Strategy Proof Voting Mechanisms

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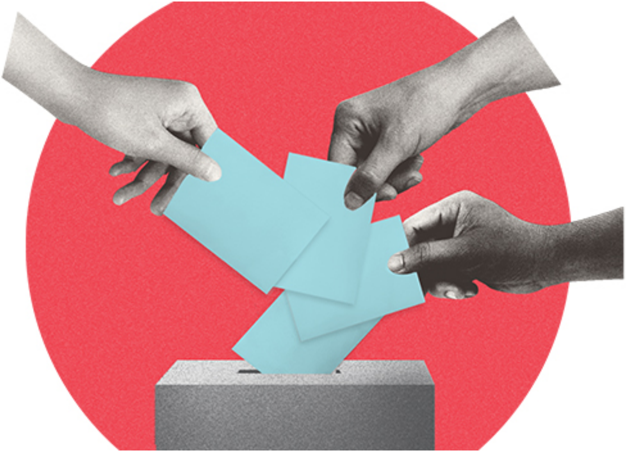
Outline

A Proof Beyond Words

THEOREM 1 (Groves and Loeb [5]): *A Groves mechanism is s.i.i.c.*

PROOF: For any $w_{-i}(\cdot) \in V_{-i}$ and any $w_i(\cdot) \in V_i$, $u_i(w_{-i}(\cdot),\ v_i(\cdot);\ GM) - u_i(w_{-i}(\cdot),\ w_i(\cdot);\ GM) = v_i(K^*(w_{-i}(\cdot),\ v_i(\cdot))) + \Sigma w_{-i}(K^*(w_{-i}(\cdot),\ v_i(\cdot))) + h_i(w_{-i}(\cdot)) - v_i(K^*(w_{-i}(\cdot),\ w_i(\cdot))) - \Sigma w_{-i}(K^*(w_{-i}(\cdot),\ w_i(\cdot))) - h_i(w_{-i}(\cdot)) = \max_{K \in \mathscr{K}}[v_i(K) + \Sigma w_{-i}(K)] - [v_i(K^*(w_{-i}(\cdot),\ w_i(\cdot))) + \Sigma w_{-i}(K^*(w_{-1}(\cdot),\ w_i(\cdot)))] \geqslant 0.$ *Q.E.D.*

Motivation



I_n: Voting Committee

$I_n = \{$All Voting Eligible American$\}$ (US Electorate)

$I_2 = \{$Rafe, Rob$\}$ (Aryan's Comps Committee)

$I_{20} = \{$Carleton Mathematics Professors$\}$ (Math Department)

All $i \in I_n$ have preferences R_i over S_m

Setting Up the Problem: Alternatives

S_m: Set of Alternatives

$S_3 = \{$Democrat, Republican, Independent$\}$ (US President)

$S_2 = \{$Pass, Fail$\}$ (Aryan's Comps)

$S_{20} = \{$Halloween Candies$\}$ (Halloween Candy)

Setting Up the Problem: Voting

Each voter casts a ballot, $\mathbf{B_i}$, which is a weak ordering of the alternatives:

$$X > Y > Z$$

$$Y > X \geq Z$$

$$Z \geq Y > X$$

Ballot Sets

$$\pi_m = \{\text{All possible ballots}\}$$

Assume $m = 3$ and $S_3 = \{X, Y, Z\}$

$$\begin{aligned}\pi_3 = \{&(X > Y > Z), (X > Z > Y), (Y > X > Z),\\ &(Y > Z > X), (Z > X > Y), (Z > Y > X),\\ &(X \geq Y > Z), ..., (X \geq Y \geq Z), ..., (Z \geq Y \geq X)\}\end{aligned}$$

Ballot Sets

$$\pi_m^n = \{\text{All possible ballots for all voters}\}$$

Assume $m = 3$, $n = 5$, and $S_3 = \{X, Y, Z\}$

$$\begin{aligned}\pi_3^5 = \{&\{B_1, B_2, B_3, B_4, B_5\},\\ &\{B_1^*, B_2^*, B_3^*, B_4^*, B_5^*\},\\ &\{B_1', B_2', B_3', B_4', B_5'\}, \ldots\}\end{aligned}$$

where $B_i \in \pi_m$

Social Choice Function

The **social choice function ($\mathbf{u^{nm}}$)** maps the collection of individual ballots to a **social choice**: a societal weak ordering of alternatives:

