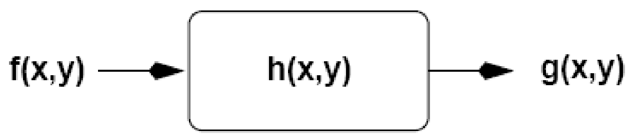
Frequency Domain Filters

Frequency Domain Methods

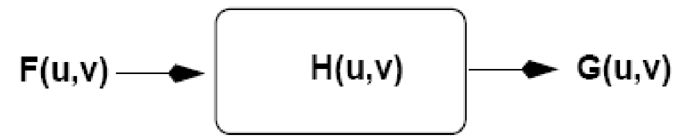
Spatial Domain



$$g(x,y) = f(x,y) * h(x,y)$$

h(x,y) : impulse response

Frequency Domain

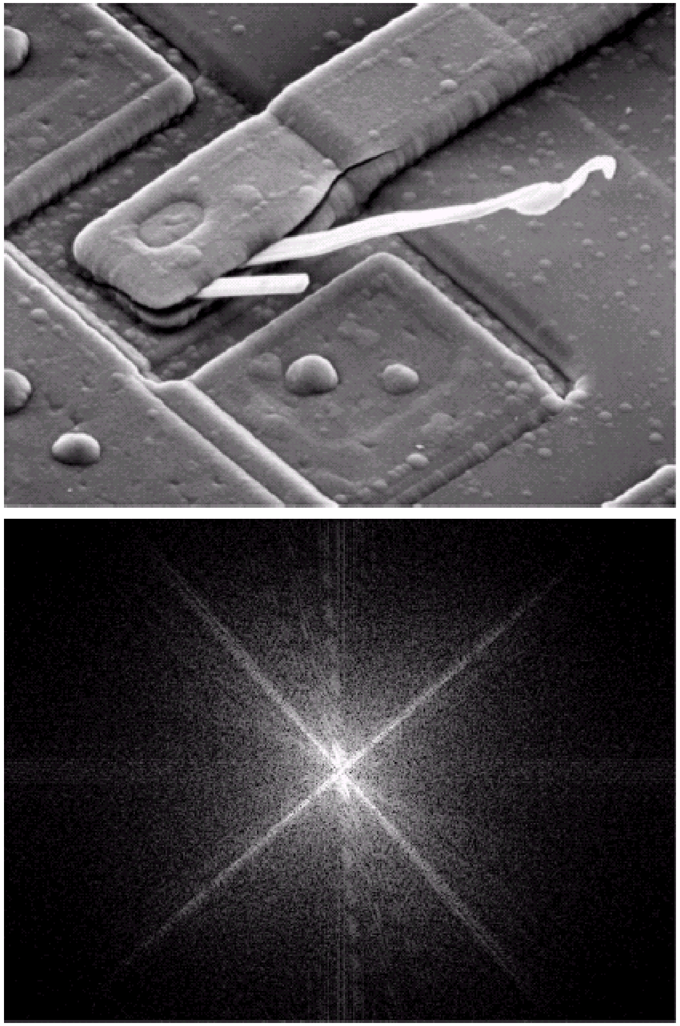


$$G(u,v) = F(u,v)\,H(u,v)$$

$$\left(g(x,y) = \mathbf{F}^{-1}\left(F(u,v)\,H(u,v)\right) \right)$$

H(u,v): transfer function

Filtering in the Frequency Domain



a
b

FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

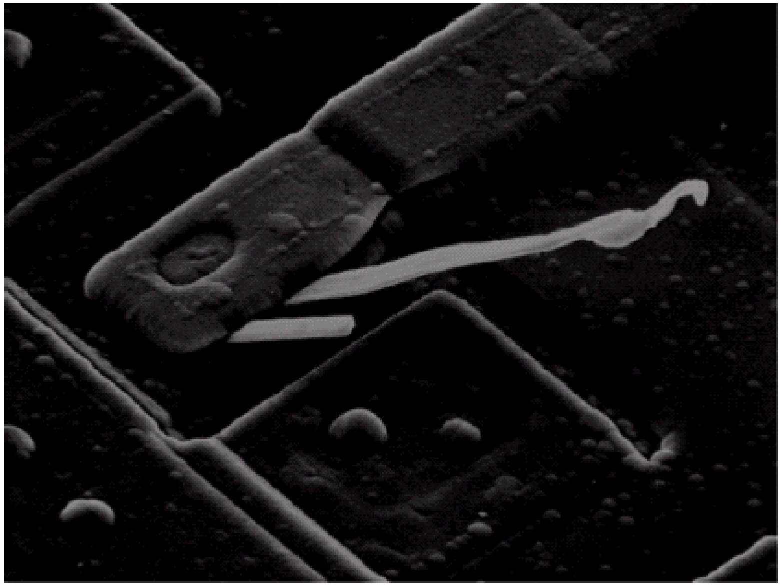
Some Basic Filters and Their Functions

- Multiply all values of $F(u,v)$ by the filter function (notch filter):

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise.} \end{cases}$$

  – All this filter would do is set $F(0,0)$ to zero (force the average value of an image to zero) and leave all frequency components of the Fourier transform untouched.

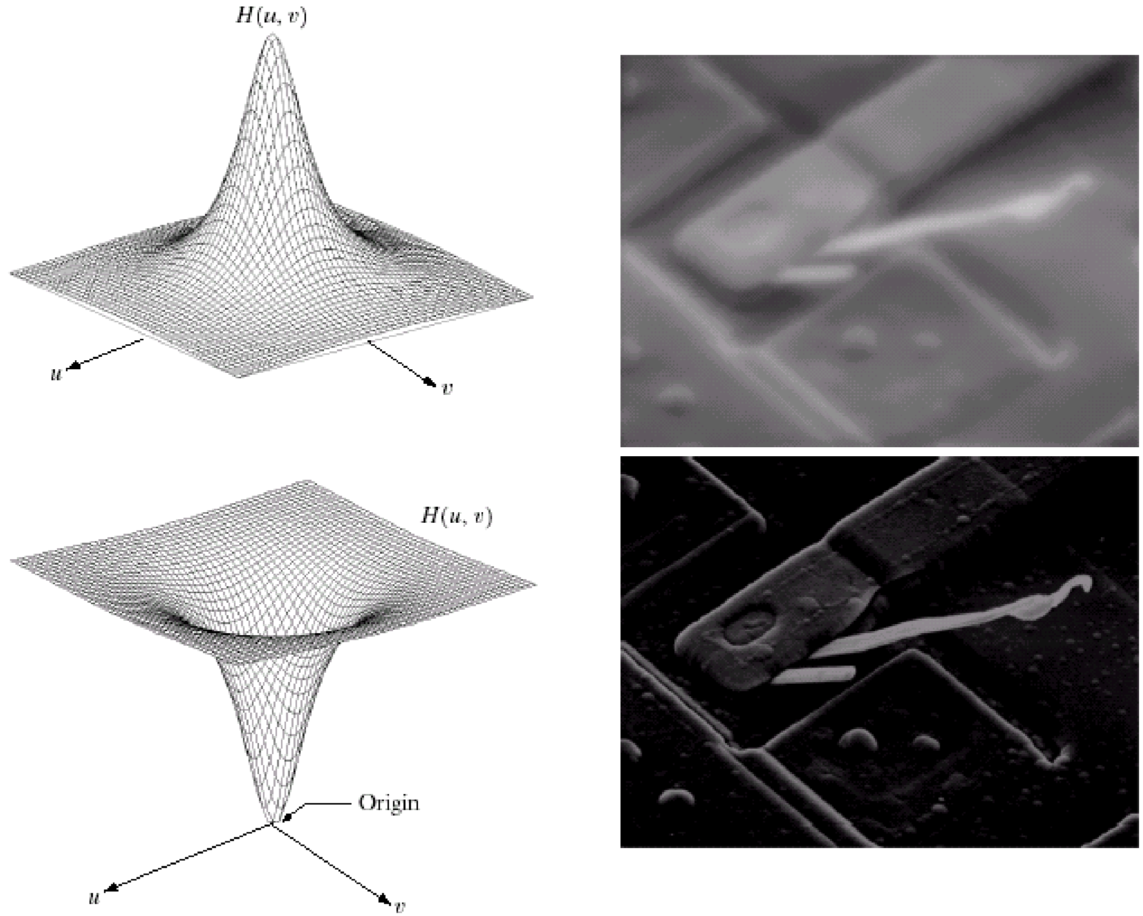
FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0,0)$ term in the Fourier transform.



