Section 10

Wednesday, November 2, 2022 5:58 PM

Oct-03, lec-10: law of the unconscious statistician, moments/variance
Oct-05, lec-11: MGF (part 1), the cumulative distribution function CDF
Oct-10, lec-12: continuous random variables, uniform, exponential, expected value, variance
Oct-12, lec-13: MGF (part 2), the gamma distribution
Oct-17, lec-14: the normal distribution
Oct-19, lec-15: transformations of continuous rvs: CDF method, joint distributions - jointly discrete
Oct-24, lec-16: joint distributions - jointly continuous, independent random variables
Oct-26, lec-17: sums of independent random variables, convolution
→ Oct-31, lec-18: MGF (part 3), conditional distributions - discrete case
Nov-02, lec-19: conditional distributions - continuous cases → Today's topic (not on exam!)

Reminders

(1) Always mention the support of your pdf / distribution.
(2) When in doubt, use definitions or try to visualize the problem.
(3) $\frac{75 \text{ min}}{10 \text{ ques}} = 7.5$ min / question

Convolutions – Discrete case

(Th) If X, Y are independent, discrete r.v.'s with pmfs p_X, p_Y, then $X+Y \sim p_{X+Y}(k) = \sum_x p_X(x)\, p_Y(k-x)$
$= \sum_y p_Y(y)\, p_X(k-y)$

(Pf) Now, $\mathbb{P}[X+Y=k] = \sum_x \mathbb{P}[X=x, X+Y=k]$ (law of total probability)
$= \sum_x \mathbb{P}[X=x, Y=k-x]$
$= \sum_x \mathbb{P}[X=x]\,\mathbb{P}[Y=k-x]$ (independence of X, Y)

(Ex) Say, $X \sim \text{Poisson}(a)$, $Y \sim \text{Poisson}(b)$. Compute $p_{X+Y}(k)$.

Here, $p_X(x) = \frac{e^{-\alpha}\alpha^x}{x!}$, $p_Y(y) = \frac{e^{-b} b^y}{y!}$. Then, $\sum_{x=0}^{k} \frac{e^{-\alpha}\alpha^x}{x!} \frac{e^{-b} b^{(k-x)}}{(k-x)!}$

$= e^{-(\alpha+\beta)} \sum_{x=0}^{k} b^{(k-x)} \frac{k(k-1)\cdots(k-(x+1))}{k!} \frac{\alpha^x}{x!}$

$= \frac{e^{\alpha+\beta}}{k!} \sum_{x=0}^{k} b^{(k-x)} \alpha^x \frac{k!}{(k-x)!\,x!}$

$= \frac{e^{\alpha+\beta}}{k!} (\alpha+b)^k$

Continuous case

(Th) If X, Y are independent, discrete r.v.'s with pmfs f_X, f_Y, then $X+Y \sim f_{X+Y}(k) = \int f_X(x)\, f_Y(k-x)\, dx$
$= \int f_Y(y)\, f_X(k-y)\, dy$

(Pf) If X, Y are cts r.v.'s, and independent, then let's apply the "CDF method":

$\Rightarrow F_{X+Y}(\mu) = \mathbb{P}[X+Y \le \mu]$
$= \int_{X+Y \le \mu} f_{X,Y}(x,y)\, dA$
(express as double integral + invoke independence) $= \int_{-\infty}^{\infty}\int_{-\infty}^{u-x} f_X(x)\, f_Y(y)\, dy\, dx$

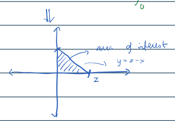
$\Rightarrow \frac{\partial}{\partial u} F_{X+Y}(u) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(u-x)\, dx$

(Ex) $X, Y \sim \exp(\lambda)$. Compute f_{X+Y}.

Now, $f_{X+Y}(z) = \int_0^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx$

why is this the support? $= \int_0^{z} \lambda^2 e^{-\lambda z}\, dx = \boxed{\lambda^2 z e^{-\lambda z}}$

⇓



MGF method

(Th) Say, X and Y are independent R.V.'s with MGF $M_X(\theta), M_Y(\theta)$ respectively.
Then, $M_{X+Y}(\theta) = M_X(\theta) M_Y(\theta)$.

(Pf) $M_{X+Y}(\theta) = \mathbb{E}[e^{\theta(X+Y)}] = \mathbb{E}[e^{\theta X + \theta Y}] = \mathbb{E}[e^{\theta x} e^{\theta y}]$
$= \mathbb{E}[e^{\theta x}]\,\mathbb{E}[e^{\theta y}]$
$= M_X(\theta) M_Y(\theta)$

(Q) where is independence used?

