Section 4

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1 New Things!

Having moved farrrr beyond naive probability, we have been learning how to probabilistically model novel varieties of questions and real-life experiments. We have started to think in terms of events - an event being a subset of your sample space - which has naturally brought out some set theory. This week, I'll review some set theoretical notions, and give a review of the new modelling techniques we have learnt - namely, conditional probability, law of total probability, Bayes' Theorem, and independence.

1.1 The importance of set Theory

Recall that we think of probability spaces as triples $(\Omega, \mathcal{F}, P)$ where,

1. Ω is our sample space (all possible coin toss sequences if tossing a coin 5 times - $HHHHH, TTTTT, \ldots.$
2. $\mathcal{F}$ is the set of events in the sample space (so, every 5-coin-toss sequence)
3. P is a probability measure on $\mathcal{F}$ (for any sequence in our space, the probability would be $\frac{1}{2^5}$

Because we see events as subsets of a bigger set, set theory becomes a very natural tool to invoke here. Venn diagrams and usual set operations (unions and intersections) are probably familiar to you. But I want to give a new way of looking at sets that (at least for me) makes it easier to do complex operations on them, and to understand something like De Morgan's laws.

1.1.1 Reminder of Set Theory Properties

The basic definitions of complements, unions, intersections:

1. $A \cup B := \{x \in A \text{ or } x \in B\}$
2. $A \cap B := \{x \in A \text{ and } x \in B\}$
3. $A^c = \{x \in \Omega, x \notin A\}$

From above, we can derive many properties. One useful property is distributivity.

1. $A \cap (B \cup C) := \{x \in A \ \underline{\text{and}}\ (x \in B \text{ or } x \in C\} = \{x \in A \text{ and } x \in B \ \underline{\text{or}}\ x \in A \text{ and } x \in C\}$

   However, the natural "reversal" of operations isn't true.

2. $A \cup (B \cap C) := \{x \in A \ \underline{\text{or}}\ (x \in B \ \text{ and } x \in C\} \neq \{x \in A \text{ or } x \in B \ \ \underline{\text{and}}\ x \in A \text{ or } x \in C\}$.

   You might note from the above that intersections distribute over unions in general. This is a useful intuition - to see intersections as analogous to multiplication, and unions as analogous to addition. It can speed up calculations!

   For example, you can expand $(A \cup B) \cap (C \cup D)$ similarly to $(A + B) \times (C + D)$.

1.1.2 De Morgan's Laws

When taking the complement of a union (or intersection), take the complement of the sets in consideration and flip the operation to an intersection (or union).

1. $(A_1 \cup A_2 \cup A_3 \cup ...)^c = A_1^c \cap A_2^c \cap A_3^c \cap ...$
2. $(A_1 \cap A_2 \cap A_3 \cap ...)^c = A_1^c \cup A_2^c \cup A_3^c \cup ...$

1.2 New Theorems

1. We have **conditional probability** from before,

$$P[B|A] = \frac{P[B \cap A]}{P[A]}$$

2. We have the **law of total probability** - it follows from countable addivitity of probability measures. So, whenever $B_1 ... B_n$ are a disjoint partitition of your sample space,

$$P[A] = P[A|B_1] + P[A|B_2] + P[A|B_3] + ... + P[A|B_n]$$

3. The result of the above two is **Bayes' Rule**. This is a very interesting result in probability (we will see why later today) and is a result of the definition of conditional probability. Specifically,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

4. And lastly, we have the definition of independence: A and B are independent if $P(A \cap B) = P(A)P(B)$. Equivalently, if $P(A|B) = P(A)$.

2 Questions

1. Give examples, if they exist, of events A and B in an arbitrary sample space such that,

   a) A and B are independent and mutually exclusive.

   b) A and B are not independent and mutually exclusive.

   c) A and B are independent and not mutually exclusive.

   d) A and B are not independent and not mutually exclusive.

2. There are 2 jellybeans - they can be either red, blue or green. If one of them is red, what is the probability that the other is red also?

3. You are a scientist observing your bacteria culture when you notice a glow under your microscope.

Now, bacteria glowing is kinda rare - it is known that approximately 0.2% of all bacteria glow. Moreover, even if the bacteria were to glow in actuality, your experiment was crudely designed - and so, the probability of witnessing a glow was only 5%.

In fact, what makes you more doubtful is that your lab also works with fluorescence. If a fluorescent contaminant got into your culture, you know it would glow approximately 99% of the time. However, there is only a 3% chance of contamination.

BUT - it might be that your instrument is malfunctioning. The probability of that happening is 0.01. Moreover, even it does malfunction, only 25% of users observe a glow.

You assume that the probability of multiple factors coinciding (i.e. malfunctioning instrument and contamination) is negligible.

Thus, what is the probability that your bacteria glow, given you observed a glow?