Image Registration

Why Perform Registration?

- Medical imaging modalities provide information about pathology and associated anatomy of the human body.
  - Differences in the spatial properties of anatomical structures between imaging studies can make it difficult for a clinician to mentally fuse all the image information accurately.

What is Registration?

- Registration is the process of bringing two or more images into spatial correlation.
  - It is sometimes known as matching.
- Physicians often wish to compare two images of the same anatomical region acquired under different circumstances.
  - For instance a pair of images might be of the same patient and the same modality, or different modalities. Also, images of two different patients might be compared.

The Process of Registration

- The process of registration can be separated into three basic elements:
  1. The spatial transformation defines the spatial relationship between the images.
  2. The registration basis characterizes the type of features used to establish the correspondence between the images.
  3. Optimization is used to calculate the optimal transformation parameters.

Categories of Registration

Intermodal Registration

- Intermodal (multimodal, crossmodal) registration involves matching images of the same patient acquired from different modalities.
  - This category can be further divided on a modality basis into anatomical-anatomical, functional-anatomical and functional-functional registration.
  - Multi-anatomical registration deals with the registration of imaging studies depicting different aspects of tissue morphology.

    e.g. CT-MR

Intermodal Registration

– The correlation of functional studies with anatomical studies allows for the anatomical localisation of functional parameters, as functional studies often lack morphological information.

  e.g. SPECT-MR

– Functional studies correlate differing types of functional information

  e.g. ^{201}TI or ^{99m}Tc-MIBI SPECT study depicting perfusion with a metabolic 18F-FDG PET study

Intramodal Registration

- Intramodal (monomodal, isomodal) registration involves matching images of one or more patients and the same modality acquired at different times.
  - This allows quantitative comparison for longitudinal monitoring of disease progression/recession and postoperative follow-up.
  - Intramodal registration is well suited to tasks relating to detection: monitoring growth, comparative studies using contrast agents, and subtractive imaging (DSA, MRA, CTA).

Intramodal Registration

- There are three interrelated subcategories of intramodal registration:
  - interstudy
  - intrastudy
  - intersubject

Intramodal Registration:

Inter/Intra-Subject Registration

- When all the images involved in registration are acquired from a single patient, this is known as intrasubject registration.
- If the registration is accomplished using images from different patients, this is referred to as intersubject registration. This implies matching studies taken from different patients but from the same modality.
  - *Spatial normalization*: which minimises shape variability

Intramodal Registration:
Interstudy Registration

- Matching images from the same patient and the same modality, but from different studies is known as interstudy registration and is synonymous with temporal or time-series registration.
  - Interstudy registration is performed in situations such as quantisation of bone growth (long time interval), monitoring tumor growth (medium to long interval), quantisation of cardiac motion (short interval), or observing the movement of a contrast agent through vasculature (ultra-short interval).

Intramodal Registration:
Intrastudy Registration

- Matching images from the same patient and from within the same study is called intrastudy registration.
  - For example when acquiring a contrast-study, two or more separate images are acquired in the same session. The first of these is known as the pre-contrast image and is acquired before the administration of a contrast agent (a pharmaceutical used to enhance particular types of tissue). After the contrast-agent has been administered one or more post-contrast images is acquired.

Template-Atlas Registration

- Template-atlas registration involves matching a study acquired from a patient with an image obtained from an image information database generated using imaging studies from a number of patients.
  - It involves the alignment of individual studies, from different subjects, with different morphology for the analysis of inter-individual variability of anatomical structures.

Intra-Procedural Registration

- Intra-procedural registration requires matching images to real-world operative space and may include matching a pre-procedural study (one taken prior to the procedure) with either an intra-procedural study (one taken during the procedure), or directly to the patient.
  - Prior to a surgical, or other interventional procedure, 3D anatomical (CT or MR) images are usually obtained.
  - During the procedure spatial information might be available from, perhaps, a surgical localiser, intra-operative video (e.g. endoscopy), intra-operative ultrasound, x-ray fluoroscopy, interventional CT or MR, x-ray, laser range scanner, or surgical microscope.

Applications for Registration in Medicine

- Applications can be broadly grouped into two interrelated categories; clinical (detection and diagnosis) and surgical. For example:
  - Radiation therapy
  - Interventional radiology
  - Diagnostic radiology
  - Image-augmented surgery
  - Preprocedural planning and simulation
  - Minimally invasive procedures

Spatial Transformation Models

- Spatial transformation models play a central role in any image registration process
  - The models impose mathematical constraints on the type of spatial transformation that can be imposed during the registration process

Spatial Transformations

- A spatial transformation in *d* dimensions, denoted T(x,y), is composed of *k* mapping functions *k*=1,…,d such that:

$$T(\vec{x}) = \left[\ f_1(\vec{x}), \ldots, f_k(\vec{x}), \ldots, f_d(\vec{x})\right]$$

where $f_1(\vec{x})$ represents the mapping function in the first dimension etc.

Registration Transformations

- Rigid or affine transformations are suitable for intrasubject registration tasks with little physical deformation or motion

  e.g. head, spine, pelvis

- Nonrigid or elastic transformation models are suitable for intersubject registration tasks

  e.g. anatomical variability across subjects or soft-tissues that deform over time

Registration Basis

- A registration basis describes the types of features used during the registration process
  - point-based features
  - contour or surface-based features
  - pixel-based features (intensities, edges, texture)

Homogeneous Coordinates

NOTE:

- Homogeneous coordinates allow two-dimensional points (x,y) to be represented using a three-dimensional vector, with the third dimension equal to 1.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rigid Model

- The rigid model is the most constrained spatial transformation model.
  - A spatial transformation is considered rigid if the spatial distance between any consecutive points is preserved.
  - A rigid transformation can be decomposed into a translation and/or a rotation.