My suggestion:

Is function sum of r.v.'s?
- YES ($X+Y$) → Do r.v.'s have MGF?
  - YES (M_X, M_Y exist) → MGF
  - NO (M_X / M_Y don't exist) → CDF and convolution
  - (M_{X+Y} is weird / doesn't simplify to known form → CDF and convolution)
- NO (X^2, e^X) → CDF method (after this exam → method of Jacobian)

(why am I suggesting this?)

Conditional distribution

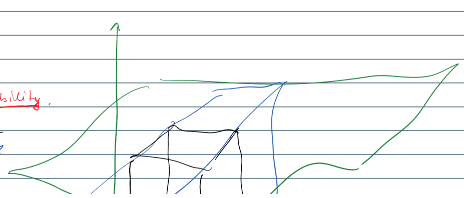
Discrete case

(1) $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

Continuous case

(i) Conditioning event has nonzero probability.

(2) $F_{X|Y}(x|y) = \frac{F_{X,Y}(x,y)}{F_Y(y)} =$



think,
$\mathbb{P}[X \le A \mid Y \le B]$

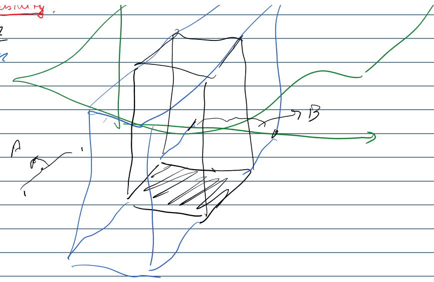
(i) conditioning event has nonzero probability.

(2) $F_{X|Y}(x|y) = \dfrac{F_{X,Y}(x,y)}{F_Y(y)} = \dfrac{\text{~~~}}{\text{~~~}}$

think,

$\mathbb{P}[X \in A \mid Y \in B]$

$= \dfrac{\mathbb{P}(X \in A, Y \in B)}{\mathbb{P}(Y \in B)}$



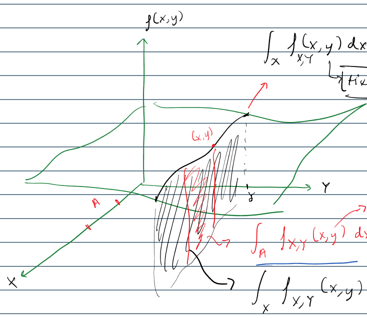
(ii) Conditioning event has zero probability, but non-zero density.

Then, we have the conditional density:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

What's the idea behind this construction?

(1) $\displaystyle\int_A f_{X|Y}(x|y)\,dx = \int_A \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$ (density) $= \mathbb{P}[(X,Y) \in (A_x \times \{y\}) \mid Y \in \{y\}]$



$\displaystyle\int_X f_{X,Y}(x,y)\,dx = f_Y(y)$

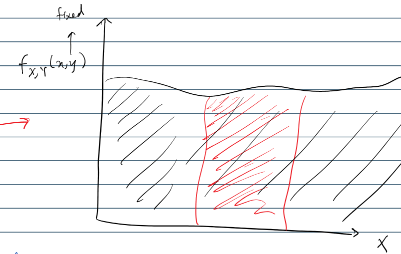
Fixed y

again, fixed

$\displaystyle\int_A f_{X,Y}(x,y)\,dx$ (numerator)

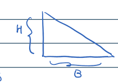
$\displaystyle\int_X f_{X,Y}(x,y)\,dx = f_Y(y)$ (denominator)

$\displaystyle = \int_A f_{X|Y}(x,y)\,dx$



$\dfrac{\text{~~~}}{\text{~~~}}$ → what you're interested in. → what you know has already occurred.

Questions



① You are drawing triangles with randomly uniform dimensions, H and B

Say, $H \sim \text{uniform}(0,5)$ (Assume you always draw right-angled triangles)

$B \sim \text{uniform}(0,2)$

(and H, B are independent)

(a) Find the probability that the area of your triangle is at least 2 units^2.

(b) Given that the area of your triangle is ≥ 1 units^2, what is the probability it is ≥ 2 units?

(c) Given that the base is 1 unit^2, what is the probability that area ≥ 2 units?

② Say, $X, Y, Z \sim \exp(1)$ iid. Compute $\mathbb{P}[X + Z < Y]$.

(Use fact that if $X \sim \exp(1)$, $Y \sim \exp(2)$, then $X + Y \sim \text{Gamma}(2,1)$,)

and $f_{X+Y}(v) = ve^{-v}$, $v > 0$

+ Law of total probability in cts case ⇓

$$\mathbb{P}[X \in A] = \int \mathbb{P}[X \in A \mid Y = y]\, f_Y(y)\,dy$$

(3) A dartboard is modelled by a disk $x^2+y^2 \le 1$.
A dart lands on the board at a point uniformly distributed about its area. Let (X,Y) be its coordinates.

(1) Compute marginal PDFs of X and Y

(2) Are X, Y independent?