Section 11

Wednesday, November 9, 2022 11:45 PM

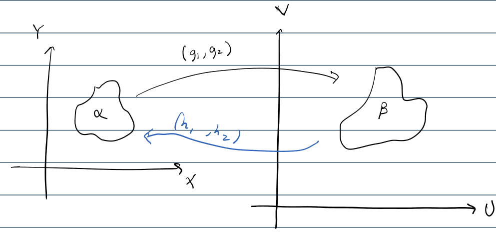
(Th)

• Say, X and Y are jointly continuous w/ pdf $f_{X,Y}(x,y)$ on support α, and

$u = g_1(x,y)$

$v = g_2(x,y)$

is one-to-one transformation of α to β.



• Then, having inverse transformation $T^{-1}: \begin{cases} \beta \longrightarrow \alpha \\ (u,v) \longmapsto (x(u,v), y(u,v)) \end{cases}$

the joint pdf of $U = g_1(X,Y)$, $V = g_2(X,Y)$ is,

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| , \quad \text{where } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} .$$

should be injective

$$\begin{tikzcd} (\Omega, \mathcal{F}, P) \arrow[r] \arrow[d] & (X \times Y, \mathbb{R}^2, \mathcal{R}^2) \arrow[dl, bend right, "(g_1,g_2)"'] \\ (U \times V, \mathbb{R}^2, \mathcal{R}^2) \arrow[ur, bend right, "(h_1,h_2)"'] & \end{tikzcd}$$

Then, this is bijective if it is defined on image of (g_1, g_2).

(Pf) Now, we show that $\iint_{\alpha_1} f_{X,Y}(x,y)\,dx\,dy = \iint_{\beta_1} f(h_1(u,v), h_2(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$,

where $(h_1, h_2)(\beta_1) = \alpha_1$, and (h_1, h_2) define one-to-one transformation.

↳ This is true by our derivation of the Jacobian.

Therefore, we may simply "read" off the joint of U, V:

$$\iint_{\alpha_1} f_{X,Y}(x,y)\,dx\,dy = \iint_{\beta_1} \underbrace{f(h_1(u,v), h_2(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|}_{f_{U,V}} du\,dv \quad \text{(as desired!)}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix}$$

Quick exercise

(Q) Say, X and Y are r.v's. Are following transformation one-to-one?

(i) $U = X$, $V = Y$, $\begin{pmatrix} X \in \mathbb{R} \\ Y \in \mathbb{R} \end{pmatrix}$

(ii) $U = \sqrt{X^2 + Y^2}$, $V = X^2$, $(X, Y \in (-1, 1))$

(iii) $U = \sqrt{X^2 + Y^2}$, $V = X^2$ $(X, Y \in (0,1))$

(iv) $U = e^X$, $V = Y^2$ $(X, Y \in \mathbb{R})$

(Rm) Support matters!

(Q) Ex3, pg 212

Maybe X = waiting time for purple on day 1

Y = " for day 2

$\boxed{\text{Say, } X \text{ and } Y \sim \exp(1) \text{ iid. Find density of } U = \frac{X}{Y}.}$

Recall, $Exp(\lambda)$ represents waiting time until first event occurs, given occurrence rate of λ occurs/time.. → $\mathbb{E}[X] = \frac{1}{\lambda}$

(So, if you expect to receive 2 phone calls per hour, Expected waiting time is $\frac{1}{2}$ hours for first call)

Ordered statistics

(Df) Given $\underbrace{X_1, X_2 \ldots X_n}_{\text{our sample}} \sim f_X, F_X$ iid as random variables, let $Y_{(j)}$ be the random variable representing the j^{th} smallest value. We call $Y_{(j)}$ the j^{th}-order statistic.

