Unit 1 :- Mathematical Logic

* Statements and Notation

Units of object language called Primary / Primitive / Atomic statements.

Object language contains a set of declarative sentences which can not be further broken down or analyzed into simpler sentences.

Declarative sentences will be admitted in object language which have one and only one of two possible values called "truth values".

The truth values are true and false are denoted by the symbols T and F respectively. They are also denoted by the symbols 1 and 0.

Statement having one of the following values.

	Symbol	Binary
True	T	1
False	F	0

Declare statements using two values are called as two value logic.

We do not admit any other types of sentences.

such as exclamatory, interrogative, etc. in object language.

There are two types of Declarative sentences:

① Primitive object language, these are denoted by distinct symbols from the capotial letters A, B, ---- P, Q, ------ Z

② Second type are obtained from the premitive ones by using certain symbols, called connectives and certain punctuation marks, such as parantheses.

Declarative sentences which is possible to assign one and only one two possible truth values, are called statements.

Statement do not contain any connectives are called atomic (primary, premitive) statements.

Examples:

① Canada is countary (T)

② Mumbai is capital of India (F)

③ This statement is true

The given ① and ② have truth values true & false statement ③ is not according to our definition. So we can't find any proper thing true or false.

④ $1 + 101 = 110$

The given statement is true or false depend upon condition.

if it is binary then true,
if decimal then false

⑤ Toronto is an old city
It is true for some condition and false for some condition

* Connectives

To construct statements from simpler statements by using certain connecting words or expression known as "sentential connectives".

The resulting statements are called molecular or compound statements. The atomic statements are those which do not have any connectives.

In Declarative statement we use also different connectives :
① Conjunction (Anding) $\wedge$
② Disjunction (Oring) $\vee$
③ Conditional (if - then) $\rightarrow$
④ Biconditional (if and only if) $\longleftrightarrow$
⑤ Negation $(\neg)$ - $\sim$

* Types of statements

① Atomic or Primary statements :

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The statement doesn't contain any connective called Primary / Atomic statements.

② Molecular / compound Statement :

The statement which are containing no of primary statements called as molecular / compound statements.

Some connective in English language :

① And
② Or
③ But

* Constant :

The use of some symbols for denoting statement as well as connection called constant

E.g., P, Q, ----

P ⇒ It is raining today

Q ⇒ It is snowing

* Negation :

The negation of a statement formed by introducing the word "not"

If "P" denotes a statement, then the negation of "p" is written as "¬P" and read as "not P".

The symbol "¬" has been used to denote negation.

If the truth value of "p" is T, then truth value of "¬p" is F. Also if the truth value of "p" is F, the truth value of "¬p" is T.

Ex: p ⇒ London is city

¬p ⇒ London is not city

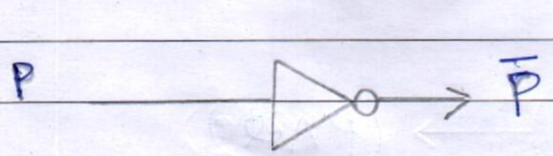
Truth table of negation :

p	¬p
T	F
F	T

Negation is unary operation which operates on a single statement, or a variable.

* "¬" used for denote negation alternative symbol "~", "a bar", "NOT".
i.e., ~p, ¬p, $\bar{p}$, NOTP having same meaning

Input	output
p	($\bar{p}$)
1	0
0	1



* Conjunction :

The conjunction of P and Q statement P ∧ Q which is read as "P and Q".

The statement P ∧ Q has the truth value T whenever both P and Q have truth value T ; otherwise it has the truth value F.

Ex :

P ⇒ It is raining today
Q ⇒ There are 20 tables in room

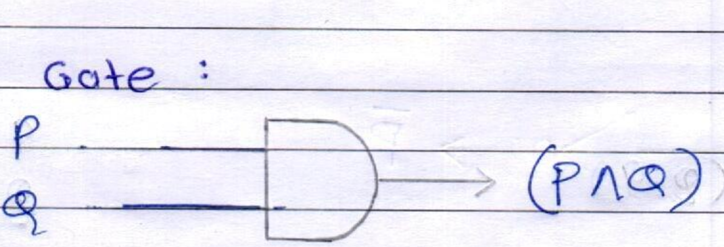
P ∧ Q :

It is raining today and there are 20 tables in room.

Truth table :

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

AND Gate :



Depend on sequences we can change the statemen-ts in given :

① Roses are Red & violets are blue it can be change
② He opened books & stored to write it can't be changed

conjunction is binary operator because it joins two seprate sentences.

conjunction is denoted by "&", "a dot", "AND"
It follows both Associative & Commutative laws

* Disjunction :

The Disjunction of P and Q statement is $P \vee Q$ which is read as "P or Q".

The statement $P \vee Q$ has truth value F only when both P and Q have the truth value F. otherwise it is true.

Truth table :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

① The OR should be Exclusive OR
Ex: I shall watch the game on television or go to the same.

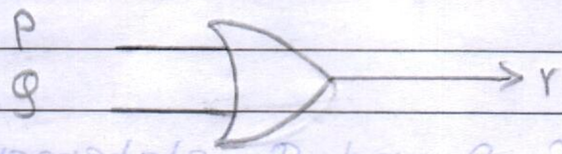
Any one statement from given is true or both are false.

② Inclusive OR:

The two statements are true or one of them are true.

Ex: There is something wrong with bulb or wiring.