Rigid Model

- A translation is a constant displacement over space. $x' = x + t_x$ $\quad\quad$ $y' = y + t_y$

  where t_x and t_y are the translation parameters in the x and y axes

- A rotation is

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

  where θ defines the clockwise rotation

Rigid Model

- These elementary transformations can be written in homogeneous coordinate matrix formulation as follows:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rigid Model

- A rigid transformation involving a rotation followed by a translation can be defined as:

$$T_{rigid} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & t_x \\ -\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  – The coefficients are found by solving the matrix using simple matrix inversion or Singular Value Decomposition (SVD)

Rigid Model

- Applications of the rigid model:
  - Registration using a rigid model is appropriate only when the anatomical structure is itself rigid, or which for diagnostic and therapeutic purposes may be considered rigid

    e.g. pelvis, femur, skull, and other bones.
  - Normally justified for images of the head as the skull is rigid and constrains the motion of the brain.

Affine Transformation

- An affine transformation can be decomposed into a linear transformation and a translation.
  - An affine model maintains spatial relationships between points and asserts that lines that are parallel before transformation remain parallel after transformation .
- The general affine model has six independent parameters in 2D:

$$x' = a_{11}x + a_{12}y + t_x$$
$$y' = a_{21}x + a_{22}y + t_y$$

Affine Transformation

- Used to correct for scaling and shearing differences

  e.g. shear caused by the gantry-tilt of CT scanner

$$T_{affine} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Perspective Transformation

- The perspective model has eight independent parameters in 2D.
  - This model asserts that lines that are parallel before transformation can intersect after transformation.
  - In 2D there are two parameters that control perspective t_x and t_y:

$$T_{perspective} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ p_1 & p_2 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rigid vs. Nonrigid Models

- In a spatial transformation based on a rigid model, structures retain their shape and form during matching.
- A nonrigid or deformable model is one in which a structure may not necessarily retain its shape or form during transformation.

Polynomial Models

- A bivariate polynomial is a global spatial transformation model.
  - It can be defined as follows:

$$x' = \sum_{i=0}^{m}\sum_{j=0}^{i}\alpha_{ij}x^{j}y^{i-j} \qquad y' = \sum_{i=0}^{m}\sum_{j=0}^{i}\beta_{ij}x^{j}y^{i-j}$$

where α_{ij} and β_{ij} are the constant polynomial coefficients. The order of the polynomial is represented by m.

Polynomial Models

- For example: A polynomial of degree 2 (quadratic), (m=2):

$$x' = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2$$

$$y' = b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2$$

where for convenience sake, the coefficients α_{ij} and β_{ij} have been replaced respectively by a_k and b_k for $k = 0,\ldots,l-1$ with

$$l = \sum_{i=0}^{m} \sum_{j=0}^{i} 1$$

Polynomial Models

- The coefficients of the polynomial are calculated using least squares approximation:

$$\begin{bmatrix} n & \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} y_i & \sum_{i=0}^{n} x_i y_i & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} y_i^2 \\ \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i y_i & \sum_{i=0}^{n} x_i^2 y_i & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i y_i^2 \\ \sum_{i=0}^{n} y_i & \sum_{i=0}^{n} x_i y_i & \sum_{i=0}^{n} y_i^2 & \sum_{i=0}^{n} x_i y_i^2 & \sum_{i=0}^{n} x_i^2 y_i & \sum_{i=0}^{n} y_i^3 \\ \sum_{i=0}^{n} x_i y_i & \sum_{i=0}^{n} x_i^2 y_i & \sum_{i=0}^{n} x_i y_i^2 & \sum_{i=0}^{n} x_i^2 y_i^2 & \sum_{i=0}^{n} x_i^3 y_i & \sum_{i=0}^{n} x_i y_i^3 \\ \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^2 y_i & \sum_{i=0}^{n} x_i^3 y_i & \sum_{i=0}^{n} x_i^4 & \sum_{i=0}^{n} x_i^2 y_i^2 \\ \sum_{i=0}^{n} y_i^2 & \sum_{i=0}^{n} x_i y_i^2 & \sum_{i=0}^{n} y_i^3 & \sum_{i=0}^{n} x_i y_i^3 & \sum_{i=0}^{n} x_i^2 y_i^2 & \sum_{i=0}^{n} y_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} u_i \\ \sum_{i=0}^{n} x_i u_i \\ \sum_{i=0}^{n} y_i u_i \\ \sum_{i=0}^{n} x_i y_i u_i \\ \sum_{i=0}^{n} x_i^2 u_i \\ \sum_{i=0}^{n} y_i^2 u_i \end{bmatrix}$$

Spatial Transformations: Local vs. Global

- Spatial transformations can be classified on the basis of their *scope*, or domain.
- The *scope* refers to its region of influence and can be either local or global.
  - A spatial transformation is deemed to be global if $f_k(\vec{x})$ is dependent on all the control-points, regardless of their distance from $\vec{x}$.
  - If $f_k(\vec{x})$ does not depend on control-points more than a certain distance from $\vec{x}$, the spatial transformation is considered to be local. Only control points sufficiently close, or perhaps weighted by their proximity, influence the mapping function.

