Section 14

Wednesday, November 30, 2022 7:25 PM

Today

① New kind of r.v.'s — "conditional" r.v.'s

$X|Y$ $\quad X|Y=y \quad X|Y\geq y \quad X|Y\leq y$

$X|A$ (an event)

$\mathbb{E}[X|Y]$

② Inequalities

(1) Markov

(2) Chebyshev

(3) WLLN

- Today, suppose X and Y are two r.v.'s, such that X is dependent on Y, so that $X|Y=y \sim f_{X|Y=y}$

- Now, $X|Y=y$ is a random variable.

(Ex1) $Y \sim \text{Binomial}(n,p)$, "# of heads in 10 tosses"

$X|Y=y \sim \text{Bernoulli}\left(\frac{y}{n}\right)$, "you win money, with winning prob $\propto$ no. of heads"

(Ex2) $H \sim \text{Normal}(\mu, \sigma^2)$, "height of person"

$W|H=h \sim \text{Normal}(2h, \sigma^2)$, "weight of people, given they have height h"

(Ex3) $Y \sim \text{Bernoulli}\left(\frac{1}{n}\right)$, "odds of picking fair coin amongst $\underline{n-1}$ unfair coin"

$X|Y=y \sim \text{Geometric}\left(\left(\frac{1}{2}\right)^y \left(\frac{1}{3}\right)^{1-y}\right)$ "number of tosses to get one head"

The above demonstrate that something like "$X|Y$" is a random variable in its own right, w/ the conditioning r.v. Y specifying a parameter of the distribution, or something we already know about X.

$\Downarrow$

Thus, we can compute statistics like <u>mean</u>, <u>variance</u> etc.

(Ques 1) Say, $Y \sim \text{Binomial}(n,p)$, "# of heads in 10 tosses"

$X|Y=y \sim \text{Bernoulli}\left(\frac{y}{n}\right)$, "you win money, with winning prob $\propto$ no. of heads"

What is $\mathbb{E}[X|Y]$? What does it denote? Is it a random variable or a deterministic value?

(Hint: if $Z \sim \text{Bernoulli}(\theta)$, $\mathbb{E}Z =$ _)

(Formally, $\mathbb{E}[X|Y] = \int_X x f_{X|Y}(x)\, dx$)

(Ques 2) Say, $Y \sim \text{Bernoulli}\left(\frac{1}{n}\right)$, "odds of picking fair coin amongst $\underline{n-1}$ unfair coin"

$X|Y=y \sim \text{Geometric}\left(\left(\frac{1}{2}\right)^y \left(\frac{1}{3}\right)^{1-y}\right)$ "number of tosses to get one head"

What is $\mathbb{E}[X|Y]$? What does it denote? Is it a random variable or a deterministic value?

(Hint, if $Z \sim \text{Geom}(p)$, $\mathbb{E}Z =$ _)

Another thing we may do is compute $\mathbb{E}[X]$, i.e. <u>unconditional expectation</u>, from the conditional expectation.

(Ques) Say, $Y \sim \text{Binomial}(n,p)$, "# of heads in 10 tosses"

$X|Y=y \sim \text{Bernoulli}\left(\frac{y}{n}\right)$, "you win money, with winning prob $\propto$ no. of heads"

What is $\mathbb{E}[X]$? What does it denote? Is it a random variable or a deterministic value?

(Formally,

since $\mathbb{E}[X|Y] = g(Y)$,

$$
\begin{aligned}
\mathbb{E}[\mathbb{E}[X|Y]] &= \mathbb{E}[g(Y)] \\
&= \int_Y g(y) f_Y(y)\, dy \\
&= \int_Y \int_X x f_{X|Y=y}(x)\, dx\, f_Y(y)\, dy \\
&= \int_Y \int_X x f_{X|Y=y}(x) f_Y(y)\, dx\, dy \\
&= \int_Y \int_X x f_{X,Y}(x,y)\, dx\, dy \\
&= \int_X x f_X(x)\, dx = \mathbb{E}[X]
\end{aligned}
$$
)

