Unit No - 5

Lattice & Boolean Algebra

* Lattice and Boolean Algebra

* Lattice as Partially ordered set :

Algebraic system was introduced by George Boole in 1854 and called it as Boolean Algebra.

Boolean Algebra is a special Lattice.

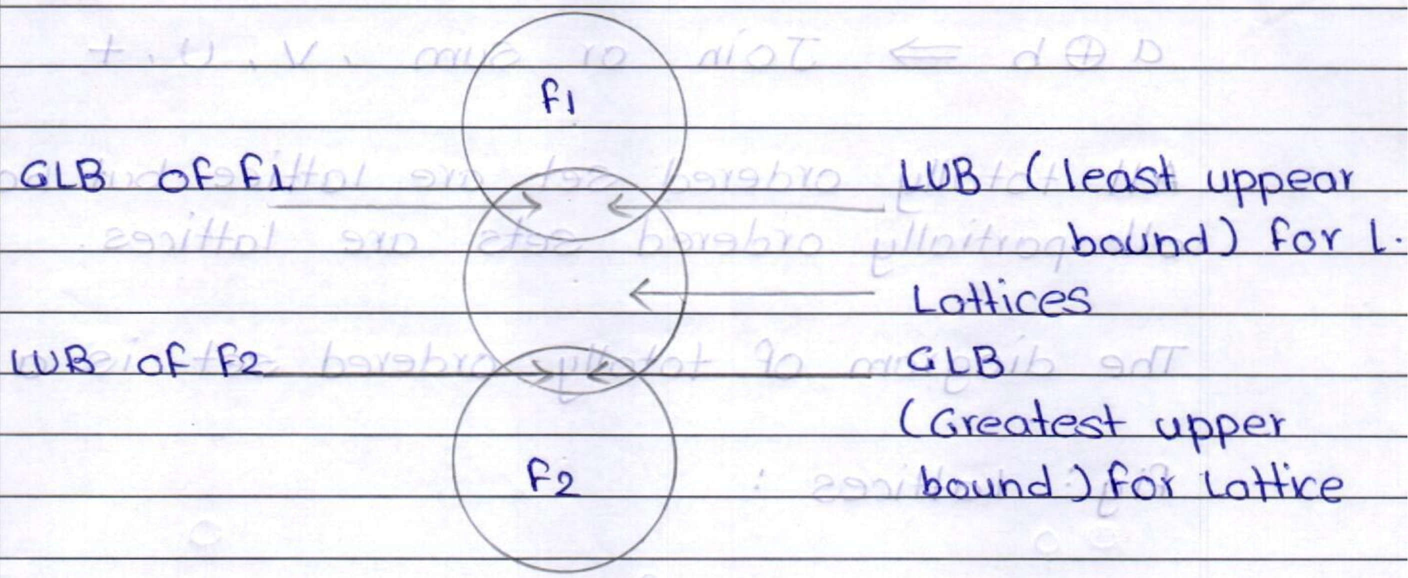


fig: Lattice with LUB and GLB

A lattice is a partially ordered set $<L, \leq>$ in which every pair of elements $a, b \in L$ has LUB and GLB

GLB is denoted by $a * b$

LUB is denoted by $a \oplus b$

$a * b \Rightarrow$ Meet or product

$\wedge$ in mathematical logic

$\cap$ intersection in set.

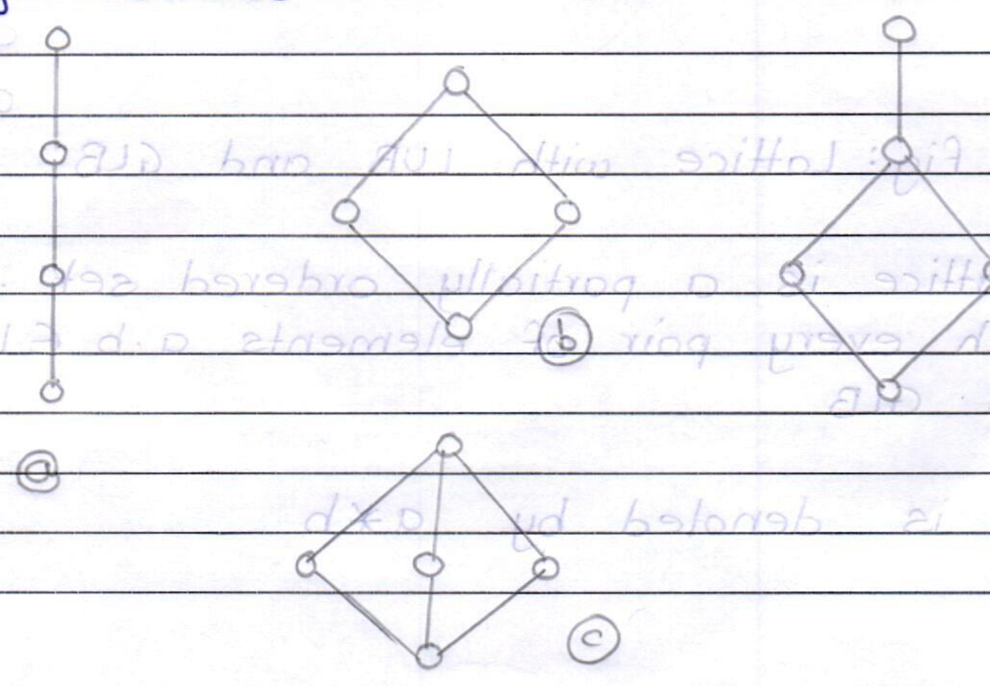
$\cdot$ multiplication

$a \oplus b \Rightarrow$ Join or sum , $\vee$, $\cup$, $+$

All totally ordered set are lattice but not all partially ordered sets are lattices.