Point Matching

- In point-matching a set of n control-point pairs $(\vec{p}_i, \vec{q}_i)$ are selected from the two images.
  - For instance, if $\vec{p}_1$ is the coordinate of a feature in the first image, then $\vec{q}_1$ is the corresponding point of the same feature in the second image.
  - A control-point is a unique positional descriptor derived from an anatomical feature-point.
  - A spatial transformation mathematically describes the spatial relationship between corresponding control-points.

Radial Basis Functions

- A Radial Basis Function (RBF) is a scattered data interpolation method where the spatial transformation is a linear combination of radially symmetric basis functions, each centred on a particular control-point.
  - The choice of a radial function reflects the fact that the scattered data has no preferred orientation.
  - RBFs provide smooth deformations with easily controllable behavior.

Radial Basis Functions

- An RBF in two-dimensions is composed of two mapping functions each of which is decomposed into a global component and a local component.
  - Although the two components are distinct they are evaluated almost simultaneously, giving rise to a single transformation.

Radial Basis Functions

- Given n corresponding control-points, each of the (k=1,2) mapping functions of the RBF has the following general form:

$$f_k(\vec{x}) = P_{mk}(\vec{x}) + \sum_{i=1}^{n} A_{ik} g(r_i)$$

- The first component is a polynomial of degree m, or is not present. This global linear transformation assures a certain degree m of polynomial precision (accounting for global affine differences). In general, a linear polynomial (m=1), ie. $P_{1k}(x, y) = a_{0k} + a_{1k}x + a_{2k}y$ is used → affine transformation

Radial Basis Functions

- The latter component, is the sum of a weighted elastic or nonlinear basis function $g(r_i)$, where r_i denotes the Euclidean norm such that:

$$r_i = [\vec{x} - \vec{x}_i]^{\frac{1}{2}} \qquad \text{or} \qquad r_i = \| \ \vec{x} - \vec{x}_i \ \|$$

where $\vec{x} = (x, y)$

- Thus the mapping function $f_k(\vec{x})$ is a linear combination of a radially symmetric function $g(r_i)$ and a low degree polynomial.

Radial Basis Functions

- The RBF transformation is determined by coefficients in each dimension:
  - The coefficients of the function $f_k(\vec{x})$ are determined by requiring that $f_k(\vec{x})$ satisfy the interpolation conditions:

    $$f_1(\vec{x}_j) = u_j \ , \ f_2(\vec{x}_j) = v_j \quad \text{for} \quad j = 1, \ldots, n$$

    giving n linear equations together with the additional compatibility constraints:

    $$\sum_{i=1}^{n} A_{ik} = \sum_{i=1}^{n} A_{ik} x_i = \sum_{i=1}^{n} A_{ik} y_i = 0$$

Radial Basis Functions

- The coefficients for both mapping functions are calculated by placing the *n* corresponding control-point pairs into the RBF and solving the linear system:

$$\mathrm{W} = \mathrm{L}^{-1}\mathrm{Y}$$

as given by

$$\mathrm{L} = \begin{bmatrix} \mathrm{G} & \mathrm{P} \\ \mathrm{P}^{\mathrm{T}} & 0 \end{bmatrix}$$

$$\mathrm{G} = \begin{bmatrix} g(r_{11}) & g(r_{12}) & \cdots & g(r_{1n}) \\ g(r_{21}) & g(r_{22}) & \cdots & g(r_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ g(r_{n1}) & g(r_{n2}) & \cdots & g(r_{nn}) \end{bmatrix} \qquad \mathrm{P}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

Radial Basis Functions

$$\mathrm{W}^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} & a_{01} & a_{11} & a_{21} \\ A_{12} & A_{22} & \cdots & A_{n2} & a_{02} & a_{12} & a_{22} \end{bmatrix}$$

$$\mathrm{Y}^{\mathrm{T}} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n & 0 & 0 & 0 \\ v_1 & v_2 & \cdots & v_n & 0 & 0 & 0 \end{bmatrix}$$

where where ^{T} is the matrix transpose operator, **0** is a 3×3 matrix of zeros, and $r_{ij} = [\vec{x}_i - \vec{x}_j]^{\frac{1}{2}}$

- The linear system is solved using Singular Value Decomposition (SVD) to determine the coefficients.

Basis Functions

- The basis function for the MQ determines the effect of neighboring control-points on the RBF.
- The range of influence of the basis function can be controlled by adjusting the parameters of the RBF.
- Some RBFs have a global behavior (e.g. Thin-Plate Spline) whilst others have a localised influence (e.g. Gaussian).

Basis Functions

Basis Function	$g(r_i)$	*Parameters*	*Scope*
Thin-Plate Spline	$r_i^2 \log r_i^2$	-	global
Multiquadric	$(r_i^2 + \delta)^{\pm\mu}$	$\delta > 0,\ \mu \neq 0$	local
Gaussian	$e^{(-r_i^2/\sigma)}$	$\sigma > 0$	local
Shifted-LOG	$\log(r_i^2 + \delta)^{\frac{3}{2}}$	$\delta \geq 1$	local
Cubic Spline	$\Vert r_i \Vert^3$	-	global

Thin-Plate Spline

- The Thin-Plate Spline (TPS), introduced by Harder and Desmarais, is so-called because it models the shape of a thin steel plate deformed by point loads.

$$g(r_i) = r_i^2 \log r_i^2$$

- Note that when specifying the component matrix **G** in the 2D linear system:

$$g(r_{ij}) = \begin{cases} r_{ij}^2 \log r_{ij}^2 & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j \end{cases}$$

Multiquadric

- The Multiquadric (MQ) RBF, was introduced in 1971 by Hardy.