Some Basic Filters and Their Functions

Lowpass filter

Highpass filter



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Correspondence between Filtering in the Spatial and Frequency Domain

- Convolution theorem:
  - The discrete convolution of two functions *f*(*x*,*y*) and *h*(*x*,*y*) of size *M*X*N* is defined as

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

  - Let *F*(*u*,*v*) and *H*(*u*,*v*) denote the Fourier transforms of *f*(*x*,*y*) and *h*(*x*,*y*), then

$$f(x,y)*h(x,y)\Leftrightarrow F(u,v)H(u,v) \quad \text{Eq. (4.2-31)}$$

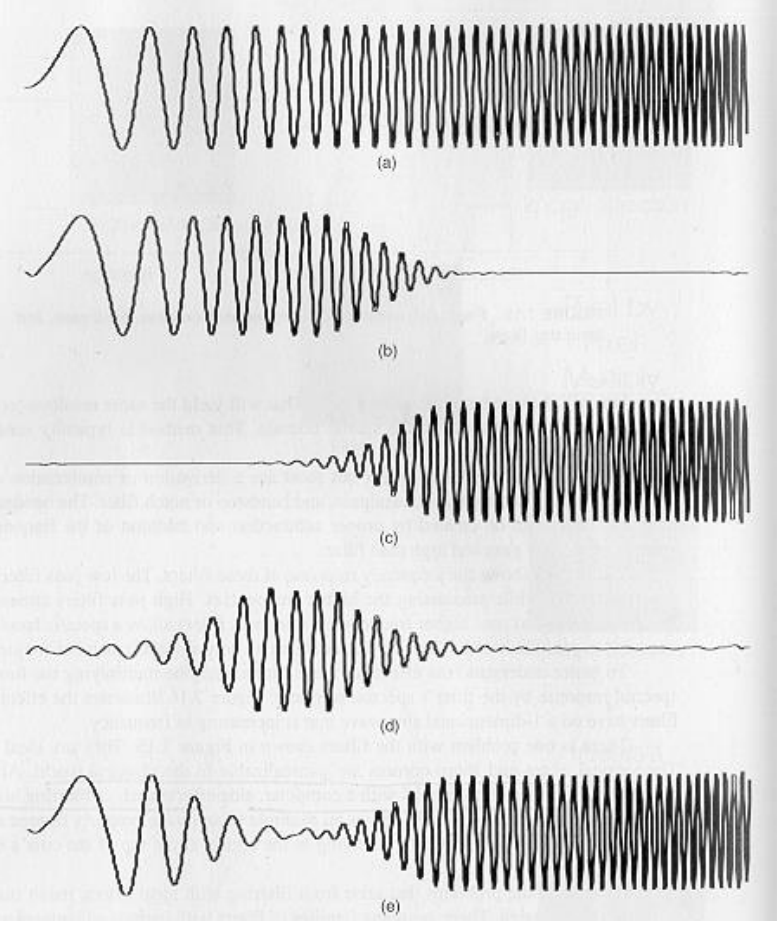
$$f(x,y)h(x,y)\Leftrightarrow F(u,v)*H(u,v) \quad \text{Eq. (4.2-32)}$$

Major filter categories

- Typically, filters are classified by examining their properties in the frequency domain:

  (1) Low-pass
  (2) High-pass
  (3) Band-pass
  (4) Band-stop

Example



Original signal

Low-pass filtered

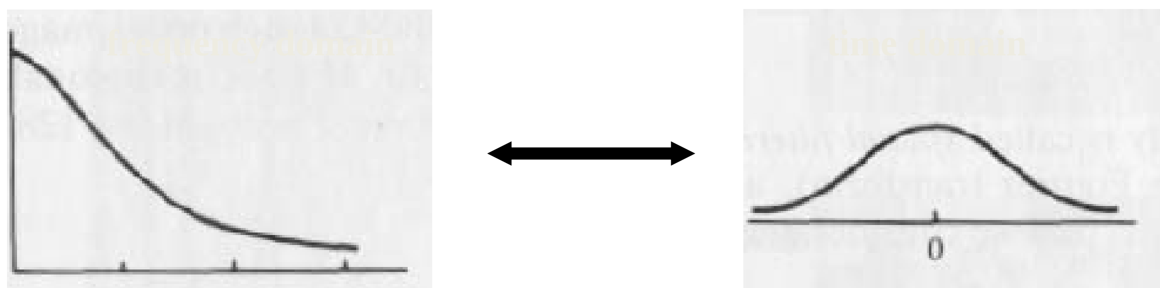
High-pass filtered

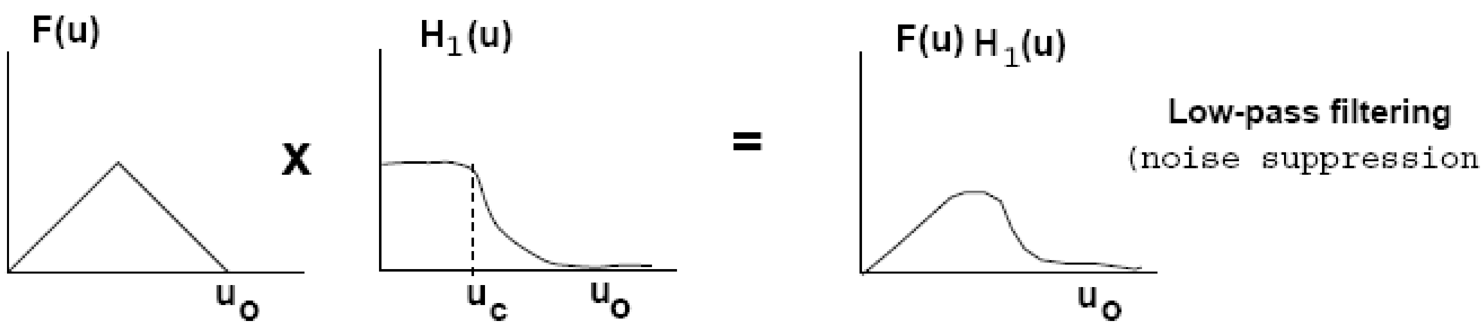
Band-pass filtered

Band-stop filtered

Low-pass filters
(i.e., smoothing filters)

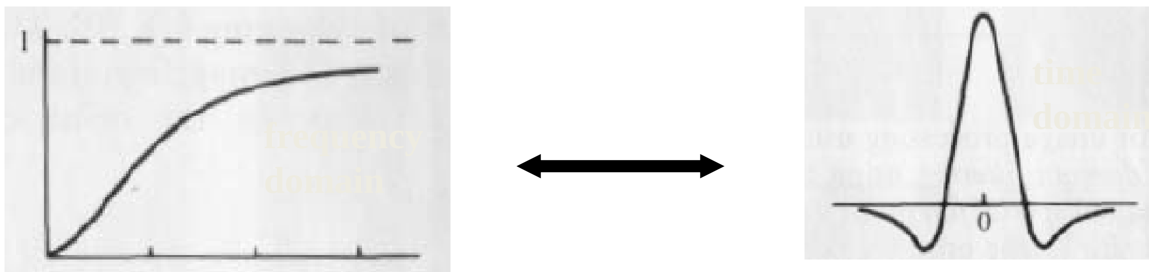
- Preserve low frequencies - useful for noise suppression

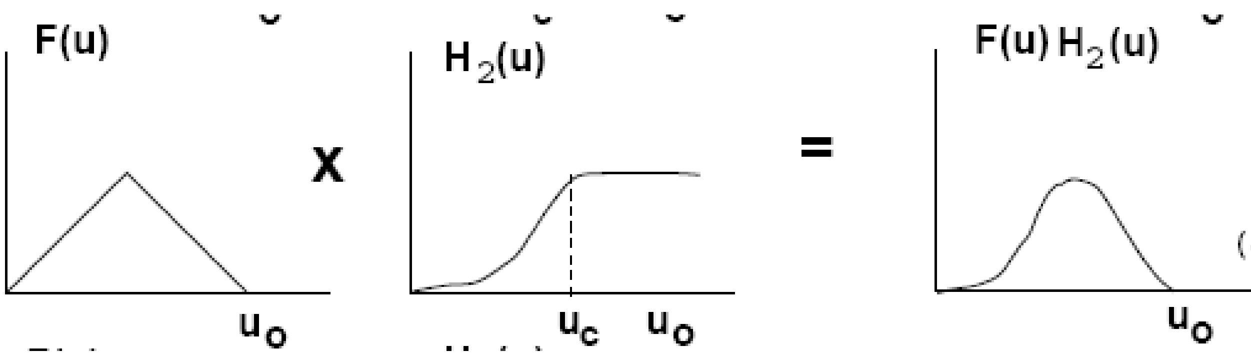




High-pass filters
(i.e., sharpening filters)

- Preserves high frequencies - useful for edge detection

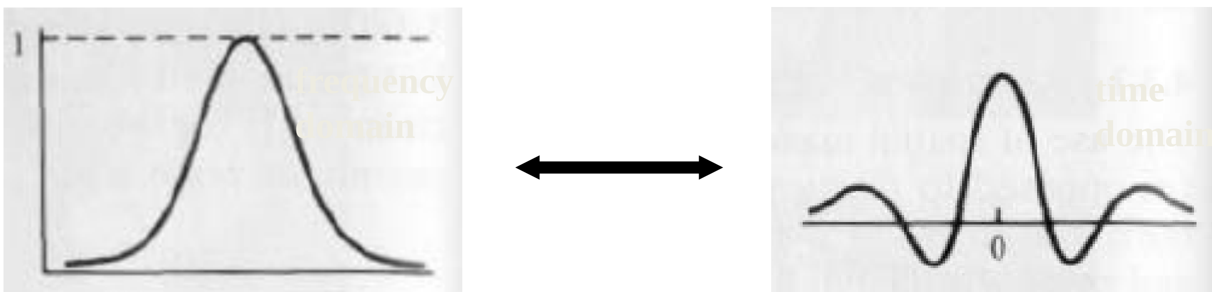


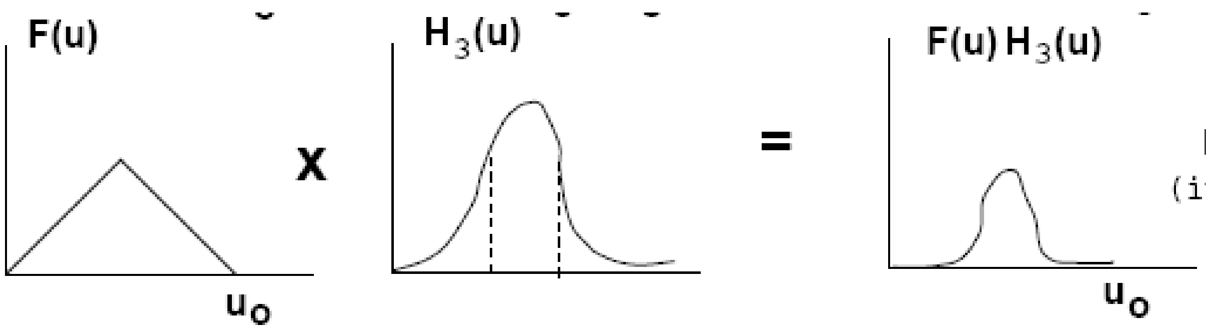


High-pass filtering
(edge detection)

