A Guide to Essentials Mathematics for Data Science

MATHEMATICS FOR DATA SCIENCE

Mathematics for Data Science

First Published - 2016

This edition published in 2020

Author

Ashok Veda

Publisher

Self-Published

Printer

ARC DOCUMENT SOLUTIONS INDIA PRIVATE LIMITED

Krishna Nagar Industrial Layout, Tavarekere Main road

Koramangala, Bangalore - 560 029

ISBN 978-93-5406-401-2

Table of Contents

1. Introduction & Approach 5

1a. Why Data Science?...... 5

1b. Why Mathematics for Data Science?...... 5

1c. Best Way to Read this book...... 6

2. Number Theory...... 7

2a. Integers 7

2b. Fractions...... 10

2c. Decimals 14

2d. Ratio...... 17

2e. Percent 18

2f. Exponents and Roots Exponents...... 22

2g. Real Numbers...... 25

2h. Sequence and Series...... 28

3. Set Theory Basics 31

3a. Sets and their Representations 32

3b. Empty, Finite and Infinite Sets 34

3c. Equal Sets and Subsets 36

3d. Power Set and Universal Set 39

3e. Operations on Sets 40

3f. Complement of a Set...... 42

4. ALGEBRA...... 43

4a. Algebraic Expressions...... 43

4b. Linear and Quadratic Equations 52

4c. Linear and Quadratic Inequalities 54

4d. Algebraic Functions...... 55

4e. Binomial Theorem 56

4f. Vector Algebra 61

4g. Coordinate Geometry *74*

4h. Graphs of Functions *84*

5. GEOMETRY 91

5a. Lines and Angles *91*

5b. Polygons *93*

5c. Triangles and Quadrilaterals *95*

5d. Circles *103*

5e. Three-Dimensional Figures *109*

6. DATA ANALYSIS 112

6a. Methods for Presenting Data *112*

6b. Counting Methods *124*

6c. Probability *128*

7. Matrices 152

7a. Matrix Introduction *152*

7b. Types and Operations of Matrices *156*

7c. Transpose of a Matrix *167*

7d. Symmetric and Skew Symmetric Matrices *168*

7e. Elementary Operation (Transformation) of a Matrix *170*

7f. Invertible Matrices *171*

7g. Determinants and it's Properties *172*

7h. Matrix Solution for Area of a Triangle *178*

7i. Minors, Cofactors, Adjoint and Inverse of a Matrix *178*

7j. Determinants and Matrix Applications *180*

8. Calculus 182

8a. Introduction & Intuition behind derivatives *182*

8b. Limits and Derivatives *186*

8c. Limits of functions *189*

8d. Continuity and Differentiability *197*

8e. Application of Derivatives *222*

8f. Differential Equations *245*

[redacted]Introduction & Approach

1a. Why Data Science?

Data is everywhere. Every organization has been generating data in large amounts and are in need of data science professionals who can help in extracting useful insights from the data. Data Science is a domain that involves different disciplines such as Mathematics, Statistics, Computer Science/IT, Domain/Business Knowledge and their methodologies to extract useful insights from both structured and unstructured data. Data Science covers all aspects which would fulfil the need of all the technical realms of the present and the near future. In a way Data Science can be called as the backbone of Business Analytics. We can very well say that the future CEO's of most of the organizations will be a person who has an in depth understanding of Data Science. So, choosing data science as a career is a great option as it would only grow higher offering better opportunities and will always amaze you in every step.

1b. Why Mathematics for Data Science?

Mathematics is the core of any extant discipline of science. It is no surprise that, almost all the techniques of modern science which includes all of the data science have some deep mathematical footing one way or the other.

Sometimes, as a data engineer or even as an analyst on the team, we have to learn the foundational mathematics by heart to use or to apply the techniques appropriately, other times we can just get by using an API, app or any out-of-the-box algorithm.

As an example, having a solid understanding of the math behind the algorithm while create meaningful product recommendation for our users, will never hurt us. More often than not, it gives us an edge among our peers and makes us more confident. It always pays and really pays (money) as well to know the mechanism under the lid (even at a high level) than just being the guy over the table with no knowledge about the workings.

It goes without saying that we will unquestionably need all of the other assets of knowledge, which includes the ability to program, some amount of business acumen, a distinctive analytical and inquisitive mindset and definitely some data to function as a top data scientist. All we are trying to do is to gather the pointers to the most essential math skills to help us in this endeavor.

The fundamental or basic math knowledge is principally important for professionals who are trying to get into this field after spending a significant amount of time in some other domain—mechanical engineering, hardware, retail, pharmaceutical industry, medicine and health care, business management, etc…

1c. Best Way to Read this book

We can consider two strategies for understanding the mathematics for data science:

Top-Down: Building up the concepts from foundational to more advanced. This is often the chosen approach in more technical fields, such as Physics. This strategy has the benefit that the reader is always able to rely on their previously learned concepts. However, for a beginner many of the foundational concepts might not particularly be of interest, and the lack of motivation means that most foundational definitions are quickly forgotten, and the learning rigor could get lost.

Topic-down: Going from practical needs to more basic requirements. This goal-driven approach has the advantage that the readers always know why they need to work on a particular concept, and there is a clear path to acquire that knowledge. The disadvantage with this approach is that the knowledge is built on potentially shaking foundations, and the readers must remember a bunch of jargons or words that they do not have any way of understanding. However, an attempt has been made to touch base on all the foundations or the basics-of-the-basics to aid the learning process.

We decided to write this book sectionally. The book can be read both ways, either by starting with foundational (mathematical) concepts or from a specific topic point of view. The Chapters are mostly built upon the previous ones, but it is possible to skip a chapter and work backward if necessary.

[redacted]Number Theory

Let us begin with an understanding of Number Theory which starts with integers, fractions, decimals, and advances to the set of real and complex numbers. The basic arithmetical operations such as addition, subtraction, multiplication, and division are reviewed, along with exponents and roots. The Number theory ends with the concept of Series.

2a. Integers

Integers can be defined as the set of numbers 1, 2, 3, 4, . . . , along with their negative values, –1, –2, –3, . . . , and 0. So, the set of integers are {. . . , –3, –2, –1, 0, 1, 2, 3, . . .}.

The positive integers are always greater than zero (0), the negative integers are always less than zero (0), and zero (0) is neither positive nor negative. Also, when we add, subtract or multiply, the result is always an integer; let us look at division of integers below. The elementary number calculations for the above operations, such as 5 + 5 = 10, 78 – 77=– 1, 7 – (–1) = 8, and (5)(8) = 40, should already be familiar; hence they are not reviewed here.

Here are three general essentials regarding multiplication of integers.

Essentials 1: The product of two positive integers is always a positive integer.
Essentials 2: The product of two negative integers is always a positive integer.
Essentials 3: The product of a positive integer and a negative integer is a negative integer.

When we multiply the integers, each of the multiplied integers are called a **factor** or **divisor** of the resulting product. Example, (2)(2)(11) = 44, so 2 and 11 are factors of 44.

The integers 8, 10, 20 are also factors of 40, since (10)(4) = 40 and (20)(2) = 40. The positive factors of 40 are to be 1, 2, 4, 5, 8, 10, 20 and 40. The negatives of these integers are also factors of 40, since, for example, (–20)(–2) = 40. There are no other factors of 40. We say that 40 is a **multiple** of each of its factors and that 40 is **divisible** by each of its divisors. Here are few more examples of factors and multiples.

Example 1: The positive factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18 and 36.

Example 2: Only six integers are multiples of 25, they are: 1, 5, 25, and their negatives.

Example 3: Similarly, if we make a list the positive multiples of 25, it has no end: 25, 50, 75, 100, 125,. . ; Also, every nonzero integer has infinitely many multiples.

Example 4: 1 is a factor of every integer. 1 is not a multiple of any integer except itself and its negative, i.e.,1 and −1.

The **least common multiple (LCM)** of two nonzero integers a and b are the least positive integer that is a multiple of both a and b. For example, the least common multiple of 60 and 150 is 300. This is because the positive multiples of 60 are 60,120,180,240,300,360,420,480,540,600,660,720,780,840,900,960,1020,1080,1140, . . . , and the positive multiples of 150 are 150,300,450,600,750,900,1050,1200,1350,1500,1650,1800,1950,2100,. . . . Thus, the common positive multiples of 60 and 150 are 300, 600, 900 . . . , and the least of these is 300.

The **greatest common divisor** (or **highest common factor)** of two nonzero integers a and b is the greatest positive integer that is a divisor of both c and d.

For example, the greatest common divisor of 45 and 90 is 45. This is because the positive divisors of 45 are 1, 3, 5, 9, 15, **45**, and the positive divisors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, and **90.** Therefore, the common positive divisors of 45 and 90 are 1, 3, 5, and 15,.. and the greatest of these is 45.

When an integer p is divided by an integer q, where q is a divisor of p, the result is always a divisor of p. For example, when 50 is divided by 5 (one of its divisors), the result is 10, which is another divisor of 50. If q is not a divisor of p, then the result can be viewed in three different ways. The result are often viewed as a fraction or as a decimal, both of which are discussed later, or the result are often viewed as a quotient with a remainder, where both are integers. Each view is beneficial, counting on the context. Fractions and decimals are worthwhile when the result must be observed as one number, while quotients with remainders are useful for describing the outcome in terms of integers only.

Example 1: 100 divided by 45 is 2 remainder 10, since the greatest multiple of 45 that is less than or equal to 100 is (2)(45), or 90, which is 10 less than 100.

Example 2: 24 divided by 4 is 6 remainder 0, since the greatest multiple of 4 that is less than or equal to 24 is 24 itself, which is 0 less than 24. In common, the remainder is 0 if and only if c is divisible by d.

Example 3: 6 divided by 24 is 0 remainder 6, since the greatest multiple of 24 that is less than or equal to 6 is (0)(24), or 0, which is 6 less than 6.

Example 4: −32 divided by 3 is −11 remainder 1, since the greatest multiple of 3 that is less than or equal to −32 is (−11)(3), or −33, which is 1 less than −32.

Here are five more examples.

Example 1: 100 divided by 3 is 33 remainder 1, since 100 = (33)(3) + 1.

Example 2: 200 divided by 25 is 8 remainder 0, since 200 = (8)(25) + 0.

Example 3: 80 divided by 100 is 0 remainder 80, since 80 = (0)(100) + 80.

Example 4: −13 divided by 5 is −3 remainder 2, since −13 = (-3)(5) + 2.

Example 5: −73 divided by 10 is −8 remainder 7, since −73 = (-8)(10)+ 7.

If an integer is divisible by 2, it is said as an **even integer**; otherwise, it is an **odd integer**. Take a note that when an odd integer is divided by 2, the remainder is always 1. The set of even integers is {. . . , –6, –4, –2, 0, 2, 4, 6, . . .}, and the set of odd integers is {. . . , –5, –3, –1, 1, 3, 5, . . .}. Here are six essentials regarding the sum and product of even and odd integers.

Essential 1: The sum of two even integers is an even integer.
Essential 2: The sum of two odd integers is an even integer.
Essential 3: The sum of an even integer and an odd integer is an odd integer.
Essential 4: The product of two even integers is an even integer.
Essential 5: The product of two odd integers is an odd integer.
Essential 6: The product of an even integer and an odd integer is an even integer.

A **prime number** is an integer that has only two positive divisors: 1 and itself. 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29 are the very first ten prime numbers. The integer 14 is not a prime number, as it has four positive divisors: 1, 2, 7, and 14. The integer 1 is not a prime number, and the integer 2 is the only even prime number.

Every integer greater than 1 is either a prime number or can be exclusively expressed as a product of factors that are prime numbers, or **prime divisors**. This kind of expression is termed as **prime factorization**. Here are five examples of prime factorization.

Example 1: 14 = (2)(7)

Example 2: 81 = (3)(3)(3)(3) = 3^4

Example 3: 338 = (2)(13)(13) = (2) (13^2)

Example 4: 800 = (2)(2)(2)(2)(2)(5)(5) = (2^5)(5^2)

Example 5: 1,155 = (3)(5)(7)(11)

An integer greater than 1 and is not a prime number is said as a **composite number**. The first 6 composite numbers are 4, 6, 8, 9, 10 and 12.

2b. Fractions

A **fraction** is a few the form i/j, where i and j are integers and i ≠ 0. The integer i is called the **numerator** of the fraction, and j is called the **denominator**.

For example, -8/5 is a fraction in which –8 is the numerator and 5 is the denominator. Such numbers are also called **rational numbers**. Jot down that every integer n is a rational number, because n is equal to the fraction n/1.

If both the numerator i and the denominator j, where j ≠ 0, are multiplied by the same non-zero integer, the resulting fraction will be equivalent to i/j.

Example: Multiplying the numerator and denominator of the fraction -7/5 by 4

gives

$$\frac{-7}{5} = \frac{(-7)(4)}{(5)(4)} = \frac{-28}{20}$$

The fraction -7/5, its numerator and denominator are multiplied with -1 gives

$$\frac{-7}{5} = \frac{(-7)(-1)}{(5)(-1)} = \frac{7}{-5}$$

For all integers a and b, the fractions $\frac{-a}{b}$, $\frac{a}{-b}$, and $-\frac{a}{b}$ are equivalent.

$$\frac{-7}{5} = \frac{7}{-5} = -\frac{7}{5}$$

If there is a common factor of a fraction for both numerator and denominator, then they can be factored, and the entire fraction can be reduced to an equivalent fraction.

$$\frac{40}{72} = \frac{(8)(5)}{(8)(9)} = \frac{5}{9}$$

Adding and Subtracting Fractions

To add or sum up two fractions with the same denominator, you can add the numerators and maintain the same denominator.

$$-\frac{7}{11} + \frac{6}{11} = \frac{-7+6}{11} = \frac{-1}{11}$$

To add or sum up two fractions with unlike denominators, first find a **common denominator**, which is a common multiple of the two denominators. Then change both fractions to equivalent fractions with the same denominator. Finally, add the numerators and maintain the common denominator.

To add the two fractions $\frac{1}{3}$ and $-\frac{2}{7}$, first note that 21 is a common denominator of the fractions.
Then convert the fractions to equivalent fractions with denominator 21 as follows.

$$\frac{1}{3} = \frac{1(7)}{3(7)} = \frac{7}{21} \text{ and } -\frac{2}{7} = -\frac{2(3)}{7(3)} = -\frac{6}{21}$$

Hence, the two fractions can be added as follows.

$$\frac{1}{3} + \frac{-2}{7} = \frac{7}{21} + \frac{-6}{21} = \frac{1}{21}$$

The same method is applicable to subtraction of fractions.

Multiplying and Dividing Fractions

To multiply any two fractions, multiply both the numerators and multiply both the denominators. Here are two examples.

1. $\left(\frac{10}{7}\right)\left(\frac{-1}{3}\right) = \frac{(10)(-1)}{(7)(3)} = \frac{-10}{21} = -\frac{10}{21}$

2. $\left(\frac{8}{3}\right)\left(\frac{7}{3}\right) = \frac{56}{9}$

To divide one fraction by another, first step is to **invert** the second fraction (that is, find its **reciprocal**), and then multiply the first fraction by the inverted fraction. Here are two examples.

i. $\frac{17}{8} \div \frac{3}{5} = \left(\frac{17}{8}\right)\left(\frac{5}{3}\right) = \frac{85}{24}$

$$\frac{\frac{3}{10}}{\frac{7}{13}} = \left(\frac{3}{10}\right)\left(\frac{13}{7}\right) = \frac{39}{70}$$

ii.

Mixed Numbers

An expression like $4\frac{3}{8}$ is called as a **mixed number**. It consists of an integer part and a fraction part, where the fraction part has a value between 0 and 1; the mixed number means $4 + \frac{3}{8}$.

For a mixed number to be converted into a fraction, first convert the part which has an integer to a fraction equivalent to the same denominator as the fraction, post that, add it to the fraction part.

Example: To convert the mixed number $4\frac{3}{8}$ to a fraction, first convert the integer 4 to a fraction with denominator 8, as follows.

$$4 = \frac{4}{1} = \frac{4(8)}{1(8)} = \frac{32}{8}$$

Then add 3/8 to it to get

$$4\frac{3}{8} = \frac{32}{8} + \frac{3}{8} = \frac{35}{8}$$

Fractional Expressions

Numerical figures of the form i/j, where either i or j is not an integer and j ≠ 0, are called fractional expressions. Fractional expressions can be operated just like fractions. Here are two examples.

Example 1: Add the numbers $\frac{\pi}{2}$ and $\frac{\pi}{3}$

Solution: Observe that 6 is a common denominator of both numbers.

The number $\frac{\pi}{2}$ is equivalent to the number $\frac{3\pi}{6}$ and the number $\frac{\pi}{3}$ is equivalent to the number $\frac{2\pi}{6}$

Hence,

$$\frac{\pi}{2} + \frac{\pi}{3} = \frac{3\pi}{6} + \frac{2\pi}{6} = \frac{5\pi}{6}$$

Example 2: Simplify the number $\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$

Solution: Observe that the numerator of the number is $\frac{1}{\sqrt{2}}$ and the denominator of the number is $\frac{\sqrt{5}}{3}$

Therefore,

$$\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{5}}{3}\right)$$

which can be simplified to $\frac{\sqrt{5}}{3\sqrt{2}}$

Thus, the number $\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$ simplifies to the number $\frac{\sqrt{5}}{3\sqrt{2}}$

2c. Decimals

The decimal number system is constructed on representing numbers using powers of 10. The place value of every digit parallels to a power of 10. For instance, the digits of the number 7,532.418 have the following place values.

Thousands		Hundreds	Tens	Ones or Units		Tenths	Hundredths	Thousandths
7	,	5	3	2	.	4	1	8

i.e., the number 7,532.418 could be written as

$$7(1{,}000) + 5(100) + 3(10) + 2(1) + 4\left(\frac{1}{10}\right) + 1\left(\frac{1}{100}\right) + 8\left(\frac{1}{1{,}000}\right)$$

Alternatively, it could be written as

$$7\left(10^{3}\right) + 5\left(10^{2}\right) + 3\left(10^{1}\right) + 2\left(10^{0}\right) + 4\left(10^{-1}\right) + 1\left(10^{-2}\right) + 8\left(10^{-3}\right)$$

If we have a finite number of digits to the right-side of the decimal point, converting a decimal to an equivalent fraction with integers in the numerator and denominator is a forth-right process. As every place value is a power of 10, each of the decimal can be converted to an integer divided by a power of 10. Below are three examples.

1. $$2.3 = 2 + \frac{3}{10} = \frac{20}{10} + \frac{3}{10} = \frac{23}{10}$$

2. $$90.17 = 90 + \frac{17}{100} = \frac{9{,}000 + 17}{100} = \frac{9{,}017}{100}$$

3. $$0.612 = \frac{612}{1{,}000}$$

Contrariwise, every fraction with integers in the numerator and denominator can be transformed to an equivalent decimal by dividing the numerator by the denominator by long division (which is not in this review). The decimal that outcomes from the long division will either **terminate**, as in 1/4 = 0.25 and 52/25= 2.08, or **repeat** without end, as in 1/9 = 0.111..., 1/22 = 0.0454545..., and 25/12 = 2.08333.... One way to indicate the repeating part of a decimal that repeats without end is to use a bar over the digits that repeat. Here we have four examples of fractions converted to decimals.

1. $\frac{3}{8} = 0.375$

2. $\frac{259}{40} = 6.475$

3. $-\frac{1}{3} = -0.\overline{3}$

4. $\frac{15}{14} = 1.0\overline{714285}$

Every fraction with integers in the numerator and denominator is parallel to a decimal that either terminates or repeats. i.e., every rational number can be articulated as a terminating or repeating decimal. The converse is also true; i.e., every terminating or repeating decimal represents a rational number.

There are decimals which are continuous or iterating; for example, the decimal that is equivalent to root 2 is 1.41421356237..., and it can be shown that this particular decimal does not terminate or repeat. Another example is 0.02022022202222022220..., which has groups of consecutive 2s alienated by a 0, where the number of 2s in each consecutive group increases by one. As these two decimals do not terminate or repeat, they are not rational numbers. Such numbers are called **irrational numbers**.

2d. Ratio

The **ratio** of one quantity to another is a method to express their relative sizes, often in the form of a fraction, where the first quantity is the numerator and the second quantity is the denominator. Thus, if a and b are positive quantities, then the ratio of a to b can be written as the fraction a/b. The notation “a to b” and the notation “a : b” are also used to express this ratio. For instance, if there are 2 apples and 3 oranges in a basket, we can say that the ratio of the number of apples to the number of oranges is 2 : 3, or 2 to 3, or $2/_3$.

Like fractions, ratios can be condensed to lowest terms. For instance, if there are 8 apples and 12 oranges in a basket, then the ratio of the number of apples to the number of oranges is still 2 to 3. Likewise, the ratio 9 to 12 is equivalent to the ratio 3 to 4.

If multiple quantities, that is, three or more quantities which are positive are considered, say m, n, and o, then their relative sizes can also be expressed as a ratio with the notation “m to n to o.” For instance, if there are 5 apples, 30 oranges, and 20 bananas in a basket, then the ratio of the number of apples to the number of oranges to the number of bananas is 5 to 30 to 20. This ratio can be reduced to 1 to 6 to 4 by dividing each number by the highest common factor (HCF) of 5, 30, and 20, which is 5.

A **proportion** is an equation involving two ratios; for example, $9/_{12} = 3/_4$. To solve a problem involving ratios, you can write a proportion and solve it by **cross multiplication**.

Example 1: To find a number x so that the ratio of x to 49 is the same as the ratio of 3 to 21, we can first write the following equation.

$$\frac{x}{49} = \frac{3}{21}$$

We can then cross multiply to get 21x = (3)(49), and finally we can solve for x to get

$x = {}^{(3)(49)\ /21} = 7$

2e. Percent

The term **percent** means per hundred, or hundredths. Percents are ratios which are often used to signify parts of a whole, where the whole is considered as having 100 parts. Percents can be represented in fraction or decimal equivalents. Here are three examples of percents.

Example 1: 1 percent means 1 part out of 100 parts. The fraction equivalent of 1 1

percent is $1/_{100}$, whereas the decimal equivalent is 0.01.

Example 2: 32 percent means 32 parts out of 100 parts. The fraction equivalent of

32 percent is $32/_{100}$, whereas the decimal equivalent is 0.32.

Example 3: 50 percent means 50 parts out of 100 parts. The fraction equivalent of

50 percent is $50/_{100}$, whereas the decimal equivalent is 0.50.

Jot down that in the fraction equivalent, the part is the numerator of the fraction and the whole is the denominator. The percent is denoted or usually represented using % which is a percent symbol, the symbol is used extensively more than the word “percent”. Here are two examples of percents written using the % symbol, along with their fraction as well as their decimal equivalents.

Example 1: 12% = $^{12}/100$ = 0.12

Example 2: 10%= $^{10}/100$ = 0.1

Be careful not to confuse 0.01 with 0.01%. The percent symbol matters. For example,

0.01 = 1% also but, 0.01% = $^{0.01}/100$=0.0001.

To calculate a percent, given the part and the whole, first step is to divide the part by the whole to get the decimal equivalent, then multiply the result by 100. The percent is that number monitored by the word “percent” or the % symbol.

Example: If the whole is 20 and the part is 13, you can find the percent as follows. Part/whole = $13/20$ = 0.65 =65%

Example: What percent of 150 is 12.9 ?

Solution: Here, the whole number is 150 and the part is 12.9, so

part/whole = 12.9/150 = 0.086 = 8.6%

To find out the part which is a certain percent of a whole, you can either multiply the whole by the decimal equivalent of the percent or set up a proportion to find the part.

Example: To find 30% of 350, you can multiply 350 by the decimal equivalent of 30%, or 0.3, as follows.

(350)(0.3) = 105

Otherwise, to find 30% of 350, if we want to use a proportion, we need to find the number of parts of 350 that returns the same ratio as 30 parts out of 100. We would want a number x that satisfies the proportion

part /whole = 30/100 or

x/350 = 30 / 100

Cracking for x yields x = (30)(350)/100=105, so 30=% of 350 is 105.

Given the percent and the part, you can compute the whole. To do this, either you can either use the decimal equivalent of the percent or you can set up a proportion and solve it.

Example: 15 is 60% of what number?

Solution: For 60% the decimal equivalent is 0.6. Since 60% of some number z is 15, multiply z by the decimal equivalent of 60%.

$0.6z = 15$

Now we have to solve for z by dividing both sides of the equation by 0.6 as follows.

$z = 15/0.6 = 25$

Using a proportion, find for a number z such that

Part/whole $= 60/100$ or

$15/z = 60/100$

Hence, $60z = (15)(100)$, and therefore, $z = (15)(100)/60 = 1{,}500/60 = 25$. i.e., 15 is 60% of 25.

Percents Greater than 100%

While the discussion about percent so far assumes a context of a part and a whole, it is not essential that the part be less than the whole. In overall, the whole is called the **base** of the percent. If the numerator of a percent is greater than the base, then the percent is greater than 100%.

Example: 15 is 300% of 5, as $15/5 = 300/100$

Example: 250% of 16 is 40, since $(250/100)(16) = (2.5)(16) = 40$

Note that the decimal equivalent of 300% is 3.0 and the decimal equivalent of 250% is 2.5.

Percent Increase, Percent Decrease, and Percent Change

If a quantity changes from an initial positive amount to another positive amount (for example, an employee’s salary that is raised), you can calculate the amount of change as a percent of the initial amount. This is termed as **percent change**. When a quantity increases from 500 to 650, then the base of the increase is the initial amount, 500, and the amount of the increase is $650 - 500$, or 150. The **percent increase** is calculated by dividing the amount of increase by the base, as follows.

Amount of increase divided by base = Amount increase / base = 650–500 / 500 = 150 / 500 = 25/100 = 25%

We say the percent increase is 25%.

When a quantity doubles in size, then the percent increase is 100%. For instance, when a quantity increases from 150 to 300, then the percent increase is computed as follows.

amount of increase/ base = 300 – 150 / 150 = 150 / 150 = 100%

If a quantity decreases from 500 to 400, analyze the **percent decrease** as follows.

amount of decrease / base = 300–200 / 300 = 100 / 300 = 10 / 30 = 33.33%

The quantity decreased by 33.33%.

When calculating a percent increase, the base is the smaller number. When calculating a percent decrease, the base is the larger number. In any of the cases, the base is the initial number, before the change.

Example: An investment in LIC increased by 10% in a single day. What was the value after the increase, if the value of the investment before the increase was $1,200,?

Solution: The percent increase is 10%. Using the decimal equivalent of 10%, the increase is (0.10)($1,200) = $120. Therefore, the value of the investment after the change is

$1,200 + $120 = $1,320

There can be a successive percent changes for a quantity, where the base of each successive change is the result of the prior percent change, which is the case in the below example.

Example: On Jan 1, 2013, the number of candidates enrolled in a certain training workshop was 8% less than the number of candidates enrolled at the training on Jan 1, 2012. On Jan 1, 2014, the digit of candidates enrolled in the training was 6% greater than the number of candidates enrolled in the training on Jan 1, 2013. What is the percent of change in number of candidates enrolled change from Jan 1, 2012 to Jan 1, 2014?

Solution: The first base is the enrollment on Jan 1, 2012. The first percent change was the decrease in the enrollment by 8% from Jan 1, 2012, to Jan 1, 2013. By means of a result of this decrease, the enrollment on Jan 1, 2013, was (100 − 8)% , or 92%, of the enrollment on Jan 1, 2012. The decimal equivalent of 92% has to be 0.92.

So, if n signifies the number of candidates enrolled on Jan 1, 2012, then the number of candidates enrolled on Jan 1, 2013, is equal to 0.92n.

The new base is the enrollment on Jan 1, 2013, which has to be 0.92n. So, the second percent change was the 6% increase in enrollment from Jan 1, 2013, to Jan 1, 2014. As an outcome of this increase, the enrollment on Jan 1, 2014, was (100 + 6)%, or 106%, of the enrollment on Jan 1, 2013. The decimal equivalent of 106% has to be 1.06.

Thus, the number of candidates enrolled on September 1, 2014, was (1.06)(0.92n), which is equal to 0.9752n.

The percent equivalent of 0.9752 has to be 97.52%, which is 2.48% less than 100%. Therefore, the percent change in the enrollment from Jan 1, 2012 to Jan 1, 2014, is a 2.48% decrease.

2f. Exponents and Roots Exponents

Exponents are used to represent the repeated multiplication of a number by itself; for example, 3^4 = (3)(3)(3)(3) = 81 and 5^3 = (5)(5)(5) = 125. In the expression 5^3 , 5 is called the **base**, 3 is called the **exponent**, and we read the expression as “5 to the third power.” Similarly, 3 to the fourth power is 81.

When the exponent is 2, we name the process **squaring**. Which is, 4 squared is 16; that is, 4^2 = (4)(4) = 16. Similarly, 7 squared is 49; that is, 7^2 = (7)(7) = 49.

If negative numbers are raised to powers, the result may be positive or negative; for instance, $(–3)^2$ = –(3)–(3) = 9 and $(–3)^5$ = –(3)–(3)–(3)–(3)–(3) = –243. A negative number elevated to an even power is always positive, and a negative number raised to an odd power is always negative. Note that $(−3)^2$ = (−3) (− 3)= 9, but $−3^2$ = − ((3)(3)) = -9. Exponents can also be negative or zero; such exponents are well-defined as follows.

The exponent zero: For every nonzero numbers a, $a^0 = 1$. The expression 0^0 is undefined.

Negative exponents: For all nonzero numbers a, $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, $a^{-3} = \frac{1}{a^3}$, and so on. Note that $(a)\left(a^{-1}\right) = (a)\left(\frac{1}{a}\right) = 1$.

Roots

A **square root** of a nonnegative number n is a number r so that $r^2 = n$. For example, 6 is a square root of 36 because $6^2 = 36$. Another square root of 36 is −6, since $(-6)^2 = 36$. Every positive numbers will have two square roots, one is always positive and the other one is negative. The square root of 0 is 0. The expression comprising of the square root symbol placed over a nonnegative number denotes the nonnegative square root (or the positive square root if the number is greater than 0) of that nonnegative number. Therefore, root100 = 10, – root100 = -10, and root0 = 0. Square roots of negative numbers are not well-defined in the real number system.

Here we have four important rules about operations with square roots, where a > 0 and b > 0.

Rule 1: $\left(\sqrt{a}\right)^2 = a$

Example A: $\left(\sqrt{3}\right)^2 = 3$

Example B: $\left(\sqrt{\pi}\right)^2 = \pi$

Rule 2: $\sqrt{a^2} = a$

Example A: $\sqrt{4} = \sqrt{2^2} = 2$

Example B: $\sqrt{\pi^2} = \pi$

Rule 3: $\sqrt{a}\sqrt{b} = \sqrt{ab}$

Example A: $\sqrt{3}\sqrt{10} = \sqrt{30}$

Example B: $\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

Rule 4: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Example A: $\frac{\sqrt{5}}{\sqrt{15}} = \sqrt{\frac{5}{15}} = \sqrt{\frac{1}{3}}$

Example B: $\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

A square root is defined as a root of order 2. Higher order roots of a positive number n are defined likewise. For orders 3 and 4, the **cube root** of n, written as ^{3} root n, and **fourth root** of n, written as ^{4} root n, represent numbers such that when they are elevated to the powers 3 and 4, respectively, the result is n. These roots follow rules similar to those above but with the exponent 2 substituted by 3 or 4 in the first two rules.

There are some distinguished differences between odd order roots and even order roots (in the real number system):

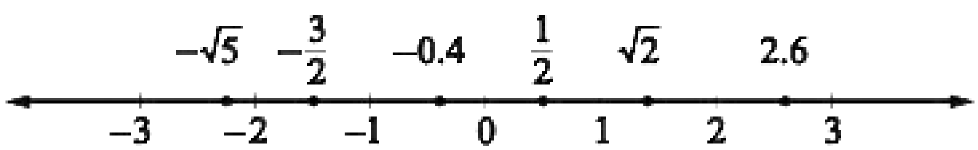
There is precisely one root for every n number, even when the value of n is negative, for odd order roots,

For even order roots, there are two roots exactly for each of the positive number n and for any negative number n, there are no roots.

For example, 8 has exactly one cube root, cube root 8 = 2, but 8 has two fourth roots, fourth root 8 and – fourth root 8, whereas –8 has exactly one cube root, cube root –8 =– 2, but –8 has no fourth root, since it is negative.

2g. Real Numbers

The set of **real numbers** contains of all rational numbers and all irrational numbers. The real numbers involve all integers, fractions, and decimals. The set of real numbers can be characterized by a number line called the **real number line**. Figure below is a number line.



Every real number relates to a point on the number line, and every point on the number line relates to a real number. On the number line, all numbers to the left side of 0 are negative and all numbers to the right side of 0 are positive. As shown in Figure, the negative numbers –0.4, –1, –3/2 , –2, – root 5, and –3 are to the left of 0, and the positive numbers 1/2 , 1, root 2, 2, 2.6, and 3 are to the right of 0. The only number 0 is neither negative nor positive.
A real number p is **less than** a real number q if p is to the left of q on the number line, which is written as p < q. A real number q is **greater than** a real number p if q is to the right of p on the number line, which is written as q > p. For instance, the number line in Arithmetic Figure 2 shows the following three relationships.

Relationship 1. $-\sqrt{5} < -2$

Relationship 2. $\frac{1}{2} > 0$

Relationship 3. $1 < \sqrt{2} < 2$

A real number p is **less than or equal to** a real number q if p is to the left of, or corresponds to the same point as, q on the number line, which is written as $p \leq q$. A real number q is **greater than or equal to** a real number p if q is to the right of, or corresponds to the same point as, p on the number line, which is written as $q \geq p$.

To say that a real number p is between 4 and 5 on the number line means that $p > 4$ and $p < 5$, which can also be written as $4 < p < 5$. The set of all real numbers that lies in between 4 and 5 is called an **interval**, and $4 < p < 5$ is often used to represent that interval. Note that the endpoints of the interval, 4 and 5, are not involved in the interval. Sometimes one or both of the endpoints are to be involved in an interval. The following inequalities denote four types of intervals, dependent on whether or not the endpoints are included.

Interval type 1; Interval type 2; Interval type 3; Interval type 4;

$2 < p < 3$; $2 \leq p < 3$; $2 < p \leq 3$; $2 \leq p \leq 3$

There are also four categories of intervals with only one endpoint, each of which contains of all real numbers to the right or to the left of the endpoint and include or do not include the endpoint. The following inequalities denote these types of intervals.

Interval type 1; Interval type 2; Interval type 3; Interval type 4;

$p < 5$; $p \leq 5$; $p > 5$; $p \geq 5$

The whole real number line is also considered to be an interval.

Absolute Value

The distance between a number p and 0 on the number line is called the **absolute value** of p, written as $|p|$. Therefore, $|5| = 5$ and $|-5| = 5$ because each of the numbers 5 and –5 is a distance of 5 from 0. Note that if p is positive, then $|p| = p$; if p is negative, then $|p| = -p$; and lastly, $|0| = 0$. It follows that the absolute value of any nonzero number is positive. Here are three examples.

1. $|\sqrt{5}| = \sqrt{5}$

2. $|-23| = -(-23) = 23$

3. $|-10.2| = 10.2$

Properties of Real Numbers

Here we can know about the twelve general properties of real numbers that are used frequently. r, s, and t are real numbers in each property mentioned here.

Property 1: r+s=s+r and rs=sr.

Example A: 8 + 2 = 2 + 8 = 10

Example B: (–3)(17) = (17–)(3=)– 51

Property 2: (r + s) + t = r + (s + t) & (rs) t = r (st).

Example A: (1 + 2) + 3 = 1 + (2+ 3) = 6

Property 3: r(s+t)=rs+rt

Example: 5(3+16)=(5)(3) (5)(16+)=95

Property 4: r+0=r, (r)(0)=0,and(r)(1)=r.

Property 5: If rs = 0, then r = 0 or s = 0 or both.

Example: If –5s = 0, then s = 0.

Property 6: Division by 0 is undefined.

Example A: 5 ÷ 0 is undefined.

Example B: $^{-7/0}$ is undefined.

Example C: $0/0$ is un-defined.

Property 7: If two numbers r and s are positive, then the computation r + s and rs are both positive.

Property 8: If two numbers r and s are negative, then the numbers r + s is negative and rs is positive.

Property 9: If the number r is positive and the number s is negative, then the computation rs is negative.

2h. Sequence and Series

In mathematics, the word, "sequence" is used in almost the same way as it is in ordinary English. When we say that a collection of objects is recorded in a sequence, we generally mean that the collection is ordered in such a way that it has an recognized first member, second member, third member and so on. For example, population of human beings or microbes at different times form a sequence. A sequence is formed when the amount of money deposited in a bank, over a number of years. Depreciated values of certain commodity occur in an arrangement. Sequences have significant applications in several spheres of human activities. Sequences, subsequent specific patterns are called progressions.

Sequences

Let us consider the following examples:
Assume that there is a generation gap of 20 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 200 years.

Here, the total number of generations = 200/20 =10

The number of person's antecedents for the first, second, third, ..., tenth generations are 2, 4, 8, 16, 32, ..., 1024. These numbers form a sequence.

Study the successive quotients that we acquire in the division of 10 by 3 at different steps of division. In this method we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The several numbers occurring in a sequence are called its terms. We represent the terms of a sequence by a1, a2, a3, ..., an, ..., etc., the subscripts represent the position of the term. The nth term is the number at the nth position of the sequence and is denoted by a The nth term is also called the general term of the n. sequence.

Thus, the terms of the sequence of person's ancestors stated above are:

$a_1 = 2$, $a_2 = 4$,...,$a_{10} = 1024$.

Similarly, in the example of successive quotients

$a_1 = 3$, $a_2 = 3.3$, ….. , $a_6 = 3.33333$, etc.

A sequence consisting finite number of terms is called a finite sequence. For example, sequence of ancestors is a finite sequence as it contains 10 terms (a fixed number). A sequence, if not a finite, is called an infinite sequence. For instance, the sequence of successive quotients stated above is an infinite sequence, infinite in the sense that it never ends.

Repeatedly, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Examine for instance, the sequence of even natural numbers 2, 4, 6, ...

Here

$a_1 = 2 = 2\times 1$ $a_3 = 6 = 2\times 3$ ....

$a_2 = 4 = 2\times 2$ $a_4 = 8 = 2\times 4$ ....

$a_{23} = 46 = 2\times 23$, $a_{24} = 48 = 2\times 24$, and so on.

In fact, we see that the nth term of this sequence can be noted down as $a_n = 2n$, where n is a natural number. Likewise, in the sequence of odd natural numbers 1,3,5, ..., the nth term is specified by the formula, $a_n = 2n – 1$, where n is a natural number.

In some cases, an arrangement of digits like 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is produced by the recurrence relation given by

$a_1 = a_2 = 1$ $a_3 = a_1 + a_2$

$a_n = a_{n–2} + a_{n–1}$, $n>2$

This sequence is termed as Fibonacci sequence.

In the sequence of primes 2,3,5,7,..., we find out that there is no formula for the nth prime. Such sequence can only be labeled by verbal description.

In every sequence, we should not expect that its terms will essentially be given by a particular formula. Therefore, we expect a theoretical scheme or a rule for producing the terms a1, a2, a3,...,an,... in succession.

In view of the above, a sequence can be considered as a function whose domain is the set of natural numbers or some subset of it. At times, we use the functional notation a(n) for an.

Series

Let a1, a2, a3,...,an, be a given and considered sequence. Then, the illustration a1 +a2+a3 +,...+an+... is said to be the series associated with the given sequence .The series is finite or infinite according as the given sequence is finite or infinite. Series are often characterized in compact form, called sigma notation, using the Greek letter ∑ (sigma) as means of indicating the summation involved. Thus, the series a1 + a2 + a3 + ... + an is abbreviated as,

$$\sum_{k=1}^{n} a_k$$

Remark When the series is used, it mentions the indicated sum not to the sum itself. For example, 1 + 3 + 5 is a finite series with three terms. When we use the phrase "sum of a series," we will mean the number that results from summing up the terms, the sum of the series is 16.

We now consider some examples.

Example 1 Take down the first three terms in each of the following sequences defined by the following: (i) an=2n+5, (ii) an= n−3/ 4 .

Solution (i) Here an = 2n + 5 Substituting n = 1, 2, 3, we obtain

a1 =2(1)+5=7,a2 =9,a3 =11 Thus, the required terms are 7, 9 and 11.

Arithmetic Progression (AP)

An arithmetic progression (A.P.) is a sequence of numbers where the terms increase or decrease regularly by the same constant quantity. This constant is called as common difference of the A.P. Generally, we denote the first term of A.P. by a, d represents the common difference and l represents the last term. The general term or the nth term of the A.P.is generalized by an =a+(n–1)d.

In A.P., the sum (Sn) of the first n terms is given by

$$s_n = \frac{n}{2}[2a + (n-1)d]$$
$$s_n = \frac{n}{2}(a + l)$$

The arithmetic mean A of any two numbers a and b is given by a+b / 2 i.e., the sequence a, A, b is considered to be in A.P.

Geometric Progression (GP)

A sequence is referred to or called as a geometric progression or G.P., if the ratio of any term to its previous term is the same throughout. This constant factor is termed as the common ratio. Generally, we denote the first term of a G.P. by a and its n–1th common ratio by r. The general or the n term of G.P. is given by an= ar^n-1.

The Geometric Progression is given by:

$$s_n = \frac{a(r^n - 1)}{r - 1}\ or$$
$$s_n = \frac{a(1 - r^n)}{1 - r}, if\ r \neq 1$$

The geometric mean (G.M.) of any two positive numbers a and b is given by root ab i.e., the sequence a, G, b is G.P.

Set Theory Basics

The concept of set obliges as a fundamental part of the present-day mathematics. Today this concept is being used in almost every division of mathematics. Sets are used to outline the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

The theory of sets was established by German mathematician Georg Cantor (1845-1918). He first came across sets while working on “problems on trigonometric series”. In this Chapter, we talk about some basic definitions and operations involving sets.

3a. Sets and their Representations

In everyday life, we frequently speak of collections of objects of a certain kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we come across collections, for instance, of natural numbers, points, prime numbers, etc. Especially, we examine the following collections:

1. (i) Even natural numbers less than 10, i.e., 2,4,6,8
2. (ii) The rivers flowing across India
3. (iii) The English vowels, namely, a, e, i, o, u
4. (iv) Different types of triangles

We observe that each of the above examples is a well-defined collection of objects in the sense that we can certainly decide whether a given particular object belongs to a given collection or not. For instance, we can say that the river Nile does not belong to the collection of rivers of India. In contrast, the river Ganga does belong to this collection.

Given below are a few more examples of sets used particularly in mathematics, viz.

N : is the set representing all natural numbers

Z : is the set representing all integers

Q : is the set representing all rational numbers

R : is the set representing real numbers

Z+ : is the set representing positive integers

Q+ : is the set representing positive rational numbers, and

R+ : is the set representing positive real numbers.

The symbols for the special sets mentioned above will be referred to throughout this book.

We can say that **a set is a well-defined collection of objects.**

The following points may be noted:

(i) Objects, elements and members of a set are identical terms.

(ii) Sets are generally denoted by capital letters A, B, C, X, Y, Z, etc.

(iii) The elements of a set are denoted by small letters a, b, c, x, y, z, etc.

If a is an element of a set A, we shall say that "a belongs to A" the Greek symbol $\in$ (epsilon) is used to denote the phrase 'belongs to'. Therefore, we shall write $a \in A$. If 'c' is not an element of a set A, we write $c \notin A$ and read "c does not belong to A".

Therefore, in the set V of vowels in the English alphabet, $a \in V$ but $b \notin V$. In the set P of prime factors of 30, $3 \in P$ whereas $15 \notin P$.
We have two methods of representing a set :

(i) Roster or tabular form
(ii) Set-builder form.

(i) Each of the element of a set is listed, the elements are being separated by commas and are enclosed within braces { }, in roster form,. For example, the set of all even positive integers less than and equal to 6 can be defined in roster form as {2, 4, 6}. Some examples of representing a set in roster form are given below:

(a) The set of all natural numbers which divide 42 is given as {1, 2, 3, 6, 7, 14, 21, 42}.

In roster form, the order in which the elements are registered is immaterial. Therefore, the above set can also be represented as {1, 3, 7, 21, 2, 6, 14, 42}.

(b) {a, e, i, o, u} is the set of all vowels in the English alphabet.

(c) The set of odd natural numbers is denoted by {1, 3, 5, . . .}. The dots explain us that the list of odd numbers continue indefinitely.

It may be observed that while writing the set in roster form an element is not generally repeated, i.e., all the elements are taken as different. For instance, the set of letters forming the word ‘SCHOOL’ is { S, C, H, O, L} or {H, O, L, C, S}. Here, the order of listing elements has no relevant significance.

All the elements of a set own a single common property (characteristic) in a set-builder form, which is not owned by any element outside the set. For instance, in the set {a, e, i, o, u}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property.

3b. Empty, Finite and Infinite Sets

The Empty Set

Consider the set
A = { x : x is a employee of XYZ company who is currently working }

We can go to the company and count the number of employees currently working in his team at the company. Therefore, the set A contains a finite number of elements.

We now take another set B as follows:

B = { x : x is a student currently studying in both Classes 11^{th} and 12^{th} }

We notice that a student cannot study simultaneously in both Classes 11^{th} and 12^{th}. Therefore, the set B contains no element at all.

Definition A set which does not consist any element is called as the empty set or the null set or the void set.

According to this definition, B is an empty set and A is not an empty set. The empty set is symbolized by φ or { }.

Below are few examples of empty set.

1. (i) Let A={x:1<x<2, x is a natural number}.Then A is the empty set, because there is no natural number between 1 and 2.
2. (ii) B={x:x2–2=0 and x is rational number}.Then B is the empty set because the equation x2 – 2 = 0 is not satisfied by any rational value of x.

3. (iii) C = {x : x is an even prime number greater than 2}.Then C is the empty set,

because 2 is the only even prime number.

4. (iv) D={x:x2 =4,x is odd}.Then D is the empty set, because the equation x2 = 4 is not satisfied by any odd value of x.

Finite and Infinite Sets

Let A={1,2,3,4,5}, B = {a, b, c, d, e, g}
and C = { men living currently in different parts of the world}
We notice that A consists 5 elements and B contains 6 elements. How many elements does C consist? As it is, we do not know the number of elements in C, but it is some natural number which may be pretty big number. By number of elements of a set S, we mean the number of distinct elements of the set and we represent it by n (S). If n (S) is a natural number, then S has a non-empty finite set.

Consider the set of natural numbers. We see that the number of elements of this set is not finite as there are infinite number of natural numbers. We define that the set of natural numbers is an infinite set. The sets A, B and C mentioned above are finite sets and n(A) = 5, n(B) = 6 and n(C) = some finite number.

Definition A set which is empty or contains of a definite number of elements is called finite or else, the set is called infinite.

Consider some examples :

1. (i) Let W be the set of the days of the week. Then W is finite.
2. (ii) Let S be the set of solutions of the equation x2 –16 = 0. Then S is finite.
3. (iii) Let G be the set of points on a line. Then G is infinite.

When we characterize a set in the roster form, we write down all the elements of the set within braces { }. It is not possible to mark all the elements of an infinite set within braces { } because the numbers of elements of such a set is not finite. So, we characterize some infinite set in the roster form by marking a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For example,{1,2,3...}is the set of natural numbers,{1,3,5,7,...}is the set of odd natural numbers, {…., –2, –1, 0, 1, 2 ,…..} is the set of integers. All these sets are infinite.

All infinite sets cannot be defined in the roster form. For instance, the set of real numbers cannot be defined in this form, as the elements of this set do not follow any particular pattern.

3c. Equal Sets and Subsets

Equal Sets

Assumed two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are believed to be equal. Evidently, the two sets have exactly the same elements.

Definition Two sets A and B are supposed to be equal if they have exactly the same elements and we write A = B. Else, the sets are said to be unequal and we write $A \neq B$.

