2 Player Zero Sum Quantum Games

An Introduction to Zero Sum Games & their Quantum Counterparts

Introduction

Background

What is a Game

“Games” are a common trope among all people these days. It is the one of the few things we know all humans have at some time collectively engaged with. Although the contemporary use of the word “Game” is usually loosely defined, and its definition in everyday life is malleable and wide ranging. It usually relates to a piece of entertainment that humans can actually impact. Culturally, games have been given the broadest of definitions, such as “a way to entertain the mind in times of boredom”, or “a voluntary attempt to overcome unnecessary obstacles.” However, even movies and books seemingly satisfy this definition. Over the past 100 years, mathematicians have described exactly what a game is, the kind of game that this paper will be exploring.. Let's start with the broad definition of a game that needs to be something to entertain the mind. An aspect missing from this definition is the actual interactivity that is intrinsic to games of all forms. Furthermore, we would also like games to have a goal, otherwise this isn't a game, but instead it is a toy, such as a barbie doll. Continuing, we see that games also need to have multiple agents to compete with or against, otherwise this isn't a game, but a puzzle, like a rubik's cube. Likewise, it is required that a game’s agents be allowed to interact with each other in order to influence another’s outcomes, otherwise this isn't a game, but a contest, like a foot-race. If all the criteria above are met, then the form of entertainment for the mind can be called a full fledged game. So, all in all, a game needs to be a form of play that is entertaining, interactive, and involves multiple agents who can interfere with and influence each other's outcomes.

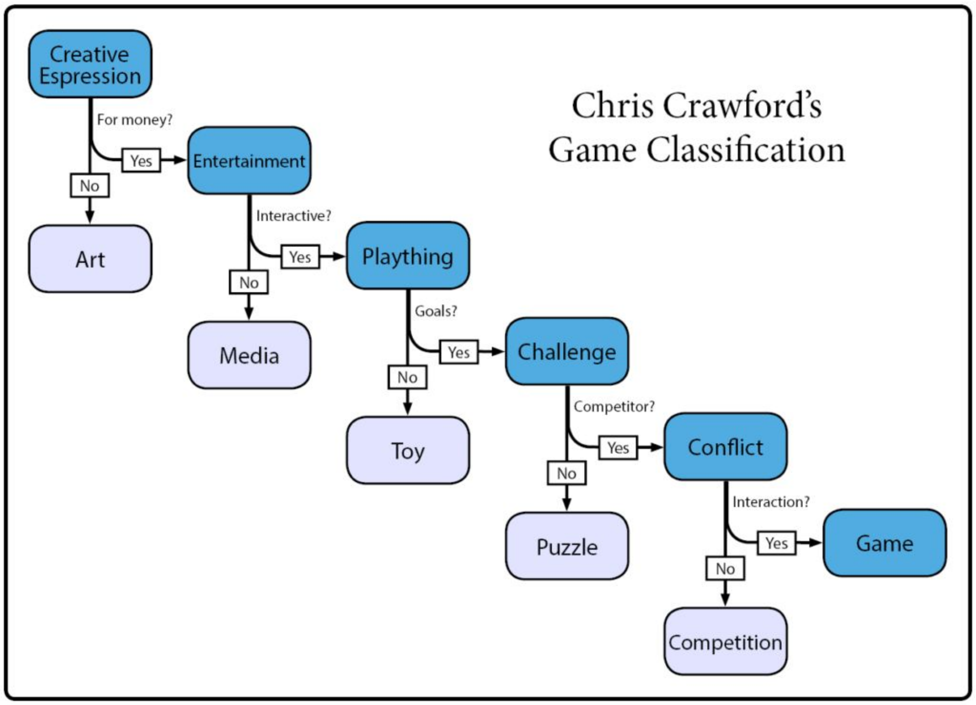


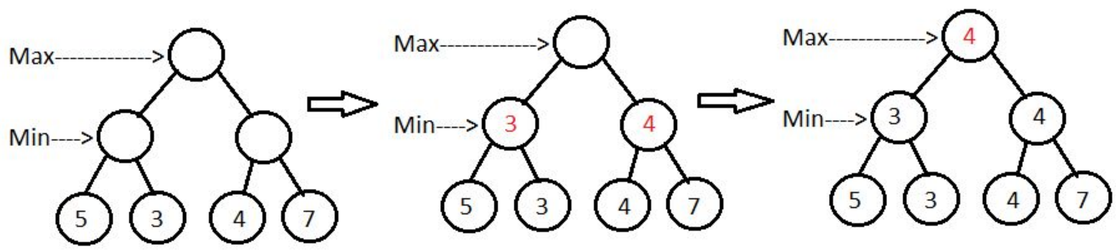
Fig: Crawford Exact Definition of a "Game"

What does Zero-Sum mean?

