DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

MA601: Graphs and Matrices — Quiz IV

Instructor: Bikash Bhattacharjya — March 22, 2021

Time: 45 Minutes [including PDF making and uploading time] — Maximum Marks: 8

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1. Let Q be the $(0, 1, -1)$-incidence matrix of a tree on n vertices.
   Show that $\det(Q'Q) = n$. $\boxed{\mathbf{2}}$

   Solution: Let T be a tree with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$.

   Let Q_i be the matrix obtained by deleting i-th row of Q. We know that $\det(Q_i) = \pm 1$.

   By Cauchy-Binet Formula, we have

   $$\det(Q'Q) = \sum_S \det Q[S|\{1, \ldots, n-1\}]^2,$$

   where the summation is over all subsets S of $\{1, 2, \ldots, n\}$ with $|S| = n - 1$. Note that, if $|S| = n-1$ and $i \notin S$, then $Q[S|\{1, \ldots, n-1\}] = Q_i$. Therefore

   $$\det(Q'Q) = \sum_S \det Q[S|\{1, \ldots, n-1\}]^2 = \sum_{i=1}^{n} \det(Q_i)^2 = \sum_{i=1}^{n} 1 = n.$$

   <u>**Marking Scheme**</u>:

   1 mark is deducted for any minor mistake or missing a step.

2. Write the fundamental cut-matrix of the following graph with respect to the spanning tree shown by the **thick edges**. [Only write the matrix, no justification needed.] $\boxed{\mathbf{2}}$

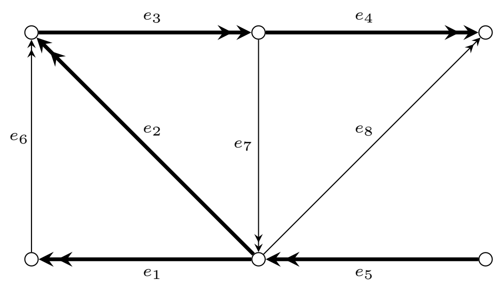

Figure 1: A spanning tree is indicated by the thick edges

Solution: The required matrix is

$$\left[\begin{array}{ccccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Marking Scheme:

1 mark is deducted for one minor mistake.

3. Prove or disprove: If the fundamental cycle-matrix of a simple graph G with respect to a spanning tree T_1 is equal to the fundamental cycle-matrix of another simple graph H with respect to a spanning tree T_2, then the graphs G and H are isomorphic to each other. $\boxed{4}$

Solution: Consider the graphs G and H shown in Figure 2.

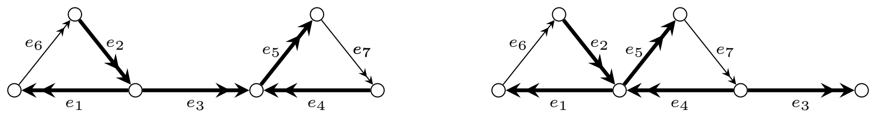

Figure 2: Two non-isomorphic graphs with equal fundamental cycle-matrix

These graphs are clearly non-isomorphic to each other, as H has a vertex of degree 1, while G has no vertex of degree 1.

The fundamental cycle matrix $C(G)$ of G with respect to the spanning tree, **shown by thick edges**, is

$$C(G) = \left[\begin{array}{ccccc|cc} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

The fundamental cycle matrix $C(H)$ of H with respect to the spanning tree, **shown by thick edges**, is

$$C(H) = \left[\begin{array}{ccccc|cc} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

We see that $C(G) = C(H)$, even though $G \ncong H$. Hence the statement is disproved by this counterexample.

Note: Many such counterexample can be constructed.

Marking Scheme:

1 or 2 marks is deducted for any minor mistake or missing a step.