A = \{3\}, \mathcal{B}_B = \{1, 3\} \to \mathcal{B}_A = \{1, 2, 3\}, \mathcal{B}_B = \{1, 3\}$$

This behavior is consistent for all values of n for $n \geq 3$. Thus, we can establish our second observation:

Theorem 3. *Betting set strategies are* ***nondominant****.*

And, by extension of concepts established in the previous section,

Theorem 4. *Cutoff strategies are* ***nondominant****.*

Notably, we can see that when one's opponent plays in a safe manner, such as $\mathcal{B}_B = \emptyset$, it is in our favor to play risky, specifically by betting on all cards $\mathcal{B}_A = \{1, 2, 3\}$.

Conversely, when one's opponent plays risky, it is in our favor to play slightly less risky ($\mathcal{B}_B = \{1, 2, 3\}$, $\mathcal{B}_A = \{2, 3\}$). This counter strategy maximizes one's expected payout opportunities on higher card values, and minimizes losses generally incurred by betting on lower card values.

5 Exploring Bluffing Strategies

Up until now, the strategies we have dealt with were largely deterministic. That is, should a card be a part of one's strategy, they should bet. However, let us now consider a probabilistic take on betting. That is, for a card c_x, one should bet with probability p_X. Then, a player's strategy can be expressed as a probability vector $\mathcal{P}_x = \vec{p_i} = (p_1, p_2, p_3, \ldots, p_n)$[2].

5.1 Cutoff-Bluffing Strategies

From the set of all bluffing strategies, we can consider the subset of strategies which follow the pattern $\mathcal{P}_x = \vec{p_i} = (1, 1, 1, \ldots, p_{n-3}, p_{n-2}, p_{n-1}, p_n)$[3]. For the case $n = 3$, we get the strategies $(1, 1, 1), (p_1, 1, 1), (p_1, p_2, 1), (p_1, p_2, p_3)$. We can interpret each of these strategies as a

[2]For the purposes of this text, we will refer to Player A's strategy as $\mathcal{P}_x = \vec{p_i} = (p_1, p_2, p_2, \ldots, p_n)$ and Player B's strategy as $\mathcal{P}_B = \vec{q_i} = (q_1, q_2, q_3, \ldots, q_n)$.

[3]We refer to strategies not found in this subset as **betting-bluffing strategies**.

Definition 6. ***Cutoff-bluffing strategy***, *where a Player Bets with probability $p = 1 \iff c_x \geq k_x$, and bluffs (bets) with probability $p = p_x$ otherwise.*

For cutoff-bluffing strategies,

- If $c_A < c_B$, then Player A has a expected payout of

$$\mathbb{E}_{c_A<c_B}(\mathcal{P}_A, \mathcal{P}_B) = -2p_i \cdot q_i + p_i(1-q_i) - (1-p_i)q_i - (1-p_i)(1-q_i) = -1 + 2p_i - 3p_iq_i$$
$$= (-1)(1 - 2p_i + 3p_iq_i)$$

- If $c_A > c_B$, then Player A has a expected payout of

$$\mathbb{E}_{c_A>c_B}(\mathcal{P}_A, \mathcal{P}_B) = 2p_i \cdot q_i + p_i(1-q_i) - (1-p_i)q_i + (1-p_i)(1-q_i)$$
$$= 1 - 2q_i + 3p_iq_i$$

Which results in six distinct sub-cases, which we can sum to derive the total expected payout of Player A:

$$\mathbb{E}_A = \frac{\mathbb{E}_{1<2} + \mathbb{E}_{1<3} + \mathbb{E}_{2<3} + \mathbb{E}_{3>2} + \mathbb{E}_{3>1} + \mathbb{E}_{2>1}}{6}$$

$$= \frac{1}{6}((-1 + 2p_1 - 3p_1q_2) + (-1 + 2p_1 - 3p_1q_3) + (-1 + 2p_2 - 3p_2q_3)$$
$$+(1 - 2q_1 + 3p_2q_1) + (1 - 2q_1 + 3p_3q_1) + (1 - 2q_2 + 3p_3q_2)) \implies$$

$$\mathbb{E}_A = \frac{3(p_2q_1 + p_3q_1 + p_3q_2 - p_1q_2 - p_1q_3 - p_2q_3) + 2(2p_1 + p_2 - 2q_1 - q_2)}{6}.$$

Let us perform case-by-case analysis in order to derive the expected payout of Player A given Player B's strategy.