The diagram of totally ordered set is a chain

Fig: Lattices :



if $S = \{a, b, c\}$

$\pi(S) = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$

Where,

$\pi_1 = \{\overline{a, b, c}\}$

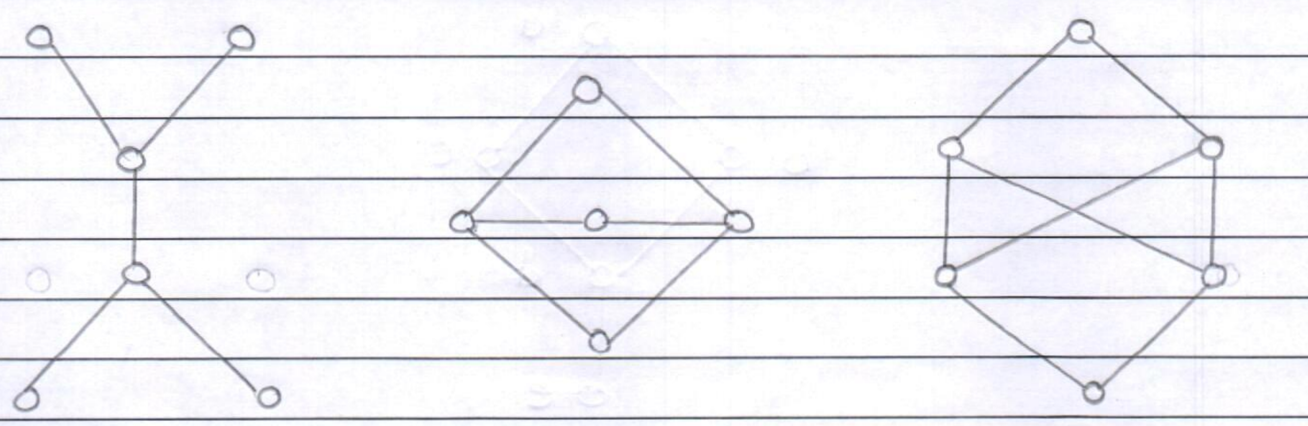
$\pi_2 = \{\bar{a}, \overline{b, c}\}$

$\pi_3 = \{\overline{a, b}, \bar{c}\}$

$\pi_4 = \{\overline{a, c}, \bar{b}\}$

$\pi_5 = \{\bar{a}, \bar{b}, \bar{c}\}$

If there are 4 elements then 15 partitions
5 elements then 52 partitions



(a) (b) (c)

Fig: Partially ordered sets which are not lattices.

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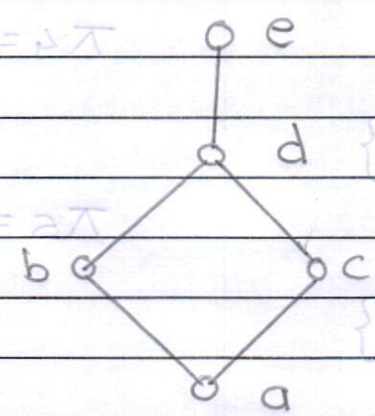
Principle of Duality:

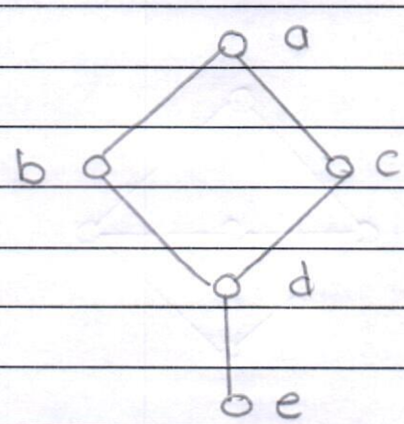
If $\le$ is partially ordered then $\ge$ is also partially ordered.

If $<S, \le>$ having the hasse diagram.



Then

$<S, \ge>$ having the inverting hasse diagram to original.



$<S, \le>$ & $<S, \ge>$ called deals of each other. Also $\oplus$ and $*$ are dual of each other $<L, \le>$ and $<L, \ge>$ also dual of each other.

GLB

*	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

LUB

$\oplus$	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

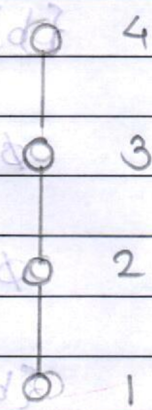


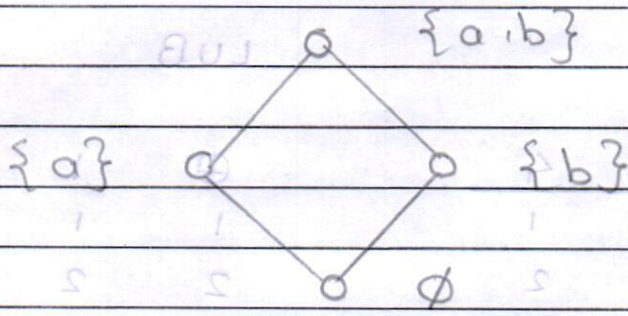
fig : Totally ordered set [chain]

Ex : Consider set $P = \{a, b\}$ and the relation is of conclusion (subset) for the power set. Draw the hasse diagram for the POSET $< P(P), \subseteq >$ and check it wheather lattice or Not.

→ Now, set $P = \{a, b\}$

$P(P) = 2^2 = 4$ elements.

$\wp(P) = \{ \phi, \{a\}, \{b\}, \{a,b\} \}$



GLB ∩

*	∅	{a}	{b}	{a,b}
∅	∅	∅	∅	∅
{a}	∅	{a}	∅	{a}
{b}	∅	∅	{b}	{b}
{a,b}	∅	{a}	{b}	{a,b}

LUB ∪

⊕	∅	{a}	{b}	{a,b}
∅	∅	{a}	{b}	{a,b}
{a}	{a}	{a}	{a,b}	{a,b}
{b}	{b}	{a,b}	{b}	{a,b}
{a,b}	{a,b}	{a,b}	{a,b}	{a,b}

* Some properties of Lattices :

If $\langle L, \leq \rangle$ is lattices having two binary operations $*$ and $\oplus$

Where $a, b, c \in L$ then

① $a * a = a$

(Idempotent)

$a \oplus a = a$

② $a * b = b * a$

(commutative)

$a \oplus b = b \oplus a$

③ $(a * b) * c = a * (b * c)$

(Associative)

$(a \oplus b) \oplus c = a \oplus (b \oplus c)$

④ $a * (a \oplus b) = a$

(Absorption)

$a \oplus (a * b) = a$

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* Theorem :
Let $\langle L, \leq \rangle$ is a lattice in which * and $\oplus$ denotes the operations of meet and join respectively for any $a, b \in L$

$$a \leq b \iff a * b = a \iff a \oplus b = b$$

To prove above equalities we will first assume $a \leq b$
i.e. a is having return with b
We also know $a \leq a$ because POSET is reflexive.

i.e. $a \leq a * b$ because the GLB operation of a & b with return answer a.

we also have $a * b = a$
Simply we can prove

$$a \leq b \iff a \oplus b = b$$

* Lattice as an algebric system

A Lattice an algebric system $\langle L, *, \oplus \rangle$ with two binary operation $*$ and $\oplus$ on elements of L which are both.

