Random variables

Thursday, October 6, 2022 10:09 AM

Df) Real-valued fns $X: \Omega \to \mathbb{R}$, which give us a sense of the probabilities $\mathbb{P}[X=x]$ in terms of the probability measure defined on Ω.

HOW?

We say, for $x \in \mathbb{R}$, that $\mathbb{P}[X=x] := \mathbb{P}[\omega \in \Omega : X(\omega) = x]$

this means that the LHS is defined as being the RHS.

We will now see some examples of r.v.'s, focussing on the general experiment they model, their pmf, and how to derive their pmf from the experiment they model.

probability mass function, assigning $\mathbb{P}[X=x]$ to x in $\mathbb{R}$.

⓪ Bernoulli

· The most fundamental and 'simple' random variable

(DF) $\boxed{X \sim \text{Bernoulli}(p) \text{ when } \mathbb{P}[X=x] = \begin{cases} p, & x=1 \\ 1-p, & x=0 \end{cases}}$

X is the 'indicator' function of A — if $X(\omega)=1$, we learn that $\omega \in A$, and vice versa.

As a function, we can think of it like,

$X:$ $\omega \in$

where $P(A) = 1$, and $P(A^c) = 0$.

, such that $\omega \in A \Rightarrow X(\omega) = 1$. $\omega \in A^c \Rightarrow X(\omega) = 0$

Deriving PMF

So, $\mathbb{P}[X=1] = \mathbb{P}[\omega : X(\omega) = 1]$
$= \mathbb{P}[A]$
$= p$

(similarly for $\mathbb{P}[X=0]$)

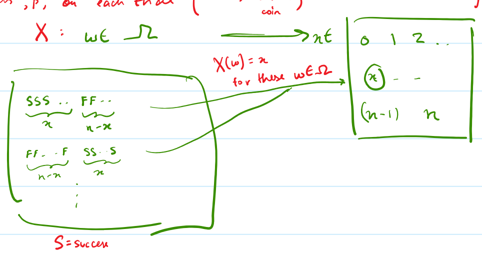
① Binomial

(sampling w/ replacement & considering # of successes given finitely many trials)

(DF) $\boxed{X \sim \text{Binomial}(n,p) \text{ when } \mathbb{P}[X=x] = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 \le x \le n}$

As an example, say we repeat an experiment n times, with probability of success, p, on each trial (for ex., toss a coin). Let X = # of successes (S) obtained. Then, as a function:

$X: \omega \in \Omega \longrightarrow x \in$



(& there are $\frac{n!}{x!(n-x)!}$ such toss sequences leading to $x \in \mathbb{R}$)

Remember, (Ω is meant to represent our experiment space)

S = success
F = failure

, and $\mathbb{P}[X=x]$

Deriving PMF

$= \mathbb{P}[\omega : X(\omega) = x]$

$= \underbrace{\frac{n!}{x!(n-x)!}}_{\text{\# of ways to get } x \text{ successes in } n \text{ trials.}} \ \underbrace{p^x(1-p)^{n-x}}_{\mathbb{P}[x \text{ successes}, \ n-x \text{ fails}] \text{ for one such outcome}}$

② Hypergeometric

(sampling w/o replacement)

(DF) $\boxed{X \sim \text{Hypergeometric}(N, M, n) \text{ when } \mathbb{P}[X=x] = \frac{\binom{M}{x}\binom{N-M}{N-x}}{\binom{n}{N}}, \quad 0 \le x \le N}$

As an experiment, say we have $\underline{N}$ objects, out of which M are 'successes', and we proceed to sample n without replacement.

$X: \omega \in \Omega \longrightarrow x \in \{0, 1, \ldots, (n-1), n\}$

$\underbrace{SF\ldots SS}_{n}$

$\underbrace{FS\ldots FS}_{n}$

$\vdots$

Now, to find $\mathbb{P}[X=x]$, note that

$$\mathbb{P}[\omega \in \Omega : X(\omega) = x] = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$\binom{M}{x}$: total ways to draw exactly x successes out of M total

$\binom{N-M}{n-x}$: total ways to draw exactly $(n-x)$ failures out of $(N-M)$ total.