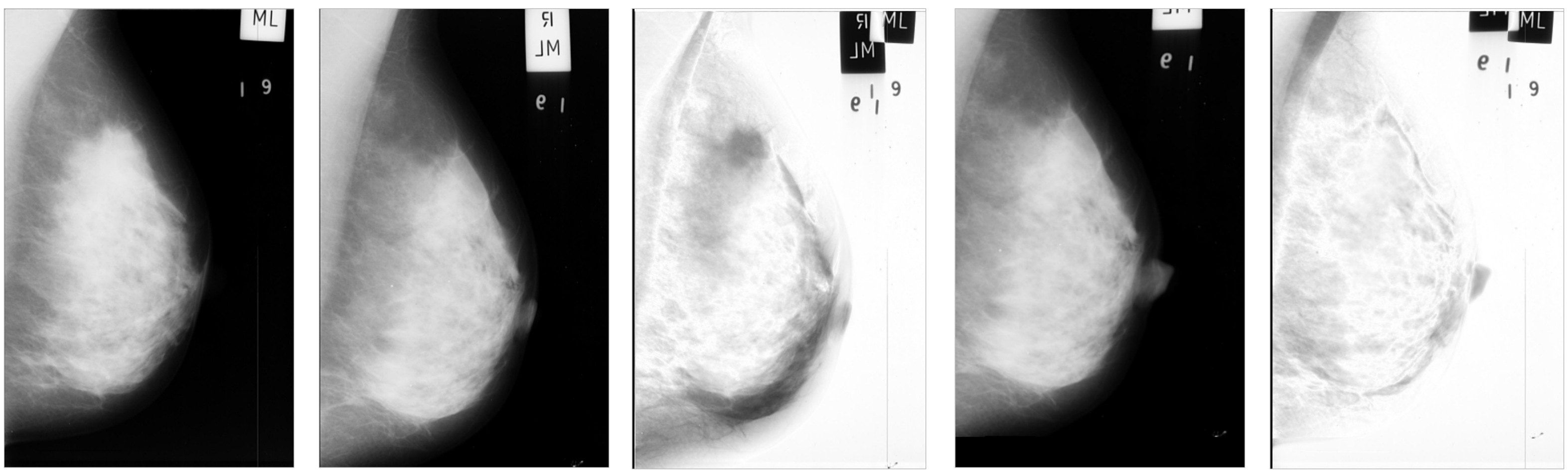
$$g(r_i) = (r_i^2 + \delta)^{\pm\mu}$$

with $\delta > 0$, $\mu \neq 0$.

- When the exponent μ is negative, the RBF is known as an Inverse Multiquadric.

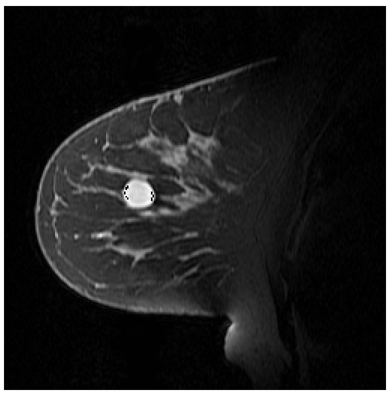
Mammogram Registration

Reference Comparative Pre-Registration Transformed Post-
Registration
Image Image Difference Image Image Difference
Image

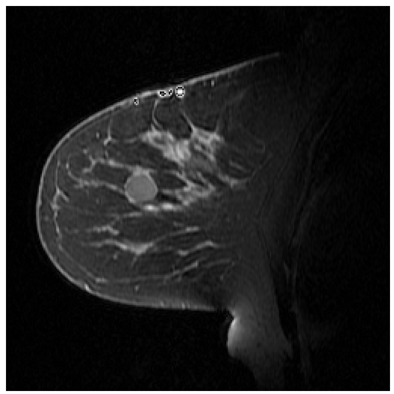


Breast MR Registration

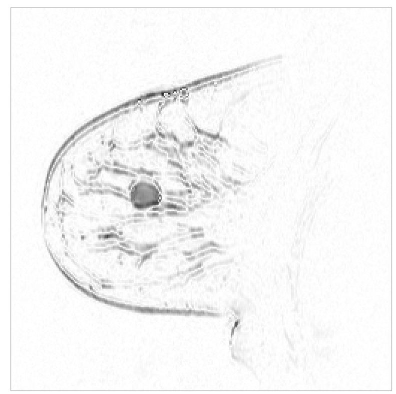
Reference Image



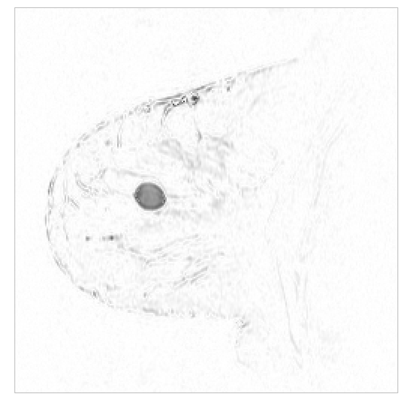
Comparative Image



Pre-Registration Difference Image



Post-Registration Difference Image



Locality Parameters

- Localised RBFs usually incorporate a locality parameter which is used to control the behavior of the basis function, and how it is influenced by neighboring control-points.
  - The locality parameter gives less weight to distant control-points and more weight to neighboring control-points.

