EE208 EXPERIMENT 6

STATE FEEDBACK CONTROLLER DESIGN WITH MATLAB

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Objective

To design the state feedback gain matrix for a given analog state-space system to satisfy required performance specifications.

Given

The analog (linearized) system of an autonomous underwater vehicle (AUV) is given as follows:

$$\mathbf{A} = \begin{bmatrix} -0.14 & -0.69 & 0.0 \\ -0.19 & -0.048 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} 0.056 \\ -0.23 \\ 0.0 \end{bmatrix} \quad ; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The *sway speed*, *yaw angle*, and *yaw rate* are state variables x_1, x_2, x_3 respectively.

The input is the *rudder angle*, and the outputs are *sway speed* and *yaw angle*.

Specifications

Design state feedback gain matrices $\boldsymbol{K}$ such that:

1. Settling times of individual states are retained at those of the nominal eigenvalues.
2. Maximum magnitude of all eigenvalues is as in the nominal set.
3. The CL system is always observable and controllable.

As far as possible, all three requirements are to be met simultaneously.

Tasks

- Record and discuss the combination of feedback gains, showing how they ensure the performance required as above.
- In situations where the no designed value of the gain matrix is possible, explain and discuss why this is so.

MATLAB Functions Used

```
ss, set, sminreal, eig, pzmap, step, place, acker, fsurf, eye, vpa, collect,
solve, isstable, tf, ss2tf, poles, unique, sort, combvec, fsf (custom function)
```

Analyzing the Given system

We create our system with the **ss** command:

```
%% Creating the Analog System
A = [-0.14   -0.69   0.0;
     -0.19  -0.048   0.0;
       0.0     1.0   0.0];
B = [0.056;
     -0.23;
       0.0];
C = [1 0 0;
     0 1 0];
D = 0;
system1 = ss(A,B,C,D);
%% Setting some names
set(system1, 'inputname',  {'rudder angle'}, ...
             'outputname', {'sway speed' 'yaw angle'}, ...
             'statename',  {'sway speed' 'yaw angle' 'yaw rate'});
```

Eigenvalues of the A matrix

```
>> eig(A)
ans =
         0
    0.2710
   -0.4590
```

- One of the eigenvalues is in the RHP, and therefore the OL system is <u>unstable</u>.
- As a result, the first Specification does not make sense – the step responses of the individual states in open-loop are **unstable**, so their settling times are **undefined**.
- We will however keep one pole at -0.4590 and another at the origin, so that Spec. 1 can be met to some extent, if we interpret settling times to mean the settling times of individual modes instead of states. We will then place the third pole so that Spec. 3 is met.
- Also, we must keep in mind to not place any poles with a magnitude greater than 0.4590 in order to meet Spec. 2.

Controllability and Observability

We check Controllability and Observability with the relevant commands:

```
>> rank(ctrb(A,B)) % Controllable if rank = 3
ans =
     3
>> rank(obsv(A,C)) % Observable if rank = 3
ans =
     2
```

- The system is Controllable but not Observable. The Feedback Gain Matrix ***K*** can change the observability of the system, but it cannot change the controllability.
- In order to meet specification three, it is necessary to find the condition for observability with respect to the Gain Matrix.

Condition for Observability in terms of CL poles

The gain matrix **K** is given by $[k_1 \quad k_2 \quad k_3]$. We thus obtain the new A matrix as follows:

$$\mathrm{A_{CL}} = A - (\mathrm{B} \times \mathrm{K})$$

Then we can obtain the Observability matrix:

$$Ob = \begin{bmatrix} \mathrm{C} \\ \mathrm{CA_{CL}} \\ \mathrm{CA_{CL}}^2 \end{bmatrix} = \begin{bmatrix} \mathrm{C} \\ \mathrm{C(A-BK)} \\ \mathrm{C(A-BK)^2} \end{bmatrix}$$

We can then find a relation between the rank of the *Ob* matrix and the gain matrix **K.**

Let us form the symbolic *Ob* matrix in MATLAB:

```
%% Condition for Observability
syms k1 k2 k3
K = [k1 k2 k3];
A_cl = A-B*K;
Ob = [C;
      C*A_cl;
      C*(A_cl^2)];
```

Our task is to now compute the rank of the matrix. The rank of a matrix is equal to the number of nonzero rows in the matrix after reducing it to the row echelon form.

Ob is a 6×3 matrix with long terms, so finding it's rank by hand is very hard. We shall use this site to calculate the rank by inputting the *Ob* matrix obtained from MATLAB, since the *rref* command in MATLAB cannot give us an answer in terms of the gain coefficients.

We obtain:

$$\mathrm{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-14\cdot k_1 - 35}{250} & \frac{-28\cdot k_2 - 345}{500} & \frac{-7\cdot k_3}{125} \\ \frac{23k_1 - 19}{100} & \frac{115k_2 - 24}{500} & \frac{23k_3}{100} \\ \frac{784k_1^2 - 35755k_1 - 3220k_1k_2 + 2660k_2 + 37675}{250000} & \frac{9660k_1 - 3220k_2^2 + 784k_1k_2 - 37043k_2 - 14000k_3 + 32430}{250000} & \frac{784k_1k_3 - 3220k_2k_3 - 37715k_3}{250000} \\ \frac{-644\cdot k_1^2 - 1630k_1 + 2645k_1k_2 - 2185k_2 + 1786}{50000} & \frac{-39675\cdot k_1 + 13225k_2^2 - 3220k_1k_2 - 2860k_2 + 57500k_3 + 33351}{250000} & \frac{-644\cdot k_1k_3 + 2645k_2k_3 - 20k_3}{50000} \end{pmatrix} = \begin{cases} 3 & k_3 \neq 0 \\ 2 & k_3 = 0 \end{cases}$$

Hence, the condition for Observability is:

$$\boldsymbol{k_3 \neq 0}$$

Now we shall state this condition in terms of CL poles.