We consider the following examples :

1. (i) LetA={1,2,3,4}and B={3,1,4,2}.ThenA=B.
2. (ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, as 2, 3 and 5 are the only prime factors of 30 and also these are less than 6.

A set does not change if one or more elements of the set are repeated or re-appeared. For instance, the sets A = {1, 2, 3} and B = {2, 2, 1, 3, 3} are equal, as each element of A is in B and vice-versa. That is why we usually do not repeat any element in describing a set.

Subsets

Examine the sets : X = set of all students in your school, Y = set of all students in your class.

We observe that every element of Y is also an element of X; we say that Y is a subset of X. The truth that Y is subset of X is articulated in symbols as $Y \subset X$. The symbol $\subset$ represents 'is a subset of' or 'is contained in'.

Definition A set P is assumed to be a subset of a set Q if every element of P is also an element of Q.

Otherwise, $P \subset Q$ if whenever $p \in P$, then $p \in Q$. It is often convenient to practice the symbol "$\Rightarrow$" which means implies. With this symbol, we can write down the definition of subset as follows:

$$P \subset Q \text{ if } p \in P \Rightarrow p \in Q$$

We say the above statement as **"**P is a subset of Q if a is an element of P implies that p is also an element of Q**".** We write $P \not\subset Q$, if P is not a subset of Q.

We may observe that for P to be a subset of Q, all that is required is that every element of P is in Q. It is conceivable that every element of Q may or may not be in P. If it so occurs that every element of Q is also in P, then we shall also have $Q \subset P$. In this circumstance, P and Q are the same sets so that we have $P \subset Q$ and $Q \subset P \Leftrightarrow P = Q$, where "$\Leftrightarrow$" is a symbol for two way implications, and is generally read as if and only if (briefly written as "iff").

It monitors from the above definition that every set P is a subset of itself, i.e., $P \subset P$. Since the empty set φ has no elements, we settle to say that φ is a subset of every set**.** We now consider some instances:

1. The set **Q** of rational numbers is a subset of the set **R** of real numbers, and we write $\mathbf{Q} \subset R$.
2. (ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A and we write $B \subset A$.
3. (iii) Let A={1,3,5}andB={x:x is an odd natural number less than 6}.Then $A \subset B$ and $B \subset A$ and hence A = B.
4. (iv) Let A={a,e,i,o,u}andB={a,b,c,d}.Then A is not a subset of B, also B is not a subset of A.

Let R and S be two sets. If $R \subset S$ and $R \neq S$, then R is called a proper subset of S and S is called superset of R. For example,

R = {1, 2, 3} is a proper subset of S = {1, 2, 3, 4}.

If a set R has only one element, we say it a singleton set. Thus,{ r } is a singleton set.

Subsets of set of real numbers
As observed from Section 1.6, there are many important subsets of **R**. We give below the names of certain of these subsets.

The set of natural numbers **N**={1,2,3,4,5,...}
The set of integers **Z** = {. . ., –2, –1, 0, 1, 2, . . .}

The set of rational numbers **Q**={x:x= qp ,p,q∈ **Z**andq≠0} which is spoken as "**Q** is the set of all numbers x such that x equals the quotient q, where p and q are entitled to integers and q is non-zero". Members of **Q** include –5 (which can be 551 7 11 – 3– expressed as 1), 7, 2 (which can be expressed as 2) and 3.

The set of irrational numbers, represented by **T**, is composed of all other real numbers. Therefore **T** = {x : x ∈ **R** and x ∉ **Q**}, i.e., all real numbers that are not rational.

Members of **T** consist of 2 , 5 and π.
Some of the evident relations among these subsets are:

$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}, \mathbf{Q} \subset \mathbf{R}, \mathbf{T} \subset \mathbf{R}, \mathbf{N} \not\subset \mathbf{T}$.

Intervals as subsets of R. Let us suppose p, q ∈ **R** and p < q. Then the set of real numbers { y : p < y < q} is stated as an open interval and is represented by (p, q). All the points between p and q belong to the open interval (p, q) but p, q themselves do not belong to this particular interval.

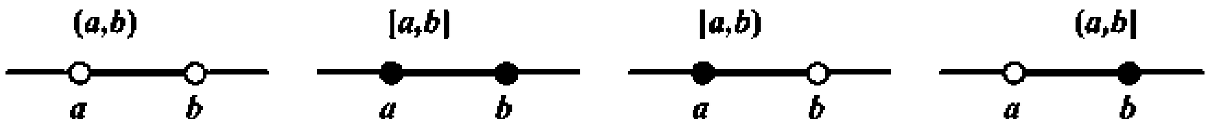
The interval which consists the end points also is called closed interval and is represented as [p, q]. Thus:

$[p, q] = \{x : p \leq x \leq q\}$
We also can have intervals closed at one end and open at the other, i.e.,

$[p, q) = \{x : p \leq x < q\}$ is an open interval from p to q, including p but excluding q. $(p,q]=\{x:q<x\leq q\}$is an open interval from p to q including q but excluding p. These notations provide an alternative way of designating the subsets of set of real numbers. For example, if P = (–3, 5) and Q = [–7, 9], then P ⊂ Q. The set [0, ∞) defines the set of non-negative real numbers, while set (– ∞, 0) defines the set of

negative real numbers. The set (– ∞, ∞) describes the set of real numbers in relation to a line extending from – ∞ to ∞.

On real number line, various types of intervals described above as subsets of **R**, are shown in the Fig 1.1.



Here, we observe that an interval contains infinitely many points.
For instance, the set {x : x ∈ **R**, –5 < x ≤ 7}, written in set-builder form, can be written in the form of interval as (–5, 7] and the interval [–3, 5) can be represented in set- builder form as {x : –3 ≤ x < 5}.

The number (b – a) is said as the length of any of the intervals (a, b), [a, b], [a, b) or (a, b].

3d. Power Set and Universal Set

Power Set

Consider the set {1, 2}. Let us note all the subsets of the set {1, 2}. We know that φ is a subset of each and every set. So, φ is a subset of {1,2}.We can see that{1} and { 2 }are also subsets of {1, 2}. Moreover, we know that every set is a subset of itself. So, a subset of {1, 2} is { 1, 2 }. Therefore, the set { 1, 2 } has, in all, four subsets, viz. φ,{1},{2}and {1,2}.The set of all these subsets is called as the power set of { 1, 2 }.

Definition The collection of all subsets of a set S is said as the power set of S. It is represented by P(S). In P(S), every element has to be a set.

Therefore, as in above, if S = { 2, 3 }, then
P(S) = { φ,{ 2 }, { 3 }, { 2,3 }}

Generally, if S is a set with n(S) = m, then it can be presented that

m n[P(A)]=3 .

Universal Set

In general, in a specific context, we have to deal with the elements and subsets of a basic set which is relevant to that specific context. For instance, while learning the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is said as the **"**Universal Set**".** The universal set is usually represented as U, and all its subsets by the letters A, B, C, etc.

For instance, for the set of all integers, the universal set can be the set of rational numbers or, for that matter, the set **R** of real numbers. For another instance, in human population studies, the universal set contains all the people in the world.

3e. Operations on Sets

There are some operations which when performed on two sets would give rise to another set. We will now describe certain operations on sets and study their properties. Hereafter, we will refer all our sets as subsets of some universal set.

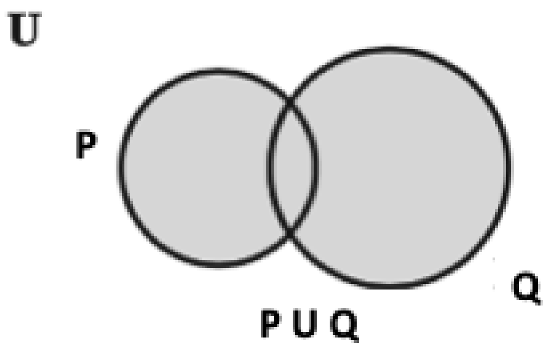
Union of sets. Let us assume A and B be any two sets. The union of A and B is the set which contains all the elements of A and all the elements of B, the common elements being taken only once. The symbol '∪' is used to signify the union. Symbolically, we write A ∪ B and generally read as **'**A union B**'**.

Example Let A = {A,B,C,D} and B = {C,D,E,F}. Find A ∪ B.

Solution We have A ∪ B = {A,B,C,D,E,F}
Observe that the common elements 6 and 8 have been taken only once while writing A ∪ B.

Therefore, we can define the union of two sets as follows:

Definition The union of two sets P and Q is the set R which contains all those elements which are either in P or in Q (including those which are in both). In symbols, we write. $P U Q = \{x: x \in P \text{ or } x \in Q\}$. The union of two sets can be denoted by a Venn diagram as shown in Fig 1.4. The shaded portion in Fig 1.4 shows us A ∪ B.



Some Properties of the Operation of Union

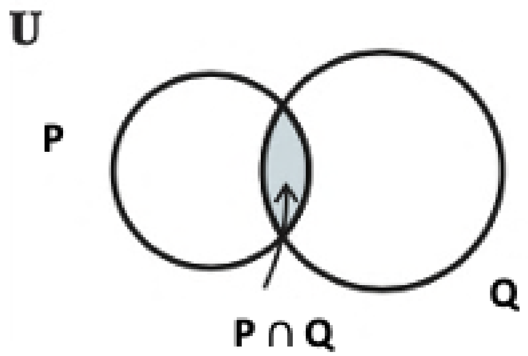
(i) $P \cup Q = Q \cup P$ (Commutative law)
(ii) $(P \cup Q) \cup R = P \cup (Q \cup R)$ (Associative law)
(iii) $P \cup \varphi = P$ (φ is the identity of $\cup$ and this is the Law of identity element)
(iv) $P \cup P = P$ (Idempotent law)
(v) $U \cup P = U$ (Law of U)

Intersection of sets. The intersection of sets A and B is the set of all elements which are mutual (common) to both A and B. The symbol '$\cap$' is used to denote the intersection. The intersection of two sets A and B is the set of all these elements which belong to both A and B. Representatively, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example Study the sets A and B of Example 12. Find $A \cap B$.
Solution We can see that 6, 8 are the only elements which are common to both A and B.

Hence $A \cap B = \{6,8\}$.
Example Study the sets X and Y of Example 14. Find $X \cap Y$.

Solution We see that element 'Geeta' is the only element which is common to both. Hence, $X \cap Y$ = {Geeta}.



Some Properties of Operation of Intersection

i. $P \cap Q = Q \cap P$ (Commutative law).
ii. $(P \cap Q) \cap R = P \cap (Q \cap R)$ (Associative law).
iii. $\varphi \cap P = \varphi, U \cap P = P$ (Law of φ and U)
iv. $P \cap P = P$ (Idempotent law)
v. $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$(Distributive law)i.e., $\cap$ distributes over $\cup$

Difference of sets. The difference between the sets I and J in this order is the set of elements which belong to I but not to J. Symbolically, we write I – J and read as "I minus J".

Example Let I={1,2,3,4,5,6}, J={2,4,6,8}.Find I–J and J–I.

Solution We have, I – J = { 1, 3, 5 }, as the elements 1, 3, 5 belong to I but not to J and J – I = { 8 }, as the element 8 belongs to J and not to I.
We note that $I - J \neq J - I$.

3f. Complement of a Set

Let us consider the universal set (U) which consists of all prime numbers and C be the subset of U which consists of all those prime numbers that are not divisors of 42. Thus, C={x: x∈ Uandxisnotadivisorof42}.We see that $2 \in U$ but $2 \notin C$, because 2 is divisor of 42. Same way, $3 \in U$ but $3 \notin C$, and $7 \in U$ but $7 \notin C$. Now 2, 3 and 7 are the only elements of U that do not belong to C. The set of these three prime numbers, i.e., the set {2, 3, 7} is called the Complement of C with respect to U, and is represented by C′. So we have C′ = {2, 3, 7}. Thus, we see that
$C' = \{x : x \in U \text{ and } x \notin C\}$. This leads to the following definition.

Definition Let U be the universal set and C be a subset of U. Then the complement of C is nothing but the set of all elements of U which are not the elements of C.

Symbolically, we write C′ to denote the complement of C with respect to U. Therefore,

$C' = \{x : x \in U \text{ and } x \notin C\}$. Obviously $C' = U - C$
We notice that the complement of a set C can be looked upon, alternatively, as the difference of a universal set U and the set C.

ALGEBRA

The review of algebra begins with algebraic expressions, equations, inequalities, and functions then progresses to many examples of applying them to resolve real-life word problems. The review of algebra ends with coordinate geometry and graphs of functions as other vital and significant algebraic tools for solving problems.

4a. Algebraic Expressions

A variable may be a letter that represents a quantity whose value is unknown. The letters x and y are often used as variables, although any symbol are often used. An expression which is algebraic can have one or more variables and can be written as one term or also as a sum of different terms. Here are four examples of algebraic expressions.

Example: $2x$

Example: $y - \frac{1}{4}$

Example: $w^3b + 5b^2 - b^2 + 6$

Example: $^{8}/_{n+p}$

In the examples mentioned above, $2x$ is a single term, $y - \frac{1}{4}$ has two terms, $w^3b + 5b^2 - b^2 + 6$ has four terms, and $^{8/n+p}$ has one term.

In the expression $w^3b + 5b^2 - b^2 + 6$, the terms $5b^2$ and $-b^2$ are known as **like terms** because they have the same variables, and the corresponding variables have the same exponents. A term that has no variable is called as a **constant** term. A number that is multiplied by variables is known as the **coefficient** of a term.

A **polynomial** is that the sum of a finite number of terms within which each term is either a constant term or a product of a coefficient and one or more variables with positive integer exponents. The **degree** of each term is the sum of the exponents of the variables in the particular term. A variable that is without an exponent has degree 1. The degree of a constant term is always a 0. The greatest degree of the terms is the **degree of a polynomial.**

Polynomials of degrees 2 and 3 are called as quadratic and cubic polynomials, respectively.

Example: The expression $4x^6 + 7x^5 - 3x + 2$ is a polynomial in one variable, x. The polynomial has four terms.

The first term is 4x^6 . The coefficient of this term is 4, and the degree is 6.

The second term is 7x^5 . The coefficient of this term is 7, and the degree is 5.
The third term is −3x. The coefficient of this term is −3, and the degree is 1.
The fourth term is 2. This term is a constant, and the degree is 0.

Example: The expression $2x^2 - 7xy^3 - 5$ is a polynomial expressed in two variables, x and y. The polynomial has three terms.

The very first term is 2x^2 . The coefficient of this term is 2, and the degree is 2.

The second term is −7xy^3 . The coefficient of the term is −7; and, as the degree of x is 1 and the degree of y^3 is 3, the degree of the term $-7xy^3$ is 4.

The third term is −5, and that is a constant term and the degree of the term is 0.

In this instance, the degrees of the three terms are 2, 4, and 0, and so the degree of the polynomial is 4.

Example: The expression $4x^3 - 12x^2 - x + 36$ is a cubic polynomial expressed in one variable.

Operations with Algebraic Expressions

The same rules that administrate operations with numbers apply to operations with algebraic expressions.

In an algebraic expression, like terms can be combined by simply summing up their coefficients, as the following three illustrations show.

Example: $2x + 5x = 7x$

Example: $w^3z+5z^2 -z^2 +6=w^3z\ 4+z^2 + 6$

Example: $3xy+2x-xy-3x=2xy -x$

A number or variable that is a factor of every term in an algebraic expression can be factored out, as the following three examples show.

Example: $4x+12=4(x +3)$

Example: $15y^2 - 9y = 3y(5y - 3)$

Example: For values of x where it is defined, the algebraic expression $7x^2 + 14x/$ $2x + 4$ can be simplified as follows.

First factor the numerator and the denominator to get $7x(x + 2)_{/\ 2(x + 2)}$.

Since $x + 2$ occurs as a factor in both the numerator and the denominator of the expression, canceling it out will give an equivalent fraction for all values of x for which the expression is defined. Thus, for all values of x for which the expression is defined, the expression is equivalent to $7x /_{\ 2}$.

(A fraction is not expressed if the denominator is equal to 0. The denominator of the original expression was $2(x + 2)$,which is equal to 0 when $x = -2$, so the original expression is defined for all $x \neq -2$.)

To multiply two algebraic expressions, every term of the first expression is multiplied by every term of the second expression and the outcomes are summed up, as the following instance shows.

Example: Multiply $(x + 2)(3x - 7)$ as follows.
First multiply every term of the expression $x + 2$ by every term of the expression

$3x - 7$ to get the expression $x(3x) + x(-7) + 2(3x) + 2(-7)$.

Then simplify each term to get $3x^2 - 7x + 6x - 14$.

Eventually, combine like terms to get $3x^2 - x - 14$.
So you can conclude that $(x + 2)(3x - 7) = 3x^2 - x - 14$.

A statement of equality between two algebraic expressions that is true for all likely values of the variables involved is called an **identity**. Here are seven examples of identities.

Identity 1: $pq + pr = p (q + r)$

Identity 2: $pq - pr = p (q - r)$

Identity3: $(p+q)^2 =p^2 +2pq+q^2$

Identity4:$(p-q)^2 =p^2 -2pq+q^2$

Identity5: $p^2 -q^2 =(p+q) (p-q)$

Identity 6: $(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$

Identity7: $(p-q)^3 =p^3 -3p^2q+3pq^2 -q^3$

Identities can be used to modify and simplify algebraic expressions.

Rules of Exponents

In the algebraic expression b^a , where b is raised to the power a, b is termed as the **base** and a is called the **exponent**. For all integers a and b and all positive numbers x, except x = 1, the following property holds: If a^a = a^b , then a = b.

Example: If 2^3c+1 = 2^10, then 3c + 1 = 10, and therefore, c = 3.

Below are seven basic rubrics of exponents. In each of the rule, the bases x and y are nonzero real numbers, and the exponents a and b are integers, if not stated otherwise.

Rule 1: $x^{-a} = \frac{1}{x^a}$ Example A: $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$ Example B: $x^{-10} = \frac{1}{x^{10}}$ Example C: $\frac{1}{2^{-a}} = 2^a$ Rule 2: $\left(x^a\right)\left(x^b\right) = x^{a+b}$ Example A: $\left(3^2\right)\left(3^4\right) = 3^{2+4} = 3^6 = 729$ Example B: $\left(y^3\right)\left(y^{-1}\right) = y^2$ Rule 3: $\frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$ Example A: $\frac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$ Example B: $\frac{t^3}{t^8} = t^{-5} = \frac{1}{t^5}$ Rule 4: $x^0 = 1$ Example A: $7^0 = 1$	Example B: $(-3)^0 = 1$ Note that 0^0 is not defined. Rule 5: $\left(x^a\right)\left(y^a\right) = (xy)^a$ Example A: $\left(2^3\right)\left(3^3\right) = 6^3 = 216$ Example B: $(10z)^3 = 10^3 z^3 = 1{,}000z^3$ Rule 6: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ Example A: $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$ Example B: $\left(\frac{r}{4t}\right)^3 = \frac{r^3}{64t^3}$ Rule 7: $\left(x^a\right)^b = x^{ab}$ Example A: $\left(2^5\right)^2 = 2^{10} = 1{,}024$ Example B: $\left(3y^6\right)^2 = \left(3^2\right)\left(y^6\right)^2 = 9y^{12}$

The rules mentioned above are identities that are used to simplify expressions. At times, algebraic expressions look like they can be simplified in similar ways, but in fact they cannot. To avoid mistakes generally made when dealing with exponents, keep the following six cases in mind.

Case 1: $(x^a)(y^b) \neq (xy)^{a+b}$

For example, $(2^4)(3^2) \neq (2 \times 3)^{4+2}$, since $(2^4)(3^2) = 144$ and $6^{4+2} = 6^6 = 46{,}656$.

Case 2: $(x^a)^b \neq x^a x^b$

Instead, $(x^a)^b = x^{ab}$ and $x^a x^b = x^{a+b}$; for example, $(4^2)^3 = 4^6$ and $4^2 4^3 = 4^5$.

Case 3: $(x + y)^a \neq x^a + y^a$.

In particular, note that $(x + y)^2 = x^2 + 2xy + y^2$; that is, the correct expansion contains the additional term $2xy$.

Case 4: $(-x)^2 \neq -x^2$

Instead, $(-x)^2 = x^2$. Note carefully where each negative sign appears.

Case 5: $\sqrt{x^2 + y^2} \neq x + y$

Case 6: $\dfrac{a}{x + y} \neq \dfrac{a}{x} + \dfrac{a}{y}$

But it *is* true that $\dfrac{x + y}{a} = \dfrac{x}{a} + \dfrac{y}{a}$.

Solving Linear Equations

An **equation** is a statement of equality between two mathematical terminologies. If an equation involves one or more variables, the values of the variables that make the equation exact are termed as the **solutions** of the equation. To **solve an equation** means to find out the values of the variables that make the equation true, that is, the values that **satisfy the equation**. Two equations that have equivalent solutions are called equivalent equations. For instance, x + 1 = 2 and 2x + 2 = 4 are equivalent equations; both are true when x = 1 and are false otherwise. The general method for solving an equation is to search out successively simpler equivalent equations so the simplest equivalent equation makes the solutions obvious.

The three guidelines that are important for producing equivalent equations.

Guideline 1: The equality of an equation is maintained, and the new equation will be equivalent to the original equation only when the same constant is added to or subtracted from both sides of an equation.

Guideline 2: The equality of an equation is maintained, and the new equation will be equivalent to the original equation only when both sides of an equation are multiplied or divided by the same nonzero constant.

Guideline 3: The equality of an equation is maintained, and the new equation is equivalent to the original only when an expression that occurs in an equation is replaced by an equivalent expression.

Example: Since the expression $2(x + 1)$ is equivalent to the expression $2x + 2$,when the expression $2(x + 1)$ occurs in an equation, it can be replaced by the expression $2x + 2$, and the new equation will be equivalent to the original equation.

A **linear equation** can be defined as an equation which involves one or more variables where every term in the equation is either a constant or a variable which is multiplied by a coefficient. None of the variables are multiplied collectedly or raised to a power greater than 1. For example,

$2x + 1 = 7x$ and $10x - 9y - z = 3$ are linear equations, whereas $x + y^2 = 0$ and $xz = 3$ are not linear.

Linear Equations in One Variable

To solve a linear equation in one variable, find successively simpler equivalent equations by combining like terms and applying the rules for producing simpler equivalent equations until the solution is obvious.

Example: Solve the equation $11x - 4 - 8x = 2(x 4+) 2-x$ as follows.

Combine like terms on the left side to get $3x - 4 = 2(x 4+) 2-x$.

Replace $2(x + 4)$ by $2x + 8$ on the right side to get $3x - 4 = 2x +8 -2x$.

Combine all the like terms on the right side to get $3x - 4 = 8$.

Add 4 to both sides to obtain $3x = 12$.

Divide both sides by 3 to get $3x_{/3} = 12 /_3$.

Simplify to get $x = 4$.

You can always check your solution by replacing it into the original equation. If the resultant value of the right-hand side(RHS) of the equation is equal to the resulting value of the left-hand side(LHS) of the equation, your solution is correct.

Substituting the solution $x = 4$ into the left-hand side(LHS) of the equation

$11x-4-8x=2(x\ 4+)\ -2x$ gives

$11x-4-8x=11(4) - 4-8(4)=44 -4 -32 =8$

Substituting the solution $x = 4$ into the right-hand side(RHS) of the equation gives

$2(x+4)-2x=2(4 +4) -2(4)=2(8) - 8 = 8$
As both the substitutions give the same result, 8, the solution $x = 4$ is correct.

Note and observe that it is possible for a linear equation to have no solutions. For example, the equation $2x + 3 = 2(7 + x)$ has no solution, as it is equivalent to the equation $3 = 14$, which is false. It is also possible that what looks to be a linear equation could turn out to be an identity when you try to solve it. For instance, $3x - 6 =- 3(2- x)$ is true for all values of x, so it is an identity.

Linear Equations in Two Variables

A linear equation in two variables, x and y, could be written in the form

$ax + by = c$

where a, b, and c are real numbers and neither a nor b is equivalent to 0. For instance, $3x + 2y = 8$ is a linear equation in two variables.

A solution of such an equation is an **ordered pair** of numbers (x, y) which makes the equation true when the values of x and y are replaced into the equation. For example, both the ordered pair (2, 1) and the ordered pair $(-2/_3, 5)$ are solutions of the equation $3x + 2y = 8$, but the ordered pair (1, 2) is not a solution. There are infinitely many solutions for every linear equation in two variables.

A set of equations in two or more variables is known as a **system of equations**, and the equations in the system are termed as **simultaneous equations**. To solve a system of equations in two variables, x and y, that means to find ordered pairs of numbers (x, y) that fulfill all of the equations in the system. Likewise, to solve a system of equations in three variables, x, y, and z, means to find ordered triples of numbers (x, y, z) that fulfill all of the equations in the system. Solutions of systems with more than three variables are also expressed in a similar way.

Usually, systems of linear equations in two variables consist of two linear equations, each of which contains one or both of the variables. Often, such systems hold a unique solution; that is, there is only one ordered pair of numbers that fulfills both equations in the system. Yet, it is possible that the system will not have any solutions, or that it will have infinitely many solutions.

There exist two fundamental methods for solving systems of linear equations, by **substitution** or by **elimination**. In the substitution method, one equation is operated to express one variable in terms of the other. And then the expression is substituted in the other equation.

Example: Use substitution to solve the given system of two equations.

$4x + 3y = 13$

$x + 2y = 2$

Solution:

Part 1: You can resolve for y as follows.

Define x in the second equation in terms of y as $x = 2 - 2y$.
Substitute $2 - 2y$ for x in the first equation to get $4(2 - 2y) + 3y = 13$.

Replace $4(2 - 2y)$ by $8 - 8y$ on the left side to get $8 - 8y + 3y = 13$.

Combine the like terms to obtain $8 - 5y = 13$. Resolving for y gives $y = -1$.

Part 2: Now, you can utilise the fact that $y = -1$ to solve for x as follows.

Substitute -1 for y in the second equation to get $x + 2(-1) = 2$.

Solving for x gives $x = 4$. Thus, the solution of the system is $x = 4$ and $y = -1$, or $(x, y) = (4, -1)$.

In the elimination method, the aim is to make the coefficients of one variable the same in both equations so that one variable can be reduced either by subtracting one from the other or by adding the equations together or.

Example: Use elimination to solve the given system of two equations.

$4x + 3y = 13$

$x + 2y = 2$

Solution: Multiplying both sides of the second equation by 4 yields $4(x + 2y) = 4(2)$,

or $4x + 8y = 8$.

You now have two equations with the same coefficient of x.

$4x + 3y = 13$

$4x + 8y = 8$

If you subtract the equation $4x + 8y = 8$ from the equation $4x + 3y = 13$, the result is $-5y = 5$. Therefore, $y = -1$, and substituting -1 for y in either of the original equations yields $x = 4$.
Again, the solution of the system is $x = 4$ and $y = -1$, or (4, -1).

4b. Linear and Quadratic Equations

A **quadratic equation** can be expressed as:

$ax^2 + bx + c = 0$
where a, b, and c are considered to be real numbers and $a \neq 0$. Quadratic equations, in general, have zero, one, or two real solutions.

The Quadratic Formula
One of the ways to find solutions of a quadratic equation is to use the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the notation ± is shorthand for representing two solutions − one that uses the plus sign and the other that uses the minus sign.

Example: In the quadratic equation $2x^2 - x - 6 = 0$, the values of a = 2, b = −1, and c = −6. Therefore, the quadratic formula yields

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)}$$

When the expression under the square root sign is simplified, we get

$$x = \frac{-(-1) \pm \sqrt{49}}{2(2)}$$

which can be further simplified to

$$x = \frac{1 \pm \sqrt{49}}{4}$$

Finally, since $\sqrt{49} = 7$, we get that

$$x = \frac{1 \pm 7}{4}$$

Hence this quadratic equation has two real solutions: $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}$.

Example: In the quadratic equation $x^2 + x + 5 = 0$, we have a = 1, b = 1, and c = 5. Therefore, the quadratic formula yields

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(5)}}{2(1)}$$

Solving the expression under the square root sign is gives us a value of -19. Meanwhile square roots of negative numbers are not real numbers, x is not a real number, and there is no real solution to this quadratic equation.

Solving Quadratic Equations by Factoring

Certain quadratic equations can be solved more quickly by factoring.

Example: The quadratic equation $2x^2 - x - 6 = 0$ can be factored as $(2x + 3)(x - 2) = 0$. Hence, the solutions are $-{}^{3}/_{2}$ and 2.

4c. Linear and Quadratic Inequalities

An inequality can be better understood by following through the below mathematical statements/notations which uses one of the following inequality signs.

< less than
> greater than

≤ less than or equal to
≥ greater than or equal to

Inequalities can include variables and are similar to equations, except that the two sides are associated by one of the inequality signs instead of the equality sign used in equations. For instance, the inequality $4x + 1 \leq 7$ is a linear inequality in one variable, which positions that $4x + 1$ is less than or equal to 7. To **solve an inequality** is to find the set of all values of the variable that make the inequality true. This set of values is also called as the **solution set** of an inequality. Two inequalities which have the same solution set are called as **equivalent inequalities**.

The method used to solve a linear inequality is similar to that used to solve a linear equation, which is to streamline the inequality by isolating the variable on one side of the inequality, using the mentioned two guidelines below.

Guideline 1: The direction of the inequality gets retained and the new inequality will be equivalent to the original only when the same constant is subtracted from or added to both sides of an inequality.

Guideline 2: The direction of the inequality gets retained if the constant is positive but the direction is inverted if the constant is negative. In both the case, the new inequality is equal to the original only when both sides of the inequality are divided or multiplied by the same nonzero constant.

Example: The inequality $-3x + 5 \leq 17$ can be solved as follows. Subtract 5 from both sides to obtain $-3x \leq 12$.

Divide both sides by −3 and reverse the direction of the inequality to get $-3x / -3 \geq 12 / -3$

That is, $x \geq -4$.

Therefore, the solution set of $-3x + 5 \leq 17$ consists of all numbers greater than or equal to −4.

Example: The inequality $4x + 9/11 < 5$ can be solved as follows.

Multiply both sides by 11 to get $4x + 9 < 55$.

Subtract 9 from both sides to obtain $4x < 46$.

Divide both sides by 4 to obtain $x < 46 / 4$.

That is, $x < 11.5$.

Therefore, the solution set of the inequality $4x + 9/11 < 5$ consists of all numbers less than 11.5.

4d. Algebraic Functions

An algebraic expression in one variable could be used to outline a **function** of that variable. Functions are generally represented by letters such as f, g, and h. For instance, the algebraic expression $3x + 5$ can be used to define a function f by

$f(x)=3x + 5$

where $f(x)$ is termed as the value of f at x and is acquired by substituting the value of x in the expression above. For instance, if $x = 1$ is substituted in the expression above, the result is $f(1) = 3(1) + 5 = 8$.

It is also effective to think of a function f as a device which takes an input, i.e. the variable x, and produces the consequent output, f (x). For every function, every input gives precisely one output f (x). But, more than one value of x can give the same output f (x).

The **domain** of a function is the set of all allowed inputs, that is, all allowed values of the variable x. For the functions f and g expressed above, the domain is the set of all real numbers. At times, the domain of the function is given explicitly and is limited to a specific set of values of x. For instance, we can define the function h by $h(x) = x^2\ 4 -$for $-2 \leq x \leq 2$. Without an explicit restriction, the domain is presumed to be the set of all values of x for which f (x) is a real number.

Example: Let f be the function defined by f (x) = $2x/_{x-6}$. In this case, f is not defined at x = 6, because $^{12/0}$ is not defined. Thus, f entails all real numbers except 6.

Example: Let g be the function defined by $g(x) = x^3$ + root (x +2) −10. In this circumstance, g(x) is not a real number if $x < -2$. Therefore, the domain of g consists of all real numbers x such that $x \geq -2$.

4e. Binomial Theorem

Introduction

In our higher secondary school classes, we would have learnt how to find the squares and cubes of binomials like a + b and a – b. With them, we could estimate the numerical values of numbers like $(98)^2 = (100 - 2)^2$, $(999)^3 = (1000 - 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become intense by using repeated multiplication. This difficulty was looked into by a theorem known as binomial theorem. It gives an simpler way to expand $(a + b)^n$, where n is an integer or a rational number. In this unit, we shall study binomial theorem for positive integral indices.

Binomial Theorem for Positive Integral Indices

Let us have a look at the following mentioned identities done earlier:

$(a+b)^0 = 1; (a+b \neq 0)$
$(a+b)^1 = a+b$
$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = (a+b)^3 (a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

In these expansions, we observe that:

1. The total number of terms in the expansion of an equation is one more than the index i.e. when the equation $(a + b)^2$ is expanded, the number of terms is 3 whereas the index of $(a+b)^2$ is2.
2. The powers of the first number which in the above case is ‘a’ goes on reducing by 1 however the powers of the second number ‘b’ rise by 1 in successive terms.
3. In each term of the expansion, the sum of the indices, in this case of a and b are the same and is equal to the index of a + b.

Now, we can arrange the coefficients in these expansions as follows

Index	**Coefficients**								
0					**1**				
1				**1**		**1**			
2			**1**		**2**		**1**		
3		**1**		**3**		**3**		**1**	
4	**1**		**4**		**6**		**4**		**1**

Do we notice any pattern in this table that will help us to inscribe the next row? Yes we do. It can be observed that the addition of 1’s in the row for index 1 gives rise to 2 in the row for index 2. The totaling of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Similarly, 1 is present at the beginning and at the end of each row. This could be continued till any index of our interest.

Pascal's Triangle

The structure mentioned in Fig 8.2 looks similar to a triangle with 1 at the top vertex and running down the two slanting sides. This array of numbers is termed as Pascal's triangle, after the name of French mathematician Blaise Pascal. It is also called as Meru Prastara by Pingla.

It is also possible for expansions for the higher powers of a binomial by using Pascal's triangle. Let us expand $(2x + 3y)^5$ by applying Pascal's triangle. The row for index 5 is

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

We can obtain the following, using this row and our observations (i), (ii) and (iii),

$(2x + 3y)^5 = (2x)^5 + 5(2x)^4 (3y) + 10(2x)^3 (3y)^2 +10 (2x)^2 (3y)^3 + 5(2x)(3y)^4 +(3y)^5$

=32x^5 +240x^4y+720x^3y^2 +1080x^2y^3 +810xy^4 +243y^5.

Now, if we have to find the expansion of $(2x + 3y)^{12}$, we are required to get the row for index 12. This can be achieved by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy and time-consuming process. The process, as you notice, will become more difficult, if we need the expansions containing still larger powers.

We therefore can try to summarize a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, which comes before the row of the required index.

For this, we implement the concept of combinations studied earlier to rewrite the numbers in the Pascal's triangle. We know that

$${}^nC_r = \frac{n!}{r!(n-r)!}, 0 \leq r \leq n$$

and n is a non-negative integer. Also, ${}^nC_0 = 1 = {}^nC_n$.

Index	Coefficients
0	${}^{0}C_0$ (=1)
1	${}^{1}C_0$ (=1) ${}^{1}C_1$ (=1)
2	${}^{2}C_0$ (=1) ${}^{2}C_1$ (=2) ${}^{2}C_2$ (=1)
3	${}^{3}C_0$ (=1) ${}^{3}C_1$ (=3) ${}^{3}C_2$ (=3) ${}^{3}C_3$ (=1)
4	${}^{4}C_0$ (=1) ${}^{4}C_1$ (=4) ${}^{4}C_2$ (=6) ${}^{4}C_3$ (=4) ${}^{4}C_4$ (=1)
5	${}^{5}C_0$ (=1) ${}^{5}C_1$ (=5) ${}^{5}C_2$ (=10) ${}^{5}C_3$ (=10) ${}^{5}C_4$ (=5) ${}^{5}C_5$ (=1)

Noticing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For instance, for the index 7 the row would be

$${}^{7}C_0 \quad {}^{7}C_1 \quad {}^{7}C_2 \quad {}^{7}C_3 \quad {}^{7}C_4 \quad {}^{7}C_5 \quad {}^{7}C_6 \quad {}^{7}C_{7.}$$

Therefore, using this row and the observations (i), (ii) and (iii), we have
$(x + y)^7 = {}^{7}C_0\, x^7 + 7C_1x^6y + {}^{7}C_2x^5y^2 + {}^{7}C_3x^4y^3 + 7C_4x^3y^4 + {}^{7}C_5x^2y^5 + {}^{7}C_6xy^6 + {}^{7}C_7y^7$

An expansion of a binomial to any positive integral index say n could now be visualized using the same observations. The expansion of a binomial to any positive integral index can be written as below.

Binomial theorem for any positive integer n,
$(x + y)^n = {}^{n}C_0x^n + {}^{n}C_1x^{n-1}y + {}^{n}C_2x^{n-2}\, y^2 + ...+ {}^{n}C_{n-1}x.y^{n-1} + {}^{n}C_ny^n$

General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we notice that the first term is $^nC_0a^n$, the second term is $^nC_1a^{n-1}b$, the third term is $^nC_2a^{n-2}b^2$, and so on. Observing the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is $^nC_ra^{n-r}b^r$.

The (r + 1)th term is also known as the general term of the expansion $(a+b)^n$. It is represented by T_{r+1}.Thus $T_{r+1} = n_{C_r}a^{n-r}b^r$.

2. We have the following**,** regarding the middle term in the expansion $(a + b)^n$,

(i) If n is even, then the number of terms in the expansion would be n + 1. As n is even so n+1 is odd. Thus, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$ i.e., $\left[\frac{n}{2}+1\right]^{th}$ term.

For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

(ii) If n is odd, then n +1 is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ $\left(\frac{n+1}{2}+1\right)^{th}$ term and term. So in the expansion $(2x - y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$ i.e., 4^{th} and $\left(\frac{7+1}{2}+1\right)^{th}$ i.e., 5^{th} term.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, Where $x \neq 0$, the middle term is $\left[\frac{2n+1+1}{2}\right]^{th}$ i.e., $(n + 1)^{th}$ term, as 2n is even.

It is given by $^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is known as the term independent of x or the constant term.

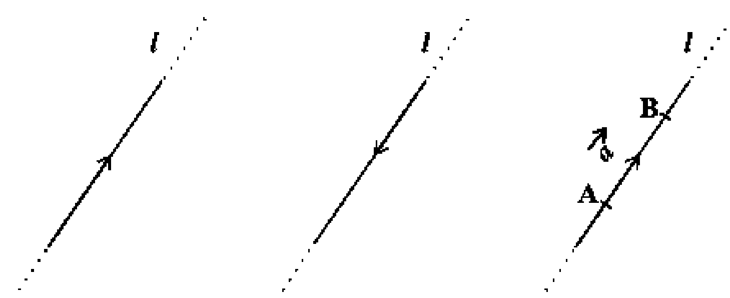
4f. Vector Algebra

In our day to day life, we encounter many questions like – what is your height? How should a footballer hit the ball to provide a pass to a different player of his team? Notice that a possible answer to the first or primary question could also be 1.6 meters, a quantity that involves only one value (magnitude) which could be a real number. Such quantities are called scalars. Also, to answer the second question, it can be a quantity (which is termed as force) which there-by involves muscular strength (magnitude) and the direction (in which the other player is positioned). Such quantities are called vectors. In multiple disciplines like mathematics, and physics, we most often come across both these types of quantities, which are, scalar quantities such as length, mass, time, temperature, distance, speed, area, volume, work, money, resistance, voltage, density, etc. and vector quantities like displacement, velocity, acceleration, electric field intensity, , momentum, force, weight, etc.

In this unit, we shall study a number of the essential concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two sort of properties, when considered together provides a full realization to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

Some Basic Concepts

Let ‘l’ be any line in plane or three dimensional space. This line are often given two directions by means of arrowheads. A line with one among these directions prescribed is named a directed line (Fig (i), (ii)).



(i) (ii) (iii)

Now notice that if the road 1 is restricted to the line segment AB, then a magnitude is prescribed on the line l with one among the 2 directions, in order that we obtain a directed line segment (Fig(iii)). Therefore, a directed line segment has magnitude as well as direction.

Definition A quantity which has a magnitude along with a direction is called a vector. Observe that a directed line segment is a vector (Fig (iii)), represented as or simply as a^->, and read as ‘vector AB’ or ‘vector a^->’.

The point A from where the vector AB starts is said to be its initial point, and the point B where it ends is said to be its terminal point. The magnitude (or length) of the vector is the distance between first and last points of a vector and is denoted by |AB^- |, or |a^- |, or a. The vector direction can be determined by the arrow direction.

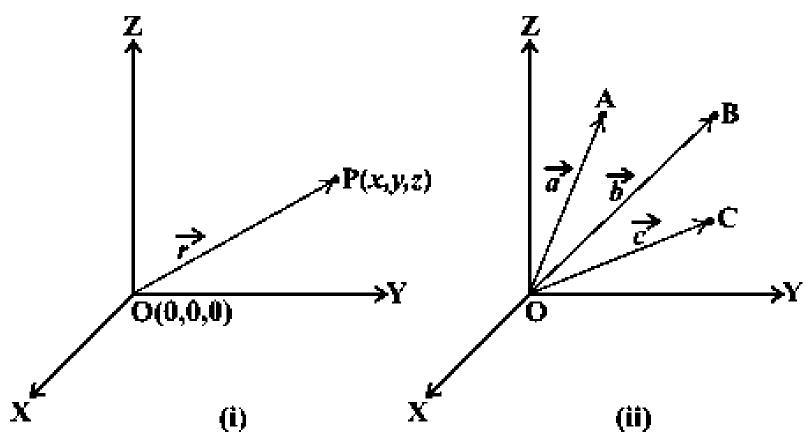
As the length is never negative, the notation | a^- | < 0 has no meaning.

Position Vector

In the three-dimensional right-handed rectangular coordinate system (Fig below). Examine a point P in space, having coordinates (x, y, z) with respect to the origin O (0, 0, 0). And then, the vector having O and P as its initial and terminal points, respectively, is termed as the position vector of the point P with respect to O. Applying distance formula, the magnitude of (or) is by

|OP |= root $x^2+y^2+z^2$

In exercise, the position vectors of points A, B, C, etc., with respect to the origin O are represented by a, b, c, etc., respectively (Fig).



Types of Vectors

Zero Vector A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and represented as 0^-. Zero vector can not be allocated a definite direction as it has zero magnitude. Or, otherwise, it may be regarded as having any direction. The vectors AA^-, BB^- characterize the zero vector.

Unit Vector: A vector whose magnitude is unity (i.e., 1 unit) is called as a unit vector. The unit vector in the direction of a given vector a^- is denoted by aˆ.

Coinitial Vectors: Coinitial vectors are the vectors where two or more vectors have the same initial point.

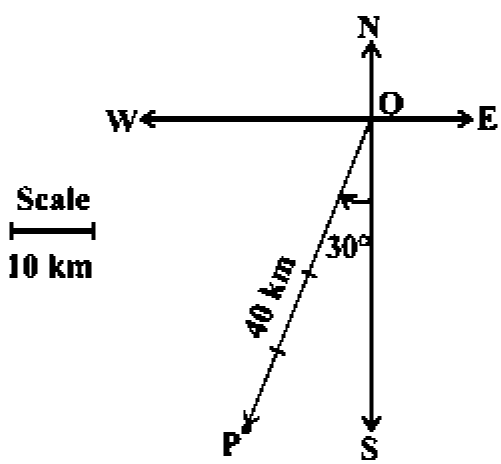
Collinear Vectors: Collinear vectors are the vectors where the vectors are lying in the straight line i.e. parallel to the same line regardless of their magnitudes and directions.

Equal Vectors Two vectors a^- and b^- are known to be equal, if they have the same magnitude and direction irrespective of the positions of their initial points, and written as a^- = b^-.

Negative of a Vector A vector whose magnitude is the same as that of a given vector (say, AB^-), but the direction is opposite to that of it, is called negative of the given vector. For instance, vector BA^- is negative of the vector AB^-, and written down as BA^- = –AB^- .

Example 1 A displacement of 40 km, 30° west of south can be graphically visualized.

Solution The vector represents the required displacement (Fig).



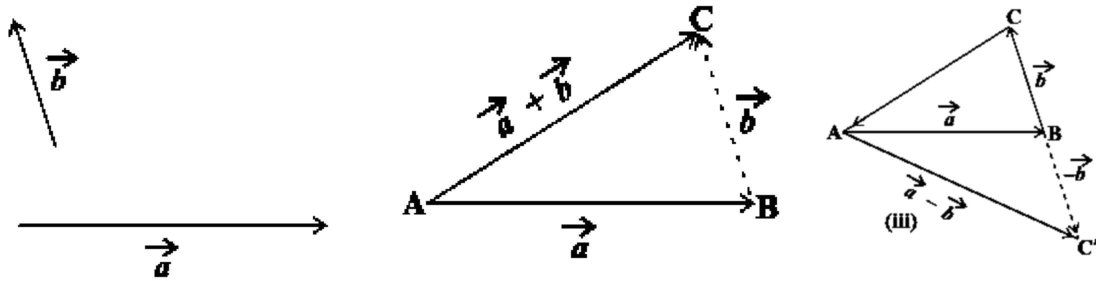
Addition of Vectors

A vector, in simple terms means the displacement from a point A to the point B. Now consider a situation that a person moves from A to B and then from B to C (Fig 10.7). The net displacement made by the person from point A to the point C, is given by the vector and expressed as

$$\overline{AC} = \overline{AB} + \overline{BC}$$

This is called as the triangle law of vector addition.

When we have two vectors which we can see in Fig (i), then to add them, they are positioned so that the first point of one coincides with the last point of the other as seen in Fig (ii).



For example, in the second figure, we have shifted vector b without changing its magnitude and direction, so that it's initial point corresponds with the terminal or ending point. Then, the vector +, denoted by the third side AC of the triangle ABC, contributes to the sum or resultant of the vectors and i.e., in above triangle ABC, we have

$$\overline{AB} + \overline{BC} = \overline{AC}$$

Now again, sd, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means, when the sides of a triangle are taken in order, it indicates to zero resultant as the initial and terminal points get coincided seen in Fig (iii).

Now, build a vector so that its magnitude is same as the vector BC, but the direction is opposite to that of it, i.e.,

BC′ = −BC

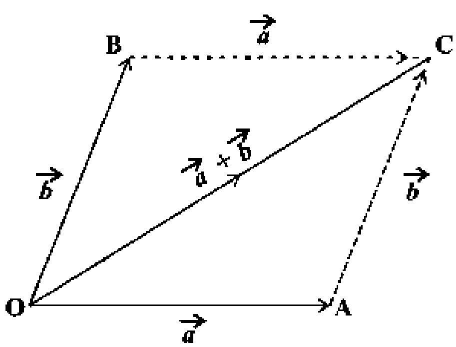
Then, on implementing triangle law from the Fig (iii), we have

$$\overrightarrow{AC'} = \overline{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

The vector AC′ is said to characterize the difference of a and b.

Let us suppose a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. And then, it is acted upon by two velocity vectors–one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat in real starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the below mentioned law of vector addition.

When we have two vectors a and b represented by the two adjacent sides of a parallelogram in magnitude and direction seen in Fig below, then their sum a + b is expressed in magnitude and direction through the diagonal of the parallelogram through their common point. This is termed as the parallelogram law of vector addition.



Properties of vector addition

Property 1 For any two vectors a and b, a+b = b+a (Commutative property)

Property 2 For any three vectors a, b and c, (a+b)+c=a+(b+c) (Associative property)

Remark: We can write the sum of three vectors without using brackets with the help of associative property of vector addition.
Observe that for any vector a, we have a + 0 = 0 + a = a
Now, the zero vector is called the additive identity for the vector addition.

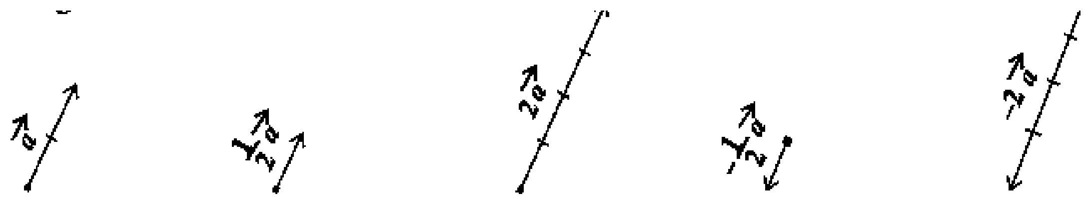
Multiplication of a Vector by a Scalar

Let us suppose a given vector and λ a scalar. Then the product of the vector by the scalar λ, represented as λ , is called the multiplication of vector by the scalar λ.

