

Assignment

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Question: Suppose that $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ are independent and identically distributed random vectors each having $N_p(\boldsymbol{\mu}, \Sigma)$ distributions, where Σ is non-singular, $p > 1$ and $n > 1$. If $\mathbf{X} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i$ and $\mathbf{Y} = \frac{1}{n} \sum_{i=1}^n \mathbf{Y}_i$, then which one of the following statements is true?

- (a) There exists $c > 0$ such that $c(\mathbf{X} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})$ has χ^2 -distribution with p degrees of freedom.
- (b) There exists $c > 0$ such that $c(\mathbf{X} - \mathbf{Y})^T \Sigma^{-1} (\mathbf{X} - \mathbf{Y})$ has χ^2 -distribution with $(p - 1)$ degrees of freedom.
- (c) There exists $c > 0$ such that $c \sum_{i=1}^n (\mathbf{X}_i - \mathbf{X})^T \Sigma^{-1} (\mathbf{X}_i - \mathbf{X})$ has χ^2 -distribution with p degrees of freedom.
- (d) There exists $c > 0$ such that $c \sum_{i=1}^n (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y})^T \Sigma^{-1} (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y})$ has χ^2 -distribution with p degrees of freedom.

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Solution:

We are given that,

$$\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n \sim N_p(\boldsymbol{\mu}, \Sigma) \quad (1)$$

Also,

$$\mathbf{X} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i \quad (2)$$

$$\mathbf{Y} = \frac{1}{n} \sum_{i=1}^n \mathbf{Y}_i \quad (3)$$

The mean of \mathbf{X} is given by:

$$\boldsymbol{\mu}_X = E(\mathbf{X}) \quad (4)$$

$$= \frac{1}{n} \sum_{i=1}^n E(\mathbf{X}_i) \quad (5)$$

$$= \boldsymbol{\mu} \quad (6)$$

Similarly,

$$\boldsymbol{\mu}_Y = \boldsymbol{\mu} \quad (7)$$

The covariance of \mathbf{X} is given by:

$$\Sigma_{\mathbf{X}} = E \left[(\mathbf{X} - \boldsymbol{\mu}) (\mathbf{X} - \boldsymbol{\mu})^T \right] \quad (8)$$

$$= E \left[\left(\frac{1}{n} \sum_{i=1}^n \mathbf{X}_i - \boldsymbol{\mu} \right) \left(\frac{1}{n} \sum_{i=1}^n \mathbf{X}_i - \boldsymbol{\mu} \right)^T \right] \quad (9)$$

$$= \frac{1}{n^2} E \left[\sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\mu}) (\mathbf{X}_i - \boldsymbol{\mu})^T \right] \quad (10)$$

$$= \frac{1}{n^2} \left[\sum_{i=1}^n E (\mathbf{X}_i^2 + \boldsymbol{\mu}^2 - 2\boldsymbol{\mu} \mathbf{X}_i) \right] \quad (11)$$

$$= \frac{1}{n^2} \left[\sum_{i=1}^n E (\mathbf{X}_i^2) + \boldsymbol{\mu}^2 - 2\boldsymbol{\mu} \sum_{i=1}^n E (\mathbf{X}_i) \right] \quad (12)$$

$$= \frac{1}{n^2} \left[n\Sigma + n\boldsymbol{\mu}^2 + \boldsymbol{\mu}^2 - 2\boldsymbol{\mu}^2 \right] \quad \left[\because E (\mathbf{X}_i^2) = \Sigma_{\mathbf{X}_i} + E (\mathbf{X}_i)^2 \right] \quad (13)$$

$$= \frac{\Sigma}{n} \quad (14)$$

Similarly,

$$\Sigma_{\mathbf{Y}} = \frac{\Sigma}{n} \quad (15)$$

To check option (A):

let us say,

$$\mathbf{A} = c(\mathbf{X} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu}) \quad (16)$$

$$(17)$$

And,

$$\Sigma^{-1} = \mathbf{F}^T \mathbf{F} \quad (18)$$

$$\mathbf{y} = \mathbf{F} (\mathbf{X} - \boldsymbol{\mu}) \quad (19)$$

$$\implies \mathbf{A} = c\mathbf{y}^T \bar{\mathbf{y}} \quad (20)$$

$$= c\|\mathbf{y}\|^2 \quad (21)$$

Equation (21) shows that \mathbf{A} can have χ^2 -distribution.

