Section 10

This week, we'll be exploring **path and line integrals**. I have omitted writing a content review because I think Professor Wang's lecture notes already capture all the relevant insights and intuitions for these topics.

Justifying path integrals

Given a path $\mathbf{c} : [a, b] \to \mathbb{R}^n$ taking $t \mapsto (x_1(t), x_2(t)...x_n(t))$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. the **path integral** of f along c is...

$$\int_{\mathbf{c}} f ds := \int_a^b f(x_1(t), x_2(t)...x_n(t)) \; ||\mathbf{c}'(t)||dt = \int_a^b f(\mathbf{c}(t)) \; ||\mathbf{c}'(t)||dt$$

where $||\mathbf{c'(t)}|| = \sqrt{x_1'(t)^2 + ... + x_n'(t)^2}$.

Thus, the procedure to compute a path integral is 3-step...

1. Compute $f(\mathbf{c}(t))$.
2. Compute $\mathbf{c}'(t)$, and its magnitude $||\mathbf{c'(t)}||$.
3. Plug into the integral, and evaluate the integral.

You can do all this without knowing what you're doing, of course, but it's essential to remember what you're really finding: **the surface area of a fence that swerves according to $\mathbf{c}(t)$ and has height according to f, or the mass of an object shaped according to $\mathbf{c}(t)$ with density according to f.**

Line integrals

If $\mathbf{F}$ is a vector field on R^n, and $\mathbf{c} : [t_0, t_1] \to \mathbb{R}^n$ is a path, the **line integral of F along c** is...

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_{t_0}^{t_1} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

Thus, to compute it, you...

1. Compute $\mathbf{F}(\mathbf{c}(t))$.
2. Compute $\mathbf{c}'(t)$.
3. Compute their dot product.
4. Integrate their dot product.

Again, you can do all this without knowing what you're doing, but the underlying idea is essential. What you're really finding is the **work done by a force field F on a particle following the path c**.

Some natural questions to ask here include...

1. Does the integral depend on the way in which we express the curve $\mathbf{c}(t)$?
2. What happens if the curve given by $\mathbf{c}(t)$ is 'closed' (i.e. loops back in on itself)?
3. What happens if $\mathbf{F}$ is a gradient vector field? Since we are integrating a 'derivative', do we have something analogous to $\int_a^b f' = f(b) - f(a)$?

Q1) You are a fence setter-upper. Today, you're setting up 3 fences. You choose to design fences whose height at a point (x, y) is $h(x, y) = 2 - x^2 - y^2$. Compute the surface area of your fence if your fence is shaped as:

1. the unit circle centered at $(0,0)$.
2. portion of the line $y = 0$ that runs from $x = -1$ to $x = 1$
3. the curve $c(t) = (\sin(t), t)$ from $t = -1$ to $t = 1$.

Q2) Suppose that $\mathbf{F}(x,y) = (2x, 2y)$. Sketch the vector field and the following curves, then compute the line integral along them.

1. the unit circle centered at $(0,0)$.
2. portion of the line $y = 0$ that runs from $x = 0$ to $x = -1$.
3. the curve $c(t) = (t, t + \sin(t))$ from $t = -1$ to $t = 1$.

Q3) Given two vectors $\mathbf{v}, \mathbf{w}$, we can find the cosine of the angle between them as $\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \cdot ||\mathbf{w}||}$. However, if we have time-varying vectors, then the angle θ depends on time too, given by the formula $\cos(\theta(t)) = \frac{\mathbf{v(t)} \cdot \mathbf{w(t)}}{||\mathbf{v(t)}|| \cdot ||\mathbf{w(t)}||}$

Following this, given a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ and a path $\mathbf{c} : [t_0, t_1] \to \mathbb{R}^n$ we define the following integral,

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \frac{\mathbf{F} \cdot \mathbf{c'}(t)}{||\mathbf{F}|| \cdot ||\mathbf{c}'(t)||} dt$$

This gives us the **average cosine of the angle between the vector field and the path**. Let's call it the **angular path integral**. Express the angular path integral for (1), (2) and (3) from question 2, and compute it if you can. Does it match your intuition?