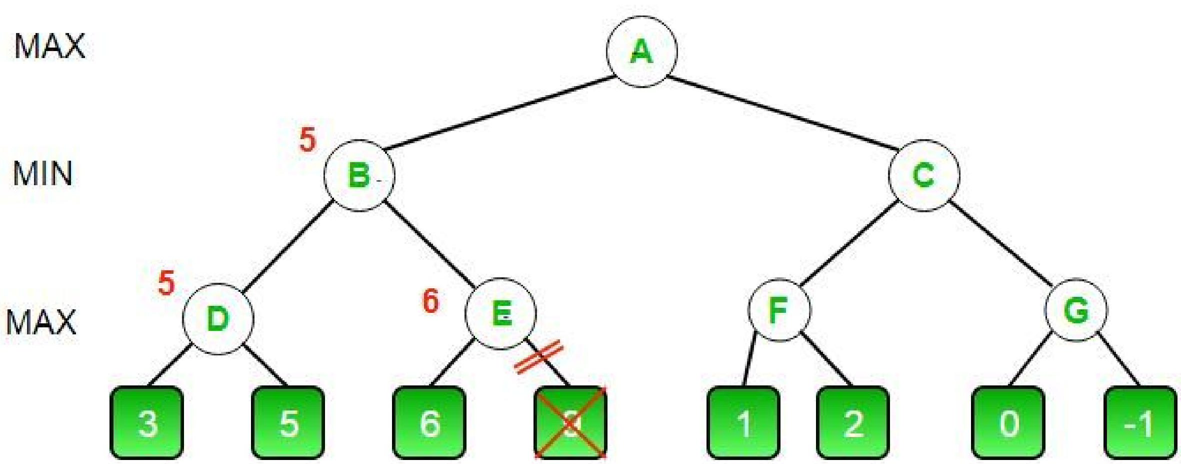
As was demonstrated in the previous section, games require multiple interactive agents that interfere and/or influence everyone's outcomes. This paper is only going to focus on the case where there are 2 players. At every timestep, any game can be represented by a game state, which describes all the most recent and relevant information about the game that is required for the perfect replication of the current rendition of the game with no loss of information. Zero sum games are a subset of games that require each gamestate to be Pareto optimal. This essentially means that at each game state, any action that increases the utility of one player must lead to the opposing player losing that much utility, and vice-versa. An example of a simple zero sum game would be checkers: any utility gained by making a move in checkers means that the opponent lost the exact same amount of utility as the other player gained. More complex examples include financial options and futures, where one group's profit comes from the loss of the complement of this set within the same game.

Classical methods of solving games

Players (not necessarily human ones) of a game are called agents. A “rational agent” is defined to be an agent that always takes actions that maximize expected utility. Mathematicians define a game to be “solved”, when given a state the agent knows and enacts the set of moves a rational agent would make from that state. In simple words, in a solved game, given the starting state we know the exact path (set of moves one after the other until the end of the game) that a player playing *perfectly* (anticipating all possible future opponent moves) would play. This set of moves is often referred to as a strategy. In this paper we shall be focusing on non-stochastic games, where a strategy is a set of moves that are each played definitievely, as opposed to stochastic game where the opponent is allowed to play a probability dist over the possible actions available. We will see how this assumption deals with the quantum extension later in this paper. Currently the state of the art classical technique for solving deterministic games is called the Minimax algorithm with αβ-pruning. The figure below demonstrates an example:



As shown above, in a zero sum game, the rational agent attempts to maximize his utility at each turn he is involved in. He/she also assumes that the opposing agents are trying to maximize **their** utility, which is the same as minimizing his, during **their** turn. This results in the maximizing agent always playing the move that maximizes his possible utility. αβ-pruning improves this technique to explore branches of the tree that are known to have lower utility than other available options. An example is shown below