The characteristic equation for the CL system is given by $\boldsymbol{det}(sI - (A - BK))$.

We use MATLAB to obtain this equation:

```
>> syms s
>> CharEq = vpa(collect(det(s*eye(3) - A_cl),s),3)
 CharEq =
s^3 + (0.056*k1 - 0.23*k2 + 0.188)*s^2 + (0.161*k1 - 0.0428*k2 - 0.23*k3 -
0.124)*s - 0.0428*k3
```

Thus, the Characteristic Equation is

$$s^3 + (0.056k_1 - 0.23k_2 + 0.188)s^2 + (0.161k_1 - 0.0428k_2 - 0.23k_3 - 0.124)s - 0.0428k_3 = 0$$

Now, let us compare this equation to the general 3rd order polynomial equation – this is the desired characteristic equation.

$$(s-a)(s-b)(s-c) = 0$$
$$\Rightarrow s^3 + (-a-b-c)s^2 + (ab+ac+bc)s - abc = 0$$

Comparing the constant terms of both the equations, we obtain the following condition for k_3:

$$abc = 0.0428k_3$$

We know that $\boldsymbol{k_3 \neq 0}$ to satisfy Observability. This implies that $a, b, c \neq 0$.

Thus –

- To obtain an Observable system, there can be no CL poles at the origin, i.e., all CL eigenvalues must be nonzero.
- As a result, Specification 1 cannot be achieved for the pole at origin (which is present in OL) if we are to meet Specification 3.
- Overall, we cannot meet Specification 1 for $\lambda = 0.2710$ because it is unstable, and for $\lambda = 0$ since it will make meeting Specification 3 impossible.

We shall explore both cases for the sake of completeness – meeting Spec. 3 and violating it.

Pole Placement

- We place poles at various locations, see the variation of gain matrix elements, and view the step responses for select pole locations.
- We shall place one pole at -0.4590 so that the settling time for at least one mode remains the same in closed loop.
- Specification 2 will be met by not placing any poles greater than -0.4590 in magnitude for the closed loop.
- Specification 3 is violated in one case and met in another case – this is done for the sake of completeness, and also because we would like to avoid disturbing the linearized dynamics too much.

Thus, our objective for pole placement is two-fold:

1. Stabilize the system by moving the unstable eigenvalue to the LHP.
2. Meet specification 3 by not placing any pole at the origin.

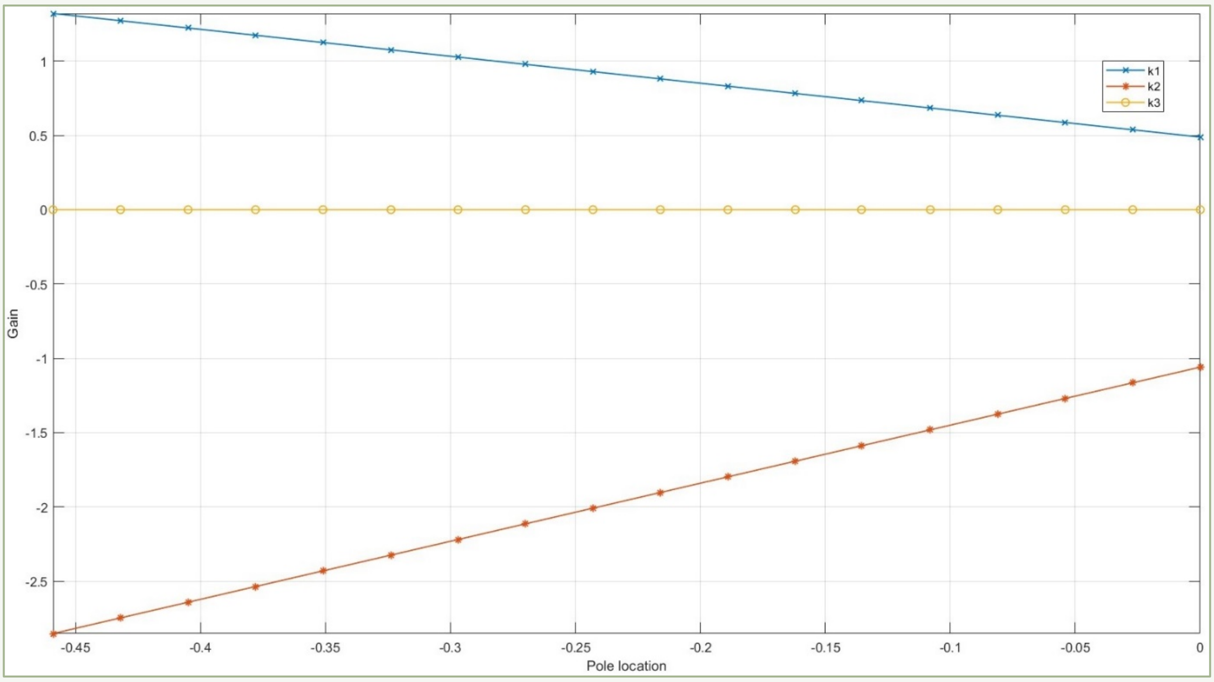
The system is 3rd order, so we can have 3 real poles or one real and two complex conjugate poles. Let us explore these two cases sequentially.

