Assignment

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Question: Find the sum of n terms of GP where common ratio is r using Contour Integration.
Solution:

Parameter	Value	Description
$x(n)$	$x(0)r^n u(n)$	n^{th} term of GP
$x(0)$	$x(0)$	1^{st} term of GP
r	r	common ratio
$s(n)$	$\sum_{k=0}^{n} x(k)$	sum of n terms of GP

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{1}$$
$$= \sum_{n=-\infty}^{\infty} x(0) r^n u(n) z^{-n} \tag{2}$$
$$= \sum_{n=0}^{\infty} x(0) r^n z^{-n} \tag{3}$$
$$= \frac{x(0)}{1 - rz^{-1}} \tag{4}$$
$$U(z) = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{5}$$
$$S(z) = \sum_{n=-\infty}^{\infty} s(n) z^{-n} \tag{6}$$
$$s(n) = x(n) * u(n) \tag{7}$$
$$S(z) = X(z) U(z) \tag{8}$$
$$= \left(\frac{x(0)}{1 - rz^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right), |z| > 1 \tag{9}$$

Now we will perform inverse Z transform on S(z) using contour integration to find s(n)

$$s(n) = \frac{1}{2\pi j} \oint_C S(z) z^{n-1} \, dz \tag{10}$$
$$= \frac{1}{2\pi j} \oint_C \frac{x(0) z^{n-1}}{(1 - rz^{-1})(1 - z^{-1})} \, dz \tag{11}$$
$$= \frac{1}{2\pi j} \oint_C \frac{x(0) z^{n+1}}{(z - r)(z - 1)} \, dz \tag{12}$$
$$= \frac{x(0)}{r - 1}\left(\frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z - r} \, dz - \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{z - 1}\right) dz \tag{13}$$

we already know;

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z - a)^m f(z)\right) \tag{14}$$

Now for first contour integral,

$$R_1 = \frac{1}{(1-1)!} \lim_{z \to a} \left((z - a) f(z)\right) \tag{15}$$
$$= \lim_{z \to r} \left((z - r) \frac{z^{n+1}}{z - r}\right) \tag{16}$$
$$= \lim_{z \to r} \left(z^{n+1}\right) \tag{17}$$
$$= r^{n+1} \tag{18}$$

for second contour integral,

$$R_2 = \frac{1}{(1-1)!} \lim_{z \to a} \left((z - a) f(z)\right) \tag{19}$$
$$= \lim_{z \to 1} \left((z - 1) \frac{z^{n+1}}{z - 1}\right) \tag{20}$$
$$= \lim_{z \to 1} \left(z^{n+1}\right) \tag{21}$$
$$= 1 \tag{22}$$

So finally the sum of n terms of the GP is given by:

$$s(n) = \frac{x(0)}{r - 1} (R_1 - R_2) \tag{23}$$
$$= \frac{x(0)}{r - 1} \left(r^{n+1} - 1\right) \tag{24}$$