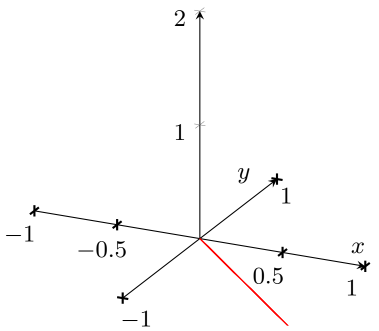
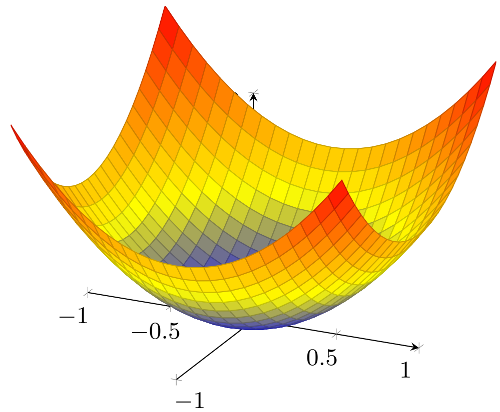
Section 10

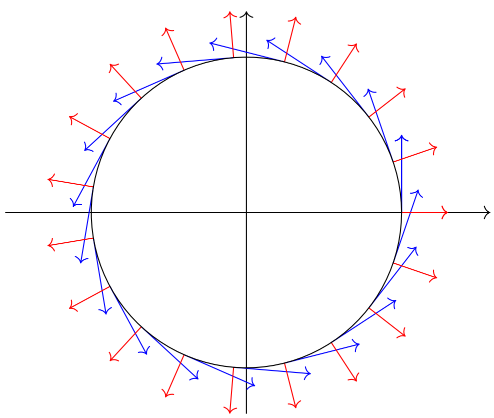
Today, we'll be exploring **surface parametrizations and surface integrals**.

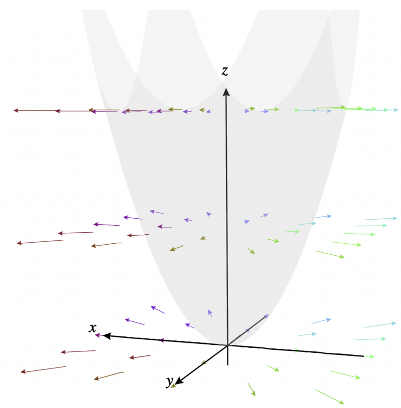
Recall that, last section, we were integrating real-valued functions over curves to get 'surface area of a fence', and integrating vector fields dotted with trajectories to get 'work done on a particle'. In each case, we were integrating along a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$.

Today, we'll do the same, but integrate instead along surfaces, given to us as 'paths' $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$.









Surface Integral for Scalar Functions

Given a surface parameterized by differentiable map $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ and a function $f : \mathbb{R}^3 \to \mathbb{R}$, the integral of f over the **surface** $\Phi(D)$ is...

$$\int\int_D (f \circ \Phi)(u, v)||\mathbf{T}_u \times \mathbf{T}_v||dudv$$

where $\mathbf{T}_u$ is the vector consisting of derivatives of components of Φ w.r.t. u, and similarly for v. Think of this as the mass of the surface $\Phi(D)$ whose density at a point is given by f.

The special case when $f = 1$ yields the surface area of $\Phi(D)$. In another special case, when Φ is a 'graph', that is, it is defined as $\Phi(u, v) = (u, v, g(u, v))$, we get a simplification: $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{1 + ||\nabla g||^2}$.

Surface Integral for Vector-Valued Functions

In the same setting of a surface S parameterized by differentiable map $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, if we are given a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, the surface integral of $\mathbf{F}$ over S is...

$$\int\int_D \mathbf{F}(\Phi(u,v)) \cdot d\mathbf{S} := \int\int_D \mathbf{F}(\Phi(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) du dv$$

If we think of $\mathbf{F}$ as the velocity field of a fluid in 3D space, this gives us the rate of flow of the fluid across the surface.

Surface Parameterizations

We introduced these concepts while sweeping the idea of a surface parameterization under the rug. Now, a **surface parameterization** is to a surface (a 2D object) what a **path parameterization** is to a curve (a 1D object).

Therefore, you want to be able to accomplish two things...

1. given a surface parameterization $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, develop a sketch of its image surface, $\Phi(D)$.
2. Reciprocally, given a surface S, find a map $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ whose image is this surface (which we call its **parameterization**).

Questions

Q1) Sketch the surfaces obtained from the following parameterizations:

1. $\Phi : \begin{cases} [0,1] \times [0,1] \to \mathbb{R}^3 \\ (x,y) \mapsto (x, y, x+y) \end{cases}$
2. $\Phi : \begin{cases} [0,2] \times [0,2\pi] \to \mathbb{R}^3 \\ (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta), r) \end{cases}$

Q2) Find the parameterization of the following surfaces:

1. surface bounded by $z = 4$ and $z = x^2 + y^2$.
2. surface bounded by $z = 4 - x^2 - y^2$ and $z = 0$.

Q3) A Pringles chip is shaped according to a hyperbolic paraboloid, whose equation in 3D is given by $z = \frac{x^2}{2} - \frac{y^2}{2}$, for $x, y \in [0,1]$. It's density at a point (x,y) is given by $d(x,y) = 4$. Can you find an integral to express its mass?