OR Gate:



Symbol : or, $\vee$

* conditional statement:

If p & q are two statements then statement $p \rightarrow q$ read as, "if p then q" is called conditional statements.

p is called 'antecedent'
q is called 'consequence.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex : Write following statement in $P \rightarrow Q$

P : The sun is shining today

Q : $5>4$

solution : the sun is shining today, then $5>4$

② The crop will be destroyed if there is a flood.

Solution:

C : The crop will be destroyed

F : There is a flood.

Truth table of $(P \rightarrow Q) \wedge (Q \rightarrow P)$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

* Biconditional :

If P and Q are two statements $P \rightleftarrows$ which is read as ("P if and only if Q".

Read or abbrivated as P if Q called biconditional statements.

If the truth values of both P and Q are true or false i.e., identical then it is true.

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

* Statement formulas and Truth table

The statement which do not contain any connecti -ves called atomic or simple or primary statements

The statement which are containing one or more primary statements are called molecular or composite or compound statements.

Ex: P and Q any two statements.

$\neg P$, $P \vee Q$, $(P \vee Q) \vee (\neg P)$, $P \wedge (\neg Q)$

If the given formula consist any paranthesis, algebra then solve innermost formula first then remaining.

① $\neg(P \wedge Q)$ means negation of $(P \wedge Q)$

② $(P \wedge Q) \vee R$ means Disjunction of $(P \wedge Q)$ and R

In case of negation we can solve it in such a way firstly solve the inner term & give negation to given

Ex: $\neg P \wedge Q$

In this case firstly false negation of P and then disjunction with q

The table containing such values called truth table.

If the statements in given information are 2 then No of combinations are $2^2 = 4$

if n statements then 2^n combinations of truth value.

Ex: ① $P \vee \neg q$
② $P \wedge \neg q$
③ $(P \vee q) \vee \neg P$

① $P \vee \neg q$

P	q	$\neg q$	$P \vee \neg q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Method 1

P	q	P	$\vee$	$\neg$	q
T	T	T	T	F	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	F	T	T	F
step number		1	3	2	1

method 2

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② $P \wedge \neg P$

P	¬P	P ∧ ¬P
T	F	F
F	T	F

method 1

P	P	∧	¬	P
T	T	F	F	T
F	F	F	T	F
step number	1	3	2	1

method 2

* Well - formed Formulas

Well formed formulas are not statements although called as well formed formulas

The statement formula is an expression which is string consisting of variable [capital letters with or without subscripts] parantheses & connective symbols

Rules for well formed formulas:

① A statement variable standing alone is well formed formula

② If A is well formed formula then ¬A is well formed formula.

③ if A & B are well formed then $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \rightleftarrows B$ are well formed formula.

④ A string of symbols containing the statement variable connective and paranthesis is well formed formula, if it can be obtained by firstly applications of the rule 1,2,3,4

According to definition following are well formed formula:

$\neg(P \wedge Q)$, $\neg(P \vee Q)$, $(P \rightarrow (P \vee Q))$, $(P \rightarrow (Q \rightarrow R))$, $(((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R))$

The following are not well formed formulas:

① $\neg P \wedge Q$. P and Q are well formed Formula it may be either $(\neg P \wedge Q)$, $\neg(P \wedge Q)$

② $(P \rightarrow Q) \rightarrow (\wedge Q)$ not well formed formula

③ $(P \rightarrow Q$ is not well formed formula because paranthesis is missed

④ $(P \wedge Q) \rightarrow Q)$ is not WFF because paranthesis is missed.

In well formed formula we can omitte the outer paranthesis by using both of them to replace.

Eg: ① $P \wedge Q$ instead of $(P \wedge Q)$

② $(P \rightarrow Q) \wedge Q$ instead of $((P \rightarrow Q) \rightarrow Q)$

* Tautologies :

The truth table of resulting statement is depend upon the other statement in variables

There are some table which having always truth value true called as "Tautology".

The table which containing False value called "cordradiction (identically false)

The tautology called :

① universally valid form
② logical truth
③ identically true

If in given formula number of distinct variables in formula is 'n' then number of truth table is 2^n

conjunction - disjunction of tautology is always true i.e tautology

Ex: $P \vee \neg P$

Tautology contradiction

P	¬P	P∨¬P	P∧¬P
T	F	T	F
F	T	T	F

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* Equivalence of Formulas :

If A and B are two statements formulas and P_1, P_2 ----- P_n are all variables occuring in A & B.

consider an assignment of truth values to P_1, P_2, ----- P_n & resulting truth values of AB if truth values of A equal to truth values of B in every 2^n possible sets are truth value assigned to equivalent.

If the final values of A & B columns are technical then called as equivalent.

Ex : ① $\neg\neg P \Leftrightarrow P$ equivalent

② $P \vee P \Leftrightarrow P$

③ $(P \wedge \neg P) \vee Q \Leftrightarrow Q$

④ $(P \vee \neg P)$ equivalent to $(Q \vee \neg Q)$

The equivalance can be indicated by '$\Leftrightarrow$' read as "is equivalant to"

"$A \Leftrightarrow B$" can be read as

"A" $\Leftrightarrow$ "B" or,

$A \Leftrightarrow B$

A is equivalant to B can be read as B is equivalant to A and they having transitive relation.

Ex : if $A \Leftrightarrow B$ and $B \Leftrightarrow C$ then $A \Leftrightarrow C$

i.e., equivalance is transitive

Ex ① Prove: $(P \to Q) \Leftrightarrow (\neg P \land Q)$

P	Q	$P \to Q$	$\neg P$	$\neg P \land Q$
T	T	T	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

* Duality Law:

By using connective $\land, \lor, \neg$ we can change the formula.

