Section 9

Tuesday, October 25, 2022 6:34 PM

Oct 17-19	Oct 24-26
(i) Normal distribution	(i) Joint distributions
(ii) Functions of r.v's	· Continuous
· CDF methods	(ii) Independent r.v.'s
(iii) Joint distributions	(iii) MGF method (?)
· Discrete	

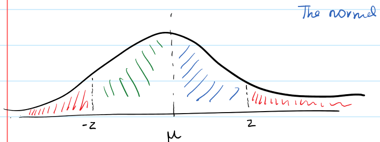
$\boxed{\text{Normal distribution}}$

① Using $\phi(z)$ and its properties to compute $\mathbb{P}[Z \le z]$, $\mathbb{P}[a < Z \le b]$ etc...

The normal density is symmetric about its mean!

(i.e. $f_X(\mu+a) = f_X(\mu-a)$) $\forall a \in \mathbb{R}$

why? $f_X(\mu \pm a) = \dfrac{e^{-\frac{1}{2}\left(\frac{\cancel{\mu} \pm a) - \cancel{\mu}}{\sigma}\right)^2}}{\sqrt{2\pi\sigma}} = \dfrac{e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2}}{\sqrt{2\pi\sigma}}$

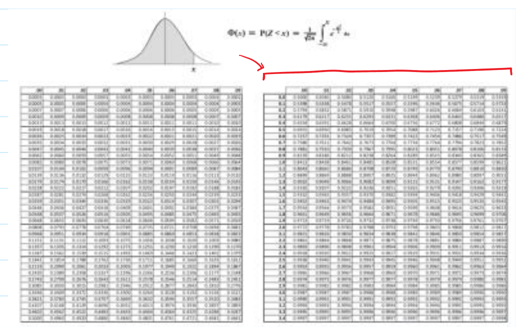


(i) $\forall z \in \mathbb{R}$, $\mathbb{P}[Z \le z] = \phi(z)$ (by defn)

(ii) $\phi(-z) = 1 - \phi(z)$ Left = Right   So, if you know CDF for positive val, you know it for negative val.

(iii) $\mathbb{P}[-z \le Z \le z] = \phi(z) - \phi(-z) = 2\phi(z) - 1$ So, you know CDF for intervals if you know it for one end.

(Can you think of more?)



② Linear combo. of normals is normal (including affine transformations)

(1) If $X \sim N(\mu, \sigma^2)$, $aX + b \sim N(a\mu + b, a^2\sigma^2)$

(2) If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, and they are independent*, $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

③ z-score : $\dfrac{X - \mu_X}{\sigma_X}$ (We do this because, if X follows some normal distribution w/ mean μ_X & variance σ_X, $\dfrac{X-\mu_X}{\sigma_X} \sim N(0,1)$ $\boxed{\text{WHY?}}$)

(Q1) Say, $X_1 \sim N(82, 4^2)$ is the distribution of homework 1 scores.
$X_2 \sim N(90, 2^2)$ is distribution of homework 2 scores.
Suppose you choose student randomly from your class. Compute:

(i) $\mathbb{P}[X_1 \le 90]$ (iii) $\mathbb{P}[X_1 \ge 80]$

(ii) $\mathbb{P}[X_2 \le 98]$ (iv) $\mathbb{P}[84 \le X_2 \le 92]$

$\boxed{\text{CDF method}}$ $\Rightarrow$ 1st way to find densities of functions of r.v.'s.

With an example...

(i) Your original r.v. and function of interest

$X \sim \mathcal{N}(0,1)$

$Y = aX + b$, $(a > 0, b \in \mathbb{R})$

(ii) Write CDF of Y in terms of that of X $\longrightarrow$ aim: to differentiate it to get density $f_Y(y)$

Now, $\mathbb{P}[Y \leq y] = \mathbb{P}[X \leq \frac{y-b}{a}] = \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx$

(iii) Differentiate

Here, we invoke Leibniz rule:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(x,y)\, dx = g(b(y), y)\, b'(y) - g(a(y), y)\, a'(y) + \int_{a(y)}^{b(y)} \frac{\partial g}{\partial y}(x,y)\, dy$$

So, $$\frac{d}{dy}\mathbb{P}[Y=y] = \frac{d}{dy}\int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{\exp\left(-\frac{1}{2}\left(\frac{y-b}{a}\right)^2\right)}{\sqrt{2\pi a^2}} = \text{pdf of } \mathcal{N}(b, a^2)$$

(Q's) If $Z \sim \mathcal{N}(0,1)$, find pdf of $Y = Z^2$.

If $X \sim \exp(1)$, find pdf of $Y = 1/X$.

If $X \sim N(\mu, \sigma^2)$, find pdf of $Y = e^X$ (lognormal(μ, σ^2))

$\boxed{\text{Special case}}$

(Q) What if your function, $g(X)$, is $\Uparrow$ strictly monotone?