Case 1: $\mathcal{P}_B = (1, 1, 1)$.

In this case, Player B always bets. We plug this into $\mathbb{E}_A$:

$$\mathbb{E}_A = \frac{3p_2 + 3p_3 + 3p_3 - 3p_1 - 3p_1 - 3p_2 + 2(2p_1 + p_2 - 3)}{6} = \frac{-2p_1 + 2p_2 + 6p_3 - 6}{6}.$$

Because of the negative coefficient on the p_1 term, as well as the natural bound on probabilities, the optimal value for p_1 that maximizes $\mathbb{E}_A$ is 0. Following the same logic, the optimal values for p_2 and p_3 are both 1. Thus, the optimal response is $(p_1, p_2, p_3) = (0, 1, 1)$. This results in an expected payout of

$$\frac{2 + 6 - 6}{6} = \frac{1}{3},$$

which is consistent with our earlier (deterministic) calculation.

Case 2: $\mathcal{P}_B = (q, 1, 1)$.

In this case, Player B always bets on cards 2 and 3, but bluffs with probability q on card 1. Solving for $\mathbb{E}_A$, we get:

$$\mathbb{E}_A = \frac{3p_2q + 3p_3q + 3p_3 - 3p_1 - 3p_1 - 3p_2 + 2(2p_1 + p_2 - 2q - 1)}{6} \implies$$

$$\mathbb{E}_A = \frac{p_1 + (3q-1)p_2 + (3q+3)p_3 - 4q - 2}{6}.$$

Should $q < \frac{1}{3}$, the optimal value for $p_2 = 0$. Otherwise, we should take $p_2 = 1$. Additionally, we take $p_1 = 0$ and $p_3 = 1$. This gives us:

$$\mathbb{E}_A = \begin{cases} \frac{1-q}{6}, & q < \frac{1}{3} \\ \frac{q}{3}, & q \geq \frac{1}{3}. \end{cases}$$

Case 3: $\mathcal{P}_B = (q, q, 1)$.

In this case, Player B bets on card 3 and bluffs with probability q on cards 1 and 2. Solving gives us

$$\mathbb{E}_A = \frac{3p_2q + 3p_3q + 3p_3q - 3p_1q - 3p_1 - 3p_2 + 2(2p_1 + p_2 - 2q - q)}{6} \implies$$

$$\mathbb{E}_A = \frac{(1-3q)p_1 + (3q-1)p_2 + 6qp_3 - 6q}{6}.$$

Now, we see that if $q < \frac{1}{3}$, we should take $p_2 = 0$ and $p_1 = 1$. Otherwise, we take $p_2 = 1$ and $p_1 = 0$. Also, should always bet on card 3, or in other words, $p_3 = 1$. This gives us:

$$\mathbb{E}_A = \begin{cases} \frac{1-3q}{6}, & q < \frac{1}{3} \\ \frac{3q-1}{6}, & q \geq \frac{1}{3}. \end{cases}$$

Case 4: $\mathcal{P}_B = (q, q, q)$.

In this case, Player B always bluffs with probability q on cards $1, 2$, and 3. We plug this into $\mathbb{E}_A$:

$$\mathbb{E}_A = \frac{3p_2q + 3p_3q + 3p_3q - 3p_1q - 3p_1q - 3p_2q + 2(2p_1 + p_2 - 2q - q)}{6} \implies$$

$$\mathbb{E}_A = \frac{(4-6q)p_1 + 2p_2 + 6qp_3 - 6q}{6}.$$

Now, we see that if $q < \frac{2}{3}$, we should take $p_1 = 1$ and $p_1 = 0$ otherwise. Also, we should take $p_2 = p_3 = 1$. This gives:

$$\mathbb{E}_A = \begin{cases} 1-q, & q < \frac{2}{3} \\ \frac{1}{3}, & q \geq \frac{2}{3}. \end{cases}$$

Thus, we have found all of the expected payouts assuming cutoff-bluffing strategies.

