Section 1 - Probability

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1 Structure Of The Class

Welcome to 553.620 Introduction to Probability! It is a class that (at least at the undergraduate level) is notorious for its difficulty. In my humble opinion, however, it is quite interesting and, with practice, will form part of how you naturally think about things in everyday life (such as counting the number of different meals you can plan from a finite number of ingredients - even though all of them may not be so appetising).

The class is centered around computing the likelihood of events. We'll approach this central theme through five phases of the material:

1. **Combinatorics** - we begin by seeing probability in its most apparent form around us, in collections of object(s) (dice, books) upon which we perform some experiment (rolling it 5 times, or arranging them in a certain way). Here, we exploit the structure of our sample space to determine the number of favourable outcomes (getting three 6's, or having the math books grouped together), which enables us to compute their likelihood.

   I call this "naive" probability - not because it is easy (I think it is one of the trickiest things in the course), but because it is often the sort of probability that one first comes across in life, when wondering about the chances of things happening.

2. **Beyond "Naive" Probability** - here, we move beyond "naive" probability by abstraction. We consider some axioms of probability that make rigorous our intuitions about it. We also think about doing probability in more complicated scenarios - for example, in computing the likelihood of outcomes when we are already given some information about the experiment (in our above example, suppose we know we have gotten one 6 already). How might this update the probability of getting three 6's?

3. **Random variables** - being prepared with the theoretical axioms of probability, as well as knowing how to compute probabilities for various experiments, we begin to notice symmetries across different experiments. For example, we realise that getting 3 heads in 5 coin tosses is analogous to you enjoying on-campus food 3 out of 5 days (the probability of the latter arguably lower than that of the former). Indeed, many of these symmetries have been named by famous mathematicians, and we'll learn a bunch of them to develop our probabilistic language.

   At this point, we also start to talk about probabilities of events when the probability of a sample point is 0. We dive into the continuous realm! For example, sampling a number uniformly randomly from the interval $(0, 1)$, what is the probability of getting a number less than 0.5? We'll learn about the glorious normal distribution here!

4. **Messing around with random variables** Armed with knowledge of random variables of various types and their probability distributions, we'll start "riffing" with them. We'll take functions of random variables (which will also turn out to be random variables), study how one random variable might co-vary with another, and other things along this line.

5. **General theorems** Finally, we'll speak about the about the asymptotic behavior of sums of random variables - what happens if you take the sums of infinitely many random variables that have the same distribution (you might have heard of the central limit theorem)? We also consider inequalities that describe behaviour of very broad classes of probability distributions.

2 Topics Discussed

This week was the start of phase 1 - combinatorics. We saw the basic counting rule, permutations, anagrams, and combinations.

2.1 Basic Counting Rule

Given $n_1, n_2...n_r$ distinct choices for distinct positions $1, 2...r$, there are $n_1 n_2 n_3 ... n_r = \prod_i n_i$ choices.

Pf. We prove this by induction on the number of distinct positions, r. When $r = 1$, there are clearly n_1 sequences of length 1, corresponding to a choice of one of n_1 objects for the first position.

Then, assume that for fixed $r \in \mathbb{N}$, there are $\prod_{i=1}^{r} n_i$ possible distinct sequences.

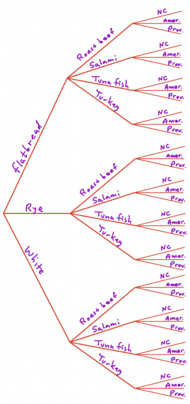
Finally, for $r+1$ positions, we have (by the induction hypothesis) $\prod_{i=1}^{r} n_i$ possible sequences for the first r positions. Fixing one of these r-length sequences, there are n_{r+1} choices for the $(r+1)^{th}$ position. This is true for any arbitrary sequence. Therefore, there are $n_{r+1} \prod_{i=1}^{r} n_i = \prod_{i=1}^{r+1} n_i$ possible sequences.

Ex. Suppose you are choosing a committee with one student, one teacher and one parent position. If there are 5 students, 7 teachers and 10 parents to choose from, how many different committees can you form?

