Theory Assignment II

Discrete Structures Monsoon 2024, IIIT Hyderabad

August 30, 2024

Total Marks: 30 points | Due date: **18/09/24 11:59 pm**

General Instructions: All symbols have the usual meanings (example: $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on). Every problem is mapped to the following course outcomes: CO 1, CO 2, CO 3, and CO 4.

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1. [2+2+2+4 points] Professor Infiniti has a magical drawing board that extends forever in all directions. She challenges her students with a puzzle:

   "On this infinite board, I want you to draw as many shapes as possible without any of them touching or overlapping."

   She gives four different challenges:

   a) Draw straight line segments

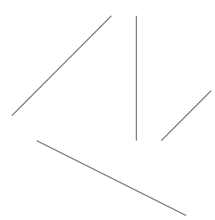
   

   Figure 1: Filling up a plane with line segments

   b) Draw hollow squares (just the outlines)

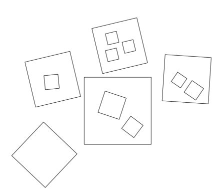
   

   Figure 2: Filling up a plane with squares

c) Draw filled-in squares (hence now a square includes the outline as well as the area inside of it)

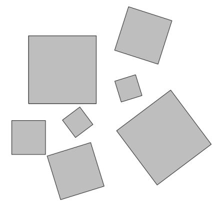


Figure 3: Filling up a plane with solid squares

d) Draw infinity symbols (∞)

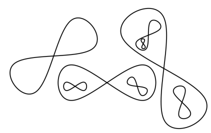


Figure 4: Filling up a plane with infinities

Note that you are free to choose any size for any of your shapes, you just need to ensure that the shapes don't touch or intersect with each other. For each shape, your task is to figure out:

a) Is it possible to draw an **uncountable** number of these shapes?

b) If yes, describe how you would do it (Just a simple construction would be enough, along with an explanation of why it works)

c) If no, explain why it's impossible. (Provide a rigorous proof)

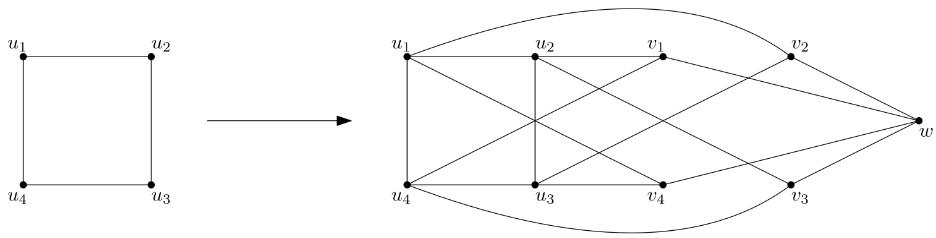
Remember, Professor Infiniti's board is infinitely large, so you have all the space you need. But can you fit an "uncountable" number of each shape? (**Hint:** You might have to utilize the fact that rational numbers *are a dense subset* of the real numbers)

2. [5 points] Given an undirected graph G, with n vertices $u_1, u_2 \dots u_n$, we construct a new graph G' as follows.

   1. Add $n+1$ new vertices $v_1, v_2 \dots v_n$ and w
   2. Add all of the edges from G in G'
   3. In addition, for all v_i, connect it to all neighbours of u_i
   4. Finally, connect all of v_i to w

   Prove that if G does not have any cycle of length 3 (a triangle), then G' also does not have any cycle of length 3.

Given below is the construction of G' from G for an example graph



G G'

3. [3 points] We define that a fraction is irreducible if its numerator and denominator are co-prime. For example $\frac{4}{7}$ is irreducible, however $\frac{4}{6}$ isn't irreducible, since 4 and 6 are not co-prime. Count the number of **proper irreducible** fractions, where the product of the numerator and denominator is 30!

4. [2 points] If $A_1, A_2, \ldots, A_n$ are n sets in a universe Q of N elements, then prove that the number N_m of elements in exactly m sets and the number N_m^* of elements in at least m sets are given by

   (a) $N_m = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} + \cdots + (-1)^{k-m} \binom{k}{m} S_k + \cdots + (-1)^{n-m} \binom{n}{m} S_n$

   (b) $N_m^* = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} + \cdots + (-1)^{k-m} \binom{k-1}{m-1} S_k + \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n$

   Where S_m is the sum of the sizes of all m-tuple intersections of the A_is.

   (**Hints:** (a) Use the fact that $\binom{k}{m}\binom{r}{k} = \binom{r}{m}\binom{r-m}{k-m}$. (b) Is there a relation between N_m and N_m^* that you can capitalize on?)

5. [3 points] Let n be a positive integer. Show that if you have n integers, then either one of them is a multiple of n or a sum of several of them is a multiple of n using the pigeonhole principle.

6. [3 points] Let A be a non-empty set. Show that the following are equivalent:

   (a) A is countable.

   (b) There exists a surjection $f : \mathbb{N} \to A$.

   (c) There exists an injection $g : A \to \mathbb{N}$.

7. [2 points] How can it be demonstrated that the power set of positive integers $\mathbb{Z}^+$ and the set of real numbers $\mathbb{R}$ have the same cardinality, i.e., $|P(\mathbb{Z}^+)| = |\mathbb{R}| = c$, where c represents the cardinality of the set of real numbers?

8. [2 points] Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n.