To confirm that we will find the mean and covariance-matrix of $\bar{\mathbf{y}}$.

$$\boldsymbol{\mu}_{\mathbf{y}} = E (\mathbf{y}) \quad (22)$$

$$= E (\mathbf{F} (\mathbf{X} - \boldsymbol{\mu})) \quad (23)$$

$$= \mathbf{F} [E (\mathbf{X}) - E (\boldsymbol{\mu})] \quad (24)$$

$$= \mathbf{F} [\boldsymbol{\mu} - \boldsymbol{\mu}] \quad \text{from (6)} \quad (25)$$

$$= 0 \quad (26)$$

And,

$$\Sigma_{\mathbf{y}} = E \left[(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) (\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}})^T \right] \quad (27)$$

$$= E \left[(\mathbf{F} (\mathbf{X} - \boldsymbol{\mu})) (\mathbf{F} (\mathbf{X} - \boldsymbol{\mu}))^T \right] \quad (28)$$

$$= E \left[\mathbf{F} (\mathbf{X} - \boldsymbol{\mu}) (\mathbf{X} - \boldsymbol{\mu})^T \mathbf{F}^T \right] \quad (29)$$

$$= \mathbf{F} \left[E \left[(\mathbf{X} - \boldsymbol{\mu}) (\mathbf{X} - \boldsymbol{\mu})^T \right] \right] \mathbf{F}^T \quad (30)$$

$$= \mathbf{F} \Sigma \mathbf{F}^T \quad (31)$$

since,

$$\Sigma^{-1} = \mathbf{F}^T \mathbf{F} \quad (32)$$

$$\Sigma \Sigma^{-1} = \Sigma \mathbf{F}^T \mathbf{F} \quad (33)$$

$$\mathbf{I} = \Sigma \mathbf{F}^T \mathbf{F} \quad (34)$$

$$\mathbf{I} \mathbf{F}^{-1} = \Sigma \mathbf{F}^T \quad (35)$$

$$\mathbf{F} \mathbf{F}^{-1} = \mathbf{F} \Sigma \mathbf{F}^T \quad (36)$$

$$\mathbf{I} = \mathbf{F} \Sigma \mathbf{F}^T \quad (37)$$

So using (37),

$$\Sigma_{\mathbf{y}} = \mathbf{I} \quad (38)$$

Hence \mathbf{A} has χ^2 -distribution with p degrees of freedom.

So option (A) is correct.

To check option (B):

Let us say,

$$\mathbf{B} = c(\mathbf{X} - \mathbf{Y})^T \Sigma^{-1} (\mathbf{X} - \mathbf{Y}) \quad (39)$$

And,

$$\Sigma^{-1} = \mathbf{F}^T \mathbf{F} \quad (40)$$

$$\mathbf{y} = \mathbf{F} (\mathbf{X} - \mathbf{Y}) \quad (41)$$

$$\implies \mathbf{B} = c \mathbf{y}^T \bar{\mathbf{y}} \quad (42)$$

$$= c \|\mathbf{y}\|^2 \quad (43)$$

Equation (43) shows that \mathbf{B} can have χ^2 -distribution.

