Section 3 - Probability

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1 The Last of Combinatorics - Stars and Bars

Stars & Bars

Counting number of distinct multisets of length n, with elements originating from a set with r elements, by using stars-and-bars.

Stars and bars

non-distinct bars (because we don't want to count permutation of it-stuff like $\overset{2}{|}\ \overset{3}{|}\ \overset{5}{|} \ldots \overset{1}{|}$)

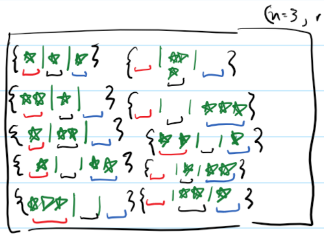
$\underbrace{\star\star\star\ldots\star}_{n \text{ (indistinct) stars}}$ $\quad \underbrace{\sqcup \overset{1}{|} \sqcup \overset{2}{|} \sqcup \overset{3}{|} \ldots \overset{r-1}{|} \sqcup}_{r \text{ (distinct) boxes}}$

And so, how many diff. anagrams exist?

$$\binom{n+r-1}{r-1} = \frac{(n+r-1)!}{n!\,(r-1)!} = \binom{n+r-1}{n} \quad \left(= \# \text{ of non-negative solutions to } n_1 + n_2 + \ldots + n_r = n \right)$$

This counts sample space like:

$(n=3,\ r=3)$

$\Omega =$ 

$\left(\# \text{ of positive solutions } (n_i > 0) \text{ to } n_1 + n_2 + \ldots + n_r = n \right) \Rightarrow$ every box already has 1 star. Then,

$\underbrace{\star\star\ldots\star}_{(n-r) \text{ stars}}$ $\quad \underbrace{\overbrace{|\ |}^{r-1 \text{ bars}}}_{r \text{ boxes}} \Rightarrow \binom{n-1}{r-1}$

2 Axioms of Probability

With the end of counting techniques comes the beginning of axiomatic probability. In particular, you saw the following definition of a probability measure:

A **probability measure** $\mathbb{P}$ on a sample space Ω is a set function defined on the set of events in Ω, $\mathcal{F}$, into $[0,1]$ such that,

1. $\mathbb{P}[\Omega] = 1$. That is, the probability measure of the sample space is 0
2. $\mathbb{P}[A] \geq 0$. That is, you can't have any event with negative probability.
3. Given disjoint $\{A_i\}_{i=1}^{\infty}$, we have that $\mathbb{P}[\cup A_i] = \sum \mathbb{P}[A_i]$. That is, given finite or countably infinite set of disjoint events, the sum of their probability = the probability of their union.

The above axioms are really just fancy ways of saying things you intuitively realise already. For example, how can something be more probable than 100% likely to happen? Nothing too novel here!

There are other useful properties to know, too, which are a result of the above axioms (Try to prove some of them!):

1. Complementarity: $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$
2. Monotonicity: If $A \subset B \subset \Omega$, then $\mathbb{P}[A] \leq \mathbb{P}[B]$.
3. Subadditivity: Given arbitrary $\{A_i\}_{i=1}^{\infty}$ (not necessarily disjoint), we have that $\mathbb{P}[\cup A_i] \leq \sum \mathbb{P}[A_i]$.
4. Inclusion-Exclusion Principle: If $A, B \subset \Omega$, then $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$

And one last thing you saw was this new notion of conditional probability: specifically,

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cup B]}{\mathbb{P}[B]}$$

Again, all intuitive things! The useful thing here are the ways in which they make computing probabilities easier. Today's section will be dedicated to seeing just that. Try to think about what sort of problem lends itself to what sort of technique of solving it. For example, why might direct counting not work? Or why might the complement event be easier? How might you change the problem to suit one technique over another?

3 Discussion Questions

1. Suppose you are throwing 8 chocolates and 16 candy bars at 8 children. How many different ways can the children get chocolates and candies?

   BONUS: What is the probability that each child gets twice as many candies as chocolate bars? Assume that a child getting 0 candies and 0 chocolate bars satisfies this condition. Think about an easier and equivalent way of counting this.

STARS/BARS for chocolate $\leftarrow \binom{15}{7} \cdot \binom{23}{7} \rightarrow$ for candies

h_i = # choco. given to child i

BONUS

$$\frac{\binom{15}{7}}{\binom{15}{7} \cdot \binom{23}{7}} = \frac{1}{\binom{23}{7}}$$

(Given assignment of chocolates $(h_1, h_2 \dots h_8)$, there's a unique assignment $(c_1, c_2 \dots c_8)$ of candies satisfying our condition)

2. How many distinct rolls of three 6-sided die are possible? What about four die? Can we express this as a formula for n die?