Applications and Foreword

The applications of games and methods of solving them are useful in all situations reducible to zero sum games. As mentioned above, bonds and futures are one such example. Other examples are games like black-jack, chess, etc. Techniques like this can also be used to analyze statistics about sports, and can thus be used as a method to improve strategy and technique. In this paper we will explore the quantum counterparts of these classical 2 player zero sum games. It will investigate how the principles of superposition and collapse dramatically expand the size of the state-space, leading to vastly different and more numerous outcomes. From education to benchmarking, there are a wide variety of plausible applications.

Technical Content

Quantum Games

In order to extend the world of classical games into the quantum realm, we need to introduce and define what quantum principles are we allowed to use in a quantum game, while still maintaining the definitions of what a game is described as above. While analyzing fundamental principles of quantum mechanics, a prudent question always seems to arise: which principles are compatible with the definitions of a game set forth in the introduction? Quantum game theorists answered this by adding the following extensions to Zero-Sum games:

- Players can play a superposition of multiple moves and/or play moves that force the game state into superposition, or edit that superposition.

The details of this are up to the rule maker, the rules may limit the total amount of moves allowed to be played in as basis states, or even define a minimum/maximum amplitude of a given **initial** basis state. If we make the minimum amplitude $\frac{1}{\sqrt{2}}$, then only a superposition of 2 moves is allowed. If we allow for a minimum amplitude of 1, we are left with a classical deterministic game.

- Moves can, but do not necessarily need to be, beentangle states, and if this resulting entanglement leads to the violation of other rules intrinsic to games, the entanglement must be collapsed before such inconsistencies occur during the next turn.

This happens when agents play moves that result in the current action to be entangled with actions (moves) played in previous iterations of the game. When this happens, it often results in only one set of moves played upon measurement. It is often the case that the game cannot proceed without forcing a collapse when entanglement occurs. This is usually the case when not collapsing the state would result in a violation of the definition of a game.

- Measurement is the action that commits utility to a player, thus upon measurement, any utility gained by one player, must be lost by the other (in order for the game to be zero sum)

This rule defines what exactly it means to be "zero sum" in a quantum game. Mathematically the rule can be rewritten as the following.

let Ψ be the complete quantum gamestate wavefunction.
At every turn of the game, the following must hold
$\sum_{i=1}^{n} \Delta U_i(\Psi) = 0$
or in the case of two players
$\Delta U_1(\Psi) + \Delta U_2(\Psi) = 0$

This looks not too different from the classical definition. Here, when the game state wave function is measured, no matter what basis it collapses to, we are ensuring that in every possible measurement outcome, the utility gained by player 1 is lost by player 2. This extends to multiple players by saying the change in utility of all the players, in all the possible measurement outcomes, sums to zero. Make any game that follows all the rules of a classical game, along with these extensions, and it is now considered a quantum game.

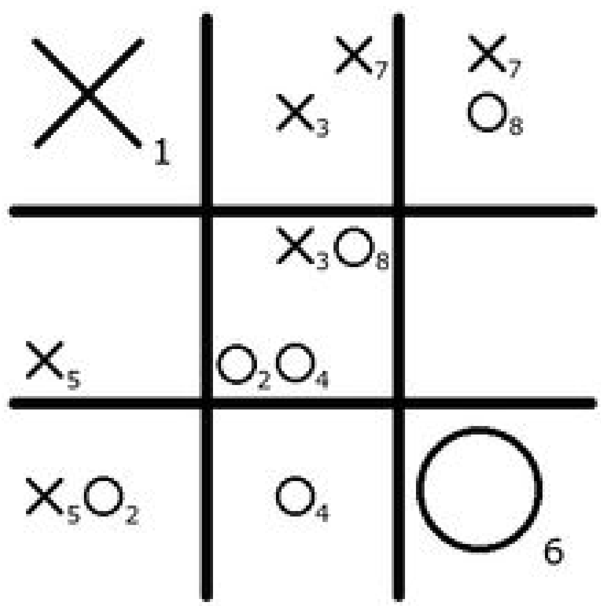


Fig: Example of a quantum game, Quantum Tic Tac Toe, later addressed in this paper

Examples of Quantum Zero Sum Games

Although the rules above describe quantum games quite well, they seem overwhelming. It is in high opinion that quantum mechanics, and hence quantum games, are too complicated, but upon closer inspection that is hardly the case. The following are examples of how the quantum games are easy to understand, but also demonstrate how the quantum postulates actually make the games more interesting to play.

