Section 9: Midterm Review

Our first question is an overview of how to compute integrals by substitution in higher dimensions. This is commonly referred to as **the method of Jacobians**, or the **change of variables** formula. To start, recall how integration by substitution works with an example...

Compute $\int_0^2 xe^{x^2}dx$.

A1) Let $x = \sqrt{u}$. That is, we now have x dependent on u, such that $x(u) = \sqrt{u}$. Then, $\frac{dx}{du} = \frac{1}{2\sqrt{u}}$. Now, we are integrating $x \in D = [0, 2]$ w.r.t. x. Upon substitution, we will integrate over $x \in D^* = [0, 4]$.

A2) Then,

$$\int_D f(x)dx = \int_0^2 xe^{x^2}dx =_{\text{subs.}} \int_0^4 \sqrt{u}e^u \frac{1}{2\sqrt{u}}du = \int_{D^*} f(x(u))\frac{dx}{du}du$$

i.e. to compute something here...

...we can instead compute in a 'simpler' regime, and 'scale' to account for area deformation.

The exact same idea holds for higher dimensions...

A1) Suppose you are given $f(x, y)$ to integrate over some region $D \subset \mathbb{R}^2$. That is, you want $\int_D f(x, y)dxdy$. However, it is not easy to compute integral of, but you know that some substitution can simplify it.

A2) Define your desired substitution[1]...

$$T : \begin{cases} D^* \subset R^2 \to D \\ (u, v) \mapsto (x(u, v), y(u, v)) \end{cases}$$

A3) Now, you can integrate $f(x(u, v), y(u, v))$ over the 'simpler' regime D^*, and 'scale' by area-deforming factor, $\frac{\partial(x,y)}{\partial(u,v)} = \det \mathrm{D}T(u, v)$. That is,

$$\int_D f(x, y)dxdy = \int_{D^*} f(x(u, v), y(u, v))\frac{\partial(x, y)}{\partial(u, v)}dudv$$

Practice Questions

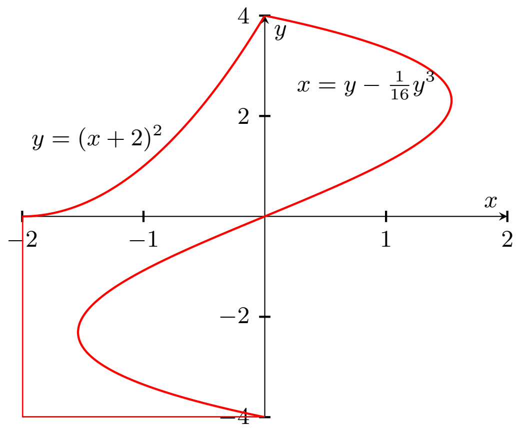
(Q1) Let E be the set of points satisfying $x^2 + y^2 \leq 2, 0 \leq z \leq 3$. Evaluate $\int\int\int_E ze^{x^2+y^2}dzdxdy$.

...

[1] T should be one-to-one (i.e. injective) and have continuous first derivatives for (A3) to work.

(Q2) Find the minimum and maximum of $f(x, y, z) = x^2y^2z$ subject to $x^2 + 2y^2 + 3z^2 \leq 1$.

(Q3) Integrate $f(x, y) = 2x+5y$ over the region bounded by $y = (x+2)^2, x = -2, y = -4,$ and $x = y - \frac{1}{16}y^3$. Write the integral in both the $dxdy$ and $dydx$ direction, then integrate the easier one.



(Q4) Reverse the order of integration of the following integral

$$\int_{\pi/2}^{5\pi/2} \int_{\sin x}^{1} g(x, y)\,dy\,dx$$

(Q5) Evaluate the volume of the region formed by the intersection of the cylinders $x^2+y^2 = 4$ and $x^2+z^2 = 4$.

(Q6) Utilize a double integral to prove that the area beneath a Standard Normal Distribution is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$$