Two formulas A and A^* said to be duals of each other, if either one can be obtained from the other by replacing $\land$ by $\lor$, $\lor$ by $\land$.

The connectives $\land$ & $\lor$ are called connectives of each other. i.e., dual connective.

If A is formula contain special variable T or F then dual could be, A^* is F or T.

Ex: ① $A \Leftrightarrow (P \lor Q) \land R$

$A^* \Leftrightarrow (P \land Q) \lor R$

② $(P \land Q) \lor T$ its dual,

$(P \lor Q) \land F$

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Theorem 1 :

Let A & A^* be duals formulas and let $P_1, P_2, \ldots P_n$ be all atomic variables occur in A and A^* it may be write as,

$$A(P_1, P_2, \ldots P_n)$$
$$A^*(P_1, P_2, \ldots P_n)$$

By de morgan's law.

$$(P \wedge Q) \Leftrightarrow \neg(\neg P \vee \neg Q)$$
$$(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q)$$

Now we have to show

$$A \Leftrightarrow A^*$$

Now

$$\neg A(P_1, P_2, \ldots P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, \ldots \neg P_n) \quad \text{—①}$$

$$A(\neg P_1, \neg P_2, \ldots \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2, \ldots P_n) \quad \text{—②}$$

From ① and ② we get

$$A \Leftrightarrow A^*$$

Theorem 2 :

Let $P_1, P_2, \ldots P_n$ be all atomic variables apperaing in formulas A & B given that $A \Leftrightarrow B$ means, $A \rightleftarrows B$ is a tautology. then following are tautology.

$$A(P_1, P_2, \ldots P_n) \rightleftarrows B(P_1, P_2, \ldots P_n)$$
$$A(\neg P_1, \neg P_2, \ldots \neg P_n) \rightleftarrows B(\neg P_1, \neg P_2, \ldots \neg P_n)$$

by using 2^{nd} theorem,

$$A^* \Leftrightarrow B^*$$

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* Tautological Implication :

For any statement formula $P \rightarrow Q$, the statement formula $Q \rightarrow P$ is called its converse, $\neg P \rightarrow \neg Q$ is called its inverse and $\neg Q \rightarrow \neg P$ is called its contrapositive.

The $\wedge, \vee, \rightleftarrows$ are symmetric connectives.

① $P \wedge Q \Leftrightarrow Q \wedge P$ will be have same meaning
$Q \wedge P \Leftrightarrow P \wedge Q$
$P \vee Q \Leftrightarrow Q \vee P$
② $P \rightleftarrows Q \Leftrightarrow Q \rightleftarrows P$
③ $P \rightarrow Q$ is not equivalent to $Q \rightarrow P$

If any formula $P \rightarrow Q$
① converse $Q \rightarrow P$
② inverse $\neg P \rightarrow \neg Q$
③ contrapositive $\neg Q \rightarrow \neg P$
④ Orginal formula a contrapositive have same meaning

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

A statement 'A' is said to be tautologically imply statement 'B' if $A \rightarrow B$ is tautology

Denoted by $A \Rightarrow B$ read as A implies B

$A \Longleftrightarrow B$ states A and B are equivalent or that $A \rightleftarrows B$ is tautology

$A \Rightarrow B$ states that $A \rightarrow B$ is a tautology read as A is tautologically implies B

Tautologically implies can be only read as implies.

Implications :

$P \wedge Q \Rightarrow P$

$P \wedge Q \Rightarrow Q$

$P \Rightarrow P \vee Q$

$\neg P \Rightarrow P \rightarrow Q$

$Q \Rightarrow P \rightarrow Q$

$\neg(P \rightarrow Q) \Rightarrow P$

$\neg(P \rightarrow Q) \Rightarrow \neg Q$

$P \wedge (P \rightarrow Q) \Rightarrow Q$

$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$

$\neg P \wedge (P \vee Q) \Rightarrow Q$

$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow P \rightarrow R$

$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Rightarrow R$

There are several important facts about implications and equivalance they are, If a formula is equivalent to a tatuology then it must be a tautology.

Both implications and equivalance are transitive i.e., if $A \Leftrightarrow B$, $B \Leftrightarrow C$ then $A \Leftrightarrow C$

* Functionally complete sets of connectives

Any set of connectives in which every formula can be expressed in terms of equivalant formula containing the connectives from this set is called a functiona-lly complete set of connectives

It does not contain any redundant connectives. ie., connective which can be expressed in terms of the other connectives.

$$P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$$

there the equivalance don't contain any bi conditi-onal in second formula

Ex: Write the formula in such a way that it should not contain biconditional

$P \wedge (Q \rightleftarrows R) \vee (R \rightleftarrows P)$

$P \wedge ((Q \rightarrow R) \wedge (R \rightarrow Q) \vee (R \rightarrow P) \wedge (P \rightarrow R))$

Hence it is proved

Write the equivalance formula for $P \wedge (Q \rightleftarrows R)$ without ① biconditional ② conditional

① $P \wedge (Q \rightleftarrows R)$

$P \wedge ((Q \rightarrow R) \wedge (R \rightarrow Q))$

$\therefore (Q \rightarrow R) \leftrightarrow (\neg Q \vee R)$

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② $P \wedge ((\neg Q \vee R) \wedge (\neg R \vee Q))$

In above cases we firstly replace biconditional by conditional and then conditional by $\wedge$ or $\vee$ when given stmt contain only $\{\wedge, \neg\}$ or $\{\vee, \neg\}$ then and then only the stmt called as functionally complete set of connectives

In the given five connective no any connective should be functionally i.e., No single
$\{\neg\}$, $\{\vee\}$, $\{\wedge\}$ etc

out of given five connectives it may be contain at least two connective to become functionally complete set.