(Ex) (1) $Y_{(1)} = \min(X_1 \ldots X_n)$

(2) $Y_{(n)} = \max(X_1 \ldots X_n)$

(3) $Y_{(n)} - Y_{(1)} = \max(X_1 \ldots X_n) - \min(X_1 \ldots X_n)$ = "sample range"

(4) $Y_{\lceil \frac{n}{2} \rceil}$ = sample median (if n is odd)

What is the pdf of the j^{th}-order statistic?

$$\Rightarrow f_{Y_j}(v) = \binom{n}{j-1,\, 1,\, n-j} \left(F_X(v)\right)^{j-1} f_X(v) \left(1 - F_X(v)\right)^{n-j}, \quad \text{for } v \in \text{support}(X).$$

What is joint distribution of $Y_{(1)} \ldots Y_{(n)}$?

$$\Rightarrow f_{Y_1 \ldots Y_n}(v_1, \ldots v_n) = n!\, f(v_1) \ldots f(v_n), \quad v_1 < v_2 \ldots < v_n \text{ and } v_i \in \text{support}(X).$$

In lecture, you noted...

(1) If $X_1 \ldots X_n \sim$ uniform$(0,1)$ iid, $Y_{(j)} \sim$ Beta$(k, n-k+1)$.

(Q) Are ordered statistics independent r.v.'s?

(Q) 30 firecrackers are ignited simultaneously, each of whose ignition is ~ exponentially distributed with mean 1 minute. Find the probability that the first firecracker to stop does so within the first 10 seconds.

(Q) (Functions of ordered stats)

Let $X_1 X_2 X_3 \sim \text{Uniform}(0,1)$. Find density of $U = \frac{\min\{X_i\}}{\max\{X_i\}}$.

Exchangeability

(Df) A collection of R.V.'s is <u>exchangeable</u> if $(X_1, X_2 \dots X_n) \overset{\mathcal{D}}{=} (X_{i_1}, X_{i_2} \dots X_{i_n})$,

where $(i_1, i_2 \dots i_n)$ is a permutation of $(1, 2 \dots n)$.

(Df) $f: \mathbb{R}^n \to \mathbb{R}$ is a <u>symmetric function</u> <u>if</u> $f(x_1 \dots x_n) = f(x_{i_1} \dots x_{i_n})$, where $\{i_j\}_{j=1}^{n}$ is as above.

(Lm) A collection $X_1 \dots X_n$ of r.v.'s is exchangeable <u>if and only if</u> their joint pmf/pdf is a symmetric function.

(Th) If $X_1 \dots X_n$ are exchangeable, then for all $1 \le k \le n$,
$(X_1 \dots X_k) \overset{\mathcal{D}}{=} (X_{i_1} \dots X_{i_k})$, where $\{i_j\}_{j=1}^{k}$ is a subset of $\{1 \dots n\}$ of length k.

(Lm) iid r.v.'s are exchangeable, but not all exchangeable r.v.'s are iid.

(Q) You draw randomly & uniformly, <u>with replacement</u> students from a sample of 100. (100 students)
Let Z_i = index of student you draw on i^{th} trial.
Compute:

$S_1 \ S_2 \ S_3$ ... S_{100}

(1) $\mathbb{P}[Z_1 = 1]$
(2) $\mathbb{P}[Z_2 = 1]$
(3) $\mathbb{P}[Z_1 = 1, Z_2 = 2]$
(4) $\mathbb{P}[Z_8 = 1, Z_{99} = 2]$
(5) $\mathbb{P}[Z_3 = 1, Z_4 = 3, Z_9 = 2]$
(6) $\mathbb{P}[Z_8 = 9, Z_{11} = 9]$

(Q) Same as above, but now you draw without replacement. What changes?
Now, <u>prove that $Z_1 \dots Z_{100}$ are exchangeable.</u>

(Q) Say, X_1, X_2 & $X_3 \sim \mathcal{N}(0,1)$ iid.
Compute:
(1) $\mathbb{P}[X_1 > X_2]$
(2) $\mathbb{P}[X_1 > X_2 > X_3]$
(3) $\mathbb{P}[X_3 > X_1]$

(Q) Say, $X_1 \sim \text{unif}(0,1)$, $X_2 \sim \text{unif}(0,2)$ are independent. Are they exchangeable?