Q1. Let the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be given by

$$f(x,y) = \frac{x \sin(x^2+2y^2)}{x^2+y^2}.$$

Is it possible to define $f(0,0)$ so that f is continuous at $(0,0)$?

Soln: Note that for $(x,y) \neq (0,0)$,

$$|f(x,y)| = \frac{|x|\,|\sin(x^2+2y^2)|}{x^2+y^2}$$

$$\leq \frac{|x|\,(x^2+2y^2)}{x^2+y^2}$$

$$\leq 2|x| \qquad (\because x^2+2y^2 \leq 2(x^2+y^2))$$

$$\leq 2\sqrt{x^2+y^2}$$

So, given $\varepsilon > 0$, if we choose $\delta = \varepsilon/2$, then

$0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - 0| \leq 2\sqrt{x^2+y^2} < 2\delta = \varepsilon$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

If we define $f(0,0) = 0$, then $f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$ which implies f is continuous at $(0,0)$.

Question 2: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} \frac{xy(x^2+y^4)}{x^4+y^2} & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Discuss the differentiability of f at $(0,0)$. (5)

Solution: Note that,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\cdot 0(h^2+0)}{h^4+0} - 0}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0\cdot k(0+k^4)}{0+k^2} - 0}{k} = 0.$$

We know that f is differentiable at $(0,0)$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = 0.$$

Thus, f is differentiable at $(0,0)$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{hk(h^2+k^4)}{(h^4+k^2)(\sqrt{h^2+k^2})} = 0.$$

Therefore, to show that f is not differentiable at $(0,0)$ it is sufficient to show that

$$\lim_{(h,k)\to(0,0)} \frac{hk(h^2+k^4)}{(h^4+k^2)(\sqrt{h^2+k^2})} \neq 0.$$

If (h,k) approaches to $(0,0)$ along the curve $k = h^2$ then $(h,k) \to (0,0) \Leftrightarrow h \to 0$. Further, along that curve $k = h^2$ we have

$$\begin{aligned} \lim_{(h,k)\to(0,0)} \frac{hk(h^2+k^4)}{(h^4+k^2)\sqrt{h^2+k^2}} &= \lim_{h\to 0} \frac{hh^2(h^2+h^8)}{(h^4+h^4)\sqrt{h^2+h^4}} = \lim_{h\to 0} \frac{h^5(1+h^6)}{2h^4|h|\sqrt{1+h^2}} \\ &= \lim_{h\to 0} \frac{1+h^6}{2\sqrt{1+h^2}} \cdot \lim_{h\to 0} \frac{h}{|h|} = \frac{1}{2} \lim_{h\to 0} \frac{h}{|h|} \end{aligned}$$

Since $\lim_{h\to 0^+} \frac{h}{|h|} = 1$ and $\lim_{h\to 0^-} \frac{h}{|h|} = -1$, $\lim_{(h,k)\to(0,0)} \frac{hk(h^2+k^4)}{(h^4+k^2)\sqrt{h^2+k^2}}$ does not exist. In particular, $\lim_{(h,k)\to(0,0)} \frac{hk(h^2+k^4)}{(h^4+k^2)(\sqrt{h^2+k^2})} \neq 0$, and hence f is not differentiable at $(0,0)$.

Given.

$$f(x,y) = 2e^{x+y} - \sin(xy)$$

$f_x(x,y) = 2e^{x+y} - y\cos(xy)$	$f_x(0,0) = 2$
$f_{xx}(x,y) = 2e^{x+y} + y^2 \sin(xy)$	$f_{xx}(0,0) = 2$
$f_{yx}(x,y) = 2e^{x+y} - \cos(xy) + xy\sin(xy) = f_{xy}(x,y)$	$f_{yx}(0,0) = 2-1 = 1$
$f_y(x,y) = 2e^{x+y} - x\cos(xy)$	$f_y(0,0) = 2$
$f_{yy}(x,y) = 2e^{x+y} + x^2\sin(xy)$	$f_{yy}(0,0) = 2$

