Section 7: Vector Fields

(1) Our focus this week is two-fold: on **arc lengths** and **vector-valued functions**. First, we spend some time talking about a quick and dirty derivation of the arc-length formula (this is just meant as a heuristic!).

(2) Now, let's talk about **vector-valued functions** i.e. functions $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2$, that take on vector values. So much can be said about these functions, but on the auspicious occasion of π-day, we focus on one idea:

An inversion problem

When can we conclude that $\mathbf{F}$ is the gradient of some real-valued function $f : \mathbb{R}^n \to \mathbb{R}$?

Why care about this? Well, consider the differentiable functions $f : \mathbb{R}^3 \to \mathbb{R}$. These have gradients $\nabla f(x)$ defined at each point. Therefore, we can define the vector-valued function $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$, taking $x \mapsto \nabla f(x)$. This means that, for every differentiable function, there is a corresponding vector-valued function.

A natural question to ask, then, is the 'inversion' - if I gave you a vector-valued function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, can you give me a function whose gradient is $\mathbf{F}$?

The answer is no!

(Def. 1) For $n-$dimensions, let $\nabla = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} ... \frac{\partial f}{\partial x_n}]$. We call this the 'del' operator. For example, in $\mathbb{R}^3, \nabla = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$. We'll use it in this form herein.
(Def. 2) Thinking of ∇ as a vector, we can dot it and cross it with other vectors in $\mathbb{R}^3$. Let's do this with a vector-valued function, $\mathbf{F}$, to derive the **divergence** and **curl** of a vector-valued function.

$$\operatorname{div} \mathbf{F}_{(x,y,z)} = \nabla \cdot \mathbf{F}_{(x,y,z)}$$

$$\operatorname{curl} \mathbf{F}_{(x,y,z)} = \nabla \times \mathbf{F}_{(x,y,z)}$$

Given these ideas[1], we have the following theorem.

(Th. 1 (Gradients are curl-free)) If $\nabla \times \mathbf{F} \neq 0$, then $\mathbf{F}$ is not the gradient of any function.

An easy proof involving brute computation can be found in Dr. Wang's lecture notes on divergence and curl. .

[1]We won't consider deeply into what these notions signify, because we'll do so later in the course. For the curious, however, I suggest 3Blue1Brown's video.

There is a more intuitive condition to check. Suppose that $\mathbf{F}$ is the gradient of some function f, i.e. $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = [F_1, F_2, F_3]$. Then, since mixed partials are equal, we'd have that $\frac{\partial f}{\partial x \partial y} = \frac{\partial F_1}{\partial y} = \frac{\partial f}{\partial y \partial x} = \frac{\partial F_1}{\partial x}$.

Therefore, if mixed partials are unequal, i.e. $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_1}{\partial x}$, then F can't be a gradient. Thus, we get the following...

(Th. 2 (Identity of mixed partials)) If $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$, then $\mathbf{F}$ is not the gradient of any function.

(3) Unfortunately, most vector fields are not gradients. However, there is a weaker sense in which this is possible - specifically, it may be that vector fields are gradients of a function along some path...

(Def. 3) Given $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, if there is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ such that $\mathbf{c'}(t) = \mathbf{F}(\mathbf{c}(t))$, then $\mathbf{c}$ is called a **flow line** for $\mathbf{F}$.

Q1. The **arc-length** function, $s(t)$, for a path $c(t)$ is given by $s(x) = \int_{t_0}^{x} ||c'(t)||dt$. It represents the distance travelled until time x. Suppose you are walking on a helix staircase, so that your path is $c(t) = (\cos(t), \sin(t), t)$. Compute $s(x)$ as you walk along this path.

Q2. Consider $f(x, y) = x^2 + y^2$. Sketch f, the gradient vector field of f, and some additional flow lines. What do you observe?

Q3. Is the curl of $\mathbf{F}$ perpendicular to $\mathbf{F}$? Consider $\mathbf{F} = (-y, x, 1)$.