Other properties

(1) $\mathbb{E}[Xh(Y)|Y] = h(Y)\mathbb{E}[X|Y]$

(2) $\mathbb{E}[aX + bY|Z] = \mathbb{E}[aX|Z] + \mathbb{E}[bY|Z]$

$= a\mathbb{E}[X|Z] + b\mathbb{E}[Y|Z]$

(3) If X, Y independent, $\mathbb{E}[X|Y] = \mathbb{E}[X]$

(4) If $c \in \mathbb{R}$, $\mathbb{E}[c|Y] = c$

If these r.v.'s have calculable expected values, they must have calculable variances too!
Indeed:

$$Var(X|Y=y) = \mathbb{E}\left[\left(X - \mathbb{E}[X|Y=y]\right)^2 \,\middle|\, Y=y\right]$$, Variance of $X|Y=y$

Law of total variance

$$Var(X) = \mathbb{E}(Var(X|Y)) + Var(\mathbb{E}(X|Y))$$

(Ques) Compute the variance of X, given

$Y \sim Binomial(n,p)$, "# of heads in 10 tosses"

$X|Y=y \sim Bernoulli\left(\frac{y}{n}\right)$, "you win money, with winning prob $\propto$ no. of heads"

Inequalities

- Finitely many r.v.'s $\xrightarrow[\text{Markov, Chebyshev}]{}$ • Infinite sequence of r.v.'s. WLLN, SLLN, CLT
- Behaviour for finite sample size • Asymptotic behaviour

(Ques) Prove Markov's inequality, explicitly stating which assumption is used where:
If X is non-negative, continuous random variable, then for $a > 0$,

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}X}{a}$$

For $a > 0$, $\mathbb{P}[X \geq a] = \int_a^\infty a f_X(x)\, dx \leq$ ________ $\leq \int_0^\infty x f_X(x)\, dx = \mathbb{E}X$

(Ques) Prove Chebyshev's inequality, explicitly stating which assumption is used where:
For arbitrary r.v. X with finite mean μ, $\forall \epsilon > 0$

$$\mathbb{P}[|X-\mu| \geq \epsilon] \leq \frac{Var(X)}{\epsilon^2}$$

$\mathbb{P}[|X-\mu| \geq \epsilon] = \mathbb{P}[|X-\mu|^2 \geq \epsilon^2] \leq$ ________ $= \frac{Var(X)}{}$

$$\frac{\quad}{\varepsilon^2}$$

$$\mathbb{P}[\,|X-\mu| > \varepsilon\,] = \mathbb{P}[\,|X-\mu|^2 > \varepsilon^2\,] \leq \underline{\qquad\qquad} = \frac{\mathrm{Var}(X)}{\varepsilon^2}$$

The above theorems are very cool because they give bounds on the distribution of a <u>large</u> class of r.v's – they tell us that, just because your mean is finite, there's only so much probability that you can deviate from it.

(Ex) Suppose the mean on a probability exam is 75.
With just this information,

(i) Find an upper bound for probability that someone scored above 90.

(ii) If I told you that 85% of students scored ≥ 90, would that be possible or impossible? Why? What about 75%?

(iii) Suppose you gain some information – specifically, the standard deviation was 5. Answer (i) and (ii) again.

(Ex) Suppose that students at Hopkins take an average of 16 credits/semester, with standard deviation of 1.3.
Compute a bound for the probability that a student is taking ≥ 15 and ≤ 17 credits.

(Ques) Prove the following version of "The Weak Law of Large Numbers"
Given $X_1 \dots X_n \sim$ iid with finite mean μ, & finite variance σ^2, let $\bar{X} = \frac{\sum X_i}{n}$, "sample mean".

Then, $\forall \varepsilon > 0$, $\mathbb{P}\left[\,|\bar{X}-\mu| > \varepsilon\,\right] \to 0$ as $n \to \infty$.

(Hint, consider the r.v. $\bar{X}$ and apply Chebyshev)