Oct 17-19
(i) Normal distribution
(ii) Functions of r.v.'s
- CDF methods
(iii) Joint distributions
- Discrete

Oct 24-26
(i) Joint distributions
- Continuous
(ii) Independent r.v.'s
(iii) MGF method (?)

Normal distribution

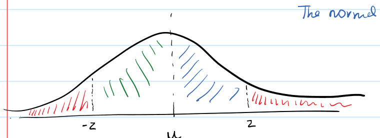
① Using $\phi(z)$ and its properties to compute $\mathbb{P}[Z \le z]$, $\mathbb{P}[a < Z \le b]$ etc...

The normal density is symmetric about its mean!

(i.e. $f_X(\mu+a) = f_X(\mu-a)$, $\forall a \in \mathbb{R}$)

why?

$$f_X(\mu \pm a) = \frac{e^{-\frac{1}{2}\left(\frac{\cancel{\mu} \pm a - \cancel{\mu}}{\sigma}\right)^2}}{\sqrt{2\pi\sigma}} = \frac{e^{-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2}}{\sqrt{2\pi\sigma}}$$

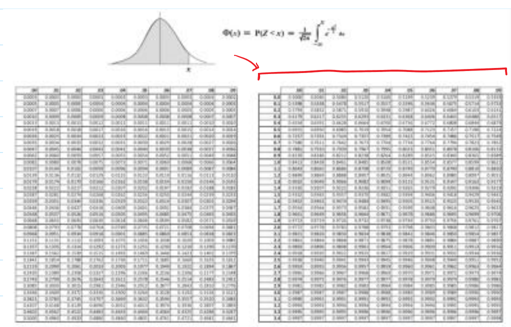


(i) $\forall z \in \mathbb{R}$, $\mathbb{P}[Z \le z] = \phi(z)$ (by defn)

(ii) $\phi(-z) = 1 - \phi(z)$ or $\phi(-z) + \phi(z) = 1$. So, if you know CDF for positive val, you know it for negative val.

(iii) $\mathbb{P}[-z \le Z \le z] = \phi(z) - \phi(-z) = 2\phi(z) - 1$ So, you know CDF for intervals if you know it for one end.

(Can you think of more?)



② Linear combo. of normals is normal (including affine transformations)

(1) If $X \sim N(\mu, \sigma^2)$, $aX + b \sim N(a\mu + b, a^2\sigma^2)$

(2) If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, and they are independent*, $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

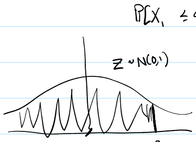
③ z-score : $\frac{X - \mu_X}{\sigma_X}$ (We do this because, if X follows some normal distribution w/ mean μ_X & variance σ_X, $\frac{X - \mu_X}{\sigma_X} \sim N(0,1)$ $\boxed{\text{WHY?}}$)

(Q1) Say, $X_1 \sim N(82, 4^2)$ is the distribution of homework 1 scores.
$X_2 \sim N(90, 2^2)$ is distribution of homework 2 scores.
Suppose you choose student randomly from your class. Compute:

(i) $\mathbb{P}[X_1 \le 90]$

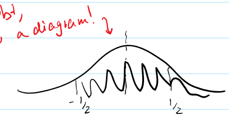
$\mathbb{P}[X_1 \le 90] = \mathbb{P}\left[\frac{X_1 - 82}{4} \le \frac{90-82}{4}\right]$

$= \mathbb{P}[Z \le 2] \approx 0.9772$ $(\phi(2))$



(iii) $\mathbb{P}[X_1 \ge 80]$

when in doubt, draw a diagram!



$= 1 - \mathbb{P}[X_1 \le 80]$

$= 1 - \mathbb{P}\left[\frac{X_1 - 82}{4} \le \frac{80-82}{4}\right]$

$= 1 - \mathbb{P}[Z \le -\frac{1}{2}] = 1 - \phi(-\frac{1}{2}) = 1 - (1 - \phi(\frac{1}{2}))$

$= \phi(\frac{1}{2}) \approx 0.6915$

(ii) $\mathbb{P}[X_2 \le 98]$

Same idea as (i)

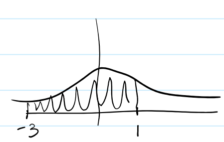
(iv) $\mathbb{P}[84 \le X_2 \le 92]$

$= \mathbb{P}\left[\frac{84-90}{2} \le \frac{X_2 - 90}{2} \le \frac{92-90}{2}\right]$

$= \mathbb{P}[-3 \le Z \le 1]$

$= \phi(1) - \phi(-3)$

$= \phi(1) - (1 - \phi(3))$



$\boxed{\text{CDF method}}$ $\Rightarrow$ 1st way to find densities of functions of r.v.'s.

With an example...