3 Real poles, $k_3 = 0$

In this case we place one pole at the origin, one at -0.4590. The third pole can be placed in the region $[-0.4590, 0]$.

We will first plot the variation of $k_1,\ k_2, and\ k_3$ wrt the third pole, which we shall refer to as **c**.

```
%% Plotting for k1 k2 k3
syms a b c
% Create equations
eq1 = 0.0428*k3 == a*b*c;
eq2 = 0.056*k1-0.23*k2+0.188 == -a-b-c;
eq3 = 0.161*k1-0.0428*k2-0.23*k3-0.124 == a*b+a*c+b*c;
% Solve, truncate to 3 decimal poins, a=-0.4590, b=0
[k1, k2, k3 ] = solve(eq1,eq2,eq3);
k1 = subs(vpa(k1,3),[a b], [-0.4590 0]);
k2 = subs(vpa(k2,3),[a b], [-0.4590 0]);
k3 = subs(vpa(k3,3),[a b], [-0.4590 0]);
% Plot in the relevant range
fplot([k1 k2 k3],[-0.459 0]), grid on
```



- We can see from the graph that the gains range from -2.8524 to 1.3205, which are achievable. Had there been large gain values, the gain matrix could have become difficult to realize.
- The gain values vary linearly wrt. Pole location.

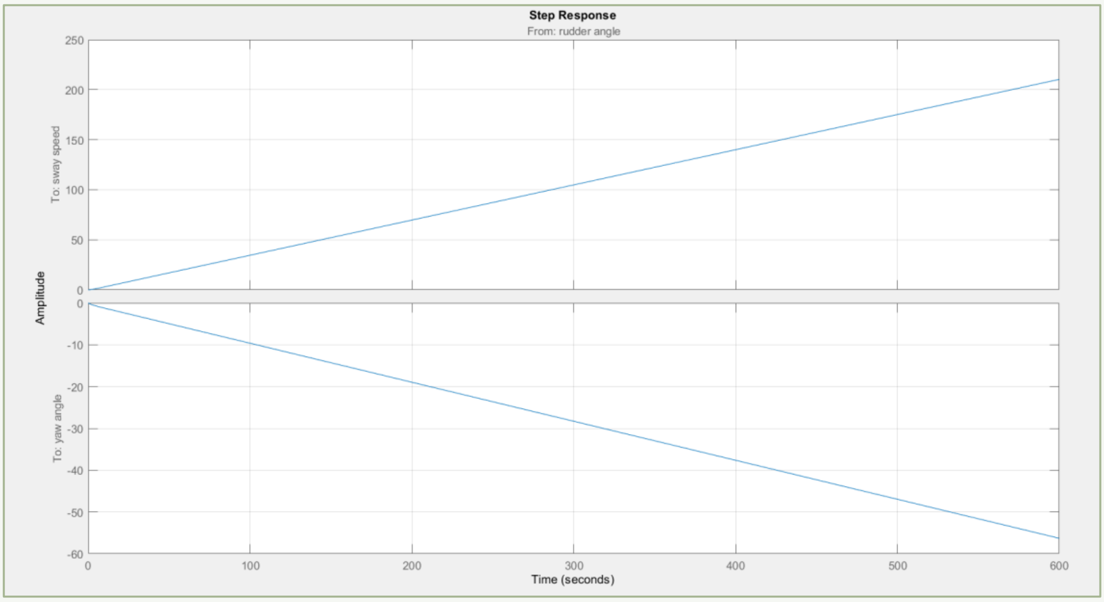
c = 0, double pole at origin

We use our custom function **fsf** to obtain the system in closed loop.

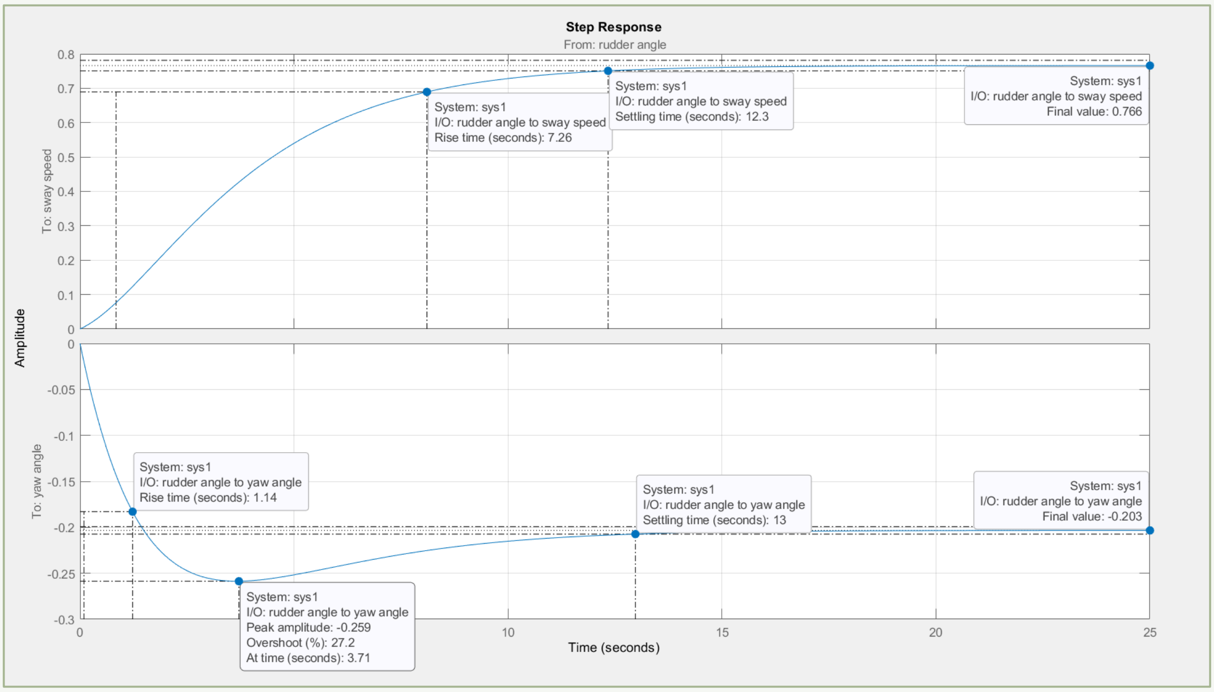
```
>> sys1 = fsf(system1,[0 0 -0.459]);
>> eig(sys1)
ans =
    0.0000
   -0.4590
         0
>> isstable(sys1)
ans =
  logical
  0
```