Locality Parameters

e.g. For a Multiquadric (μ=0.5):

– As the locality parameter δ approaches 0, the basis function $g(r_i)$ approaches the linear RBF. As the parameter δ grows, the region of influence at each control-point decreases and the effect of the basis function becomes more localised.

RBF's with Compact Support

- Basis functions with compact support (locally bounded)
  - One plausible extension to RBFs is to limit the range of influence of a basis function so that evaluation stops at some distance from the control-points.

    e.g. A Multiquadric where ***R*** is the limit of the range of influence.

$$g(r_i) = \begin{cases} (r_i^2 + \delta)^{\pm\mu} & \text{if} \quad 0 \le r_i \le R \\ 0 & \text{if} \quad r_i > R \end{cases}$$

RBF Approximation

- To take into account control-point localisation errors, the interpolation condition of the RBF can be weakened.
  - This can be achieved by combining an approximation parameter, λ, with the RBF mapping functions.
  - If λ is small, a solution with good approximating behavior is obtained, in that as λ approaches 0 ($\lambda \to 0$) we have an interpolating transformation.

RBF Approximation

– If λ is large, a smooth transformation is obtained with little adaptation to local deformations. In the limit if $\lambda \to \infty$ we get a polynomial of order up to *m*-1 which has no smoothness criteria.

- This is achieved by adding λ to the diagonal of the matrix G:

$$\mathrm{G} + \lambda \mathrm{I}$$

$$\mathbf{G} = \begin{bmatrix} g(r_{11}) + \lambda & g(r_{12}) & \cdots & g(r_{1n}) \\ g(r_{21}) & g(r_{22}) + \lambda & \cdots & g(r_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ g(r_{n1}) & g(r_{n2}) & \cdots & g(r_{nn}) + \lambda \end{bmatrix}$$

Similarity Measures

- Similarity-measures are a form of registration based on the intensity properties of the images.
  - Similarity measure use cost-functions which quantify the degree of similarity between two images.
  - Registration models based on these cost-functions simply adjust the parameters of an appropriate spatial-transformation model until the cost-function reaches a local optimum.

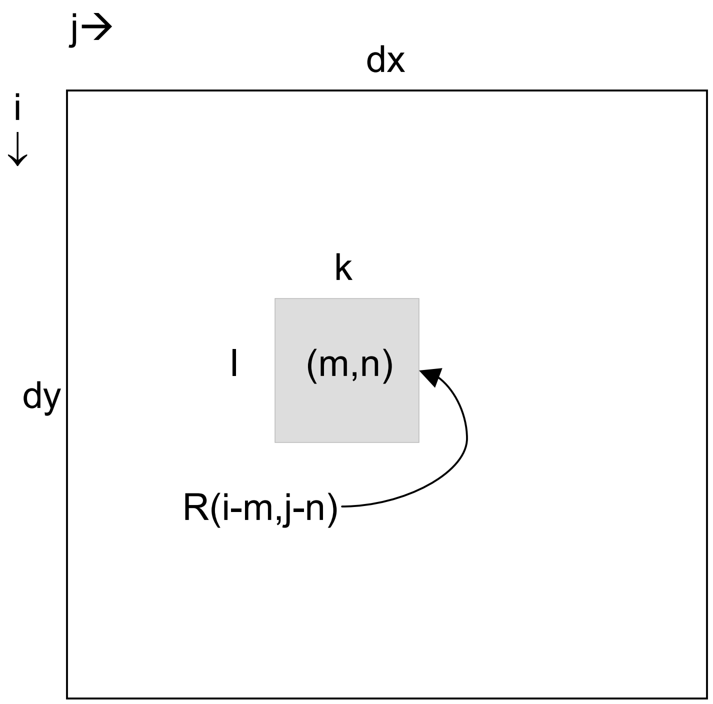
Similarity Measures: Search Strategies

Algorithm: Similarity measure search strategies

1. Define the reference and search windows
2. Define a similarity measure cost-function
3. For all the positions of the reference window in the search window, calculate the similarity measure
4. Select the optimal location

Similarity Measures: Search Strategies

- R(x,y) → reference window, C(x,y) → search window
  - k/2<m<i-k/2
  - l/2<n<j-l/2



Cross-Correlation

- Cross-correlation was one of the first cost-functions used for image registration.
  - This cost-function multiplies the intensities of the two images at each pixel and sums the resulting product.
  - The product is then normalized by dividing it by the product of the root-mean squared intensities of each of the two images.
  - Used for monomodal, temporal registration

Cross-Correlation

$$C(m,n) = \sum_{i}^{dx} \sum_{j}^{dy} C(i,j) \bullet R(i-m, j-n)$$

$$CN(m,n) = \frac{\sum_{i}^{dx} \sum_{j}^{dy} C(i,j) \bullet R(i-m, j-n)}{\left[\sum_{i}^{dx} \sum_{j}^{dy} C(i,j)^2 \right]^{0.5} \left[\sum_{i}^{dx} \sum_{j}^{dy} R(i-m, j-n)^2 \right]^{0.5}}$$