This is perhaps the easiest to understand version of a quantum game. This game is used as an example to learn the quantum constraints #1 and #3. As #2 does not force entanglement, this game chooses to ignore this constraint. For those unfamiliar, the classical "Coin Game" is played by the following rules, as described by the QuisKit team at IBM:

1. The game starts with the coin showing Heads.
2. Player A starts and may either turn the coin or leave it as is.
3. The moves are hidden, i.e. not revealed to the other player.
4. B may now also turn the coin or leave it as is.
5. A then has the third and final move.
6. Now the coin gets revealed.
7. If it shows Heads, A wins; if it shows Tails, B wins.

In a quantum version of the game, we simply change step 4 to not only allow for a flip (like an X-gate) but to also allow for Hadamard gate and Z-gate operations on the coin. Furthermore, the moves are no longer hidden, both players can see the state of the coin but in the end, instead of “showing the coin” it is instead measured in the computational basis. Both rules are very visibly satisfied. Both players can use the H and Z gates to edit the GameState wavefunction, satisfying constraint 1. Furthermore, upon measurement one player wins (utility +1) and the other must lose (utility -1), satisfying constraint #3 and hence proving this is a quantum zero-sum game.

Solving Quantum Games

Just like the Minimax algorithm works to solve classical 2-player games, there exists a version of Minimax that manifests itself into the world of quantum games. Fuyuhiko Tanaka and his research group at Osaka University, in 2014, published a research paper proving that the Minimax theorem from statistics had a valid extension into quantum statistical mechanics. Following this discovery, Andreas Gilyen and his team from the Center for Mathematics and Information Science discovered an algorithm for solving Quantum Games, similar to one that solves stochastic games (using a payoff matrix). Furthermore, just as a linear program is used to solve stochastic classical games, a quantum linear program can be used to solve quantum games. Quantum linear programs, just like linear programs, are in essence solved by the simplex algorithm, which checks all the solutions to the system on linear constraints, and the solution with the highest objective is the global max of the objective given the constraints. The optima are found by finding the solution to the system of linear equations derived from the constraints. In essence, if we can solve this system of linear equations efficiently, we shall be able solve 2 player zero sum games as well. Earlier, in

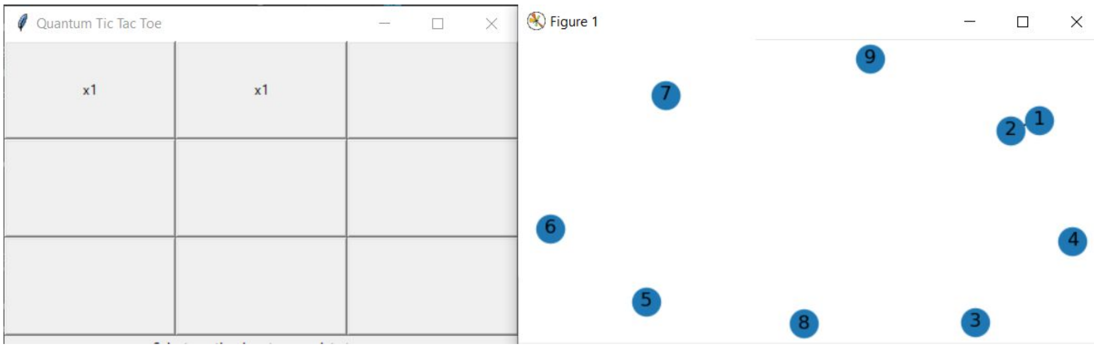
2009, Seth Lloyd and team showed that we can efficiently encode the system of linear equations into Hamiltonian operators, and thereby a quantum system, whose eigenvalues are the energies of a quantum system, and eigenstates are the corresponding eigenvectors. This is essentially an efficient way to diagonalize a matrix, which can hence be easily used to find the inverse matrix in a system of linear equations. Lloyd et al showed that this can be done in O(log(N)) runtime, where N is the number of variables in the system. When this is incorporated into the simplex algorithm, and then applied to quantum games, we achieve a runtime of O(m/epsilon) where m is the largest dimension of the matrix, and epsilon is the sparsity of the matrix. The classical minimax algorithm works in exponential time, and quantum computers, as expected, give us an exponential speedup on this. While it is true that methods such as alpha-beta pruning--mentioned in the background--do exist, their use is not necessary as the linear runtime makes finding an optimal strategy incredibly tractable. The caveat of course is that this game solver can only work on a quantum computer due to the fact that finding the inverse of a matrix is sublinear on a quantum computer. If one were to try and use classical eigenvector solvers for this, the resulting algorithm would, in essence, be the same as minimax. This is yet another example of quantum supremacy in action!