The formula having same meaning as original but different connectives.

* Other connectives

In functionally complete set's those formulas are only defined which as needed but.

To simplefy more we define sepretely

If we have given a 2 formulas P and Q then by using exclusive OR we can write it like, $P \bar{\vee} Q$

exclusive OR also called exclusive Disjuction.

① $p \bar{\vee} q \Leftrightarrow q \bar{\vee} p$ (symmetric)

② $(p \bar{\vee} q) \bar{\vee} R \Leftrightarrow p \vee (q \bar{\vee} R)$ (associative)

③ $p \wedge (q \bar{\vee} R) \Leftrightarrow (p \wedge q) \bar{\vee} (p \wedge \bar{R})$ (Distribute)

④ $(p \bar{\vee} q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

⑤ $(p \bar{\vee} q) \Leftrightarrow \neg (p \rightleftarrows q)$

P	Q	$P \bar{\vee} Q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive Disjunction

NAND : AND + NOT
combination of AND & NOT
symbol ↑

NOR : OR + NOT
combination of OR & NOT
symbol ↓

Ex : $p \uparrow q \Leftrightarrow \neg (p \wedge q)$

$p \downarrow q \Leftrightarrow \neg (p \vee q)$

↑↓ these two are functionally complete it will be sufficient to set of connectives
$\{\wedge, \neg\}$ $\{\vee, \neg\}$

$p \uparrow q \Leftrightarrow \neg (p \wedge q)$

$p \uparrow p \Leftrightarrow \neg (p \wedge p) \Leftrightarrow \neg p \vee \neg p \Leftrightarrow \underline{\underline{\neg p}}$

$(p \uparrow q) \uparrow (p \uparrow q) \Leftrightarrow \neg (p \uparrow q) \Leftrightarrow \underline{\underline{(p \wedge q)}}$

Like NAND same for NOR
$\uparrow$ & $\downarrow$ called as minimal functionally complete set or called minimal set.

① $P \uparrow Q \Leftrightarrow Q \uparrow P$ commutative
$P \downarrow Q \Leftrightarrow Q \downarrow P$

② $P \uparrow (Q \uparrow R) \Leftrightarrow P \uparrow \neg(Q \land R)$
$P \uparrow (Q \uparrow R) \Leftrightarrow \neg[P \land \neg(Q \land R)]$
$\Leftrightarrow \neg P \lor (Q \land R)$

③ $\uparrow$ (NAND) is not associative so
$P \uparrow Q \uparrow R \Leftrightarrow Q \uparrow P \uparrow R$ Not same things

* Normal forms & Principal Normal forms

Let $A(P_1, P_2, --- P_n)$ be a statement formula where $P_1, P_2, --- P_n$ are the atomic variable.

If we assign all truth values for formula and get truth table for it then truth table contain 2^n rows

If $P_1, P_2, --- P_n$ having truth values True then called tautology

if $\neg (P_1, P_2 --- P_n)$ called contradiction.

If any of given row having truth value true called as "satisfible".

The problem of determing the given formula is tautology ~~is~~ or contradiction given finite state called decision problem

Every stmt having the decision problem i.e., some solution but in case of other may not be have.

* Disjunctive Normal Form (sum of product (minterms))

We use "product" in place of "conjunction and "sum" in place of "disjunction".

Elementry product:
A product of a variable and their negations in a given formula is called Elementry product.

Elementary sum:
Sum of the variable and their negation is called Elementary sum.

Example of elementary product: P, $P \wedge Q$, $P \wedge \neg Q$ etc

Example of elementary sum: P, $P \vee Q$, $\neg P \vee Q$ etc

Any part of elementary sum or product which is itself elementary sum or product is called factor of the original elementary sum or product,

Ex: P, $Q \wedge \neg P$ are factors of $P \wedge Q \wedge \neg P$

Necessary Condition

A necessary and sufficient condition for an elementary product to be identically false is that it contain at least one pair of factors in which one is the negation of the other

A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

A formula which is equivalent to a given formula and which consist of a sum of elementary product is called disjunctive normal form of the given formula.

Here $\rightarrow$ & $\rightleftarrows$ connectives are replaced by $\vee, \wedge, \neg$ connectives.

Ex: ① $P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

② $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$

$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$

The disjunction normal form of given formula is not unique.

Depend on distribution low it will change the o/p.

The disjunctive normal form have, identically lot of results but to avoid this we can use, principal Disjuctive Normal Form.

* Conjuctive Normal Form

A formula which is equivalent to a given formula and which consist of a product of elementary sums is called a conjuctive normal form of given formula.

It is not a unique solution.

Given formula is identically true if every elementry sum in it conjuctive normal form is identically true

Ex: ① $P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$ required Form

② $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$

$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee (\neg P \wedge \neg Q))$

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \wedge Q \vee \neg P) \wedge \neg(P \wedge Q) \vee \neg Q)$

So it is principal conjuctive Normal Form.

* Principal Disjuctive Normal Form

If P & Q be two statement variables. constant conjunctions of P & Q & $\neg P$ & $\neg Q$

These are 2 stmt so no of conjunctions are equals to $2^2 = 4$

i.e., $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ + $\neg P \wedge \neg Q$

Are called minterms or Boolean conjunction of P & Q

Truth table for above terms as follows.

