a)

$F(k) = \sum f(x) \exp(-2\pi i k x / N)$

we should take the Fourier Transform of Correlation, so we have:

$$F(k) = \sum_{x=0}^{N-1} ((C - |x|) \exp(-2\pi i k x / N)$$

$$= \underbrace{\sum_{x=0}^{N-1} C \exp\left(-2\pi i \frac{kx}{N}\right)}_{C \frac{1 - \exp(-2\pi i k)}{1 - \exp\left(-2\pi i \frac{k}{N}\right)}} - \sum_{x=0}^{N-1} x \exp(-2\pi i kx/N)$$

but we have to calculate the second term with this trick:

$$\sum_{x=0}^{N-1} x a^x = a \frac{\partial}{\partial a} \sum_{x=0}^{N-1} a^x =$$

$$a \frac{\partial}{\partial a} \left(\frac{1 - a^N}{1 - a} \right) = a \frac{a^N (N-1) - N a^{N-1} + 1}{(1-a)^2}$$

Let's calculate!

$$F(k) = \frac{c(1-\exp(-2\pi i k))}{1-\exp(-2\pi i \frac{k}{N})} + \underbrace{\frac{-a^{N+1}(N-1)+Na^{N}-a}{(1-a)^{2}}}$$

$$\frac{-(N-1)\exp(-2\pi i k(1+\frac{1}{N})) + N\exp(-2\pi i k) - e^{\frac{-2\pi i k}{N}}}{(1-\exp(-2\pi i \frac{k}{N}))^{2}}$$

$$F(k) = \frac{c - c e^{\frac{-2\pi i k}{N}} - c e^{-2\pi i k} + c e^{-2\pi i k(1+\frac{1}{N})}}{(1-\exp(-2\pi i \frac{k}{N}))^{2}}$$

$$\frac{-(N-1)\exp(-2\pi i k(1+\frac{1}{N})) + N\exp(-2\pi i k) - \exp(-\frac{2\pi i k}{N})}{(1-\exp(-2\pi i \frac{k}{N}))^{2}}$$

$$c - c e^{-\frac{2\pi i k}{N}} - c e^{-2\pi i k} + c e^{-2\pi i k(1+\frac{1}{N})}$$

$$-N e^{-2\pi i k(1+\frac{1}{N})} + e^{-2\pi i k(1+\frac{1}{N})} + N e^{-2\pi i k} - e^{-\frac{2\pi i k}{N}}$$

$$\rightarrow e^{-2\pi i k/N}(-c-1) + e^{-2\pi i k}(-c+N)$$

$$+ e^{-2\pi i k(1+1/N)}(c-N+1) + c$$

$$\cancel{F(k) = e^{-2\pi i k/N}(-c-1) + e^{-2\pi i k}(-c+N)}$$

$$F(k) = \frac{e^{-2\pi i k/N}(-c-1) + e^{-2\pi i k}(N-c)}{(1-\exp(-2\pi i \frac{k}{N}))^2}$$

$$+ \frac{e^{-2\pi i k(1+1/N)}(c-N+1) + c}{(1-\exp(-2\pi i \frac{k}{N}))^2}$$

$$P = |F(k)|^2 =$$

$$(1-\exp(-2\pi i \tfrac{k}{N}))^{-2}(1-\exp(2\pi i \tfrac{k}{N}))^{-2}$$

$$\left(e^{-2\pi i \frac{k}{N}}(-c-1) + e^{-2\pi i k}(N-c) + e^{-2\pi i k(1+\frac{1}{N})}(c-N+1) + c\right)$$

$$\times\left(e^{2\pi i \frac{k}{N}}(-c-1) + e^{2\pi i k}(N-c) + e^{2\pi i k(1+\frac{1}{N})}(c-N+1) + c\right)$$

$$\rightarrow \left[1 - e^{2\pi i \frac{k}{N}} - e^{-2\pi i \frac{k}{N}} + 1\right]^{-2} \times$$

$$\Big[(c+1)^2 - (N-c)(c+1)e^{2\pi i k(1-\frac{1}{N})} - (c+1)(c-N+1)e^{2\pi i k}$$

$$- c(c+1)e^{-2\pi i \frac{k}{N}} - (c+1)(N-c)e^{2\pi i k(\frac{1}{N}-1)} + (N-c)^2$$

$$+ (N-c)(c-N+1)e^{2\pi i k/N} + c(N-c)e^{-2\pi i k}$$

$$-(c+1)(c-N+1)e^{-2\pi i k} + (c-N+1)(N-c)e^{-2\pi i k/N}$$

$$+ (c-N+1)^2 + c(c-N+1)e^{-2\pi i k(1+\frac{1}{N})}$$

$$- c e^{2\pi i \frac{k}{N}}(c+1) + c(N-c)e^{2\pi i k} + c(c-N+1)e^{2\pi i k(1+\frac{1}{N})}$$

$$+ c^2\Big]$$

$$P_k = \left[2 - e^{2\pi i \frac{k}{N}} - e^{-2\pi i \frac{k}{N}}\right]^{-2}_{x}$$

$$\Big[c^2 + (c+1)^2 + (N-c)^2 + (c-N+1)^2$$

$$- (c+1)(N-c)\left(e^{2\pi i k(1-\frac{1}{N})} + e^{-2\pi i k(1-\frac{1}{N})}\right)$$