(i) Your original r.v. and function of interest

$X \sim \mathcal{N}(0,1)$

$Y = aX + b$, $(a>0, b \in \mathbb{R})$

(ii) Write CDF of Y in terms of that of X $\longrightarrow$ aim: to differentiate it to get density $f_Y(y)$

Now, $\mathbb{P}[Y \le y] = \mathbb{P}[X \le \frac{y-b}{a}] = \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx$

(iii) Differentiate

Here, we invoke Leibniz rule:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(x,y)\, dx = g(b(y),y)\, b'(y) - g(a(y),y)\, a'(y) + \int_{a(y)}^{b(y)} \frac{\partial g}{\partial y}(x,y)\, dy$$

So, $\frac{d}{dy}\mathbb{P}[Y \le y] = \frac{d}{dy}\int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$

$$= \frac{\exp\left(-\frac{1}{2}\left(\frac{y-b}{a}\right)^2\right)}{\sqrt{2\pi a^2}} = \text{pdf of } \mathcal{N}(b, a^2)$$

(Q's) If $Z \sim \mathcal{N}(0,1)$, find pdf of $Y = Z^2$.

If $X \sim \exp(1)$, find pdf of $Y = 1/X$.

If $X \sim N(\mu, \sigma^2)$, find pdf of $Y = e^X$ (lognormal(μ, σ^2))

$\boxed{\text{Special case}}$

(Q) What if your function, $g(X)$, is strictly $\Uparrow$ monotone?

$\Downarrow$

Now, following our steps, (Since g is monotone)

(i) $\mathbb{P}[g(X) \le y] = \mathbb{P}[X \le g^{-1}(y)]$ (If g is strictly increasing)

$= \int_a^{g^{-1}(y)} f_X(x)\, dx$

(ii) Then, $\frac{\partial}{\partial y}\int_a^{g^{-1}(y)} f_X(x)\, dx = f_X(g^{-1}(y))\, g^{-1\,\prime}(y)$

① Example 1 (Random number generation)

Say, $Y \sim \exp(\lambda)$. The inverse cdf is a function: $\begin{cases} [0,1] \to \mathbb{R} \\ k \to x \end{cases}$

where x is such that $\mathbb{P}[X \le x] = k$.

① Compute the inverse cdf of Y as a function $f(x)$

Now, $\mathbb{P}[Y \le y] = \int_0^y \lambda e^{-\lambda x} dx = 1 - e^{-\lambda y}$

Then, if cdf is $1 - e^{-\lambda y} = \mathbb{P}[Y \le y]$, then it's inverse is found by:

$x \in (0,1]$

$x = 1 - e^{-\lambda y}$

$\Rightarrow e^{-\lambda y} = 1 - x$

$\Rightarrow -\lambda y = \ln(1-x)$

$\Rightarrow \boxed{y = \frac{-1}{\lambda}\ln(1-x)}$

"give me some probability x, and I will give you $y_x > 0$ s.t. $\mathbb{P}[Y \le y_x] = x$"

② If $X \sim$ uniform$(0,1)$ and $Y = f(X)$, find pdf of Y.

Now, $\mathbb{P}[Y \le y] = \mathbb{P}[-\frac{1}{\lambda}\ln(1-X) \le y] = \mathbb{P}[X \le 1 - e^{-\lambda y}]$

$= \int_0^{1-e^{-\lambda y}} 1\, dx = 1 - e^{-\lambda y}$

Then, $\frac{d}{dy}(1 - e^{-\lambda y}) = \lambda e^{-\lambda y}$ $\Rightarrow$ pdf of $\exp(\lambda)$! Given what we did in 1, this is no coincidence!

(2) Example 2 (Practicing caution when using CDF method)

this fxn isn't injective on the support of the density.

Say, $X \sim \exp(\lambda)$. Consider $g(X) = (X-1)^2$. Find its pdf.

Now, $f_X(x) = \lambda e^{-\lambda x}$

Then, for $y > 0$, $\mathbb{P}[g(X) \le y] = \mathbb{P}[(X-1)^2 \le y] = \mathbb{P}[1-\sqrt{y} \le X \le 1+\sqrt{y}]$

$$= \int_{1-\sqrt{y}}^{1+\sqrt{y}} f_X(x)\, dx$$

note, $y=1$ needs to be accounted in the second case, since $\text{supp}(f_X) = (0,\infty)$.

- Now, if $y < 1$, then $(1-\sqrt{y}, 1+\sqrt{y}) \subset \mathbb{R}^+$ i.e. it is contained in support of X.
- But, if $y \ge 1$, then $(1-\sqrt{y}, 1+\sqrt{y}) \cap \mathbb{R}^+ = (0, 1+\sqrt{y})$.