Designing a Quantum Game: Q-Tic-Tac-Toe

Designing the game

Our team decided to also study what it means to play a quantum game, and that this would solidify many of the principles set forth above. To this end, we designed a fully functional version of Quantum Tic Tac Toe. Note, although we did draw inspiration from the traditional “Quantum Tic Tac Toe” game, we designed this game by adding the 3 quantum principles to regular tic tac toe. This is also the reason that some of our rules differ from the aforementioned “traditional” Quantum Tic Tac Toe.

As mentioned above, we started with regular Tic Tac Toe, and built upon it. For future reference, the squares on the board will be labeled 1-9 row by row, and the state X represents an X in the referred square. Likewise, an O represents an O, and a B represents a blank. To address the constraint 1, we say that a player can play his/her move in a superposition of exactly 2 locations. Due to the way our gamestate representation was decided, it is more convenient to view this as a move that “forces the gamestate into superposition” rather than a “superposition of possible moves”. Below is a screenshot from out team’s simulated game portraying what that looks like for a human agent.



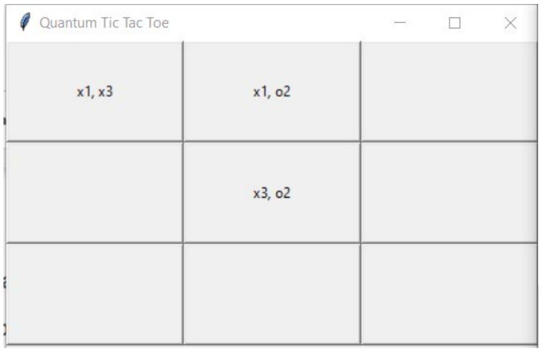
On the left there is an image of the board, and it shows that move 1 was played by agent X, and the game board (expressed as a joint state of location wave-functions) is now in a superposition of states:

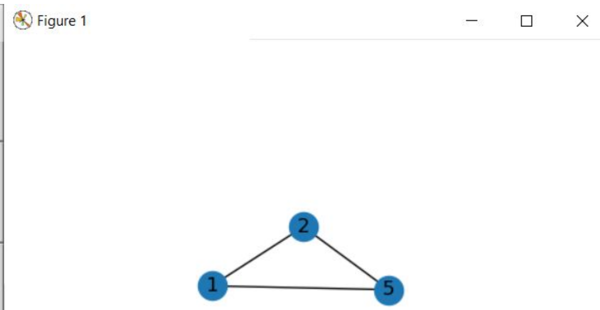
$$\Psi = \frac{1}{\sqrt{2}}|XB...B\rangle + \frac{1}{\sqrt{2}}|BX...B\rangle$$

We can quickly see how this directly segways into constraint #2. When viewed on a location by location basis, we see that location 1 is entangled with location 2. This is also reiterated by the graph visualization on the right side, which shows locations as being in a connected component if they are entangled. Locations 1 and 2 are shown as connected, and are in the entangles state:

$$\Psi = \frac{1}{\sqrt{2}}|XB\rangle + \frac{1}{\sqrt{2}}|BX\rangle$$

As hinted at above, this can eventually lead to inconsistencies within the game. In our case inconsistencies are a result of entanglement cycles. Let's consider the following scenario:

