MATHEMATICS

February 16, 2024

1 Matrices

1. In the interval $\pi/2 < x < \pi$, find the value of x for which the matrix $\begin{pmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{pmatrix}$ is singular.

2. Express the matrix $A = \begin{pmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{pmatrix}$ as the sum of a symmetric and skew symmetric matrix.

3. A trust fund has ₹35,000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to orphanage and second bond pays 10% interest per annum which will be given to an N.G.O. (Cancer Aid Society). Using matrix multiplication, determine how to divide ₹35,000 among two types of bonds if the trust fund obtains an annual total interest of ₹3,200. What are the values reflected in this question?

4. If $A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

5. Using properties of determinant, solve for x: $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

2 probability

6. Four cards are drawn successively with replacement from a well shuffled deck of 52 cards. What is the probability that
   (a) all the four cards are spades?
   (b) only 2 cards are spades?

7. A pair of dice is thrown 4 times, If getting a doublet is considered a success, find the probabbility distribution of number of successes. Hence find the mean of the distribution.

8. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who gusses at the answer will be correct with probability **1**3, what is the probability that the student knows the answer given that he answered it correctly?

3 Algebra

9. Prove that $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$.

10. Solve for x: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$, $x > 0$.

4 vector

11. Write the direction ratios of the vector $3\mathbf{a} + 2\mathbf{b}$ where $\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

12. Find the projection of the vector $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ on the vector $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

13. Write the vector equation of the line passing through $(1, 2, 3)$ and perpendicular to the plane $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + 9 = 0$.

14. Prove that $[\mathbf{a}, \mathbf{b} + \mathbf{c}, \mathbf{d}] = [\mathbf{a}, \mathbf{b}, \mathbf{d}] + [\mathbf{a}, \mathbf{c}, \mathbf{d}]$.

15. Find the shortrest distance between the following lines:
$\mathbf{r} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$ and $\mathbf{r} = 7\mathbf{i} - 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$.

16. Find the vector and cartesian equations of the plane passing through the line of intersection of planes
$\mathbf{r} \cdot (2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 7, \quad \mathbf{r} \cdot (2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 9$
such that the intercepts made by the plane on x-axis and z-axis are equal.

5 Differentiation

17. For what value of λ the function defined by $f(x) = \begin{cases} \lambda\left(x^2 + 2\right) & , if\, x \leq 0 \\ 4x + 6 & , if\, x > 0 \end{cases}$ is continuous at $x = 0$? Hence check the differentiability of $f(x)$ at $x = 0$.

18. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

19. if $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m + n)\frac{dy}{dx} + mny = 0$.

20. The sum of surface areas of a spere and a cuboid with sides $\frac{x}{3}$, x and $2x$ is constant . Show that the sum of their volumes is minimum if x is equal to three times the radius of sphere.

6 Integration

21. Find the solution to the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$

22. Write the integrating factor of the differential equation $\sqrt{x}\frac{dy}{dx} + y = e^{-2\sqrt{x}}$

23. Find $\int \frac{x}{(x^2+1)(x-1)} dx$.

24. Find $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$.

25. Find $\int \frac{x+3}{\sqrt{5-4x-2x^2}} dx$.

26. Solve the differential equation:
$\left(x \sin^2\left(\frac{y}{x}\right) - y\right) dx + x dy = 0$ given $y = \frac{\pi}{4}$ when $x = 1$.

27. Solve the differential equation $\frac{dy}{dx} - 3y \cot x = \sin 2x$ given $y = 2$ when $x = \frac{\pi}{2}$.

7 Linear forms

28. Find the area of the region $\left\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\right\}$, using integration.

29. Using integration, find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

8 Functions

30. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$ is an equivalence relation. Write all the equivalence classes of R.

9 Optimization

31. A manufacturer produces nuts and bolts. It takes 2 hours work on machine A and 3 hours work on machine B to produce a package of nuts. It takes 3 hours on machine A and 2 hours on machine B to produce a package of bolts. He earns a profit of ₹24 per package on nuts and ₹18 per package on bolts. How many packages of each should be produced each day so as to maximize his profit, if he operates his machines for at the most 10 hours a day. Make an L.P.P. from above and solve it graphically.