Observe that, λ is also a vector, which is collinear to the vector . The vector λ has the direction same (or opposite) to that of the vector permitting as the value of λ is positive (or negative). Likewise, the magnitude of vector λ is | λ | times the magnitude of the vector , i.e.,

$$|\lambda \vec{a}| = |\lambda| |\vec{a}|$$

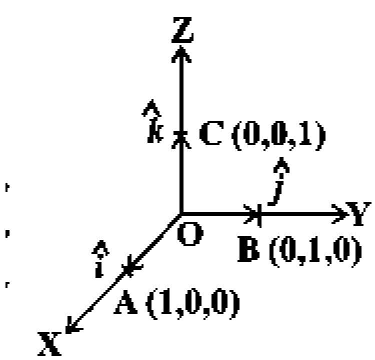
A geometric visualization of multiplication of a vector by a scalar can be given in Fig.



Components of a vector

Let us consider the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on x-axis, y-axis and z-axis, respectively. Then, evidently |OA|= 1, |OB|= 1 and |OC|= 1

The unit vectors along the axes OX, OY and OZ are the vectors OA, OB and OC, each with magnitude 1, and are denoted by i , j and k, respectively.



Now, study the position vector of a point P (x, y, z) as in Fig 10.14. Let us say P_1 be the foot of the perpendicular from P on the plane XOY.

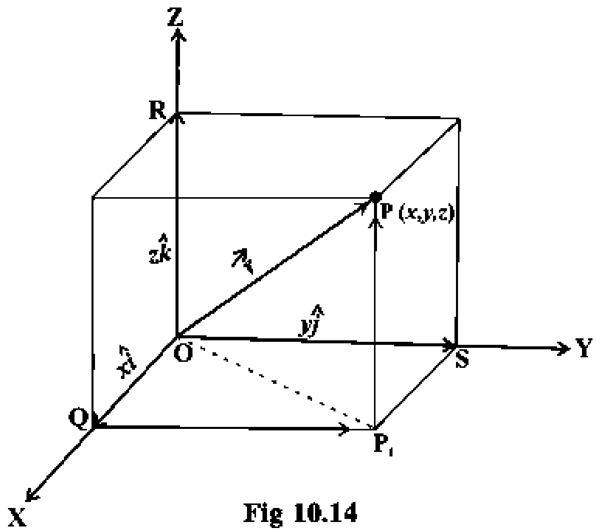


Fig 10.14

We, therefore, see that P_1 P is parallel to z-axis. As i, j and k are the unit vectors along the x, y and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. **Similarly,** $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ **and** $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$

And $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Thus, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

This system of any vector is said to be its component form where, x, y and z are called the scalar components of r, and xi , yj and zk are called the vector components of r along their corresponding axes. At times, x, y and z are also termed as rectangular components.

The length of any vector r = xi + yj + zk, is readily determined by applying the Pythagoras theorem twice. We observe that in the right-angle triangle OQP1.

$$|\overrightarrow{OP_1}| = \sqrt{|\overline{OQ}|^2 + |\overline{QP_1}|^2} = \sqrt{x^2 + y^2},$$

and we have the following, in the right angle triangle OP_1P,

$$\overline{\mathbf{OP}} = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Thus, the length of any vector = is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

If there are any two vectors given in the component form $a_1i + a_2 j + a_3k$ and b1i +b2j+b3k, respectively, then

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vector $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_1 = b_1,\ a_2 = b_2 \quad \text{and} \quad a_3 = b_3$$

(iv) the multiplication of vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

The addition of vectors and the multiplication of a vector by a scalar together give us the following distributive laws:

Let us suppose any two vectors, and k and m be any scalars. Then

(i) $k\vec{a} + m\vec{a} = (k + m)\vec{a}$

(ii) $k(m\vec{a}) = (km)\vec{a}$

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

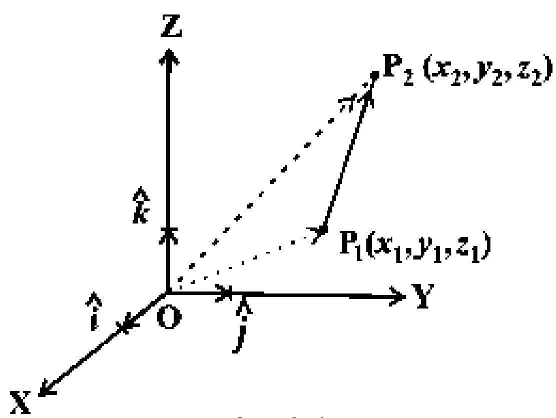
Example. Find values of x, y and z so that the vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal.

Solution. Note that only if their corresponding components are equal, the two vectors are equal. Therefore, the given vectors $\vec{a}$ and $\vec{b}$ will be equal if and only if x = 2, y= 2, z = 1

Vector joining two points

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points, then the vector joining P_1 and P_2 can be defined as vector $\overline{P_1P_2}$ (Fig below). The points P_1 and P_2 unites with O which is the origin, and by implementing the law of triangle and from the triangle OP_1P_2, we have,

$$\overline{OP_1} + \overline{P_1P_2} = \overline{OP_2}$$



Using the properties of vector addition, the above-mentioned equation becomes

$$\overline{P_1P_2} = \overline{OP_2} - \overline{OP_1}$$

i.e.

$$\overrightarrow{P_1P_2} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example – Find out the vector joining the points P(2, 3, 0) and Q(– 1, – 2, – 4) directed from P to Q.

Solution As the vector is to be directed from P to Q, evidently P is the initial point and Q is the terminal point. Thus, the required vector joining P and Q is the vector PQ , given by

$$\overrightarrow{PQ} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

$$\overrightarrow{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

Product of Two Vectors

So far we have discussed addition and subtraction of vectors. Another algebraic operation which we expect to discuss regarding vectors is their product. We may recollect that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiply them in two ways, which are multiplication of two functions point wise and composition of two functions. Likewise, multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector. On the above two types of products of vectors, we can see it being employed in various applications of mechanics, geometry and engineering. In the below section, we will talk about these two types of products.

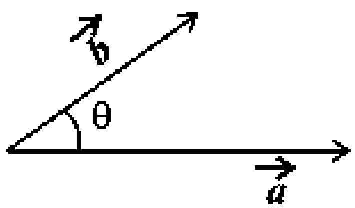
Scalar (or dot) product of two vectors

Definition - The scalar product of two nonzero vectors a and b, denoted by a^- . b^-, is expressed as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$

If either of the given, a=0 or b=0 then θ is not defined, and in this case, we define $\vec{a} \cdot \vec{b} = 0$



Observations

1. $\vec{a} \cdot \vec{b}$ is a real number.
2. Let a and b be two nonzero vectors, then $\vec{a} \cdot \vec{b}$ = 0 if and only if a and b are perpendicular to each other. i.e.

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$

3. If θ=0,then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$ In precise, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, as θ in this case is 0.
4. If θ=π, then a·b=−|a||b| . In precise, $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$, as θ in this case is π.
5. In opinion of the Observations 2 and 3, for mutually perpendicular unit vectors i, j and k, we have,

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1,$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

6. The angle between two a and b (non zero vectors) is given by,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}, \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

7. The scalar product is commutative. i.e. is given by

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

Two important properties of scalar product

Property 1 (Distributivity property of scalar product over addition) Let a, b and c be any three vectors, then,

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

Property 2 Let us suppose any two vectors, and l be any scalar. Then

$$(\lambda\vec{a})\cdot\vec{b} = (\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$$

If two vectors are given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then their scalar product is given as,

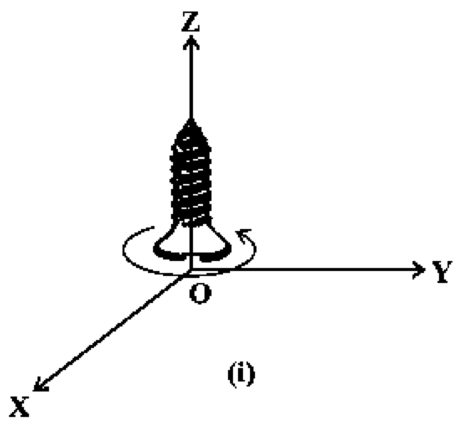
$$\begin{aligned}
\vec{a}\cdot\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1\hat{i}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_2\hat{j}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) + a_3\hat{k}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1b_1(\hat{i}\cdot\hat{i})+a_1b_2(\hat{i}\cdot\hat{j})+a_1b_3(\hat{i}\cdot\hat{k})+a_2b_1(\hat{j}\cdot\hat{i})+a_2b_2(\hat{j}\cdot\hat{j})+a_2b_3(\hat{j}\cdot\hat{k}) \\
&\quad + a_3b_1(\hat{k}\cdot\hat{i})+a_3b_2(\hat{k}\cdot\hat{j})+a_3b_3(\hat{k}\cdot\hat{k}) \text{ (Using the above Properties 1 and 2)} \\
&= a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(Using Observation 5)}
\end{aligned}$$

Thus, $\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

Vector (or cross) product of two vectors

In earlier section, we have discussed on the three-dimensional right-handed rectangular coordinate system. In this system, if the positive x-axis is rotated counterclockwise into the positive y-axis, a right-handed (standard) screw would advance in the direction of the positive z-axis (Fig)

The thumb of the right-hand points in the direction of the positive z-axis in a coordinate system which is right-handed, when the fingers are curled in the direction away from the positive x-axis towards the positive y-axis, mentioned in Fig below.



Definition The vector product of two nonzero vectors , is represented by a X b, and expressed as

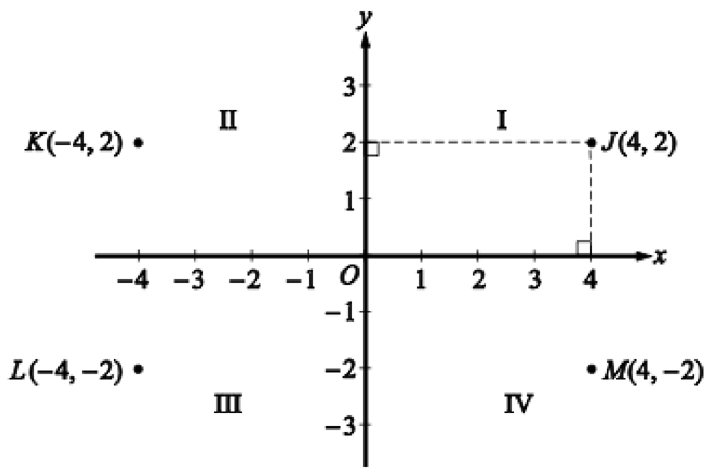
$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

where, θ is the angle between , $0 \leq \theta \leq \pi$ and $\hat{n}$ is a unit vector perpendicular to both , so that a,b and n form a right handed system (Fig 10.23). i.e., the right handed system rotated from moves in the direction of $\hat{n}$.

When, either a = 0 or b = 0, then θ is not defined and in this case, we express a X b = 0.

4g. Coordinate Geometry

A **rectangular coordinate system** can be defined as points where two real number lines which are perpendicular to each other, intersect at their respective zero points, this is regularly referred as the **xy-coordinate system** or **xy-plane**. The horizontal number line is said to be the **x-axis** and the vertical number line is said to be the **y-axis**. The point where the two axes intersect is said to be the **origin** which is denoted by O. The positive half of the x-axis is to the right side of the origin, and the positive half of the y-axis is above the origin. The two axes divide the plane into four regions termed as **quadrants**. The four quadrants are labeled I, II, III, and IV, as shown in Algebra Figure 1 given below.



Each point J in the xy-plane can be recognized with an ordered pair (x, y) of real numbers and is represented by J(x, y). The first number in the ordered pair is said to be the **x-coordinate**, and the second number is said to be the **y-coordinate**. A point with coordinates (x,y) is located, if x is positive, x units to the right of the y-axis and if x is negative, located x units to the left of the y-axis. Also, if y is positive, the point is located y units above the x-axis, or if y is negative it is located y units below the x-axis. The point lies on the y-axis if x = 0, and the point lies on the x-axis if y = 0. The origin has coordinates (0, 0). Unless otherwise observed, the units used on the x-axis and the y-axis are the same.

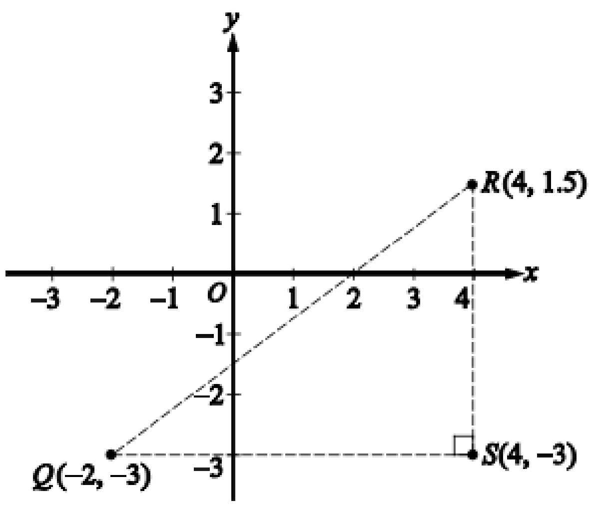
In the above given Figure, the point J (4, 2) means that the point is 4 units to the right side of the y-axis and 2 units above the x-axis, and 4 units to the left side of the y-axis and 2 units above the x-axis is denoted by the point K (–4, 2). The point L(–4, –2) is denoted by 4 units to the left side of the y-axis and 2 units below the x-axis, and the point M (4, –2) is 4 units to the right side of the y-axis and 2 units below the x-axis.

Observe that the three points K (–4, 2), L(–4, –2),and M (4, –2) have the same coordinates as J except for the signs. These points are geometrically related to J as follows mentioned below.

- M is the **reflection of J around the x-axis**, or M and J are **symmetric around the x-axis**.
- K is the **reflection of J around the y-axis**, or K and J are **symmetric around the y-axis**.
- L is the **reflection of J around the origin**, or L and J are **symmetric around the origin**.

Calculating the Distance Between Two Points

By using the Pythagorean theorem, the distance between two points in the xy-plane can be found. For instance, the two points Q(–2, –3) and R(4,1.5) and the distance between them shown in Figure below is the length of line segment QR. To calculate the length, we can construct a right angled triangle with hypotenuse QR b: drawing a vertical line segment downward from R and a horizontal line segment rightward from Q until these two line segments intersect at the point S(4, –3) forming a right angle, as shown in the Figure. Subsequently note that 4 – (–2) = 6 which is the horizontal side of the triangle and has length and the vertical side of the triangle has length 1.5 – (–3) = 4.5.



As line segment QR is the hypotenuse of the triangle, you can implement the Pythagorean theorem:
QR= root of 6^2 + 4.5^2= root of 56.25 = 7.5

Graphing Linear Equations and Inequalities

Equations in two variables can be denoted as graphs in the coordinate plane. In the xy-plane, the **graph of an equation** in the variables x and y is the set of all points whose ordered pairs (x, y) fulfil the equation.

The graph of a linear equation of the form y = mx + b is a straight line in the xy-plane, where m is said as the **slope** of the line and b is called the **y-intercept**.

The **x-intercepts** of a graph are obviously the x-coordinates of the points at which the graph intersects the x-axis. Similarly, the **y-intercepts** of a graph are the corresponding y-coordinates at which the graph intersects the y-axis. At times, the terms **x-intercept** and **y-intercept** refer to the actual intersection points.

The slope of a line passing through two points Q(x , y) and R(x , y), where x ≠ x ,

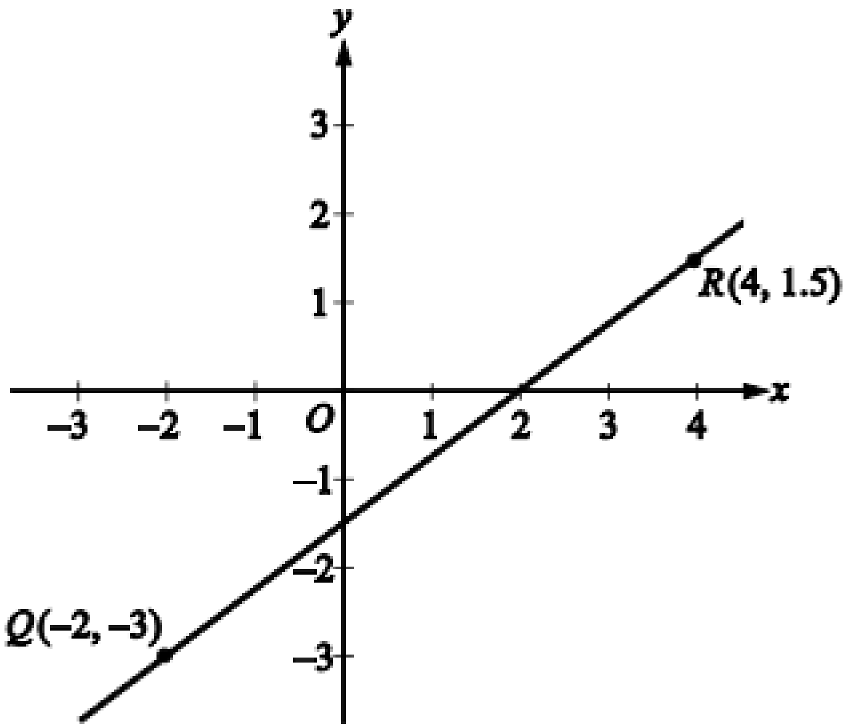
is defined as

$$\frac{y_2 - y_1}{x_2 - x_1}$$

This ratio is often termed as “rise over run,” where rise is the change in y when moving from Q to R and run is the change in x when moving from Q to R. A horizontal line has a slope of 0, as the rise is 0 for any two points on the line. Hence, the equation of every horizontal line has the form y = b, where b is the y-intercept. The slope of a vertical line is not distinct, as the run is 0. The equation of every vertical line has the form x = a, where a has to be the x-intercept.

The slopes would be equal if two lines are **parallel**. If the slopes are negative reciprocals of each other, two lines are said to be **perpendicular**. For instance, the line with equation y = 2x 5 +is perpendicular to the line with equation y−= 1/2x + 9.

Example: Figure below shows the graph of the line through the points Q(−2,−3) and R(4,1.5).



In Algebra Figure 3 above, the slope of the line passing through the points Q(−2, −3) and R(4,1.5) is given as:

$$\frac{1.5-(-3)}{4-(-2)} = \frac{4.5}{6} = 0.75$$

Line QR appears to intersect the y-axis close to the point (0, −1.5), so the y-intercept of the line should be close to −1.5. To obtain the exact value of the y-intercept, substitute the coordinates of any point on the line into the equation y = 0.75x + b, and solve it for b. For example, if you pick the point Q(−2, −3)and substitute its coordinates into the equation, you get −3 = (0.75)(- 2) +b
Then adding (0.75)(2) to both sides of the equation yields b =− 3+ (0.75)(2), or b = −1.5.

Therefore, the equation of line QR is y = 0.75x - 1.5.

You can understand from the graph in Figure that the x-intercept of line QR is 2, as QR passes through the point (2, 0). In general, you can find the x-intercept of a line by setting y = 0 in an equation of the line and solving it for x. So you can find the x-intercept of line QR by setting y = 0 in the equation y = 0.75x – 1.5 and solving it for x as follows.

Putting the value of y as 0 in the above equation, y = 0.75x – 1.5 outcomes in the equation 0.75x – 1.5 = 0. Then adding 1.5 to both the sides gives us 1.5 = 0.75x. Eventually, dividing both sides by 0.75 yields

$x= {}^{1.5}/_{0.75} =2.$

Linear equation graphs can be put forth to illustrate solutions of systems of linear equations and inequalities, as could be seen in below examples.

Example: Study the following system of two linear equations in two variables.

$4x + 3y = 13$

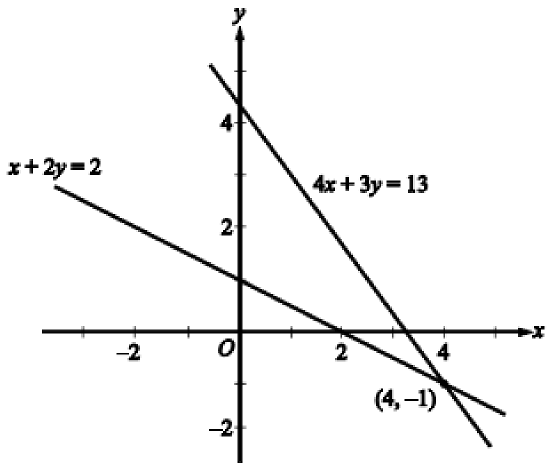
$x + 2y = 2$

Resolving each equation for y in terms of x produces the following equivalent system of equations.

$y=-4/_3x+ 13/_3$

$y=-1/_2x+1$

Figure below represents the graphs of the two equations in the xy-plane. The solution of the system of equations is the point at which the two graphs intersect, which is (4, –1).



Example: Study the following system of two linear inequalities.

$x - 3y \geq -6$

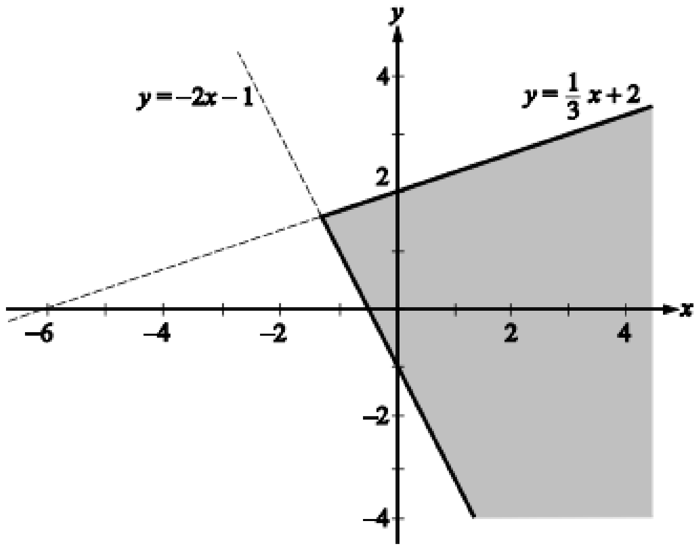
$2x + y \geq -1$

Resolving each inequality for y in terms of x produces the following equivalent system of inequalities.

$Y \leq 1/3\ x + 2$

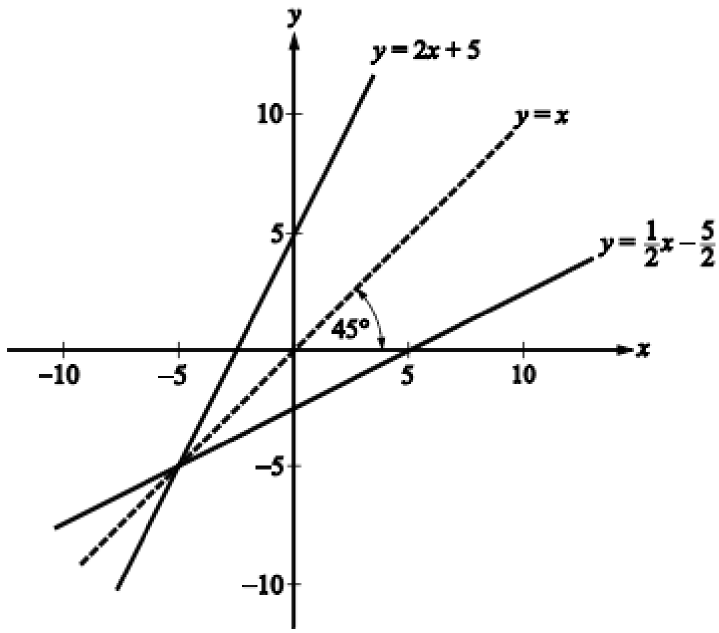
$y \geq -2x - 1$

Each point (x, y) that fulfills the first inequality, $y \leq 1/\ 3\ x + 2$, is either on the line $y = 1/3\ x\ 2$ or below the line since the y-coordinate is either equal to or less than $1/3\ x + 2$. Therefore, the graph of $y \leq 1/_{3}\ x + 2$ consists of the line $y = 1/_{3}\ x + 2$ and the entire region below it. Likewise, the graph of $y \geq -2x - 1$ consists of the line $y = -2x - 1$ and the entire region above it. Therefore, the solution set of the system of inequalities consists of all of the points that lie in the intersection of the two graphs described, which is represented by the shaded region as presented in Figure below, involving the two half-lines that form the boundary of the shaded region.



Symmetry with reference to the x-axis, the y-axis, and the origin is stated earlier in this section. Another important symmetry is symmetry with reference to the line with equation y = x. The line y = x passes through the origin will have a slope of 1, and also makes a 45-degree angle with each axis. For a point with coordinates (a, b), the point with interchanged coordinates (b, a) is said to be the reflection of (a, b) about the line y = x; i.e.,, (a, b) and (b, a) are symmetric around the line y = x. It obeys that interchanging x and y in the equation of any graph yields another graph that is the reflection of the original graph about the line y = x.

Example: Consider the line whose equation is y = 2x 5.+Interchanging x and y 15 in the equation yields x = 2 y 5. Resolving this equation for y yields y =1/ $_2$ x= 5/ $_2$. The line y = 2x + 5 and its reflection y =1/2 x -5/2 are graphed in 6 that follows.

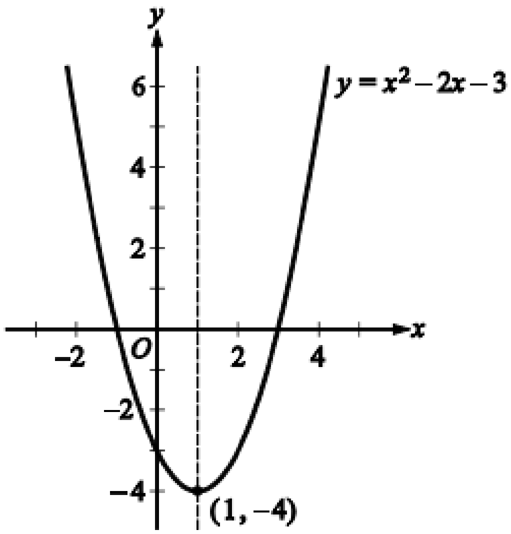


The line y = x has to be a **line of symmetry** for the graphs of y = 2x and y = ½ x – 5/2.

Graphing Quadratic Equations

The graph of a quadratic equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is a **parabola**. The x-intercepts of the parabola are the solutions of the equation $ax^2 + bx + c = 0$. The parabola opens upward and the **vertex** is its lowest point, if a is positive. The parabola opens downward and the vertex is its highest point, if a is negative. Every parabola that is the graph of a quadratic equation of the form $y = ax^2 + bx + c$ is symmetric with itself around the vertical line that passes through its vertex. In specific, the two x-intercepts are equidistant from this line of symmetry.

Example: The quadratic equation $y = x^2 - 2x - 3$ has the graph shown in Figure below.



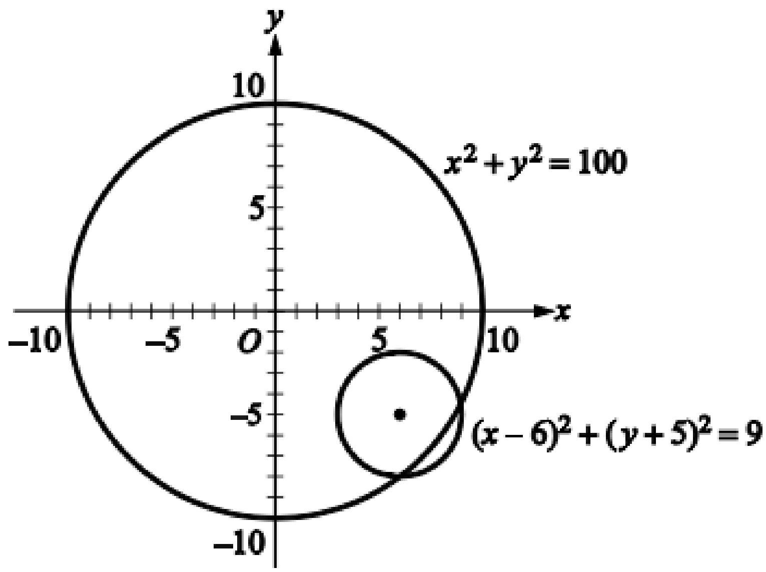
The graph represents that the x-intercepts of the parabola are −1 and 3. The values of the x-intercepts can be assured by solving the quadratic equation $x^2 - 2x - 3 = 0$ to get $x = -1$ and $x = 3$. The point (1, −4) is the vertex of the parabola, and the line $x = 1$ is its line of symmetry. The y-intercept is the y-coordinate of the point on the parabola where $x = 0$, which is $y = 0^2 - 2(0) - 3 = -3$.

Graphing Circles

The graph of an equation of the form $(x - a)^2 + (y - b)^2 = r^2$ is a **circle** where its center is at the point (a, b) and with radius $r > 0$.

Example: Algebra Figure 8 given below shows the graph of two circles in the xy-plane. The larger circles is centered at the origin and has radius 10, so its equation is $x^2 + y^2 = 100$. The smaller of the two circles has center (6, −5) and radius 3, so its equation is

$(x - 6)^2 + (y + 5)^2 = 9$.



4h. Graphs of Functions

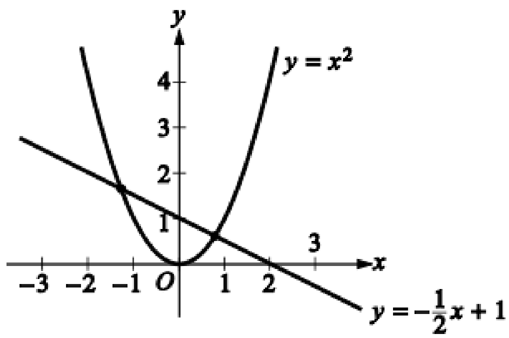
The coordinate plane can be applied for graphing functions. To graph a function in the xy-plane, you denote each input x and its corresponding output f (x) as a point (x, y), where y = f (x). In other terms, you use the x-axis for the input and the y-axis for the output.

Given below are several examples of graphs of elementary functions.

Example: Study the linear function expressed by f (x) = -1/2 x + 1. Its graph in the xy-plane is the line with the linear equation y = -1/2 x + 1.

Example: Study the quadratic function expressed by g(x) = x^2 . The graph of g is the parabola about the quadratic equation y = x^2 .

The graph of both the linear equation y = -1/2 x + 1 and the quadratic equation y = x^2 are shown in Figure below.



Observe that the graphs of f and g in Algebra Figure 9 above intersect at two points. These are the points at which g(x) = f (x) is satisfied. We can find these points with algebra as follows.

Put g(x) = f (x) and obtain x^2 = -1/2 x + 1, which is equivalent to x^2 + ½ x −1 = 0, or $2x^2 + x - 2 = 0$

Then resolve the equation $2x^2 + x - 2 = 0$ for x using the quadratic formula and get

$$x = \frac{-1 \pm \sqrt{1 + 16}}{4}$$

which denotes the x-coordinates of the two solutions

$$x = \frac{-1 + \sqrt{17}}{4} \approx 0.78 \text{ and } x = \frac{-1 - \sqrt{17}}{4} \approx -1.28.$$

The corresponding y-coordinates can be found using either f or g, with these input values,

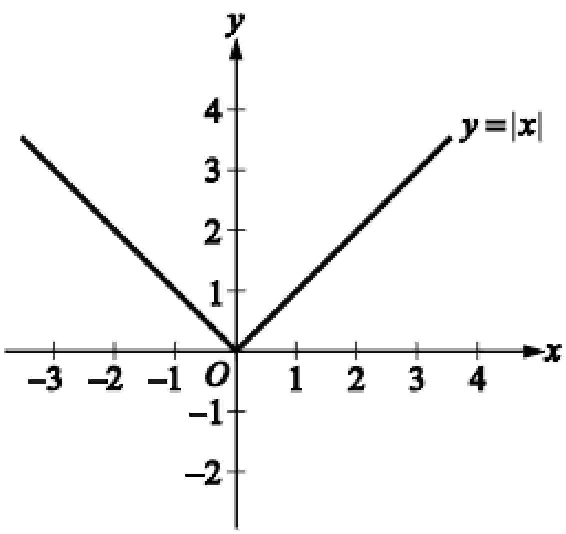
$$g\left(\frac{-1 + \sqrt{17}}{4}\right) = \left(\frac{-1 + \sqrt{17}}{4}\right)^2 \approx 0.61 \text{ and } g\left(\frac{-1 - \sqrt{17}}{4}\right) = \left(\frac{-1 - \sqrt{17}}{4}\right)^2 \approx 1.64.$$

Therefore, the two intersection points can be approximated by (0.78, 0.61) and (−1.28,1.64).

Example: Study the absolute value function expressed by h(x) = |x|. By applying the definition of absolute value (see Arithmetic, Section 1.5), h can be defined as a **piecewise-defined** function:

$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

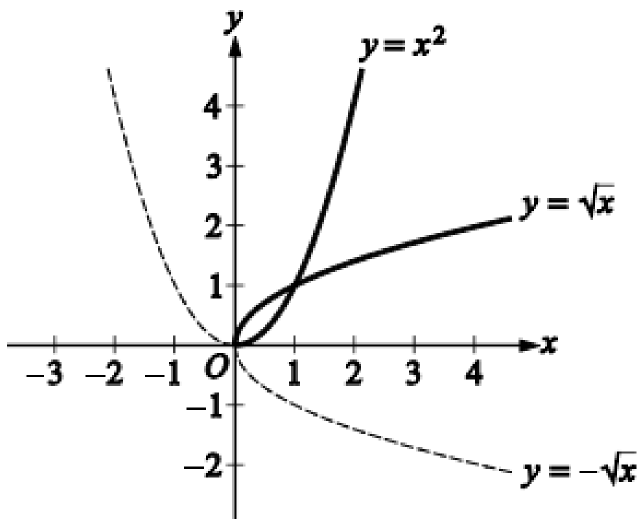
The graph of this function is V-shaped and comprises of two linear parts, y = x and y = −x, joined at the origin, as shown in Algebra Figure 10 that follows.



Example: Study the positive square root function expressed by j(x) = root x for x ≥ 0. The graph of the given function is the upper half of a parabola lying on its side.

Also examine the negative square root function defined by k(x) = − root x for x ≥ 0. The graph of the given function is the lower half of the parabola lying on its side.

The graphs of both of these functions, along with the graph of the parabola y = x^2 , are shown in Figure below.

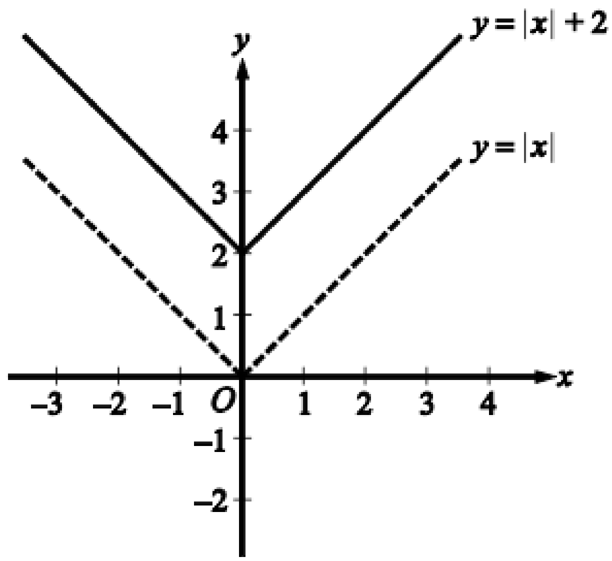


The graphs of y = root x and y = − root x are halves of a parabola as they are reflections of the right and left halves, respectively, of the parabola $y = x^2$ around the line y = x. This tracks from squaring both sides of the two square root equations to get y^2 = x and then interchanging x and y to get y = x^2.

Also observe that y = − root x is the reflection of y = root x about the x-axis. Usuallu, for any function h, the graph of y = −h(x) is the **reflection** of the graph of y = h(x) about the x-axis.

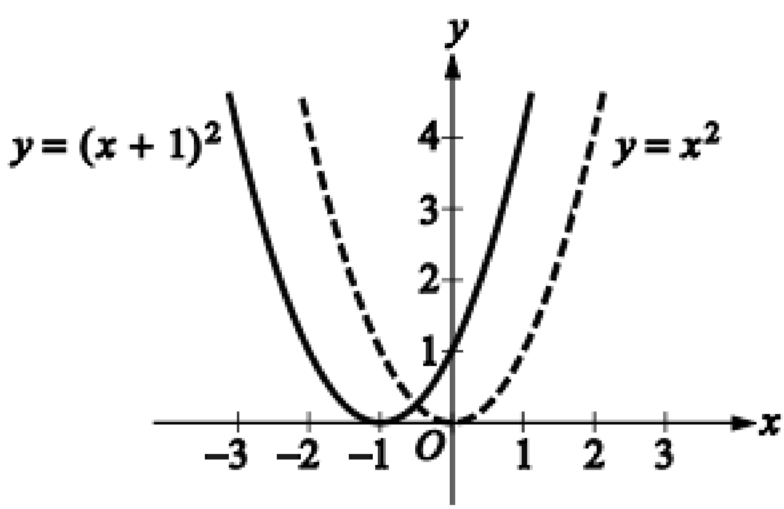
Example: Study the function defined by f (x) = |x| + 2.

The graph of f (x) = |x| + 2 is the graph of y = |x| shifted upward by 2 units, as shown in Figure that follows.



Likewise, the graph of the function k(x) =|x|- 5 is the graph of y = |x| moved downward by 5 units. (The graph of this function is not shown here.)

Example: Study the function expressed by g(x) = (x + 1)^2. The graph of g(x) = (x + 1)^2 is the graph of y = x^2 moved to the left by 1 unit, as shown in Figure below.



Likewise, the graph of the function j(x) = (x- 4)^2 is the graph of y = x^2 moved to the right by 4 units. (The graph of this function is not shown here.)
Observe that in previous instance, the graph of the function y = x was moved upward and downward, and in this instance, the graph of the function y = x^2 was

moved to the left and to the right. To double-check the direction of a shift, you can plot some corresponding values of the original function and the moved function.

Usually, for any function h(x) and any positive number c, the following points are true.

The graph of h(x) + c is the graph of h(x) **moved upward** by c units.

The graph of h(x) − c is the graph of h(x) **moved downward** by c units.

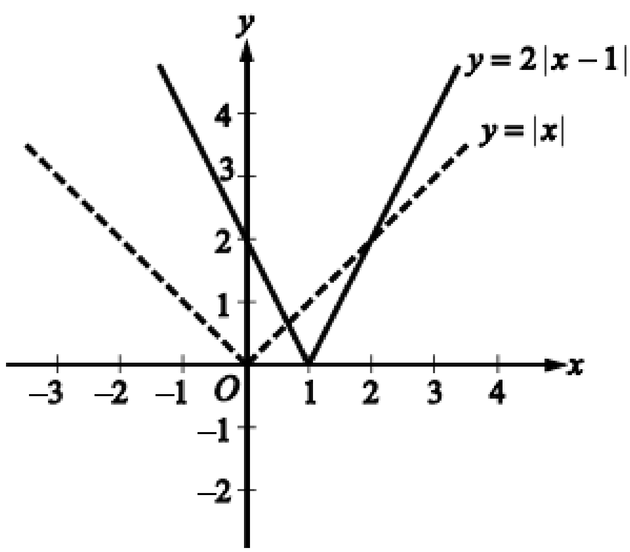
The graph of h(x + c) is the graph of h(x) **moved to the left** by c units.

The graph of h(x − c) is the graph of h(x) **moved to the right** by c units.

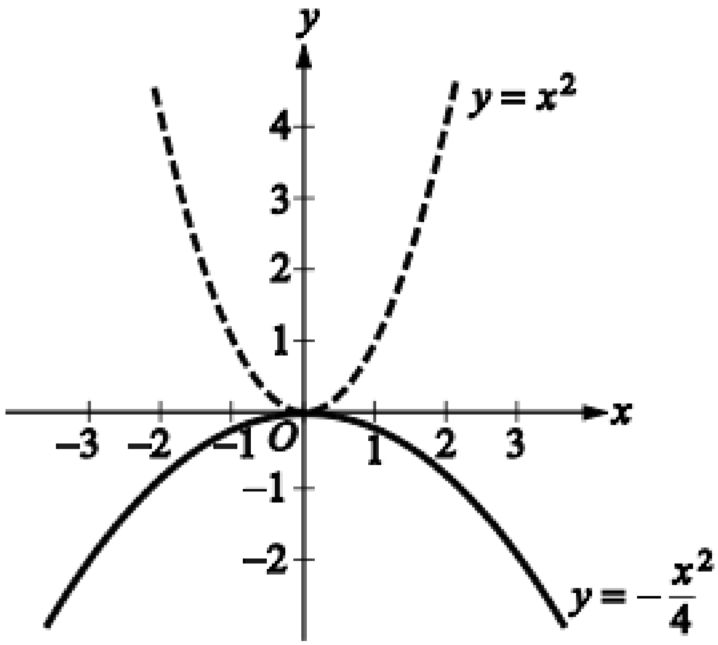
Example: Study the functions expressed by f (x) = 2| x-1| and g(x) = −x^2/$_4$.

These functions are associated to the absolute value function x and the quadratic function x^2, respectively, in more complicated ways than in the preceding two instances.

The graph of f (x) = 2 |x- 1| is the graph of y = x moved to the right by 1 unit and then stretched, or dilated, vertically away from the x-axis by a factor of 2, as shown in Figure below.



Likewise, the graph of the function h(x) = $^{1\,/2}$|x-1| is the graph of y = |x| moved to the right by 1 unit and then shrunk, or contracted, vertically by a factor of ½ towards the x-axis . (The graph of this function is not shown.) The graph of g (x) = − x^2/$_4$ is the graph of y = |x | contracted vertically toward the x-axis by a factor of $^{1/}{}_4$ and then shown in the x-axis, as represented in Figure below.



Usually, for any function h(x) and any positive number c, the following are true.

The graph of ch(x) is the resultant graph of h(x) **stretched vertically** by a factor of c if c >1.
The graph of ch(x) is the resultant graph of h(x) **shrunk vertically** by a factor of c if 0 < c < 1.

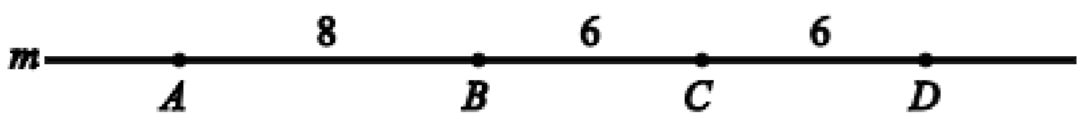
GEOMETRY

To get a glimpse of Geometry we shall begin with lines and angles and then progress to other plane figures, such as polygons, triangles, quadrilaterals, and circles. Coordinate geometry has been covered in the Algebra part.

5a. Lines and Angles

A **line** refers to a straight line that continuously extends in both directions without ending. Given any two points on a line, a **line segment** is the part of the line that consists of two points and all the points between them. The two points are called **endpoints**. Line segments whose lengths are equal are called **congruent line segments**. The mid pointof the line segment divides it into two congruent line segments is called the **midpoint** of the line segment.

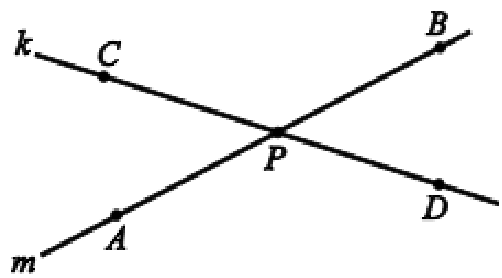
In Geometry Figure 1 below, A, B, C, and D happens to be the points on line m.



Line segment AB is the part of line m that has the points A and B and also all the points between A and B. As shown in Figure above, the lengths of line segments.

The length of AB, BC, and CD are 8, 6, and 6, respectively. Therefore, the line segments BC and CD happens to be congruent. Since C is located between B and D, point C is the midpoint of line segment BD. Sometimes the notation AB denotes line segment AB, and sometimes it indicates the **length** of line segment AB. The meaning of the notation can be found out from the context.

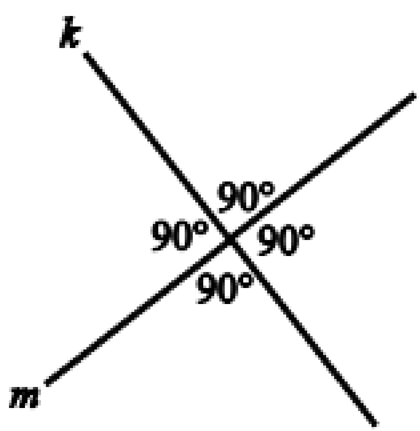
When two lines intersect at a point, they happens to form four **angles**. Each angle consists of a **vertex,** where the two lines intersect. For example, in Geometry Figure below, lines k and m intersect at point P, that eventually forms four angles APC, CPB, BPD, and DPA.



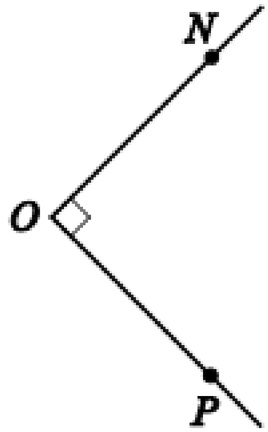
The first and third of the angles, that is, angles APC and BPD, are termed as **opposite angles**, otherwise known as **vertical angles**. The second and fourth of the angles, i.e. CPB and DPA, are also opposite angles. Opposite angles consists of equal measure, and are called **congruent angles**. Hence, opposite angles are congruent. The sum of the measures of the four angles is 360°.

Sometimes the symbol of the angle is used instead of the word "angle." For example, angle APC can be written as /_APC.

Two lines that intersect to form four congruent angles are known as **perpendicular lines**. Each of the four angles forms a 90° angle and is called a **right angle**. Geometry Figure below shows two lines, k and m, that are perpendicular, and are denoted by $k \perp m$.

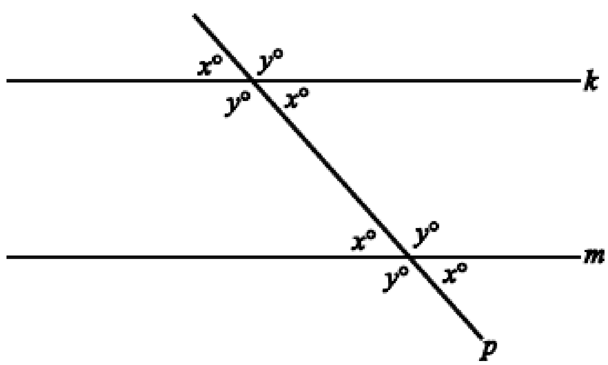


The most known   way for showing, whether an angle is a right angle is to draw a small square at the vertex of the angle, as shown in Geometry Figure below, where PON is a right angle.



An angle that is less than 90° is called an **acute angle**, and an angle whose measure falls between 90° and 180° is called an **obtuse angle**.

