Playing around with random variables

distribution	pdf and domain	$E(X)$	$Var(X)$	mgf $M(\theta)$
$uniform(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{\theta b}-e^{\theta a}}{(b-a)\theta}$
$Exp(\lambda)$	$f(x)=\lambda e^{-\lambda x},\ x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$(1-\frac{\theta}{\lambda})^{-1}$ $\theta<\lambda$
$Gamma(\alpha,\beta)$	$f(x)=\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)},\ x>0$	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta\theta)^{-\alpha}$ $\theta<1/\beta$
$Normal(\mu,\sigma^2)$	$f(x)=\frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sqrt{2\pi\sigma^2}},\ -\infty<x<\infty$	μ	σ^2	$e^{\mu\theta+\frac{\sigma^2\theta^2}{2}}$

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Why care?

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- Because they model real-life experiments (or thought experiments) that we may find interesting.
- Today, we'll motivate the various things you've learnt through these experiments

① Suppose you are kicking a soccer ball. The distance you kick it is uniformly distributed between 10 ft and 70 ft. Your dog is 40 ft away from you and is willing to move 20 ft to catch the ball.

(Describing the distribution)

(1) What is the expected distance you kick the ball? $\mathbb{E}[X] =$

(2) What is the standard deviation of the distance? $Var(X) =$

(3) What is the probability that your dog gets the ball? $\mathbb{P}[\ ....\ ]$

② Say, $X \sim Unif(0,1)$. Express $\mathbb{E}[X^k]$, for $k>-1$. (Why does it not work for $k \leq -1$?)

(Moments)

$\mathbb{E}[X^k] =$

(3) Derive the moment-generating function of X, if $X \sim \text{Unif}(a,b)$.

(MGF)

$M_X(\theta) =$

Indicator function

$I_{(a,b)}(x) = \begin{cases} 1, & x \in (a,b] \\ 0, & \text{else} \end{cases}$

<u>The exponential random variable</u>

$X \sim \exp(\lambda)$ <u>if</u> $f_X(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$

the average rate of occurence is λ, the Exp(λ) distribution represents the waiting time until the first event occurs when events occur at a rate of λ per unit time. Note that this is the same description that we gave the Geom(p) random variable, but now in continuous time. This is not a coincidence! The exponential random variable can be seen as a

$\mathbb{E}[X] = \frac{1}{\lambda}$ [if success occur at rate of λ per unit time, then time until first success $= \frac{1}{\lambda}$] expected

(4) The length of train rides is exponetially distributed w/ mean 5 hours.

(a) Find $\mathbb{P}$[train ride is longer than 10 hours]

(b) Find time that 90% of train rides are shorter than.

<u>The gamma r.v.</u>

It is motivated by the Gamma function,

$\boxed{\Gamma(\alpha) = \int^{\infty} x^{\alpha-1} e^{-x} dx}$ $(\alpha > 1)$

$$\boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx} \quad (\alpha > 1)$$

Cool properties

(i) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, when $\alpha > 1$

(5) Using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, derive,

(i) $\Gamma\left(\frac{7}{2}\right)$

(ii) $\Gamma\left(\frac{9}{2}\right)$

(6) Compute $\int_0^\infty x^3 e^{-x}\,dx$.

(7) Compute $\int_1^\infty \frac{(\ln(x))^{a-1}}{x^2}\,dx$, $a > 0$.