The system has the CL poles are desired, but it is unstable – because of a double pole at the origin.

The step response is therefore similar to a ramp function.



c = -0.459, double pole at stable eigenvalue

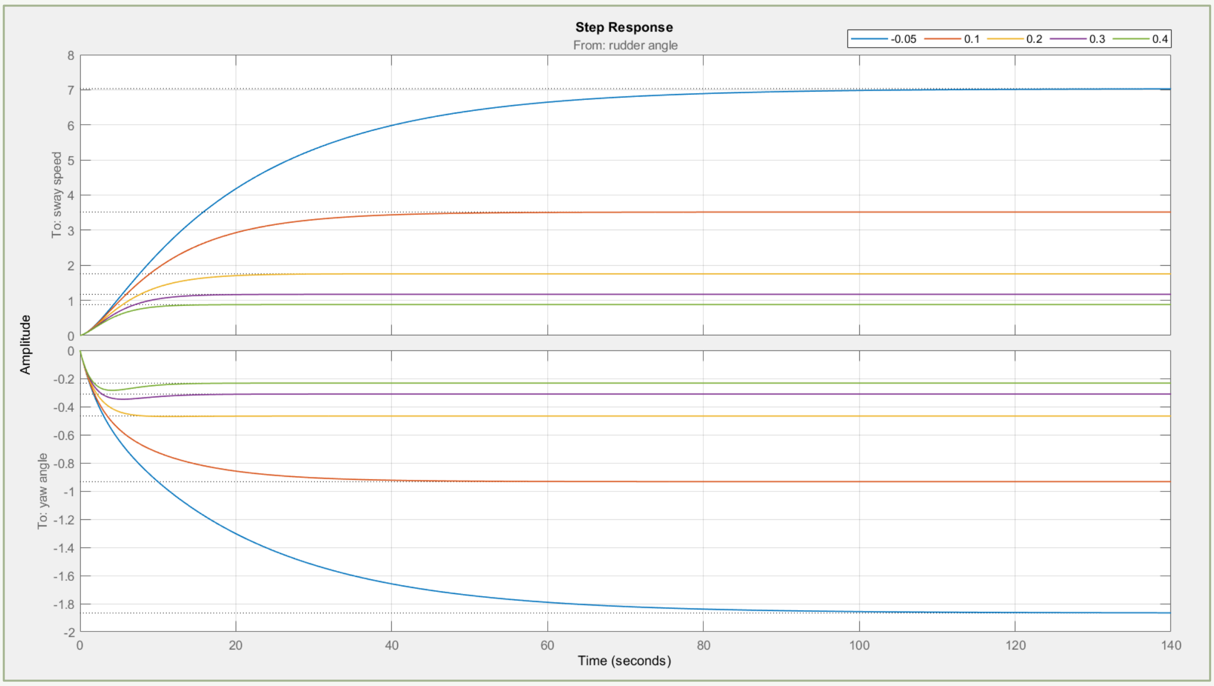


As expected, we obtain a stable response, however pole placement is not exact (due to the double pole), and we obtain an error message:

```
Warning: Pole locations are more than 10% in error.
```

We should avoid placing poles of Multiplicity>1 using the **place** or **acker** functions, due to numerical errors.