For n die : $\binom{n+5}{5} = \binom{n+5}{n}$

each (so, total of 40 representatives)

3. Suppose you are designing a committee. You have 10 representatives from 4 continents (Asia, Europe, Africa and South America), with each also being from a distinct country. You choose 10 at random. What is the probability that your committee is missing a member from at least one ~~country~~ continent?

(so, 40 countries)

Use PIE

Let A_i = event that we miss person from country i, $i \in \{1,2,3,4\}$.

Then, by PIE,

$$P\left(\bigcup_{i=1}^{4} A_i\right) = \underbrace{\sum_{i=1}^{4} P(A_i)}_{\Downarrow} - \underbrace{\sum\sum_{\substack{i,j=1 \\ i\neq j}}^{4} P(A_i \cap A_j)}_{\Downarrow} + \sum\sum\sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{4} P(A_i \cap A_j \cap A_k) - \underbrace{\sum\sum\sum\sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l}}^{4} P(A_i \cap A_j \cap A_k \cap A_l)}_{\Downarrow \; 0}$$

$$= 4 \cdot \frac{\binom{30}{10}}{\binom{40}{10}} - \binom{4}{2} \cdot \frac{\binom{20}{10}}{\binom{40}{10}} + \binom{4}{3} \cdot \frac{\binom{10}{10}}{\binom{40}{10}}$$

4. Suppose you are picking up pencils numbered 1 through 6. What is the probability that one of your last 2 draws is an even-numbered pencil?

- Let E_i = i^{th} draw is even no. $i \in \{1,2,3 \dots 6\}$
- We are interested in $P(E_5 \cup E_6) = 1 - P(E_5^c \cap E_6^c)$ (De Morgan's law)

① complementarity

$$P\left(\begin{matrix}\text{both last draws} \\ \text{are odd}\end{matrix}\right) = \frac{4! \times \binom{3}{1} \times \binom{2}{1}}{6!} = \frac{\cancel{3 \times 2}}{\cancel{6} \times 5} = \frac{1}{5}$$

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$\underbrace{\_ \_ \_ \_}_{4!\text{ permutation}}$; choose 1 out of 3 odds $\binom{3}{1}$; choose 1 out of 2 odds $\binom{2}{1}$; their permutation $(2!)$

② PIE

$$P(E_5 \cup E_6) = \underbrace{P(E_5)}_{\Downarrow \; 1/2} + \underbrace{P(E_6)}_{\downarrow \; 1/2} - \underbrace{P(E_5 \cap E_6)}_{\Downarrow \; \frac{4!}{6!} \times \binom{3}{1} \times \binom{2}{1} = \frac{1}{5}}$$

③ Direct

$|\Omega| = 6!$

$$|A|, \text{ favourable outcomes} = \underbrace{(3 \times 5!)}_{5^{th}\text{ draw is even}} + \underbrace{(3 \times 5!)}_{6^{th}\text{ is even}} - \underbrace{4! \times 3 \times 2}_{\text{to prevent double counting}}$$

$\frac{|A|}{|\Omega|}$ is our result

5. You are composing sequences of 104 letters (A,B,C...Z) and suddenly wonder how many sequences are possible with the alphabet in order at least once. So, how many are there?

# of letters in alphabet = 26

$|A|$ "

So, we can fit,

1 sequence of alphabet $= 26^{78} \cdot \frac{79!}{78!\,1!}$ — 78, 26 letters as 1 block

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2 " " " $= 26^{52} \cdot \frac{54!}{52!\,2!}$ — 52, 1, 1

$+$

3 " " " $26^{26} \cdot \frac{29!}{26!\,3!}$ — 26, 1, 1, 1

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4 " " " $\frac{4!}{4!} = 1$

(Q) Why are we adding & subtracting the way we are?

6. (Conditional Probability) What is the probability that 4 randomly drawn cards from a deck are $\underbrace{\text{different values}}_{V}$ given they are $\underbrace{\text{different suits}}_{S}$?

$$\mathbb{P}[V \mid S] = \frac{\mathbb{P}[V \cap S]}{\mathbb{P}[S]} \rightarrow \frac{(13 \times 12 \times 11 \times 10) \times \cancel{4!}}{(13 \times 13 \times 13 \times 13) \times \cancel{4!}}$$