linear ordinary differential equations of higher order

Differential equation: An equation containing the derivatives of one or more dependent variables (or unknown functions) with respect to one or more independent variables is called a differential equation.

ex: 1) $\frac{dy}{dx} + 5y = 0$ 2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

3) $\frac{dy}{dx} + 4x\sin y = e^x$ 4) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

5) $u_{xx} + u_{yy} = 2xy$ 6) $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$

Ordinary differential equation (ODE): A differential equation involving ordinary derivatives of one or more dependent variables wrt a single independent variable is called an ODE. Examples (1) – (3) are ODEs.

The alternative (more than one independent variable) is called a partial differential equation (PDE). Examples (4) – (6) are PDEs.

Order of differential eq: It is defined as highest derivative appearing in the equation.

Degree of differential eq: It is defined as the positive integral power of the highest ordered derivative appearing in the equation

Note: In order to obtain degree of differential equation, the equation must be expressed in a form free from the radicals and fractional powers as far as derivatives are concerned.

ex: 1) $\frac{d^2y}{dx^2} + 16\frac{dy}{dx} + 2y = 0$: Order − 2, Degree − 1

2) $\left[\frac{d^2y}{dx^2} + 2y\right]^{3/2} = \frac{dy}{dx} + 2$: Order − 2, Degree − 3

3) $\frac{d^4y}{dx^4} + \sin y''' = 0$: Order − 4, Degree − not defined

<u>Linear differential equation</u> (LDE):

A differential equation is said to be linear if the dependent variable and all its derivatives are of first degree and they are not multiplied together.

Otherwise, the D.E. is said to be non-linear.

Examples:- 1) $\frac{dy}{dx} + 2y = e^x \longrightarrow$ Linear

2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = \cos x \longrightarrow$ Linear

3) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \rightarrow$ Non-Linear

4) $y''' + yy'' + 2y\tan x = 0 \longrightarrow$ Non-Linear

5) $y'' + 2xy' = \sin y \longrightarrow$ Non-Linear

because $\sin y$ involves powers of y.

Initial value problem (IVP):

Differential equation subject to the condition prescribed at one point is called an IVP.

ex: 1) $\frac{dy}{dx} + x^2 y = e^{2x}$; $y(0) = 1$

2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6y = \sin x$; $y(1) = 1$, $y'(1) = 2$

Boundary value problem (BVP):

Differential equation subject to the condition prescribed at more than one point is called boundary value problem.

ex: 1) $\frac{d^2y}{dx^2} + y = \cos 2x$; $y(0) = 1$, $y(\pi) = 0$

2) $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 6y = 2$; $y(0) = 1$, $y(2) = \pi$

Linear ODE of higher order

A linear ODE of n^{th} order is of the form:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = g(x)$$

or $(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = f(x) \rightarrow$ (1)

where $a_0, a_1, \dots a_n$ are constants or functions of x alone, $D = \frac{d}{dx}$ and $a_0 \neq 0$.

Let $F(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$ be a differential operator then (1) can be written as

$$F(D) y = g(x) \rightarrow (2)$$

If $g(x) = 0$, then (1) is said to be homogeneous otherwise it is said to be non-homogeneous.

If $y_1, y_2, \ldots y_n$ are n linearly independent solutions of n^{th} order homogeneous differential eq. then $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ where $c_1, c_2, \ldots c_n$ are arbitrary constants is also a solution of homogeneous equation.

If y_p is a particular function which satisfies non-homogeneous linear differential eq. then the general solution of (2) is $\boxed{y = y_c + y_p}$

where $y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is called as complementary function (CF) of corresponding homogeneous D.E. and y_p is called particular integral of (2) involving no arbitrary constants

Homogeneous LDE with constant coefficients

Consider a n^{th} order homogeneous ODE,

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n) y = 0 \rightarrow (*)$$

where a_i's are constants.

Method to find solution of (*)

Consider 2^{nd} order homogeneous LDE,

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \rightarrow (1)$$

Let $y = e^{mx}$ be a trial solution of (1)

Then (1) $\Rightarrow$ $a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$

$$\Rightarrow (a_0 m^2 + a_1 m + a_2) e^{mx} = 0$$

$$\Rightarrow a_0 m^2 + a_1 m + a_2 = 0 \rightarrow (2) \quad (\because e^{mx} \neq 0, \forall x)$$

(2) is called Auxiliary equation (A.E.) or characteristic equation of (1)

The independent solutions of (1) depend on the nature of the roots of above quadratic equation.

Case 1: Roots are real and distinct ($m_1 \neq m_2$, real)

If m_1 and m_2 are 2 distinct real roots, then the linearly independent solutions are $e^{m_1 x}$ and $e^{m_2 x}$

The general solution of (1) is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case 2: Roots are real and equal ($m_1 = m_2$)

If the roots of auxiliary eq. are real and repeated say $m_1 = m_2 = m$ then the general solution of (1) is

$$\boxed{y = c_1 e^{mx} + c_2 x e^{mx} = (c_1 + c_2 x) e^{mx}}$$

Case 3: Roots are complex ($m_1 = a+ib$, $m_2 = a-ib$)

The general solution of (1) is

$$\boxed{y = e^{ax} (c_1 \cos bx + c_2 \sin bx)}$$

Note: If the ODE is 4^{th} order and roots of auxiliary eq. are complex repeated say $m = a \pm ib$, $m = a \pm ib$ then the general solution of (1) is

$$\boxed{y = e^{ax} \left[(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx \right]}$$