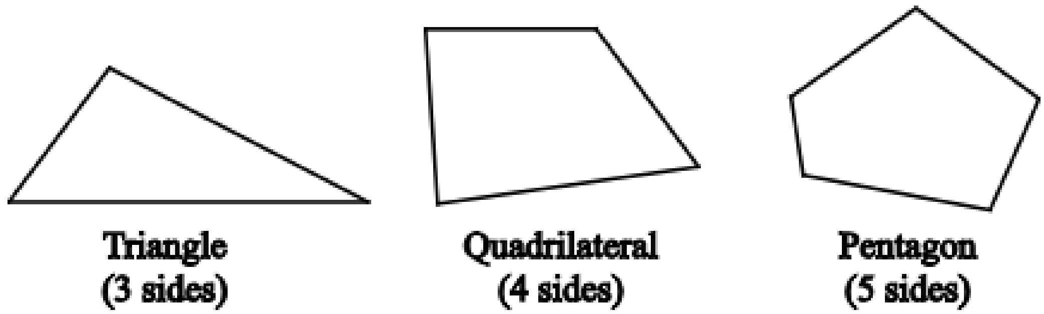
Two lines in the same plane whose intersection is impossible are called **parallel lines**.
Geometry Figure below depicts the two lines, k and m, that are parallel, denoted by k Π m. The two lines are intersected by a third line, p, to form eight angles.



Note that four of the eight angles in Figure have the measure x°, and the remaining four angles have the measure y°, wherein x + y = 180.

5b. Polygons

A **polygon** is a closed figure formed by three or more line segments that are in the same plane. The line segments are known as **sides** of the polygon. Each side is joined to two other sides at its endpoints, and the endpoints are called as**vertices**. In this discussion, the term “polygon” mainly refers to the “convex polygon,” i.e.,a polygon in which the each measure of interior angle is less than 180°. The Figure below consists of the examples of a triangle, a quadrilateral, and a pentagon, all of which are convex.



The simplest polygon happens to be **triangle**. Note that a **quadrilateral** can be divided into 2 triangles by drawing a line diagonally; and a **pentagon** can be divided into 3 triangles by selecting one of the vertices and drawing 2 line segments that connects the selected vertex to the two nonadjacent vertices, as shown in Geometry Figure below.

If a polygon has n sides, then it is possible to be divided into n – 2 triangles. Since the interior angles of a triangle has a sum of 180°, then it impliesthat the sum of the interior angles of an n-sided polygon is (n – 2)(180°). For example, since a quadrilateral consists of 4 sides, then the sum of the measures of the interior angles of a quadrilateral is (4 – 2)(180°)= 360°, and since a **hexagon** consists of 6 sides, the sum of the measures of the interior angles of a hexagon is (6 – 2)(180°)= 720°.

A polygon in which all sides happens to be congruent and all interior angles are congruent is called as a **regular polygon**. For example,an**octagon** has 8 sides, then the sum of the measures of the interior angles of an octagon is (8 – 2)(180°)= 1,080°. Therefore, in a **regular** 1,080°**octagon** the measure of each angle is (8-2)(180°)= 135°.

The **perimeter** of a polygon happens to be the sum of the lengths of its sides. The **area** of a polygon can be understood as an area of the region enclosed by the polygon.

In the next two sections, we shall look at some basic properties of triangles and quadrilaterals.

5c. Triangles and Quadrilaterals

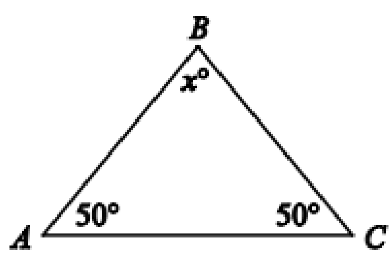
Triangles

Every triangle consists of three sides and three interior angles. The measures of the interior angles sums up to 180°. The length of each side should be less than the sum of the lengths of the other two sides. For example, the sides of a triangle cannot consist the lengths 4, 7, and 12 because 12 is greater than 4 + 7.

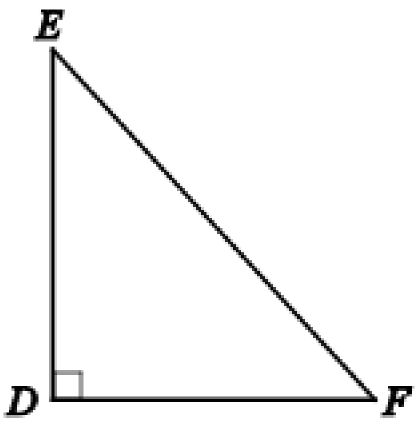
The 3 types of special triangles are explained in the following.

Type 1: A triangle with three congruent sides is known as an **equilateral triangle**. The measures of the three interior angles of such a triangle happens to be always equal, and each measure is 60°.

Type 2: A triangle with at least two congruent sides is known an **isosceles triangle**. If a triangle has two congruent sides, then the angles opposite the two congruent sides are congruent. The converse is also true. For example, in triangle ABC in Figure below, the measure of angle A is 50°, the measure of angle C is 50°, and the measure of angle B is x°. Since both angle A and angle C are 50°, it means that the length of AB is equal to the length of BC. Also, since the sum of the three angles of a triangle is 180°, it follows that 50 + 50 + x = 180, and the measure of angle B is 80°.



Type 3: A triangle that consists of an interior right angle is known a **right triangle**. The side that is falls oppositeof the right angle is called the **hypotenuse**; the other two sides are called **legs**. For example, in right triangle DEF in Geometry Figure below, side EF is the side opposite right angle D; therefore EF is the hypotenuse and DE and DF are legs.

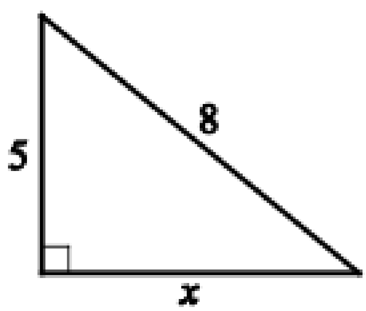


The **Pythagorean theorem** saya that in a right triangle, the square of the length of thehypotenuse is equal to the sum of the squares of the lengths of the other two sides. Therefore, for triangle DEF in Figure,

$(EF)^2 =(DE)^2 + (DF)^2$

This relationship can be used for finding the length of one side of a right triangle if the lengths of the other two sides are already known. For example, Let's consider a right triangle with hypotenuse of length 8, a leg of length 5, and another leg of unknown length *x*, as shown in Figure below.

The Pythagorean Theorem

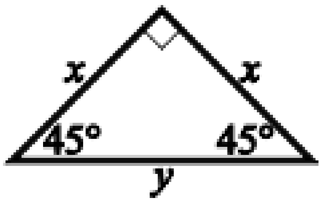


By the Pythagorean theorem, $8^2= 5^2+ x^2$. Therefore, $39=x^2$.

And $64= 25 + x^2$.

Since $x^2 = 39$ and x should be positive, it follows that x = 39, or approximately 6.2.

The Pythagorean theorem can be used in determining the ratios of the lengths of the sides of two special right triangles. One special right triangle happens to be an isosceles right triangle, as shown in Figure below.

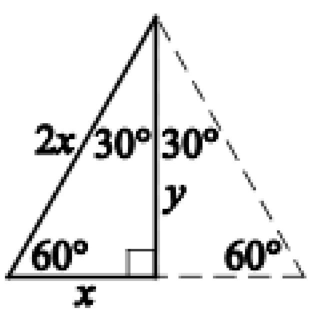


In Figure, the hypotenuse of the right triangle happens to be of length y, both sides are of length *x*, and the angles opposite the legs are both 45-degree angles.

Applying the Pythagorean theorem to the isosceles right triangle in Figure 11 above tells us that the lengths of its sides are in the ratio 1 to 1 to root 2, as follows.

By the Pythagorean theorem, y^2= x^2+x^2. Thus y^2 = 2x^2 and y =root 2*x*. Therefore, the lengths of the sides are in the ratio *x* to *x* to root 2 *x*, which is same as the ratio 1 to 1 to root 2.

The other special right triangle is a 30°-60°-90° right triangle, which is half of an equilateral triangle, as shown in Figure below.



Note that the length of the horizontal side, x, happens to be one-half the length of the hypotenuse, 2*x*. By applying the Pythagorean theorem to the 30°-60°-90° right triangle shows that the lengths of its sides are in the ratio 1 to root 3 to 2, as follows.

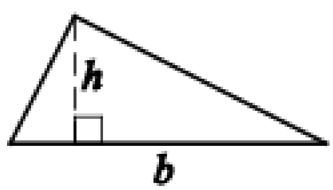
By applying Pythagorean theorem, $x^2 + y^2 = (2x)^2$, that breaks down to $x^2 + y^2 = 4x^2$. Subtracting x^2 from both sides gives $y^2 = 4x^2 - x^2$ –or $y^2 = 3x^2$. Therefore,

y =root 3 x. Hence, the ratio of the lengths of the three sides of a 30°-60°-90° right triangle is x to root 3x to 2x, which is the same as the ratio 1 to root 3 to 2.

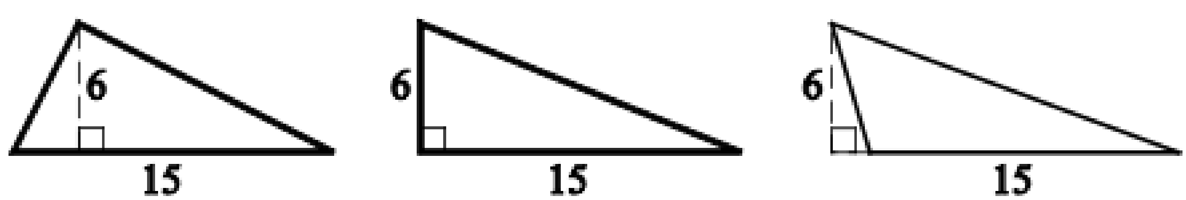
The Area of a Triangle
The formula for the Area of a Triangle can be given as:

$A = {}^{bh}/_{2}$

Wherein b happens to be the length of a base, and h is the length of the corresponding height. Figure below shows a triangle: the length of the horizontal base of the triangle is denoted by b and the length of the corresponding vertical height is denoted by h.



Any side of a triangle can be utilized as a base; The perpendicular line segment from the opposite vertex to the base (or an extension of the base) is the height that corresponds to the base. Depending on the context, the term “base” also refers to the length of a side of the triangle, and the term “height” refers to the perpendicular line segment’s length from the opposite vertex to that side. The examples in Figure below depicts the three different configurations of a base and the corresponding height.



In all three triangles in Geometry Figure above, the area is $(15)(6)/_{2}$, or 45.

Congruent Triangles and Similar Triangles

Two triangles that consists of the same shape and size are called **congruent triangles**. Precisely, two triangles are congruent when their vertices can be matched up so that the corresponding angles and the corresponding sides are congruent.

By convention, the statement "triangles PQR and STU are congruent" does not just tell you that the two triangles are congruent, but also tells you regarding what the corresponding parts of the two triangles are. In particular, as the letters in the name of the first triangle have been mentioned in the order PQR, and the letters in the name of the second triangle are mentioned in the order STU , the statement tells you that angle P is congruent to angle S, angle Q is congruent to angle T, and angle R is congruent to angle U. It also conveys that the sides PQ, QR, and PR in triangle PQR are congruent to sides ST , TU , and SU in triangle STU , respectively.

The following three propositions can be used in determining whether two triangles are congruent by comparing some of their sides and angles.

Proposition 1: If the three sides of one triangle happens to be congruent to the three sides of another triangle, then the triangles can be called as congruent. This proposition is known as Side- Side-Side, or SSS, congruence.

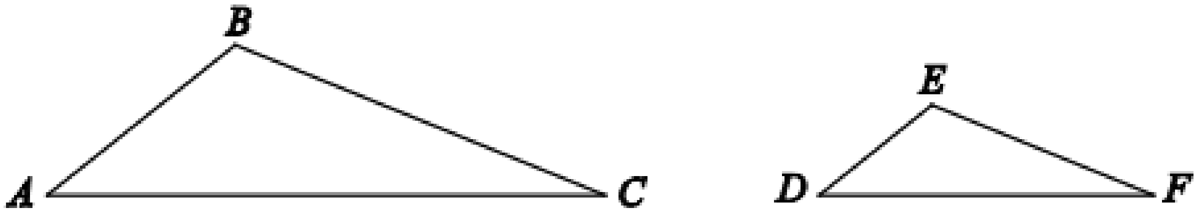
Proposition 2: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another, then the triangles can be called congruent. This proposition is called Side-Angle-Side (SAS) congruence.

Proposition 3: If two angles and the included side of one triangle happens to be congruent to two angles and the included side of another triangle, then the triangles can be called congruent. This proposition is known as Angle-Side-Angle, or ASA, congruence. Also note that if two angles of one triangle are congruent to two angles of another triangle, then the remaining angles alsohappens to be congruent to each other, as the sum of the angle measures in any triangle amounts to be 180 degrees. Therefore, a similar proposition, called Angle-Angle-Side, or AAS, congruence, continues from ASA congruence.

Two triangles that consists of the same shape but not necessarily the same size are known as **similar triangles**. To be precise, two triangles are similar if their vertices can be matched up so that the corresponding angles are congruent or equivalently, the lengths of the corresponding sides have the same ratio, known as the **scale**

factor of similarity. For example, all the triangles that has 30°-60°-90° right triangles are the same, even though they may differ in size.

Figure below depicts two similar triangles, triangle ABC and triangle DEF.



As with the principle for congruent triangles, the letters in similar triangles ABC and DEF indicate their corresponding parts.

Since triangles ABC and DEF happens to be similar, we have AB/DE= BC/EF= AC/DF. By cross

multiplication, we can get other proportions, such as AB $_{/BC}$ =DE $_{/EF}$.

Quadrilaterals

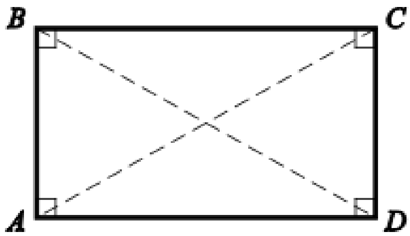
Every quadrilateral consists of four sides and four interior angles. The measures of the interior angles sums up to360°.

Special Types of Quadrilaterals

The four special types of quadrilaterals are explained in the following:-

Type 1: A quadrilateral that consists of four right angles is known as a **rectangle**. Opposite sides of a rectangle are parallel and congruent, and the two diagonals are also are congruent.

Figure below depicts the rectangle ABCD in which, the opposite sides AD and BC are congruent and parallel, the opposite sides AB and DC are parallel and congruent, and diagonal AC is congruent to diagonal BD.



Type 2: A rectangle with four congruent sides is known as a **square**.

Type 3: A quadrilateral in which both pairs of opposite sides are parallel is known as a parallelogram. In a parallelogram, opposite sides and the opposite angles are congruent and.

Note that all rectangles are parallelograms.
Geometry Figure 17 below shows parallelogram PQRS. In parallelogram PQRS:

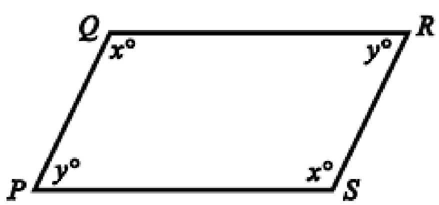
The opposite sides SR and PQ are parallel and congruent.

The opposite sides PS and QR are parallel and congruent.

The opposite angles S and Q are congruent.

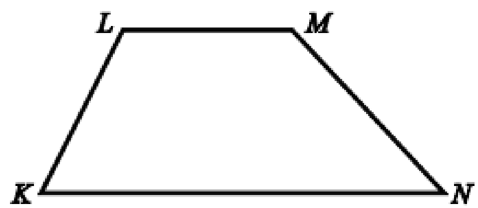
Opposite angles P and R are congruent.

In the figure, angles Q and S are both labeled x°, and angles P and R are both labeled y°.



Type 4: A quadrilateral in which at least one pair of opposite sides is parallel is known as a **trapezoid**. Two opposite, parallel sides of the trapezoid forms it's **bases** .

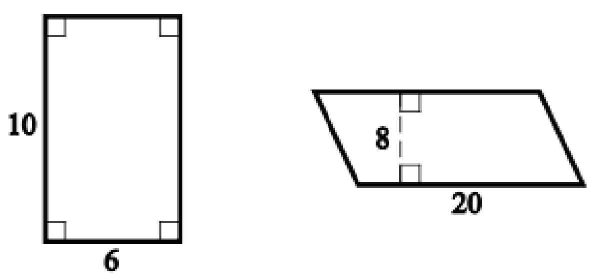
Figure below shows trapezoid KLMN. In trapezoid KLMN, the horizontal side KN is parallel to the horizontal side LM . Sides LM and KN happens to be the bases of the trapezoid.



The Areas Concerning the Special Types of Quadrilaterals

For every parallelograms, that includes rectangles and squares, the **area** A is given by the formula A = bh where b is the length of a base and h is the length of the corresponding height.

Any side of a parallelogram can be utilised as a base. The height that corresponds to the base is the perpendicular line segment from any point on the side opposite the base to the base (or an extension of that base). Depending on the context, the term “base” also refers tothe length of a side of the parallelogram, and the term “height” refers to the length of the perpendicular line segment from that side, up to the opposite side. Examples of finding the areas of a rectangle and a parallelogram are shown in Geometry Figure below.



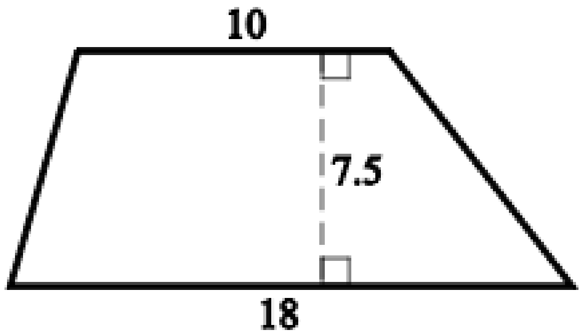
$A = (6)(10) = 60$ $\qquad$ $A = (20)(8) = 160$

The trapezoid **area** A of a is given by the formula:

$$A = \frac{1}{2}(b_1 + b_2)(h)$$

where b_1 and b_2 are the trapezoid’s bases lengths, and h is the corresponding height. For example, for the trapezoid in Figure 20 below with bases of length 10 and 18 and a height of 7.5, the area is

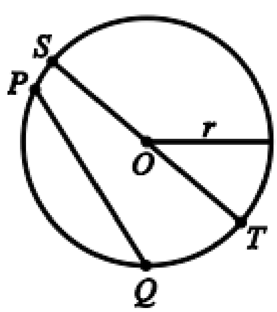
$$\frac{1}{2}(10 + 18)(7.5) = 105$$



5d. Circles

Given a point O in a plane and a positive number r, the set of points in the plane that arein a distance of r units from that of O is called a **circle**. The point O is known as the **center** of the circle and the distance r is termed as the **radius** of the circle. The line that is twice the radius is called as the **diameter** of the circle. Two circles that has equal radii are called **congruent circles**.

Any line segment that joins two points on the circle is known as a**chord**. The terms "radius" and "diameter" can also refer to line segments: A **radius** is any line segment joining a point on the circle from the center of the circle, and a **diameter** is a chord that happens to pass through the center of the circle. In Figure 21 below, O is at the center of the circle, r is length of a radius, PQ is a chord, and ST is a diameter, as well as a chord.



The distance around a circle is known as the **circumference** of the circle, which is analogous to the perimeter of a polygon. The ratio of the circumference C to the diameter d happens to be the same for all circles and is indicated by the Greek letter pi ; that is,

$^{C/}{}_{d}$ = pi

The value of p comes approximately to 3.14 and can also be approximated by the fraction $22/_{7}$.

If radius of a circle is r, then $^{C/d}=^{C/2r}=$ pi, and so the circumference is related to the radius by the equation

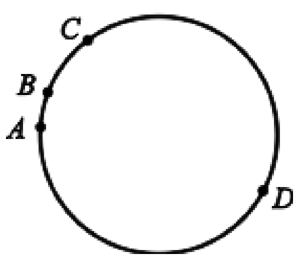
$C = 2\pi r$

For example, if a circle consists of a radius of 5.2, then its circumference is

$(2)(\pi)(5.2)=(10.4)(\pi)\approx(10.4)(3.14)$ which is approximately 32.7.

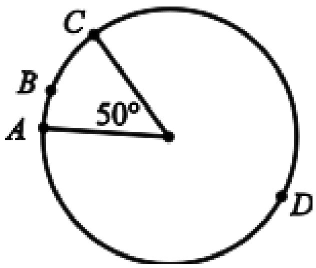
Given any two points on a circle, an **arc** is the part of the circle that consists of thetwo points and all the points between them. The endpoints of two arcs are two points on a circle. An arc is frequently identified by three points to avoid confusion.

In Figure below, there are four points on a circle. If we go clockwise around the circle, the four points are A, B, C, and D. Between points A and C there are two different arcs: arc ABC is the shorter arc between A and C, and arc ADC is the longer arc between A and C.



A **central angle** of a circle is the angle which has it’s vertex at the center of the circle. The **measure of an arc** is the measure of it’s central angle, that is the angle formed by two radii which connects the center of the circle to the two endpoints of the arc. The circle as a whole is considered to be an arc with measure 360°.

In Figure below, there are four points on a circle: points A, B, C, and D. Given that the radius of the circle as 5.



In Figure, the measure of the shorter arc between points A and C, i.e. arc ABC, happens to be 50°; and the measure of the longer arc, that is between points C and A, the arc ADC, is 310°.

To find the **length of an arc** of a circle, note that the ratio of the length of an arc to the circumference must be equal to the ratio of the degree measure of the arc to 360°. For example, since the radius of the circle in Figure is 5, the circumference is 10π. Therefore,

$$\frac{\text{length of arc } ABC}{10\pi} = \frac{50}{360}$$

Multiplying both sides by 10π gives

$$\text{length of arc } ABC = \left(\frac{50}{360}\right)(10\pi)$$

Then, since

$$\left(\frac{50}{360}\right)(10\pi) = \frac{25\pi}{18} \approx \frac{(25)(3.14)}{18} \approx 4.4$$

it shows that the length of arc ABC happens to be approximately 4.4.

The **area** of a circle which has a radius r is πr . For example, since the radius of the circle in Figure 23 above is 5, the area of the circle is $\pi(5^2) = 25\pi$.

A **sector** of a circle is a region which is bounded by an arc of the circle and two radii. In order to find the **area of a sector**, the ratio of the area of a sector of a circle to that of the the area of the entire circle must be equal to the ratio of the degree measure of its arc to 360°. For example, in the circle in Figure the region which is bounded by arc ABC and the two radii is a sector with central angle 50°, and the radius of the circle is 5. Therefore, if the area of the sector with central angle 50° is represented by S, then

$$\frac{S}{25\pi} = \frac{50}{360}$$

Multiplying both sides by 25π gives

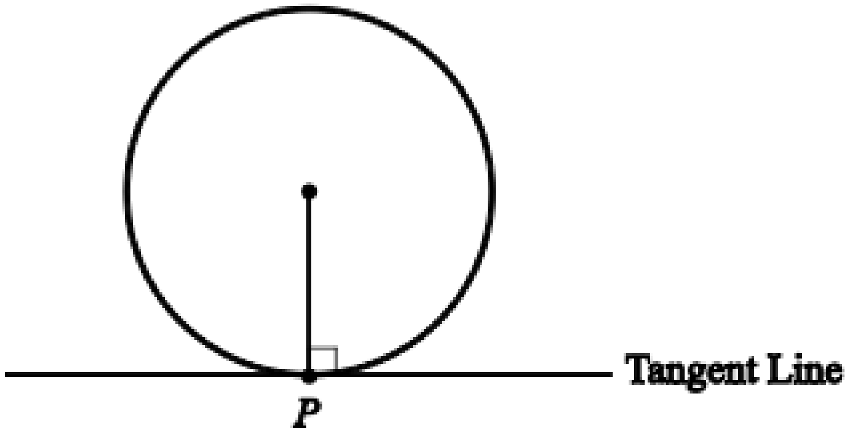
$$S = \left(\frac{50}{360}\right)(25\pi)$$

Then, since

$$\left(\frac{50}{360}\right)(25\pi) = \frac{125\pi}{36} \approx \frac{(125)(3.14)}{36} \approx 10.9$$

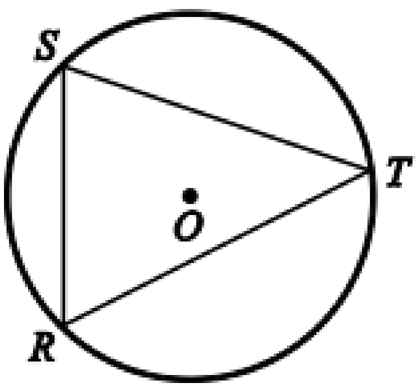
it follows that the area of the sector with central angle 50° is approximately 10.9.

The line that lies in the same plane as the circle is known as the tangent and which intersects the circle at exactly one point, called the **point of tangency**. If a line is tangent to a circle, then a radius drawn to the point of tangency must be always perpendicular to the tangent line. The other way also holds true; i.e., if a radius and a line intersect at a point on the circle and the line is perpendicular to the radius, then the line should be a tangent to the circle at the point of intersection. The Figure below depicts a circle, a line tangent to the circle at point P, and a radius drawn to point P.



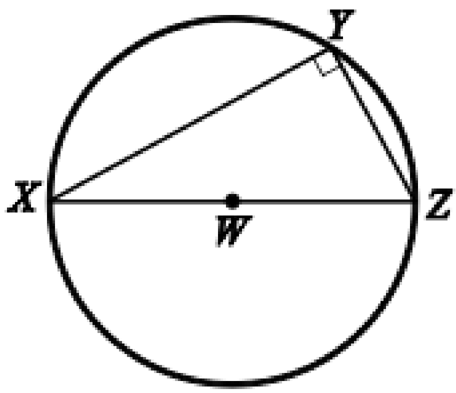
The circle is **circumscribed** about the polygon if for a polygon all its vertices lie on the circle and is **inscribed** in a circle, or equivalently.

Geometry Figure below shows triangle RST inscribed in a circle with center O. Inside the triangle we can see the center of the circle or circle center.

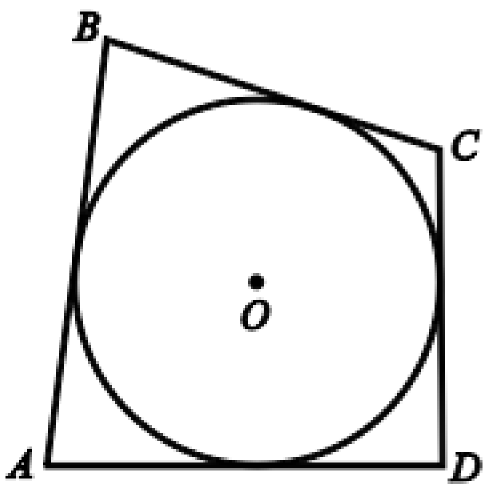


It is not always necessary that if a triangle is inscribed in a circle, the center of the circle is inside the inscribed triangle. It is possible that the center of the circle be outside the inscribed triangle, or on one of the sides of inscribed triangle. Note that if the center of the circle is on one of the sides of the inscribed triangle, that side is called the diameter of the circle.

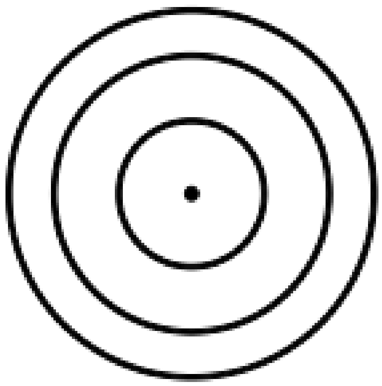
If one side of an inscribed triangle is a diameter of the circle, then the triangle happens to be a right triangle. Conversely, if an inscribed triangle is a right triangle, then one of its sides happens to be a diameter of the circle. The Figure below shows right triangle XYZ inscribed in a circle with center W. In triangle XYZ, side XZ is the diameter of the circle and angle Y is the right angle.



A polygon is circumscribed about a circle if each side of the polygon happens to be a tangent to the circle, or equivalently, the circle is inscribed in the polygon. The Figure below depicts the quadrilateral ABCD circumscribed about a circle with center O.



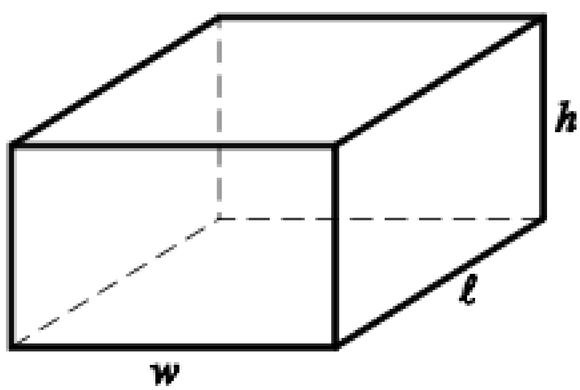
Two or more circles which has the same center are called **concentric circles**, as depicted in Figure below.



5e. Three-Dimensional Figures

Basic three-dimensional figures mainly include rectangular solids, cubes, cylinders, spheres, pyramids, and cones. In this section, we shall look at some properties of rectangular solids and right circular cylinders.

A **rectangular solid**, or **rectangular prism**, consists of 6 rectangular surfaces known as **faces**, as depicted in Figure 29 below. Adjacent faces are perpendicular to each other. Each line segment that is the intersection of two faces is known as an **edge**, and each point at which the edges intersect is known as a **vertex**. There are 12 edges and 8 vertices. The dimensions of a rectangular solid are the length ℓ, the width w, and the height h.



A rectangular solid with six square faces is known as a **cube**, in which case = w λ h.=

The **volume** V of a rectangular solid isobtained by multiplying the values of its three dimensions, or

$V = lwh$

The **surface area** A of a rectangular solid is obtained by adding the areas of the six faces, or

$A=2(lw+lh+wh)$

For example, if a rectangular solid whose length , width, and height are 8.5, 5, and 10 respectively, then its volume

is $V =(8.5)(5)(10) = 425$

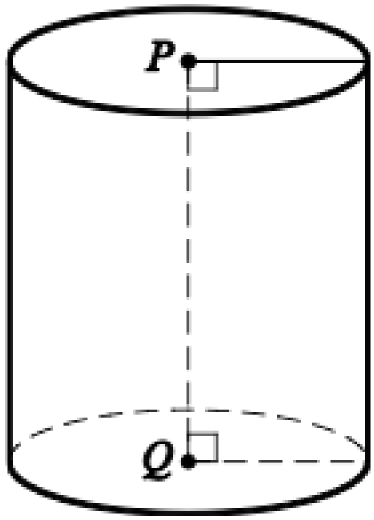
and its surface area is

$A = 2((8.5)(5)(+8.5)(10)(+5)(10))=355$

A **circular cylinder** consists of two bases that are congruent circles which lies in the parallel planes and a **lateral surface** which are made of all line segments that join points on the two circles and that are parallel to the line segment which joins the centers of the two circles. The latter line segment is known as the **axis** of the cylinder.

A **right circular cylinder** is the one whose axis is perpendicular to its bases. The perpendicular distance between the two bases is the height of the cylinder. Because the axis of a right circular cylinder is perpendicular to both the bases, the height of a right circular cylinder is equal to the length of the axis.

The right circular cylinder depicted in Figure 30 below has circular bases with centers P and Q. Line segment PQ happens to be the axis of the cylinder and is perpendicular to both bases. The length of PQ is the height of the cylinder.



The **volume** V of a right circular cylinder whose height h and a base with radius r is the product of the height and the area of the base, or

$V = pi\ r^2 h$

The **surface area** A of a right circular cylinder is equal to the sum of the areas of the two bases and the area of its lateral surface, or

$A=2(pi\ r^2)+ 2pi\ rh$

For example, if a right circular cylinder has height 6.5 and a base with radius 3, then its volume calculated as follows:-

$V = pi\ (3^2)(6.5) = 58.5\ pi$

and its surface area is

$A=2pi\ (3^2) + 2pi\ (3)(6.5)= 57pi$

[redacted]DATA ANALYSIS

The review of data analysis commences with the methods for presenting data, followed by counting methods and probability, and then progresses to distributions of data, random variables, and probability distributions. The review of data analysis winds up with examples of data interpretation.

6a. Methods for Presenting Data

Data can be presented in different ways. The most commonly used ones are Tables, and there are many other graphical and numerical methods as well. In this section, we shall discuss about tables and some common graphical methods for presenting and summarizing data.

In data analysis, a variable is any characteristic that can range from a population of individuals or objects. Variables can be quantitative, such as the age of individuals. Variables can also be non-numerical, such as the eye color of individuals.

Data is collected from a population by examining one or more variables. The **distribution of a variable**, or **distribution of data**, indicates how frequently different numerical data values are observed in the data.

Example: In a population of students in a sixth-grade classroom, a variable that can be observed is the height of each student. Note that the variable in this example is in numerical form.

Example: In a population of voters in a city's mayoral election, a variable that can be observed is the candidate that each voter voted for. Note that the variable in this example is in non numerical.

The **frequency**, of a particular numerical value is the number of times that the category or numerical value appears in the data. A **frequency distribution** is a table or graph that presents the numerical values along with their corresponding frequencies. The **relative frequency** of a numerical value is the corresponding frequency divided by the total number of data value. Relative frequencies can be expressed in terms of percents, fractions, or decimals. A table or graph that presents the relative frequencies of the categories or numerical values is called as relative frequency distribution.

Tables

Tables are used for presenting that includes a wide variety of data, including frequency distributions and relative frequency distributions. The rows and columns provide clear relations between categories and data. A frequency distribution is often presented as a 2-column table in which the categories or numerical values of the data are listed in the first column and the corresponding frequencies are listed in the second column. A relative frequency distribution table consists of the same layout but with relative frequencies instead of frequencies. When data include a large number of categories or numerical values, the categories or values are often grouped together in a smaller number of groups and the corresponding frequencies are given.

Example: An inspection was taken to find the number of kids in each of 25 families. A 25 values list was collected in the inspection as follows.

1	2	0	4	1
3	3	1	2	0
4	5	2	3	2
3	2	4	1	2
3	0	2	3	1

Here are tables that present the resulting frequency distribution and relative frequency distribution of the data.

Frequency Distribution

Number of Kids	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Relative Frequency Distribution

Number of Kids	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

In the relative frequency distribution table the relative frequencies are expressed as percents and that the total for the relative frequencies amounts to 100%. If the relative frequencies were expressed in the form of decimals or fractions instead of percents, then the total would be 1.

Example: Thirty graduates took a statistics exam. Below is a list of the 30 scores on the exam, from least to greatest.

62	63	68	70	72	72	72	75	76	76	76	76	78	78	82
82	85	85	85	85	85	86	87	88	91	91	92	95	97	100

The 30 graduates who achieved 18 different scores in the test. When we display the frequency distribution of this many different scores then that would make the frequency distribution table very large, so instead we group the scores into four groups: the scores range from 61 to 70, 71 to 80, 81 to 90, and from 91 to 100. Given below is the frequency distribution of the scores with these groups.

Score	Frequency
61 to 70	4
71 to 80	10
81 to 90	10
91 to 100	6

In addition to being used to present frequency and relative frequency distributions, tables are used for displaying a wide variety of other data. Let's look at two examples.

Example: The below table displays the yearly per capita income of a certain state, from 1930 to 1980.

Year	Annual Per Capita Income
1930	$656
1940	$680
1950	$1,717
1960	$2,437
1970	$4,198
1980	$10,291

Example: The below table illustrates the closest and farthest distance of the eight planets from the Sun, in millions of kilometers.

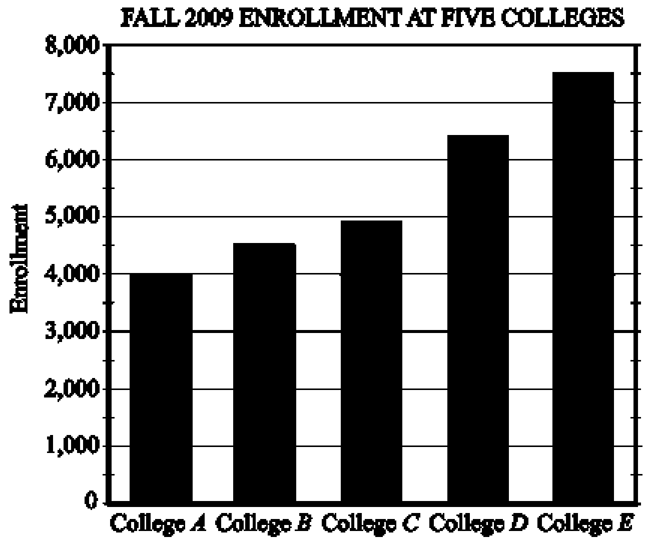
Planet	Closest Distance from the Sun (in millions of kilometers)	Farthest Distance from the Sun (in millions of kilometers)
Mercury	46	70
Venus	107	109
Earth	147	152
Mars	205	249
Jupiter	741	817
Saturn	1,350	1,510
Uranus	2,750	3,000
Neptune	4,450	4,550

Bar Graphs

A frequency distribution or relative frequency distribution of data that is collected from a population by observing one or more variables can be presented with the help of a **bar graph**, or **bar chart**. In a bar graph, each of the categories of data or numerical values is presented with the help of a rectangular bar, and the height of each bar is proportional to the corresponding frequency or relative frequency. All of the bars are drawn with t same width, and the bars can be presented either horizontally or vertically. When data consists a large number of different categories of numerical values, the categories or values are often grouped together in several groups and the corresponding frequencies or relative frequencies are given. Bar graphs helsus compare across several categories more easily than tables do. For example, in a bar graph it is easy to identify the category with the greatest frequency by locating the bar with the greatest height.

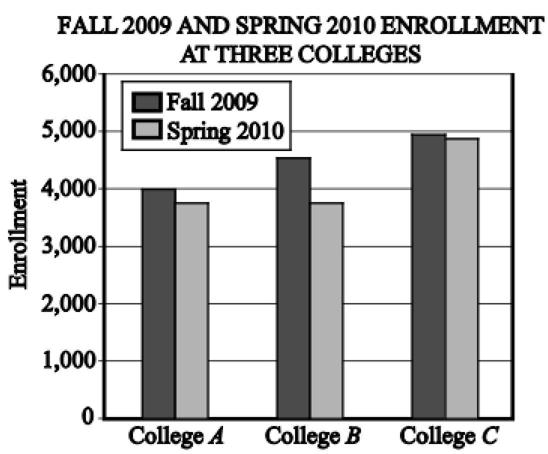
Let’s look at two examples of frequency distributions presented as bar graphs.

Example: Below Figure is a bar chart with vertical bars. It shows the frequency distribution of one variable, fall 2009 enrollment. The variable has been taken for five data categories, Colleges A, B, C, D, and E.



From the graph, we can arrive at the conclusion that the college with the greatest fall 2009 enrollment was College E and the college which had the least enrollment was College A. Also, we can arrive at an estimation that the enrollment for College D was about 6,400.

Example: In below Figure, we can see a bar graph with vertical bars which depicts the frequency distributions of two variables, fall 2009 enrollment and spring 2010 enrollment. Both variables are pertaining to three data categories, Colleges A, B, and C.

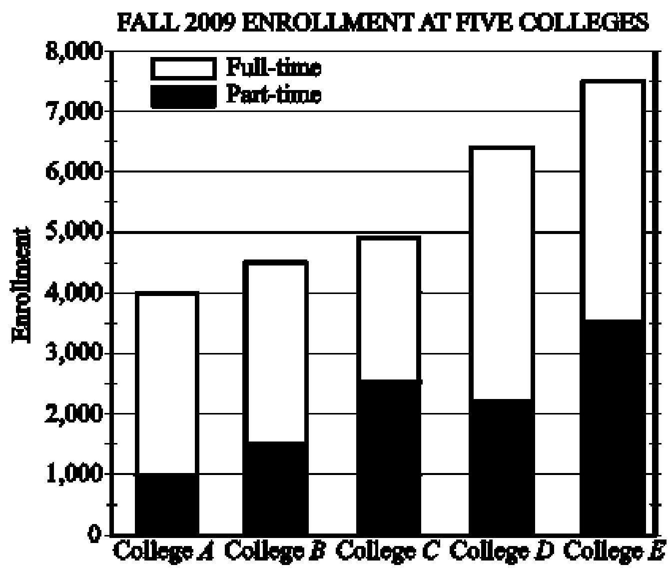


Observe from the above chart that for all three colleges, the fall 2009 enrollment was greater than the spring 2010 enrollment. Also, a substantial decrease in the enrollment from fall 2009 to spring 2010 occurred at College B.

Segmented Bar Graphs

A **segmented bar graph**, or **stacked bar graph**, is similar to a regular bar graph except that in a segmented bar graph, each rectangular bar is divided,into smaller rectangles that explain how the variable is “separated” into other related variables. For example, rectangular bars which represent enrollment can be divided into two smaller rectangles, one representing full-time enrollment and the other representing part-time enrollment, as shown in the following example.

Example: In the Figure below, the fall 2009 enrollment at the five colleges shown in Figure is presented again, but this time each bar ismade into two segments: one representing full-time enrollment and one representing part-time enrollment.



The enrollments (full-time, part-time and the total enrollment) are estimated from the segmented bar graph in above Figure. For example, in case of College D, the total enrollment was a little below 6,500, or approximately 6,400 students; the part-time enrollment was approximately 2,200; and the full-time enrollment came approximately to 6,400 – 2,200, or 4,200 students.

Bar-graphs are occasionally used to compare numerical data which could be displayed in a table, typical examples are for temperatures, amount of dollars, percentages, heights and the weights. Do note that bar graphs are most frequently used for comparing frequencies, as in the examples above. Also, the categories at times are numerical in nature, such as years or other time intervals.

Histograms

When a list of data is colossal and consists of many different values of a numerical variable, it is more apt, to organize the data by grouping the values into intervals, often called classes. For this, first we need to divide the entire interval of values into smaller intervals of equal length and then count the values that happen to fall into each interval. In this way, each interval consists frequency and a relative frequency. The intervals and their frequencies (or relative frequencies) are often displayed in with the help of an **histogram**. Histograms are graphs of frequency distributions that are similar to bar graphs, but they must have a number line for the horizontal axis, which represents the numerical variable. Any spaces between bars in a histogram is a sign that there are no data in the intervals represented by the spaces and in a histogram, there are no regular spaces between the bars.

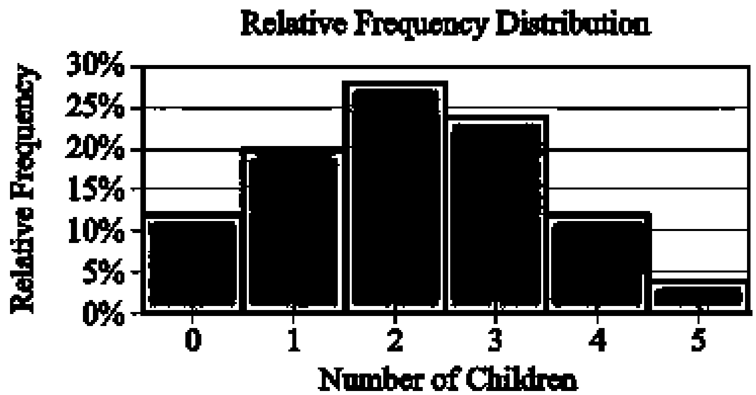
Numerical variables with just a few values can be displayed with the help of histograms, where the frequency or relative frequency of each value is represented with the help of bar centered over the value.

Example: In certain previous example, the relative frequency distribution of the number of kids of each of 25 families has been represented as a 2-column table. For convenience, the table is repeated below.

Relative Frequency Distribution

Number of Kids	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

This relative frequency distribution can be displayed in the form of a histogram, as shown in the following figure.



Histograms are useful in identifying the general shape of a data distribution. Also it is evident that the “center” and degree of “spread” of the distribution, as well as high- frequency and low-frequency intervals. From the histogram in Figure above, we understand that the distribution is shaped like a mound with one peak; that is, the data are frequent in the middle and sparse at both ends. The central values are 2 and 3, and the distribution is almost near to being symmetric about those values, as the bars all consists of the same width, the area of each bar is proportional to the amount of data it represents. Therefore, the areas of the bars indicate where the data are concentrated and where they are not.

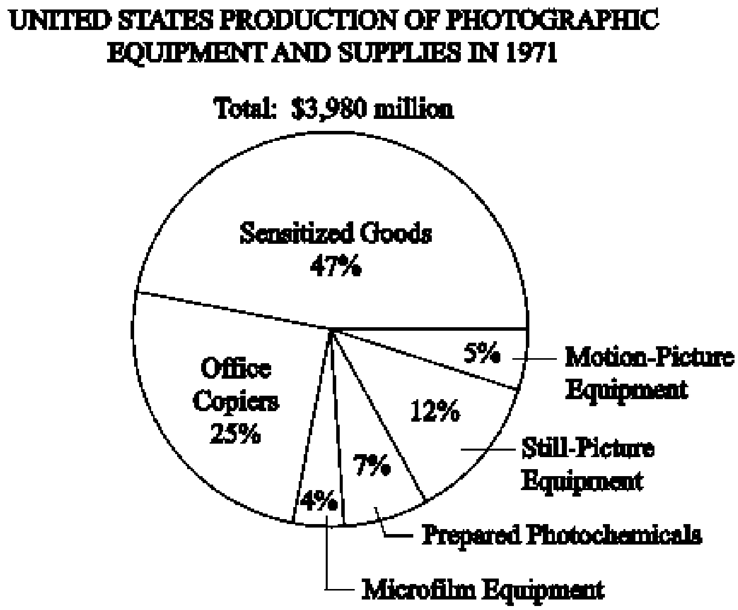
The fact which is central to the discussion of probability distributions are to be noted. That is, each bar consists of a width of 1, the sum of the areas of the bars equals the sum of the relative frequencies, which is 100% or 1, depending on whether percents or decimals are being used.

Circle Graphs

Circle graphs, are otherwise known as **pie charts**, and are used for representing r data that have been separated into smaller of categories. They show, how a whole is separated into parts. The data is presented in a circle in such a way that the area of the circle representing each category is proportional to the part of the whole that the category represents.

A circle graph can be used for representing a frequency distribution or a relative frequency distribution. Generally, a circle graph may represent any total amount that is distributed into a small categories, as in the following example.

Example:



From the above graph we can see that Sensitized Goods is the category with the greatest dollar value.

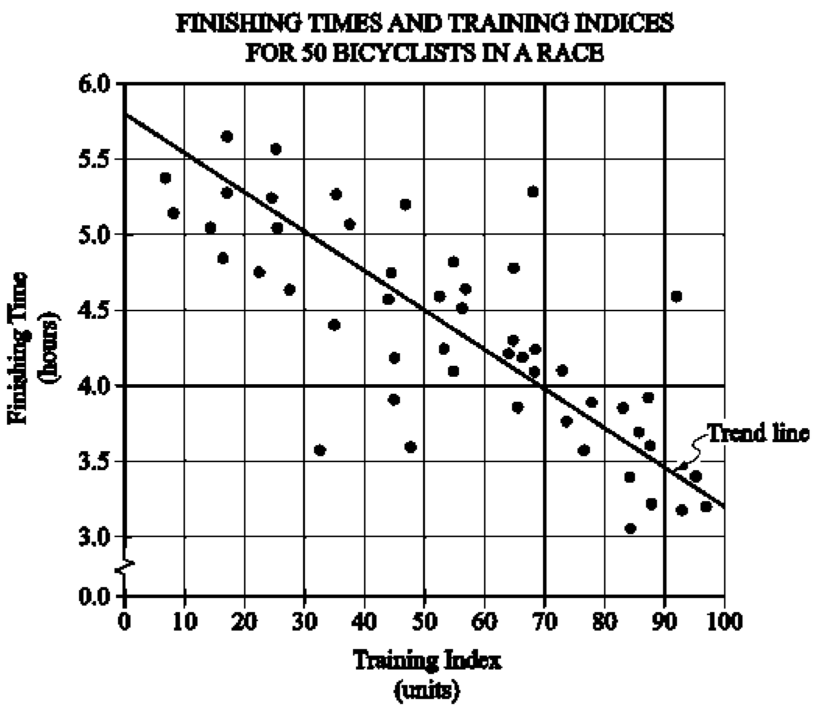
The area of each sector is proportional to the percent of the whole of that the sector, for the percent of 360 degrees that the sector represents the measure of the central angle of a sector is proportional. Each part of this circular graph is known as a **sector**. For instance, the measure of the central angle of the sector representing the category Prepared Photo chemicals is 7 percent of 360 degrees, or 25.2 degrees.