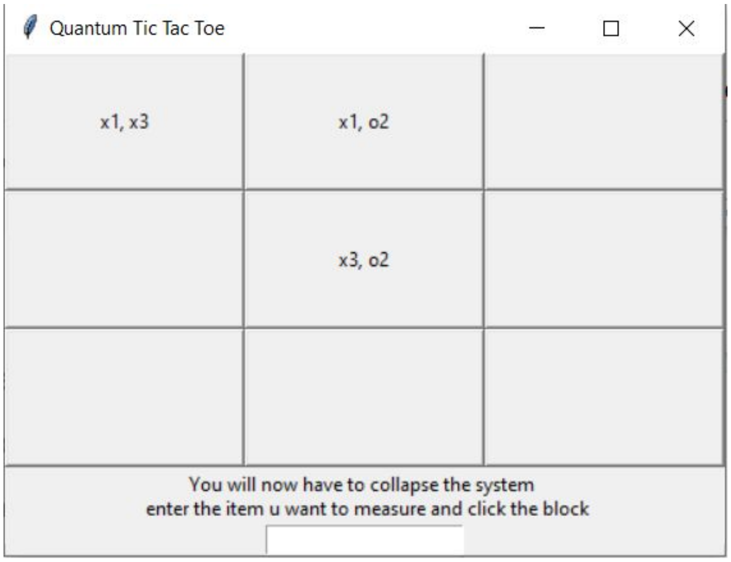




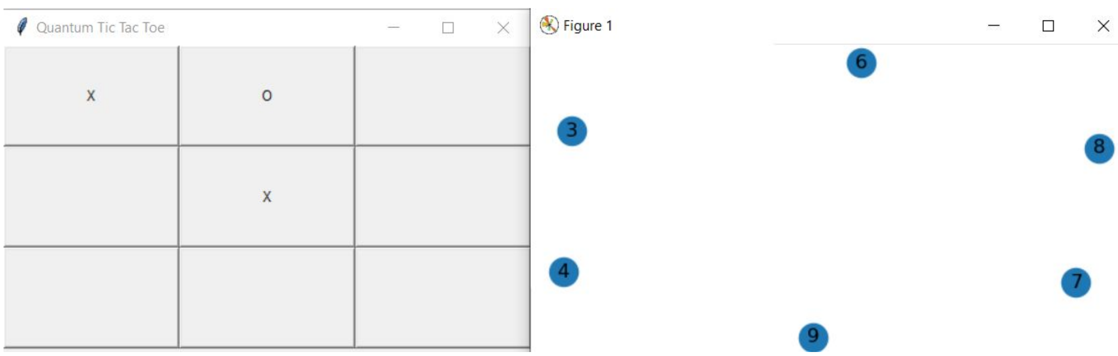
Here we see the board on the left, which seems innocuous at first. On the right we see that the entanglement graph has been put into a cycle. In order to see the inconsistency, let's write out the full wavefunction of the non-blank squares (joint state of 1, 2, 5 in that order):

$$\Psi = \frac{1}{\sqrt{2}}|XOX\rangle + \frac{1}{\sqrt{2}}|XXO\rangle$$

Notice however, if the wave function is not collapsed right now, agent O could now (according to our rules) choose squares 2 and 5 as his move. However, this would mean that the agents have played 4 moves on 3 locations. When measured, this would imply that we would need to collapse 4 characters into 3 squares, and by the pigeonhole principle is not possible. Thus, in accordance with constraint #2, we added the following rules to the game. If the moves result in a cyclic entanglement, then the agent that caused the entanglement must choose a specific state of the entangled connected component to amplify the amplitude of, and then partially measure the squares in said entangled connected component. Since this rule takes place immediately after an entanglement cycle is created, this avoids any inconsistencies. No agent can play any other moves until the component is collapsed. An example is shown in the prompt of the image below.



Once the entanglement cycle is amplified and measured, no agents can play on those squares, as shown in the following.



Notice that locations 1, 2, 5 are no longer nodes in the entanglement graph on the right due to the fact they have been measured, and can not be entangled with any other locations again.

