# Section 12

Today, we'll be exploring **Green's and Stokes' theorem**. These are theorems that give us a beautiful connection between line integrals and surface integrals. The versions we are seeing today are special cases of Generalized Stokes Theorem, which one encounters in a class on calculus on manifolds.

In the following, let $\delta D$ denote the **boundary** of $D$.

**(Th. 1: Green)** Given a continuously differentiable vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and a closed region $D \subset \mathbb{R}^2$, we have the equality...

$$\int_{\delta D} \mathbf{F} \cdot d\mathbf{s} = \int \int_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k} dA$$

That is, if $\mathbf{F} = (P(x,y), Q(x,y))$, $\int_{\delta D} Pdx + Qdy = \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy$

**(Th. 2: Stokes)** Given a continuously differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and a surface $S \subset \mathbb{R}^3$, we have the equality...

$$\int_{\delta S} \mathbf{F} \cdot d\mathbf{s} = \int \int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{A}$$

That is, if $\mathbf{F} = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ and $S$ is parameterized by $\Phi(u,v) : D \subset \mathbb{R}^2 \to S$, then

$$\int_{\delta D} F_1 dx + F_2 dy + F_3 dz = \int \int_D \operatorname{curl} \mathbf{F}(u,v) \cdot (\mathbf{T}_u \times \mathbf{T}_v) dudv$$

## What do we mean by boundary?

To truly appreciate the definition of a boundary requires the language of manifolds. However, as we haven't explored that in this class, we can instead think about it more intuitively. Suppose you want to determine the boundary of a region $A \subset \mathbb{R}^2$ or $\mathbb{R}^3$. To do so, imagine living on $A$. Travelling across the entirety of $A$, at each point you consider neighbourhoods of radius $r > 0$ that lie on $A$.

Now, at point $x \in A$, if you find a radius $r_x > 0$ such that the neighbourhood of radius $r_x$ is on $A$ completely, you're not at the boundary and so you continue moving forward. However, if you stumble at a point where you cannot do so, i.e. cannot find a neighbourhood of any radius that is entirely contained on $A$, then you have discovered a boundary point.

### Examples of objects and their boundaries

*(To be filled in section)*

1. Suppose a particle moves along the rectangle with vertices $(2, 0, 2), (2, 2, 0), (0, 2, 0)$ and $(0, 0, 2)$ in the *clockwise* direction. It is being acted on by a force field $\mathbf{F} = (x + y^2, y + z^2, z + x^2)$. Using Stokes' theorem, compute the work done by the force field on the particle.

2. By computing both the line and double integral, verify Green's theorem for $\int_C (xy^2 + x^2)dx + (4x - 1)dy$, where $C$ is the triangle with vertices $(-3, 0), (0, 0)$ and $(0, 3)$, oriented counterclockwise.

3. Use Stokes' Theorem to evaluate $\int \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (3yx^2 + z^3, y^2, 4yz^2)$ and $C$ is triangle with vertices $(0, 0, 3), (0, 2, 0)$ and $(4, 0, 0)$.