Question 1: $a_n = \frac{(-1)^n(n+5)}{n}$

Consider $\alpha_n = \sup\{a_k : k \geq n\}$

$$= \begin{cases} \frac{(n+1)+5}{n+1} & \text{if } n \text{ is odd} \\ \frac{n+5}{n} & \text{if } n \text{ is even} \end{cases}$$

$\because$ $\{\alpha_n\}$ is non-increasing and $\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} \frac{2n+5}{2n} = 1$, we get

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n = 1.$$

Similarly, we get $\beta_n = \inf\{a_k : k \geq n\}$

$$= \begin{cases} -\left(\frac{n+5}{n}\right), & \text{if } n \text{ is odd} \\ -\left(\frac{(n+1)+5}{n+1}\right) & \text{if } n \text{ is even} \end{cases}$$

and $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \beta_n = -1.$

Question 2: Let $(x_n)_{n\geq 1}$ be a sequence defined by

$$x_1 = 1 \text{ and } x_{n+1} = 3 + \frac{1}{x_n} \quad \text{ for } n \in \mathbb{N}.$$

Show that the sequence $(x_n)_{n\geq 1}$ is Cauchy, and find the limit of the sequence.

Solution: Given a sequence $(x_n)_{n\geq 1}$ such that

$$x_1 = 1$$

and

$$x_{n+1} = 3 + \frac{1}{x_n}.$$

The given relation implies that $x_n > 3$ for all $n > 1$.
Also, using the reccurrence relation we get,

$$|x_{n+2} - x_{n+1}| = \left|\frac{1}{x_{n+1}} - \frac{1}{x_n}\right| = \left|\frac{x_{n+1} - x_n}{x_{n+1}x_n}\right|,$$

which gives

$$|x_{n+2} - x_{n+1}| < \frac{1}{9}\left|x_{n+1} - x_n\right|,$$

for all $n \in \mathbb{N}$.

Continuing this further we get,

$$|x_{n+2} - x_{n+1}| \leq \frac{1}{9^{n-1}}\left|x_3 - x_2\right|.$$

Let $m \in \mathbb{N}$ and $m > n$.

Then

$$\begin{aligned}
|x_m - x_n| &\leq |x_m - x_{m-1}| + \ldots + |x_{n+1} - x_n| \\
&\leq \left(\frac{1}{9^{m-3}} + \frac{1}{9^{m-4}} + \ldots + \frac{1}{9^{n-2}}\right)|x_3 - x_2| \\
&= \frac{1}{9^{n-2}} \cdot \frac{1 - \frac{1}{9^{m-n}}}{1 - \frac{1}{9}} \cdot |x_3 - x_2| \\
&< \frac{1}{8 \cdot 9^{n-2}}\left|x_3 - x_2\right|
\end{aligned}$$

Let $\epsilon > 0$. Since $0 < \frac{1}{9^{n-2}} < 1$, the sequence $(\frac{1}{9^{n-2}})$ is convergent. Hence there exists $k \in \mathbb{N}$ such that

$$\frac{1}{8 \cdot 9^{n-2}} |x_3 - x_2| < \epsilon,$$

for all $n \geq k$. It follows that $|x_m - x_n| < \epsilon$ for all $n \geq k$. This implies $(x_n)_{n \geq 1}$ is a Cauchy sequence.

Let l be the limit of the sequence $(x_n)_{n \geq 1}$. From the given reccurrence relation proceeding to limit as $n \to \infty$ we get, $l = 3 + \frac{1}{l}$. This gives

$$l = \frac{1}{2}(3 + \sqrt{13}),$$

since each element of the sequence is positive.

Question 3 :– Let $x_n = (2^n+3^n)^{\frac{1}{n}}$ for $n \in \mathbb{N}$. Discuss whether the sequence $(x_n)_{n\geq 1}$ is convergent or not? If the sequence $(x_n)_{n\geq 1}$ is convergent then find its limit.

Solution 1 :– Here $x_n = (2^n+3^n)^{\frac{1}{n}} = \left[3^n\left(\frac{2^n}{3^n}+1\right)\right]^{\frac{1}{n}} = 3\left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}}$

Now, $0 < \left(\frac{2}{3}\right)^n < 1 \quad \forall\ n \in \mathbb{N}$

$$\Rightarrow 1 < 1+\left(\frac{2}{3}\right)^n < 2 \quad \forall\ n \in \mathbb{N}$$

$$\Rightarrow 1 < \left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}} < 2^{\frac{1}{n}}$$

$$\Rightarrow 3 < 3\left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}} < 3\cdot 2^{\frac{1}{n}} \quad \forall\ n \in \mathbb{N}.$$

Let $y_n = 3$ and $z_n = 3\cdot 2^{\frac{1}{n}} \quad \forall\ n \in \mathbb{N}$.

Then $y_n < x_n < z_n \quad \forall\ n$ and

$$\lim_{n\to\infty} y_n = 3 \quad \text{and} \quad \lim_{n\to\infty} z_n = 3 \lim_{n\to\infty} 2^{\frac{1}{n}} = 3\cdot 1 = 3.$$

Thus by Sandwich theorem, the sequence $(x_n)_{n\geq 1}$ is convergent and converges to 3, i.e., $\lim_{n\to\infty} x_n = 3$.

Solution 2 :- Let $a_n = (2^n + 3^n)$ $\forall\, n \in \mathbb{N}$. Then $x_n = a_n^{\frac{1}{n}}$.

* Since $(a_n)_{n\geq 1}$ is a sequence of non-zero real numbers, $\liminf\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \liminf\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Now, $$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n} = \frac{3^{n+1}}{3^n} \frac{1 + \left(\frac{2}{3}\right)^{n+1}}{1 + \left(\frac{2}{3}\right)^n} = 3\, \frac{1 + \left(\frac{2}{3}\right)^{n+1}}{1 + \left(\frac{2}{3}\right)^n}$$ for all $n \in \mathbb{N}$.

Since $0 < \frac{2}{3} < 1$, $\lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

Thus, $$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3\, \frac{1+0}{1+0} = 3$$

Therefore, $$\liminf_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 3$$

i.e., $$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = 3 \quad \left[\because x_n = a_n^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}\right]$$

Thus the sequence $(x_n)_{n\geq 1}$ is convergent, and $\lim\limits_{n\to\infty} x_n = 3$.

Question 4) Let $a_n = \sqrt{n^4+1} - \sqrt{n^4-1}$ for $n \in \mathbb{N}$.
Discuss whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or not.

Solution. Given

$$a_n = \sqrt{n^4+1} - \sqrt{n^4-1}.$$

By rationalising, one can observe that

$$a_n = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

Let $b_n = 1/n^2$. Then

$$\frac{a_n}{b_n} = \frac{2n^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \frac{2n^2}{n^2\left(\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right)}$$

$$= \frac{2}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}}$$

$$\longrightarrow 1.$$

Thus by limit compairision test, $\sum a_n$ converges iff $\sum b_n$ converges. But, we know that $\sum 1/n^p$ converges $\Leftrightarrow p>1$. Therefore the series $\sum 1/n^2$ converges and hence the given series $\sum a_n$ converges.

Another solution. After rationatising (as in the first solution), observe that

$$a_n \le 2/n^2.$$

Since $\sum 1/n^2$ is convergent, it follows from compairision test that the given series $\sum a_n$ converges.