Band-pass filters

- Preserves frequencies within a certain band



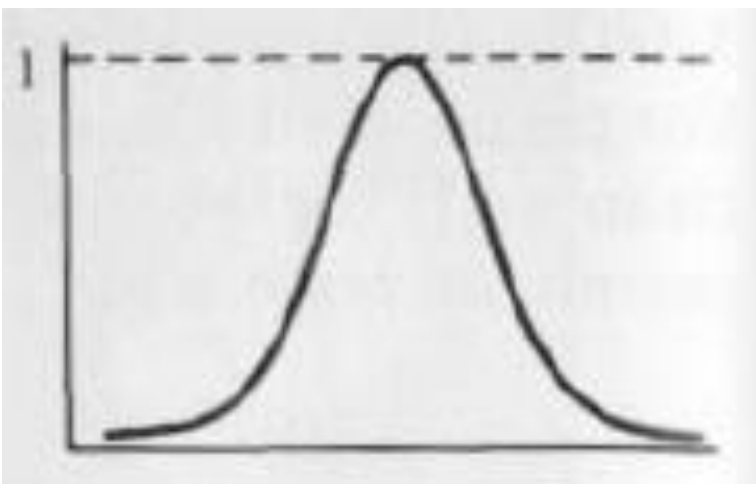


Band-pass filtering
(image restoration)

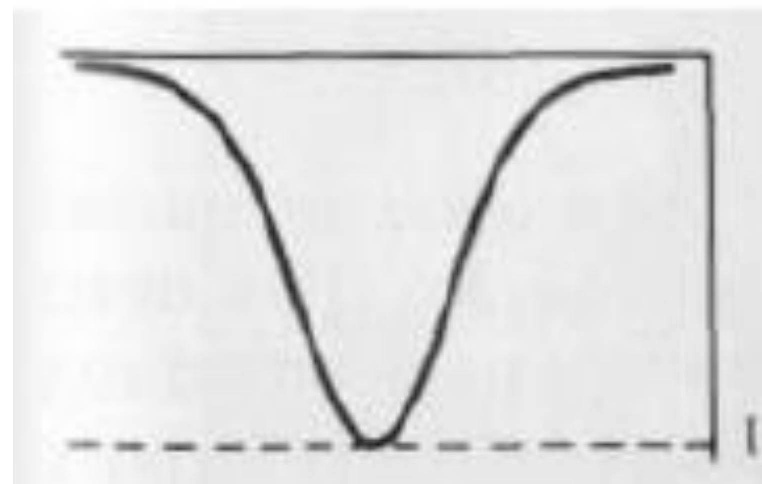
Band-stop filters

- How do they look like?

Band-pass



Band-stop



Frequency Domain Methods

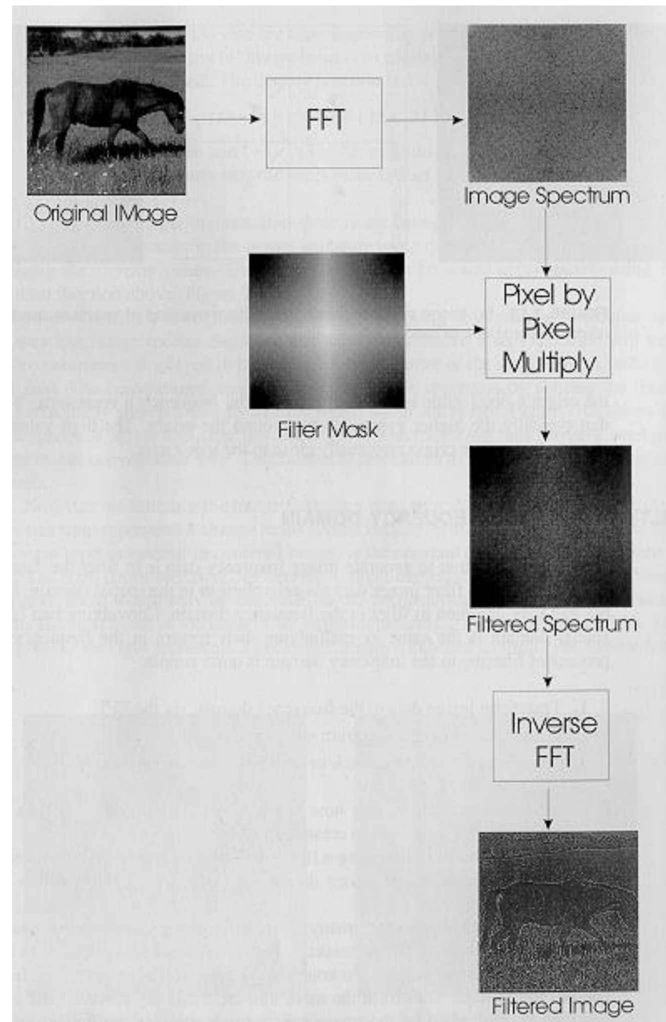
$$f(x,y) * h(x,y) = g(x,y)$$

$$\Updownarrow$$

$$F(u,v)\; H(u,v) = G(u,v)$$

<u>**Case 1:**</u> h(x,y) is given in the spatial domain.

<u>**Case 2:**</u> H(u,v) is given in the frequency domain.



Basic Filtering in the Frequency Domain

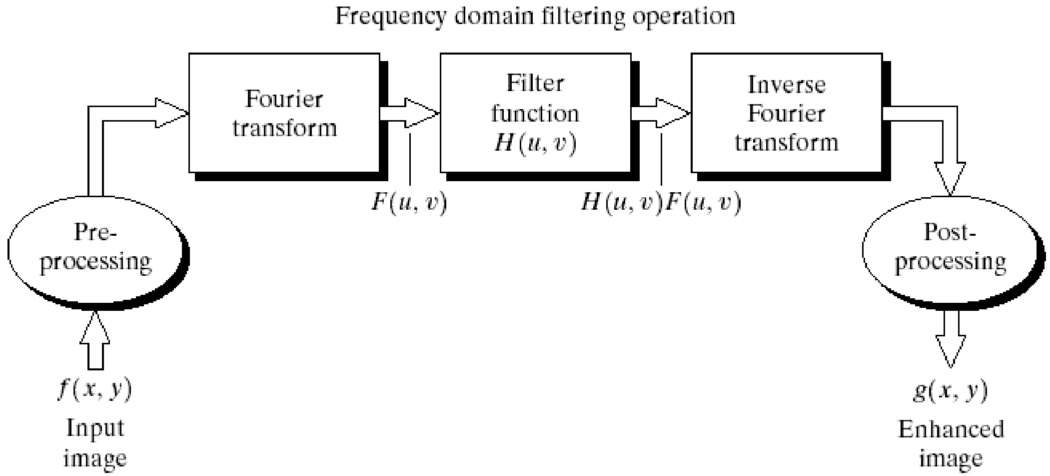
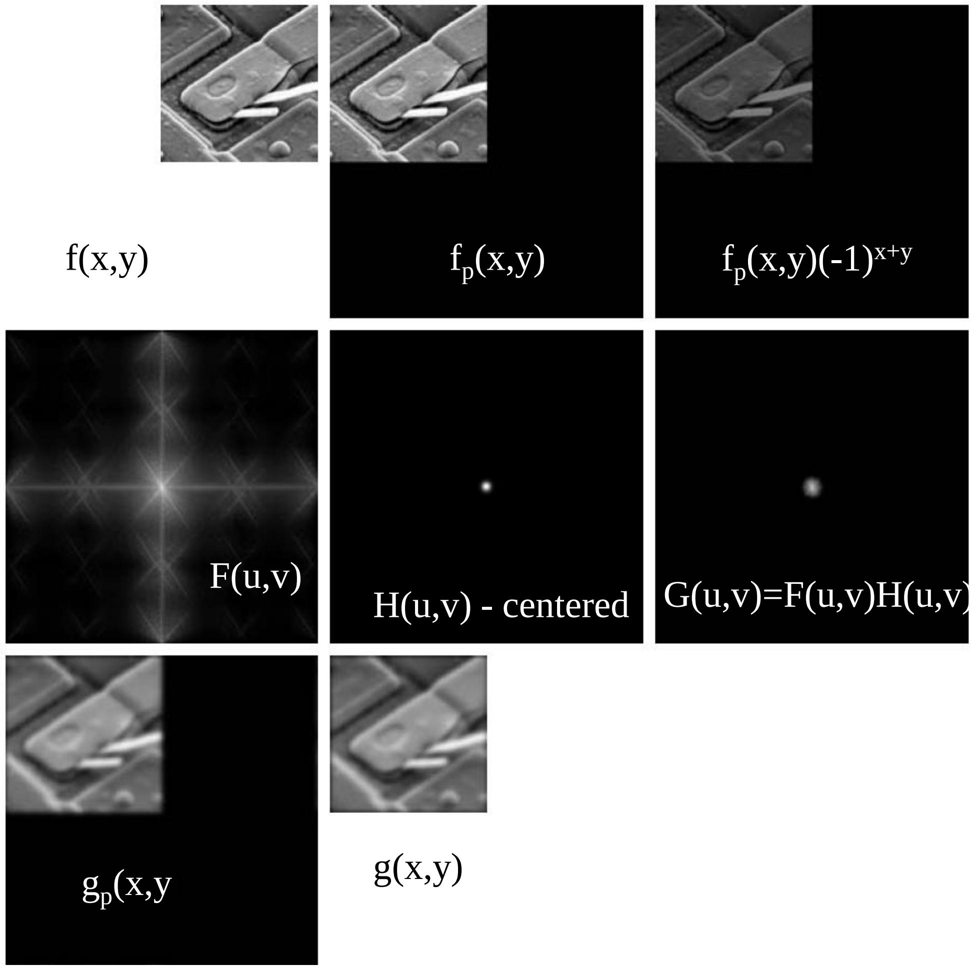


FIGURE 4.5 Basic steps for filtering in the frequency domain.