Thus, $y<1 \Rightarrow \dfrac{\partial}{\partial y}\displaystyle\int_{1-\sqrt{y}}^{1+\sqrt{y}} f_X(x)\,dx = \lambda e^{-\lambda(1+\sqrt{y})} \cdot \frac{1}{2\sqrt{y}} + \underbrace{\frac{\lambda e^{-\lambda(1-\sqrt{y})}}{2\sqrt{y}}}_{}$

$y \ge 1 \Rightarrow \dfrac{\partial}{\partial y}\displaystyle\int_{0}^{1+\sqrt{y}} f_X(x)\,dx = \lambda e^{-\lambda(1+\sqrt{y})} \cdot \frac{1}{2\sqrt{y}}$

$\Downarrow$ more density here when $y<1$ (but not twice as much) $\longrightarrow$ WHY?

Joint distributions

(Df) When r.v.'s $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ are defined on a common sample space Ω, we have the joint pmf (or pdf)*,

$$P_{X,Y}(x,y) := \mathbb{P}[X=x, Y=y]$$

$$(f_{X,Y}(x,y) := f(X=x, Y=y))$$

The marginals are distribution of some subset of r.v.'s,

$$P_X(x) := \mathbb{P}[X=x]$$

$$(f_X(x) := f(X=x))$$

$\boxed{\text{Properties}}$

(1) Joints are pmfs / pdfs

i.e. especially, $\iint f_{X,Y}\, dx\, dy = 1$

$$\left(\sum\sum P_{X,Y}\, dx\, dy = 1\right)$$

(2) We can get marginal from joint

"Integrate out other variable" $\quad f_X(x) = \int f_{X,Y}(x,y)\, dy$

$$(P_X(x) = \sum_y P_{X,Y}(x,y)$$

+ all other properties (monotonicity, ctable additivity, ctable subadditivity ...)

$\boxed{\text{Independence}}$

Soon, we'll care about conditional probability!

(Df) A collection of r.v.'s are independent if joint distribution = product of their marginals

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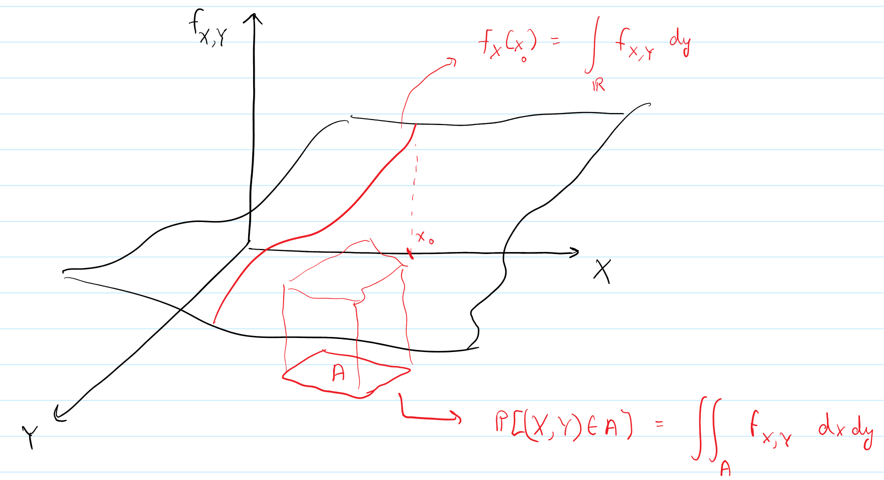
conditional probability!

<u>Visual representations</u>

	$X=1$	$X=2$	$X=3$	
$Y=1$	0.1	0.1	0.1	0.3
$Y=2$	0.2	0.2	0	0.4
$Y=3$	0.15	0	0.15	0.3
	0.45	0.30	0.25	1

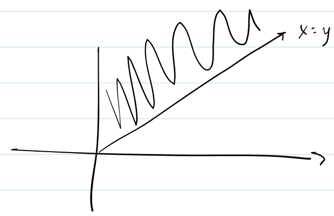


<u>Example 1</u>

Let $f_{X,Y}(x,y) = \begin{cases} e^{-y} & , 0 < x < y < \infty \\ 0 & , \text{else} \end{cases}$

(1) <u>Show this is a valid joint pdf.</u>

Show $\displaystyle\int_0^\infty\int_0^y e^{-y}\, dx\, dy = 1$



(2) <u>Compute the marginal of X and Y.</u> (Are they independent?)

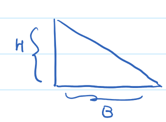
$$f_X = \int_x^\infty e^{-y}\, dy \quad , \quad f_Y = \int_0^y e^{-y}\, dx$$

Not independent

<u>Example 2</u> (Event of interest into some integral over set of interest)

Say, you walk to the taxi stop everyday at a time uniformly between 8 and 9am.
So does your crush.
However, after arriving, the person waits ~10 minutes until a taxi picks them.
What is the probability you see your crush?

<u>Example 3</u>



You are drawing triangles with randomly uniform dimensions, H and B

Say, $H \sim$ uniform $(0, 5)$

$B \sim$ uniform $(0, 2)$. Find the probability that the area of your triangle is at least 2 units^2.

(Assume you always draw right-angled triangles)