P	Q	$P \wedge Q$	$\neg P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	F	F	T

No minterm are equivalent

The equivalent formula which consisting only disjunctio-n of minterms only are called principal disjunctive normal form.

Also called sum of product canonical form. If given two formulas are equivalent then they must have "principal disjunctive normal form."

P	Q	$P \to Q$	$\neg(P \wedge Q)$	$P \vee Q$)
T	T	T	F	T
T	F	F	T	T
F	T	T	T	T
F	F	T	T	F

Event contradiction every principal disjunctive normal form having equal minterms.

If the no of varioble are given as 3 then minterms = 2^3 = 8 terms

$P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$,

$\neg P \wedge Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge \neg R$,

$\neg P \wedge \neg Q \wedge R$, $\neg P \wedge \neg Q \wedge \neg R$ are these 8

It will be possible to dicide weather given formula is tautology we can find out tautology by principal disjunctive normal form.

To create principal disjunctive normal form first replace conditional and biconditional into $\neg, \wedge, \vee$

Ex: $\neg P \vee Q$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$\forall (A \wedge T \Leftrightarrow A)$ identity

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \neg P)$

distributive low $P \vee P \Leftrightarrow P$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

commutative low $P \vee P \Leftrightarrow P$

* Principal conjuctive Normal form

The formula in principal conjunctive Normal form called maxterm.

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If P and Q are two terms that maxterm for them mainly consist of following,

$2^2 = 4$

$P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$

P	Q	$P \vee Q$	$P \vee \neg Q$	$\neg P \vee Q$	$\neg P \vee \neg Q$
T	T	T	T	T	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	T

In this case there are only single truth value as a false in given statement otherwise all are true

consisting conjunction of maxterm so called principal conjunctive normal form

Ex : $(P \vee Q) \wedge (P \vee \neg Q)$ etc

Also called product of sum canonical form.

If the principal disjunctive normal form of a given formula containing n variables is known then principal disjunctive normal form of $\neg A$ will consist of the disjunction of remaining minterms which do not appear in principal disjunctive normal form of A.

Ex : $A \Leftrightarrow \neg\neg A$

disjunction — Minterm

conjunction — Maxterm

Ex: $(\neg P \to R) \wedge (Q \rightleftarrows P)$

$\Leftrightarrow (P \vee R) \wedge (Q \to P) \wedge (P \to Q)$

$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \vee \neg R)) \wedge$
$(\neg P \vee Q \vee (R \wedge \neg R))$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge$
$(\neg P \vee Q \vee \neg R)$

* Difference Between Principal disjuctive normal Form and Principal conjuctive normal Form

Principal disjuctive normal Form	Principal conjuctive normal Form
Minterms	Maxterms
2^n	2^n
$P \wedge Q$ (minterm)	$P \vee Q$ (maxterm) etc
$(P \wedge Q) \vee (\neg P \wedge Q)$	$(P \vee Q) \wedge (\neg P \vee Q)$
Having only one value as true	Having only one value as False
Sum of product canonical Form	Product of sum canonical Form

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* Ordering and uniqueness of normal form

If the given stmt consisting n statement variables then . They are arranged in some fixed order.

1. Alphabetically they are arranged
2. Ex : A, B, --- Z . A1, A2 - - - -

Ex: if P_1, Q, R_3, S_1, T_1 & Q_8 may be variables then they could be arranged like $Q, P_1, S_1, T_1, R_8, S_3$ once it arranged then it will be possible to designite

If n variables are given then they will be arranged in such a way that, $m_0, m_1, - - - - m_2^{n-1}$, i.e., 2^n minterms

If there are n variables then 2^n minterms like if 3 variables then $m_0, m_1, - - - - m_7$ variables so subscript 5 can be write as binary 101

If $P_1, P_2, - - - - P_6$ are 6 variables then it will be contain 2^6 minterms i.e., 64 minterms [conjuction]
$m_0, m_1, m_2, - - - - - - - m_{63}$
To get minterm 36 we can write 100100 & written as
$P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge \neg P_4 \wedge \neg P_5 \wedge \neg P_6 \wedge \neg P_7$

* Principal conjunction

If they contain n variables then written as.
$M_0, M_1, - - - - - M_2^{n-1}$

Here the condition is opposite

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Ex : 36 will be represent as 100100 & represent
$\neg P_1 \vee P_2 \vee P_3 \vee \neg P_4 \vee P_5 \vee P_6$

If maxterms are $M_0, M_1, \dots M_7$ then corresponding of 3 variables P, Q, R are
$P \vee Q \vee R$, $P \vee Q \vee \neg R$, $P \vee \neg Q \vee R$,
$\neg P \vee Q \vee R$, $\neg P \vee \neg Q \vee R$, $\neg P \vee Q \vee \neg R$,
$P \vee \neg Q \vee \neg R$, $\neg P \vee \neg Q \vee \neg R$

The further simplification is introduced by Π to denote conjunction (product) of maxterms

Π, i, j, k represents maxterms M_i, M_j, M_k

* Completely Parenthesized Infix Notation and Polish Notations

Here logical expression or statement formula translated into machine language or assembly language

Generate truth table for well formed formula (wff)

We can generating only necessary paranthesis
Ex : $(((P \rightarrow Q)) \rightarrow Q)$ is equivalent to
$\Leftrightarrow (P \rightarrow Q) \rightarrow Q$

If A is well formed formula then $\neg A$ is well formed formula.