1. Multiply the input image by $(-1)^{x+y}$ to center the transform
2. Compute F(u,v), the DFT of the image from (1)
3. Multiply F(u,v) by a filter function H(u,v)
4. Compute the inverse DFT of the result in (3)
5. Obtain the real part of the result in (4)
6. Multiply the result in (5) by $(-1)^{x+y}$

Example



a b c
d e f
g h

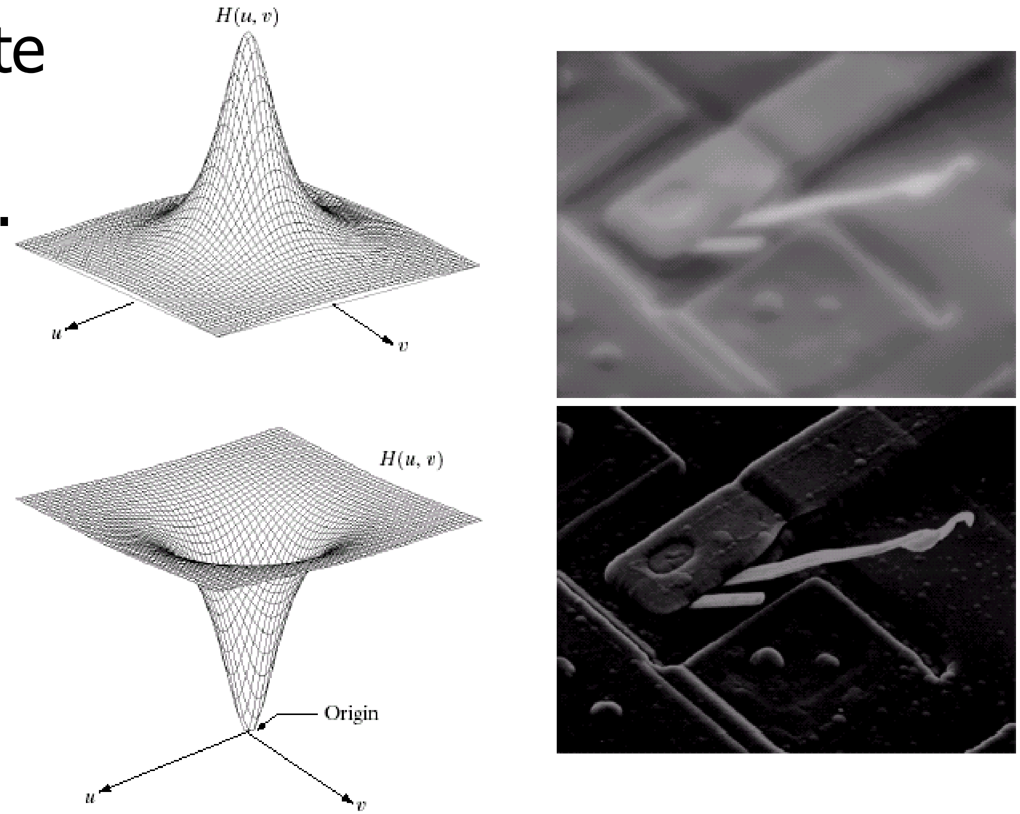
FIGURE 4.36
(a) An $M \times N$ image, f. (b) Padded image, f_p of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$. (f) Spectrum of the product HF_p. (g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p. (h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

$$g_p(x, y) = \left\{\text{real}\left[\Im^{-1}[G(u, v)]\right]\right\}(-1)^{x+y}$$

Low-pass and High-pass Filters

Low Pass Filter attenuate high frequencies while "passing" low frequencies.

High Pass Filter attenuate low frequencies while "passing" high frequencies.



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Low Pass (LP) Filters

- Ideal low-pass filter (ILPF)
- Butterworth low-pass filter (BLPF)
- Gaussian low-pass filter (GLPF)

Smoothing Frequency Domain, Ideal Low-pass Filters

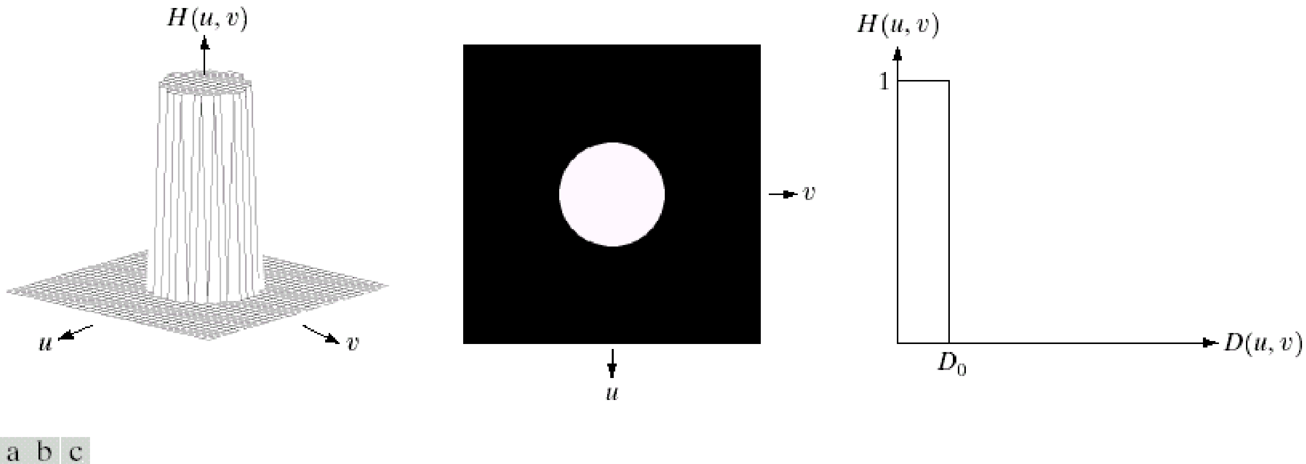
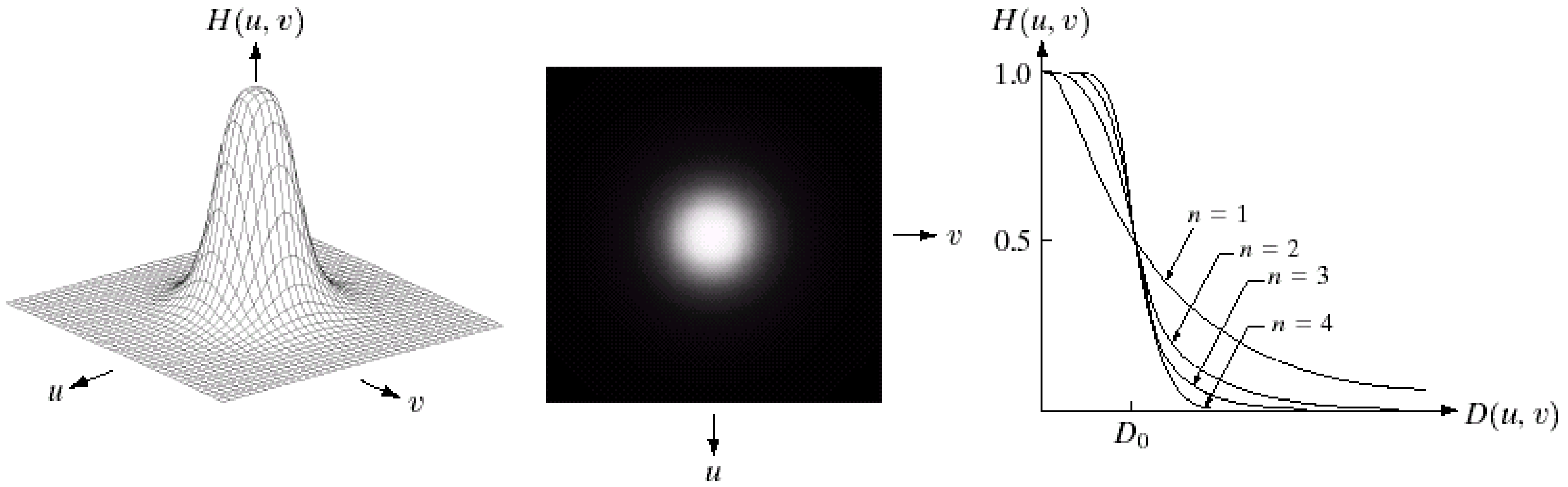