① Commutative
② Associative
③ Absorption

* sublattices :

let $\langle L, *, \oplus \rangle$ be a lattice and $S \subseteq L$ be a subset of L then the algebra $\langle S, *, \oplus \rangle$ is a sublattice of $\langle L, *, \oplus \rangle$ iff S is closed under both operations $*$ & $\oplus$

Sublattic itself is a lattice.

If $\langle P, \leq \rangle$ is POSET and $Q \subseteq P$ then $\langle Q, \leq \rangle$ is also POSET.

For a lattice $\langle L, *, \oplus \rangle$ and for any two elements $a, b, \in L$ such that $a \leq b$ i.e. closed interval $[a, b]$ consisting of the elements $x \in L$ such that

$a \leq x \leq b$ is sublattice of L.

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✰ Direct method :

Let $<L, *, \oplus>$ & $<S, \cap, \vee>$ be two lattices. Then the algebric system $<L \times S, \cdot, +>$ in which binary operation $+$ & $\cdot$ on $L \times S$ are such that any $<a_1, b_1>$ and $<a_2, b_2>$ in $L \times S$.

$$<a_1, b_1> \cdot <a_2, b_2> = <a_1 * a_2, b_1 \cap b_2>$$

$$<a_1, b_1> + <a_2, b_2> = <a_1 \oplus a_2, b_1 \vee b_2>$$

is called the direct (method) product of lattice $<L, *, \oplus>$ and $<S, \cap, \vee>$

The direct product satisfy :
1. commutative law
2. Associative law
3. Absorption law

✰ Lattice Homomorphism :

If $<L, *, \oplus>$ & $<S, \wedge, \vee>$ be two lattices. A mapping g' $L \rightarrow S$ is called a lattice homomorphism from the lattice $<L, *, \oplus>$ to $<S, \cap, \vee>$ if for any $a, b \in L$

$$g(a * b) = g(a) \wedge g(b)$$

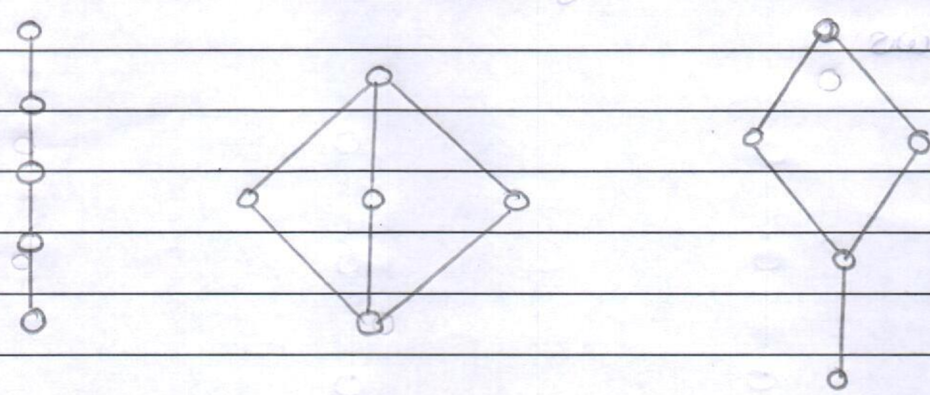
$$g(a \oplus b) = g(a) \vee g(b)$$

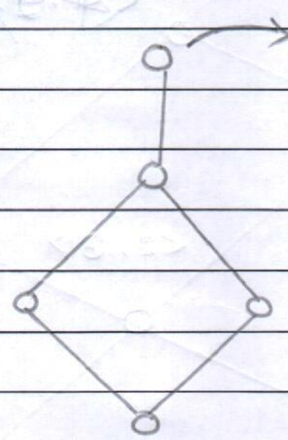
If two lattices Lattice L & S is called bijective or one to one onto mapping

* order of lattice :

The number of nodes present in the Lattice is the order of Lattice

Eg: example of lattices with order 5.





No of nodes present in lattice called order of lattice.

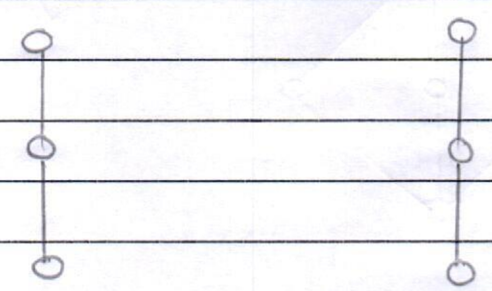
① Ex : consider the chains of divisions of 4 and 9 i.e.,