$$u^{nm} : B \in \pi_m^n \to \pi_m$$

$$B = (B_1, ..., B_n)$$

Social Choice Function Example

$$u^{5,3}(B) = u^{5,3}\begin{pmatrix} B_1: & X > Y > Z \\ B_2: & X > Y > Z \\ B_3: & X > Y > Z \\ B_4: & X > Y > Z \\ B_5: & X > Y > Z \end{pmatrix} = (X > Y > Z) \in \pi_m$$

where $B = \{B_i\}, 1 < i < n, B_i \in \pi_3$

Arrow's Impossibility Theorem

Theorem

For $n > 2$ and $m > 3$ any social choice function that obeys Rationality, Pareto Optimality, and Independence of Irrelevant Alternatives, must be Dictatorial.

Arrow's Impossibility Theorem Example

$$I_3 = \{i_1, i_2, i_3\} \qquad S_3 = \{X, Y, Z\}$$

$$B_1 : X > Y > Z$$
$$B_2 : Y > Z > X$$
$$B_3 : Z > X > Y$$

Arrow's Impossibility Theorem Example

$$I_3 = \{i_1, i_2, i_3\} \qquad S_3 = \{X, Y, Z\}$$

$$B_1 : \mathbf{X} > \mathbf{Y} > Z$$
$$B_2 : Y > Z > X$$
$$B_3 : Z > \mathbf{X} > \mathbf{Y}$$

Arrow's Impossibility Theorem Example

$$X > Y$$

Arrow's Impossibility Theorem Example

$$I_3 = \{i_1, i_2, i_3\} \qquad S_3 = \{X, Y, Z\}$$

$$i_1 : X > \mathbf{Y} > \mathbf{Z}$$
$$i_2 : \mathbf{Y} > \mathbf{Z} > X$$
$$i_3 : Z > X > Y$$

Arrow's Impossibility Theorem Example

$$X > Y > Z$$

Arrow's Impossibility Theorem Example

$$X > Y > Z \implies X > Z$$

Arrow's Impossibility Theorem Example

$$I_3 = \{i_1, i_2, i_3\} \qquad S_3 = \{X, Y, Z\}$$

$$\begin{aligned} i_1 &: X > Y > Z \\ i_2 &: Y > \mathbf{Z} > \mathbf{X} \\ i_3 &: \mathbf{Z} > \mathbf{X} > Y \end{aligned}$$

Arrow's Impossibility Theorem

Theorem

For $n > 2$ and $m > 3$ any social choice function that obeys **Rationality**, *Pareto Optimality, and Independence of Irrelevant Alternatives, must be Dictatorial.*

Rationality/Universality

Definition (Rationality)

A social choice function should account for all individual preferences and provide the same ranking every time voter's preferences are presented the same way.

Arrow's Impossibility Theorem

Theorem

For $n > 2$ and $m > 3$ any social choice function that obeys Rationality, **Pareto Optimality**, *and Independence of Irrelevant Alternatives, must be Dictatorial.*

Pareto Optimality

Definition (Pareto Optimality)

If all *voters* rank alternative X before alternative Y, the social choice function should provide a ranking that has X ranked before Y.

$$u^{5,3}(B) = u^{5,3}\begin{pmatrix} B_1: & X > Y > Z \\ B_2: & X > Y > Z \\ B_3: & X > Y > Z \\ B_4: & X > Y > Z \\ B_5: & X > Y > Z \end{pmatrix} = (X > Y > Z) \in \pi_m$$

Arrow's Impossibility Theorem

Theorem

For $n > 2$ and $m > 3$ any social choice function that obeys Rationality, Pareto Optimality, and **Independence of Irrelevant Alternatives** *must be Dictatorial.*

Independence of Irrelevant Alternatives (IIA)

Definition (Independence of Irrelevant Alternatives)

$$u^{5,3}(B) = u^{5,3}\begin{pmatrix} B_1: & X > Y > Z \\ B_2: & X > Y > Z \\ B_3: & X > Y > Z \\ B_4: & X > Y > Z \\ B_5: & X > Y > Z \end{pmatrix} : (X > Z)$$

$$u^{5,3}(B') = u^{5,3}\begin{pmatrix} B'_1: & X > Z > Y \\ B'_2: & X > Z > Y \\ B'_3: & X > Z > Y \\ B'_4: & X > Z > Y \\ B'_5: & X > Z > Y \end{pmatrix} : (X > Z)$$

Arrow's Impossibility Theorem

Theorem

For $n > 2$ and $m > 3$ any social choice function that obeys Rationality, Pareto Optimality, and Independence of Irrelevant Alternatives must be Dictatorial.