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$

Smoothing Frequency Domain, Butterworth Low-pass Filters



a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

$$H(u,v) = \frac{1}{1 + \left[D(u,v)/D_0\right]^{2n}}$$

Smoothing Frequency Domain, Butterworth Low-pass Filters

Butterworth Low-pass Filter: n=2

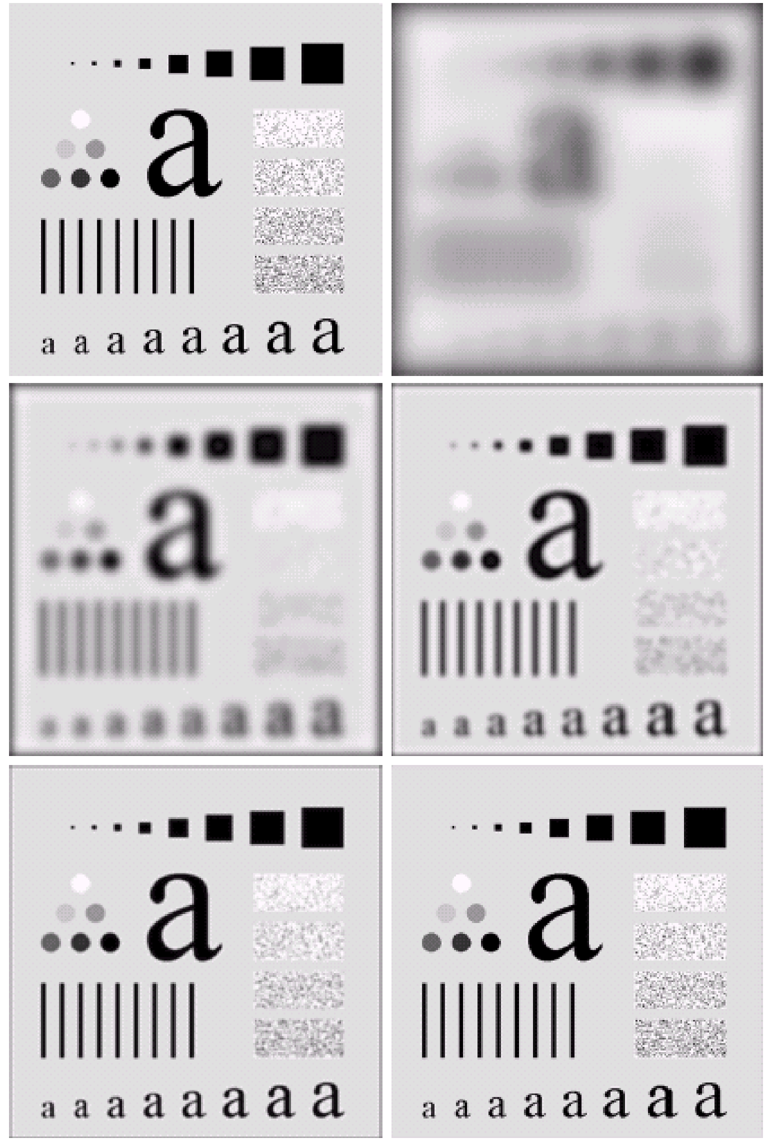
Radii= 5

Radii= 15

Radii= 30

Radii= 80

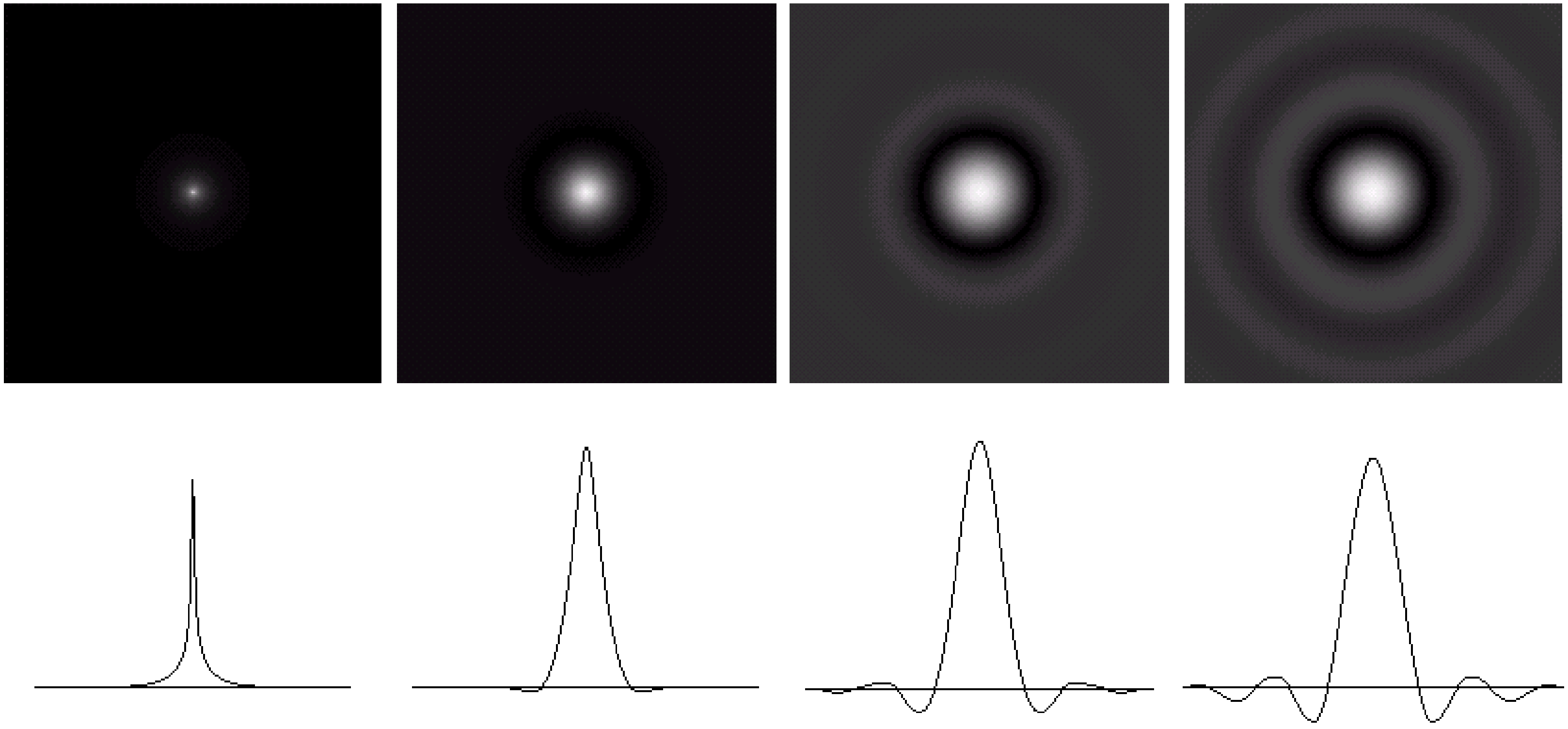
Radii= 230



a	b
c	d
e	f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Smoothing Frequency Domain, Butterworth Low-pass Filters



a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Smoothing Frequency Domain, Gaussian Low-pass Filters