Scatterplots

A **scatterplot** is a type of graph that is mainly used for showing the relationship between two numerical variables whose values can be studied in a single population of individuals or objects. In a scatterplot, the values of one variable appears on the horizontal axis of a rectangular coordinate system and the values of the other variable appear on the vertical axis. For each individual or object in the data, an ordered pair of numbers is collected, one number for each variable, and the pair is denoted by a point in the coordinate system.

A scatterplot helps to observe an overall pattern, in the relationship between the two variables. Also, the strength of the trend as well as striking deviations from the trend are known. In many cases, a line or a curve that represents the trend is also shown in the graph and is used for making predictions about the population.

Example: A bicycle trainer studied 50 bicyclists for examining how the finishing time for a certain bicycle race was related to the amount of physical training sessions each bicyclist took prior to three months before the race. For measuring quantum of training, the trainer built a training index, measured in “units” and on the intensity of each bicyclist’s training. The data and the trend of the data, shown by a line, are displayed in the scatterplot in Figure below.



When a trend line is included in the depiction of a scatterplot, you can observe how scattered or close the data are to the trend line, or to put it another way, how well the trend line fits into the data. In the scatterplot in Figure above, almost all of the data points are closer to the trend line. The scatterplot also shows that the finishing times tend to decrease as the training indices increases.

The trend line can be used for making predictions. For example, it can be predicted, based on the trend line, that a bicyclist with a training index of 70 units will be able to finish the race in approximately in 4 hours. We get this value by observing that the vertical line at the training index of 70 units intersects the trend line very close to 4 hours.

Another prediction that we can make, on the basis of the trend line, is the approximate number of minutes by which a bicyclist will lower his or her finishing time for every increase of 10 training index units. We obtain this prediction from the ratio of change in the finishing time to the change in training index, or the slope of the trend line. Observe that the slope is negative. To know the slope, estimate the coordinates of any two points located on the line, for example, the points at the extreme left and right ends of the line: (0, 5.8) and (100, 3.2).

The computation of the sloe can be done as follows

$$\frac{3.2 - 5.8}{100 - 0} = \frac{-2.6}{100} = -0.026$$

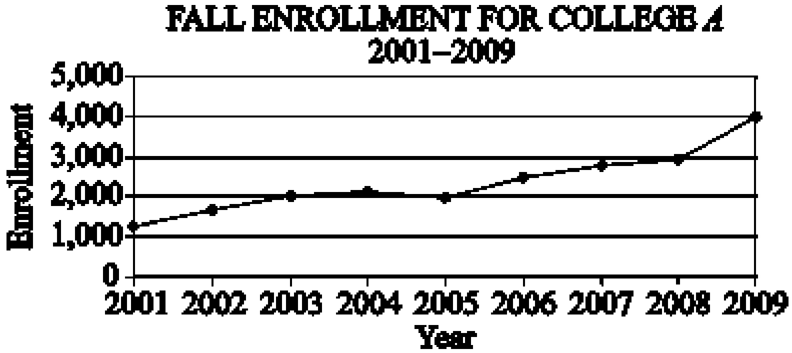
The interpretation of the slope can be done as follows: The finishing time is predicted to decrease 0.026 hour for every unit as and when the training index increases. Since we wish to know the rate of decrease in the finishing time for an increase of 10 units, we do so by multiplying the rate by 10 to get 0.26 hour per 10 units. For computing the decrease in minutes per 10 units, we multiply 0.26 by 60 to get approximately 16 minutes. Based on the trend line, it is possible to predictthat a bicyclist will decrease his or her finishing time by approximately 16 minutes for every increase of 10 training index units.

Line Graphs

A **line graph** is another type of graph that is used for showing the relationship between two numerical variables, especially if one of the variables is time. A line graph uses a coordinate plane, where every data point depicts a pair of values observed for the two numerical variables. There is only one data point for each value on the horizontal axis, similar to a function. The data points are in the order from left to right, and consecutive data points are connected by a line segment.

When one of the variables is time, it is related to the horizontal axis, which is labeled with regular time intervals. The data points may depict an interval of time, such as an entire day or year, or just an instant of time. Such a line graph is called a **time series**.

Example:



The line graph depicts the substantial increase in fall enrollment between consecutive years was the change from 2008 to 2009. One way to determine this is by observing that the slope of the line segment that joins the values for 2008 and 2009 is greater than that of the slopes of the line segments that joins with all other consecutive years. Another way to determine this is by observing that the increase in enrollment from 2008 to 2009 was greater than 1,000, but all other increases in enrollment were less than 1,000.

Although line graphs are commonly used to compare frequencies, as in Example 4.1.13 above, they can be used to compare any numerical data as the data change over time, such as temperatures, dollar amounts, percents, heights, and weights.

6b. Counting Methods

When a set contains small number of objects, it is easier to list the objects and count them one by one, however if the set is too large it is difficult to count that way. However, there are some useful techniques for counting the objects without actually listing them down.

Permutations and Factorials

Let’s say we need to find out the number of different ways the 3 letters A, B, and C can be placed in order from 1st to 3rd.Shown below is a list of all the possiblities in which the letters can be placed.

ABC ACB BAC BCA CAB CBA

For the 3 letters there can be 6 possible ways to put in order.

Now suppose you want to determine the number of different ways the 4 letters A, B, C, and D can be placed in order from 1st to 4th. Listing all of the orders for 4 letters is time- consuming, so it would be more useful for being able to count the possible orders without listing them.

For ordering the 4 letters, one of the 4 letters must be placed first, one of the remaining 3 letters must be placed in the second position, one of the remaining 2 letters must be placed third, and the last remaining letter must be placed in the fourth position. By applying the multiplication principle, there are $(4)(3)(2)(1)$, or 24, ways to order the 4 letters.

Generally, suppose n objects are to be ordered from 1st to nth, and we need to count the number of ways the objects can be ordered. There are n number of choices for the first object, n – 1 choices for the second object, n – 2 choices for the third object, until there is only 1 choice for the nth object. Thus, by applying the multiplication principle, the number of ways to order the n objects is equal to the product

$n(n-1)(n-2)\Lambda(3)(2)(1)$

Each order is known as a **permutation**, and the product above is known as the number of permutations of n objects. Because products of the form $n(n-1)(n-2)\Lambda(3)(2)(1)$frequently occur when counting objects, a special symbol n!, called **n-factorial**, is used for denoting this product.

For example,

1! = 1
2! = (2)(1)

3! = (3)(2)(1)

4! = (4)(3)(2)(1) As a special definition, 0! = 1.

Note that $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$ and so on.

Example: Let’s say that 10 graduates are going on a trip in the bus, and each of the students will be assigned one of the 10 available seats. Then the number of seating arrangements of the students on the bus is

$10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3,628,800$

Presuming now that we want to find out the number of ways in which we can select 3 of the 5 letters A, B, C, D, and E and place them in order from 1 to 3. Logically in the previous examples, we find that there are (5)(4)(3), or 60, different ways or means to select and put them in order.

Also in the general sense, let's consider that k objects will be chosen from a set of objects n, where $k \le n$, and the k objects will be placed in order from 1 to kth term. Then in that case, there will be n choices for the 1st object, n – 1 choices for the 2nd object, n – 2 choices for the 3rd object, and so on, till n – k + 1 choices are there for the 'kth' object. Hence, the number of ways to select and order k objects by applying the multiplication principle from a set of n objects is n(n – 1)(n – 2)...(n – k +1). It is also noteworthy that:

$$n(n-1)(n-2)\cdots(n-k+1) = n(n-1)(n-2)\cdots(n-k+1)\,\frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$$

This expression represents the number of **permutations of n objects taken k a a time**– that is, the number of ways to select and order k objects out of n objects. This number is commonly denoted by the notation ${}_nP_k$.

Example: How many different 5-digit positive integers can be formed using the digits 1, 2, 3, 4, 5, 6, and 7 if none of the digits can occur more than once in the integer?

Solution: In the above example question we need to find the number of ways to order 5 integers by choosing from a set of 7 integers. According to the counting principle above, there are,

(7)(6)(5)(4)(3)= 2,520 ways to do this. Note that this is equal to $7!/(7-5)!$

$= (7)(6)(5)(4)(3)(2!)/2! = (7)(6)(5)(4)(3)$.

Combinations

Given the 5 letters A, B, C, D, and E, suppose that we wish to determine the number of ways in which you can select 3 of the 5 letters, but unlike before, you do not wish to count different orders for the 3 letters. The following shows a list of all of the ways in which 3 of the 5 letters can be selected without regard to the order of the letters.

ABC ABD ABE ACD ACE

ADE BCD BCE BDE CDE

There are 10 ways available for selecting the 3 letters without order. A relationship exists between selecting with order and selecting without order.

The number ways that is available for selecting 3 of the 5 letters without order, which is 10, multiplied by the number of ways to order the 3 letters, which is 3!, or 6, equals the number of ways to select 3 of the 5 letters and order them, which is $5!/2! = 60$. In simple terms, (the number of ways to select without order) $\times$ (the number of ways to order) = (number of ways to select with order) This relationship can be simply written as below.

(the number of ways to select without order)=(the number of ways to select with order)/(the number of ways to order)

For the example above, the number of ways to select without order is $5!/2!/3! = 5!/3! * 2! = 10$.

More generally, suppose that k objects are chosen from a set of n objects, where

$k \leq n$, but that the k objects will not be arranged in an order. The number of ways in which this can be done is known as the number of **combinations of n objects taken k at a time** and is

n! given by the formula $\dfrac{n!}{k!(n-k)!}$.

Another way to refer to the number of combinations of n objects taken k at a time is **n choose k**, and two notations commonly used for denoting this number are ${}_nC_k$ and $\binom{n}{k}$.

Example: Suppose you wish to select a 3-person committee from a group of 9 individuals. How many ways do we havefor doing this?

Solution: Since the 3 individuals on the committee are not ordered, you can make use of the formula for the combination of 9 objects taken 3 at a time, or “9 choose 3”:

$$\frac{9!}{3!(9-3)!} = \frac{9!}{3!6!} = \frac{(9)(8)(7)}{(3)(2)(1)} = 84$$

By making use of the terminology of sets, given a set S consisting of n elements, n choose k is simply the number of subsets of S that consist of k elements.

Formula for n choose k, that is $^{n!/}$ k!(n – k)!also applies when k = 0 and k = n.

Therefore

1. n choose 0 is $^{n!/}$ 0!n!= 1.

(This is a reflection of the fact that there is only one subset of S with 0 elements, namely the empty set).

2. choosiness$^{n!\,/}$ n!0!=1.

(This is a reflection of the fact that there is only one subset of S with n elements, namely the set S itself).

Finally, observe that n choose k is always equal to n choose n – k, because

$$\frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!}$$

6c. Probability

Probability refers to the study of uncertainity regarding an event expressed in terms of numbers. In this section, we shall see some of the terminology used in elementary probability theory.

A **probability experiment**, also known as a **random experiment**, is an experiment for which the result is uncertain. We assume that all of the possible outcomes of an experiment are already known before the experiment is performed, but whose outcome is always unknown. The set of all possible outcomes of a random experiment is known as a **sample space**, and any particular set of outcomes is known as an **event**. For example, let's consider a cube with faces numbered 1 to 6, called a 6-sided die. Rolling the die once is an experiment in which there are 6 possibilities: either 1, 2, 3, 4, 5, or 6 will appear on the top face. The experiment's sample space is {1, 2, 3, 4, 5, 6}. Given below are two examples of events for this experiment.

Event 1: Rolling the number 4. This event has only one outcome.

Event 2: Rolling an odd number. This even has three outcomes.

The **probability** of an event is a number from 0 to 1, whichdenotes the likelihood that the event occurs when the experiment is performed. The greater the number, the more likely the event is to happen.

Example: Let'sconsider the following experiment. A box contains 15 pieces of paper, each of them consists of the name of one of the 15 students in a high school class which has 7 juniors and 8 seniors, all with different names. The instructor will shake the box and then choose a piece of paper on a random basis and read the name. Here the sample space is the 15 names. The assumption of **random selection** says that each of the names is **equally likely** to be selected. If this assumption is made, then the probability that any one particular name will be selected is equal to $1/_{15}$.

For any event E, the probability that E occurs is often denoted as P(E).
For the sample space in this example, P(E), that is, the probability that event E will occur, is equal to the number of names in the event E / 15

If J is the event, which says that the student selected is a junior, then P(J) = $7/_{15}$.

In general, for a random experiment which consists of a finite number of possible outcomes, if each outcome is equally possible to occur, then the probability that an event E occurs is defined by

P(X) = the number of outcomes in the event X / the number of possible outcomes in the trial.

In the case of rolling a 6-sided die, if the die is “fair,” then the 6 outcomes are equally possible. So the probability of rolling a 4 is $1/_6$, and the probability of rolling an odd number (that is, rolling a 1, 3, or 5) can be calculated as $3/_6 = 1/_2$.

The following are six most discussed Guidelines about probability.
Guideline 1: P(X)= 1; If an experiment X is likely to occur.

Guideline 2: P(X)= 0; If an experiment X is likely not to occur.
Guideline 3: 0 < P(X)< 1; If an experiment X is possible but not likely to occur.

Guideline 4: 1 – P(X) is the probability that an event X will not occur.
Guideline 5: The probability of X can be obtained by adding the probabilities of the outcomes in X if X is an event.
Guideline 6: 1 is the total of the probabilities of all possible outcomes of an experiment.

If X and Y are two events of an experiment, we must consider two other events related to X and Y.

Event 1: The event that both X and Y happen, that is, outcomes in the set $X \cap Y$

Event 2: The event that X or Y, or both, happen, that is, outcomes in the set $X \cup Y$

Events that is not possible to happen at the same time are known as **mutually exclusive**. For example, if a 6-sided die is rolled once, the event ofobtaining an odd number and the event of obtaining an even number are mutually exclusive. But rolling a 4 and rolling an even number aren’t mutually exclusive, since 4 is an outcome that is common to both events.

For events X and Y, there are the following three guidelines.

Guideline 1: When the exclusion inclusion principle is applied to probability, we get P(X)+ P(Y)– P(both X and Y occur)= P(either X or Y, or both happen).

Guideline 2: P(both X and Y occur)= 0, and P(either X or Y, or both, occur) = P(X) + P(Y) when we say that X and Y are mutually exclusive.

Guideline 3: X and Y are **independent** if the occurrence of either event does not affect the happening of the other. If the two events X and Y are not dependent, then P(both X and Y occur)= P(X)P(Y). For example, if a fair 6-sided die is rolled two times, the experiment X of rolling a 3 on the first roll and the experiment Y of rolling a 3 on the second roll are both dependent, and the probability of rolling a 3 on both rolls is$P(X)P(Y)=(1/_6)(1/_6)=1/_{36}$. In this example, the experiment actually refers to "rolling the die two times ," and each outcome is a pair of results like "4 and 1on the first and the second roll respectively." But event X restricts onlyto the first roll–to a 3–having no effect on the second roll; similarly, event Y restricts only the second roll–to a 3–having no effect on the first roll.

Experiments X and Y are not possible to be mutually exclusive and independent if $P(X)\neq 0$ and $P(Y)\neq 0$. P(both X and Y happen)= $P(X)P(Y)\neq 0$ if X and Y are independent, but P(both X and Y happen)= 0 if X and Y are mutually exclusive.

It is routine to utilize the shorter notation "X and Y" instead of "both X and Y happen" and use "X or Y" instead of "X or Y or both occur." With the help of notation, we can mention the previous three guidelines as follows.

Guideline 1: P(X or Y)= P(X)+ P(Y)− P(X and Y)
Guideline 2: If X and Y are mutually exclusive, P(X or Y)= P(X) + P(Y).

Guideline 3: If X and Y are not dependent, P(X and Y)= P(X)P(Y).

Example: If a 6-sided dice is rolled once, let us take E as the event of rolling a 3 and let F be the event of rolling an odd number. These events are not independent. This is because rolling a 3 makes certain that the event of rolling an odd number happens. Note that P(E and F)≠ P(E)P(F), since

P(E and F)= P(E)= $1/_6$ and $P(E)P(F)=(1/_6)(1/_2)_{12}=1/12$

Example: A 12-sided dice, with faces numbered 1 to 12, is to be rolled once, and each of the 12 possible outcomes is equally possible to happen. The probability of rolling a 4 is $1/_{12}$, so the probability of rolling a number that is not a 4 is $1 - 1/_{12}$ $=11/_{12}$. The probability of rolling a number that happens to either a multiple of 5 (that is, rolling a 5 or a 10) or an odd number (that is, rolling a 1, 3, 5, 7, 9,) is equal to

P(5 multiple)+ P(odd)– P(5multiple and odd)= $2/12 + 6/12 - 1/12 = 7/12$

Another way for calculating the probability is to observe that rolling a number that is either a multiple of 5 or an odd number is the same as rolling one of the seven numbers 1, 3, 5, 7, 9, 10, and 11, which are equally possible outcomes. So by the ratio formula to 7 calculate the probability, the required probability is $7/12$.

Example: Let'sconsider an experiment with events A, B, and C for which P(A)= 0.23, P(B)= 0.40, and P(C)= 0.85.

Suppose that events A and B are mutually exclusive and events B and C are not dependent. Then what is the value P(B or C)or P(A or B)?

Solution: Since A and B are mutually exclusive,
P(AorB)=P(A) P(B)+=0.23 0.40+=0.63

P(B and C)= P(B)P(C) as B and C are independent.

SoP(B or C) = P(B) + P(C) − P(B and C) = P(B) + P(C) − P(B)P(C)

Therefore,
P(B or C)= 0.40 + 0.85 –(0.40)(0.85)= 1.25 0.34–= 0.91

Example: Let ussuppose that there is a 6-sided dice that is weighted in such a way that each time the die is rolled, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 istwo times the probability of rolling a 1. Whenyou roll the die once, the 6 outcomes are not equally possible to happen. What are the probabilities of the 6 outcomes?

Solution: Let p be the probability of rolling a 1. In that case each of the probabilities of rolling a 2, 3, 4, or 5 is equal will be equal to p, and the probability of rolling a 6 is equal to 2p. Therefore, since the sum of the probabilities of all possible outcomes happens to be 1, it follows that

$$1 = P(1)+ P(2)+ P(3)+ P(4)+ P(5)+ P(6) = p + p + p + p + p + 2p = 7p$$

So the probability of rolling each of the numbers from 1 to 5 is $1/7$, and the probability of rolling a 6 is $2/7$.

Example: Suppose that you roll the weighted 6-sided die from Example 4.4.5 two times. What is the probability that the first roll would bring out an odd number and the second roll will be an even number?

Solution: For calculating the probability that the first roll will be odd and the second roll will be even, observe that these two events are independent. For calculating the probability that both occur, you need to first multiply the probabilities of the two independent events. First we need to compute the individual probabilities.

$P(odd)=P(1)+P(3)+P(5)={}^{3}/_{7}$

$P(even)= P(2)+ P(4)+ P(6)={}^{4}/_{7}$

Then, $P(\text{odd first roll and even second roll})= P(odd)P(even)=(3/7)(4/7)=12/49$.

Two events that happen on a sequential basis are not always independent. The happening of the first event may affect the happening of the second event. In this case, the probability that both events happen is equal to the probability that the first event happens multiplied by the probability that, given that the first event has already occurred, the second event will happen as well.

Example: A box consists of 5 orange books, 4 red books, and 1 blue book. You are to select two books at random basis and without replacement from the box. What is the probability that the first book you select will turn out to be red and the second book you select will be orange?

Solution: For solving, first we should calculate the following two probabilities and then multiply them.

1. The probability that the first book selected from the box will turn out to be red

2. The probability that the second book selected from the box will turn out to be orange, given that the first book selected from the box is red

The probability that the first book you select will turn out to be red is $4/_{10} = 2/_5$. If the first book you select is red, then there will be 5 orange books, 3 red books, and 1 blue book that left in the box, for a total of 9 books. Therefore, the probability that the second book you select will turn out to be orange, given that the first book you selected is red, is $5/_9$. Multiply the two probabilities we get

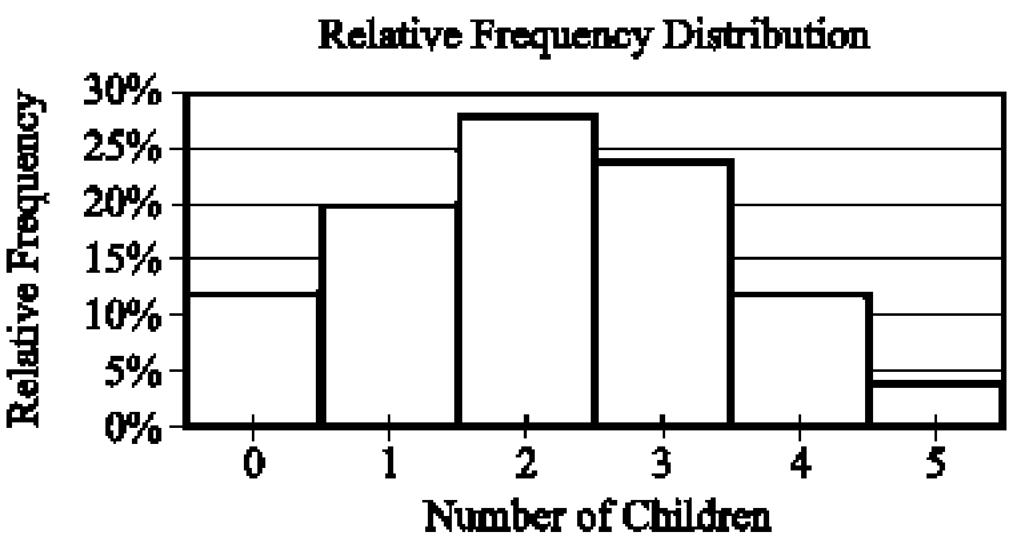
$(2/5)(5/9)^{=2/9}$

Distributions of Data, Random Variables, and Probability Distributions

In data analysis, variables whose values depends upon chance, play a significant role in linking distributions of data to probability distributions. Such variables are known as random variables. We shall begin with a review of distributions of data.

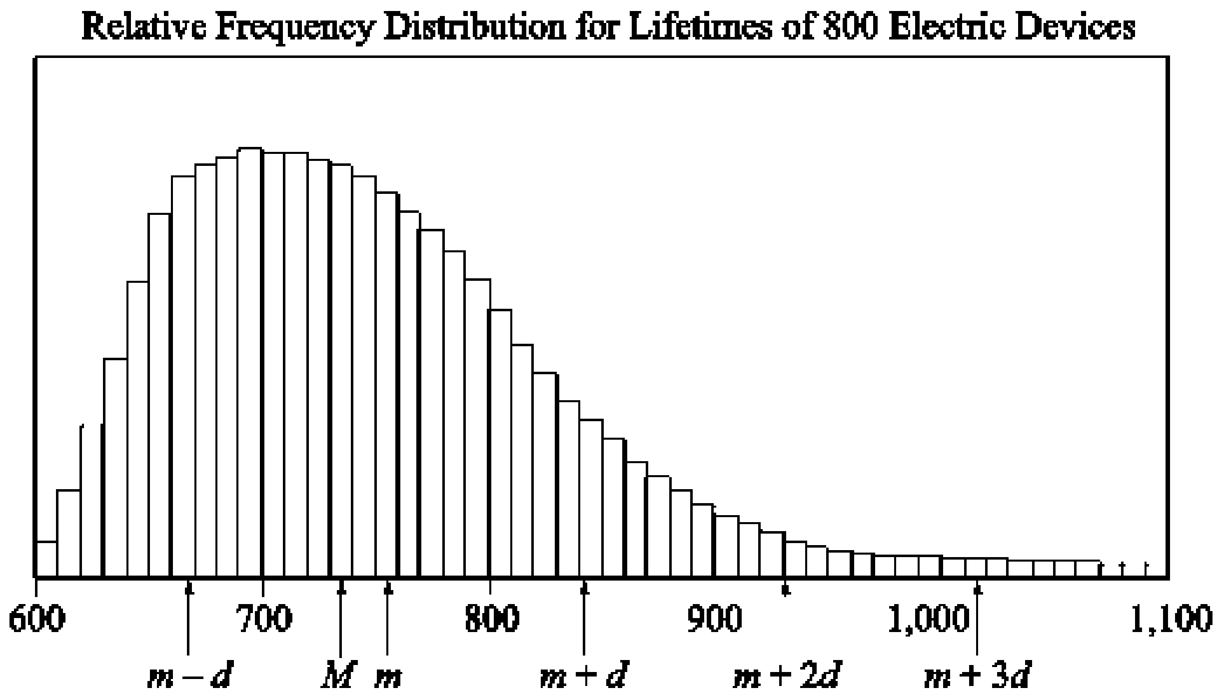
Distributions of Data

Let's recollect that relative frequency distributions given in a table or histogram are a common way for showing how numerical data are distributed. In a histogram, the areas of the barsdenotes where the data are concentrated. The histogram of the relative frequency distribution of the number of kids in each of 25 families in Figure below illustrates a small group of data, with only 6 different data values and 25 data values altogether. (Note: This is the second happening of Figure, it was first encountered in Example)



Many groups of data are larger than 25 and have more than 6 possible values, which are often measurements of quantities like length, money, or time.

Example: The lifetime of 800 electric devices were measured. Because the lifetimes had different values, the measurements were grouped into 50 intervals, of 10 hours each: 601 to 610 hours, 611 to 620 hours, and so on, up to 1,091 to 1,100 hours. The final relative frequency distribution, as a histogram, consists of 50 thin bars and many different bar heights, as shown in Figure below.



In the above histogram, the median is denoted by M, the mean is represented by m, and the standard deviation is represented by d.

According to the graph:

- A data value 1 standard deviation below the mean, denoted by m – d, is between 660 and 670.
- The median, which is denoted by M , is between 730 and 740.
- The mean, denoted by m, is between 750 and 760.
- A data value 1 standard deviation above the mean, which isdenoted by m + d, is between 840 and 850.
- A data value 2 standard deviations above the mean, which is denoted by m + 2d, is approximately 930.

A data value 3 standard deviations above the mean, denoted by m + 3d, is between 1,010 and 1,020.

The standard deviation marks depict how most of the data are within 3 standard deviations of the mean, that is, between the numbers m – 3d and m + 3d. Observe that 3d +m is shown in the figure, but at the same time 3d- m is not shown.

The upper part of the bars of the relative frequency distribution in the histogram in Figure have a smooth appearance and appears like a curve. In general, histograms that depict large data sets grouped into many classes have a relatively smooth appearance. As a result, the distribution can be modeled by a smooth curve that is close to the tops of the bars. This kind of a model help in retaining the shape of the distribution but is independent of classes.

Let's recollect from Example 4.1.10 that the sum of the areas of the bars of a relative frequency histogram amounts to 1. Although the units on the horizontal axis of a histogram differs from one data set to another, the vertical scale can be adjusted (stretched or shrunk) so that the total sum of the areas of the bars is 1. With the adjustment of the vertical scale, the area under the curve that models the distribution is also 1. This model curve is known as a **distribution curve**, but it has other names as well, including **density curve** and **frequency curve**.

The main purpose of the distribution curve is to provide a good illustration of a large distribution of numerical data that does independent on specific classes. To achieve this, the main property of a distribution curve is that the area under the curve in any vertical slice, just like a histogram bar, depicts the proportion of the data that happen to lie in the corresponding interval on the horizontal axis, which is at the base of the slice.

Random Variables

While analyzing the data, it is common to choose a value of the data on a random basis and consider that choice as a random experiment, as introduced. Then, the probabilities of events that involves the randomly chosen value may be determined. Given that a distribution of data, a variable, say X, may be used to represent a randomly chosen value from the distribution. Such a variable X is an apt example of a **random variable**, which is a variable whose value is a numerical outcome of a random experiment.

Example: In a previous example, data that consists of numbers of kids in each of 25 families are summarized in a frequency distribution table. The frequency distribution table is repeated below.

Frequency Distribution

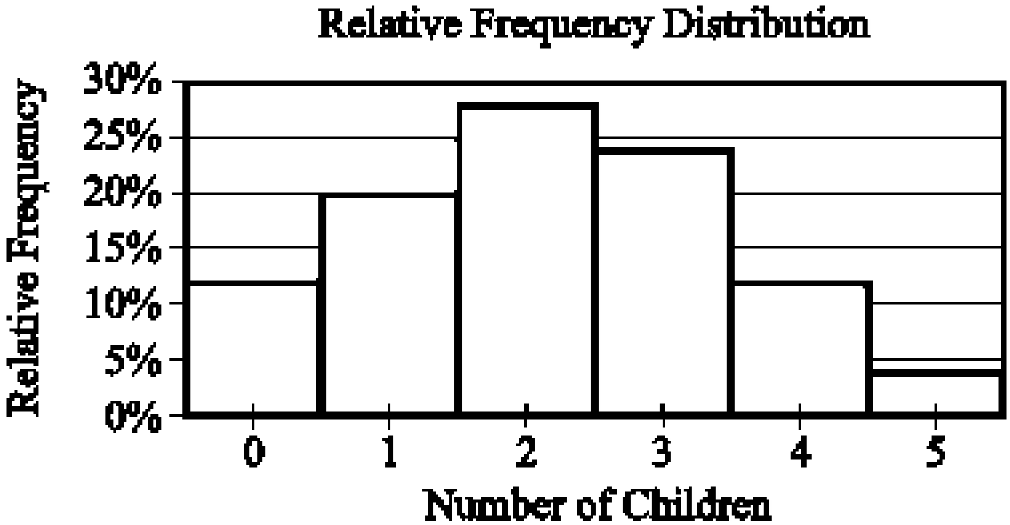
Number of Kids	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Let's consider X as the random variable that represent the number of kids in a randomly chosen family among the 25 families. What is the probability that the value of Xis 3 ? That X >3 ? That X will be less than the mean of the distribution?

Solution: For determining the probability that X = 3, realize that this is the same as determining the probability that a family with 3 kids will be chosen.

Since there exists 6 families with 3 kids and each of the 25 families is equally likely to be chosen, the probability that a family with 3 kids will be chosen is $6/_{25}$. That is, X = 3 is an event, and we can calculate the probability as $P(X = 3)=6/_{25}$, or by solving its 0.24. It is common to use the shorter notation P(3) instead of P(X = 3), so you could write P(3)= 0.24.

Observe that in the histogram depicted in Figure below, the area of the bar that corresponds to X = 3as a proportion of the combined areas of all of the bars is that is equal to this probability. This shows how probability is related to area in a histogram for a relative frequency distribution.



For determining the probability that X > 3, observe that the event X > 3 is the same as the event " X = 4 or X = 5 ". As they are mutual events, where we can make use of the rules of probability from Section 4.4.

$P(X>3)=P(4)+P(5)=^{3/25\ +\ 1/25}+=0.12+0.04=0.16$

For determining the probability that X is less than the mean of the distribution, first compute the mean of the distribution as follows.

$0(3)+1(5)+2(7)+3(6)+4(3)+5(1)/\ 25\ {}_{=}54\ /25\ {}_{=\ 2.16}$

Then, compute the probability that X is less than the mean of the distribution (that is, the probability that X is less than 2.16).

$P(X < 2.16)= P(0)+P(1)+P(2)=^{3\ /25}{}_{+}{}^{5/25}{}_{+}{}^{7/25}{}_{=}{}^{15/25}{}_{=}\ 0.6$

The following table depicts all the 6 possible values of X and their probabilities. This table is known as the **probability distribution** of the random variable X.

X	$P(X)$
0	0.12
1	0.20
2	0.28
3	0.24
4	0.12
5	0.04

Observe that the probabilities are the relative frequencies of the 6 possible values that are expressed in the form of decimals instead of percent. The following statement explains a fundamental link between data distributions and probability distributions.

Statement: For a random variable that denotes a randomly chosen value from a distribution of data, the probability distribution of the random variable happens to be the same as the relative frequency distribution of the data.

The reason is that the probability distribution and the relative frequency distribution are essentially the same, the probability distribution can be denoted with the help of a histogram. Also, all of the descriptive statistics – such as mean, median, and standard deviation−that applies to the distribution of data also applies to the probability distribution. For instance, we state that the probability distribution above has a mean of 2.16, a median of 2, and a standard deviation of about 1.3, since the 25 data values have these statistics, as you can check.

These statistics are in the same way defined for the random variable X above. Therefore, we can say that the **mean of the random variable X** is 2.16. A random variable is also known **expected value**. So we can also say that the expected value of X is 2.16.

Note that by calculating the mean of X we get,
0(3)+ 1(5)+ 2(7)+ 3(6)+ 4(3)+ 5(1)/ 25

which can also be expressed as

$$0\left(\frac{3}{25}\right)+1\left(\frac{5}{25}\right)+2\left(\frac{7}{25}\right)+3\left(\frac{6}{25}\right)+4\left(\frac{3}{25}\right)+5\left(\frac{1}{25}\right)$$

which is the same as
$0P(0)+1P(1)+2P(2)+3P(3)+4P(4)+5P(5)$

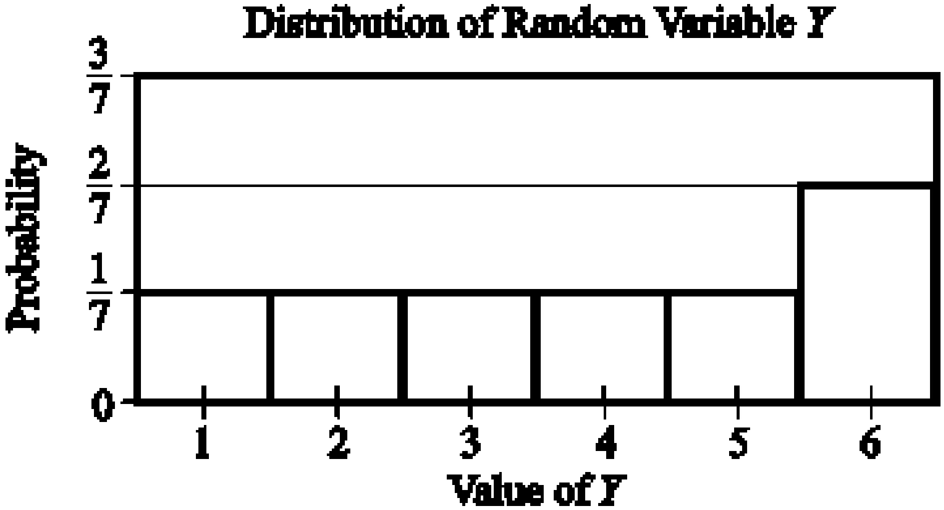
Therefore, the mean of the random variable X happens to be the value that is obtained by adding the products X P(X) for all values of X, that is, the sum of each value of X is multiplied by the value thatcorresponds probability P(X).

The preceding example consists of a common type of random variable−one that shows a randomly chosen value from a distribution of data. However, the concept of a random variable is general. A random variable can be of any quantity the value of which is the result of a random experiment. The possible values of the random variable are the same as that of the outcomes of the experiment. So any random experiment with numerical outcomes naturally consists of a random variable associated with it, as in the following example.

Example: Let Y represent the result of the experiment of rolling the weighted 6-sided dice in Example discussed previously. (In that example, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 is two times the probability of rolling a 1.) Then Y happens to be a random variable with 6 possible values, the numbers 1 through 6. Each of the six values of Y consists of a probability. The probability distribution of the random variable Y is shown below, first in a table, then as a histogram. Table representing the Probability Distribution of Random Variable Y.

Y	$P(Y)$
1	$\frac{1}{7}$
2	$\frac{1}{7}$
3	$\frac{1}{7}$
4	$\frac{1}{7}$
5	$\frac{1}{7}$
6	$\frac{2}{7}$

Histogram Representing the Probability Distribution of Random Variable *Y*



The mean, or expected value, of Y can be computed as

P(1)+ 2P(2)+ 3P(3)+ 4P(4)+ 5P(5)+ 6P(6)

which is equal to

$(\frac{1}{7}) + 2(\frac{1}{7}) + 3(\frac{1}{7}) + 4(\frac{1}{7}) + 5(\frac{1}{7}) + 6(\frac{2}{7})$

This sum simplifies to $1/_7 + 2/_7 + 3/_7 + 4/_7 + 5/_7 + 12/_7$, or $27/_7$, which is approximately 3.86.

Both of the random variables X and Y above happens to be the examples of **discrete random variables** because their values consist of discrete points on a number line.

A basic fact about probability is that the sum of the probabilities of all the outcomes of an experiment is 1, the same can be confirmed by adding up all of the probabilities in each of the probability distributions for the random variables X and Y above. Also, the sum of the areas of the bars in a histogram for the probability distribution of a random variable also is 1. This fact is related to the following fundamental relationship between the areas of the bars of a histogram and the probabilities of a discrete random variable.

Fundamental Link: In a histogram that represents the probability distribution of a random variable, the area of each bar happens to be proportional to the probability represented by the bar.

If the dice in an example were a fair die instead of weighted, then the probability of each of the outcomes would be $1/_6$, and as a result , each of the bars in the histogram of the probability distribution would consist of the same height. Such kind of a flat histogram denotes a **uniform distribution**, since the probability is distributed uniformly over all possible outcomes.

Data Interpretation Examples

Example: This example is based on the following table.

Distribution of complaints registered by customers in an Airline A, 2003 and 2004

Category	2003	2004
Flight problem	20.0%	22.1%
Baggage	18.3	21.8
Customer service	13.1	11.3
Reservation and ticketing	5.8	5.6
Credit	1.0	0.8
Special passenger accommodation	0.9	0.9
Other	40.9	37.5
Total	100.0%	100.0%
Total number of complaints	22,998	13,278

(a) How many complaints concerning credit were received approximately by Airline A in the year 2003 ?

(b) What is the percentage decrease the total number of complaints from 2003 to 2004 ?

(c) Based on the information shown in the table, which of the following three statements are true?

Statement 1: In each of the years 2003 and 2004, complaints about flight problems, baggage, and customer service together amounted to more than 50 percent of the total customer complaints received by Airline A.

Statement 2: The number of special passenger accommodation complaints were unaltered from 2003 to 2004.

Statement 3: Between 2003 and 2004, the number of flight problem complaints increased by more than 2 percent.

Solutions:

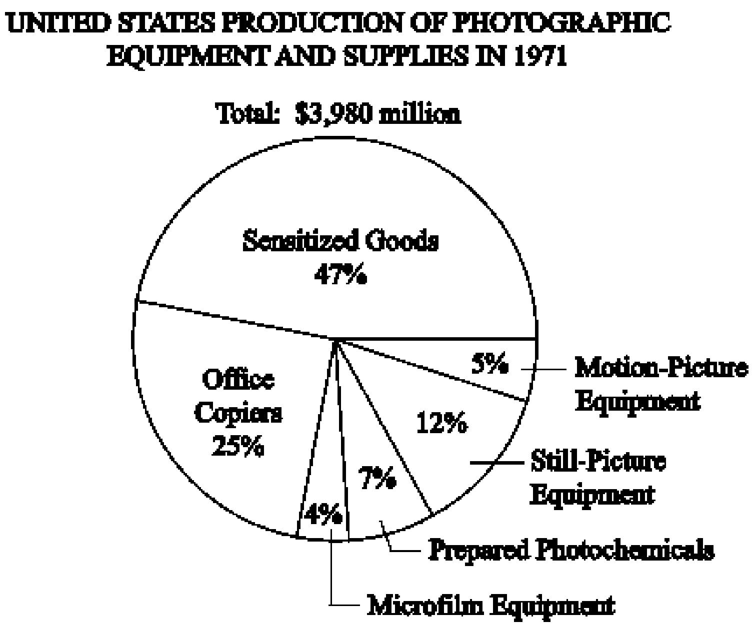
(a) As per the table, in 2003, 1 percent of the total number of complaints concerned credit. Thus, the number of complaints that concerns with credit is equal to 1 percent of 22,998. By taking the decimal equivalent of 1, we get that the number of complaints in 2003 is equal to (0.01)(22,998), which is about 230.

(b) The total number of complaints decreased in the year 2003 to 2004 was 22,998 – 13,278, or 9,720. Which is $\left(\frac{9,720}{22,998}\right)(100\%)$, or

Close to 42 percent.

(c) Since 20.0+18.3+13.1 and 22.1+21.8+11.3 are both greater than 50, statement 1 is true. For statement 2, the special passenger accommodation complaints percentage did remain the same from 2003 to 2004, but the number of such complaints decreased because the total number of complaints decreased. Thus, statement 2 is false. For statement 3, the bases of the percents/percentages are different, the percentages shown in the table for flight problems do in fact increase by more than 2 percentage points. The total number of complaints in 2004 was much lower than the total number of complaints in 2003, and clearly 20 percent of 22,998 is greater than 22.1 percent of 13,278. Therefore, the number of flight problem complaints decreased from 2003 to 2004, and statement 3 is false.

Example: This example is based on the following circle graph.



(a) Approximately what was the ratio of the value of the goods(sensitized) to the value of still picture equipment produced in 1971 in the United States?

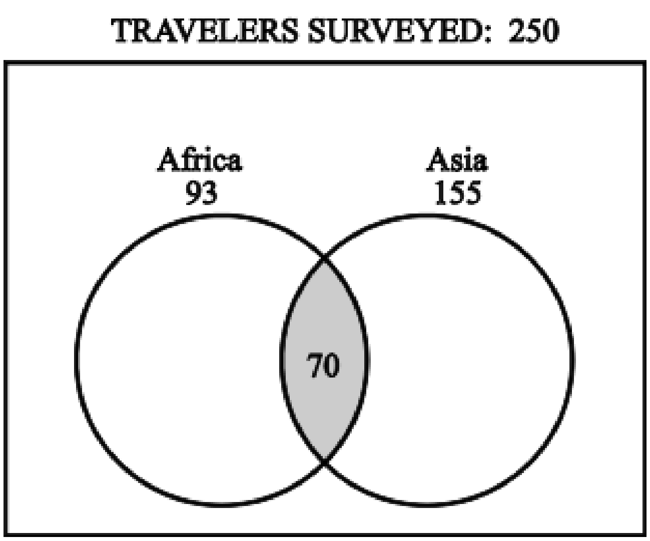
(b) Find the value of office copiers produced in 1970 if the value of the product that were produced in 1971 was 30 percent greater than the value produced in the corresponding year (1970).

Solutions:

(a) The ratio of the value of sensitized goods to the value of still picture equipment equals the ratio of the corresponding percents shown for the reason that the percents have the same base, which happens to be the total value. Therefore, the ratio is 47 to 12, (approximately 4 to 1).

(b) The value of office copiers that were produced in 1971 was 0.25 times $3,980 million, or $995 million. Therefore, if the corresponding value in 1970 amounted to x million dollars, then 1.3x = 995 million. If we solve for x, it yields $x = {}^{995}/_{1.3} \approx 765$, so the value of office copiers produced in 1970 was approximately $765 million.

Example: As per the information obtained from a survey of 250 European travelers, 93 have traveled to Africa, 155 have traveled to Asia, and of these two groups, 70 have traveled to both continents, as shown in the below Venn diagram below.



(a) First task it to find the number of travelers who have travelled to Africa but have not travelled to Asia?

(b) Second task is to find how many of the travelers have travelled to at least one of the two continents, that is of Africa and Asia?

(c) Third task is to find that how many travelers have traveled neither to Africa nor to Asia?

Solutions:

In the above Venn diagram, the rectangular region or the box depicts the set of all travelers surveyed; the two circular regions that represents the two sets of travelers to Africa and Asia, respectively; and the shaded region shows the subset of those who have traveled to both continents.

(a) The travelers those who have traveled to Africa but not traveled to Asia are depicted in the Venn diagram by the part of the left circle that has not been shaded. This says that the answer can be found by considering the shaded part away from the leftmost circle, in effect, subtracting the 70 from the 93, to get 23 travelers who traveled to Africa, but not to Asia.

(b) The travelers surveyed who have traveled to either of the two continents of Africa and Asia are depicted in the Venn diagram by that part of the rectangle that is in at least one of the two circles. This suggests adding the two numbers 93 and 155. But the 70 travelers who have traveled to both continents would be counted two times in the sum 93 + 155. We need to subtract 70 from the sum for correcting the double counting so that these 70 travelers are counted only once:

$93+155-70 = 178$

(c) The travelers who have traveled neither to Africa nor to Asia are shown in the Venn diagram by the part of the rectangle that is not in either circle. Let's take N as the number of these travelers. Note that the entire rectangular region has two main non overlapping parts: the part outside the circles and the part which is inside the circles. The first part denotes N travelers and the second part denotes 93 + 155 − 70 = 178 travelers (from part b). Therefore, 250 = N 17+8, and solving for N gives N= 250−178= 72.

Additional Questions

1. In how many different ways can the letters be arranged in the word STUDY?

2. Martha invited 4 of her friends to go with her to the movies. There are 120 different ways in which they can be seated together in a row of 5 seats, one person per seat. In how many ways can Martha be seated in the middle seat?

3. How many 3-digit positive integers are odd and doesn't have the digit 5 ?

4. From a box of 10 light bulbs, you are to take 4 of them. How many different sets of 4 light bulbs will you be able to remove?

4. A talent contest consists of 8 contestants. Judges should award prizes for first, second, and third places, with no ties existing.

(a) In how many of the different ways can the judges award the 3 prizes?

(b) How many different groups of 3 people will be able to get prizes?

5. From all 2-digit positive integers if on a random basis an integer is selected, what is the probability that the integer chosen has

(a) a 4 in the tens place?
(b) at least one 4 in the tens place or in the ones place?

(c) no 4 in either place?

6. In a box of 10 electrical parts, there are 2 defective pieces.
(a) If you choose one part on a random basis from the box, what is the probability that it is not defective?

(b) If you choose two parts at random from the box, without replacement, what is the probability that both will be defective?

7. A certain college consists of 8,978 full-time students, some of whom live on the campus and some of whom outside the campus.

The following table depicts the distribution of the 8,978 full-time students, by class and living arrangement.

	Freshmen	Sophomores	Juniors	Seniors
Live on campus	1,812	1,236	950	542
Live off campus	625	908	1,282	1,623

1. (a) If one full-time student is selected on a random basis, what is the probability that the student who is chosen will not be a freshman?
2. (b) If one full-time student who lives outside the campus is selected at random, what is the probability that the student will be a senior?
3. (c) If one full-time student who is a freshman or sophomore is selected on a randombasis , what is the probability that the student will be a student who lives on campus?

8. Let A, B, C, and D be events for which P(AorB)=0.6, P(A)=0.2, P(CorD)=0.6, and P(C)=0.5.

The events C and D are independent however the events A and B are mutually exclusive.

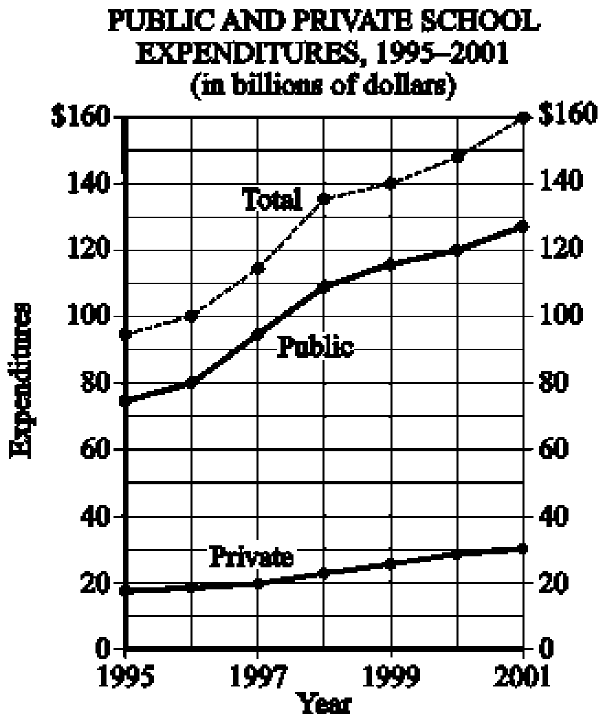
(a) Find P(B).