In order to ensure that our design is accurate to the definition of "quantum zero-sum", we need to make sure that the total change in utility of all players for any given collapse measurement is always 0. Looking at the image above we see that upon measurement, the locations in the same connected component as the cycle are (amplified and then) measured, resulting in a collapse back into a classical Tic-Tac-Toe state. Hence if we define “winning” this game as having a *measured* set of 3 characters in a row, the change in utility for any collapsable state would mirror that of a classical Tic-Tac-Toe game, which does indeed some to 0.

Designing a Quantum Rational Agent

Now that we have a game design, let us explore what it is like to design and play against a Quantum Rational Agent. Our group has designed a QAI agent that uses the minimax algorithm (since classically simulated quantum minimax is reducible from classical minimax) to play Quantum-Tic-Tac-Toe. Below we have shown how we chose to implement the minimax algorithm for our purposes (we have included alpha-beta pruning for faster runtime.

```
def minimax_helper(board, depth, isMaximizingPlayer, alpha, beta):

   print(depth)

   winner = board.is_win()
   if winner == 'X':
       return 100, "win"
   elif winner == 'O':
       return -100, "loss"
   elif winner == "nobody":
       return 0, "tie"
   elif depth <= 0:
       measured_vals = board.measured.values()
       num_x = sum([1 for val in measured_vals if val == 'X'])
       num_o = len(measured_vals) - num_x
       return num_x - num_o, "unknown"

   if isMaximizingPlayer:
       bestVal = float('-inf')
       bestMove = None
       for successor_board, successor_move in board.get_succesors():
           value = QBoard.minimax_helper(successor_board, depth - 1,
successor_board.curr_turn == 'x', alpha, beta)[0]
           if value > bestVal:
               bestVal = value
               bestMove = successor_move
           alpha = max(alpha, bestVal)
           if beta <= alpha:
               break
       return bestVal, bestMove

   else:
       bestVal = float('inf')
       bestMove = None
       for successor_board, successor_move in board.get_succesors():
           value = QBoard.minimax_helper(successor_board, depth - 1,
successor_board.curr_turn == 'x', alpha, beta)[0]
           if value < bestVal:
               bestVal = value
               bestMove = successor_move
           beta = min(beta, bestVal)
           if beta <= alpha:
               break
       return bestVal, bestMove
```

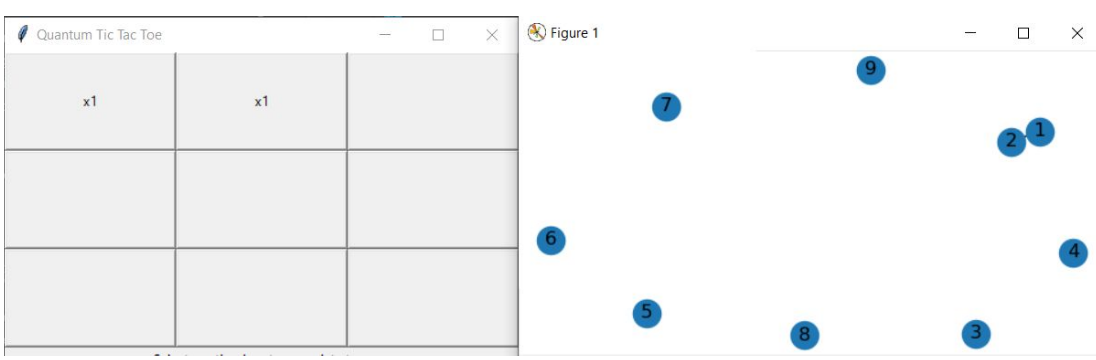
Our maximizing agent is always player X, and thus always goes first. In testing this Quantum AI, our team played 25 games against it trying to outsmart it, and the rational agent won 15 games, and tied the rest.

Game State representation & Mechanics on a Quantum Computer

Let us explore how a game state representation of such a game would manifest itself in a quantum computer, as well as how some of the game mechanics required for this to work would manifest themselves on a quantum computer. Lets see, each location can be in one of 3 basis states, either X, O, or B. Hence each of the locations can be represented by 2 qubits, for a total of 18 qubits, using the following encoding scheme:

$$\text{let} \rightarrow |00\rangle = |B\rangle$$
$$\text{let} \rightarrow |01\rangle = |X\rangle$$
$$\text{let} \rightarrow |10\rangle = |O\rangle$$

At the same time, using a few classical bits, we keep track of the entanglement graph, which will be useful during amplitude amplification. With this out of the way, we can work above this abstraction level. When a superposition (like in the following picture) of moves is played, locations and their states are now entangled with one another.