To confirm that we will find the mean and covariance-matrix of $\bar{\mathbf{y}}$.

$$\mu_{\mathbf{y}} = E(\mathbf{y}) \quad (44)$$

$$= E[\mathbf{F} (\mathbf{X} - \mathbf{Y})] \quad (45)$$

$$= \mathbf{F} [E(\mathbf{X}) - E(\mathbf{Y})] \quad (46)$$

$$= \mathbf{F} [\boldsymbol{\mu} - \boldsymbol{\mu}] \quad (47)$$

$$= 0 \quad (48)$$

And,

$$\Sigma_{\mathbf{y}} = E[(\mathbf{y} - \mu_{\mathbf{y}})(\mathbf{y} - \mu_{\mathbf{y}})^T] \quad (49)$$

$$= E[(\mathbf{F} (\mathbf{X} - \mathbf{Y})) (\mathbf{F} (\mathbf{X} - \mathbf{Y}))^T] \quad (50)$$

$$= E[\mathbf{F} (\mathbf{X} - \mathbf{Y}) (\mathbf{X} - \mathbf{Y})^T \mathbf{F}^T] \quad (51)$$

$$= \mathbf{F} [E[(\mathbf{X} - \mathbf{Y}) (\mathbf{X} - \mathbf{Y})^T]] \mathbf{F}^T \quad (52)$$

$$= \mathbf{F} [E[\|\mathbf{X} - \mathbf{Y}\|^2]] \mathbf{F}^T \quad (53)$$

$$= \mathbf{F} [E(\mathbf{X}^2) + E(\mathbf{Y}^2) - E(2\mathbf{X}\mathbf{Y})] \mathbf{F}^T \quad (54)$$

$$= \mathbf{F} \left[\frac{\Sigma}{n} + \boldsymbol{\mu}^2 + \frac{\Sigma}{n} + \boldsymbol{\mu}^2 - 2\boldsymbol{\mu}^2 \right] \mathbf{F}^T \quad [\because E(\mathbf{X}^2) = \Sigma_{\mathbf{X}} + E(\mathbf{X})^2] \quad (55)$$

$$= \frac{2}{n} \mathbf{F} \Sigma \mathbf{F}^T \quad (56)$$

$$= \frac{2}{n} \mathbf{I} \quad (57)$$

Hence \mathbf{B} has χ^2 -distribution with p degrees of freedom.

So option (B) is incorrect.

To check option (C):

let us say,

$$\mathbf{C} = c \sum_{i=1}^n (\mathbf{X}_i - \mathbf{X})^T \Sigma^{-1} (\mathbf{X}_i - \mathbf{X}) \quad (58)$$

And,

$$\Sigma^{-1} = \mathbf{F}^T \mathbf{F} \quad (59)$$

$$\mathbf{y} = \mathbf{F} \left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{X}) \right) \quad (60)$$

$$\Rightarrow \mathbf{C} = c \mathbf{y}^T \bar{\mathbf{y}} \quad (61)$$

$$= c \|\mathbf{y}\|^2 \quad (62)$$

Equation (62) shows that \mathbf{C} can have χ^2 -distribution.

To confirm that we will find the mean and covariance-matrix of $\bar{\mathbf{y}}$.

$$\mu_{\mathbf{y}} = E(\mathbf{y}) \quad (63)$$

$$= E \left[\mathbf{F} \left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{X}) \right) \right] \quad (64)$$

$$= \mathbf{F} \left[\sum_{i=1}^n (E(\mathbf{X}_i) - E(\mathbf{X})) \right] \quad (65)$$

$$= \mathbf{F} [E(X_1) - E(X) + E(X_2) - E(X) + \dots + E(X_n) - E(X)] \quad (66)$$

$$= 0 \quad (67)$$

And,

$$\Sigma_{\mathbf{y}} = E \left[(\mathbf{y} - \mu_{\mathbf{y}}) (\mathbf{y} - \mu_{\mathbf{y}})^T \right] \quad (68)$$

$$= \mathbf{F} E \left[\left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{X}) \right) \left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{X}) \right)^T \right] \mathbf{F}^T \quad (69)$$