Dictatorial

Definition (Dictatorial)

A social choice function selects the ranking of one particular voter as the social choice.

$$\begin{aligned} i_1 &: X > Y > Z \\ i_2 &: Y > Z > X \;\rightarrow (X > Y > Z) \\ i_3 &: Z > X > Y \end{aligned}$$

Arrow's Impossibility Theorem

Theorem

For $n > 2$ and $m > 3$ any social choice function that obeys Rationality, Pareto Optimality, and Independence of Irrelevant Alternatives must be **Dictatorial**.

Strategy Proof Voting Mechanisms:

Voting Mechanism

The **voting mechanism ($\mathbf{v^{nm}}$)** maps the collection of individual ballots to a **committee choice**, a single alternative $X \in S_m$:

$$v^{nm} : B \in \pi_m^n \rightarrow X \in S_m$$

$$B = (B_1, ..., B_n)$$

Social Choice Function Example

$$u^{5,3}(B) = u^{5,3}\left(\begin{array}{ll} B_1: & X > Y > Z \\ B_2: & X > Y > Z \\ B_3: & X > Y > Z \\ B_4: & X > Y > Z \\ B_5: & X > Y > Z \end{array}\right) = (X > Y > Z) \in \pi_m$$

Voting Mechanism Example

$$v^{5,3}(B) = v^{5,3}\begin{pmatrix} B_1: & X > Y > Z \\ B_2: & X > Y > Z \\ B_3: & X > Y > Z \\ B_4: & X > Y > Z \\ B_5: & X > Y > Z \end{pmatrix} = (X) \in S_m$$

Strategy Proof Voting Mechanism

Definition (Sincere Strategy)

A voter, i, employs a sincere strategy when $B_i = R_i$

Definition (Sophisticated Strategy)

A voter, i, employs a sophisticated strategy when $B_i \neq R_i$

Definition (Strategy Proof Voting Mechanism)

A voting mechanism is strategy proof if there does not exist any ballot, $B \in \pi_m^n$ such that the outcome of the voting procedure is manipulable using a sophisticated strategy

Strategy Proof Voting Mechanism

Definition (Strategy Proof Voting Mechanism)

A voting mechanism is strategy proof if no voter has an incentive to cast a ballot different from their own preferences

Example

- 49% of Americans identify as independents[1]
- In the 2020 US Presidential Election independent candidates gathered 1.9% of votes

[1] https://www.axios.com/2023/04/17/poll-americans-independent-republican-democrat

Strategy Proof Voting Mechanism

$$R_i = (Independent > Democrat > Republican)$$

Sincere: $B_i = (Independent > Democrat > Republican)$

Sophisticated: $B_i = (Democrat > Independent > Republican)$

Gibbard-Satterthwaite Theorem

Theorem (Gibbard-Satterthwaite Theorem)

Consider a voting procedure, v^{nm} with $n \geq 2$ and $m \geq 3$. The voting procedure is strategy proof if and only if it is Dictatorial.

Dictatorial $\implies$ Strategy Proof

Assume voting mechanism v^{nm} is a Dictatorial voting mechanism and voter $k \in I_n$ is the dictator.

- voter k is not incentivized to cast a sophisticated ballot
- For all $i \in \{1, ..., k-1, k+1, ..., n\}$ voter i is not incentivized to cast a sophisticated ballot.

Key Theorems

Theorem (Arrow's Impossibility)

For $n > 2$ and $m > 3$ any social choice function that obeys Rationality, Pareto Optimality, and Independence of Irrelevant Alternatives must be Dictatorial.

Theorem (Gibbard-Satterthwaite Theorem)

Consider a voting procedure, v^{nm} with $n \geq 2$ and $m \geq 3$. The voting procedure is strategy proof if and only if it is Dictatorial.

Theorem (Gibbard-Satterthwaite Correspondence Theorem)

The strategy-proofness condition for voting procedures in the Gibbard-Satterthwaite Theorem correspond precisely to Arrow's conditions for a social choice function.