$L_1 = \{1, 2, 4\}$

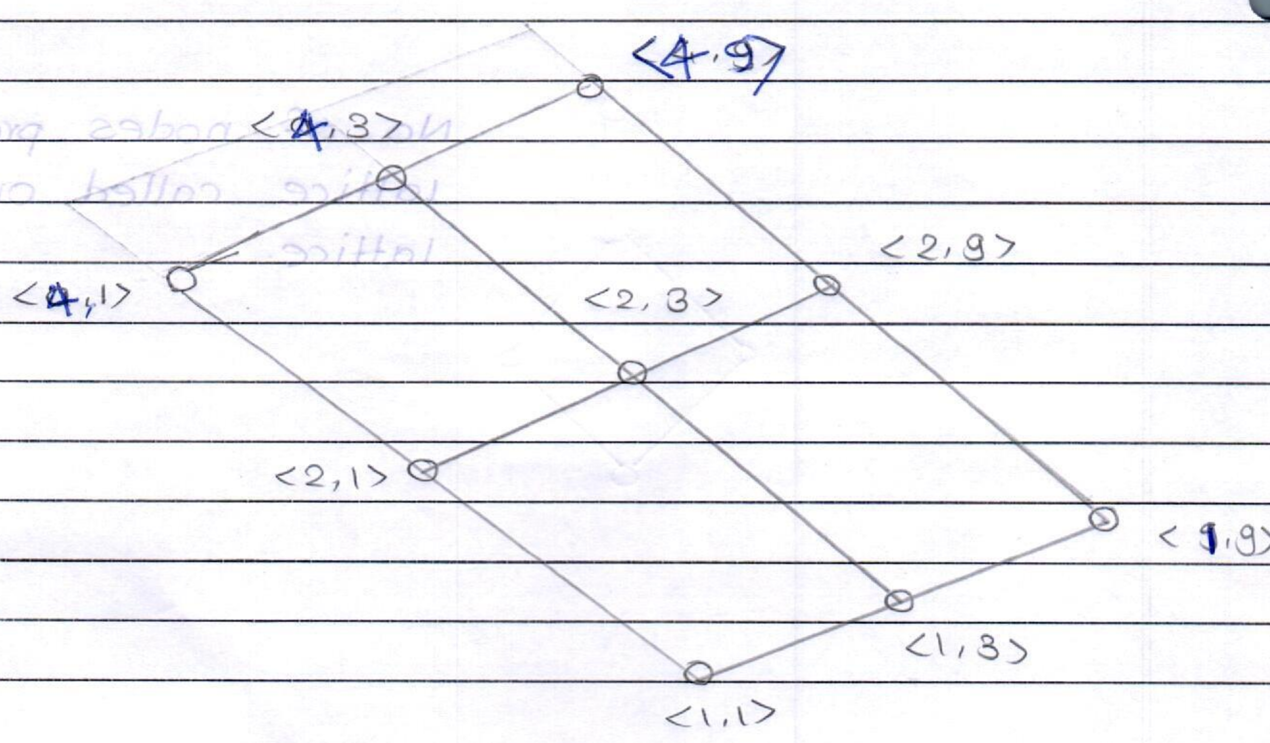
$L_2 = \{1, 3, 9\}$

and the partial order relation is of division on L_1 and L_2 find out $L_1 \times L_2$

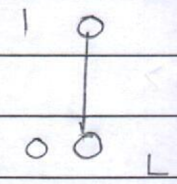
→ Given hasse diagrams for L_1 and L_2 as follows :



L1 L2

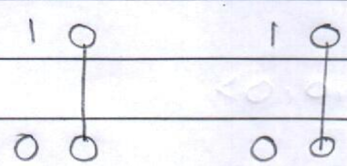


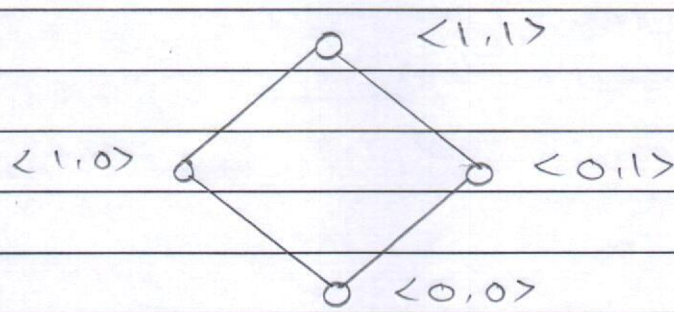
Ex ② Let $L = \{0, 1\}$ and the lattice is $\langle L, \leq \rangle$ as given find out $\langle L^2 \leq 2 \rangle$, $\langle L^3 \leq 3 \rangle$



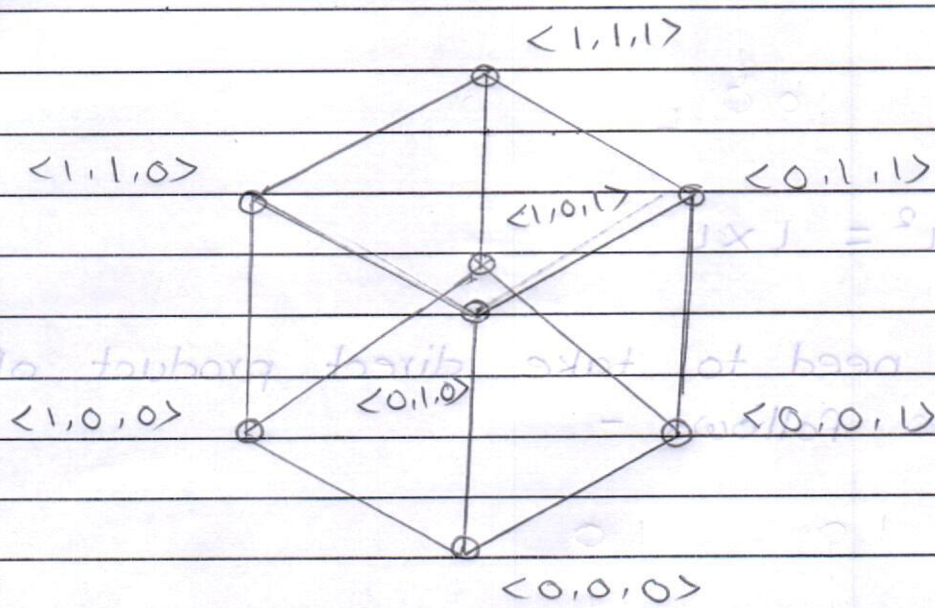
→ Given $L^2 = L \times L$

i.e., we need to take direct product of L with itself as follows -





$L^3 = L \times L \times L = L \times L^2$


<1,1,1>
<1,1,0>
<0,1,1>
<1,0,1>
<0,1,0>
<1,0,0>
<0,0,1>
<0,0,0>

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* Some special lattices

① Complete Lattice :

A lattice is called as complete if each of it's non-empty subsets has LUB and GLB i.e, $\oplus$ & $*$.

