Assignment 1

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1 Overview

We are going to implement a neural network to classify a given pair of real numbers into 2 different Gaussian's:

$$\mu_1 = 1, \sigma_1 = 0.5$$

$$\mu_2 = 2, \sigma_2 = 0.2$$

Input layer contains 2 nodes, hidden layer contains 2 nodes with tanh activation and output layer contains 2 nodes with soft-max activation.

2 Notations

A^0 : input layer values : 2x1 matrix
Z^1 : hidden layer values before tanh activation : 2x1 matrix
A^1 : hidden layer values after tanh activation : 2x1 matrix
Z^2 : hidden layer values before softmax activation : 2x1 matrix
A^2 : hidden layer values after softmax activation : 2x1 matrix
Y : Actual output values : 2x1 matrix

3 Forward Propagation

3.1 Input Layer to Hidden Layer

$$z^1 = W^1 a^0 + b^1$$

$$a^1 = tanh(z^1)$$

3.2 Hidden Layer to Output Layer

$$z^2 = W^2 a^1 + b^2$$

$${a_j}^2 = softmax({z_j}^2) = \frac{e^{{z_j}^2}}{e^{{z_1}^2} + e^{{z_2}^2}}$$

4 Backward Propagation

Backward Pass Let $Y(2 \times m)$ be our real output. Y happens to be an one hot encoded vector as, we shall be classifying two different Gaussian. Let's define a loss function , $\bar{J}$ as: $\bar{J} := \frac{1}{2m} \sum_{j=1}^{j=m} \sum_{i=1}^{i=2} (a^{[2]}ij - yij)^2 - \frac{1}{m} \sum_{j=1}^{j=m} \sum_{i=1}^{i=2} y_{ij} \log(a_{ij}^{[2]})$.
Let $J_j := \frac{1}{2} \sum_{i=1}^{i=2} (a^{[2]}ij - yij)^2 - \sum_{i=1}^{i=2} y_{ij} \log(a_{ij}^{[2]})$ and $\bar{J} = \frac{1}{m} J$

$$\frac{dJ_j}{dz_{ij}^{[2]}} = (a_{ij}^{[2]} - y_{ij}) - \sum_{i=1}^{i=2} \frac{y_{ij}}{a^{[2]}ij} \frac{daij}{dz_{ij}^{[2]}} \frac{dJ_j}{dz_{ij}^{[2]}} = (a_{ij}^{[2]} - y_{ij}) \frac{da_{ij}}{dz_{ij}^{[2]}} - (a_{ij}^{[2]} - y_{ij}) \frac{dJ_j}{dz_{ij}^{[2]}} = (a_{ij}^{[2]} - y_{ij})(softmax'(z_{ij}^{[2]}) + 1)$$

Generalising this, we get

$$\frac{dJ}{dZ}_{(2 \times m)} = (A^{[2]} - Y)(softmax'(Z^{[2]}) + 1)$$

After this, we might be a little loose on the notations, but the idea remains the same. Let's find out $\frac{d\bar{J}}{dW}$ by the chain rule of differentiation, which is $\frac{1}{m} \frac{dJ}{dZ^{[2]}} \frac{dZ^{[2]}}{dW^{[2]}}$. $\frac{dZ^{[2]}}{dW^{[2]}}$, which is simply

$$\frac{dz_i^{[2]}}{dw_{i2\times 1}^{[2]}} = ({ai_{2\times 1}}^{[1]})^T \frac{d\bar{J}}{dW^{[2]}} = \frac{1}{m} \frac{dJ}{dZ^{[2]}} (A^{[2]})^T$$

Similarly, we can find

$$\frac{dZ^{[2]}}{db_j^{[2]}} = 1 \frac{d\bar{J}}{db^{[2]}} = \frac{1}{m} \frac{dJ}{dZ^{[2]}}$$

Again,

$$\frac{d\bar{J}}{dW^{[1]}} = \frac{1}{m} \frac{dJ}{dZ^{[2]}} \frac{dZ^{[2]}}{dA^{[1]}} \frac{dA^{[1]}}{dZ^{[1]}} \frac{dZ^{[1]}}{dW^{[1]}} \frac{dZ^{[2]} 2 \times m}{dA^{[1]} 2 \times m} = W_{2\times 2}^{[2]} \frac{dA^{[1]} 2 \times m}{dZ^{[1]} 2 \times m} = sech^2(Z^{[1]})_{2\times m} \frac{dZ^{[1]} 2 \times m}{dW^{[1]} 2 \times m} = (A^{[0]})_{2\times m}^T$$

$$\frac{d\bar{J}}{dW^{[1]}} = \frac{1}{m} (W^{[2]})^T \frac{d\bar{J}}{dZ^{[2]}} \odot sech^2(Z^{[1]})(A^{[0]})^T$$

$$\frac{d\bar{J}}{db^{[1]}} = \frac{1}{m} (W^{[2]})^T \frac{d\bar{J}}{dZ^{[2]}} \odot sech^2(Z^{[1]})$$

5 Updating Parameters

Now during each iteration we update the parameters as:

$$W^{[2]} := W^{[2]} - \alpha \frac{d\bar{J}}{dW^{[2]}} b^{[2]} := b^{[2]} - \alpha \frac{d\bar{J}}{db^{[2]}} W^{[1]} := W^{[1]} - \alpha \frac{d\bar{J}}{dW^{[1]}} b^{[1]} := b^{[1]} - \alpha \frac{d\bar{J}}{db^{[1]}}$$