5.2 $n = 3$ Optimal Betting-Bluffing Strategies:

We generalize this analysis beyond cutoff strategies. Consider

$$\mathbb{E}_A = \frac{3(p_2q_1 + p_3q_1 + p_3q_2 - p_1q_2 - p_1q_3 - p_2q_3) + 2(2p_1 + p_2 - 2q_1 - q_2)}{6} \iff$$

$$\mathbb{E}_A = \frac{(4 - 3q_3 - 3q_2)p_1 + (2 + 3q_1 - 3q_3)p_2 + (3q_1 + 3q_2)p_3 - 4q_1 - 2q_2}{6}.$$

Looking at the coefficients c_i of each p_i, we can see that the expected value function is maximized by setting $p_i = 0$ if $c_i < 0$, and $p_i = 1$ otherwise. It follows that Player A should play the deterministic strategy of choosing

$$p_1 = \begin{cases} 0, & q_3 + q_2 > \frac{4}{3}, \\ 1, & q_3 + q_2 \leq \frac{4}{3}. \end{cases}$$

$$p_2 = \begin{cases} 0, & q_1 - q_3 > \frac{2}{3}, \\ 1, & q_1 - q_3 \leq \frac{2}{3}. \end{cases}$$

$$p_3 = 1.$$

It is clear for $n = 3$, Player A's maximum response strategy is deterministic. We can then compute the expected payouts depending on (q_1, q_2, q_3) as follows:

- $q_3 + q_2 \geq \frac{4}{3}$ and $q_3 - q_1 \geq \frac{2}{3}$, which gives $(p_1, p_2, p_3) = (0, 0, 1)$, and

$$\mathbb{E}_A = \frac{q_2 - q_1}{6}.$$

- $q_3 + q_2 \geq \frac{4}{3}$ and $q_3 - q_1 \leq \frac{2}{3}$, which gives $(p_1, p_2, p_3) = (0, 1, 1)$, and

$$\mathbb{E}_A = \frac{2q_1 + q_2 + 2 - 3q_3}{6}.$$

- $q_3 + q_2 \leq \frac{4}{3}$ and $q_3 - q_1 \geq \frac{2}{3}$, which gives $(p_1, p_2, p_3) = (1, 0, 1)$, and

$$\mathbb{E}_A = \frac{-q_1 - 2q_2 - 3q_3 + 4}{6}.$$

- $q_3 + q_2 \leq \frac{4}{3}$ and $q_3 - q_1 \leq \frac{2}{3}$, which gives $(p_1, p_2, p_3) = (1, 1, 1)$, and

$$\mathbb{E}_A = \frac{2q_1 - 2q_2 - 6q_3 + 6}{6}.$$

6 Identifying a Mixed Strategy Nash Equilibrium for the Three-Card Game

Now, we aim to find a Nash equilibrium. By Nash's Theorem, which states that every finite game with a finite number of players and strategies (including mixed strategies) has at least one Nash equilibrium, there must exist either a pure strategy or a mixed strategy equilibrium. From our previous analysis, we saw that no deterministic strategy results in a Nash equilibrium. Thus, we explore mixed strategy equilibria.