$$+ e^{2\pi i k}\big(-(c+1)(c-N+1) + c(N-c)\big)$$

$$+ e^{-2\pi i k}\big(-(c+1)(c-N+1) + c(N-c)\big)$$

$$+ e^{-2\pi i \frac{k}{N}}\big(-c(c+1) + (c-N+1)(N-c)\big)$$

$$+ e^{2\pi i k/N}\big(-c(c+1) + (c-N+1)(N-c)\big)$$

$$+ e^{2\pi i k(1+\frac{1}{N})}(c-N+1)c$$

$$+ e^{-2\pi i k(1+\frac{1}{N})}(c-N+1)c\Big]$$

$$P_k = \Big[4c^2 - 4cN + 2 + 2N^2 - 2N$$

$$+ 2(c+1)(N-c)\cosh\left(2\pi i k\left(1 - \tfrac{1}{N}\right)\right)$$

$$+ 2\left(-2c^2 + (N-1)(2c-1)\right)\cosh(2\pi i k)$$

$$+ 2\left(-2c^2 + 2cN - 2 + N - N^2\right)\cosh\left(2\pi i \tfrac{k}{N}\right)$$

$$+ 2\left(c^2 - cN + c\right)\cosh\left(2\pi i k\left(1 + \frac{1}{N}\right)\right)\Big]$$

$$\times \left[2 - 2\cosh\left(2\pi i \frac{k}{N}\right)\right]^{-2}$$

we know that :

$$k \to 0 \qquad \cosh k \sim 1 + \frac{k^2}{2} + \dots$$

$$P_k = \Big[\cancel{4c^2} - \cancel{4cN} + \cancel{2} + \cancel{2N^2} - \cancel{2N} + \cancel{2cN} - \cancel{2c^2} + \cancel{2N} - \cancel{2c}$$

$$- \cancel{4c^2} + \cancel{4cN} - 4c + \cancel{2N} - \cancel{2}$$

$$-4c^2 + 4cN - 4 + \cancel{2N} - \cancel{2N^2} + \cancel{2c^2} - \cancel{2cN} + \cancel{2c}$$

$$- 4\pi^2 k^2 \left(1 - \tfrac{1}{N}\right)^2 \times 2(c+1)(N-c)$$

$$- 4\pi^2 k^2 \times 2\left(-2c^2 + (N-1)(2c-1)\right)$$

$$- 4\pi^2 \frac{k^2}{N^2} \times 2\left(-2c^2 + 2cN - 2 + N - N^2\right)$$

$$- 4\pi^2 k^2 \left(1 + \frac{1}{N}\right)^2 \times 2\left(c^2 - cN + c\right)$$

$$\rightarrow \left[8\pi^2 \frac{k^2}{N^2}\right]^{-2}$$

$$\times \left[2-2\left(1-\frac{4\pi^2 k^2}{N^2}\right)\right]^{-2}$$

$$\rightarrow P_k = \left[8\pi^2\frac{k^2}{N^2}\right]^{-2} \times$$

$$\Big[-4c-4c^2+4cN-4$$

$$-\left(2cN-2c^2+2N-2c\right)4\pi^2k^2\left(1+\frac{1}{N^2}-\frac{2}{N}\right)$$

$$-\left(-4c^2+4cN-4c+2N-2\right)4\pi^2k^2$$

$$-\left(-4c^2+4cN-4+2N-2N^2\right)\frac{4\pi^2k^2}{N^2}$$

$$-\left(2c^2-2cN+2c\right)4\pi^2k^2\left(1+\frac{1}{N^2}+\frac{2}{N}\right)\Big]$$

$4\pi^2k^2$	$4\pi^2k^2/N^2$	$4\pi^2k^2/N$
$-2cN+2c^2-2N+2c$	$-2cN+2c^2-2N+2c$	$+4cN-4c^2+4N-4c$
$4c^2-4cN+4c-2N+2$	$4c^2-4cN+4$	$-4c^2+4cN-4c$
$-2c^2+2cN-2c$	$-2N+2N^2$	
	$-2c^2+2cN-2c$	

Another way:

we know that:

$k \to 0 \quad e^{x} \sim 1 + x + \cdots$

$$\to F(k) = \frac{-(C+1)(1-2\pi i k/N) + (N-C)(1-2\pi i k)}{(1-1+2\pi i k/N)^2}$$

$-N2\pi i k$

$$\to F(k) = \frac{-C-1+2C\pi i k/N + 2\pi i k/N + N - C + 2\pi i C k}{-4\pi^2 \frac{k^2}{N^2}}$$

$$F(k) = \frac{2\pi i k(C/N + 1/N + C - N) - C - 1 + N - C}{-4\pi^2 k^2/N^2}$$

$$F(k) = \frac{2\pi i k(C(1+1/N) + N(-1+1/N^2)) - 2C + N - 1}{-4\pi^2 \frac{k^2}{N^2}}$$

$$P_k = |F(k)|^2 = \left(4\pi^2 \frac{k^2}{N^2}\right)^{-2} \times$$

$$(2\pi i k(C + 1/N) + N(-1 + 1/N^2) - 2C + N - 1)$$

$$\times(-2\pi i k(C + 1/N) + N(-1 + 1/N^2) - 2C + N - 1)$$

$$P_k = (4\pi^2 k^2 (C + 1/N)^2 + 2\pi i k N (C + 1/N)(-1 + 1/N^2)$$