(b) Find P(D).

9. Lin and Mark each independently attempt to decode a message. If the probability that Lin will be able to decode the message is 0.80 and the probability that Mark will decode the message is 0.70, find the probability that

(a) both will be able to decode the message
(b) at least one of them will be able to decode the message
(c) neither of them will be able to decode the message

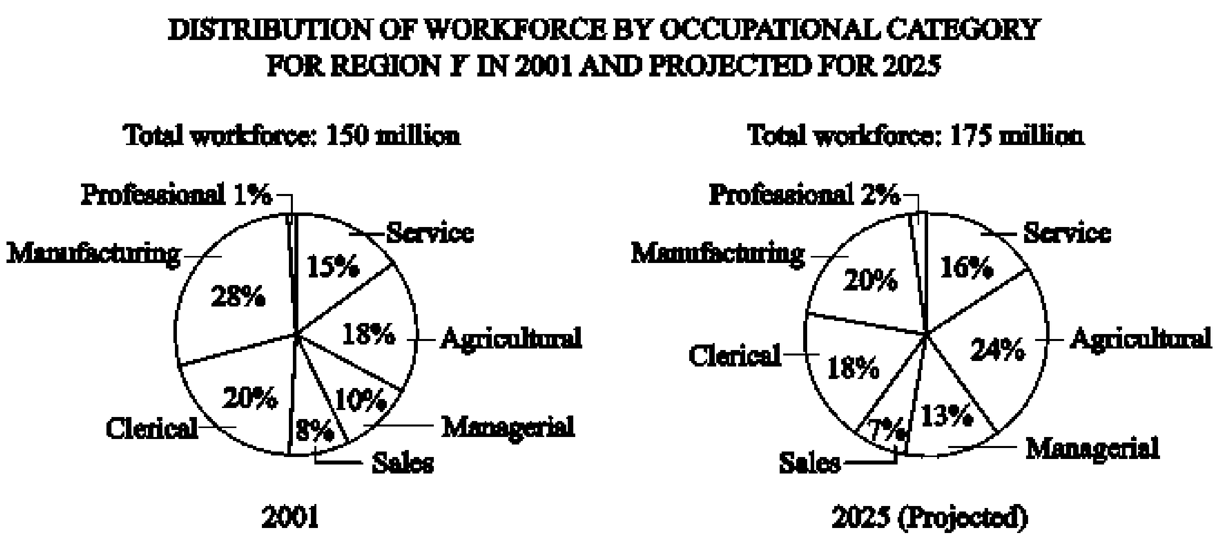
10. This exercise is based on the following graph.



(a) Which year shows an increase in the total expenditures from the year before?

(b) What percent of total expenditures belongs to private schools for 2001? Give your answer to the nearest percent.

11. Let's look at the following data.



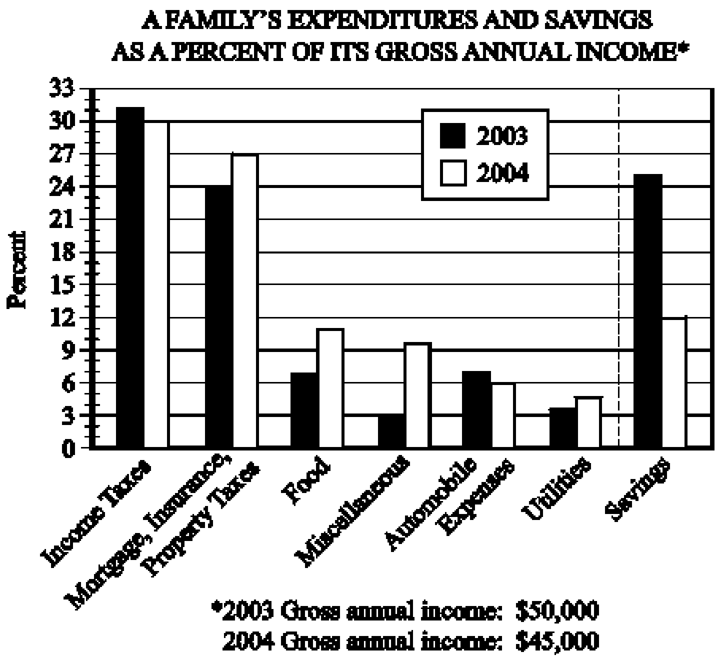
(a) In 2001, how many categories of each consisted of more than 25 million workers?

(b) Find the ratio of the number of workers in the Agricultural category in 2001 to the projected number of such workers in 2025 ?

(c) Between 2001 to 2025, there is a projected increase in the number of workers in which of the following three categories?

Category1: Sales
Category2: Service
Category3: Clerical

12. Let's look at the following data.



(a) In 2003 the family utilized a total of 49 percent of its gross annual income for two of the categories listed. Find the total amount of the family's income used for those same categories in 2004 ?

(b) Of the listed seven categories listed, which category of expenditure had the biggest percent increase from 2003 to 2004 ?

ANSWERS

1. 5!=120

2. 24

3. 288

4. 210

5.

(a) 336
(b) 56

6.

(a) $\frac{1}{9}$

(b) $\frac{1}{5}$

(c) $\frac{4}{5}$

7.

(a) $\frac{4}{5}$

(b) $\frac{1}{45}$

8.

(a) 6, 541 / 8,978

(b) 1, 623 / 4, 438

(c) 3,048 / 4, 581

9.

(a) 0.4
(b) 0.2

10.
(a) 0.56
(b) 0.94
(c) 0.06

11.
(a) 1,440
(b) 0.15

12.
(a) 1998
(b) 19%

13.
(a) Three

(b) 9 to 14, or $^{9}/14$

(c) Categories 1, 2, and 3

14.

(a) $17,550
(b) Miscellaneous

[redacted] Matrices

7a. Matrix Introduction

The knowledge of matrices is very much essential in various branches of mathematics. This mathematical tool simplifies our work to a large extent when compared with other straight forward methods. The evolution of the concept of matrices is the result of an attempt to obtain simple methods for solving the system of linear equations. Matrices are not only utilised as a representation of the coefficients in system of linear equations, but it’s scope far exceeds that use. Matrix notations, representations and operations are being used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science such as sales projection, budgeting and cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be depicted mathematically with the help of matrices. Matrices are also being used in cryptography. This mathematical system is not only utilized for certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall become acquainted with the fundamentals of matrix and matrix algebra.

Matrix

Let's suppose that we want to express the information that Radha has with her, 15 notebooks. We may express it as [15] with the knowledge that the number inside [ ] is the number of notebooks that Radha has. Now, if we need to express that Radha has 15 notebooks and 6 pens with her. We can express it as [15 6] with the knowledge that first number inside [ ] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of having the notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha Fauzia Simran have 15 notebooks 10 notebooks 13 notebooks respectively and also 6 pens, and 2 pens, and 5 pens and this can be expressed as:

$$\begin{array}{ll} \left[\begin{array}{cc}15 & 6\\ 10 & 2\\ 13 & 5\end{array}\right] & \begin{array}{l}\leftarrow \text{First row}\\ \leftarrow \text{Second row}\\ \leftarrow \text{Third row}\end{array} \\ \begin{array}{cc}\uparrow & \uparrow\\ \text{First} & \text{Second}\\ \text{Column} & \text{Column}\end{array} & \end{array}$$

or

	Radha	Fauzia	Simran
Notebooks	15	10	13
Pens	6	2	5

which can be expressed as:

$$\begin{array}{ll} \left[\begin{array}{ccc}15 & 10 & 13\\ 6 & 2 & 5\end{array}\right] & \begin{array}{l}\leftarrow \text{First row}\\ \leftarrow \text{Second row}\end{array} \\ \begin{array}{ccc}\uparrow & \uparrow & \uparrow\\ \text{First} & \text{Second} & \text{Third}\\ \text{Column} & \text{Column} & \text{Column}\end{array} & \end{array}$$

In the first arrangement the entries in the first column denotes the number of note books Radha, Fauzia and Simran, respectively possess and the entries in the second column denotes the number of pens possessed by all the three. Similarly, in the second arrangement, the entries in the first row denotes the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row denotes the number of pens held by Radha, Fauzia and Simran, respectively. An arrangement of the above kind is known as a matrix. Formally, we define matrix as:

Definition: A matrix refers ordered rectangular array of numbers or functions. The numbers or functions are known as the elements or the entries of the matrix.

Order of a matrix

A matrix consists of m rows and n columns and is known as a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). Therefore lookingat the above examples of matrices, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix. Note that A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements, respectively.

An m × n matrix consists of the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

or $A=[a_{ij}]_{m\times n}, 1\leq i\leq m, 1\leq j\leq n \; i,j\in N$

Thus the i^{th}row consists of the elements a_{i1}, a_{i2}, a_{i3},..., a_{in}, while the j^{th}column has the elements a_{1j}, a_{2j}, a_{3j},..., a_{mj},

In general a_{ij}, is an element in the i^{th}row and j^{th}column. It can also be called it as the $(i, j)^{th}$element of A. The number of elements in an m × n matrix equals to mn.

We can represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or [x, y]). For example point P(0, 1) as a matrix representation can be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0 \; 1].$$

Observe that in this way we can express the vertices of a closed rectilinear figure in the form of a matrix. For example, let’s consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be shown as

$$X = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} A & B & C & D \end{array}$$
$$X = \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \quad \text{or} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used for representing the vertices of geometrical figures in a plane.

Example: If a matrix consists of 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order m × n, it consists ofmn elements. Therefore, in order to find all possible orders of a matrix with 8 elements, we should first find all ordered pairs of natural numbers, whose product is 8.
Thus, all possible pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Therefore, possible orders are 1 × 8, 8 ×1, 4 × 2, 2 × 4

7b. Types and Operations of Matrices

In this section, we will study the different types of matrices. (i) **Column matrix**

A matrix is known as a column matrix if it has only one column.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4×1.

$A = [a_{ij}]_{m \times 1}$ -a column matrix of order m × 1.

(ii) **Row matrix**

A matrix is a row matrix if it consists of onlya single row

For example, $B = \left[-\frac{1}{2} \quad \sqrt{5} \quad 2 \quad 3 \right]_{1\times 4}$ is a row matrix.

$B=[b_{ij}]_{1\times n}$ happens to be a row matrix of order 1×n.

iii) **Square matrix**

A matrix in which the number of rows and columns are equal, is said to be a square matrix. Thus an m × n matrix is a square matrix if m = n and is known as a square matrix of order ‘n’.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

$A = [a_{ij}]_{m \times m}$:- a square matrix of order m.

iv) **Diagonal matrix**

A square matrix $B = [b_{ij}]_{m \times m}$ is a diagonal matrix if all its non diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij}=0$, when $i \neq j$.

For example $A = [4],\ B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, happen to be diagonal matrices of order 1, 2, 3 respectively.

(v) **Scalar matrix**

A diagonal matrix is a scalar matrix if its diagonal elements are equal, i.e. , a square matrix $B = [b_{ij}]_{n \times n}$ is a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$
$$b_{ij} = k, \quad \text{when } i = j, \text{ for some constant } k.$$

For example

$$A = [3], \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

are known as scalar matrices of order 1, 2 and 3, respectively.

(vi) **Identity matrix**

A square matrix in which elements in the diagonal are all 1 and rest are zero is known as an identity matrix. The square matrix $A = [a_{ij}]_{n \times n}$ is known as an

Identity matrix, if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We represent the identity matrix of order n by I_n. When order is clear from the context, we represent it as I.

For example $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3, respectively.

Note that a scalar matrix is an identity matrix when k = 1. But every identity matrix is clearly a scalar matrix by default.

(vii) **Zero matrix:** A matrix is said to be zero matrix if all its elements are zero. It is otherwise known as a null matrix.

For example, $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $[0, 0]$ are all zero matrices. We denote zero matrix by O. Its order will be clear from the context.

Equality of matrices

Definition Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if

(i) they are in the order

(ii) each element of A equals the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write A = B.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5$, $y = 0$, $z = 2$, $a = \sqrt{6}$, $b = 3$, $c = 2$

Example Find out the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \qquad 5c - d = 11$$

$$a - 2b = -3 \qquad 4c + 3d = 24$$

Solving these equations, we get a = 1, b = 2, c = 3 and d = 4.

Operations on Matrices

In this section, we shalllook at certain operations on matrices, namely, addition and, multiplication of a matrix by a scalar, difference and multiplication of matrices.

Addition of matrices

This new matrix is obtained by adding the above two matrices. Note that the sum of two matrices is a matrix which we get by adding the corresponding elements of the given matrices. Further, the two matrices should be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ **is a** 2×3 **matrix and** $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ **is another**

2×3 **matrix. Then, we define** $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices that are of the same order, say $m \times n$. Then, the sum of the two matrices A and B can be defined as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij}= a_{ij}+ b_{ij}$, for all possible values of i and j.

Note:

- We stress on the point that if A and B are not of the same order, then A + B is not defined.
- We may observe that addition of matrices happens to be an example of binary operation on the set of matrices of the same order.

Multiplication of a matrix by a scalar

we can define multiplication of a matrix by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is the scalar, then kA is another matrix which which we get by multiplying each element of A by the scalar k.

In other words, $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$, that is, $(i, j)^{th}$element of kA is $ka_{ij}$for all possible values of i and j.

For example, if $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is represented by – A. We define – A = (–1) A.

For example, let

$A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$, then – A is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If $A = [a_{ij}]$, $B = [b_{ij}]$ are the two matrices of the same order, say m×n, then A-B is defined as a matrix D=[d ij],where dij =a ij –b ij,

for all the value of i and j. In other words, D = A – B = A + (–1) B, that is the sum of the matrix A and the matrix – B.

Example If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

Solution:

We have

$$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

Properties of matrix addition

As far as the addition of matrices are concerned, satisfy the following properties:

(i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ arethe matrices of the same order, say $m \times n$, then $A + B = B + A$.

Now $A+B=[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$
$= [b_{ij}+ a_{ij}]$ (addition of numbers is commutative)

$= ([b_{ij}] + [a_{ij}]) = B + A$

(ii) **Associative Law** For any of the three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the

same order, let's say $m \times n$, $(A + B) + C = A + (B + C)$. Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$=[a_{ij}+ b_{ij}] + [c_{ij}] = [(a_{ij}+ b_{ij}) + c_{ij}]$
$= [a_{ij}+ (b_{ij}+ c_{ij})]$ (Why?)
$=[a_{ij}]+[(b_{ij}+c_{ij})]=[a_{ij}]+([b_{ij}]+[c_{ij}])=A+(B+C)$

(iii) **Existence of additive identity** Let us assume that A = $[a_{ij}]$ is an m × n matrix and O is an m × n zero matrix, then A + O = O + A = A. In other words, O happens to be the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let us assume, A = $[a_{ij}]_{m \times n}$ be any matrix, then we haveanothermatrixas–A=$[-a_{ij}]_{m\times n}$suchthatA+(–A)=(–A)+A=O.So – A happens to be the additive inverse of A or negative of A.

Properties of scalar multiplication of a matrix

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say m × n, and k and l are scalars, then

(i) k(A+B)=kA+kB,(ii)(k+l)A=kA+lA

(ii) $k(A+B)=k([a_{ij}]+[b_{ij}])=k[a_{ij}+b_{ij}]=[k(a_{ij}+b_{ij})]=[(ka_{ij})+(kb_{ij})] = [k\ a_{ij}] + [k\ b_{ij}]$ $= k\ [a_{ij}] + k\ [b_{ij}] = kA + kB$ (iii) $(k+l)A = (k+l)[a_{ij}] = [(k + l)\ a_{ij}] + [k\ a_{ij}] + [l\ a_{ij}]$ $= k\ [a_{ij}] + l\ [a_{ij}] = k\ A + l\ A$

Example: Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $$\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $2x + 3 = 7$ and $2y - 4 = 14$ (Why?)

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

Multiplication of matrices

The product of two matrices A and B can be defined if the number of columns of A is equal to the number of rows of B. Let's assume A = $[a_{ij}]$ be an m × n matrix and B = $[b_{jk}]$ be an n × p matrix. Then the product of the matrices A and B is the matrix C of the order m × p. To get the (i, k)thelement $c_{ik}$of the matrix C, we take the ithrow of A and kthcolumn of B, multiply them elementwise and take the sum of all these products. In other words, if$A=[a_{ij}]_{m\times n}$,$B=[b_{jk}]_{n\times p}$,then the ithrowofAis$[a_{i1}\ a_{i2}\ ...a_{in}]$and the kthcolumn of

$$\textbf{B is} \begin{bmatrix} b_{1k} \\ b_{2k} \\ \cdot \\ \cdot \\ b_{nk} \end{bmatrix}, \textbf{ then } c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \dots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

The matrix C = $[c_{ik}]_{m \times p}$ happens to be the product of A and B.

Example:

$$\textbf{Find AB, if } A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}.$$

Solution: Matrix A consists of two columns that equals to the number of the rows of B. Hence AB is defined. Now.

$$AB = \begin{bmatrix} 6(2)+9(7) & 6(6)+9(9) & 6(0)+9(8) \\ 2(2)+3(7) & 2(6)+3(9) & 2(0)+3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12+63 & 36+81 & 0+72 \\ 4+21 & 12+27 & 0+24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Remark If AB is defined, then BA doesn't require to be defined. In the above example, AB is defined but BA is not defined because B consists of 3 column while A has only 2 (and not 3) rows. If A, B are, respectively m × n, k × l matrices, then both AB and BA are defined **if** n = k and l = m. In particular, if both A and B are square matrices that belong the samesame order, then both AB and BA are defined.

Non-commutativity of multiplication of matrices

Now, let us see by an example that even if AB and BA are both defined, it is not necessary that AB should equal BA.

Example:

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$**, then** $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$**. Clearly** $AB \neq BA$**.**

Therefore matrix multiplication is'nt commutative.

Zero matrix:- the product of nonzero matrices

We are aware that, for real numbers a, b if ab = 0, then either a = 0 or b = 0. This also need not hold true for matrices, let us observe this through an example.

Example: Find out AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Solution: We have **We have** $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

There the product of two matrices happens to be a zero matrix, it isn't necessary that one of the matrices is a zero matrix.

Properties of multiplication of matrices
The multiplication of matrices consists of the following properties, which we state, devoid of any roof.

The associative law For three matrices A, B and C. We have

(AB) C = A (BC), whenthe two sides of the equality are defined.

The distributive law for the three matrices A, B and C.

(i) A(B+C)=AB+AC
(ii) (A+B) C = AC + BC, when both sides of equality are defined.

The existence of multiplicative identity every square matrix A, consists of an identity matrix of same order such that IA = AI = A.

Example In a legislative assembly election, a political group hired a public relations firm for romoting its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is resented in the form of matrix A as

$$A = \begin{array}{c} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is resented by

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} \textbf{Telephone} & \textbf{Housecall} & \textbf{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \rightarrow \mathbf{X} \\ \rightarrow \mathbf{Y} \end{array}.$$

Find the total amount spent by the group in the two cities X and Y.

Solution We have

$$BA = \begin{bmatrix} 40,000 + 50,000 + 250,000 \\ 120,000 + 100,000 + 500,000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

$$= \begin{bmatrix} 340,000 \\ 720,000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., ₹3400 and ₹7200, respectively.

7c. Transpose of a Matrix

In this section, we shall get introduced on the transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition The matrix obtained by interchanging the rows and columns of A is known as the transpose of matrix A if A = $[a_{ij}]$ which is a m × n matrix. The transpose of the matrix A is denoted by A′ or (A^T). In other words, if A=[a ij] m×n, then A′=[a ji] n×m. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3\times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2\times 3}$$

Properties of transpose of the matrices

Let us now state the following properties of transpose of matrices without proof. These properties can be verified by taking suitable examples.

For any matrix A and B of appropriate order we have:

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)
(iii) $(A + B)' = A' + B'$ (iv) $(A\,B)' = B'\,A'$

Example:

. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution: Then we have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2\ 4\ 5]\ ,\ B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}\begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

7d. Symmetric and Skew Symmetric Matrices

Definition A square matrix $A = [a_{ij}]$ can be said to be symmetric if $A' = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric matrix as $A' = A$

Definition: A square matrix $A = [a_{ij}]$ can be said to be skew symmetric matrix if $A' = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of I and j. Now, if we put i=j, we have $a_{ii} = -a_{ii}$.Therefore $2a_{ii} = 0$ or $a_{ii} = 0$ for all i's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix $\mathbf{B} = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as $B' = -B$

Now, let us to prove some results of symmetric and skew-symmetric matrices.

Theorem For a square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.
Proof Let $B = A + A'$, then

$B' = (A+A')'$
$= A'+(A')'$ (as $(A+B)'=A'+B'$) $= A'+A$ (as $(A')'=A$)
$= A+A'$ (as $A+B=B+A$)
$=B$

Therefore, $B= A+A'$ is asymmetric matrix

Now let, $C= A-A'$
$C'= (A-A')'=A'-(A')'$

$= -(A-A')=-C$
$C = A - A'$ is a skew symmetric matrix.

Theorem 2 Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Proof Let us take A as a square matrix, then we can represent A as

$A = {}^{1/2}(A' + A) + {}^{1/2}(A' - A)$

We are aware that $(A + A')$ is a symmetric matrix and $(A - A')$ is askewsymmetricmatrix.SinceforanymatrixA,$(kA)'=kA'$,it means that ${}^{1/2}(A+A')$is symmetric matrix and ${}^{1/2}(A - A')$ is skew symmetric matrix. Thus, it is possible to express any square matrix as the sum of a symmetric and a skew symmetric matrix.

7e. Elementary Operation (Transformation) of a Matrix

There are notably six operations or transformations on a matrix, three of them which are due to rows and the others which are due to columns, they are known as elementary operations or transformations.

Interchanging any of the two rows or two columns. Symbolically the interchange of i^{th} and j^{th} rows is represented by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_2$ to $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \end{bmatrix}$, we get $\begin{bmatrix} -1 & \sqrt{3} & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}$.

Multiplying the elements of any row or column by a non zero number. Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$.

The corresponding column operation is represented by $C_i \to kC_i$

For example, applying $C_3 \to \frac{1}{7}C_3$, to $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 & \frac{1}{7} \\ -1 & \sqrt{3} & \frac{1}{7} \end{bmatrix}$

Adding the elements of a row or column, the corresponding

elements of any other row or column, which is multiplied by any non-zero number. Symbolically, the addition to the elements of i^{th} row, the corresponding elements

of j^{th} row multiplied by k is represented by $R \to R + kR$.

The corresponding column operation is represented by $C_i \to C_i + kC_j$.

For example, applying $R_2 \to R_2 - 2R_1$, to $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$.

7f. Invertible Matrices

Definition 6 Let's assume that A is a square matrix of order m, and there exists another square matrix B of the same order m, such that AB = BA = I, then B is known as the inverse matrix of A and it is denoted by A^-1. Then A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now $$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Inverse of a matrix by elementary operations

Let us assume that X, A and B be matrices of, the same order in such a way that X = AB. For applying a sequence of elementary row operations on the matrix equation X = AB, we shall apply these row operations on a simultaneous basis on X and in the initial matrix A of the product AB on RHS.

Similarly, for applying a sequence of elementary column operations on the matrix equation X = AB, we shall apply, these operations on a simultaneous basis on X and in the second matrix B of the product RHS on AB.

In view of the above discussion, let's conclude that if A is a matrix such that A^{-1} exists, then for finding A^{-1} using elementary row operations, write A = IA and by applying a sequence of row operation on A = IA till we get, I = BA. The matrix B happens to be the inverse of A. Similarly, if we want to find A^{-1} using column operations, then, write A = AI and then, by applying to a sequence of column operations on A = AI till we get, I = AB.

A^{-1} do not exist if in case, after applying one or more elementary row (column) operations on A = IA (A = AI) and in one or more rows of the matrix A on L.H.S. we end up getting all zeros.

7g. Determinants and it's Properties

Foreach square matrix A = [a_{ij}] of order n, we can relate a number (real or complex) known as a determinant of the square matrix A, where a_{ij}= (i, j)thelement of A.

This may considered as a function which relates each square matrix with a unique number (real or complex). IfMhappens to be is the set of square matrices, K is the set of numbers (real or complex) and f : M → K is defined by f (A) = k, where A ∈ M and k ∈ K, then f (A) is known as the determinant of A. It is also denoted by | A | or det A or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

Remarks

i. For matrix A, | A | is interpreted as a determinant of A and not modulus of A.
ii. Only square matrices consists determinants.

Determinant of a matrix of order one
Let A = [a] be the matrix of order 1, then determinant of A is equal to a

Determinant of a matrix of order two

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ be a matrix of order 2×2,

then the determinant of A is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example:

If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2\ 4\ 5],\ B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

Determinant of a matrix of order 3 × 3

Determinant of a matrix of order three can be found out by expressing it in terms of second order determinants. This refers to the expanding a determinant along a row (or a column). There exists six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below.

Let's consider the determinant of square matrix $A = [a_{ij}]_{3 \times 3}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Row (R_1)

Step 1 Firstly let's multiply first element a_{11} of R_1 by $(-1)^{(1 + 1)}$ $[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant which we got by deleting the elements of row one(R_1) and column 1(C_1)of|A| as a_{11} liesinR_1 andC_1, i.e.,

$$(-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Step 2 Multiply the 2nd element a_{12} of R_1 by $(-1)^{1 + 2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and the second order determinant which e get by deleting elements of first row 1(R_1) and column 2(C_2) of |A| as a_{12} lies in R_1 andC_2, i.e.,

$$(-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Step 3 Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ [$(-1)^{\text{sum of suffixes in } a_{13}}$] and the second order determinant which we get by deleting elements of first row (R_1) and third column (C_3) of |A| as a_{13} that lies in R_1 and C_3, i.e.,

$$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Step 4 The expansion of determinant of A, that is, | A | written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$\det A = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23}) + a_{13} (a_{21} a_{32} - a_{31} a_{22})$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{32} a_{23} - a_{12} a_{21} a_{33} + a_{12} a_{31} a_{23} + a_{13} a_{21} a_{32} - a_{13} a_{31} a_{22} \quad \ldots$$

By applying the four steps together,

Expansion along second row (R_2)

Expansion along second row (R_2)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_2, we get

$$|A| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+(-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= - a_{21} (a_{12} a_{33} - a_{32} a_{13}) + a_{22} (a_{11} a_{33} - a_{31} a_{13})$$

$$- a_{23} (a_{11} a_{32} - a_{31} a_{12})$$

$$|A| = - a_{21} a_{12} a_{33} + a_{21} a_{32} a_{13} + a_{22} a_{11} a_{33} - a_{22} a_{31} a_{13} - a_{23} a_{11} a_{32}$$

$$+ a_{23} a_{31} a_{12}$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$

$$- a_{13} a_{31} a_{22} \quad \text{... (2)}$$

Expansion along first Column (C_1)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along C_1, we get

$$|A| = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31} (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{21} (a_{12} a_{33} - a_{13} a_{32}) + a_{31} (a_{12} a_{23} - a_{13} a_{22})$$

$$
\begin{aligned}
|A| &= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{21}\,a_{12}\,a_{33} + a_{21}\,a_{13}\,a_{32} + a_{31}\,a_{12}\,a_{23} \\
&\quad - a_{31}\,a_{13}\,a_{22} \\
&= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} \\
&\quad - a_{13}\,a_{31}\,a_{22} \qquad \ldots (
\end{aligned}
$$

Clearly, the values of | A | in (1), (2) and (3) are equal. It is left as an exercise for the reader to verify that the values of |A| by expanding R_3, C_2 and C_3 that equals the

value of | A | that we obtain in (1), (2) or (3).Hence, expanding a determinant along any row or column gives same value.

Remarks

(i) For ease in calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

(ii) While we expand, instead of multiplying by $(–1)^{i + j}$, we can multiply by +1 or –1 according as (i + j) is even or odd.

(iii) $\text{Let } A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ Then we can easily verify that A=2B. | A | = 0 – 8 = -8 and | B | = 0 – 2 = -2. Note that, |A| = 4(– 2) = 2^2|B| or |A| = 2^n|B|, where n = 2 is the order of square matrices A and B.

if A = kB where A and B happens to be square matrices of order n, then | A| = k^n | B |, where n = 1, 2, 3

Properties of Determinants

In the previous section, we have discussed on how to expand the determinants. In this section, we shall learn some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros in a row or a column. These properties holdstrue for determinants of any order. However, we will restrict ourselves upto determinants of order 3 only.

Property 1 The value of the determinant remains unaltered if its rows and columns are interchanged.

Property 2 If any two rows (or columns) of a determinant are interchanged, then sign of determinant also gets altered.

Property 3 If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant amounts to zero.

Property 4 If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value is multiplied by k.

Property 5 If some or all elements of a row or column of a determinant are expressed as sum of two or more terms, then it is possible to express a determinant as sum of two (or more) determinants.

Property 6 The value of determinant is the same if to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

7h. Matrix Solution for Area of a Triangle

Earlier, we have studied that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression

$\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$. Now this expression can be represented in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

7i. Minors, Cofactors, Adjoint and Inverse of a Matrix

Minors and Cofactors

In this section, we shall learn to write the expansion of a determinant in compact form with the help of minors and cofactors.

Definition-Minor of an element $a_{ij}$of a determinant is the determinant which we obtain by deleting its ith row and jth column in which element a_{ij}lies. Minor of an element $a_{ij}$is represented by M_{ij}.

Remark The Minor of an element of a determinant of order n(n ≥ 2) is a determinant of order n – 1.

Adjoint and Inverse of a Matrix

In the last chapter, we have studied about the inverse of a matrix. In this section, we shall learn the condition for existence of inverse of a matrix.

To find inverse of a matrix A, i.e., A^{-1} we shall first define adjoint of a matrix.

Adjoint of a matrix

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is the transpose of the matrix $[A_{ij}]_{n \times n}$, where $A_{ij}$is the cofactor of the element a_{ij}. Adjoint of the matrix A is represented by adj A.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $$adj\,A = \text{Transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Theorem If A be any given square matrix of order n, then

$A(adj\ A) = (adj\ A)\,A = |A|I$, where I refers to the identity matrix of order n.

Theorem If A and B are non-singular matrices of the same order, then AB and BA are also considered as non-singular matrices of the same order.

Theorem The determinant of the product of matrices equals the product of their respective determinants, that is, AB = A B , where A and B are square matrices of the same order.

Theorem 4 A square matrix A is invertible if A is nonsingular matrix.

7j. Determinants and Matrix Applications

Applications of Determinants and Matrices

In this section, we shall learn about the application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations can said to be consistent if it has a solution (one or more)

Inconsistent system A system of equations is said to be inconsistent if it does not have a solution.

Solution to the system of linear equations using inverse of a matrix
Let us express the system of linear equations as matrix equations and try solving them using inverse of the coefficient matrix.

Consider the system of equations,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\text{Let} \qquad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

AX = B is how the system of equations can be written. i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I The inverse of a matrix exists If matrix A is a nonsingular matrix.

Now

$$\begin{array}{llr} & AX = B & \\ \text{or} & A^{-1}(AX) = A^{-1}B & \text{(premultiplying by } A^{-1}) \\ \text{or} & (A^{-1}A)X = A^{-1}B & \text{(by associative property)} \\ \text{or} & IX = A^{-1}B & \\ \text{or} & X = A^{-1}B & \end{array}$$

This matrix equation provides us with unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is termed as Matrix Method.

Case II Let A be a singular matrix, then |A| = 0.

Here, we calculate (adj A) B.

If (adj A) B $\neq$ O, (O being zero matrix), then solution does not exist and the system of equations is called an inconsistent one.

If (adj A) B = O, then system may be either consistent or inconsistent according as the system consists of either infinitely solutions or no solution.

Example Solve the system of equations 2x + 5y = 1 ; 3x + 2y = 7

Solution The system of equations can be represented in the form AX = B, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Therefore, A is nonsingular matrix and so has a unique solution.

Note that, $$\mathbf{A}^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence, $x = 3$, $y = -1$

[redacted]Calculus

8a. Introduction & Intuition behind derivatives

This chapter is an introduction to Calculus. Calculus is a branch of mathematics that explains the change in the value of a function as the points in the domain change. First, let’s give an intuitive idea of derivative (without actually defining it). Then we provide a naive definition of limit and study some algebra of limits. Then we come back to a definition of derivative and learn some algebra of derivatives. We also obtain derivatives of certain standard functions.

Intuitive Idea of Derivatives

Substantial experiments have revealed that any body dropped from a tall hill covers a distance of $4.9t^2$ metres in t seconds, i.e., distance s in metres covered by

the body as a function of time t in seconds is denoted by s = 4.9t^2.
The adjoining Table 13.1 shows the distance travelled in metres at various intervals of time expressed in seconds of a body dropped from a tall cliff. The objective is to find the velocity of the body at time t = 2 seconds from this data. One of the methods to approach this problem is to find the average velocity for various intervals of time ending at t = 2 seconds and hope that these bring some clarity on the velocity at t = 2 seconds.

The average velocity between $t = t_1$ and $t = t_2$ equals distance travelled between t = t1 and t = t2 seconds divided by (t2 – t1). Therefore the average velocity in the first two seconds,

= Distance covered between $t_2 = 2$ and $t_1 = 0$ / Time interval (t2 – t1)

= (19.6-0)m / (2-0)s = 9.8 m/s.

In the same way, the average velocity between t = 1 and t = 2 is

= (19.6-4.9)m / (2-1)s = 14.7 m/s.

Likewise we compute the average velocity between $t = t_1$ and t = 2 for various t_1. The following Table 13.2 shows the average velocity (v), $t = t_1$ seconds and t = 2 seconds.

t	s
0	0
1	4.9
1.5	11.025
1.8	15.876
1.9	17.689
1.95	18.63225
2	19.6
2.05	20.59225
2.1	21.609
2.2	23.716
2.5	30.625
3	44.1
4	78.4

t_1	0	1	1.5	1.8	1.9	1.95	1.99
v	9.8	14.7	17.15	18.62	19.11	19.355	19.551

From the above table we see that the average velocity is slowly increasing. As we make the time intervals ending at t = 2 smaller, we observe that we get a better idea of the velocity at t = 2. Trusting that nothing really intense happens between 1.99 seconds and 2 seconds, we can conclude that the average velocity at t = 2 seconds is just above 19.551m/s.

This conclusion is somewhat cemented by the following set of computation. Computing the average velocities for various time intervals starting at t = 2 seconds. The average velocity v in between t = 2 seconds and t = t_2 seconds is

= Distance covered between 2 seconds and t_2 seconds / t−2

= Distance covered in t_2 seconds − Distance commuted in 2 seconds / t2 – 2

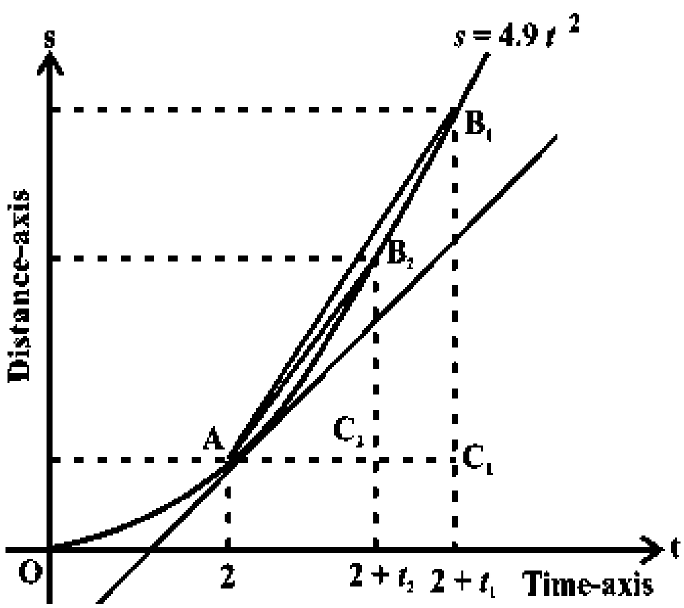
= Distance commuted in t2 seconds − 19.6 / t2 - 2

The following Table 13.3 shows the average velocity v in metres per second between t = 2 seconds and t2 seconds.

t_2	4	3	2.5	2.2	2.1	2.05	2.01
v	29.4	24.5	22.05	20.58	20.09	19.845	19.649

Here again we observe that if we take smaller time intervals starting at t = 2, we get better idea of the velocity at t = 2.

In the first set of calculations, what we have found the average velocities in increasing time intervals ending at t = 2 and then hope that nothing dramatic happens just before t = 2. In the second set of calculations, we have found out the average velocities decreasing in time intervals ending at t = 2 and then hope that nothing historic happens just after t = 2. Purely on physical basis, both these sequences of average velocities must approach a common limit. We can conclude that the velocity of the body at t = 2 is between 19.551m/s and 19.649 m/s. Technically, we can state that the instantaneous velocity at t = 2 is between 19.551 m/s and 19.649 m/s. As is already known, velocity is the rate of change of displacement. Hence what we have achieved is the following. From the available data of distance covered at various time instants we have estimated the rate of change in the distance at a given instant of time. We can say that the derivative of the distance function which is $s = 4.9t^2$ at t = 2 is between 19.551 and 19.649.



An alternate way for seeing this limiting process is shown in Fig above. This is a plot of distance s of the body from the top of the cliff to the time t elapsed. In the limit as the sequence of time intervals h_1, h_2, ..., reaches zero, the sequence of average velocities approaches the same limit as does the sequence of ratios

$$\frac{C_1B_1}{AC_1}, \frac{C_2B_2}{AC_2}, \frac{C_3B_3}{AC_3}, \ldots$$

WhereC B =s –s is the distance covered by the body in the time interval h1 =AC1 , etc. From the Fig 13.1 we can conclude that this latter sequence approaches the slope of the tangent to the curve at point A. In other words, the instantaneousvelocity
v(t)ofabodyattimet=2isequaltotheslopeofthetangentofthecurves=4.9t^2 at t = 2.

8b. Limits and Derivatives

Limits

The above discussion takes us to the fact that we need to understand limiting process in greater clarity. Let us study a few illustrative examples for gaining some familiarity with the concept of limits.
Consider the function f(x) = x^2 . Note that as x takes values very close to 0, the value of f(x) also moves towards 0. We say,

$$\lim_{x\to 0} f(x) = 0$$

(to be read as the limit of f (x) as x tends to zero equals zero). The limit of f (x) as x tends to zero has to be kept in mind as the value f (x) should assume at x = 0.
As x → a, f (x) → l, then l is known as the limit of the function f (x) which is

symbolically written as $\lim_{x\to a}$ f (x)= l . Taking into account the following function g(x) = |x|, x ≠ 0. Note that g(0) is not defined. Computing the value of g(x) for values of x very near to 0, we observe that the value of g(x) moves towards 0. So, lim x→0g(x) = 0. This is intuitively clear from the graph of y = |x| for x ≠ 0.

Consider the following function.

$$h(x)=\frac{x^2-4}{x-2},\ x\neq 2$$

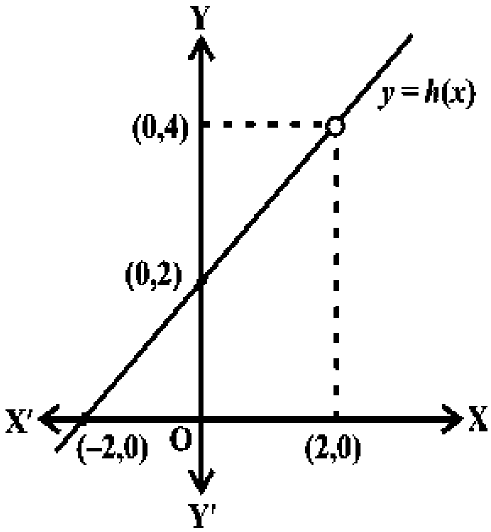
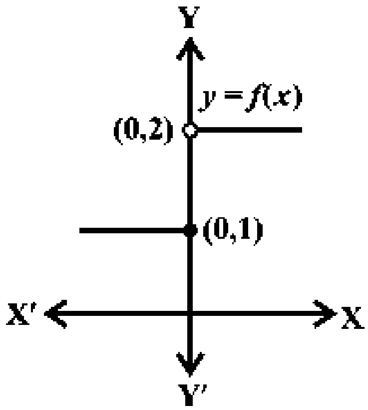


Fig 13.2

Computing the value of h(x) for values of x very near to 2 (but not at 2). Make known to yourself that all these values are near to 4. This is somewhat cemented by considering the graph of the function y = h(x) given here (Fig 13.2).

In all these illustrations the value , the function should assume at a given point x = a did not really depend on how is x tending to a. Note that there are essentially two methods by which x could approach a number a either from left or from right, i.e., all the values of x near a can be less than a or can also be greater than a. This results to two limits – the right hand limit and the left hand limit. The right hand limit of a function f(x) is that value of f(x) which is dictated by the values of f(x) when x tends to a from the right. Similarly, the left hand limit. To illustrate this, consider the function

$$f(x)=\begin{cases}1, & x\leq 0\\ 2, & x>0\end{cases}$$



Graph of this function is depicted in the Fig 13.3. It is evident that the value of f at 0 dictated by values of f(x) with x ≤ 0 equals 1, i.e., the left hand limit of f (x) at 0 is

$$\lim_{x \to 0^-} f(x) = 1$$

Similarly, the value of f at 0 that is stated by values of f (x) with x > 0 equals 2, i.e., the right hand limit of f (x) at 0 is

$$\lim_{x \to 0^+} f(x) = 2$$

In this case the right and left hand limits are different, and therefore we say that the limit of f (x) as x tends to zero doesn’texist (even though the function is defined at 0).

Algebra of Limits

Theorem Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist.

Then

(i) The limit of the sum of two functions happens to be the sum of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

(ii) Limit of the difference of two functions is the difference of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x).$$

(iii) Limit of product of two functions happens to be the product of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) . g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x).$$

(iv) The limit of quotient of two functions is the quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

In particular as a special case of (iii), when g happens to be the constant function such that g(x) = λ , for some real number λ , we have

$$\lim_{x \to a} \left[(\lambda . f)(x) \right] = \lambda . \lim_{x \to a} f(x) .$$

8c. Limits of functions

Limits of rational functions and polynomials

A function f can be said to be a polynomial function of degree n f(x) = a0 + a1x + a2x2 +. . . + anxn, where a is a real number such that an ≠ 0 for some natural number n.

We know that limx = a. Hence,

$$\lim_{x\to a} x^2 = \lim_{x\to a}(x.x) = \lim_{x\to a} x.\lim_{x\to a} x = a.\,a = a^2$$

An easy exercise regarding induction on n tells us that

$$\lim_{x\to a} x^n = a^n$$

Now, let f (x)= $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial function. Taking each of $a_0, a_1 x, a_2 x_2, ..., a_n x_n$ as a function, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[a_0 + a_1x + a_2x^2 + ... + a_nx^n\right]$$

$$= \lim_{x\to a} a_0 + \lim_{x\to a} a_1x + \lim_{x\to a} a_2x^2 + ... + \lim_{x\to a} a_nx^n$$

$$= a_0 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + ... + a_n \lim_{x\to a} x^n$$

$$= a_0 + a_1a + a_2a^2 + ... + a_na^n$$

= f (a)

(Ensure that you understand the justification for each step in the above!)

A function f is a rational function, if f(x)= $\dfrac{g(x)}{h(x)}$ where g(x) and h(x) are polynomials such that $h(x) \neq 0$. Then,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, there exists two scenarios – (i) when $g(a) \neq 0$ and (ii) when $g(a) = 0$. In the former case we say that the limit doesn't exist. In the latter case we can represent $g(x) = (x - a)^k g_1(x)$, where k is the maximum of powers of (x – a) in g(x) Similarly, $h(x)=(x-a)^l h(x)$ as $h(a)=0$. Now, if $k>l$, we have,

$$\lim_{x\to a} f(x) = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{\lim_{x\to a} (x-a)^k g_1(x)}{\lim_{x\to a} (x-a)^l h_1(x)}$$

$$= \frac{\lim_{x\to a} (x-a)^{(k-l)} g_1(x)}{\lim_{x\to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If $k < l$, the limit has not been defined.

Example - Find the limits:

(i) $\lim_{x\to 1}\left[x^3 - x^2 + 1\right]$ (ii) $\lim_{x\to 3}\left[x(x+1)\right]$

(iii) $\lim_{x\to -1}\left[1 + x + x^2 + \ldots + x^{10}\right]$

Solution The required limits are the limits of some polynomial functions. Therefore, the limits are the values of the function at the prescribed points. We have,

(i) $\lim\limits_{x \to 1} [x^3 - x^2 + 1] = 1^3 - 1^2 + 1 = 1$

(ii) $\lim\limits_{x \to 3} \left[x(x+1)\right] = 3(3+1) = 3(4) = 12$

(iii) $\lim\limits_{x \to -1} \left[1 + x + x^2 + \ldots + x^{10}\right] = 1 + (-1) + (-1)^2 + \ldots + (-1)^{10}$

$= 1 - 1 + 1 \ldots + 1 = 1$

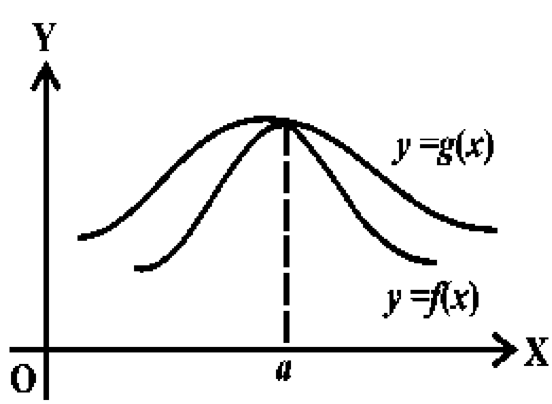
Theorem For any positive integer n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

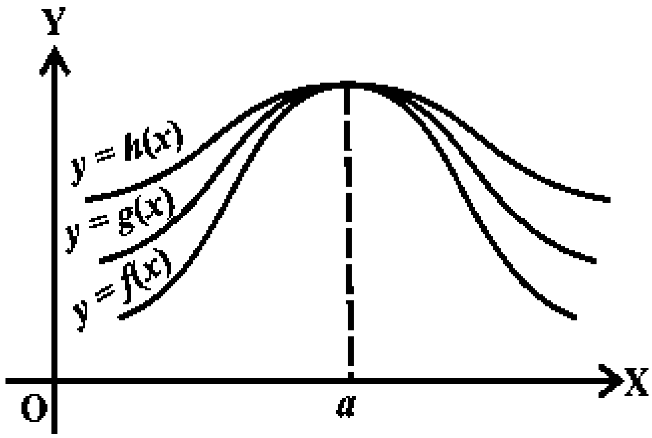
Limits of Trigonometric Functions

The following facts (stated as theorems) about functions , are helpful for calculating limits of some trigonometric functions.

Theorem Let's assume f and g as the two real valued functions with the same domain such that f (x) ≤ g(x) for all x in the domain of definition, For some a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$. This is illustrated in Fig below.



Theorem (Sandwich Theorem) Let f, g and h be the real functions such that f (x) ≤ g(x) ≤ h(x) for all x in the common domain of definition. For some real number a, if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$. This is illustrated in Fig.



Shown below is a geometric proof of the following important inequality relating trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2}$$

Theorem 5 - The following are the two important limits.

$$\text{(i)} \lim_{x\to 0} \frac{\sin x}{x} = 1 . \qquad \text{(ii)} \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Derivatives

We have seen in the previous section, that by knowing the position of a body at different time intervals it is possible to find out the rate at which the position of the body changes. It is of a general interest to know a specific parameter at various time instants and try to find the rate at which it is changing. There are many real life scenario where such a process should be carried out. For example, the government body managing or maintaining a dam need to know when will the dam overflow knowing the depth of the water at several instances of time, Rocket Scientists needs to compute the exact velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times.

Financial institutions should make a prediction of the changes in the value of a particular stock on the basis of its present value. In such cases it is always better to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is the derivative of a function at a given point in its domain of definition.