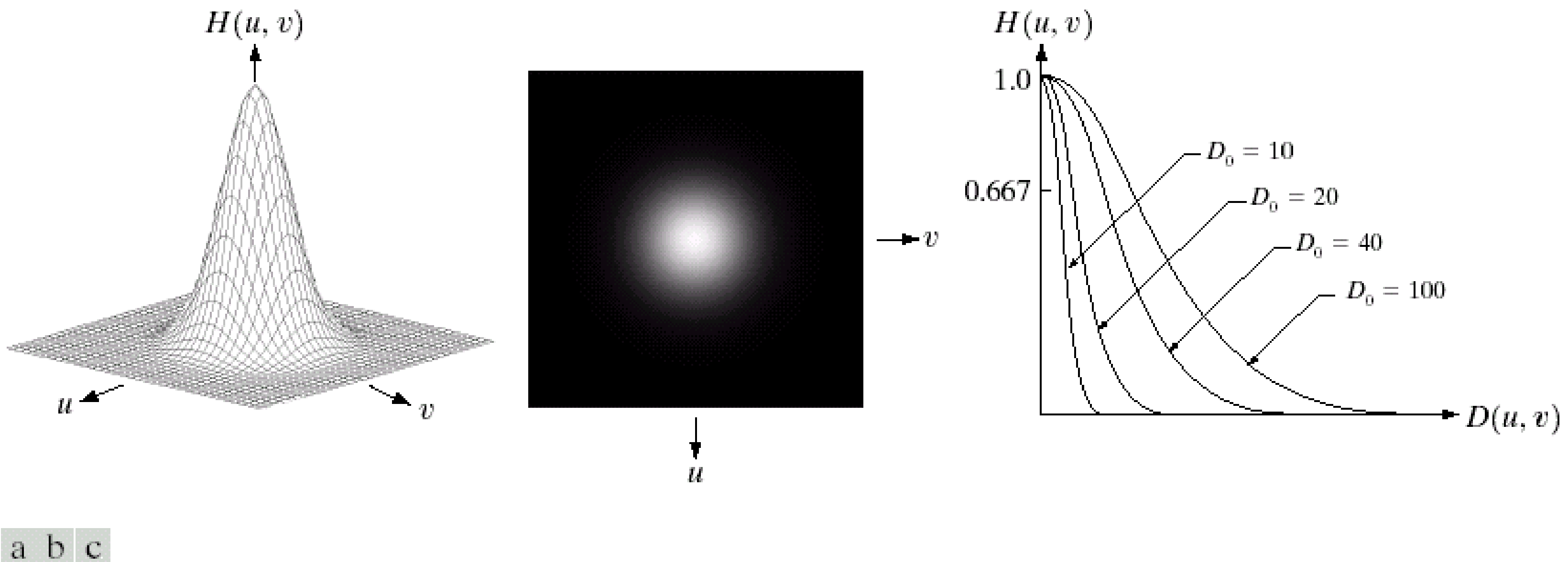


FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

Smoothing Frequency Domain, Gaussian Low-pass Filters

Gaussian Low-pass

Radii= 5

Radii= 15

Radii= 30

Radii= 80

Radii= 230

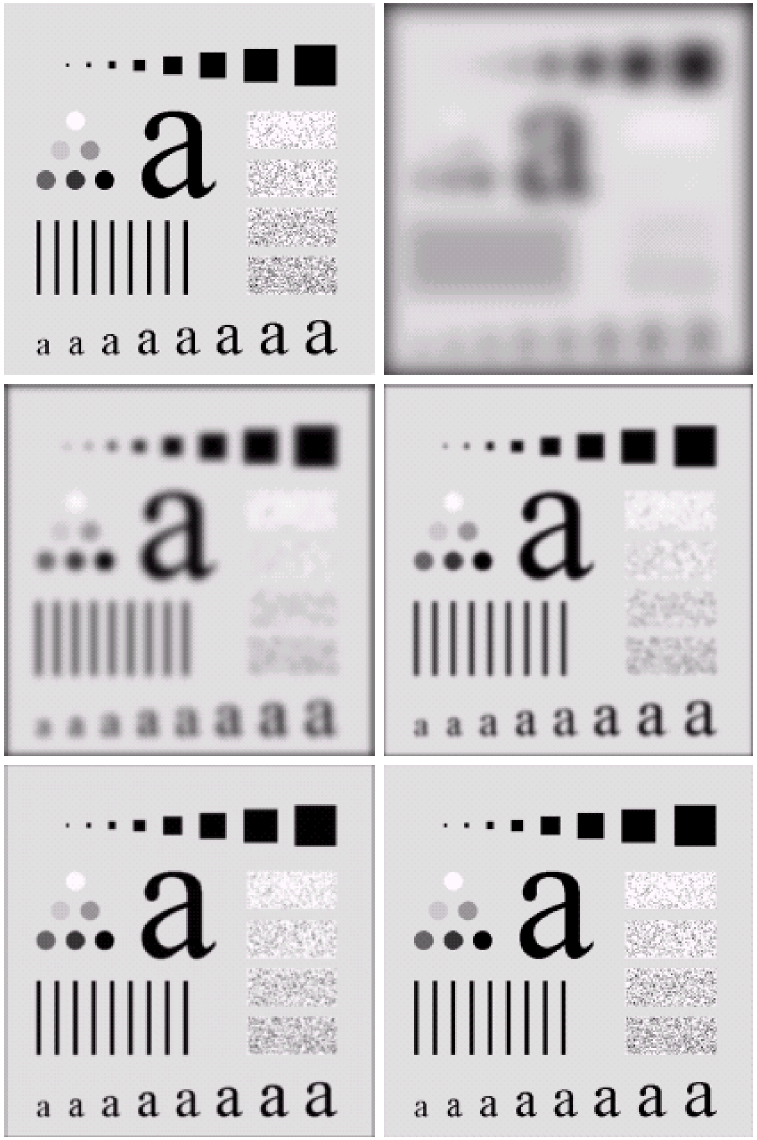


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

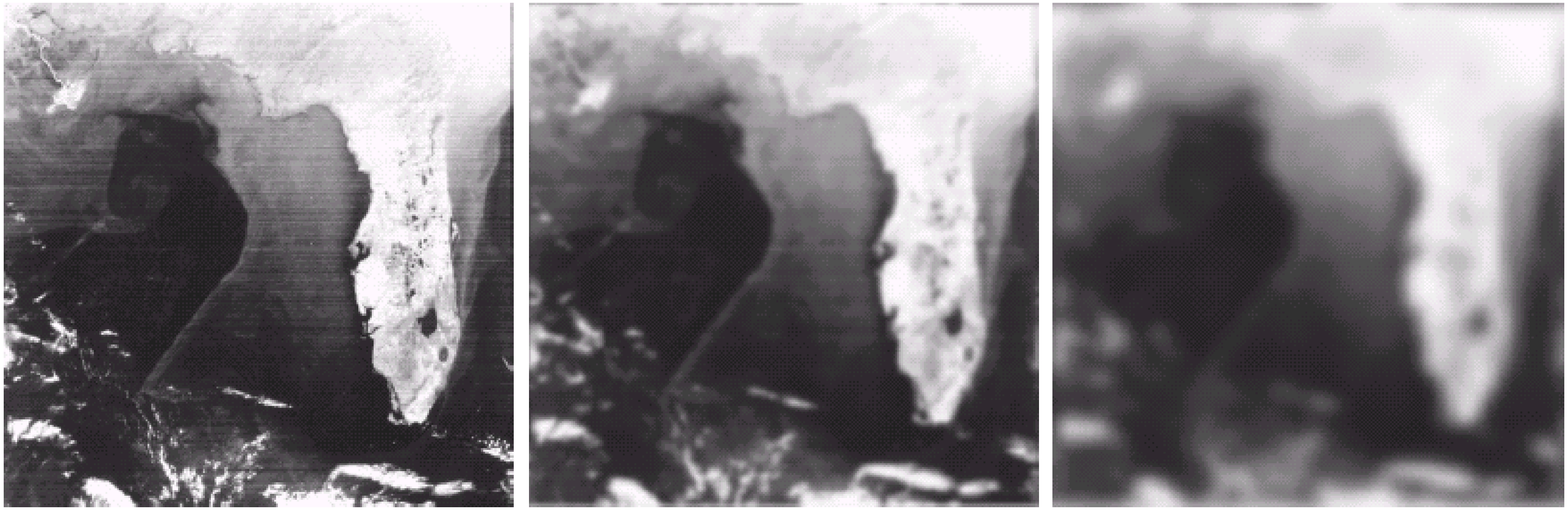
a	b
c	d
e	f

Smoothing Frequency Domain, Gaussian Low-pass Filters

a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

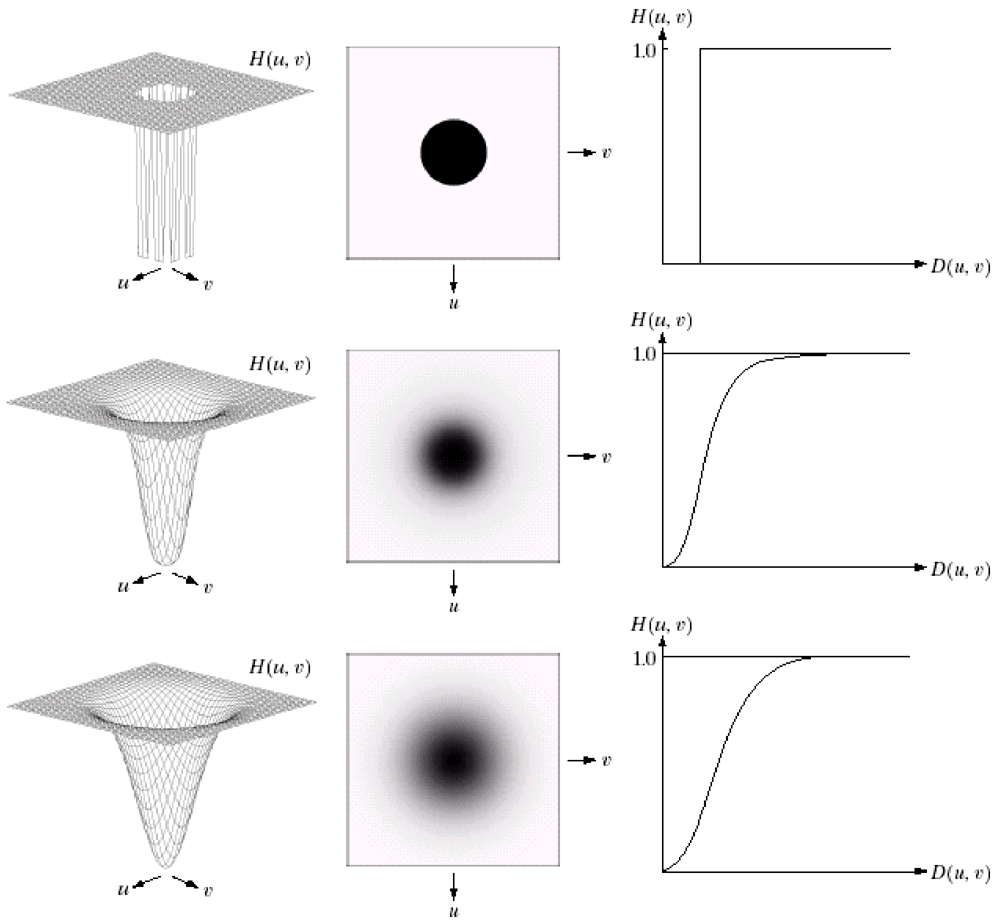
Smoothing Frequency Domain, Gaussian Low-pass Filters



a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

Sharpening Frequency Domain Filters



a b c
d e f
g h i

FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

High Pass (LP) Filters

- Ideal high-pass filter (IHPF)
- Butterworth high-pass filter (BHPF)
- Gaussian high-pass filter (GHPF)
- Unsharp Masking and High Boost filtering