This will result in forcing location 1 and 2 to be entangled together, forcing the joint state of these locations to be:

$$\Psi = \frac{1}{\sqrt{2}}|XB\rangle + \frac{1}{\sqrt{2}}|BX\rangle$$

When a board is an entanglement cycle, it is an agent to choose a state to amplify. The way this is done is that we use the classical representation of the graph to calculate the locations involved in the cyclic connected component. Then a Grover's style amplitude amplification is run on the (partial) wave function projected onto this connected component. Immediately after, these locations are measured, collapsing the wave function on these locations (but leaving the ones not part of this connected component undisturbed). As one can clearly see the size of the minimal state space representation is extremely low on a Quantum Computer as opposed to a classical one. Through the lens of Quantum-Tic-Tac-Toe we can see how most quantum games can be designed and implemented. This also demonstrates that most quantum games are not only faster to solve in runtime, but also easier to run due to the small space constraints on the number of qubits required.

Discussions

Current Uses & Potential Applications

As of recent years, quantum games--similar to the rest of the field--have been mostly theoretical. However this hasn’t stopped many from using Quantum games as a teaching tool to solidify the principles and concepts of quantum mechanics. This allows for students to understand and build intuition for the incredibly complex and eccentric realm of quantum mechanics. Beyond this, a common idea is to use quantum games as a measure to benchmark quantum computers. Due to the different nature of different types of quantum computers, from annealing to trapped-ion-gate-based, a universal benchmark is needed to compare different machines. The ability to solve quantum games of ever increasing size is a big candidate for such a benchmark. Furthermore, quantum games are a higher skill alternative to their classical counterparts. Recently, the IBM quantum team has designed a quantum version of Battleship. This twist on the classic not only helps teach the principles of Quantum Mechanics, but is also more challenging and ,as many people believe, more fun than the classical alternative. Skeptics have often said that Quantum games, due to their highly theoretical nature, have no other real applications at all, but this is what skeptics said about game theory as well when it was a nascent field. Today classical game theory is used in stock market analysis, sports medicine and sports strategy, defense, and not to mention a 135 Billion dollar gaming entertainment industry. Most experts are sure that the application of Quantum Game Theory, just like its classical predecessor, are numerous but yet to be discovered.

Potential Complications

Due to the high cost of constructing a quantum computer, it is infeasible for most people to play a quantum game with a friend on a PQC. Even if we account for mostly cloud based access to quantum computers, as of 2020, no quantum computer has enough qubits to allocate enough resources for more than a few games at a time. Hence a lot of the potential for scalability is lost due to lack of resources and accessibility. In comparison, a multiplayer version of the game “Doom” can be run on a $30 rasberry pi, whereas QCs cost millions of dollars and can only handle a few games of IBMs quantum battleship at a time. Just like it did with classical computers, it will likely take 1-2 decades before quantum games reach a similar level of accessibility as even primitive classical games like chess, checkers, and Go. Furthermore, some skeptics do believe that the added complexity of quantum games will act as a deterrent for most people. This could be the case considering since most people, when searching for a game, look for something challenging to master, yet easy to learn. Quantum Games won't fulfill this criteria until Quantum Mechanics is much more commonplace than its current state. This is also likely why Quantum Games won't be mainstream until quantum physics isn't a mandatory high school course.

Conclusion

As is evident in the paper, quantum games is still a field in its nascency, and will probably take decades to evolve to the same level of widespread adoption and applications as current classical games. But even at this stage, we can begin to see the potential that these games possess. Despite popular opinion that quantum games are too complex to be adopted, many--including our own group--were able to apply the principles of quantum mechanics to simple games like Tic Tac Toe and Battleship, elevating both the challenge and the level of gameplay. This paper also demonstrates the simplicity and efficiency of creating a quantum rational agent, and benchmarks its performance. Our group also used this opportunity to teach ourselves and build further intuition of the quantum world: one of the current major applications of quantum games. Looking to the future, our group believes--in agreement with multiple experts-- that the applications of this field, while yet to be uncovered, will be numerous and wide ranging.

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