Correspondence Theorem

Strategy:

- We can produce a v^{nm} from a u^{nm}
- We can produce a u^{nm} from a v^{nm}
- Each v^{nm} that produces a u^{nm} (and vice versa) is unique

Correspondence Theorem

A strategy proof voting procedure can be constructed from For $n > 2$ and $m > 3$ any social choice function.

$$u^{5,3}(B) = u^{5,3}\left(\begin{array}{ll} B_1 : & X > Y > Z \\ B_2 : & X > Y > Z \\ B_3 : & X > Y > Z \\ B_4 : & X > Y > Z \\ B_5 : & X > Y > Z \end{array}\right) = (X > Y > Z) \in \pi_m$$

$$\mathbf{v^{nm}} = \mathbf{max(u^{nm})} = \mathbf{X} \in \mathbf{S_m}$$

Correspondence Theorem

A social choice function can be constructed from any strategy proof voting procedure

1. Pick an arbitrary strong ordering of the alternatives
2. Define $\lambda_{X,Y}$ for $X, Y \in S_m$ where $X \neq Y$ as follows:

$$B_i = (\alpha > \beta > X > \gamma > \phi > Y)$$

$$\lambda_{X,Y}(B_i) = (X > Y > \alpha > \beta > \gamma > \phi)$$

Correspondence Theorem

3. For each ballot set $B = (B_1, ..., B_n)$ construct a binary relation, P
   3.1 For all $X, Y \in S_m$ where $X \neq Y$, $X > Y$ in P if and only if $X = v^{nm}(\lambda_{X,Y}(B_1), ..., \lambda_{X,Y}(B_n))$

Example

$$B_1 = (X > Y > Z) \qquad \lambda_{X,Y}(B_1) = (X > Y > Z)$$
$$B_2 = (Y > X > Z) \qquad \lambda_{X,Y}(B_2) = (Y > X > Z)$$
$$B_3 = (Z > X > Y) \qquad \lambda_{X,Y}(B_3) = (X > Y > Z)$$

Correspondence Theorem

$$v^{nm}(\lambda_{X,Y}(B_1), \lambda_{X,Y}(B_2), \lambda_{X,Y}(B_1)) = X$$

$$v^{nm}(\lambda_{Y,Z}(B_1), \lambda_{Y,Z}(B_2), \lambda_{Y,Z}(B_1)) = Y$$

$$v^{nm}(\lambda_{X,Z}(B_1), \lambda_{X,Z}(B_2), \lambda_{X,Z}(B_1)) = X$$

$$\mathbf{P} = (\mathbf{X} > \mathbf{Y} > \mathbf{Z})$$

4. Let μ be a function that associates P with the appropriate ballot set

Correspondence Theorem

5. if v^{nm} is strategy proof, P is a strong order
6. μ is a strict social choice function
7. μ obeys Pareto Optimality and Independence of Irrelevant Alternatives
   7.1 If v^{nm} is strategy proof, it obeys Pareto Optimality and Independence of Irrelevant Alternatives

Correspondence Theorem

5. if v^{nm} is strategy proof, P is a strong order
6. μ is a strict social choice function
7. μ obeys Pareto Optimality and Independence of Irrelevant Alternatives
    7.1 **If v^{nm} is strategy proof, it obeys Pareto Optimality and Independence of Irrelevant Alternatives**

Correspondence Theorem

v^{nm} is strategy proof $\implies$ v^{nm} is Pareto Optimal.

Assume strategy proof and not Pareto Optimal

There exists some $C \in \pi_m^n$ such that $X > Y$ for all $C_i \in C$ but $v^{nm}(C) = Y$

Because X is in the range of v^{nm} there must exist a ballot set D such that $v^{nm}(D) = X$

Correspondence Theorem

There exists some voter, k such that:

$$v^{nm}(C_1, ..., C_{k-1}, C_k, ..., C_n) = Y$$
$$v^{nm}(D_1, ..., D_{k-1}, C_k, ..., C_n) = Y$$
$$v^{nm}(D_1, ..., D_{k-1}, D_k, ..., C_n) = X$$

Correspondence Theorem

Example

v^{nm} chooses the second best alternative

$C_1 = (X > Y > Z)$
$C_2 = (X > Y > Z)$
$C_n = (X > Y > Z)$

$D_1 = (Y > X > Z)$
$D_2 = (Y > X > Z)$
$D_n = (Y > X > Z)$

Voter k:

Sincere: $(X > Y > Z)$
Outcome: Y

Sophisticated: $(Y > X > Z)$
Outcome: X

Contradiction!

A Glimmer of Hope

Vickrey Auctions:

- Sealed Bid Auction
- Winner is the second highest big

Thank you

Thank you to everyone in the Math/Stats department!