Sharpening Frequency Domain Filters

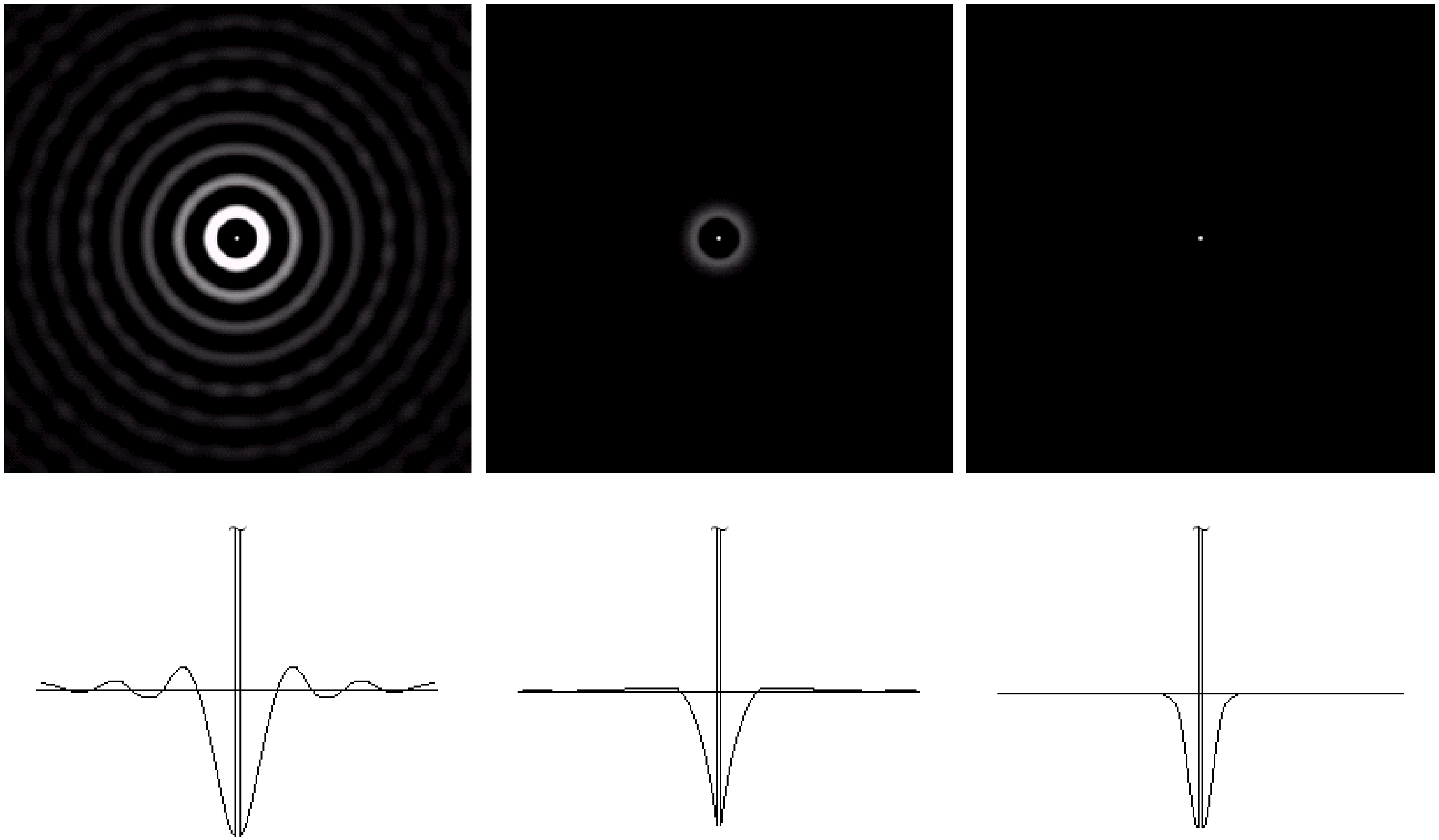
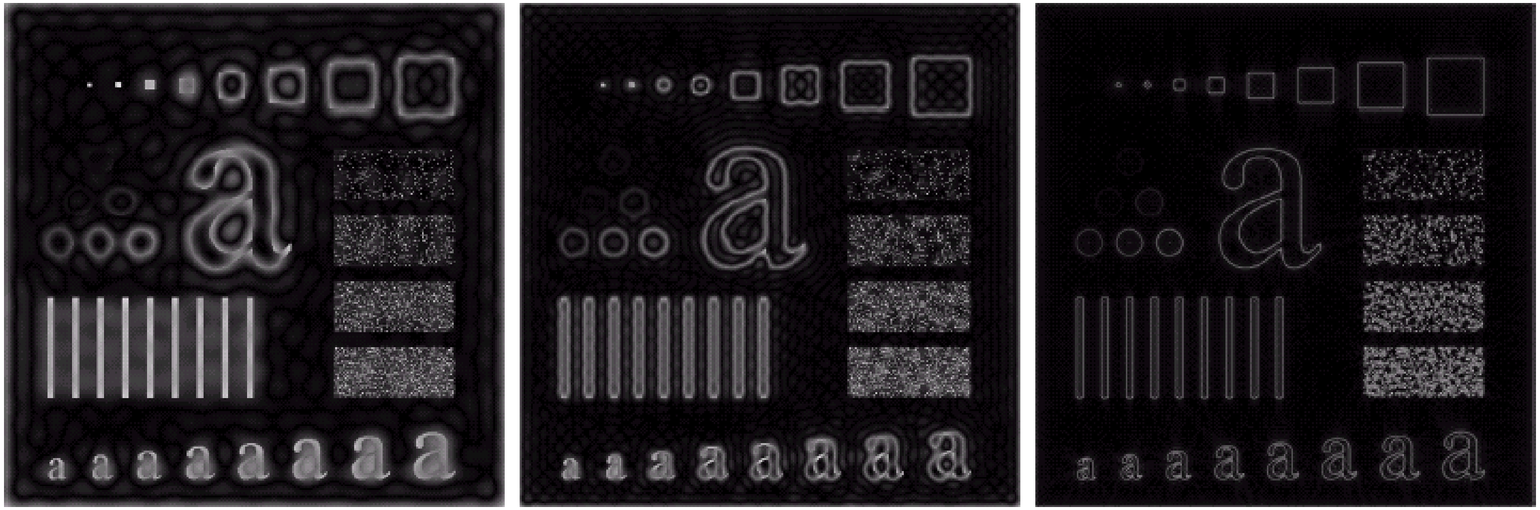


FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Sharpening Frequency Domain, Ideal High-pass Filters

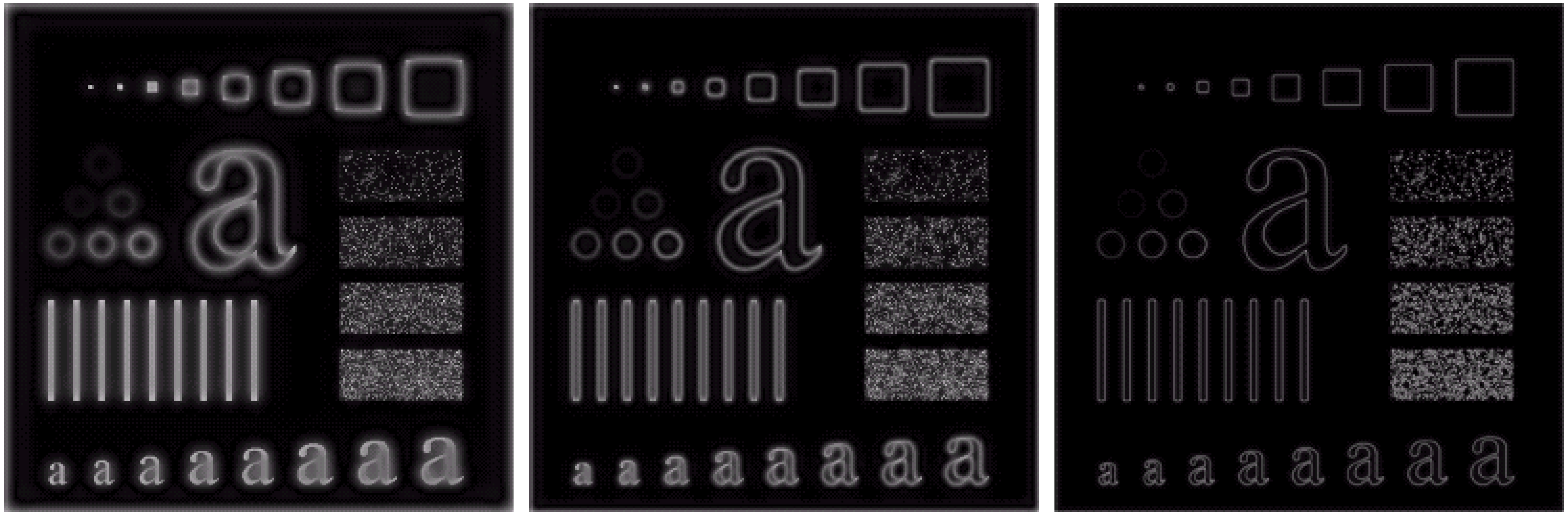


a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Sharpening Frequency Domain, Butterworth High-pass Filters