Sometimes the greatest and least elements are said to be bound p of the lattice and denoted by 0 and 1 respectively

② Bounded Lattice :

For the Lattice $\langle L, *, \oplus \rangle$ where $L = \{a_1, a_2, \ldots a_n\}$
Where L = nonempty set

$$\underset{i=1}{\overset{n}{*}} a_j = 0$$

$$\underset{i=1}{\overset{n}{\oplus}} a_i = 1$$

Then a lattice $\langle L, *, \oplus, 0, 1 \rangle$ is called as bounded Lattice.

It has following properties

$a * 1 = a$ $\qquad$ $a \oplus 1 = 1$

$a * 0 = 0$ $\qquad$ $a \oplus 0 = a$

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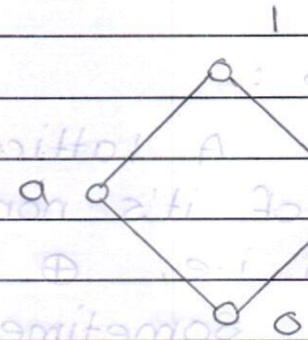
Example :



* Special Lattices :

If $\langle L, *, \oplus \rangle$ be a lattice and $S \subseteq L$ be a finite subset of L where

$S = \{a_1, a_2, \ldots, a_n\}$. The GLB and LUB can be expressed as,

$$\boxed{GLB\ S = \mathop{*}_{i=1}^{n} a_i}$$

$$\boxed{LUB\ S = \bigoplus_{i=1}^{n} a_i}$$

Where

$$\mathop{*}_{i=1}^{2} a_i = a_1 * a_2 \quad \text{--- (i)}$$

$$\mathop{*}_{i=1}^{k} a_i = \mathop{*}_{i=1}^{k-1} a_i * a_k \quad \text{--- (ii)}$$

We can perform for $\oplus$ join operation also.

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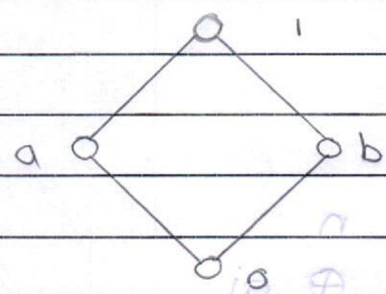
hence,

$$\underset{i=1}{\overset{n}{*}} a_i = a_1 * a_2 * \cdots a_n$$

* Complement of an element :

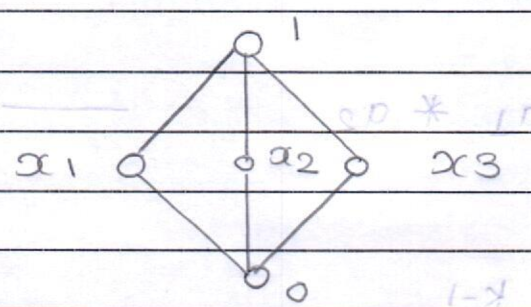
In bounded lattice $\langle L, *, \oplus, 0, 1 \rangle$ an element $b \in L$ is called as complement of an element $a \in L$ if $a * b = 0$ and $a \oplus b = 1$

Ex ① :



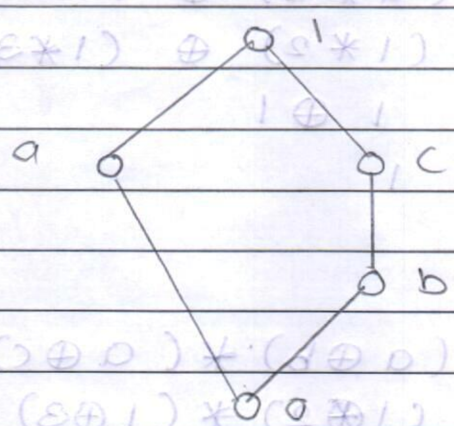
In above bounded lattice b is a complement of a or a is compliment of b

②



In above lattice,
compliment of x_1 is x_2, x_3
compliment of x_2 is x_1, x_3
compliment of x_3 is x_1, x_2

(3)



compliment of a is b, c a - b,
compliment of b is a
compliment of c is a

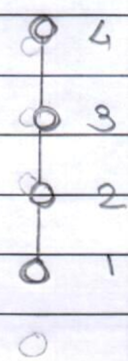
* Distributive lattice :

A lattice $\langle L, *, \oplus \rangle$ is said to be distributive lattice if for any $a, b, c \in L$

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Ex : check whether following lattice is distributive or not.



$a * (b \oplus c) = (a * b) \oplus (a * c)$

$1 * (2 \oplus 3) = (1 * 2) \oplus (1 * 3)$

$1 * 3 = 1 \oplus 1$

$1 = 1$

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

$1 \oplus (2 * 3) = (1 \oplus 2) * (1 \oplus 3)$

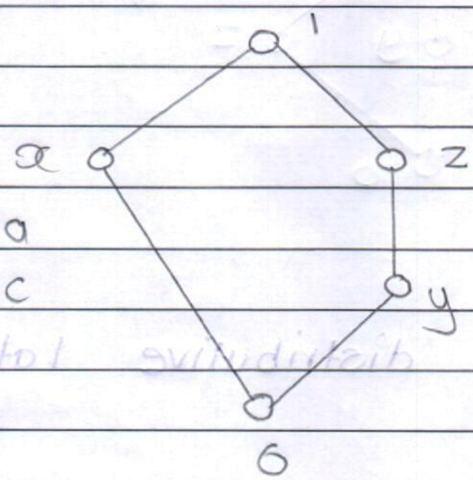
$1 \oplus 2 = 2 * 3$

$2 = 2$

Therfore above lattice is distributive lattice

* Every chain is distributive lattice

Ex ②: check whether following lattice is distributive or not.



i) c ii) a

i) a
ii) c

i) b ii) b

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

i) $x * (y \oplus z) = (x * y) \oplus (x * z)$

$x * z = 0 \oplus 0$

$0 = 0$

ii) $z * (y \oplus x) = (z * y) \oplus (z * x)$

$z * 1 = y \oplus 0$

$z = y$

consider a of z, b as y and c as x from above it is clear that the lattice is not distributive

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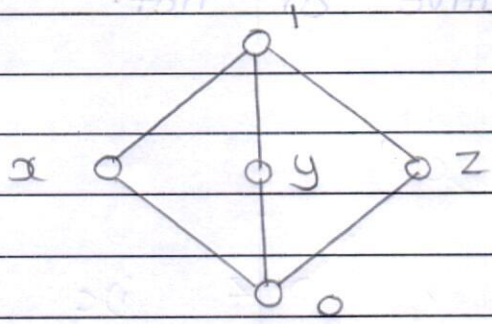

This is also not distributive lattice.

