Problem 1)

CFL Condition: $\frac{dx}{dt} \geq v$

$$\frac{f(t+dt,x) - f(t-dt,x)}{2dt} =$$

$$-v \frac{f(t,x+dx) - f(t,x-dx)}{2dx}$$

Solution : $f(x,t) = \xi^t \exp(ikx)$

ξ: function of k.

If we want our solution to be stable, $|\xi| = 1$, so we should try to find ξ (by putting the solution in the equation) and apply this condition to find what happens.

$$f(t+dt,x) = \xi^{t+dt} e^{ikx}$$

$$f(t-dt,x) = \xi^{t-dt} e^{ikx}$$

$$\Rightarrow \frac{\xi^{t+dt} e^{ikx} - \xi^{t-dt} e^{ikx}}{2dt} = -v \frac{\xi^t e^{ik(x+dx)} - \xi^t e^{ik(x-dx)}}{2dx}$$

$$\xi^{t}\,\frac{\xi^{dt}e^{ikx}-\xi^{-dt}e^{ikx}}{2dt}=-v\,\xi^{t}\,\frac{e^{ik(x+dx)}-e^{ik(x-dx)}}{2dx}$$

$$e^{ikx}\,\frac{\xi^{dt}-\xi^{-dt}}{2dt}=-v\,e^{ikx}\,\frac{e^{ikdx}-e^{-ikdx}}{2dx}$$

$$\frac{\xi^{dt}-\xi^{-dt}}{2dt}=-v\,\frac{e^{ikdx}-e^{-ikdx}}{2dx}$$

we know that : $\sin k dx = \dfrac{e^{ikdx}-e^{-ikdx}}{2i}$

$$\Rightarrow \frac{\xi^{dt}-\xi^{-dt}}{2dt}=-\frac{vi\sin k dx}{dx}$$

$$\xi^{dt}-\xi^{-dt}=-\frac{2dt}{dx}\,vi\sin k dx$$

$$\Rightarrow \xi^{2dt}-1=-\frac{2dt}{dx}\,vi\sin(k\,dx)\,\xi^{dt}$$

If this equation works for all dt, it should also work for dt = 0. Therefore, as a simplification (to be able to solve it) I assume dt to be 1.

$$\xi^2 + 2\nu i \frac{dt}{dx} \sin(kdx)\xi - 1 = 0$$

This is a second order equation and it is easy to solve for ξ:

$$\Delta = -4\nu^2 \frac{dt^2}{dx^2} \sin^2 kdx + 4$$

$$\xi = \frac{1}{\cancel{2}}\left[-\cancel{2}\nu i \frac{dt}{dx}\sin(kdx) \pm \cancel{2}\sqrt{-\nu^2\frac{dt^2}{dx^2}\sin^2 kdx + 1}\right]$$

$$\xi = \underbrace{-\nu i \frac{dt}{dx}\sin kdx}_{a} \pm \underbrace{\sqrt{1 - \nu^2\frac{dt^2}{dx^2}\sin^2 kdx}}_{b}$$

As I told previously, we need $|\xi|$ to be 1.

$$\xi = -ia \pm b \qquad \xi^* = ia \pm b$$

$$\xi\xi^* = a^2 \cancel{\mp iab} \cancel{\pm iab} + b^2$$

$$\rightarrow |\xi|^2 = a^2 + b^2 = \cancel{\nu^2\frac{dt^2}{dx^2}\sin^2 kdx} + 1 - \cancel{\nu^2\frac{dt^2}{dx^2}\sin^2 kdx}$$

$$\rightarrow |\xi|^2 = 1$$

It is always satisfied!

I may have done a mistake somewhere in my calculations, but I could not find it.