In well formed formula, $\neg$ having highest precedense as compaired to parantheses

With reading from left to right the paranthesis could be resolve the paranthesis like, following:

$$(\neg P \rightarrow P) \Leftrightarrow \neg P \rightarrow P$$

$$(\neg(P \lor Q) \land R) \Leftrightarrow \neg(P \lor Q) \land R$$

$$((P \land Q) \land R) \Leftrightarrow (P \land Q) \land R$$

Depend upon the precedence we can remove paranthesis like $\neg$, $\land$, $\lor$ is sequence of paranthesis.

If an evaluation of such an expression is to be done mechanically, the number of parenthesis should be reduced by adapting certain conventions so that an excessive number of scanning of the expression is avoided.

Binary connective are left associative while $\neg$ is right associative.

Ex: $[4 + (6 \times 3)] - 7$ will be

$4 + 6 \times 3 - 7$

$$\underbrace{\underbrace{P \lor \underbrace{Q \land R}_{1}}_{3} \lor \underbrace{S \land T}_{2}}_{4}$$

The above expression stands for

$(P \lor (Q \land R)) \lor (S \land T)$

The evaluation could be false from left to right

In some cases if we have given or in innermost paranthesis at that time we are using the innermost paranthesis

Ex : if $(P \vee ((Q \wedge R) \wedge (\neg S)))$

① ③ ② ③

Here we are solving from left to right like, $(Q \wedge R)$, $(\neg S)$ $((Q \wedge R) \wedge (\neg S))$, and then $(P \vee ((Q \wedge R) \wedge (\neg S)))$

It is called fully paranthesized expression

In these cases there is no any sequence to solve it like order $\neg, \wedge, \vee$

* Infix notation :

The operator is in between operand 1 & operand 2

Need of paranthesis

Repeated scanning is needed

Ex : $P \underline{\wedge} Q$, $P \underline{\underline{\vee}} Q$

To avoid paranthesis in the given table we can avoid the paranthesis by sufix or prefix notation

operand operator 1 operator 2 prefix

operator 1 operator 2 operand suffix / postfix

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The above notations are called polish , reverse polish or Lukasiewiczian notation. As compaired to prefix & postfix the infix notation contain :

① Paranthesis

② Extra scanning

Ex : Infix - A

postfix / sufix - A

prefix - A

A V B	A B V	V A B
A ∧ B	A B ∧	∧ A B
A V B V C	A B V C V	V V A B C

In given expression is fully paranthesis

(A V ((B ∧ C) ∧ D))

1 3 2

Here innermost subexpression 3 is firstly solved.

∧ BC

∧ BC ∧ D ⟺ ∧∧BCD

⟺ VA∧∧BCD infix to prefix called prefix polished Notation.

The right most operator having highest priority then from right to left the ∧ should be encountered before V in given format

Ex : B V A ∧ C will having

V B ∧ A C

Here reading from right to left ∧ having highest priority.

Prefix / sufix :

Scanning is one directional either left to right or right or left.

Infix :

Both side scanning

In above case, reading from right

Mention : Ex : A V B ∧ C ⇔ V A ∧ BC

Now ∧BC denoted by T1

VAT1 Again denote

T2 ⇔ VAT1

The method for prefix expression should be as follow

1. Find rightmost operator in infix to postfix
2. select two operand to right of operator
3. Perform the indicated operation
4. Replace the operator & operand with result.

Ex: A V (B ∧ C) ∧ D

prefix ∧ ∧BCD

VA ∧∧BCD

Prefix form	current operator	current operator	compound value
VA∧∧BCD	∧	B, C	T1 = T
VA∧T1D	∧	T1, D	T2 = T

VAT2	V	A, T2	T3=T
T3	-	-	

* The theory of Inference for statement calculus: validity using truth table

It provide rules of inference or principles of reasoning the theory associated with such rules known as inferences theory

When conclusion is derived form set of premices by using accepted rules of reasoning called as deduction of formal proof

In mathematics such rules are omitted or informal

Premices are used are true or false decided from faith.

*** Theorem :**

In mathematics they are obtained by rules of logic.

*** Rules of Inference :**

criteria for determining the validity of argument.

① validity
② invalidity

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* Sound :

In any argument for providing given conclusi-on as true we accept rules of logical inference called sound.

* valid conclusion :

The conclusion which is derived from rules of inference called valid conclusion

* valid Arguments :

The argument used for valid conclusion called valid arguments

* Validity using Truth table

Let A & B be two statement formulas it is read as B logically follows by A OR "B is valid premise A" iff $A \rightarrow B$ is tautology i.e., $A \Rightarrow B$

Instead of one formulae we say that set of premices. $\{H_1, H_2, ---- H_m\}$ is a conclusion C follows logically

$H_1 \wedge H_2 \wedge ------ \wedge H_m \Rightarrow C$

$H_1, H_2, ----- H_m \Rightarrow$ Premiceg

$C \Rightarrow$ conclusion

We can decide conclusion is true by constructing truth table

If P_1, P_2, P_3 are atomic variable appearing in premi- ces in premices $H_1, H_2, ---- H_m$ with conclusion C

If possible then assign all possible combinations of truth table $H_1, H_2, ------ H_m$ & also for C

True : checks table whether (i) is true the looks for table check all table row value true if true then true otherwise False

False : If any one value of $H_1, H_2, ---- H_m$ is false then it holds false F.

This technique called, "truth table technique".