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* Boolean algebra :

It is a complimented and distributive lattice

It is given by $<B, *, \oplus, ', 0, 1>$ where the original lattice is $<B, *, \oplus>$ where $\oplus$ & $*$ are meet and join respectively.

0 and 1 are the bounds and ' (dash) represents there is compliment present. Therefore for a we have a'.

The boolean algebra $<B, *, \oplus, ', 0, 1>$ satisfies following properties in which $a, b, c \in B$

(1) $<B, *, \oplus>$ is a lattice in which $*$ and $\oplus$ satisfies.

$a * a = a$ $\qquad a \oplus a = a$

$a * b = b * a$ $\qquad a \oplus b = b \oplus a$

$a * (b * c) = (a * b) * c$ $\qquad a \oplus (b \oplus c) = (a \oplus b) \oplus c$

$a * (a \oplus b) = a$ $\qquad a \oplus (a * b) = a$

(2) $<B, *, \oplus>$ is a distributive lattice

$a * (b \oplus c) = (a * b) \oplus (a * c)$

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

③ let $\langle B, *, \oplus, 0, 1 \rangle$ is a bounded lattice, in which for any $a \in B$ $0 \leq a \leq 1$

$a * 0 = 0$ $a \oplus 0 = a$

$a * 1 = a$ $a \oplus 1 = 1$

④ $(\langle B, *, \oplus, ', 0, 1 \rangle$ is a complemented lattice if $a \in B$ then $a' \in B$

$a * a' = 0$ $a \oplus a' = 1$

$a * 0' = 1$ $a \oplus 1' = 0$

$(a * b)' = a' \oplus b'$ $(a \oplus b)' = a' * b'$

⑤ There exist partial ordering relation $\leq$ on set B such that

$a * b = \text{GLB}\{a, b\}$ $a \oplus b = \text{LUB}\{a, b\}$

$a \leq b \iff a * b = a$ $a \leq b \iff a * b' = 0$

$a \leq b \iff a \oplus b = b$ $b' \leq a \iff a' \oplus b = 1$

* Boolean algebra of power set

If the s be the nonempty set and $p(s)$ is the power set

① if $s = \{1\}$

$$p(s) = \{\phi, 1\}$$

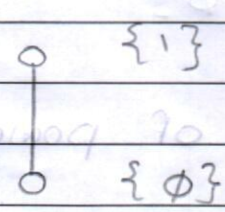

Fig: Boolean algebra of power set.

② when $s = \{a, b\}$

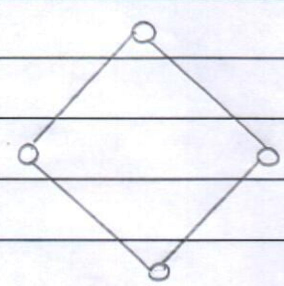

fig: Boolean algebra of power set

③ When $s = \{a, b, c\}$

$$p(s) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

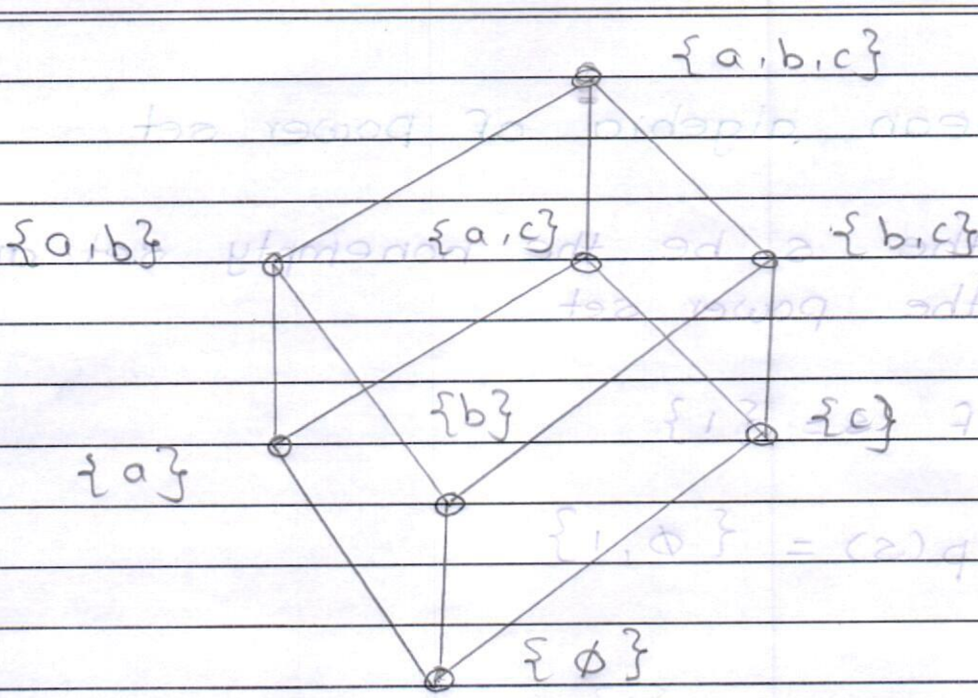

Fig: Boolean algebra of power set.

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* Boolean algebra :

Boolean algebra is an algebraic strcture defined on a set of elements is together with two binary operation. The product (or meet) and the sum (or join) provided the following Hunthingston postulates are satisfied.