Correlation Coefficient

- The correlation coefficient (CC) is an extension to correlation which assumes that the intensities in the two images are linearly related.
  - Maximized, CC=1 if correctly matched

$$CC(m,n)=\frac{1}{N}\frac{\sum_{i}^{dx}\sum_{j}^{dy}\left[C(i,j)-\bar{C}\right]\left[R(i-m,j-n)-\bar{R}\right]}{\left\{\sum_{i}^{dx}\sum_{j}^{dy}\left[C(i,j)^2-\bar{C}\right]^2\bullet\sum_{i}^{dx}\sum_{j}^{dy}\left[R(i-m,j-n)^2-\bar{R}\right]\right\}^{0.5}}$$

SSD

- The sum of squared differences (SSD)
  - Minimized, SSD=0 if correctly matched
  - Used for monomodal, temporal registration

$$SSD(m,n) = \sum_{i}^{dx} \sum_{j}^{dy} \left[C(i,j) - R(i-m, j-n) \right]^2$$

SAD

- The sum of absolute differences (SAD)
  - Minimized, SAD=0 if correctly matched
  - Used for monomodal, temporal registration

$$SAD(m,n) = \sum_{i}^{dx} \sum_{j}^{dy} \left| C(i,j) - R(i-m, j-n) \right|$$

  - Normalized SAD

$$SADN(m,n) = \sum_{i}^{dx} \sum_{j}^{dy} \left\| \left[C(i,j) - \bar{C} \right] - \left[R(i-m, j-n) - \bar{R} \right] \right\|$$

Other Similarity Measures

- There are various other similarity measures:
  - Variance of Intensity Ratios (VIR)
  - Variance of Differences (VOD)
    - The standard deviation of differences
  - Stochastic Sign Crossing (SSC)
    - Deterministic Sign Change (DSC)
  - Histogram of Differences based measures
    - Entropy of the Histogram of Differences (ENT)
    - Energy of the Histogram of Differences (EHD)

Joint Histograms

- For two images A and B, the joint histogram, $h_{AB,}$ is two-dimensional and is constructed by plotting the intensity *a* of each pixel in image A against the intensity *b* of the corresponding pixel in image B.
  - The value of each histogram position h(a,b) will therefore correspond to the number of image pixels with intensity a in image A and intensity b in image B.

Measuring Information

- Think of the registration process as trying to align the shared information between two images.
  - Uses a measure of information as the registration metric.
  - The most common measure of information: Shannon-Weiner entropy measure H:

$$H = -\sum_i p_i \log p_i$$

Measuring Information

- H is the average information supplied by a set of i symbols whose probabilities are given by $p_1, p_2, \ldots, p_i$
- Entropy has a maximum value if all symbols have equal probability

PDF

- The *joint histogram* can be normalised by dividing by the total number of pixels (N) and regarded as a joint probability distribution function or PDF ρ_{AB} of images A and B.

$$\rho_{AB} = \frac{h_{AB}}{N}$$

PDF

Algorithm: Calculating the joint PDF

1. Define an N_a by N_b array h[j,k]
2. Initialize the histogram h[j,k]=0 for all j,k
3. For each pixel $i \in A \cap B$, calculate the intensity values corresponding to a=A[i] and b=B[i] respectively and increment h[a,b]
4. Calculate $\sum_{j,k} \mathrm{h}[j,k]$
5. Normalize the histogram to calculate the PDF:

$$\rho[j,k] = \frac{\mathrm{h}[j,k]}{\sum_{j,k} \mathrm{h}[j,k]}$$

Joint Entropy

- The joint entropy H(A,B) is given by:

$$H(A,B) = -\sum_{a \in A} \sum_{b \in B} \rho_{AB}(a,b) \log \rho_{AB}(a,b)$$

  where a and b represent either image intensities or selected intensity bins.
  - A 12-bit image with 4096 intensity values will result in a very sparse PDF. Use 64 or 256 intensity bins.

Joint Entropy

- Minimizing Joint Entropy:
  - Find a spatial transformation that will produce a small number of PDF elements with very high probabilities and provide as many zero-probability elements as possible.

Mutual Information

- Mutual information (MI) can be thought of as a measure of how well one image explains the other.
  - MI is maximized at the optimal spatial correlation
  - MI normalizes the joint entropy with respect to the partial entropies of the images using the marginal probability distributions: ρ_A and ρ_B

$$H(A) = -\sum_{a \in A} \rho_A(a) \log \rho_A(a) \quad H(B) = -\sum_{b \in B} \rho_B(b) \log \rho_B(b)$$

Mutual Information

$$MI(A,B) = H(A) + H(B) - H(A,B)$$

$$= \sum_{a \in A} \sum_{b \in B} \rho_{AB}(a,b) \log \frac{\rho_{AB}(a,b)}{\rho_A(a) \cdot \rho_B(b)}$$

Mutual Information

Algorithm: Maximizing mutual information

1. Calculate the PDF ρ_{AB} from the images, A and B
2. Calculate the joint entropy, H(A,B)
3. Calculate the partial entropies H(A), and H(B)
4. Calculate the mutual information:

$$MI(A,B) = H(A) + H(B) - H(A,B)$$

Normalized Mutual Information

- Extensions to MI:

$$NMI(A,B) = \frac{2 \cdot MI(A,B)}{H(A) + H(B)}$$

$$NMI(A,B) = H(A,B) - MI(A,B)$$

$$NMI(A,B) = \frac{H(A) + H(B)}{H(A,B)}$$

Nonrigid Similarity Measures

Algorithm: Nonrigid similarity measures

1. Divide an image into a regular grid of N nodes.
2. Calculate a rigid or affine spatial transformation for a sub-image around each node.
3. Apply a similarity-measure to quantify the degree of similarity.
4. Treating each node as a spline knot, perform a nonrigid spatial transformation using a spline transformation.

    e.g. radial basis function, B-spline

Optimization

- Optimization is an iterative process which is used to calculate the best registration estimate.
- Methods for optimization include:
  - downhill descent, downhill simplex, Powell’s method

Transformation Algorithms

- How is a spatial transformation applied to an image?
  - Image intensities are defined only at integral values. However a spatial transformation may produce fractional (nonintegral) values for the transformed coordinate pair.
  - There are two basic approaches to resampling:
    - forward and backward mapping

Forward Mapping

- Forward mapping:
  - If a pixel from the input maps to a position in the output image between four pixels, then its intensity value is divided amongst the four pixels according to some interpolation criteria.
  - This approach does not guarantee all pixels will be mapped.