$$= \mathbf{F} E \left[\left(\sum_{i=1}^n \mathbf{X}_i - n\mathbf{X} \right) \left(\sum_{i=1}^n \mathbf{X}_i - n\mathbf{X} \right)^T \right] \mathbf{F}^T \quad (70)$$

$$= \mathbf{F} E \left[(n\mathbf{X} - n\mathbf{X}) (n\mathbf{X} - n\mathbf{X})^T \right] \mathbf{F}^T \quad (71)$$

$$= \mathbf{F} \mathbf{0} \mathbf{F}^T \quad (72)$$

$$= \mathbf{0} \quad (73)$$

Hence \mathbf{C} does not have χ^2 -distribution.

So option (C) is incorrect.

To check option (D):

let us say,

$$\mathbf{D} = c \sum_{i=1}^n (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y})^T \Sigma^{-1} (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y}) \quad (74)$$

And,

$$\Sigma^{-1} = \mathbf{F}^T \mathbf{F} \quad (75)$$

$$\mathbf{y} = \mathbf{F} \left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y}) \right) \quad (76)$$

$$\Rightarrow \mathbf{C} = c \mathbf{y}^T \bar{\mathbf{y}} \quad (77)$$

$$= c \|\mathbf{y}\|^2 \quad (78)$$

Equation (78) shows that \mathbf{D} can have χ^2 -distribution.

To confirm that we will find the mean and covariance-matrix of $\bar{\mathbf{y}}$.

$$\boldsymbol{\mu}_y = E(\mathbf{y}) \quad (79)$$

$$= E \left(\mathbf{F} \left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y}) \right) \right) \quad (80)$$

$$= \mathbf{F} E \left[\sum_{i=1}^n \mathbf{X}_i - \sum_{i=1}^n \mathbf{Y}_i - n\mathbf{X} + n\mathbf{Y} \right] \quad (81)$$

$$= \mathbf{F} \left[\sum_{i=1}^n E(\mathbf{X}_i) - \sum_{i=1}^n E(\mathbf{Y}_i) - nE(\mathbf{X}) + nE(\mathbf{Y}) \right] \quad (82)$$

$$= \mathbf{F} [n\boldsymbol{\mu} - n\boldsymbol{\mu} - n\boldsymbol{\mu} + n\boldsymbol{\mu}] \quad (83)$$

$$= \mathbf{0} \quad (84)$$

And,

$$\Sigma_y = E \left[(\mathbf{y} - \boldsymbol{\mu}_y) (\mathbf{y} - \boldsymbol{\mu}_y)^T \right] \quad (85)$$

$$= \mathbf{F} E \left[\left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y}) \right) \left(\sum_{i=1}^n (\mathbf{X}_i - \mathbf{Y}_i - \mathbf{X} + \mathbf{Y}) \right)^T \right] \mathbf{F}^T \quad (86)$$

$$= \mathbf{F} E \left[\left(\sum_{i=1}^n \mathbf{X}_i - \sum_{i=1}^n \mathbf{Y}_i - n\mathbf{X} + n\mathbf{Y} \right) \left(\sum_{i=1}^n \mathbf{X}_i - \sum_{i=1}^n \mathbf{Y}_i - n\mathbf{X} + n\mathbf{Y} \right)^T \right] \mathbf{F}^T \quad (87)$$

$$= \mathbf{F} E \left[(n\mathbf{X} - n\mathbf{Y} - n\mathbf{X} + n\mathbf{Y}) (n\mathbf{X} - n\mathbf{Y} - n\mathbf{X} + n\mathbf{Y})^T \right] \mathbf{F}^T \quad (88)$$

$$= \mathbf{F} \mathbf{0} \mathbf{F}^T \quad (89)$$

$$= \mathbf{0} \quad (90)$$

Hence \mathbf{D} does not have χ^2 -distribution.

So option (D) is incorrect.