① closure with respect to the operator +
closure with respect to the operator ⊙ x

② An identity element with respect to + is 0
$x+0 = 0+x$

An identity element with respect to · is 1
$x \cdot 1 = 1 \cdot x = x$

③ commutative with respect to +
$x+y = y+x$

commutative with respect to ·
$x \cdot y = y \cdot x$

④ · is distributive over +
$x(y+z) = xy + xz$

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$+$ is distributive over

$$x + (yz) = (x+y)(x+z)$$

⑤ for every element xB, there exist an $x'B$ (complement of x) such that.

$$x + x' = 1 \quad \text{and} \quad x \cdot x' = 0$$

⑥ There exists at least two elements x, y, B such that $x = y$.

Example:

show that the algebra subsets of a set and the algebra of propositions are boolean algebra.

→ let S be any set and $K = P(S)$ be the power set (and the set of all subsets) of S. Then K is the Boolean algebra. Where the meet is the intersection of two sets; the join is the union of the two sets ; the boolean is the complement of a set ; the zero element is the empty set; the unit is S.

Let L be the set of all propositions. Then L is a boolean algebra, where the meet is the conjuction of two proposition ; the join is the disjunction of two propositions ; the boolean complement is the negation of a proposition ; the zero element is a proposition F that is always false ; the unit element is a proposition T that is always true.

* Two valued boolean algebra :

A two valued boolean algebra is defined on a set of two elements. $B = \{0, 1\}$ with rules for the two binary operators + and ·

x	y	xy
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

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x	x'
0	1
1	0

These rules are exactly the same as the AND, OR, NOT operations respectively. Now show that the Hunthington postulates are valid for the set $B = \{0, 1\}$ and two binary operat[illegible] defined above

① closure :

The result of each operation is 1 or 0 and 1 · 0

② from the table an identity element with respect to + is 0 since $0+0=0$
$0+1 = 1+0 = 1$

An identity element with respect to · is 1 since $1 \cdot 1 = 1$
$1 \cdot 0 = 0 \cdot 1 = 0$

③ commutative law :

$0+1 = 1+0 = 1$ (for +)

$0 \cdot 1 = 1 \cdot 0 = 0$ (for ·)

④ Distributive law :

$\cdot$ is distributive over $+$: $x(y+z) = xy + xz$

$+$ is distributive over $\cdot$: $x + (y \cdot z) = (x+y) \cdot (x+z)$

x	y	z	$y+z$	$x\cdot(y+z)$	$x\cdot y$	$x\cdot z$	$xy+xz$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

⑤ from the complement table

$x + x' = 1$ since $0 + 0' = 0 + 1 = 1$
$1 + 1' = 1 + 0 = 1$

$x \cdot x' = 0$ since $0 \cdot 0' = 0 \cdot 1 = 0$
$1 \cdot 1' = 1 \cdot 0 = 0$

⑥ Two valued boolean algebra has two distinct element 1 and 0

* Theorems and properties of Boolean Algebra :

Theorem 1 :

a) $x + x = x$

proof :

$$
\begin{aligned}
x + x &= (x + x) \cdot 1 \quad \text{by postulate (2b)} \\
&= (x + x)(x + x') \\
&= (x + xx') \\
&= x + 0 \\
&= x
\end{aligned}
$$

b) $x \cdot x = x$

proof :

$$
\begin{aligned}
x \cdot x &= x \cdot x + 0 \\
&= x \cdot x + x \cdot x' \\
&= x(x + x') \\
&= x \cdot 1 \\
&= x
\end{aligned}
$$

Theorem 2 :

a) $x + 1 = 1$

proof :

$$
\begin{aligned}
x + 1 &= 1 \cdot (x + 1) \\
&= (x + x')(x + 1)
\end{aligned}
$$

Rajdhani
DATE / /

$$= x + x' \cdot 1$$
$$= x + x'$$
$$= 1$$

b) $x \cdot 0 = 0$ by duality

Theorem 3 :

$$(x')' = x$$

Theorem 4 :

a) $x + (y+z) = (x+y)+z$ (associative)

b) $x(yz) = (xy)z$ (associative)

Theorem 5 : (De Morgan)

a) $(x+y)' = x' \cdot y'$

b) $(xy)' = x' + y'$

proof :

x	y	$x+y$	$(x+y)'$	x'	y'	$x'y'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Theorem 6 : (Absorption)

a) $x + xy = x$

proof :

$$
\begin{aligned}
x + xy &= x \cdot 1 + xy && \text{by postulate 2 (b)} \\
&= x(1+y) && \text{by postulate 4 (a)} \\
&= x(y+1) && \text{by postulate 2 (a)} \\
&= x \cdot 1 && \text{by theorem 2 (a)} \\
&= x && \text{by postulate 2 (b)}
\end{aligned}
$$

b) $x(x+y) = x$

proof :

x	y	xy	$x+xy$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

* Boolean functions :

A binary variable can take value of 0 and 1

A boolean function is an expression formed with binary variables, the two binary operators OR and AND, the unary operator NOT, parentheses, equal sign. For a given value of the variables, the function can be either 0 or 1

Example :

$F_1 = xyz'$

$F_1 = 1$ if $x = 1$ and $y = 1$ and $z' = 1$
otherwise $F_1 = 0$

A boolean function may also be represented in a truth table. To represent a function in a truth table we need 2^n combinations of 1's and 0's of the n binary variables.

$F_2 = x + y'z$

$F_2 = 1$ if $x = 1$ or if $y = 0$ while $z = 1$

$F_3 = x'y'z' + x'yz + xy'$

$F_4 = xy' + x'z$

x	y	z	F_1	F_2	F_3	F_4
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

The number of rows : 2^n (n = number of binary variables)

A boolean function may be transformed from an algebraic expression into a logic diagram composed of AND, OR, NOT gates.

* Example :
Simplify the boolean function to a minimum number of literals.