Ex : Determine whether the conclusion C follows logically from the premices H_1 & H_2

a] $H_1 : P \rightarrow Q$ $\quad H_2 : P$ $\quad C : Q$

b] $H_1 : P \rightarrow Q$ $\quad H_2 : \neg P$ $\quad C : Q$

c] $H_1 : P \rightarrow Q$ $\quad H_2 : \neg (P \wedge Q)$ $\quad C : \neg P$

d] $H_1 : \neg P$ $\quad H_2 : P \rightleftarrows Q$ $\quad C : \neg (P \wedge Q)$

e] $H_1 : P \rightarrow Q$ $\quad H_2 : Q$ $\quad C : P$

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$P \rightleftarrows Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

Now in ⓐ

$H_1 : P \to Q$, $H_2 : \neg P$ $C : Q$

Here first row having the truth ~~(table)~~ value true for both primices so conclusion also having value T in this row. so it is valid

in ⓑ

$H_1 : P \to Q$ $H_2 : P$ $C : Q$

3^{rd} and 4^{th} row having truth value as true hence we checks for conclusion here 4^{th} row having value as False so it is not valid.

Ex :

H_1 : If canda is a country then New York is city $(P \to Q)$

H_2 : New York is city (Q)

conclusion C : canda is country (P)

so it is valid.

Eg : Determine whether the conclusion C is valid in the following then, H_1, H_2 --- are premices

a) $H_1 : P \rightarrow Q$ $H_2 : \neg Q$ $C = P$

b) $H_1 : P \vee Q$ $H_2 : P \rightarrow R$ $C = R$

c) $H_1 : \neg P$ $H_2 : P \vee Q$ $C : P \wedge C$

d) $H_1 : P \rightarrow (Q \rightarrow R)$ $H_2 : A$ $C : P$

* Rules of Inference :

In validity using truth table we just discuss the premices but in case of rules there are two rules to inference. i.e., P & T

Rule P : A premise may be introduction at any point in derivation.

Rule I : A formula may be introduced in if s in tautologically implied by any one or more of preceding formulas in derivation

Necessary Implications and equivalance

I_1 $P \wedge Q \Rightarrow P$ } simplification
I_2 $P \wedge Q \Rightarrow Q$

I_3 $P \Rightarrow P \vee Q$ } Addition
I_4 $Q \Rightarrow P \vee Q$

I5 $\neg P \Rightarrow P \to Q$

I6 $Q \Rightarrow P \to Q$

I7 $\neg(P \to Q) \Rightarrow P$

I8 $\neg(P \to Q) \Rightarrow \neg Q$

I9 $P, Q \Rightarrow P \wedge Q$

I10 $\neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism)

I11 $P, P \to Q \Rightarrow Q$ (modus ponens)

I12 $\neg Q, P \to Q \Rightarrow \neg P$ (modus tollens)

I13 $P \to Q, Q \to R \Rightarrow P \to R$ (hypothetical syllogism)

I14 $P \vee Q, P \to R, Q \to R \Rightarrow R$ (dilemma)

* Equivalences :

E1 $\neg\neg P \Leftrightarrow P$ (double negation)

E2 $P \wedge Q \Leftrightarrow Q \wedge P$ } (commulative low)

E3 $P \vee Q \Leftrightarrow Q \vee P$

E4 $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ } (associative low)

E5 $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

E6 $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ } Distributive low

E7 $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

E8 $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ } (De-Morgan's laws)

E9 $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

E10 $P \vee P \Leftrightarrow P$

E11 $P \wedge P \Leftrightarrow P$

E12 $R \vee (P \wedge \neg P) \Leftrightarrow R$

E13 $R \wedge (P \vee \neg P) \Leftrightarrow R$

E14 $R \vee (P \vee \neg P) \Leftrightarrow T$

E15 $R \wedge (P \wedge \neg P) \Leftrightarrow F$

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E16 $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

E17 $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

E18 $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

E19 $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

E20 $\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$

E21 $P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

E22 $(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Rule P : Not depend on premices is a single Formula

Rule T : Depend on different premices for solving them

Ex : Demonstrate that R is valid inference from the premices $P \rightarrow Q$, $Q \rightarrow R$ & P

Dependent Formula	Line of Derivation		Premices being Listed
$\{1\}$	(1)	$P \rightarrow Q$	Rule P
$\{2\}$	(2)	P	Rule P
$\{1,2\}$	(3)	Q	Rule T, (1),(2) & I_{11}
$\{4\}$	(4)	$Q \rightarrow R$	Rule P
$\{1,2,4\}$	(5)	R	Rule T (3),(4) & I_{11}

Ex 2 : Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$

{1}	(1)	$P \vee Q$	P
{1}	(2)	$\neg P \rightarrow Q$	T (1) E1 & E16
{3}	(3)	$Q \rightarrow S$	P
{1,3}	(4)	$\neg P \rightarrow S$	T (2)(3) & I13
{1,3}	(5)	$\neg S \rightarrow P$	T (4), E18 & E1
{6}	(6)	$P \rightarrow R$	P
{1,3,6}	(7)	$\neg S \rightarrow R$	T (5),(6) & I13
{1,3,6}	(8)	$S \vee R$	T, (7), E16 & E1

* Consistency of premices & Indirect method of proof:

A set of formulas H1, H2, --- Hm is said to be consistent if their conjunction has truth ~~table~~ value T for some assignment of the truth values to atomic variable appearing in H1, H2, --- Hm

If the at least one value is false then conjunction is identically false. The formulas H1, H2, --- Hm called inconsistent.