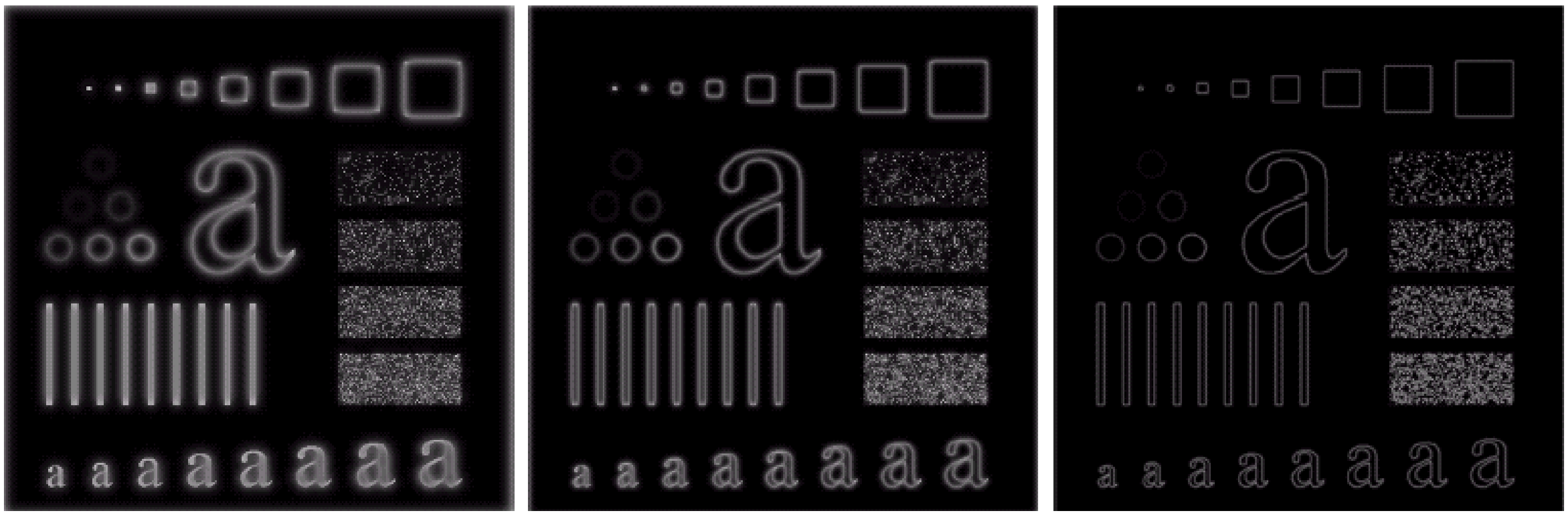


a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

$$H(u,v) = \frac{1}{1+\left[D_0 / D(u,v)\right]^{2n}}$$

Sharpening Frequency Domain, Gaussian High-pass Filters



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

Frequency Domain Analysis of Unsharp Masking and Highboost Filtering

Unsharp Masking: $$g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$$

Highboost filtering:
(alternative definition)

$$g(x,y) = f(x,y) + kg_{mask}(x,y) = f(x,y) + k(f(x,y) - f_{LP}(x,y))$$
$$= f(x,y) + kf_{HP}(x,y)$$

previous definition: $$g(x,y) = (A-1)f(x,y) + f_{HP}(x,y)$$

Frequency domain:

$$f_{LP}(x,y) = f(x,y) * h_{LP}(x,y)$$
$$\mathbf{F}(f_{LP}(x,y)) = F(u,v)H_{LP}(x,y)$$

Revisit: Unsharp Masking and Highboost Filtering

$$g(x,y) = f(x,y) + k(f(x,y) - f_{LP}(x,y))$$

$$G(u,v) = \mathbf{F}\{f(x,y) + k(f(x,y) - f_{LP}(x,y))\}$$
$$= F(u,v) + k(F(u,v) - H_{LP}(u,v)F(u,v)) =$$
$$= [1 + k(1 - H_{LP}(u,v))]F(u,v) = [1 + kH_{HP}(u,v)]F(u,v)$$

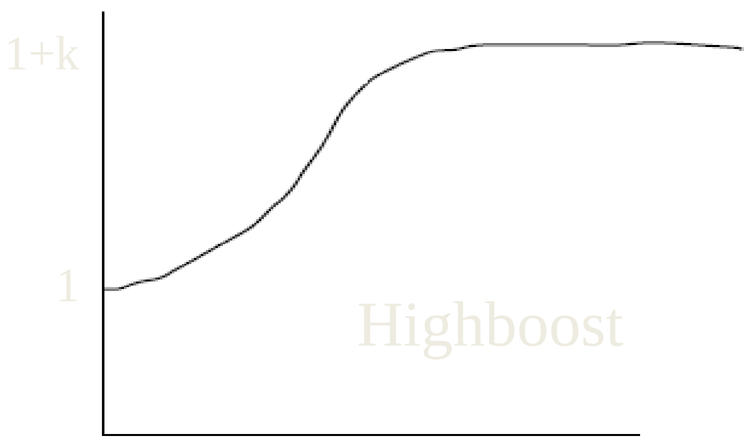
$$so: G(u,v) = [1 + kH_{HP}(u,v)]F(u,v) \;\; or \;\; g(x,y) = \mathbf{F}^{-1}\{[1 + kH_{HP}(u,v)]F(u,v)\}$$

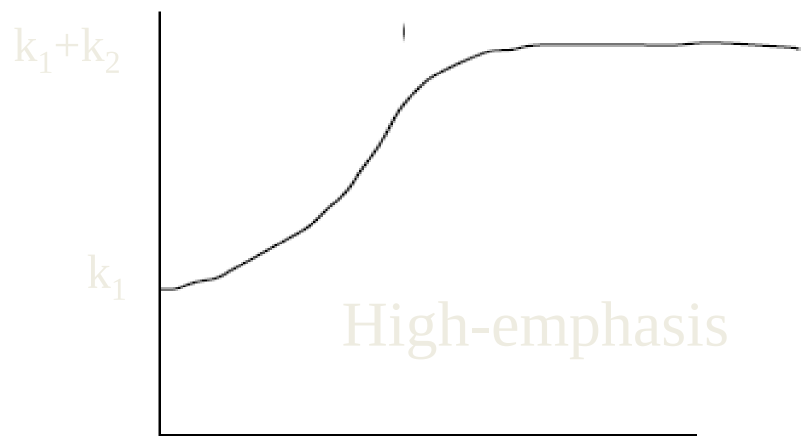
Highboost Filter

Highboost and High-Frequency-Emphasis Filters

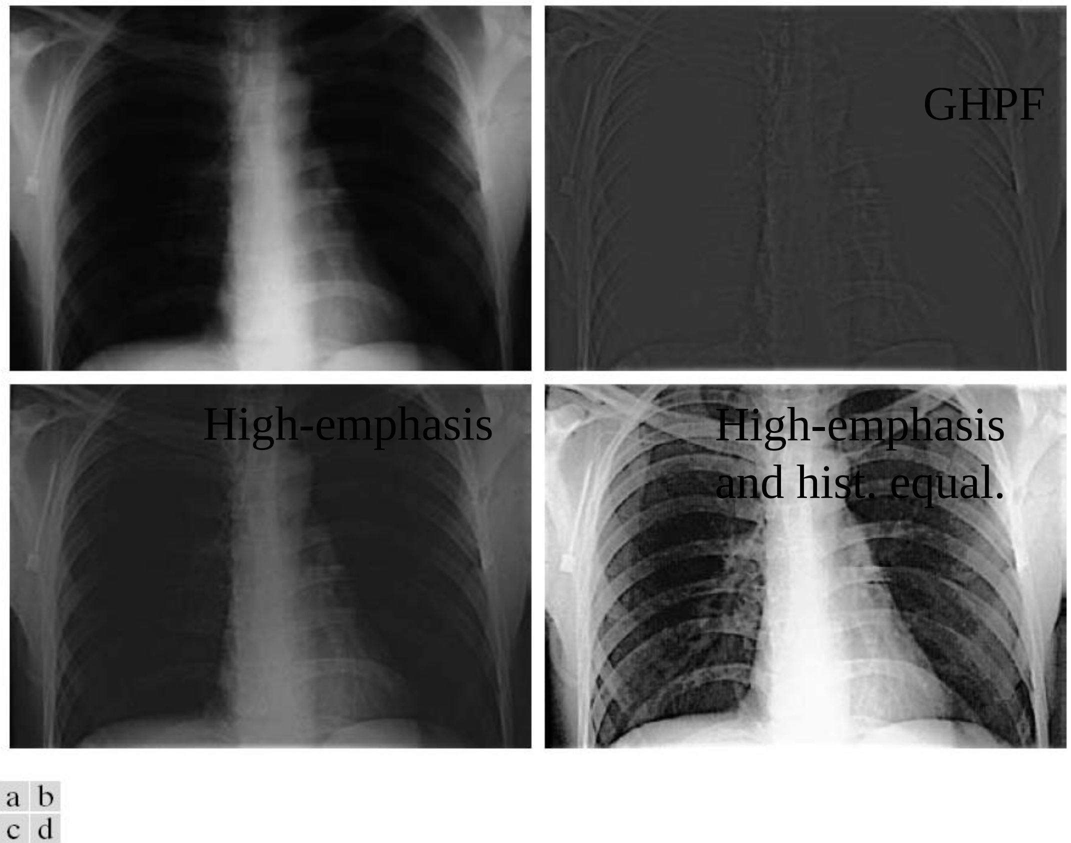
$$g(x,y) = \mathbf{F}^{-1}((1+kH_{HP}(u,v))F(u,v))$$
$$k \geq 0$$

$$g(x,y) = \mathbf{F}^{-1}((k_1+k_2H_{HP}(u,v))F(u,v))$$
$$k_1 \geq 0,\ k_2 \geq 0$$





Example



D_0=40

High-Frequency
Emphasis filtering
Using Gaussian filter
k_1=0.5, k_2=0.75

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)