① $x + x'y = (x + x')(x + y) = 1 \cdot (x + y) = x + y$

② $x(x' + y) = xx' + xy = 0 + xy = xy$

③ $x'y'z + x'yz + xy' = x'z(y' + y) + xy' = x'z + xy'$

④ $(x + y)(x'z)(y + z) = (x + y)(x' + z)$

(5) $xy + x'z + yz = xy + x'z + yz(x + x')$
$= xy + x'z + xyz + x'yz$
$= xy(1+z) + x'z(1+y)$
$= xy + x'z$

Complement of a function:

$(A+B+C)' = (A+X)'$ let $B+C = X$
$= A'X'$ De Morgan thm 5a
$= A'(B+C)'$
$= A' \cdot (B'C')$ De Morgan thm. 5a
$= A' \cdot B'C'$

Example: Find the complement of the function

→ $F_1 = x'yz' + x'y'z$ and $F_2 = x(y'z' + yz)$

$F_1' = (x'yz' + x'y'z)'$
$= (x'yz')'(x'y'z)'$
$= (x + y' + z)(x + y + z')$
$= \Gamma$

$F_2' = [x(y'z' + yz)]'$
$= x' + (y'z') + (yz)'$
$= x' + (y'z') \cdot (yz)'$
$= x' + (y+z)(y'+z')$

Rajdhani
DATE / /

Example:
Find the complement of the functions F_1 and F_2 by taking their duals and complementing each literal

→ $F_1 = x'yz' + x'y'z$

The dual of F_1 is: $(x'+y+z')(x'+y'+z)$

complement each literal: $F_1' = (x+y'+z)(x+y+z')$

$F_2 = x(y'z' + yz)$

The dual of F_2 is $x + (y'+z')(y+z)$

complement each literal: $F_2' = x' + (y+z)(y'+z')$

Rajdhani
DATE / /

* Canonical and standard forms :

A binary variable may appear in its normal form (x) or in its complement form (x')

Two binary variables x and y combined with an AND operation.

Four possible combination : $x'y'$
$x'y$
xy'
xy

Each of these terms are called MINTERM (Standard Product)

n variables can be combined to form 2^n minterms

Each minterm is obtained from an AND term of the n variable with each variable being primed if the corresponding bit of the binary number is a 0 and unprimed if a 1.

n variables forming an OR term. with each variable being primed or unprimed. provide 2^n possible combinations called MAXTERMS (or standard sums)

Each MAXTERMS is obtained from an OR term of the n variables, with each variable being unprimed if the corresponding bit is 0 and primed if a 1.

x	y	z	Minterms term	Designa -tion	Maxterms Term	Designati -on
0	0	0	$x'y'z'$	m_0	$x+y+z$	M_0
0	0	1	$x'y'z$	m_1	$x+y+z'$	M_1
0	1	0	$x'yz'$	m_2	$x+y'+z$	M_2
0	1	1	$x'yz$	m_3	$x+y'+z'$	M_3
1	0	0	$xy'z'$	m_4	$x'+y+z$	M_4
1	0	1	$xy'z$	m_5	$x'+y+z'$	M_5
1	1	0	xyz'	m_6	$x'+y'+z$	M_6
1	1	1	xyz	m_7	$x'+y'+z'$	M_7

A Boolean function may be expressed an algebraically from a given truth table by forming a MINTERM for each combination of the variable which produces a 1 in function, than taking the OR of all these terms.

Example :

x	y	z	function f_1	function f_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	

The function f_1 is determined by expressing the combinations 001, 100, 111 as $x'y'z$, $xy'z$, $xy'z'$, xyz

since each one of these minterms results in $f_1 = 1$

$$f_1 = x'y'z + xy'z' + xyz = m_1 + m_4 + m_7$$

$$f_2 = x'yz + xy'z + xyz' + xyz = m_3 + m_5 + m_6 + m_7$$

Note 1 : Any boolean function can be expressed as a sum of minterms (SUM - ORing of terms)

Complement: Read from the truth table by forming a MINTERM for each combination that produces a 0 in the function and then ORing those terms.

$$f_1' = x'y'z' + x'yz' + x'yz + xy'z + xyz'$$

If we take complement of f_1' we obtain the function f_1.

$$f_1 = (x+y+z)(x+y'+z)(x+y'+z')(x'+y+z')\cdot(x'+y'+z)$$

$$= m_0 \cdot m_2 \cdot m_3 \cdot m_5 \cdot m_6$$

Note 2:

Any boolean function can be expressed as a product of MAXTERMS (product ANDing of terms)

Procedure for obtaining the product of MAXTERMS from the truth table is:

Form a maximum for each combination of the variable which produces a 0 in the function

form the AND of all those maxterms

Rajdhani
DATE / /

Boolean functions expressed as a SUM OF MINTERMS or product of MAXTERMS are said to be in canonical form.

Sum of minterms:

It is sometimes convenient to express a boolea function in its sum of minterm form. To do this we should have each term with all variables

To expand the terms with missing variables we should AND these terms by $(x+x')$ where x is the missing variable.

Examples:

Express $F = A + B'C$ as a sum of minterms.

$$f = A + B'C$$

$$= A(B+B') + (A+A')B'C$$

$$= AB + AB' + AB'C + A'B'C$$

Note the terms AB and AB' are still back the variable C, we will expand them by $(C+C')$

$$= AB(C+C') + AB'(C+C') + AB'C + A'B'C$$

$$= ABC + ABC' + AB'C + ABC' + AB'C + A'B'C$$

Rajdhani
DATE / /

$= m_7 + m_6 + m_5 + m_4 + m_1$

$= (1, 4, 5, 6, 7)$

Product of MAXTERMS :

To express the boolean function as a product of MAXTERMs it must first, be brought into a form of OR terms. This may be done by using the distributive law :

$x + yz = (x+y)(x+z)$

Then any missing variable x in each OR term is OR ed with xx'.

Example :

Express $f = xy + x'z$ in product of maxterm form.

$f = (xy + x'z)$

$= (x' + xy)(z + xy)$

$= (x + x')(x' + y)(z + x)(z + y)$

$= (1)(x' + y)(x + z)(y + z)$

$= (x' + y + zz')(x + z + yy')(x + z + y')(y + z + x)$
$(y + z + x')$

Rajdhani
DATE / /

Reduce the terms which appear twice

$= (x' + y + z)(x' + y + z')(x + z + y')(x + z + y')$

$= (100 \quad 101 \quad 000 \quad 010)$

$= (0, 2, 4, 5)$

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DATE / /