Here H1, H2 --- Hm is inconsistent. i.e., conjunction contraduction.

$H_1 \wedge H_2 \wedge H_3 \wedge \quad \wedge H_m \Rightarrow R \wedge \neg R$

if R is any formula then $R \wedge \neg R$ is contraduction

The notation of inconsistency is used in procedure called proof by contraduction or reduction ad absurdum or indirect method of proof.

If C is conclusion logically from the premices H_1, H_2 --- H_m if we assume C as false then consider $\neg C$ as additional premise

If $\neg C$ is inconsistent then result will truely contraduction

Here if the we combine $H_1 \wedge H_2 \cdots H_m$ with $\neg C$ then result will be true

$\therefore$ C is true wherever $H_1 \wedge H_2 \cdots \wedge H_m$ is true

* Tautological Implications

1) Tautology - T
2) contridiction - F
3) contingency - T & F both

① Tautology / university true

$((p \land q) \rightarrow p)$

p	q	$(p \land q)$	$((p \land q) \rightarrow p)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Tautology

② Contridiction

$\neg((p \land q) \rightarrow p)$

Now we take negation of last column from above table we get

$\neg((p \land q) \rightarrow p)$

p	q	$(p \land q)$	$((p \land q) \rightarrow p)$	$\neg((p \land q) \rightarrow q)$
T	T	T	T	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

contradiction / university False

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* Principal CNF & DNF

DNF (sum of product):

- It contain Minterms
- Sum of minterms

Ex:

P	Q	R	A
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

Now to find minterms

$$A \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

In disjunctive normal form take the resultant truth value as true & take oring / disjunction of them

Conjunctive Normal Form (CNF)

In CNF take the term with value of F & take negation of that terms

Take maxterms

$(\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \cap (P \vee Q \vee \neg R)$

1) $(\neg P \to R) \land (Q \leftrightarrow P)$ Find PCNF & PDNF

$(P \lor R) \land ((Q \to P) \land (P \to Q))$ —— ①

$\because \neg P \to R \Leftrightarrow P \lor R$

$P \leftrightarrow Q \Leftrightarrow (P \to Q) \land (Q \to P)$

$(P \lor R) \land ((\neg Q \lor P) \land (\neg P \lor Q))$ —— ⓘⓘ

Now by Using Associative law on ⓘⓘ & Commutative law

$(P \lor R) \land (P \lor \neg Q) \land (\neg P \lor Q)$ —— ⓘⓘⓘ

product of sum (SOP) or CNF

To obtain PCNF on ⓘⓘⓘ we can apply Negation law ie $(Q \lor \neg Q) =$ & Distributive law

$(P \lor R) \lor (Q \land \neg Q) = (P \lor Q \lor R) \land (P \lor \neg Q \lor R)$

Eqⁿ ⓘⓘⓘ will become

$(P \lor Q \lor R) \land (P \lor \neg Q \lor R) \land (P \lor \neg Q \lor R) \land (P \lor \neg Q \lor \neg R)$
$\land (\neg P \lor Q \lor R) \land (\neg P \lor Q \lor \neg R)$ —— Ⓘⓥ

From eqⁿ Ⓘⓥ repeated Maxterms will be removed we will got.

$$\boxed{(P \lor Q \lor R) \land (P \lor \neg Q \lor R) \land (P \lor \neg Q \lor \neg R) \land (\neg P \lor Q \lor R) \land (\neg P \lor Q \lor \neg R)} \text{ —— ⓥ}$$

Eqⁿ ⓥ is required <u>PCNF</u> which contain 5 Maxterms in it.

In the given PCNF we have 5 Maxterms out of total 8 Maxterms.

As we have 3 statements P, Q, R in the given statement formula ie S

$S \Leftrightarrow (\neg P \to R) \wedge (Q \leftrightarrow P)$

All Possible Minterms & Maxterms are as follow.

i) Minterms	Maxterms
1) $(P \wedge Q \wedge R)$ ✓	1) $(P \vee Q \vee R)$ ✓
2) $(P \wedge Q \wedge \neg R)$ ✓	2) $(P \vee Q \vee \neg R)$
3) $(P \wedge \neg Q \wedge R)$	3) $(P \vee \neg Q \vee R)$ ✓
4) $(\neg P \wedge Q \wedge R)$	4) $(\neg P \vee Q \vee R)$ ✓
5) $(\neg P \wedge \neg Q \wedge R)$ ✓	5) $(\neg P \vee \neg Q \vee R)$
6) $(\neg P \wedge Q \wedge \neg R)$	6) $(\neg P \vee Q \vee \neg R)$ ✓
7) $(\neg P \wedge \neg Q \wedge \neg R)$	7) $(\neg P \vee \neg Q \vee \neg R)$
8) $(P \wedge \neg Q \wedge \neg R)$	8) $(P \vee \neg Q \vee \neg R)$ ✓

In the given example PCNF of S is containing 5 Maxterms

Hence by using property of S & $\bar{S}$ we can directly write

(PCNF of $\bar{S}$) = (3 Maxterms)

$$\boxed{\text{PCNF of } \bar{S} = \big((P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)\big)}$$

Now PDNF of S = Negation of PCNF of $\bar{S}$

Hence

PDNF of $S = \neg\left((P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)\right)$

$$\boxed{\text{PDNF of } S = (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)}$$

PDNF of $\bar{S}$ = Total Minterms − Minterms of S

$$\boxed{\text{PDNF of } \bar{S} = \left((P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)\right)}$$