Suppose Player B uses a mixed strategy. We know that it is always in the player's favor to bet on the highest card with probability 1. Hence, $p_3 = q_3 = 1$. Given this, we can simplify our formulas to the following:

- $q_2 \geq \frac{1}{3}$ and $q_1 \leq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (0, 0, 1)$, and
$$\mathbb{E}_A = \frac{q_2 - q_1}{6}.$$
- $q_2 \geq \frac{1}{3}$ and $q_1 \geq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (0, 1, 1)$, and
$$\mathbb{E}_A = \frac{2q_1 + q_2 - 1}{6}.$$
- $q_2 \leq \frac{1}{3}$ and $q_1 \leq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (1, 0, 1)$, and
$$\mathbb{E}_A = \frac{-q_1 - 2q_2 + 1}{6}.$$
- $q_2 \leq \frac{1}{3}$ and $q_1 \geq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (1, 1, 1)$, and
$$\mathbb{E}_A = \frac{2q_1 - 2q_2}{6}.$$

Note that if Player A had a higher expected payout by always betting or always folding on each card, then they would use that deterministic strategy. This could cause Player B to respond with a deterministic strategy as well since the cutoff values to switch strategies occur at $p_1 = 1/3$ and $p_2 = 1/3$ from the formulas above. Thus, no matter what deterministic strategy Player A uses, there exists some optimal deterministic strategy for Player B. This is a contradiction because we know that two deterministic strategies cannot form Nash equilibria.

Thus, we require that Player A is indifferent about their strategy choices for at least one card, cards 1 or 2. Looking at the cutoff values for q_i, this forces either $q_1 = 1/3$ or $q_2 = 1/3$.

- If $q_1 = 1/3$, then if $q_2 \geq \frac{1}{3}$, then
$$\mathbb{E}_A = \frac{q_2 - 1/3}{6},$$

and if $q_2 \leq \frac{1}{3}$, then

$$\mathbb{E}_A = \frac{2/3 - 2q_2}{6} = \frac{1/3 - q_2}{3}.$$

Note that over the given domains, both of these expressions have a minimum value of 0, achieved at $q_2 = 1/3$. Thus, Player B's optimal strategy is to use $(q_1, q_2, q_3) = (1/3, 1/3, 1)$. Given this strategy, Player A has a maximum payout of 0 regardless of their counter-strategy.

By symmetry, Player A must also choose $(p_1, p_2, p_3) = (1/3, 1/3, 1)$. This ensures that, just as Player A has no incentive to switch strategies, Player B also does not benefit from switching their strategy.

- The case is similar if $q_2 = 1/3$. If $q_1 \leq 1/3$, then

  $$\mathbb{E}_A = \frac{1/3 - q_1}{6},$$

  and if $q_1 \geq 1/3$, then

  $$\mathbb{E}_A = \frac{q_1 - 1/3}{3},$$

  and both of these functions are minimized at $q_1 = 1/3$, resulting in a payout of 0 for Player A.

Thus, we have shown that

$$\boxed{(p_1, p_2, p_3) = (q_1, q_2, q_3) = (1/3, 1/3, 1)}$$

is the unique Nash equilibrium for $n = 3$.

Finally, we claim that, in fact, both players must use this strategy to maximize their expected payout (which is 0). Suppose that Player B operates with some strategy $(q_1, q_2, q_3) \neq (1/3, 1/3, 1)$.. Then, from our analysis above, there exists a strategy that gives Player A a positive payout, which in turn gives Player B a negative payout. Thus, this strategy is clearly suboptimal and $(1/3, 1/3, 1)$ is the optimal strategy that either player can implement.

7 Future Directions

Our work can largely be classified as introductory and exploratory. For mathematical and notational convenience, we limit our subject to an abstraction of Poker that largely preserves the nuances of different interactions; however, there are critical deviations in the nature of our model in terms of the actual computations that must take place and optimal strategies one should adapt due to the differences in rule books.

However, this gap can be bridged. In addition to continuing our search for dominant strategies and Nash equilibrium in games with higher values of n, our goal is to evolve our model to incorporate fundamental changes such as sequential rounds, hands with multiple cards, and carryover balances. Such changes will consequentially adjust play styles, eventually resulting in the World's Simplest Poker model approximating traditional Texas Hold ‘em. We hope that this combinatorics-based excursion will progress the ever-evolving state of contemporary Poker to newer heights, being an intransitive game, by introducing new strategies to exploit weaknesses in play styles. Through the lens of Keynesian economics, just as we established the nondominance of strategies in our model, we hope to find relationships that allow players to profit against perfectly rational strategies.