c = -0.05, -0.1, -0.2, -0.3, -0.4



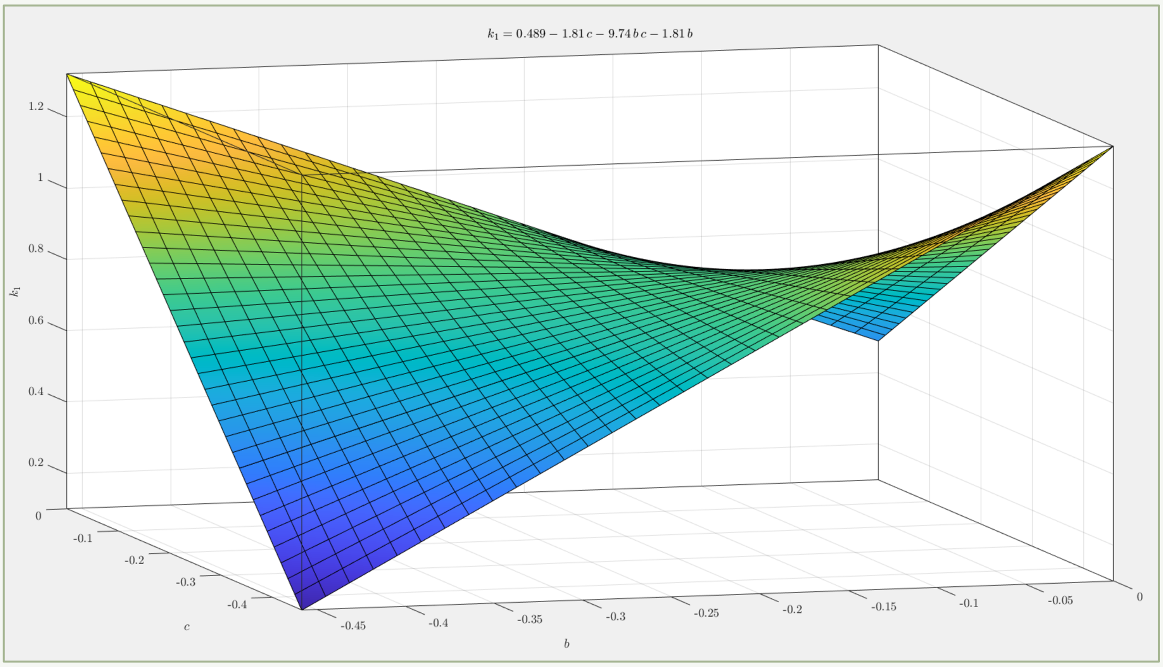
As we move the poles towards the left (i.e., more negative), our Rise Time and Settling Time decrease, which is expected. The dynamics are all lag type, with some overshoot in the lower graph for $c = -0.3, -0.4$.

3 Real poles, $k_3 \neq 0$

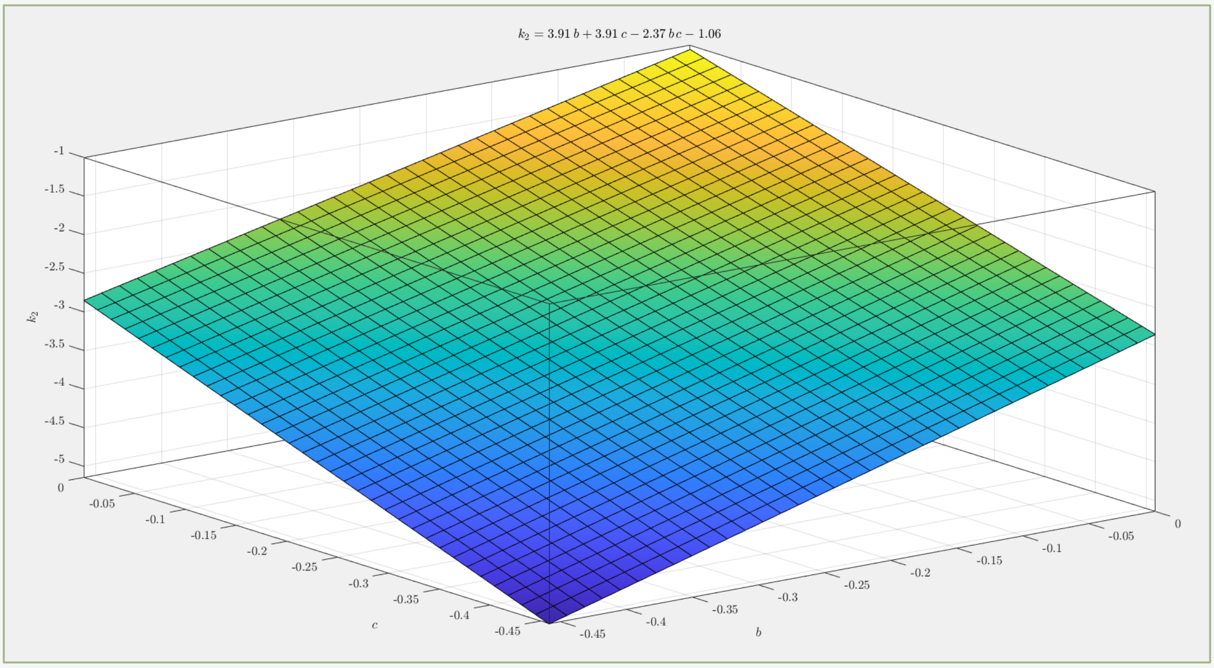
In this case, we shall place 2 poles between 0 and -0.459.

Let us visualize the variation in the gains as we move our poles around.