Backward Mapping

- Backward mapping:
  - Here the pixels in the output image are mapped back into the input image to establish their intensity values. If an output pixel falls between four input pixels, its intensity value is determined by an interpolation algorithm.
  - This approach involves defining the inverse spatial transformation.

Interpolation Schemes

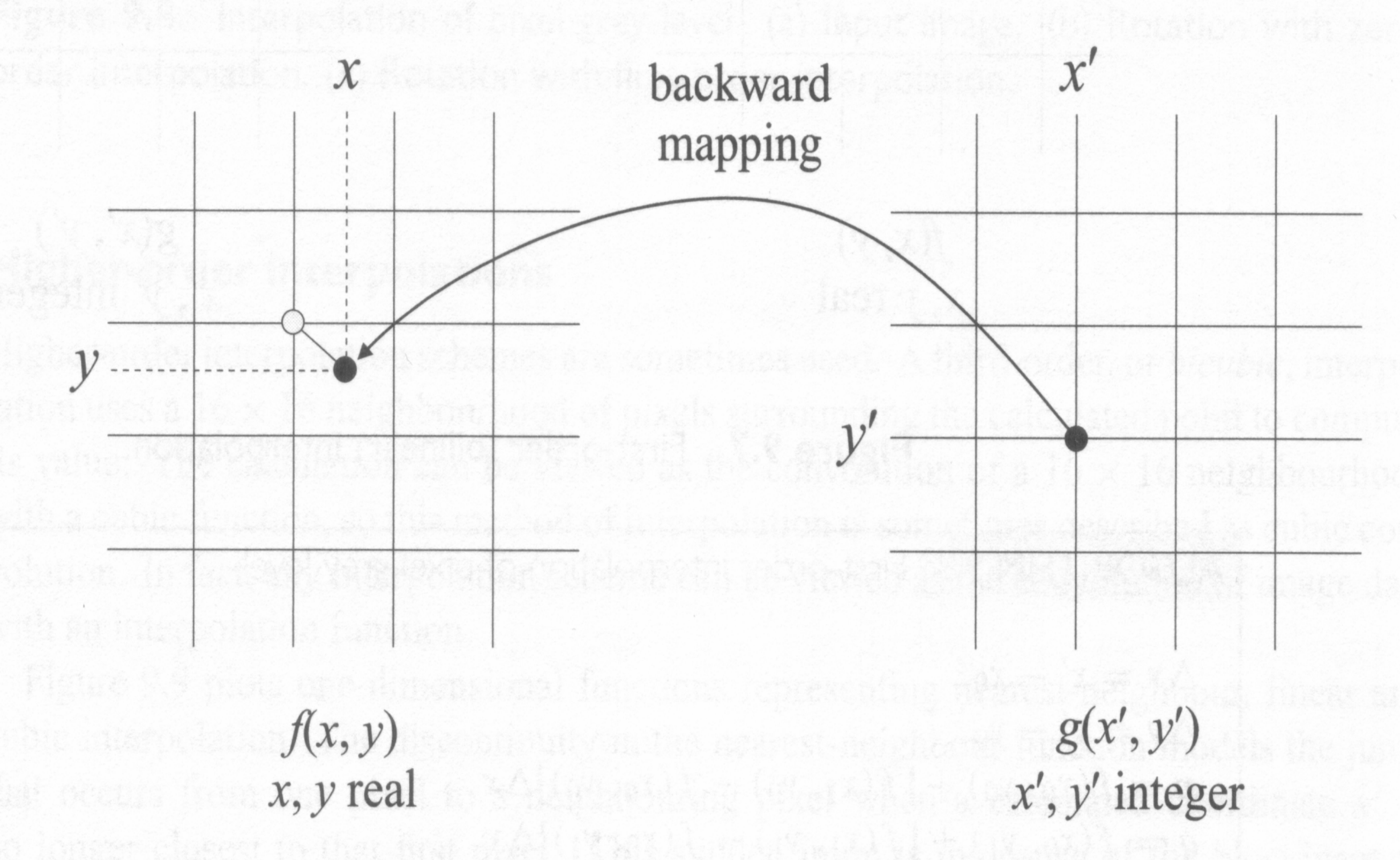
- Spatial transformations rely for their proper implementation on interpolation and image resampling.
  - Resampling is a process whereby intensity values are assigned to the pixels in the transformed image.
  - Interpolation is the process of intensity-value transformation.

Zero-order Interpolation

- The rounding of calculated coordinates (x`,y`) to their nearest integers is a process known as zero-order (or *nearest-neighbor*) interpolation.
  - Zero-order interpolation is computationally simple, but can degrade the appearance of the transformed image.

$$T'(x',y') = f(Round(x), Round(y))$$

Zero-order Interpolation



First-order Interpolation

- First-order (or *bilinear*) interpolation calculates the output intensity value as a distance-weighted function of the intensity values of the four pixels surrounding the calculated point in the input image.