Modelling our sample space as an ordered 3-tuple $\{\_s, \_t, \_p\}$, we see there are $5, 7$ and 10 choices for the first, second and third positions, respectively. Hence, there are $5 \cdot 7 \cdot 10 = 350$ total committees possible.

How to see this

1. One way to see it is as a branching tree - each branch in the i^{th} layer corresponds to a choice of object for the i^{th} position. The total number of possible sequences is the number of branches in the last layer, since a path from a node in the first layer to a node in the last layer completely and uniquely determines a sequence.



Special case

1. An experiment where you are sampling n positions objects for r positions, with replacement, is a special case of the basic counting rule, where you have $n_i = n$ for all positions $1, 2...r$.

2. An experiment where you are sampling n distinct objects for r distinct positions, without replacement, is also a special case of the basic counting rule. Applying the

theorem above gives you $n(n-1)...(n-(r+1)) = \frac{n!}{(n-r)!}$ total choices. We call these the number of **permutations** of n objects into r positions. When $r = 1$, you get $n!$ - the number of ways to order n distinct objects.

2.2 Anagrams and Combinations

Suppose we want to count the number of words we can make with the letters S, L, A, P, S. How do we do this? If we think of a word as a 5-tuple of letters, we may apply the basic counting rule (specifically, the number of ways to order n distinct objects) to get 5! as our answer.

But wait, are all the objects distinct? There are two objects in our collection of 5 objects that we might consider non-distinct when constructing words right now - they are the two S. Indeed, if we label them as S_1 and S_2, we currently count S_1LAPS_2 as a different word from S_2LAPS_1. However, for the purposes of modern English, these are the same word. We are **overcounting** here. Indeed, we want to count tuples up to some **permutation invariance** - the order of S_1 and S_2, specifically, don't matter to us in this experiment. How can we account for permutation invariance?

Th Suppose there are r distinct objects of multiplicity $m_1...m_r$ such that $\sum_{i=1}^{r} m_i = n$. Then, there exist $\frac{n!}{m_1!m_2!...m_r!}$ distinct permutation-invariant sequences of length n of these objects.

Pf This can be found as a visual argument in my PDF notes for Section 2.

Note that the "distinctiveness" of objects is a completely invented trait, decided by us for our experiment. Indeed, suppose we are choosing a committee of 3 people from 10 people. If we say that the 3 positions are distinct, there are $10 \cdot 9 \cdot 8$ choices. However, if the 3 positions are non-distinct, our experiment is just asking us how many trios of distinct people we can choose. We can apply our theorem by having 2 distinct "objects" - chosen (C) and not chosen (NC) - with multiplicity 3 and 7, respectively. Then, there are $\frac{10!}{3!7!}$ choices, which is conventionally notated as $\binom{10}{3}$ (to be explained later).

The above theorem also gives us the **multinomial coefficient**. Specifically, we introduce the notation

$$\binom{n}{m_1, m_2...m_r} = \frac{n!}{m_1!m_2!...m_r!}$$

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3 Discussion Questions

Q1) In class, you computed the number of possible full houses (one 3-of-a-kind, one 2-of-a-kind, for ex. 'KKK99') in poker. What are the total number of possible:

- straights (all 5 cards can be arranged in a consecutive ascending sequence, for ex. A,2,3,4,5 or 10,J,Q,K,A)? Note that A can be counted as both highest and lowest card of the deck.
- flushes (all 5 cards are of same suit)?

Q2) (Taken from Gabe's Intuition to Probability, Chapter 1) Suppose you are given an $m \times n$ grid of boxes. How many possible rectangles can you construct out of it? Try out these two trains of thought:

a) How can the selection of vertical and horizontal lines be uniquely associated to a rectangle?

b) What are the possible lengths and the possible widths of the rectangles we can get? For a particular length (or width), how many ways are there to get that length column-wise (or row-wise)?

Q3) How many ways are there to express a number as the product of 2 unique integers? Assume that order doesn't matter i.e. $4 \times 1 = 1 \times 4$ is the same representation.

(Hint: Think about prime factors. Try out an example with prime and with composite numbers.)