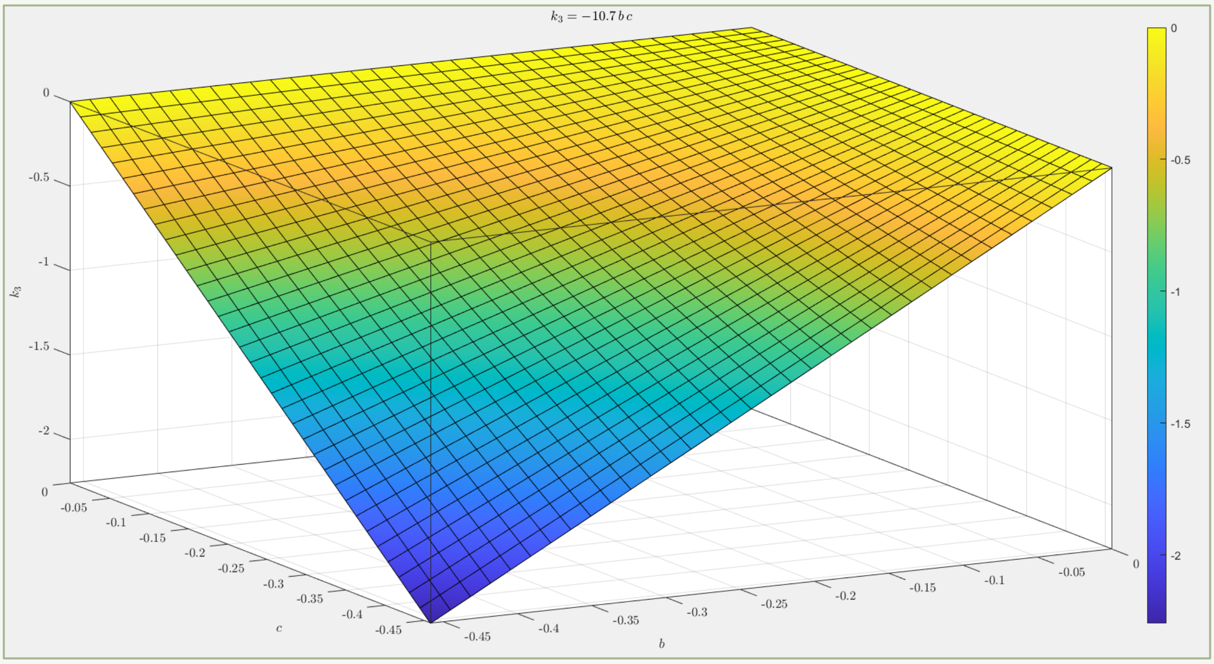
For k_1:



For k_2:

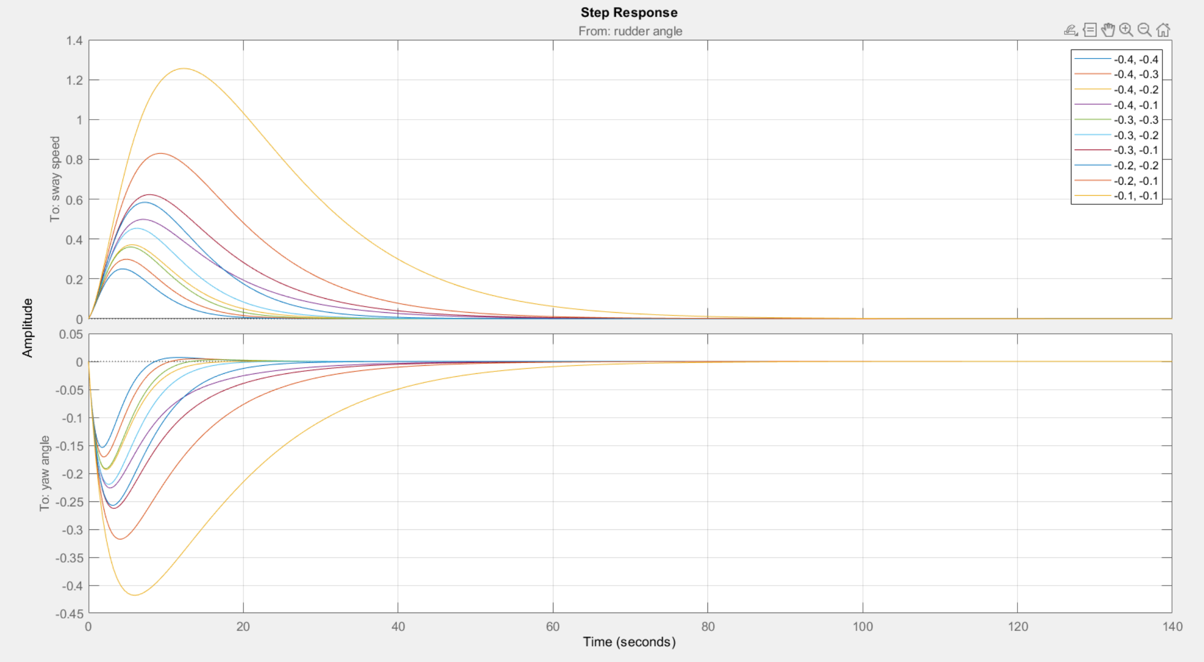


For k_3:



From these plots we can understand the variation in the gains due to pole placement. In all three plots, more negative poles are associated with smaller/more negative gains.

