Problem Set: Section 3

A foreword: All of the mathematical ideas you will be investigating today started to take their form only about 400 years ago. Before then, philosophers certainly thought about continuity and smoothness, but did not frame them in the formal terms of limits.

So, today's definitions and results may seem like an overly meticuluous attempt to say something simple. And that's because it is! Why? Well, armed with this knowledge, if your crush ever asks you about the differentiability of $\frac{e^{xy}}{x^2+y^2}$ at the origin, you can use the definition you've learnt to impress them with an unquestionably correct answer. Indeed, what you're learning herein is state-of-the-art mathematics, in all its rigor. Almost like an intellectual superpower (kind of... ok, not really).

Understanding continuity and differentiability

Here, we'll understand the ways in which continuity, partial differentiability and differentiability exist. We first introduce them as distinct notions. Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$.

Def. 1 (continuity): We say f is **continuous at** (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x) = f(x_0, y_0)$. That is, if I gave you any $\epsilon > 0$, you could find an $\delta > 0$ such that whenever $||(x, y) - (x_0, y_0)|| < \delta, (x, y) \in A$, then $||f(x, y) - f(x_0, y_0)|| < \epsilon$.
That is, f can be sketched without lifting your pencil within some region containing (x_0, y_0).

Def. 2 (partial derivatives) We say f has partial derivative with respect to the x_i^{th} component at (x_0, y_0), $\frac{\partial f}{\partial x_i}(x_0, y_0)$, if the following limit exists:

$$\lim_{h\to 0} \frac{||f((x_0, y_0) - h\vec{e}_i) - f(x_0, y_0)||}{(h)}$$

If it exists, we equate the limit to $\frac{\partial}{\partial x_i}(x_0, y_0)$. That is, f is well-approximated by a tangent line in the x_i^{th}-coordinate direction.

Def 3. (differentiability) Assuming f is partial differentiable at (x_0, y_0), consider the tangent approximation to f at (x_0, y_0)...

$$L(x, y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}\right] [(x - x_0), (y - y_0)]^T$$

Then, we say f is differentiable at (x_0, y_0) if $L(x, y)$ approximates f well near (x_0, y_0), that is...

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x, y) - L(x, y)|}{||(x, y) - (x_0, y_0)||} = 0$$

If it is differentiable, we denote its derivative at (x_0, y_0) by $\mathbf{D}f(x_0, y_0)$.
That is, f is close to its linear (i.e. tangent plane) approximation near (x_0, y_0)

And so, following the above definitions, you might notice some clear implications. For example, f is differentiable $\implies$ f is partial differentiable; f is differentiable $\implies$ f is continuous. What about all the other implications?

For our first question, we'll try fo fill some parts of this table. I've filled in one row below. Try to fill in two more using the two results we mentioned in the previous paragraph...

Continuity	Partial Differentiable	Differentiable	Possible?	Why?
Yes	Yes	Yes	Yes	
No	Yes	Yes	No	$D \implies C$
Yes	No	Yes		
No	No	Yes		
Yes	Yes	No		
No	Yes	No		
Yes	No	No		
No	No	No	Yes	

1. P.D., but not C nor D Let $f(x,y) = \frac{2xy}{x^2+y^2}$ for $(x,y) \in \mathbb{R} - 0$, and 0 when $(x,y) = (0,0)$. By using the definition, first show that f is partial differentiable at the origin. However, show thereafter that it is not continuous at the origin (*Hint: consider $f(x,y)$ on the line $x = y$, and thereafter the limit of f as $x \to 0$*).

2. P.D. and C, but not D Consider the function $f(x,y) = x^{1/3}y^{1/3}$, for $x, y \in \mathbb{R}$. Using the definition, does f have partial derivatives at $(0,0)$? Again using the definition, show that it isn't differentiable at $(0,0)$.

3. Derivatives for even functions Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and even for all $(x,y) \in \mathbb{R}^2$, i.e. $f(x,y) = f(-x,-y)$. Compute ∇f at the origin by filling in the following proof:

A1) We know that f is differentiable at the origin, so that $\nabla f = [\_\_, \_\_]$. Let's compute the first component.

A2) By definition, the first component ∇f_x is ________________.

A3) Since f is even, for any positive h, $\frac{f(h,0)-f(0,0)}{h} \geq 0$ tells us that $\frac{f(-h,0)-f(0,0)}{-h}$ ___.

A4) Similarly to (A3), if $\frac{f(h,0)-f(0,0)}{h} \leq 0$, then ____________________.

A5) Forgiving abuse of notation, recall the definition of the right-side and left-side derivatives, f' and f, -

A6) Since f is partial differentiable w.r.t. x at the origin, we can relate f' and f, by _____. By above,

3. Fun visualizations! The sphere $x^2 + y^2 + z^2 = 6$ and the ellipsoid $x^2 + 3y^2 + 2z^2 = 9$ intersect at $(2,1,1)$. Find the angle between their tangent planes at this point (for one way on how to do this, recall the previous problem set).