$\binom{N}{n}$: total ways to draw n out of N

3) Negative binomial

(Df) $\boxed{X \sim \text{neg. binom}(r, p) \text{ if } \mathbb{P}[X=x] = \binom{x-1}{r-1} p^{r-1}(1-p)^{x-r} \cdot p \quad (x \geq r)}$

As an experiment, say we repeat an experiment until we get the r^{th} success. On each trial, probability of success is p. Let $X = x$ be trial of the r^{th} success.

$X: \omega \in \Omega \longrightarrow x \in \{r, r+1, r+2, \ldots\}$

$\underbrace{SS\ldots S}_{r}$

$\underbrace{\underbrace{SS\ldots}_{r-1}\underbrace{FF\ldots}_{(x-1)-(r-1) = x-r}}_{(x-1) \text{ repeats}}\ S$ (r^{th} success on x^{th} repeat)

Ω is all possible S,F-sequences (of length infinity, even) until r^{th} success is obtained.

So, $\mathbb{P}[X=x] = \mathbb{P}[\omega \in \Omega : X(\omega) = x]$

$$= \underbrace{\binom{x-1}{r-1}}_{\substack{\text{\# of ways to get } (r-1) \text{ successes} \\ \text{in } (x-1) \text{ trials.}}} \underbrace{p^{r-1}(1-p)^{(x-1)-(r-1)}}_{\mathbb{P}[\text{one of these outcomes}]} \cdot \underbrace{p}_{\mathbb{P}[\text{success at } x^{th} \text{ trial}]}$$

4) Geometric

(Df) $\boxed{X \sim \text{Geometric}(p) \text{ if } \mathbb{P}[X=x] = p^x, \text{ where } p \in (0,1) \text{ and } x \geq 0}$

Try the above procedure yourself!

(1) What is the experiment?

(2) How might you derive pmf from experiment?

5) Poisson

(Df) $\boxed{X \sim \text{Poisson}(\lambda) \text{ if } \mathbb{P}[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \geq 0}$

Expectation and Variance

① $\mathbb{E}[X] = \sum x \mathbb{P}[X=x]$

Linearity of expectation (for sums) $\mathbb{E}[\sum X_i] = \sum \mathbb{E}[X_i]$ (always)

Products Not generally linearly expandable - we will come back to this

LOTUS

$\mathbb{E}[g(X)] = \sum g(x)\mathbb{P}[X=x]$

② $\text{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$

② $Var(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$

$\rightarrow Var(cX) = c^2 Var(X)$

$\mathbb{E}[g(X)] = \sum g(x) \mathbb{P}[X=x]$

$\rightarrow \mathbb{E}[cX] = c\mathbb{E}[X]$, for $c \in \mathbb{R}$

<u>Questions</u>

① True or false?

(i) $\mathbb{E}[X] \geq 0$ always

(ii) The following is a valid pmf,

x	0	1	2	3
$\mathbb{P}[X=x]$	1/3	1/3	1/3	1/3

(iii) $Var(X) \geq 0$ <u>always</u>

(iv) $Var(X) = 0$ <u>if and only if</u> $\mathbb{P}[X = \mathbb{E}[X]] = 1$

② Suppose you are applying to colleges.
The probability you like a college is $\frac{1}{10}$.
The probability that you are accepted by a college is $1/5$.
Assume that the above 2 are independent events.
If X_n = # of colleges you like and get into, after applying to n colleges.

① Find the distribution of X_n

② Find the mean of X_n

③ Suppose that the distribution of # errors occur in any given page on a newspaper follows Poisson(λ) distribution.

Also, assume that number of errors on different pages are independent.

(1) Find the probability that the first mistake is on page n.

(2) Find the probability that the second mistake is on page n.