First-order Interpolation

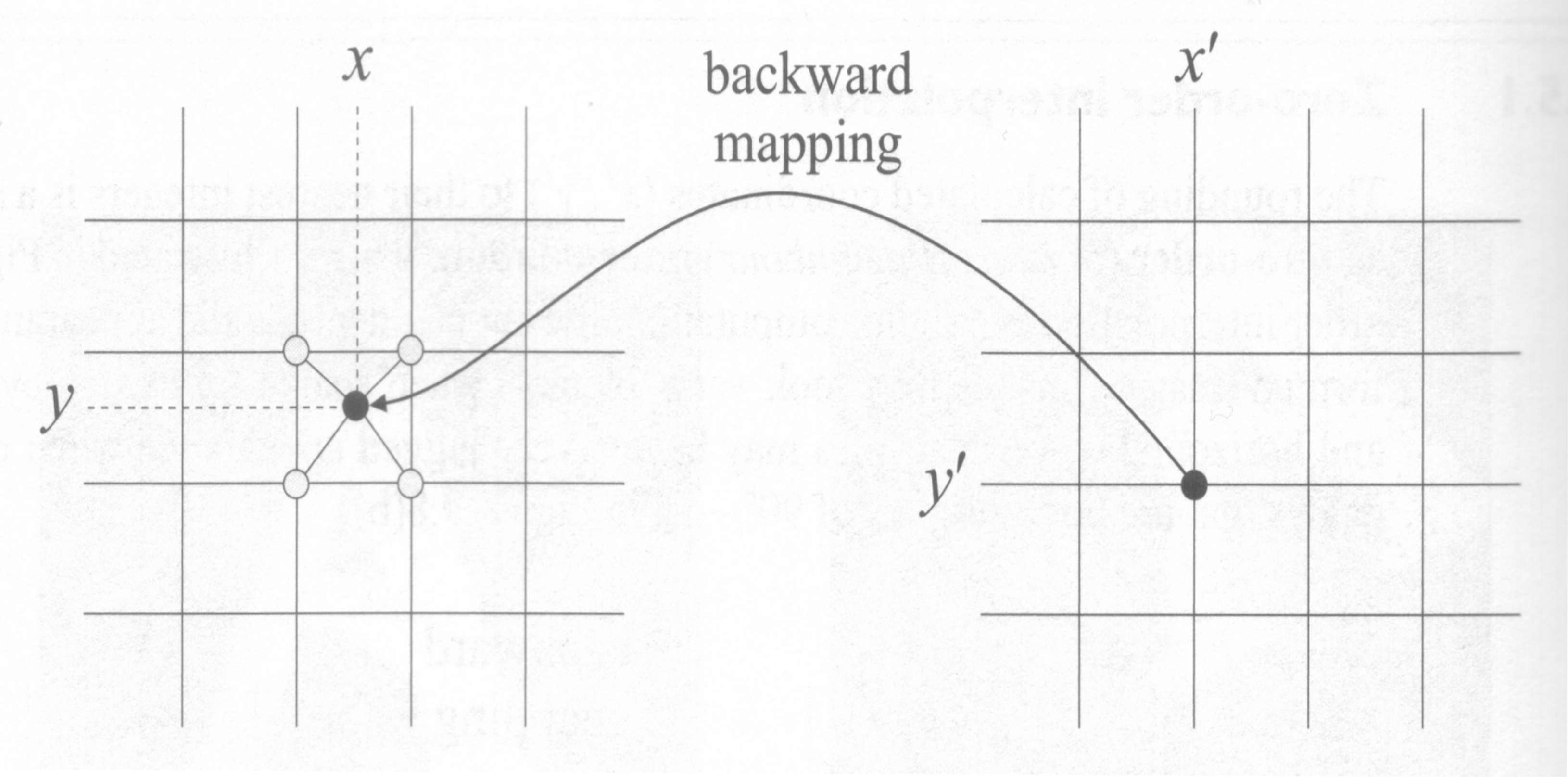
- For a calculated point (x', y') surrounded by pixels with coordinates (x_0, y_0), (x_1, y_0), (x_0, y_1), (x_1, y_1), the first order interpolation function is:

$$T'(x', y') = (1 - r_x)(1 - r_y) f(x_0, y_0) + (1 - r_x) r_y f(x_0, y_1)$$
$$+ r_x (1 - r_y) f(x_1, y_0) + r_x r_y (x_1, y_1)$$

where r_x and r_y represent the non-integer components:

$$r_x = x' - \lfloor x' \rfloor \qquad r_y = y' - \lfloor y' \rfloor$$

First-order Interpolation



Higher-order Interpolation

- A third-order, or *bicubic* interpolation uses a 16$\times$16 neighborhood of pixels surrounding the calculated point to compute its value.

Other Registration Models

- Optical Flow registration
- Surface-based registration
  - This involves the determination of corresponding surfaces in different images and the estimation of a spatial transformation between these structures. There are three popular approaches to surface matching: the Head-hat algorithm, the Hierarchical Chamfer Matching (HCM) algorithm and the Iterated Closest Points (ICP) algorithm.

Other Registration Models

- Principal-axes and Moments matching
  - Registration is performed by overlaying the centroids of the objects, and aligning the principal axes