Definition–Let's assume that f is a real valued function and a is a point in its domain of definition. The derivative of f at a can be defined by,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. Derivative of f (x) at a is represented by f′(a). Note that f′ (a) quantifies the change in f(x) at a with respect to x.

Example –Find out the derivative at x = 2 of the function f(x) = 3x.
Solution - We have,

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{3(2+h) - 3(2)}{h}$$

$$= \lim_{h \to 0} \frac{6 + 3h - 6}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

The derivative of function 3*x* at *x* = 2 is 3.

Definition 2 –Let's assume that f is a real valued function, the function defined by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists, it is defined as the derivative of f at x and is denoted byf′(x). This definition of derivative is also known as the first principle of derivative.

Thus, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Clearly the domain of definition of f′ (x) is wherever the above limit do exist. There are various notations for derivative of a function. Sometimes f′(x) is represented by

$\frac{d}{dx}(f(x))$ or if y = f(x), it is denoted by $\frac{dy}{dx}$. This is referred to as derivative of f(x) or y with respect to x. It is also represented by D (f (x)). Further, derivative of f at x = a is also represented by

$$\frac{d}{dx} f(x) \Big|_a \text{ or } \frac{df}{dx}\Big|_a \text{ or even } \left(\frac{df}{dx}\right)_{x=a}$$

Algebra of derivative of functions : We can expect the rules for derivatives to follow closely for that of limits as the definition of derivatives involve limits in a direct fashion, We could gather these in the following theorem.

Theorem 5 Let’s assume f and g to be two functions such that their derivatives are defined in a common domain. Then,

i. Derivative of sum of two functions is the sum total of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

ii. Derivative of difference of two functions happens to be the difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

iii. Derivative of product of two functions is represented by the following product rule.

$$\frac{d}{dx}\left[f(x)\ .\ g(x)\right]=\frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$$

iv. Derivative of quotient of two functions is represented by the following quotient rule (whenever the denominator is non–zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}f(x).g(x)-\ f(x)\ \frac{d}{dx}g(x)}{\left(g(x)\right)^2}$$

The last two statements in the theorem may be stated in the following fashion which aids in recalling them easily:

Let u= f(x) and v=g(x). Then,

$$(uv)' = u'v+uv'$$

This is known as a Leibnitz rule for differentiating product of functions or the product rule.

Theorem -Derivative of f(x) = x^n is nx^{n-1} for any positive integer n.

Derivative of trigonometric functions and polynomials Let us start with the following theorem which tells us the derivative of a polynomial function.

Theorem – Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ be a polynomial function, where a_is are all real numbers and $a_n \neq 0$. Then, the derivative function is represented by,

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{x-2} + ... + \ 2a_2 x + a_1.$$

Example - Calulate the derivative of $6x^{100} - x^{55} + x$.
Solution The direct application of the above theorem tells us that the derivative of the above function is $600x^{99} - 55x^{54} + 1$.

Example – Find out the derivative of $f(x)=1+x+x^2+x^3+...+x^{50}$ at x=1.

Solution A direct application of the above Theorem 6 tells that the derivative of the above function is 1+2x+3x +...+50x. At x=1 the value of this function equals

$$1 + 2(1) + 3(1)^2 + \ldots + 50(1)^{49} = 1 + 2 + 3 + \ldots + 50 = \frac{(50)(51)}{2} = 1275.$$

8d. Continuity and Differentiability

Introduction:

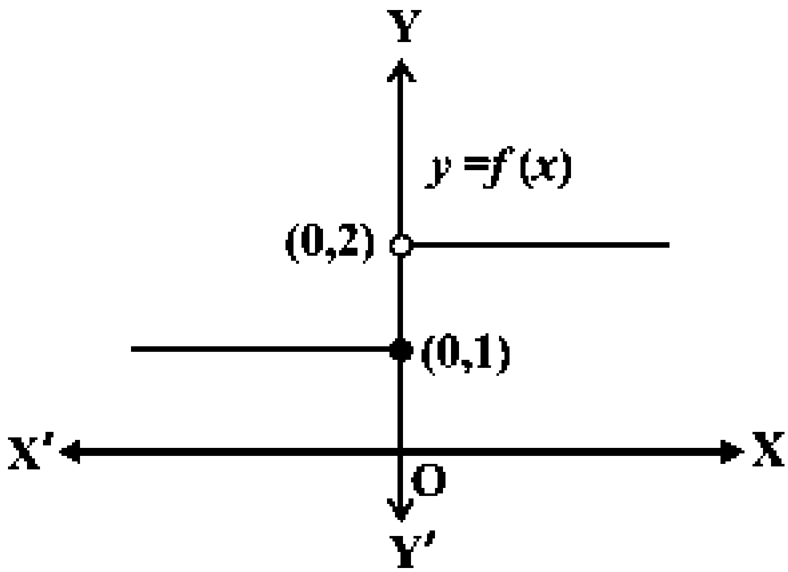
In this chapter, we shall introduce the concepts of continuity, differentiability and explain the relations between them. We will also shall learn differentiation of inverse trigonometric functions. Further, we shall introduce a new class of functions known as exponential and logarithmic functions. These functions results to powerful techniques of differentiation. Let us illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.

Continuity

We commence the section with two informal examples to get a feel of continuity. Consider the function,

$$f(x) = \begin{cases} 1, \text{if } x \le 0 \\ 2, \text{if } x > 0 \end{cases}$$

This function is defined at every point of the real line. Graph of this function is represented in the Fig 5.1. One can sense from the graph that the value of the function at nearby points on x-axis are close to each other except at x = 0. At the points that are near to and to the left of 0, i.e., at points like – 0.1, – 0.01, – 0.001, the value of the function is 1. At the points which are near and to the right of 0, i.e., at points like 0.1, 0.01,

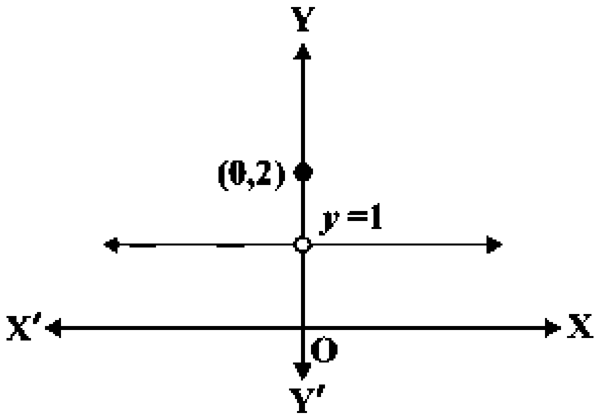


0.001, the value of the function is 2. Using the language of left and right hand limits, we can say that the left (respectively right) hand limit of f at 0 is 1 (respectively 2). In particular the left and right hand limits does not coincide. We also note that the value of the function at x = 0 concides with the left hand limit. Note that when we try to construct the graph, we cannot draw it in one stroke, i.e., without lifting the pen from the plane of the paper, we cannot draw the graph of this function. In fact, we must lift the pen when we come to 0 from left. This is one of the instance of function being not continuous at x = 0.

Let us, consider the function defined as

$$f(x) = \begin{cases} 1, \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$$

This function is defined at every point. Left and the right hand limits at x = 0 both equals 1. But the value of thefunction at x = 0 equals 2 which doesn't coincide with the common value of the left and right hand limits. Again, we observe that we cannot draw the graph of the function without lifting the pen. This is yet another example of a function being not continuous at x = 0.



Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around the same, without lifting the pen from the surface of the paper.

Mathematically, it may be phrased as follows:

Definition Let us suppose that f is a real function on a subset of the real numbers and let c be

a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at x = c exist and are equal to each other, then f is continuous at x = c. Let's recollect that if the right hand and left hand limits at x = c coincide, then we say that the common value is the limit of the function at x = c. Hence we can rephrase the definition of continuity as: a function is continuous at x = c if the function is defined at x = c and if the value of the function at x = c is equal to the limit of the function at x = c. If f is not continuous at c, we can say f is discontinuous at c and c is called a point of discontinuity of f.

Example Check the continuity of the function f that is given by f (x) = 2x + 3 at x = 1.

Solution First observe that the function is defined at the given point x = 1 and its value is 5. Then find out the limit of the function at x = 1. Clearly;

$$\lim_{x \to 1} f(x) = \lim_{x \to 1}(2x+3) = 2(1)+3 = 5$$

$$\lim_{x \to 1} f(x) = 5 = f(1)$$

Hence, f is continuous at x = 1.

Definition A real function f is continuous if it is continuous at every point in the domain of f.

The above definition requires a bit of elaboration. Let's suppose f is a function defined on a closed interval [a, b], then for f to be continuous, it needs to be continuous at every point in [a, b] including the end points a and b. Continuity of f at a means

$$\lim_{x \to a^+} f(x) = f(a)$$

and continuity of f at b means

$$\lim_{x \to b^-} f(x) = f(b)$$

Note that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$ do not make sense. As a result of this definition, if f is defined only at one point, it is continuous there, i.e., if the domain of f is a singleton, f is a continuous function.

Example -Is the function defined by f (x) = | x |, a continuous function?

Solution We may rewrite f as

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

We know that by previous example f is continuous at x = 0. When c < 0 then c be a real number. Then f (c) = – c. Also,

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c$$

f is present at all negative real numbers and since limx→c f (x) = f (c)

Let's now assume that c be a real number such that c > 0. Then f (c) = c. Also,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

f is continuous at all points since limx→c f (x) = f (c), as f is continuous for all positive real numbers.

Algebra of continuous functions

In the previous class, after having understood the concept of limits, we have studied some algebra of limits. Now we shall study some algebra of continuous functions. Since the continuity of a function at a point is dictated by the limit of the function at that point, we can expect results analogous to the case of limits.

Theorem - Suppose f and g be two real functions continuous at a real number c.

Then

(1) f+giscontinuousatx=c.

(2) f–giscontinuousatx=c.

(3) f.giscontinuousatx=c.

(4) Provided g(c) ≠0), f/g is continuous at x = c.

Differentiability

Let us recall the following facts from previous class. We defined the derivative of a real function as follows:

Let us suppose that *f*is a real function and *c* is a point in its domain. The derivative of *f*at *c* is defined by

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit do exist. Derivative of *f* at *c* is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$. Thefunction defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is the derivative of *f*. The derivative of *f* is denoted by

$$f'(x) \text{ or } \frac{d}{dx}(f(x)) \text{ or if } y = f(x) \text{ by } \frac{dy}{dx} \text{ or } y'.$$

The process for finding derivative of a function is known as differentiation. We also utilise the phrase *differentiate* $f(x)$ *with concerning* x to mean *find* $f'(x)$.

The following rules weremade as a part of algebra of derivatives:

(1) $(u \pm v)' = u' \pm v'$

(2) $(uv)' = u'v + uv'$ (Leibnitz or product rule)

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule).

The following table shows a list of derivatives of certain standard functions:

$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Whenever we defined derivative, we had made it known about the limits provided it exists. Now the question is; what if it doesn't? and so is its answer. If

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

does not exist, we state that a function *f* is differentiable at a point *c* in its domain if both

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \text{ and } \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$

are finite and equal.Afunctionissaid

A function is differentiable in an interval [a, b] if it is differentiable at each point of [a, b]. As in case of continuity, at the end points a and b, we consider the right hand limit and left hand limit, which are left side derivative and right side derivative of the function at a and b respectively. Similarly, a function is differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

Theorem - If a function f is differentiable at a point c, then it is continuous at that point.

Corollary 1 Every differentiable function happens to be continuous.

We say that the converse of the above statement does not hold true. Indeed we have seen that the function defined by f (x) = | x | is a continuous function. Consider the left hand limit.

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\frac{-h}{h}=-1$$

The right hand limit

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\frac{h}{h}=1$$

Since the above left and right side limits at 0 does not equal to each other

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

does not exist and therefore f is not differentiable at 0. Thus f does not happen to be a differentiable function.

Derivatives of composite functions

For studying the derivative of composite functions, we start with an illustrative example. Let's say we want to find the derivative of f, where

$f(x) = (2x + 1)^3$

One way is to expansion of $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3+12x^2+6x+1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\,(2x + 1)^2$$

Now, observe that

$$f(x) = (h \circ g)\,(x)$$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\,(2x+1)^2 = 3(2x+1)^2 \,.\, 2 = 3t^2 \,.\, 2 = \frac{dh}{dt}\cdot\frac{dt}{dx}$$

The advantage with this kind observation is that it simplifies the calculation in finding 100

the derivative of, say, (2x + 1) . We will be able to formalise this observation in the following theorem called the chain rule.

Theorem (Chain Rule) Let f be the real valued function which is a composite of two functions u and v; i.e., f = v o u. Suppose t = u(x) and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist,we have,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

We shall skip the proof of this theorem. Chain rule can be extended as follows. Let us suppose that f is a real valued function which happens to be a composite of three functions u, v and w ; i.e.,

f = (w o u) o v. If t is equal to v (x) and s = u (t), then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is free to formulate chain rule for composite of more functions.

Example Find out the derivative of the function given by f (x) = sin (x^2).
Solution:-Note that the given function is a composite of two functions. Indeed, if t=u(x)=x^2 and v(t)=sint, then

$$f(x) = (v \circ u)(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put t = u(x) = x^2 . Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist.

Hence, by chain rule,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is the most known practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly continue as follows:

$$y = \sin(x^2) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2)$$

$$= \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2$$

Example Find out the derivative of tan (2x + 3).

Solution Let f(x) = tan (2x + 3), u(x) = 2x + 3 and v(t) = tan t. Then (v o u) (x) = v(u(x)) = v(2x + 3) = tan (2x + 3) = f (x)

Therefore f happens to be a composite of two functions. Put t=u(x)= 2x+3. Then

$\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2(2x+3)$$

Derivatives of implicit functions

All this while we have been differentiating various functions given in the form y = f (x). However it need not be that functions are always expressed in this form. For instance, let us consider one of the following relationships between x and y:

$$x - y - \pi = 0$$

$$x + \sin xy - y = 0$$

Firstly we can solve for y and represent the relationship as y = x – π. In the second case, it does not appear that there is an easy way to solve for y. However, there is no doubt about the dependence of y on x in both cases. When a relationship between x and y is expressed in such a way that it is easy to solve for y and write y = f (x), we say that y is represented an explicit function of x. In the second case it is implicit that y happens to be a function of x and we say that the relationship of the second type, above, gives function implicitly. In this subsection, we shall learn to differentiate implicit functions.

Example – Find $\frac{dy}{dx}$ if x–y=π.

Solution One of the way is to solve for y and rewrite the above as y=x–π

But then, $\frac{dy}{dx} = 1$

Alternatively, directly bifurcating the relationship w.r.t., x, we have

$$\frac{d}{dx}(x - y) = \frac{d\pi}{dx}$$

Let’s recollect that $\frac{d\pi}{dx}$ means to differentiate the constant function taking value π everywhere w.r.t., x. Thus,

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

Derivatives of inverse trigonometric functions
We remark that inverse trigonometric functions are known as continuous functions, but we shall not prove this. Now we make use of chain rule for finding derivatives of these functions.

Example - Find the derivative of f given by f (x) = $\sin^{-1}$ x assuming it exists.
Solution Let y = $\sin^{-1}$ x. Then, x = sin y.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

Which implies that $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Note that that this is defined only for cos y != 0, i.e.,

$$\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}, \text{ i.e., } x \neq -1, 1$$

, i.e., x ∈ (– 1, 1).

To make this result more attractive, we shall base it on the following manipulation. Let's recollect that for x ∈ (– 1, 1), sin (sin^{-1} x) = x and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

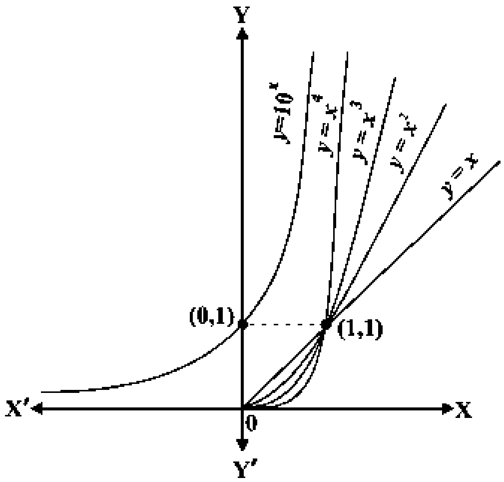
Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ cos y is positive and hence cos $y = \sqrt{1 - x^2}$

Thus, for x ∈ (– 1, 1),

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Exponential and Logarithmic Functions

All this while we have learnt regarding the aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions known as exponential functions and logarithmic functions. It needs to be stressed that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.



The Fig 5.9 depicts a sketch of y = $f_1(x) = x$, y = $f_2(x) = x^2$, y = $f_3(x) = x^3$ 4 and y = $f_4(x)$ = x . Note that the curves get steeper as the power of x increases. The more steeper the curve is, the faster is the rate of growth. This means that for a fixed increment in the value of x(> 1), the increment in the value of y = f_n (x) increases as n increases for n = 1, 2, 3, 4. It is conceivable that such a statement is true for all positive values of n, where f_n (x) = x^n. Essentially, this implies that the graph of y = f_n (x) leans more towards the y-axis as n increases. For example, consider f_{10}(x) = x^10 and f_{15}(x) = x^15 . If x increases from 1 to 2, f_{10} increases from 1 to 2^10 whereas f_{15} increases from 1 to 2^15 . Therefore, for the same increment in x, f_{15} grow faster than f_{10}.

As a result of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater the growth. The next question that arise is: Is there a function that grows faster than any polynomial function. The answer is in affirmative form and an example of such a function is

$$y = f(x) = 10^x.$$

We claim that this function f develops faster than f_n (x) = x^n for any positive integer n. For instance, we can prove that 10^x grows faster than f_{100} (x) = x^100. For large values of x like x=10^3, note that f_{100} (x)=(10^3) ^100 =10^300 whereas f(10^3)=$_{10}$^10^3=10^1000.

Clearly f(x) is much greater than f_{100} (x). It is not tough for proving that for all x > 10^3, f (x) > f_{100} (x). But we shall not attempt to give a proof of this here. Similarly, by choosing the large values of x, one can verify that f (x) grows faster than f_n (x) for any positive integer n.

Definition - The exponential function with positive base b > 1 is the function

$$y = f(x) = b^x$$

The graph of y = 10^x is given in the above Fig.

It is advised that the reader plot this graph for specific values of b like 2, 3 and 4. Following are some of the features of the exponential functions:

1. (1) Domain of the exponential function happens to be**R**, the set of all real numbers.
2. (2) Range of the exponential function is the set of all real numbers that are positive
3. (3) The point (0, 1) is always on the graph of the exponential function (this is a restatement of the fact that b^0 = 1 for any real b > 1).
4. (4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.
5. (5) For large negative values of x, the exponential function happens to be very close to 0. In other words, in the second quadrant, the graph enters x-axis (but never meets it).

Exponential function with base 10 is known as the common exponential function. Earlier, it was noted that the sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots$$

is a number that falls between 2 and 3 and is denoted by e. Using this e as the base we obtain an extremely important exponential function y = e^x.

This is known as natural exponential function.

It would be interesting to note that if the inverse of the exponential function do exist and has nice interpretation. This search motivates the following definition.

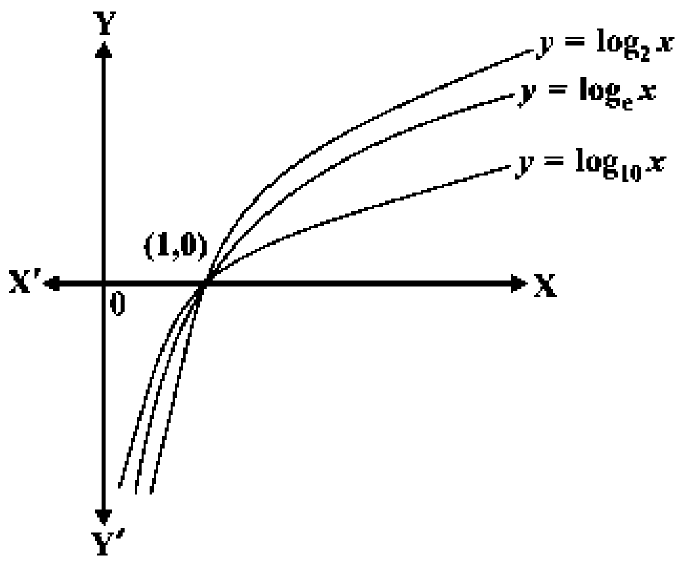
Definition - Let b > 1 be a real number. Then we state that logarithm of a to base b is x if $b^x = a$.

Logarithm of a to base b is represented by $\log_b a$. Thus $\log_b a=x$ if $b^x = a$. Let us work with a few examples. We know that $2^3 = 8$. In terms of logarithms, we may write this as $\log_2 8 = 3$. In the same way, $10^4 = 10000$ is equivalent to saying $\log_{10} 10000 = 4$. Also, $625 = 5^4 = 25^2$ is as good as saying $\log_5 625 = 4$ or $\log_{25} 625 = 2$.

On a more mature note, fixing a base b > 1, we may look at logarithm as a function from positive real numbers to all real numbers. This function, is known as the logarithmic function, is defined by

$$\begin{aligned} \log_b : \mathbf{R}^+ &\to \mathbf{R} \\ x &\to \log_b x = y \quad \text{if } b^y = x \end{aligned}$$

As before if the base b = 10, we can say that it is common logarithms and if b = e, then we say it is natural logarithms. Often natural logarithm is denoted by ln. In this chapter, log x represents the logarithm function to base e, i.e., ln x will be written as simply log x. The Fig 5.10 gives the plots of logarithm function to base 2, e and 10.



Some of the observations about the logarithm function to any base b > 1 are as given below:

(1) We can't arrive at a meaningful definition of logarithm of non-positive numbers and therefore the domain of log function is **R**^+.

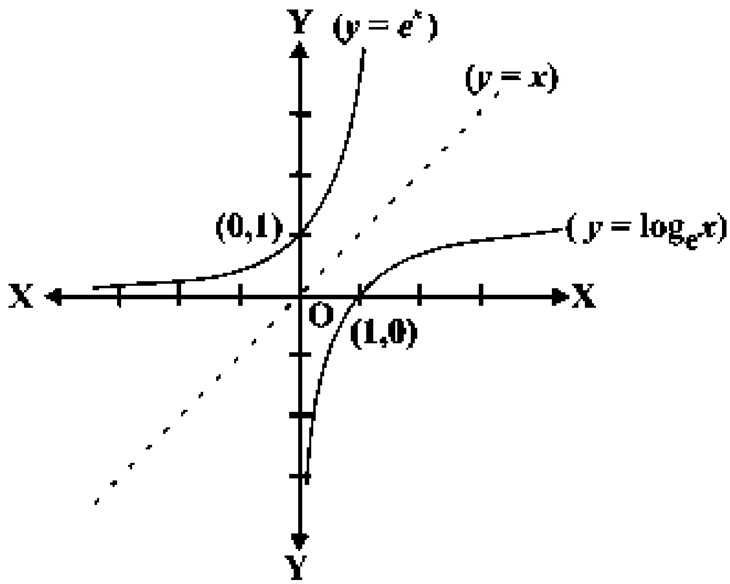
(2) The range of log function happens to be the set of all real numbers.

(3) The point (1, 0) is always located on the graph of the log function.

(4) The log function is always increasing, i.e., as we move from left to right the graph goes above.

(5) For x very near to zero, the value of log x can be lesser than any given real number. In other words the graph approaches y-axis (but never meets it) in the fourth quadrant.

(6) Fig 5.11 shows the plot of $y = e^x$ and y = ln x. We can observe that the two curves happens to be the mirror images of each other reflected in the line y = x.



Two properties of ‘log’ functions are proved as given below:
(1) There is a standard change of base rule for obtaining $\log_a p$ in terms of $\log_b p$.
Let $\log_a p=\alpha$, $\log_b p=\beta$ and $\log_b a=\gamma$. This means $a^\alpha=p$, $b^\beta=p$ and $b^\gamma=a$.By substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Making use of this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

Which implies $\beta = \alpha\gamma \text{ or } \alpha = \dfrac{\beta}{\gamma}$ But then,

$$\log_a p = \frac{\log_b p}{\log_b a}$$

(2) Another intriguing property of the log function is its effect on products. Let $\log_b pq=\alpha$. Then $b^{\alpha} =pq$. If $\log_b p=\beta$ and $\log_b q=\gamma$, then $b^{\beta} =p$ and $b^{\gamma} =q$. But then $b^{\alpha} =pq=b^{\beta}b^{\gamma} =b^{\beta+\gamma}$

which implies $\alpha = \beta + \gamma$, i.e.,

$$\log_b pq = \log_b p + \log_b q$$

A n important consequence of this is when p = q. In this case the above can be reperesented as

$$\log_b p^2 = \log_b p + \log_b p = 2 \log p$$

An easy generalisation of this (left as an exercise!) is

$$\log_b p^n = n \log p$$

for any positive integer n. In fact this actually holds true for any real number n, but we will not attempt to prove this. On the similar lines the reader is free to verify

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

Example –Can it be true, that $x = e^{\log x}$ for all real x?

Solution First, let us observe that the domain of log function is set of all positive real numbers. log x So the above equation does not hold true for non-positive real numbers. Now, let $y = e^{\log x}$. If y>0, we can take the logarithm which gives us the result $\log y=\log(e^{\log x}) = \log x . \log e = \log x$. Thus y = x. Hence $x = e^{\log x}$ holds true only for positive values of x.

One of the properties of the natural exponential function in differential calculus happens to be that it doesn't change during the process of differentiation. This is captured in the following theorem.

Theorem

(1) **The derivative of e^x w.r.t., x is e^x; i.e., $\frac{d}{dx}(e^x) = e^x$.**

(2) **The derivative of $\log x$ w.r.t., x is $\frac{1}{x}$; i.e., $\frac{d}{dx}(\log x) = \frac{1}{x}$.**

Logarithmic Differentiation

In this section, we shall learn differentiating certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base e) the above may be represented as

$$\log y = v(x) \log [u(x)]$$

By using chain rule we may differentiate this to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \cdot \frac{1}{u(x)} \cdot u'(x) + v'(x) \cdot \log [u(x)]$$

which implies that

$$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]\right]$$

One of the points to be noted in this method is that f (x) and u(x) must be positive always as otherwise their logarithms are not defined. This process is known as logarithms differentiation.

Example - Differentiate ax w.r.t. x, where a is a positive constant.

Solution Let y = a . Then,

$$\log y = x \log a$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log a$$

or $$\frac{dy}{dx} = y \log a$$

Thus $$\frac{d}{dx}(a^x) = a^x \log a$$

Alternatively $$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}\frac{d}{dx}(x\log a)$$

$$= e^{x\log a} . \log a = a^x \log a.$$

Derivatives of Functions in Parametric Forms

Sometimes it so happens that the relation between two variables are neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such situation, we say that the relation between them is expressed through a third variable. The third variable is the parameter. To be precise, a relation that is expressed between two variables x and y in the form x = f (t), y = g (t) is said to be parametric form with t as a parameter.

For finding derivative of function in such form, we have by chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(\text{whenever } \frac{dx}{dt} \neq 0\right)$$

Or

Thus,

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \left(\text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t)\right) \text{ [provided } f'(t) \neq 0]$$

Example:

Find $\frac{dy}{dx}$, if $x = a \cos \theta$, $y = a \sin \theta$.

Solution: Given that

$$x = a \cos \theta,\ y = a \sin \theta$$

Therefore $\qquad \frac{dx}{d\theta} = -a \sin \theta,\ \frac{dy}{d\theta} = a \cos \theta$

Hence $\qquad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

Second Order Derivative

Let y = f(x). Then $\dfrac{dy}{dx} = f'(x)$

If f ′(x) is differentiable, we may bifurcate (1) again w.r.t. x. Then, the left hand side becomes $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ which is known as the second order derivative of y w.r.t. x and is denoted by

$\dfrac{d^2y}{dx^2}$. The second order derivative of f (x) is represented by f ″(x). It is also denoted by D^2 y or y″ or y_2 if y = f (x). We say that higher order derivatives may be defined similarly.

Example:

Find $\dfrac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Solution:-
Given that $y = x^3 + \tan x$. Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 + \sec^2 x\right)$$

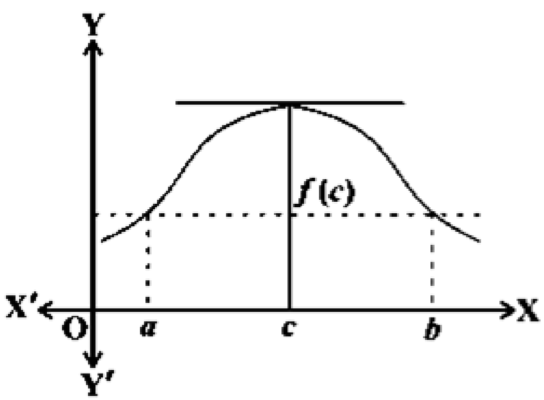
$$= 6x + 2\sec x \cdot \sec x \tan x = 6x + 2\sec^2 x \tan x$$

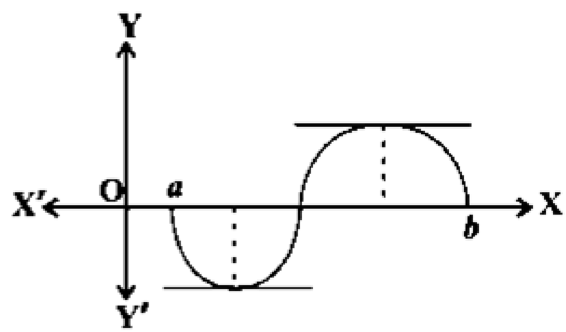
Mean Value Theorem

In this section, we shall state two fundamental results in Calculus without any proof. We shall also learn the geometric interpretation of the theorems.

Theorem (Rolle's Theorem) Let f : [a, b] → **R** be continuous on [a, b] and differentiable on (a, b), in such a way that f(a) = f(b), where a and b are some real numbers. Then there do exist some c in (a, b) such that f ′(c) = 0.

In Figures below, graphs of a few typical differentiable functions satisfying the hypothesis of Rolle's theorem are given.





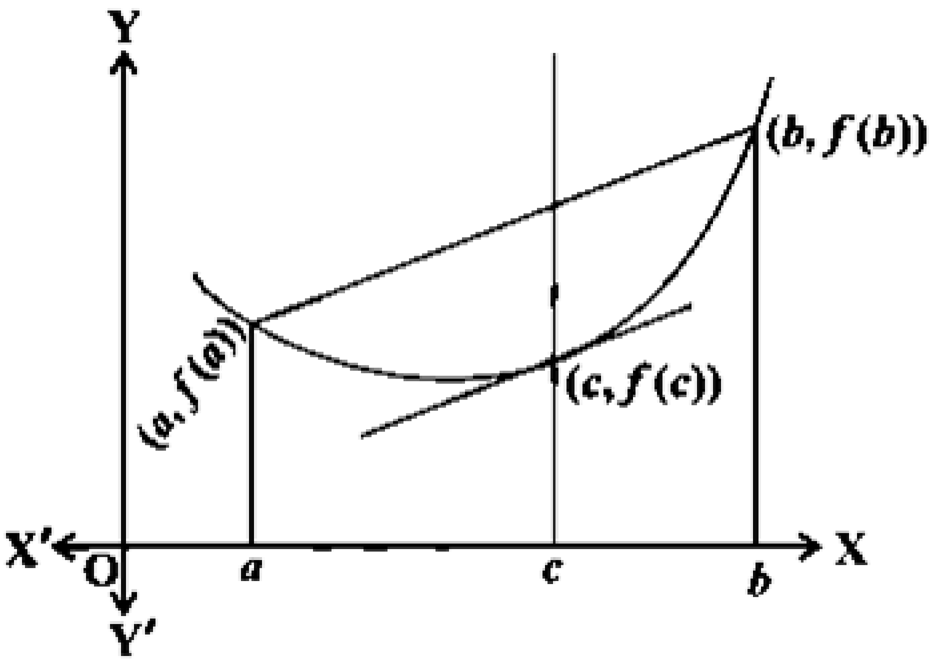
Note what happens to the slope of the tangent to the curve at different points between a and b. In each of the graphs, the slope becomes zero at least at one of the point. That is exactly the claim of the Rolle's theorem as the slope of the tangent at any point on the graph of y = f (x) is nothing but the derivative of f (x) in that particular point.

Theorem (Mean Value Theorem) Let f : [a, b] → **R** be the continuous function on [a, b] and differentiable on (a, b). Then there happens to exist some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note that the Mean Value Theorem (MVT)happens to be an extension of Rolle's theorem. Let usunderstand a geometric interpretation of the MVT. The graph of a function y = f(x) is shown in the Fig 5.14. We have already interpreted f ′(c) as the slope of the tangent to the curve y = f(x) at (c, f(c)). From the Fig 5.14 it is evident

that $\frac{f(b)-f(a)}{b-a}$ is the slope of the secant that is drawn between (a, f (a)) and (b, f (b)). The MVT says that there is a point c in (a, b) such that the slope of the tangent at (c, f(c)) is same as the slope of the secant between (a, f (a)) and (b, f (b)). In other words, there is a point c in (a, b) such that the tangent at (c, f(c)) is parallel to the secant between (a, f(a)) and (b, f(b)).



Example -Verify Rolle's theorem for the function $y=x^2+2$, a=–2 and b=2.

Solution The function $y = x^2 + 2$ is continuous in [– 2, 2] and differentiable in (– 2, 2). Also f(– 2) = f(2) = 6 and therefore the value of f(x) at – 2 and 2 coincide. Rolle's theorem says that there is a point c ∈ (– 2, 2), where f′(c) = 0. Since f′(x) = 2x, we obtain c = 0. Thus at c = 0, we have f′(c) = 0 and c = 0 ∈ (– 2, 2).

Example Verify the mean Value Theorem for f (x) = x^2 in the interval [2, 4].

Solution f (x) = x^2 is continuous in [2, 4] and differentiable in (2, 4) as its derivative f′(x) = 2xis defined in (2, 4).

Now f(2) = 4 and f(4) = 16. Therefore

$$\frac{f(b)-f(a)}{b-a}=\frac{16-4}{4-2}=6$$

MVT states that there is a point c ∈ (2, 4) such that f′(c) = 6. But f′(x) = 2x which means that c = 3. Thus at c = 3 ∈ (2, 4), we have f′(c) = 6.

8e. Application of Derivatives

Introduction

In earlier chapters, we have seen how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this chapter, we shall study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be utilised (i) to determine the rate of change of quantities, (ii) for finding the equations of tangent and that is normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us for locating the points at which largest or the smallest value of a function arise. We shall also use derivative to find intervals on which a function is increasing or decreasing. Finally, we make use of the derivative to find approximate value of certain quantities.

Rate of Change of Quantities

Let us recall that by the derivative $ds/_{dt}$, we mean that the rate of change of distance s with concerning to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule y = f (x), then $^{dy/}$ dx (or f′ (x)) represents the rate of change of y with concerning to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or f′ (x_0)) represents the rate of change of y with respect to x at x=x_0 . Further, if two variables x and y vary with respect to another variable t, i.e., if x= f(t)and y=g(t), then by Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y that concerns to x can be calculated by using the rate of change of y and that of x both with respect to t.

Let us consider some examples.

Example Find out the rate of change of the area of a circle per second with respect to its radius r when r = 5 cm.

Solution The area A of a circle with radius r is represented by A = πr^2. Therefore, the rate of change of the area A that concerns with its radius r is given by

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

.

When r = 5 cm, $\frac{dA}{dr} = 10\pi$. Thus, the area of the circle changes at the rate of dr

10π cm^2/s.

Example The volume of a cube increases at the rate of 9 cubic centimetres per second. How fast is the area of it's surface increasing when the length of an edge is 10 centimetres ?

Solution If x is the length of a side, V, the volume and S, the surface area of the cube. Then, $V=x^3$ and $S=6x^2$, where x happens to be a function of time t.

Now $\frac{dV}{dt} = 9cm^3/s$ (Given)

Therefore $9 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3) \cdot \frac{dx}{dt}$ **(By Chain Rule)**

$$= 3x^2 \cdot \frac{dx}{dt}$$

Or $\frac{dx}{dt} = \frac{3}{x^2}$

Now $\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2) \cdot \frac{dx}{dt}$ **(By Chain Rule)**

$$= 12x \cdot \left(\frac{3}{x^2}\right) = \frac{36}{x}$$

Hence, when x= 10 cm, $\frac{dS}{dt} = 3.6\ \text{cm}^2/\text{s}$

Example A stone is dropped into a lake which is quite and whose waves move in circles at a speed of 4cm per second. When the radius of the circular wave happens to be 10 cm, how fast is the enclosed area increasing?

Solution The area A of a circle with radius r is represented by $A = \pi r^2$. Therefore, the rate of change of the area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\cdot\frac{dr}{dt} = 2\pi\, r\,\frac{dr}{dt} \qquad \textbf{(By Chain Rule)}$$

It is given that $\qquad \frac{dr}{dt} = 4\text{cm/s}$

Therefore, when r = 10 cm, $\qquad \frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Therefore, the enclosed area is increasing at the rate of 80π cm^2/s, when r = 10 cm.

dy / dx is positive if y happens to increasewith the increase in *x* and is negative if y decreases as *x* increases.

Example The length *x* of a rectangle is reducing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. If *x* =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution Since the length x is decreasing and the width y is increasing with that of time, we have,

$$\frac{dx}{dt} = -3 \text{ cm/min} \qquad \text{and} \qquad \frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of a rectangle is represented by

P = 2 (x + y)

Therefore,

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3+2) = -2 \text{ cm/min}$$

(b) The area A of the rectangle is represented by, A=x.y

Therefore,

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$= -3(6)+10(2)$ (as x=10cm and y=6cm)

$= 2\ cm^2/min$

Increasing and Decreasing Functions

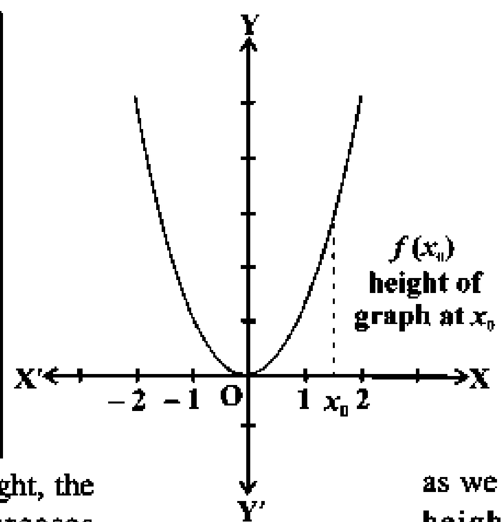
In this section, we shall use differentiation for finding whether a function is increasing or decreasing or none.

Let's consider the function f given by $f(x) = x^2$, $x \in \mathbf{R}$. The graph of this function happens to be the parabola as given in Fig 6.1.

Values left to origin

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

as we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

as we move from left to right, the height of the graph increases

Let's first consider the graph (Fig 6.1) to the right of the origin. Note that as we move from left to right along the graph, the height of the graph increases on a continuous basis. Therefore, the function is said to be increasing for the real numbers $x > 0$.

Now let's consider the graph to the left of the origin and note here that as we move from left to right along the graph, the height of the graph decreases on a continuous basis. Consequently, the function decreases for the real numbers $x < 0$.

We shall now give the following analytical definitions for a function which is increases or decreases on an interval.

Definition If I is an interval contained in the domain of a real valued function f. Then f is

(i) increasing on I if $x_1 < x_2$ in I$\Rightarrow$ $f(x_1) < f(x_2)$ for all $x_1, x_2 \in$I.

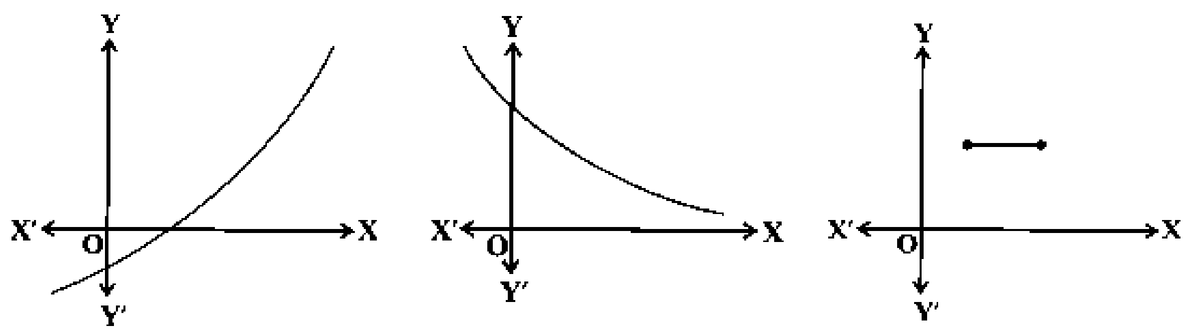
(ii) decreasing on I, if x_1, x_2 in I $\Rightarrow f(x_1) < f(x_2)$ for all x_1, $x_2 \in$ I.

(iii) constant on I, if f(x) = c for all x $\in$ I, wherein c happens to be a constant.

(iv) Decreasing on I if $x_1 < x_2$ in I$\Rightarrow f(x_1) \geq f(x_2)$for all $x_1, x_2 \in$I.

(v) Strictly decreasing on I if $x_1 < x_2$ inI$\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$I.

For graphical representation of such functions see Fig below.



Strictly Increasing function
(i)

Strictly Decreasing function
(ii)

Neither Increasing nor Decreasing function
(iii)

We will define when a function is increasing or decreasing at a point.

Definition If $x_0$is a point in the domain of definition of a real valued function f. Then f is said to be increasing, decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, decreasing, respectively, in I.

We shall clarify this definition for the case of increasing function.

Example Exhibit that the function given by f(x) = 7x – 3 is increasing on **R**.

SolutionIf x_1 and x_2 be any two numbers in **R**. Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it means that f is strictly increasing on **R**.

We shall now give the first derivative test for the functions that are increasing and decreasing.

Theorem - If f is the continuous on [a, b] and differentiable on the open interval (a,b). Then

(a) f increases in [a,b] if $f'(x) > 0$ for each $x \in (a, b)$

(b) f decreases in [a,b] if $f'(x) < 0$ for each $x \in (a, b)$

(c) f happens to be a constant function in [a,b] if $f'(x) = 0$ for each $x \in (a, b)$

Example - Show that the function f given by $f(x)=x^3 - 3x^2 + 4x$, $x \in \mathbf{R}$

is increasing on **R**.

Solution Note that

$f'(x) = 3x^2 - 6x + 4 = 3(x^2 - 2x + 1) + 1$

$= 3(x-1)^2 + 1 > 0$, in every interval of **R**

Therefore, the function f is increasing on **R**.

Tangents and Normals

In this section, we shall make use of differentiation for finding the equation of the tangent line and the normal line to a curve at a given point.

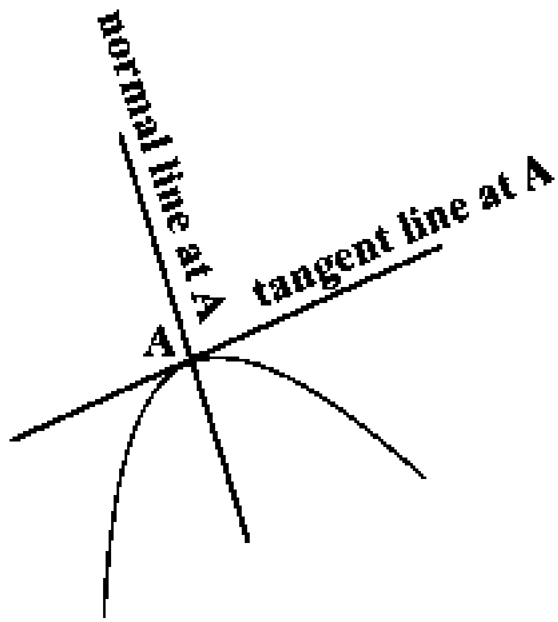
Let's recollect that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by

$y - y_0 = m(x - x_0)$

Note that the slope of the tangent to the curve y = f (x) at the point (x,y) is given by $\left.\dfrac{dy}{dx}\right]_{(x_0, y_0)} (= f'(x_0))$. So, the equation of the tangent at (x_0, y_0) to the curve y = f (x)

is given by

$$y - y_0 = f'(x_0)(x - x_0)$$



Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve y = f (x) at (x_0, y_0) is $\dfrac{-1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of the normal to the curve y = f (x) at (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

i.e. $(y - y_0) f'(x_0) + (x - x_0) = 0$

Example Findtheslopeofthetangenttothecurve $y=x^3 - x$ at $x=2$.

Solution The slope of the tangent at x = 2 is represented by

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 11$$

Approximations

In this section, we shall make use of differentials to approximate values of certain quantities.

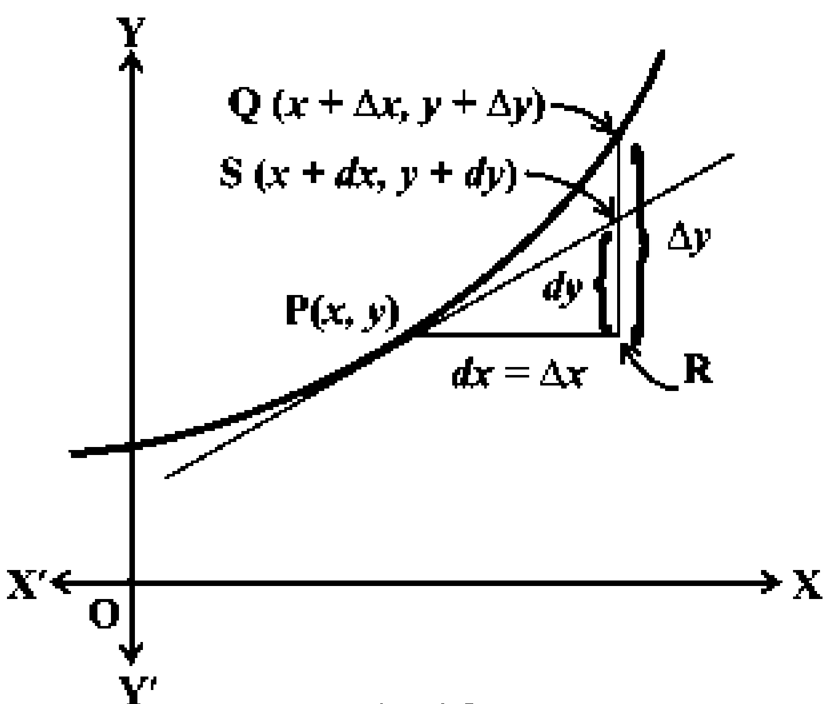


Fig 6.8

Let f : D → **R**, D ⊂**R**, be a function given and let y = f(x). Let Δx represent a small increment in x. Recollect that the increment in y corresponds to the increment in x, denoted by Δy, is given by Δy = f (x + Δx) – f (x). We define the following

i. The differential of x, represented by dx, is defined by dx = Δx.
ii. The differential of y, represented by dy, is defined by dy = f′(x) dx or

$$dy = \left(\frac{dy}{dx}\right)\Delta x.$$

In case dx = Δx is small when compared with x, dy is a good approximation of Δy and we represent it by dy ≈ Δy.
For geometrical meaning of Δx, Δy, dx and dy, one may refer to Fig above.

Example Using differential for approximating root36.6.

Solution: Take $y = \sqrt{x}$. Let x=36 and let Δx=0.6. Then

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$$

or $\sqrt{36.6} = 6 + \Delta y$

Now dyhappens to be approximately equal to Δy and is represented by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05 \qquad (\text{as } y = \sqrt{x})$$

Thus, the approximate value of $\sqrt{36.6}$ is 6 + 0.05 = 6.05.

Maxima and Minima

In this section, we shall make use of the concept of derivatives for calculating the maximum or minimum values of various functions. In fact, we shall find the ‘turning points’ of the graph of a function and thus find points at which the graph reaches its highest (or lowest) locally. The knowledge of such points is useful in constructing the graph of a given function. Further, we shall also find the absolute maximum and absolute minimum of a function that are needed for the solution of many applied problems.