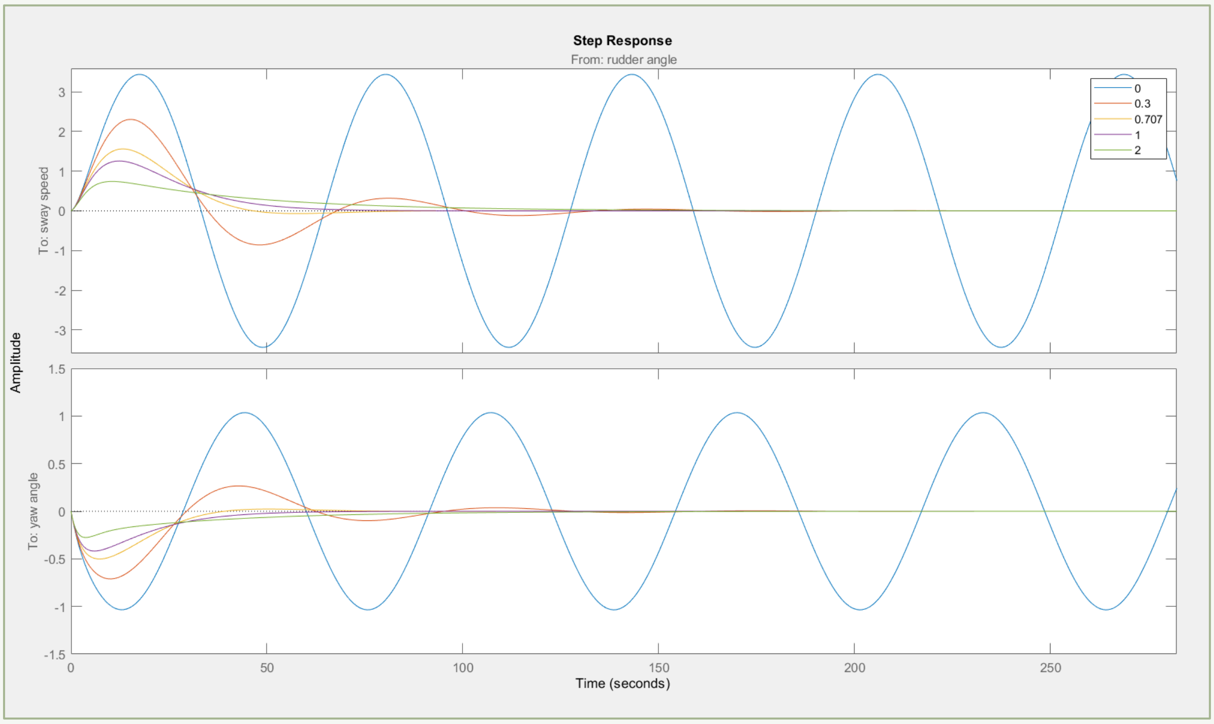
We can view the variation in step responses as well. The poles are indicated in the legend.



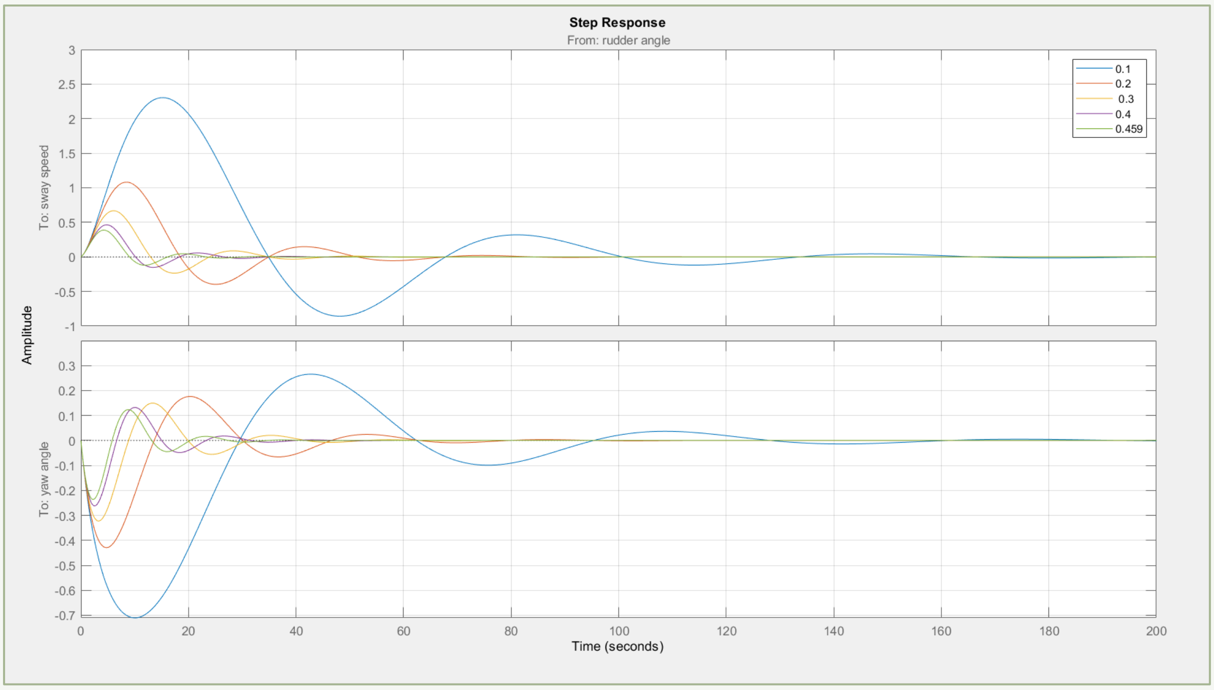
1 Real and 2 Complex conjugate poles

We have already seen various responses possible with three real poles. We shall now study the step responses for complex conjugate poles. As before, we keep one pole at -0.459, and vary the other two poles.

Let us look at the step responses for varying damping ratios. The natural frequency is 0.1 rad/s.



Similarly, we can plot step responses for varying natural frequency. The damping ratio is 0.3.



In all of these cases we viewed above, we were able to:

1. Satisfy Specification 2 by choosing poles smaller than 0.459 in magnitude,
2. Satisfy Specification 3 whenever $k_3 \neq 0$, and
3. Stabilize the system by placing poles in the LHP.

We were unable to fully satisfy Specification 1 since the original system is unstable, so the states do not have any settling time wrt step input. However, we tried to satisfy it as much as possible by fixing one pole at -0.459.

