PoPL Assignment 2

19CS30005

Aryan Agarwal

① ⓐ $(\lambda z.z)(\lambda y.yy)(\lambda x.xa)$

$\Rightarrow (\lambda z.z)(\lambda y.yy)(\lambda x.xa)$

[∵ β-reduction replace z with $\lambda y.yy$]

$\Rightarrow (\lambda y.yy)(\lambda x.xa)$

[∵ β-reduction replace y with $\lambda x.xa$]

$\Rightarrow (\lambda x.xa)(\lambda x.xa)$

[∵ β-reduction replace x with $\lambda x.xa$]

$\Rightarrow (\lambda x.xa)a$

[∵ β-reduction replace x with a]

$\Rightarrow a\ a$

ⓑ $(\lambda z.z)(\lambda z.zz)(\lambda z.zy)$

$\Rightarrow (\lambda z.z)(\lambda z.zz)(\lambda z.zy)$

[∵ β reduction replace z with $\lambda z.zz$]

$\Rightarrow \lambda z.zz\ (\lambda z.zy)$

[∵ β-reduction replace z with $\lambda z.zy$]

$\Rightarrow (\lambda z \cdot z y)(\lambda y \cdot z y)$

[$\because$ β-reduction replace z with $\lambda y \cdot z y$]

$\Rightarrow (\lambda z \cdot z y) y$

[$\because$ β-reduction replace z with y]

$\Rightarrow y y$

ⓒ $(\lambda x \cdot \lambda y \cdot x y y)(\lambda a \cdot a) b$

$\Rightarrow (\lambda x \cdot \lambda y \cdot x y y)(\lambda a \cdot a) b$

[$\because$ β-reduction replace x with $\lambda a \cdot a$]

$\Rightarrow (\lambda y \cdot (\lambda a \cdot a) y y) b$

[$\because$ β-reduction replace y with b]

$\Rightarrow (\lambda a \cdot a) b b$

[$\because$ β-reduction replace a with b]

$\Rightarrow b b$

ⓓ $(\lambda x \cdot \lambda y \cdot x y y)(\lambda y \cdot y) y$

$\Rightarrow (\lambda x \cdot \lambda y \cdot x y y)(\lambda y \cdot y) y$

[$\because$ α-conversion rename y with a]

$\Rightarrow (\lambda x \cdot \lambda a \cdot x a a)(\lambda y \cdot y) y$

$\because$ [β-reduction replace x with $\lambda y \cdot y$]

$\Rightarrow (\lambda a \cdot (\lambda y \cdot y)\, a a)\, y$

[$\because$ β-reduction replace a with y]

$\Rightarrow (\lambda y \cdot y)\, y y$

[$\because$ β-reduction replace y with y]

$\Rightarrow y y$

(e) $(\lambda x \cdot (\lambda y \cdot (x y))\, y)\, z$

$\Rightarrow (\lambda x \cdot (\lambda y \cdot (x y))\, y)\, z$

[$\because$ α-conversion rename y to a]

$\Rightarrow [\lambda x \cdot (\lambda a \cdot (x a))\, y)\, z$

[$\because$ β-reduction replace x with z]

$\Rightarrow (\lambda a \cdot (z a))\, y$

[$\because$ β-reduction replace a with y]

$\Rightarrow z y$

(f) $(\lambda x \cdot x x)(\lambda y \cdot y x)\, z$

$\Rightarrow (\lambda x \cdot x x)\, (\lambda y \cdot y x)\, z$

[$\because$ β-reduction replace x with $\lambda y \cdot y x$]

$\Rightarrow (\lambda y. y x)\ (\lambda y. y x)\ z$

[$\because$ β-reduction replace y with $\lambda y.yx$]

$\Rightarrow (\lambda y. y x)\ x\ z$

[$\because$ β-reduction replace y with x]

$\Rightarrow x\ x\ z$

② ⓐ def add f x y
if iszero y
then x
else f (succ x) (pred y)

ⓑ $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$

$Y\,add = add\ (Y\ add)$

$= \lambda f.\lambda x.\lambda y.($if iszero y then x else $f\ (succ\ x)\ (pred\ y))\ (Y\ add)$

[$\because$ β-reduction replace f with $(Y\ add)$]

$= \lambda x.\lambda y.($if iszero y then x else $(Y\ add)\ (succ\ x)\ (pred\ y))$

The function we derived takes x and y as parameters. We have to make it concrete and we will do that by evaluating above function with concrete values 1 and 1

$Y_{add}\ 1\ 1 = add(Y_{add})\ 1\ 1$

$= \lambda f. \lambda x. \lambda y. (\text{if iszero } y \text{ then } x \text{ else}$
$f\ (succ\ x)\ (pred\ y))\ \underline{(Y_{add})}\ 1\ 1$

[∴ Substitution & β-reduction replace f with Y_{add}]

$= \lambda x. \lambda y\ (\text{if iszero } y \text{ then } x \text{ else } (Y_{add}$
$(succ\ x)\ (pred\ y))\ 1\ 1$

[∴ β-reduction replace x with 1]

$\Rightarrow \lambda y\ (\text{if iszero } y \text{ then } 1 \text{ else } Y_{add}\ (succ\ 1)\ (pred\ y))\ 1$

[∴ β-reduction replace y with 1]

$= (\text{if iszero } 1 \text{ then } 1 \text{ else } Y_{add}(succ\ 1)\ (pred\ 1)).$
$= (\text{if iszero } 1 \text{ then } 1 \text{ else } Y_{add}\ 2\ 0)$

$\left[\begin{array}{l} \therefore\ pred\ 1 = 0 \\ succ\ 1 = 2 \end{array}\right]$

$= Y_{add}\ 2\ 0$
$= add\ (Y_{add})\ 2\ 0$
$= \lambda f. \lambda x. \lambda y\ (\text{if iszero } y \text{ then } x \text{ else}$
$f\ (succ\ x)\ (pred\ y))\ Y_{add}\ 2\ 0$

$= \lambda x.\lambda y.$ (if iszero y then x else Y add
(succ x (pred y))) 2 0

$\therefore$ [by βreduction replacing f with Yadd].

= (if iszero 0 then 2 else Y add (succ 2)
(pred 0))

$\therefore$ (by (i) β-reduction replacing x with 2
(ii) β-reduction replacing y with 0)

= (if iszero 0 then 2 else Yadd 3 - 1)

= 2

③ $((((\lambda f.(\lambda g.(\lambda x.((f\,x)(g\,x)))))(\lambda m.(\lambda n.(n\,m))))(\lambda n.3))\,p)$

[$\therefore$ β-reduction replace f with $(\lambda m.(\lambda n.(n\,m)))$]

$\Rightarrow (((\lambda g.(\lambda x.(((\lambda m.(\lambda n.(n\,m)))x)(g\,x))))(\lambda n.3))\,p)$

[$\therefore$ β reduction g with $(\lambda n.3)$]

$\Rightarrow ((\lambda x.(((\lambda m.(\lambda n.(n\,m)))x)((\lambda n.3)\,x)))\,p)$

[$\therefore$ β-reduction replace n with x in]

$\Rightarrow ((\lambda x.(((\lambda m.(\lambda n.(n\,m)))x)\,z))\,p)$

[$\because$ β-reduction replace m with x in]

$\Rightarrow ((\lambda x.((\lambda n.(n\,x))\,z))\,p)$

[$\because$ β-reduction replace n with z]

$\Rightarrow ((\lambda x.(z\,x))\,p)$

[$\because$ β-reduction replace x with p]

$\Rightarrow z\,p$