$\Downarrow$

Now, following our steps, (Since g is monotone)

(i) $\mathbb{P}[g(X) \leq y] = \mathbb{P}[X \leq g^{-1}(y)]$ (If g is strictly increasing)

$= \int_a^{g^{-1}(y)} f_X(x)\, dx$

(ii) Then, $\frac{\partial}{\partial y}\int_a^{g^{-1}(y)} f_X(x)\, dx = f_X(g^{-1}(y))\; g^{-1\prime}(y)$

① Example 1 (Random number generation)

Say, $Y \sim \exp(\lambda)$. The inverse cdf is a function: $\begin{cases} (0,1] \to \mathbb{R} \\ k \to x \end{cases}$

where x is such that $\mathbb{P}[X \leq x] = k$.

① Compute the inverse cdf of Y as a function $f(x)$

② If $X \sim$ uniform$(0,1)$ and $Y = f(X)$, find pdf of Y.

(2) Example 2 (Practicing caution when using CDF method)

Say, $X \sim \exp(\lambda)$. Consider $g(X) = (X-1)^2$. Find its pdf.

Joint distributions

(Df) When r.v.'s $X, Y : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ are defined on a common sample space Ω, we have the joint pmf (or pdf)*,

$$p_{X,Y}(x,y) := \mathbb{P}[X=x, Y=y]$$

$$(f_{X,Y}(x,y) := f(X=x, Y=y))$$

The marginals are distribution of some subset of r.v.'s,

$$p_X(x) := \mathbb{P}[X=x]$$

$$(f_X(x) := f(X=x))$$

$\boxed{\text{Properties}}$

(1) Joints are pmfs / pdfs
i.e. especially, $\iint f_{X,Y}\, dx\, dy = 1$

$$\left(\sum\sum p_{X,Y}\, dx\, dy = 1 \right)$$

(2) We can get marginal from joint

"Integrate out other variable" $\quad f_X(x) = \int f_{X,Y}(x,y)\, dy$

$$\left(p_X(x) = \sum_y p_{X,Y}(x,y) \right)$$

+ all other properties
(monotonicity, ctable additivity, ctable subadditivity ...)

$\boxed{\text{Independence}}$

Soon, we'll care about conditional probability!

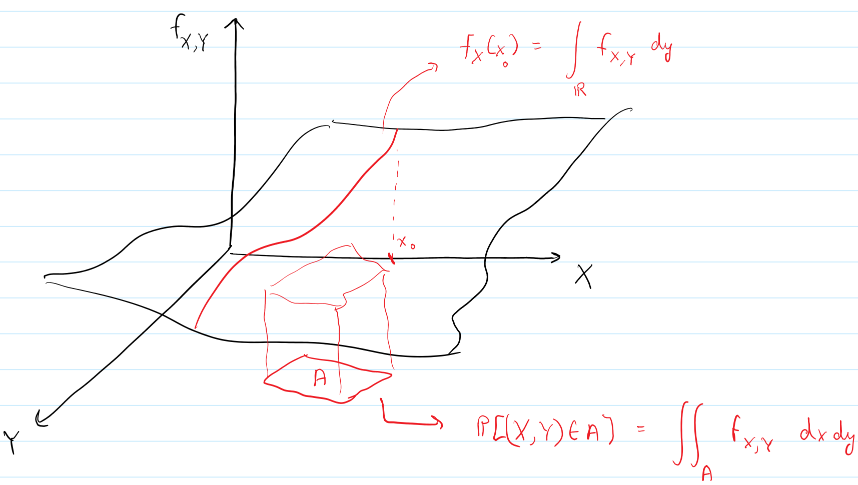
(Df) A collection of r.v.'s are independent if $\quad$ joint $\quad$ = $\quad$ product of

(Df) A collection of r.v.'s are independent if $\boxed{\text{joint distribution} = \text{product of their marginals.}}$

conditional probability!

Visual representations

	X = 1	X = 2	X = 3	
Y = 1	0.1	0.1	0.1	0.3
Y = 2	0.2	0.2	0	0.4
Y = 3	0.15	0	0.15	0.3
	0.45	0.30	0.25	1



Example 1

Let $f_{X,Y}(x,y) = \begin{cases} e^{-y} & , 0 < x < y < \infty \\ 0 & , \text{else} \end{cases}$

(1) Show this is a valid joint pdf.

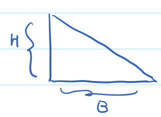
(2) Compute the marginal of X and Y. (Are they independent?)

Example 2 (Event of interest into some integral over set of interest)

Say, you walk to the taxi stop everyday at a time uniformly between 8 and 9am.
So does your crush.
However, after arriving, the person waits ~10 minutes until a taxi picks them.
What is the probability you see your crush?

Example 3

You are drawing triangles with randomly uniform dimensions, H and B



Say, $H \sim$ uniform $(0, 5)$

$B \sim$ uniform $(0, 2)$. Find the probability that the area of your triangle is at least 2 units^2.

(Assume you always draw right-angled triangles)