Minimal State-space Realization

Let us look at the original system once more:

$$\mathbf{A} = \begin{bmatrix} -0.14 & -0.69 & 0.0 \\ -0.19 & -0.048 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} 0.056 \\ -0.23 \\ 0.0 \end{bmatrix} \quad ; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

State x_3 is not directly affected by the input and it does not affect the output. This state is thus *unobservable*.

It is possible to remove this state, our input-output dynamics will not be affected by doing so using the **sminreal** function. We use this function instead of **minreal** to preserve the system's structure. The relevant code is:

```
>> system1 = sminreal(system1)
system1 =
   A =
                sway speed   yaw angle
   sway speed        -0.14       -0.69
   yaw angle         -0.19      -0.048
  B =
                rudder angle
   sway speed          0.056
   yaw angle           -0.23
  C =
                sway speed   yaw angle
   sway speed            1           0
   yaw angle             0           1
  D =
                rudder angle
   sway speed              0
   yaw angle               0

Continuous-time state-space model.
```

This corresponds to a pole-zero cancellation in the System Transfer Matrix.

The eigenvalues are:

```
>> eig(system1.A)
ans =
   -0.4590
    0.2710
```

Once again one pole is unstable, so it be should stabilized with feedback.

We can also check for Controllability and Observability:

```
>> rank(ctrb(system1)), rank(obsv(system1))
ans =
     2
ans =
     2
```

In this case the system is Controllable and Observable.

Proceeding as before, we can check if the Gain Matrix affects Observability (we already know that State-Feedback Control does not affect Controllability).

Upon creating our observability matrix and checking its rank we obtain:

$$rank\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -0.056 * k_1 - 0.14 & -0.056 * k_2 - 0.69 \\ 0.23 * k_1 - 0.19 & 0.23 * k_2 - 0.048 \end{pmatrix} = 2$$

This can be shown through simple row operations:

$R_3 \leftarrow R_3 - (-0.056 \cdot k_1 - 0.14) \cdot R_1 - (-0.056 \cdot k_2 - 0.69) \cdot R_2$

$R_4 \leftarrow R_4 - (0.23 \cdot k_1 - 0.19) \cdot R_1 - (0.23 \cdot k_2 - 0.048) \cdot R_2$

To obtain: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. This matrix clearly has a rank of 2 which is independent of **K**.

- So, the Gain Matrix does not affect Observability, allowing us to satisfy Specification 3 for all Gain Matrices **K**.
- Since the system is Controllable, we can place our poles anywhere in the s-plane – but since this is a linearized system, we should not place them too far off from the nominal eigenvalues.

Since we do not have the original nonlinear equations at hand, we cannot predict the effect of this minimal realization on the nonlinear dynamics – therefore, we will not explore this minimal realization further. Our intention is only to show a property of the given system that may be of interest in certain situations.

Summary of Observations and Analysis

- We examined the Stability, Controllability and Observability for the given system.
- We found that the System was unstable and unobservable.
- The condition for Observability using State Feedback was found.
- The condition is: To obtain an Observable system, there can be no CL poles at the origin, i.e., all CL eigenvalues must be nonzero.
- We performed Pole Placement to form the Closed-Loop system and demonstrated the various possible step responses.
- We also explained that the system can be simplified due to the presence of unobservable states –similar to a pole-zero cancellation.

Conclusion

In this experiment we explored State Feedback Matrices for a given Analog system, following certain Specifications and conditions. Our conclusions are as follows:

1. The system is Controllable, so the possible range for Pole Placement is the complete s-plane. The system is unstable and unobservable. We can use State Feedback to solve this.
2. Specification 1 cannot be met properly since the Open-Loop system is unstable, thus the individual states do not settle down in response to a step input, and thus the "settling times of individual states" are undefined. We have placed one pole at -0.459 so that at least one eigenvalue is unaffected by the Control.
3. Specification 2 can be always met without affecting the other two Specs. We have chosen to place poles in the LHP with a magnitude not greater than 0.459, which is the maximum magnitude among the nominal eigenvalues.
4. Specification 3 can be met by not placing any CL pole at the origin. The derivation for this condition has been given in the report.
5. In the absence of additional Specifications, such as required damping ratio, peak overshoot etc., we have not pursued the Gain Matrix design further, but we have shown various step responses possible by placing the poles at appropriate positions.
6. We have also shown a minimum realization of the given system.

The overall range for Pole Placement can be expressed as $|s| \leq 0.459;\ \mathrm{Re}(s) < 0$. All the poles in this range can be realized by an appropriate Gain Matrix **K**, because the system is Controllable (and Observable since we are not placing poles at the origin).

Resources Used

1. Feedback Control of Dynamic Systems, *7.5*, *7.6*
2. Modern Control Engineering, *10.2*, *10.3*, *10.4*
3. https://matrixcalc.org/en/
4. https://www.ecse.rpi.edu/courses/CStudio/RTA_lab/Lab_text_references/Swisher_Ch10.pdf
5. https://www.control.utoronto.ca/people/profs/kwong/ece410/2008/notes/chap4.pdf
6. https://www.site.uottawa.ca/~rhabash/ELG4152L5.pdfq
7. http://floatium.stanford.edu/engr210a/lectures/lecture4_2001_10_10_01.pdf
8. https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-241j-dynamic-systems-and-control-spring-2011/readings/MIT6_241JS11_chap28.pdf

MATLAB Script (2022a)

https://drive.google.com/drive/folders/18zogcTXt06osrQG9Rqk_IKpGJm3UdAIY?usp=sharing