Then the 2nd order Taylor polynomial at $(0,0)$ is,

$$P(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right)$$

$$= 2 + 2x + 2y + \frac{1}{2!}(2x^2 + 2xy + 2y^2)$$

$$= 2(1+x+y) + (x^2 + xy + y^2).$$

Now,

$$f_{yyy}(x,y) = 2e^{x+y} + x^3\sin(xy)$$

$$f_{xxx}(x,y) = 2e^{x+y} + y^3\sin(xy)$$

$$f_{xyy}(x,y) = 2e^{x+y} + 2x\sin(xy) + x^2y\cos(xy)$$

$$f_{xxy}(x,y) = 2e^{x+y} + 2y\sin(xy) + xy^2\cos(xy)$$

$$f_{yyx}(x,y) = 2e^{x+y} + 2x\sin(xy) + x^2y\cos(xy)$$

The remainder term is,

$$R_2 = \frac{1}{3!}\left(x^3 f_{xxx}(\theta x, \theta y) + 3x^2y f_{xxy}(\theta_x, \theta_y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta_x, \theta_y)\right)$$

where $0<\theta<1$.

When $|x|<.1$, $|y|<.1$ we get,

$$|R_2| \leq \frac{1}{3!}\Big(|x|^3 |f_{xxx}(\theta x, \theta y)| + 3|x|^2|y| \, |f_{xxy}(\theta x, \theta y)| + 3|x||y|^2 |f_{xyy}(\theta x, \theta y)| + |y|^3 |f_{yyy}(\theta x, \theta y)|\Big)$$

$$\leq \frac{1}{3!}\Big(2(.1)^3\left(2e^{.2} + 1.1^3\right) + 6(.1)^3\left(2e^{.2} + 2(.1) + (.1)^3\right)\Big)$$

Q4. Find the points on the surface given by $z^2 = xy + 4$ closest to the origin.

Soln: $f(x,y,z) = x^2 + y^2 + z^2, \qquad g(x,y,z) = xy + 4 - z^2.$

By Lagrange Multiplier we have: $\exists\ \lambda \in \mathbb{R}$

$$\nabla f = \lambda \nabla g.$$

$\Rightarrow \quad \underset{\to(1)}{2x = \lambda y}, \qquad \underset{\to(2)}{2y = \lambda x} \quad \& \quad \underset{\to(3)}{2z = -2\lambda z} \Rightarrow \quad z = -\lambda z.$

First Observation: If $\lambda = 0 \Rightarrow (x,y,z) = (0,0,0)$ & since $g(0,0,0) \neq 0$. (NOT possible)

$\Rightarrow$ ~~WL~~. We assume that $\lambda \neq 0$.

Method 1: using (1) & (2):

$$2x = \lambda\left(\frac{\lambda}{2}x\right) \Rightarrow 4x = \lambda^2 x$$

$$\Rightarrow (\lambda^2 - 4)\,x = 0.$$

Then $x = 0$ or $\lambda = \pm 2$

Case-I. $x = 0 \Rightarrow y = 0$, $z = \pm 2$ (using Eq. $g(x,y,z) = 0$).

$\Rightarrow (0, 0, \pm 2)$.

Case-II: $x \neq 0$, & $\lambda = \pm 2$.

$\Rightarrow y = \pm x \quad \left(2x = \frac{2y}{x}x \Rightarrow x^2 = y^2\right)$.

Also, $z = 0$ (

$\Rightarrow$ The critical points are $(\pm 2, \mp 2, 0)$.

$f(0,0,\pm 2) = 4, \qquad f(\pm 2, \mp 2, 0) = 8.$

So, $(0,0,\pm 2)$ are points closest to the origin.

Method 2:

$z = -\lambda z.$

Suppose $z \neq 0 \Rightarrow \lambda = -1.$

$$\left.\begin{aligned} 2x + y &= 0 \\ 2y + x &= 0 \end{aligned}\right\} \Rightarrow y = 0, \; x = 0.$$

$\Rightarrow z^2 = 4 \Rightarrow z = \pm 2$

$\Rightarrow (0, 0, \pm 2)$ are the points.

Other possibilities:

$z = 0.$ Then, since $\lambda \neq 0 \Rightarrow y = \pm x.$

$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$\Rightarrow (\pm 2, \mp 2, 0)$ are the points.

Hence, $(0, 0, \pm 2)$ are the points closest to origin //.