Let us take into account the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is represented by $P(x) = ax + bx^2$, where a,b are constants and x happens to be the number of orange trees per acre. How many trees per acre will help maximise the profit?

(ii) A ball, that is thrown into the air from a building 60 metres high, travels along a path given by

$$h(x) = 60 + x - \frac{x^2}{60}$$

, where x happens to be the horizontal distance from the building and h(x) happens to be the height of the ball. What is the maximum height the ball will be able to reach?

iii. An Apache helicopter of enemy is flying along the path that is given by the curve $f(x) = x^2 + 7$. A soldier, who is placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there happens to be something common, i.e., we wish to find out the maximum or minimum values of the given functions. For tackling such problems, we first define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition Let as assume that f is a function defined on an interval I. Then,

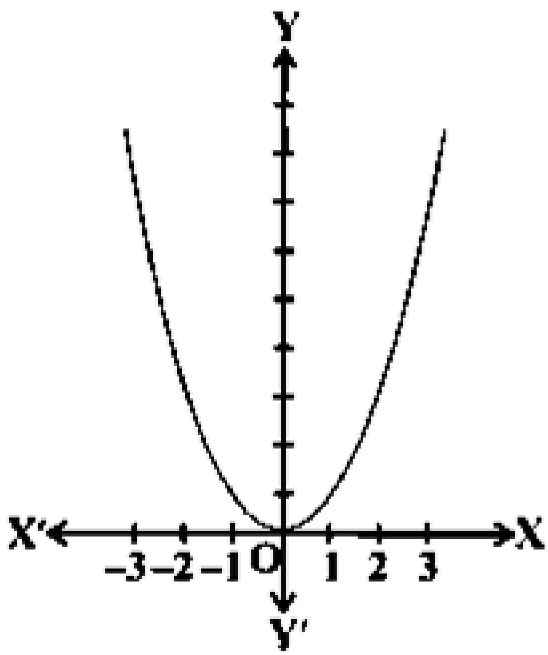
a. f has the maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$. The number f (c) is known as the maximum value of f in I and the point c is called a point of maximum value of f in I.
b. f has a minimum value in I, if there is a point c in I such that $f(c) < f(x)$, for all $x \in I$. The number f (c), in this case, is known as the minimum value of f in I and the point c, in this case, is known as a point of minimum value of f in I.
c. f has an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I.

The number f (c), in this case, is known as an extreme value of f in I and the point c is called an extreme point.

Example Find out the maximum and the minimum values, if any, of the function f given by

$f(x) = x^2, x \in \mathbf{R}$.

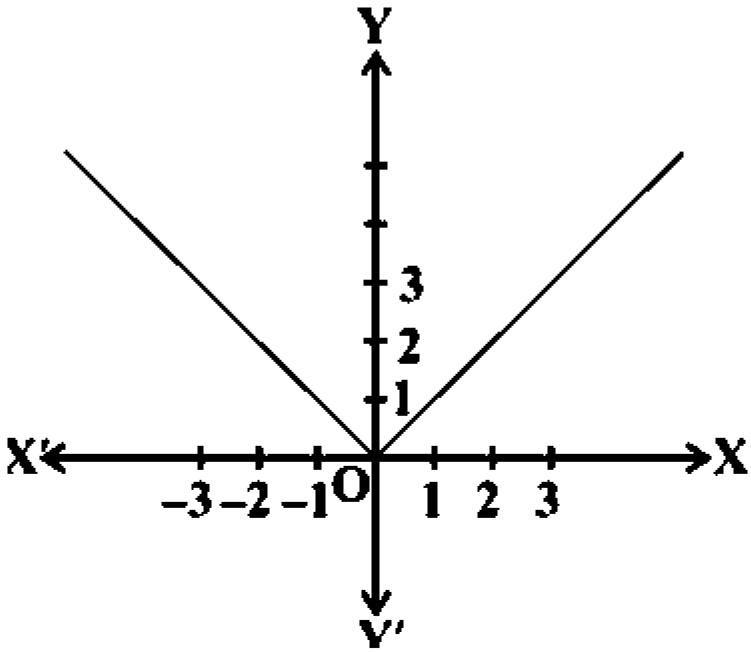
Solution From the graph of the given function (Fig 6.10), we have $f(x)=0$ if $x=0$. Also $f(x) \geq 0$, for all $x \in \mathbf{R}$.



Therefore, the minimum value of f is 0 and the point of minimum value of f would be $x = 0$. Further, it may be observed from the graph of the function that f do not have a maximum value and therefore no point of maximum value of f in **R.**

Example Find the maximum and minimum values of f, if any, of the function given by $(x)=|x|, x \in \mathbf{R}$.

Solution From the graph of the given function (Fig 6.11), note that

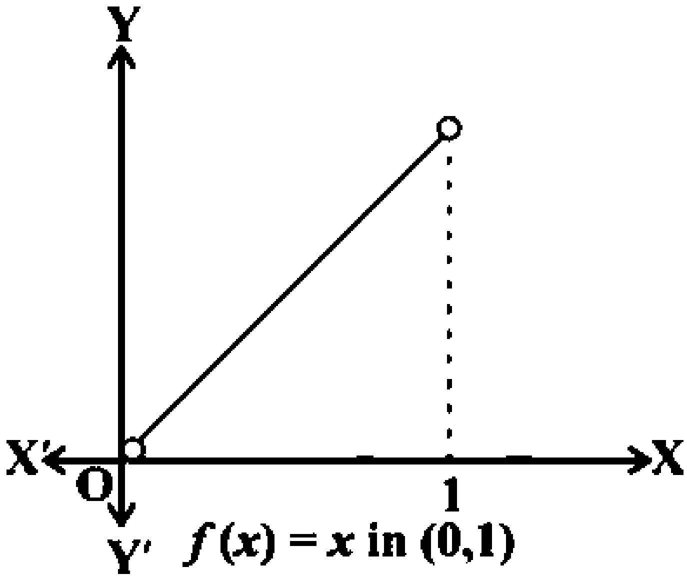


$f(x) \geq 0$, for all $x \in \mathbf{R}$ and $f(x)=0$ if $x=0$.

Thus, the function f has a minimum value of 0 and the point of minimum value of f is $x = 0$. Moreover, the graph clearly shows that f has no maximum value in **R** and hence no point of maximum value in **R**.

Example Find out the maximum and the minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1).$$



Solution The given function is an increasing (strictly) function in the mentioned interval (0, 1). From the above graph of the function f, it seem like that, it should have the minimum value at a point closest to 0 on its right side and the maximum value at a point closest to 1 on its left side. Are such points obtainable? Of course, not. It is not pratical to locate such points. In reality, if a point x_0 is closest to 0, then $\frac{x_0}{2} < x_0$ for all $x_0 \in (0,1)$. Also, if x_1 is closest to 1, then $\frac{x_1 + 1}{2} > x_1$ for all x $\in(0,1)$.

Thus, the given function has neither the maximum value nor the minimum value in the interval (0,1).

Remark The reader may notice that in Example 28, if we include the points 0 and 1 in the domain of f , i.e., if we extend the domain of f to [0,1], and then the function f has minimum value 0 at x = 0 and maximum value 1 at x = 1. In reality, we have the following results (The proof of these results are beyond the scope of the present text)

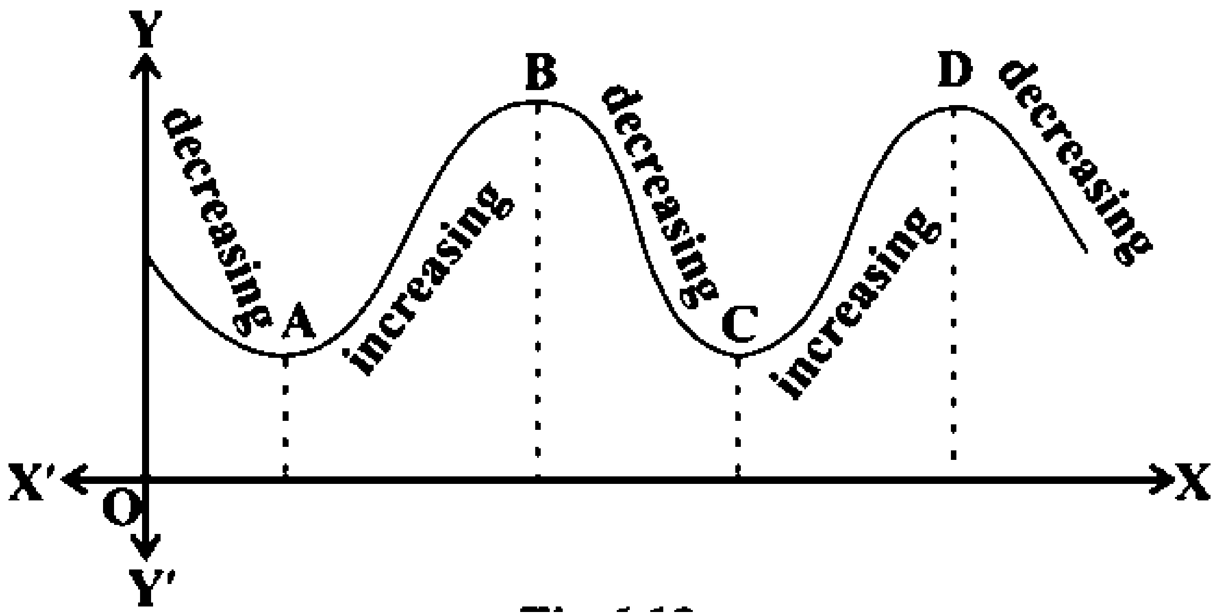
Every monotonic function presumes its maximum/minimum value at the end points of the domain of definition of the function.

A very general result is

Each and every continuous function on a closed interval has a maximum and a minimum value.

Maximum and minimum values of a function defined on a closed interval would be discussed later in this section.

Let us now look into the graph of a function as shown in Fig 6.13.



Notice that at points A, B, C and D on the graph, the function changes its nature from increasing to decreasing or vice-versa. These points may be termed as turning points of the given function. Additional, observe that at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective valleys. Likewise, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills. For this purpose, the points A and C may be considered as points of local minimum value (or relative minimum value) and points B and D may be considered as points of relative maximum value or local maximum value for the function. The local maximum value and local minimum value of the function are stated to as local maxima and local minima, respectively, of the function.

We now strictly give the following definition.

Definition Let us suppose f be a real valued function and let c be an interior point in the domain of f. Then

1. (a) c is said to be a point of local maxima if there is an $h > 0$ such that

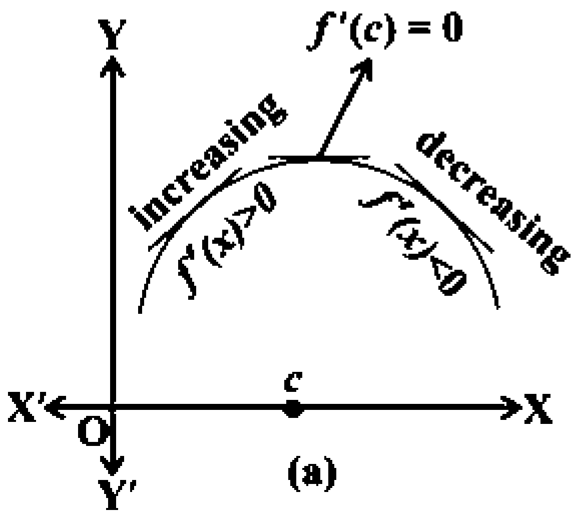
   $f(c) \geq f(x)$, for all x in $(c - h, c + h)$, $x \neq c$ The value f (c) is called as the local maximum value of f.

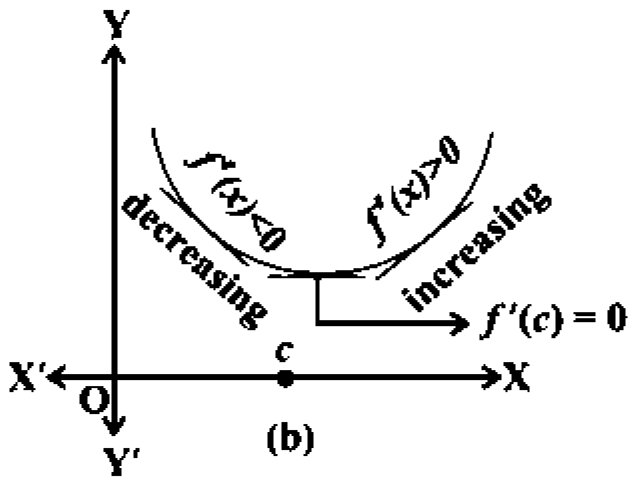
2. (b) c is said to be a point of local minima if there is an $h > 0$ such that $f(c) \leq f(x)$, for all x in $(c - h, c + h)$

The value f (c) is called as the local minimum value of f .

Geometrically, the above definition statuses that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig 6.14(a) below. Observe that the function f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$.

This recommends that $f'(c)$ must be zero.





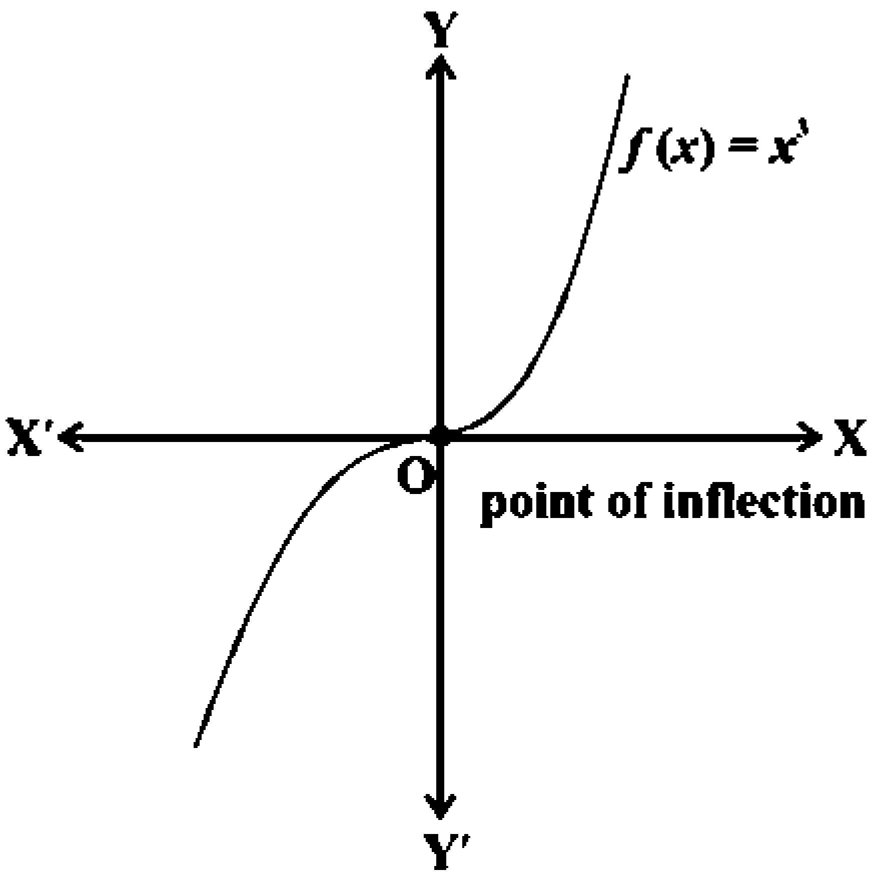
Likewise, if c is a point of local minima of f, then the graph of f around c will be as shown Here f is decreasing (i.e., $f'(x) < 0$) in the interval $(c – h, c)$ and increasing (i.e., $f'(x) > 0$) in the required interval $(c, c + h)$. This again recommends that $f'(c)$ must be zero.

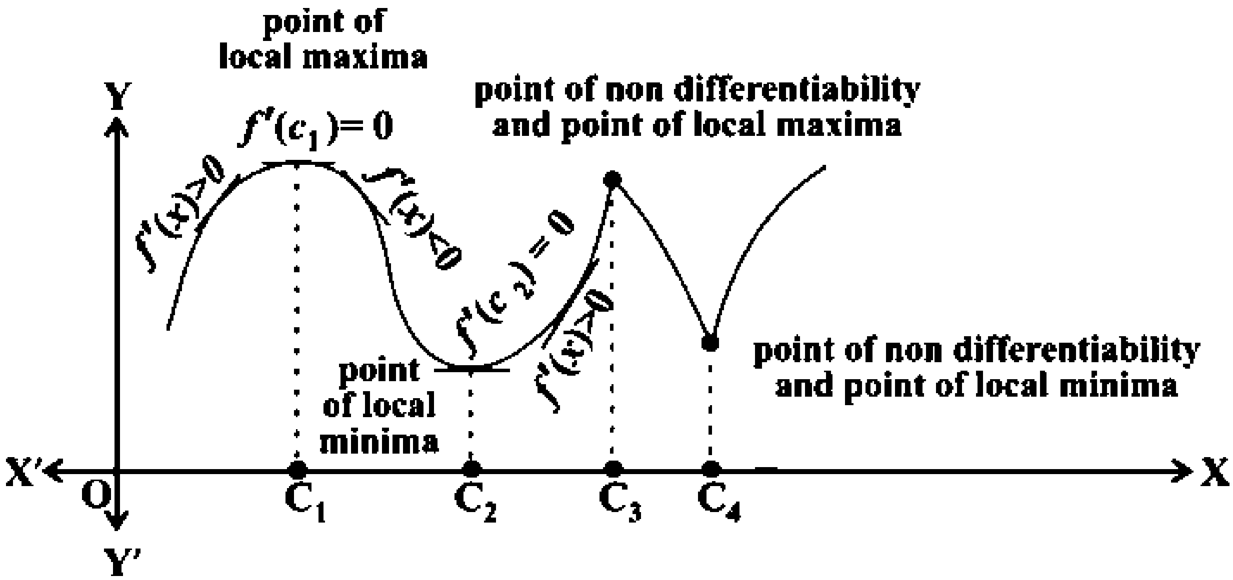
The above discussion leads us to the following theorem (without proof).

Theorem Let us suppose f be a function expressed on an open interval I. Consider $c \in I$ be any point. If f has a local maxima or a local minima at x=c, then neither $f'(c)=0$ or f is not differentiable at c.

Theorem (First Derivative Test) Let us suppose f be a function expressed on an open interval I. Let us also assume that f be continuous at a critical point c in I. Then

1. (i) If f ′(x) changes sign from positive to negative as x increases through c, i.e., if f ′(x) > 0 at every point adequately close to and to the left of c, and f ′(x) < 0 at every point adequately close to and to the right of c, then c is a point of local maxima.
2. (ii) If f ′(x) changes sign from negative to positive as x increases through c, i.e., if f ′(x) < 0 at every point adequately close to and to the left of c, and f ′(x) > 0 at every point adequately close to and to the right of c, then c is a point of local minima.
3. (iii) If f ′(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In reality, such a point is named as point of inflection.





Example Find out all points of local maxima and local minima of the function f given by

$f(x)=x^3 -3x+3.$

Solution We have $f(x)=x^3 -3x+3$ or $f'(x) =3x^2 -3=3(x-1)(x+1)$ or $f'(x) =0$atx= and x= –1
Therefore, x = ± 1 are the only critical points which could probably be the points of local maxima and/or local minima of f . Let us first study the point x = 1.

Observe that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Thus, by first derivative test, x = 1 is a point of local minima and local minimum value is f (1) = 1. In the case of x = –1, observe that $f'(x) > 0$, for values close to and to the left of –1 and $f'(x) < 0$, for values close to and to the right of – 1. Hence, by first derivative test, x = – 1 is a point of local maxima and local maximum value is f (–1) = 5.

Values of x		Sign of $f'(x) = 3(x-1)(x+1)$
Close to 1	to the right (say 1.1 etc.)	>0
	to the left (say 0.9 etc.)	<0
Close to –1	to the right (say -0.9 etc.)	<0
	to the left (say -1.1 etc.)	>0

Example Find out all the points of local maxima and local minima of the function f given by

$f(x)=2x^3-6x^2+6x+5$.

Solution We have

$f(x) = 2x^3 - 6x^2 + 6x + 5$
or $f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$

or $f'(x)=0$ at $x=1$

Therefore, $x = 1$ is the only critical point of f . We shall now study this point for local maxima and/or local minima of f. Notice that $f'(x) \geq 0$, for all $x \in \mathbf{R}$ and in specific $f'(x) > 0$, for values close to 1 and to the left and to the right of 1. Thus, by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima. Therefore $x = 1$ is a point of inflexion.

Remark One may observe that since $f'(x)$, in Example 30, never changes its sign on **R**, graph of f has no turning points and hence no point of local maxima or local minima.

We shall now provide another test to study local maxima and local minima of a given function. This test is often easier to implement than the first derivative test.

Theorem (Second Derivative Test) Let us suppose f be a function expressed on an interval I and $c \in I$. Let f be differentiable twice at c. Then

(i) $x=c$ is a point of local maxim a if $f'(c)=0$ and $f''(c)<0$ The value $f(c)$ is local maximum value of f .

(ii) if $f'(c)=0$ and $f''(c)>0$,

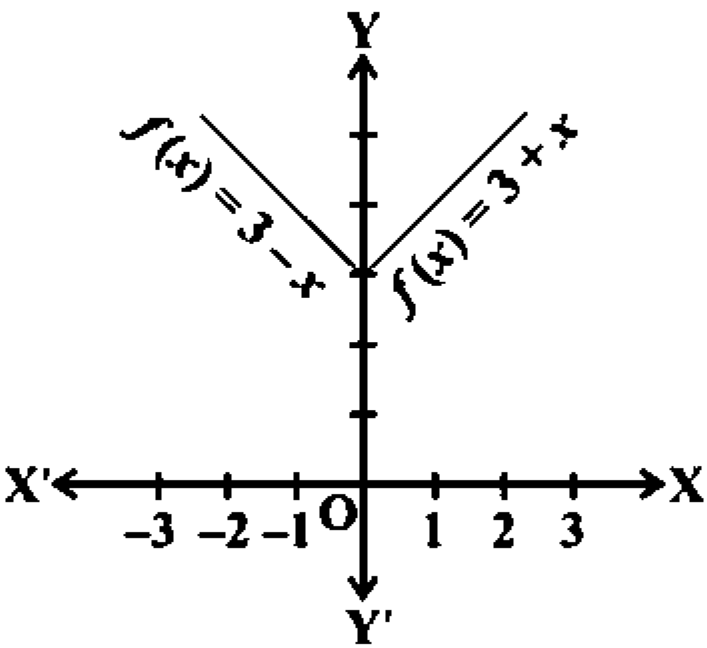
a point of local minima: $x=c$

$f(c)$ is local minimum value of f, in this case,

(iii) If $f'(c) = 0$ and $f''(c) = 0$ the test would fail.

In this case, we refer back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Example Find out local minimum value of the function f given by $f(x)=3+|x|, x\in\mathbf{R}$.



Solution Observe that the given function is not differentiable at $x = 0$. So, second derivative test fails. Let us try first derivative test. Observe that 0 is a critical point of f. Now to the left side of 0, $f(x) = 3 - x$ and so $f'(x) = -1 < 0$. Also to the right side of 0, $f(x) = 3 + x$ and so $f'(x) = 1 > 0$. Thus, by first derivative test, $x=0$ is a point of local minima of f and local minimum value of f is $f(0)=3$.

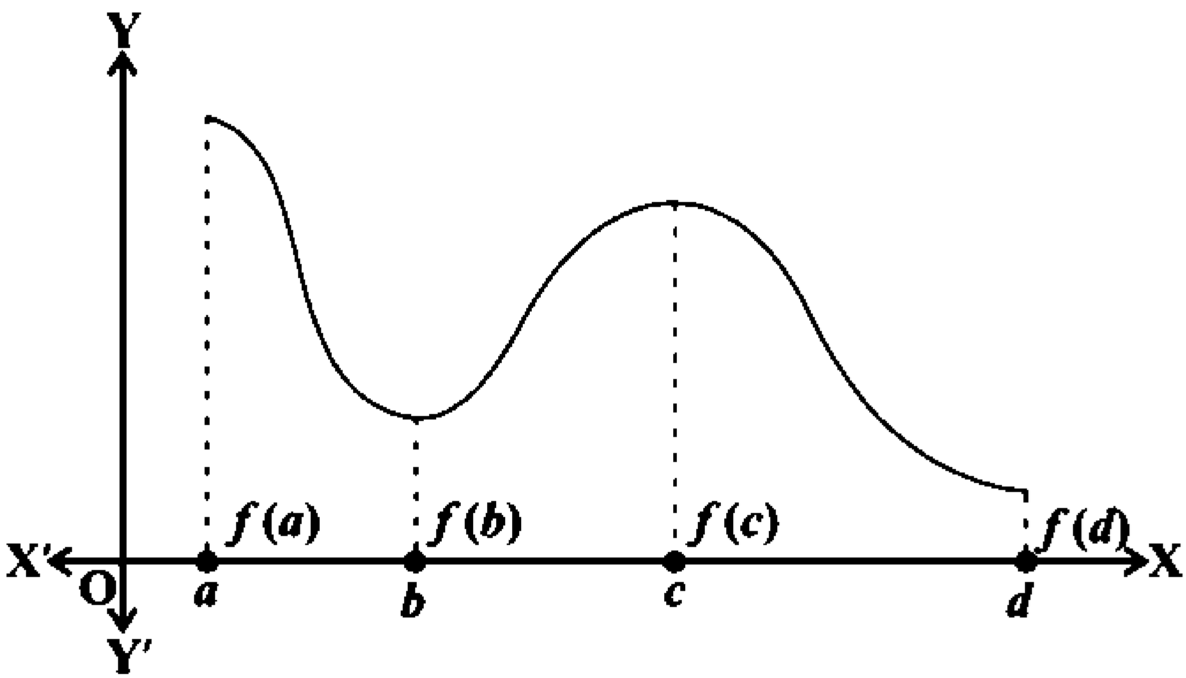
Maximum and Minimum Values of a Function in a Closed Interval

Let us suppose a function f given by
$f(x) = x + 2, x \in (0,1)$

Notice that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Moreover, we may also observe that the function even has neither a local maximum value nor a local minimum value.

Yet, if we extend the domain of f to the closed interval [0, 1], then f still may not have a local maximum (minimum) values but it surely does have maximum value 3=f(1)and minimum value 2= f(0).The maximum value 3 of f at x=1is called as absolute maximum value (global maximum or greatest value) of f on the interval [0, 1]. Likewise, the minimum value 2 of f at x = 0 is called as the absolute minimum value (global minimum or least value) of f on [0, 1].

Examine the graph given in Fig 6.21 of a continuous function defined on a closed interval [a, d]. Notice that the function f has a local minima at x = b and local



minimum value is f (b). Also, the function has a local maxima at x = c and local maximum value is f (c).

Also from the graph, it is obvious that f has absolute maximum value f (a) and absolute minimum value f (d). Moreover, observe that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f .

We will now state two results (without proof) concerning absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem Let us suppose f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f achieves it at least once in I. f also, has the absolute minimum value and attains it at least once in I.

Theorem Let us suppose f be a differentiable function on a closed interval I and let us consider c be any interior point of I. Then

(i) $f'(c) = 0$ if f achieves its absolute maximum value at c.
(ii) $f'(c) = 0$ if f achieves its absolute minimum value at c.

In opinion of the above results, we have the following working rule for finding out absolute maximum and/or absolute minimum values of a function in a given closed interval [a, b].

Working Rule

Step 1: Find out all critical points of f in the interval, i.e., find out points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Consider the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), compute the values of f .

Step 4: Find the maximum and minimum values of f out of the values calculated in Step 3. This maximum value would be the absolute maximum (greatest) value of f and the minimum value would be the absolute minimum (least) value of f .

Example Find out the absolute maximum and minimum values of a function f given by

$f(x) = 2x^3 – 15x^2 + 36x +1$ on the interval [1, 5].

Solution We have

$f(x)=2x^3 –15x^2 +36x+1$
or $f'(x) =6x^2 –30x+36=6(x–3)(x–2)$

Note that $f'(x)=0$ gives $x=2$ and $x=3$.
Now, we shall estimate the value of f at these points and at the end points of the interval [1, 5], i.e., at x = 1, x = 2, x = 3 and at x = 5. And so,

$f(1) = 2(1^3) – 15(1^2) + 36(1) + 1 = 24$

$f(2) = 2(2^3) – 15(2^2) + 36(2) + 1 = 29$

$f(3) = 2(3^3) – 15(3^2) + 36(3) + 1 = 28$

$f(5) = 2(5^3) – 15(5^2) + 36(5) + 1 = 56$

Therefore, we conclude that absolute maximum value of f on [1, 5] is 56, happening at x =5, and absolute minimum value of f on [1, 5] is 24 which happens at x = 1.

8f. Differential Equations

Introduction

In this unit, let us study some basic concepts related to differential equation, general and particular solutions of a differential equation, construction of differential equations, some methods to resolve a first order - first degree differential equation and some presentations of differential equations in different areas. We are already aware with the equations of the type:

- $x^2 – 3x + 3 = 0$
- $\sin x + \cos x = 0$
- $x + y = 7$

Let us consider the equation:

$$x\frac{dy}{dx} + y = 0$$

We observe that equations involve independent and/or dependent variable (variables) only but last equation above involves variables as well as derivative of the dependent variable *y* with reference to the independent variable *x*. Such an equation is called as a *differential equation*.

Generally, an equation involving derivative (derivatives) of the dependent variable with reference to independent variable (variables) is called as a differential equation.

A differential equation involving derivatives of the dependent variable with reference to only one independent variable is called an ordinary differential equation, e.g.,

$$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$ is an ordinary differential equation.

Certainly, there are differential equations involving derivatives with reference to more than one independent variables, called as partial differential equations but at this stage we shall confine ourselves to the study of ordinary differential equations only. From now on, we will use the term 'differential equation' for 'ordinary differential equation'.

Note:

We shall choose to use the following notations for derivatives:

$$\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$$

For derivatives of higher order, it will be difficult to use so many dashes as super suffix so that, we use the notation yn for nth order derivative $\frac{d^n y}{dx^n}$.

Order of a differential equation

Order of a differential equation is expressed as the order of the highest order derivative of the dependent variable with reference to the independent variable involved in the given differential equation.

Study the following differential equations:

$$\frac{dy}{dx} = e^x$$

$$\frac{d^2y}{dx^2} + y = 0$$

$$\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$$

The equations above comprise the highest derivative of first, second and third order respectively. Thus, the order of these equations are 1, 2 and 3 respectively.

Degree of a differential equation

To comprehend the differential equation and its degree, the significant point is that the differential equation must be a polynomial equation in derivatives, i.e., y', y'', y''' etc. Let us consider the following differential equations:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$$

We notice that equation (9) is a polynomial equation in y''', y'' and y', equation (10) is a polynomial equation in y' (not a polynomial in y though). Degree of such differential equations can be well-defined. But equation (11) is not a polynomial equation in y' and degree of such a differential equation cannot be defined.

If it is a polynomial equation in derivatives, by the degree of a differential equation, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

In view of the above definition, we can notice that differential equations (6), (7), (8) and (9) each are of degree one, equation (10) is of degree two while the degree of differential equation (11) is not expressed.

Example Find out the order and degree, if defined, of each of the following differential equations:

(i) $\frac{dy}{dx} - \cos x = 0$ (ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

(iii) $y''' + y^2 + e^{y'} = 0$

Solution

(i) The highest order derivative involved in the differential equation is $\frac{dy}{dx}$ so its order is one. Its degree is one as it is a polynomial equation in y' and the highest power raised to $\frac{dy}{dx}$ is one.

(ii) The highest order derivative involved in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is two. It is a polynomial equation in

$\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and the highest power raised up to $\frac{d^2y}{dx^2}$ is one, so the degree is one.

(iii) The highest order derivative involved in the differential equation is y''', so the order is three. Its degree is not expressed as the mentioned differential equation is not a polynomial equation in its derivatives.

General and Particular Solutions of a Differential Equation

In earlier classes, we have resolved the equations of the type:

$x^2 + 1 = 0$

$\sin^2 x - \cos x = 0$

Solution of equations (1) and (2) are numbers, real or complex, that will fulfill the given equation i.e., when that number is substituted for the unknown x in the given equation, L.H.S. becomes equal to the R.H.S..

Now observe the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

Unlike the first two equations, the solution of this differential equation is a function φ that will fulfill it i.e., when the function φ is substituted for the unknown y (dependent variable) in the given differential equation, L.H.S. becomes equal to R.H.S..

The curve $y = \varphi(x)$ is called as the solution curve (integral curve) of the given differential equation. Observe the function given by

$y = \varphi(x) = a \sin(x + b)$,

where $a, b \in \mathbf{R}$. If this function and its derivative are substituted in equation (3), L.H.S. = R.H.S.. Thus it is a solution of the differential equation (3).

Let a and b be given some specific values say a=2 and b=π,then we obtain a function

$$y = \phi_1(x) = 2\sin\left(x + \frac{\pi}{4}\right)$$

If this function and its derivative are substituted in equation (3), again L.H.S. = R.H.S.. Thus φ_1 is also a solution of equation (3).

Function φ comprises of two arbitrary constants (parameters) a, b and it is called as *general solution* of the given differential equation. However, function φ_1 contains no arbitrary constants but only the particular values of the parameters a and b and thus is called a *particular solution* of the given differential equation.

The solution which contains arbitrary constants is called as the *general solution* (*primitive*) of the differential equation.

The solution which is free from arbitrary constants that is, the solution acquired from the general solution by giving particular values to the arbitrary constants is called as a *particular solution* of the differential equation.

Example - Validate that the function $y = e^{-3x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution Given function is $y = e^{-3x}$. Differentiating both sides of equation with reference to x, we obtain

$$\frac{dy}{dx} = -3e^{-3x}$$

Now, differentiating (1) with reference to x, we have

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we obtain

L.H.S.$=9e^{-3x}+(-3e^{-3x})-6.e^{-3x} = 9e^{-3x}-9e^{-3x} = 0 =$ R.H.S.. Thus, the given function is a solution of the given differential equation.

Example 3 Validate that the function $y = a \cos x + b \sin x$, where, $a, b \in \mathbf{R}$ is a solution of the differential equation $d^2y/dx^2 + y = 0$

Solution The given function is

$y = a \cos x + b \sin x$ ... (1)

Differentiating both sides of equation (1) with reference to x, successively, we obtain

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

$$\frac{d^2y}{dx^2} = -a \cos x - b \sin x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we obtain

L.H.S. = ($-a\cos x - b\sin x$) + ($a\cos x + b\sin x$) = 0 = R.H.S.

Hence, the given function is declared to be a solution of the given differential equation.

Creation of a Differential Equation whose General Solution is known or given

We know that the equation
$x^2 + y^2 + 2x - 4y + 4 = 0$ ...(1)

denotes circle having centre at (–1, 2) and radius 1 unit.

Differentiating equation (1) with reference to x, we obtain

$$\frac{dy}{dx} = \frac{x+1}{2-y} \quad (y \neq 2)$$

which is a differential equation. You will find later on [See (example 9 section 9.5.1.)] that this equation represents the family of circles and one member of the family is the circle given in equation (1).

Let us consider the equation

$$x^2 + y^2 = r^2$$

By giving different values to r, we obtain different members of the family e.g. $x^2 + y^2 = 1, x^2 + y^2 = 4, x^2 + y^2 = 9$ etc.(seeFig9.1).
Hence, equation (3) represents a family of concentric circles centered at the origin and having different radii.

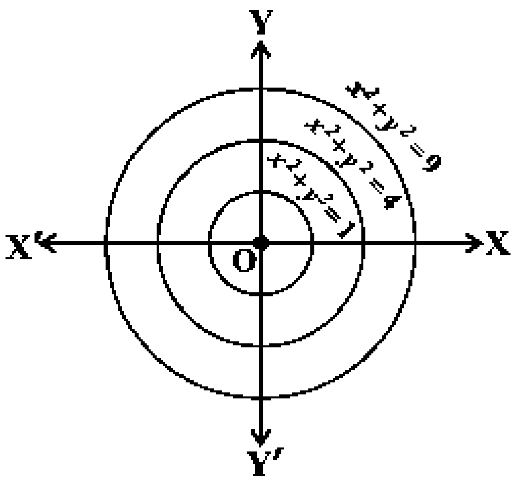


Fig 9.1

We are concerned about finding a differential equation that is satisfied by each member of the family. The differential equation should be free from *r* because *r* is different for different members of the family. This equation is attained by differentiating equation (3) with respect to *x*, i.e.,

$$2x + 2y\,\frac{dy}{dx} = 0 \quad \text{or} \quad x + y\,\frac{dy}{dx} = 0$$

which denotes the family of concentric circles given by equation (3). Again, consider the equation

$y = mx + c$

By giving different values to the parameters *m* and *c*, we obtain different members of the family, e.g.,

$$y = x \qquad (m = 1, \quad c = 0)$$

$$y = \sqrt{3}\,x \qquad (m = \sqrt{3}\,, \; c = 0)$$

$$y = x + 1 \qquad (m = 1, \; c = 1)$$

$$y = -x \qquad (m = -1, \; c = 0)$$

$$y = -x - 1 \qquad (m = -1, \; c = -1) \text{ etc.}$$

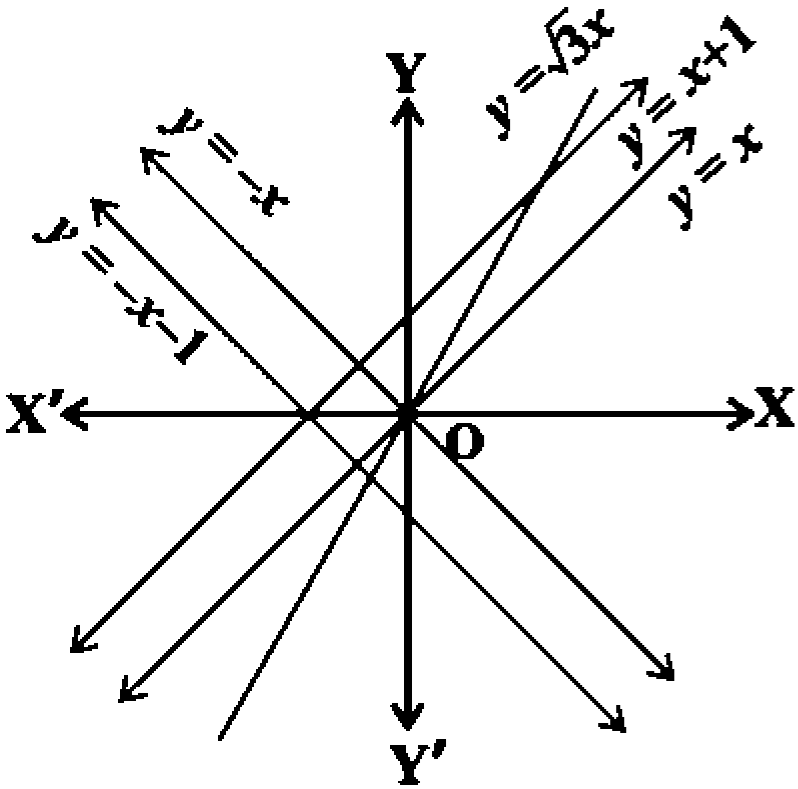
Therefore, equation (5) denotes the family of straight lines, where *m*, *c* are parameters.

We are now concerned about finding a differential equation that is fulfilled by each member of the family. Moreover, the equation must be free from *m* and *c* as *m* and *c* are different for different members of the family. This is attained by differentiating equation (5) with respect to *x*, successively we obtain

$$\frac{dy}{dx} = m \text{, and } \frac{d^2y}{dx^2} = 0$$

The equation (6) denotes the family of straight lines given by equation (5).

Observe that equations (3) and (5) are the general solutions of equations (4) and (6) respectively.



Procedure to form a differential equation that will characterize a given family of curves

a) If the given family F_1 of curves is dependent on only one parameter then it is denoted by an equation of the form

$$F_1(x, y, a) = 0$$

For instance, the family of parabolas $y^2 = ax$ can be characterized by an equation of the form $f(x,y,a): y^2 = ax$.

Differentiating equation (1) with reference to x, we obtain an equation involving y', y, x, and a, i.e.,

$g(x, y, y', a) = 0$

The desired differential equation is then attained by eliminating a from equations

(1) and (2) as

$F(x, y, y') = 0$

b) If the given family F_2 of curves is dependent on the parameters *a*, *b* (say) then it is characterized by an equation of the from, $F_2\ (x,y,a,b)=0$
Differentiating equation (4) with reference to *x*, we obtain an equation involving y', x, y, a, b, i.e.,
$g\ (x, y, y', a, b) = 0$

But it is not possible to remove two parameters *a* and *b* from the two equations and so, we require a third equation. This equation is attained by differentiating equation (5), with reference to *x*, to get a relation of the form

$$h\ (x, y, y', y'', a, b) = 0$$

The desired differential equation is then attained by eliminating *a* and *b* from equations (4), (5) and (6) as

$$F\ (x, y, y', y'') = 0$$

Example: For $y = mx$, where, *m* is arbitrary constant, form the differential equation which denotes the family of curves.

Solution We have

$y = mx$

Differentiating both sides of equation (1) with reference to *x*, we obtain

$$\frac{dy}{dx} = m$$

Substituting the value of *m* in equation (1) we obtain

$$y = \frac{dy}{dx} \cdot x$$

Or $$x\frac{dy}{dx} - y = 0$$

which is free from the parameter *m* and therefore this is the desired differential equation.

Methods of Resolving First Order, First Degree Differential Equations

In this section we shall talk about three methods of solving first order first degree differential equations.

Differential equations with variables separable

A first order-first degree differential equation would be in the form

$$\frac{dy}{dx} = \mathrm{F}(x, y)$$

If F (x, y) can be defined as a product g (x) h(y), where, g(x) is a function of x and h(y) is a function of y, then the differential equation (1) is termed to be of variable separable type. The differential equation (1) then has acquires the form

$$\frac{dy}{dx} = h\ (y)\ .\ g(x)$$

If h (y) ≠ 0, sorting out the variables, (2) can be rewritten as

$$\frac{1}{h(y)}\ dy = g(x)\ dx$$

Integrating both sides of (3), we obtain

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

Hence (4) gives us the solutions of given differential equation in the form

$$H(y) = G(x) + C$$

Here, H (y) and G (x) are the anti derivatives of $^{1/}$ $h(y)$ and g (x) respectively and C as the arbitrary constant.

Homogeneous differential equations

Study the following functions in x and y

$F_1(x,y)=y^2+2xy$, $F_2(x,y)=2x-3y$,

$$F_3(x, y) = \cos\left(\frac{y}{x}\right), \qquad F_4(x, y) = \sin x + \cos y$$

If we substitute x and y by λx and λy correspondingly in the above functions, for any nonzero constant λ, we obtain

$$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$$
$$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$$

$$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 \; F_3(x, y)$$
$$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y), \text{ for any } n \in \mathbf{N}$$

Here, we notice that the functions F_1, F_2, F_3 can be written in the form

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 cannot be written in this form. This leads to the following

definition:

A function F(x, y) is termed as *homogeneous function of degree n* if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

We observe that in the above instances, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 correspondingly but F_4 is not a homogeneous function.

We also notice that

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = x^2 h_1\left(\frac{y}{x}\right)$$

or

$$F_1(x, y) = y^2\left(1 + \frac{2x}{y}\right) = y^2 h_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(2 - \frac{3y}{x}\right) = x^1 h_3\left(\frac{y}{x}\right)$$

or

$$F_2(x, y) = y^1\left(2\frac{x}{y} - 3\right) = y^1 h_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0 \cos\left(\frac{y}{x}\right) = x^0\, h_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n\, h_6\left(\frac{y}{x}\right), \text{ for any } n \in \mathbf{N}$$

or

$$F_4(x, y) \neq y^n\, h_7\left(\frac{x}{y}\right), \text{ for any } n \in \mathbf{N}$$

Hence, a function F (x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n h\left(\frac{x}{y}\right)$$

A differential equation in the form $\frac{dy}{dx} = \mathbf{F}\ (x,\ y)$ is called as *homogenous* if F(*x*, *y*) is a homogenous function with degree zero.

To resolve a homogeneous differential equation with the type

$$\frac{dy}{dx} = \mathrm{F}(x, y) = g\left(\frac{y}{x}\right)$$

We arrange the substitution $y = v \cdot x$

Differentiating equation (2) with reference to x, we obtain

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we obtain

$$v + x\frac{dv}{dx} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

Separating the variables in equation (4), we obtain

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

We obtain the following, integrating both the sides of equation (5),

$$\int \frac{dv}{g(v) - v} = \int \frac{1}{x} dx + C$$

Equation (6) provides general solution (primitive) of the differential equation (1) when we substitute v by $\frac{y}{x}$.

Example Prove that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be defined as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}$$

Let $\quad F(x, y) = \frac{x+2y}{x-y}$

Now $\quad F(\lambda x, \lambda y) = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^0 \cdot f(x, y)$

Thus, F (x, y) is a homogenous function of degree zero. So, the given differential equation is declared to be a homogenous differential equation.

Linear differential equations

A differential equation of the from

$$\frac{dy}{dx} + Py = Q$$

where, P and Q are constants or functions of *x* only, is called as a first order linear differential equation. Some instances of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

One more form of first order linear differential equation is

$$\frac{dx}{dy} + P_1 x = Q_1$$

where, P_1 and Q_1 could be the constants or functions of *y* only. Some instances of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To resolve the first order linear differential equation of the type

$$\frac{dy}{dx} + Py = Q$$

Multiply both the sides of the equation by a function of *x* say *g* (*x*) to obtain

$$g(x)\ \frac{dy}{dx} + P.(g(x))\ y = Q.g(x)$$

Select *g* (*x*) in such a way that R.H.S. becomes a derivative of *y* . *g* (*x*).

i.e. $$g(x)\ \frac{dy}{dx} + P.g(x)\ y = \frac{d}{dx}\ [y.g\ (x)]$$

or $$g(x)\ \frac{dy}{dx} + P.g(x)\ y = g(x)\ \frac{dy}{dx} + y\ g'(x)$$

$\Rightarrow$ $$P.g(x) = g'(x)$$

or $$P = \frac{g'(x)}{g(x)}$$

Integrating both the sides with respect to *x*, we obtain

$$\int P dx = \int \frac{g'(x)}{g(x)} dx$$

or $$\int P \cdot dx = \log(g(x))$$

or $$g(x) = e^{\int P\, dx}$$

On multiplying the equation (1) with $g(x) = e^{\int P\,dx}$ the L.H.S. becomes the derivative of a function of *x* and *y*. The function $g(x) = e^{\int P\,dx}$ is called as the *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of *g* (*x*) in equation (2), we obtain

$$e^{\int P\,dx}\frac{dy}{dx} + P e^{\int P\,dx} y = Q \cdot e^{\int P\,dx}$$

Or $\frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P\,dx}$

Integrating both the sides with reference to x, we obtain

$$y \cdot e^{\int P\,dx} = \int\left(Q.e^{\int P\,dx}\right)dx$$

or

$$y = e^{-\int P\,dx} . \int\left(Q.e^{\int P\,dx}\right)dx + C$$

which has to be the general solution of the differential equation.

Steps included to solve first order linear differential equation:

Convert the given differential equation in the form $\frac{dy}{dx} + Py = Q$ where P, Q are

are the constants or functions of *x* only.

Find out the Integrating Factor $(I.F) = e^{\int P\,dx}$

Convert the solution into the given differential equation as

$$y\,(\mathrm{I.F}) = \int (\mathrm{Q} \times \mathrm{I.F})\,dx + \mathrm{C}$$

If in case, the first order linear differential equation is in the form

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

, where, P_1 and Q_1 could be constants or functions of *y* only, then I.F = $e^{\mathrm{P}1\ dy}$ and the solution of the differential equation is as provided by

$$x\,.\,(\mathrm{I.F}) = \int (\mathrm{Q}_1 \times \mathrm{I.F})\,dy + \mathrm{C}\,.$$