Solve the foll D.Es :-

1) $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$

Sol: The auxiliary eq. is $2m^2 - 5m - 3 = 0$

$$2m^2 - 6m + m - 3 = 0$$

$$2m(m-3) + 1(m-3) = 0$$

$$\therefore m = 3, -1/2$$

Roots are real and distinct

$\therefore$ The general solution is $y = c_1 e^{3x} + c_2 e^{-x/2}$

$-6m^2$

$-6m \quad m$

2) $y'' - 10y' + 25y = 0$

Sol: The auxiliary eq is $m^2 - 10m + 25 = 0$

$$\Rightarrow (m-5)^2 = 0$$

$$\therefore m = 5, 5$$

Roots are real and equal

$\therefore$ G.S. is $y = (c_1 + c_2 x) e^{5x}$

3) $y'' + 4y' + 7y = 0$

Sol: The auxiliary eq is $m^2 + 4m + 7 = 0$

$$m = \frac{-4 \pm \sqrt{16-28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2\sqrt{3}\, i}{2}$$

$$\therefore m = -2 \pm \sqrt{3}\, i$$

$$\Rightarrow m_1 = -2 + \sqrt{3}\, i, \quad m_2 = -2 - \sqrt{3}\, i$$

The general solution is

$$y = e^{-2x}(c_1 \cos\sqrt{3}\, x + c_2 \sin\sqrt{3}\, x)$$

4) $(D^3+1)y = 0$

Sol:

$$AE : m^3 + 1 = 0$$

$$(m+1)(m^2+1-m) = 0$$

$$m+1=0 , \quad m^2 - m + 1 = 0$$

$$\Rightarrow m = -1 , \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The general solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

5) $(D^4 + 8D^2 + 16)y = 0$

Sol:

$$AE : m^4 + 8m^2 + 16 = 0$$

$$(m^2+4)^2 = 0$$

$$\Rightarrow (m^2+4)(m^2+4) = 0$$

$$\Rightarrow m = \pm 2i , \pm 2i$$

The general solution is

$$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

6) $y''' - 3y' + 4y = 0$

Sol:

$$AE : m^3 - 3m^2 + 4 = 0$$

By inspection $m = -1$ is a root

-1	1	-3	0	4
	0	-1	4	-4
	1	-4	4	0

$$A.E : (m+1)(m^2 - 4m + 4) = 0$$

$$(m+1) = 0 , \quad (m-2)^2 = 0$$

$$m_1 = -1, \quad m_2 = m_3 = 2$$

$\therefore$ The general solution is

$$y = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$$

7) Solve IVP $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$, given $x(0) = 0$, $\dfrac{dx}{dt}(0) = 15$

sol: AE: $m^2 + 5m + 6 = 0$

$$(m+3)(m+2) = 0$$

$$\therefore m = -3, -2$$

The general sol. is $x = c_1 e^{-3t} + c_2 e^{-2t} \rightarrow (1)$

Using the initial conditions $x(0) = 0$

$$(1) \Rightarrow \quad 0 = c_1 + c_2 \rightarrow (2)$$

$$x' = \frac{dx}{dt} = -3c_1 e^{-3t} - 2c_2 e^{-2t}$$

$$x'(0) = 15 \Rightarrow 15 = -3c_1 - 2c_2 \rightarrow (3)$$

$$(2) \times 3 \Rightarrow 3c_1 + 3c_2 = 0 \rightarrow (4)$$

$$(3) + (4) \Rightarrow c_2 = 15$$

$$\therefore c_1 = -15$$

$$(1) \Rightarrow x = -15 e^{-3t} + 15 e^{-2t} = 15\left(e^{-2t} - e^{-3t}\right)$$

8) Solve BVP $y'' - 2y' + 2y = 0$, given $\underbrace{y(0) = 1,\ y'(\pi) = 1}_{\text{Boundary conditions}}$

Sol: A E : $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore$ G.S. is $y = e^x (c_1 \cos x + c_2 \sin x) \rightarrow (1)$

Given: $y(0) = 1$

$(1) \Rightarrow 1 = e^0 (c_1 \cos 0 + c_2 \sin 0) = 1(c_1 + 0)$

$\therefore c_1 = 1$

$(1) \Rightarrow y' = e^x (-c_1 \sin x + c_2 \cos x) + (c_1 \cos x + c_2 \sin x) e^x \rightarrow (2)$

Given : $y'(\pi) = 1$

$(2) \Rightarrow 1 = e^\pi (c_2 \cos \pi) + c_1 \cos \pi \, e^\pi$

$1 = -c_2 e^\pi - e^\pi$

$1 + e^\pi = -c_2 e^\pi$

$\therefore c_2 = -\left(\frac{1+e^\pi}{e^\pi}\right)$

$(1) \Rightarrow y = e^x \left[\cos x - \left(\frac{1+e^\pi}{e^\pi}\right) \sin x\right]$

Exercise:

1) Solve $(D^3 - 13D + 12)y = 0$ Ans: $y = c_1 e^x + c_2 e^{-4x} + c_3 e^{3x}$

2) Solve $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$ Ans: $c_1 e^{4x} + (c_2 + c_3 x + c_4 x^2) e^{-x}$

3) $(D^2+1)^2 + (D-1)^2 y = 0$ Ans: $(c_1 + c_2 x) \sin x + (c_2 + c_4 x) \cos x + (c_5 + c_6 x) e^x$

4) Solve IVP $\frac{d^2y}{dx^2} + 8y = 0$, given $y(0) = 1$, $y'(0) = 2\sqrt{2}$

Ans: $y = \cos 2\sqrt{2}\, x + \sin 2\sqrt{2}\, x$

Non-homogeneous linear ODE with constant coefficients - Inverse differential operator

The general soln of non-homogeneous ODE has 2 parts - complementary function and particular integral

The complementary function is solution of homogeneous ODE.

Method of finding particular integral

Consider $F(D)\,y = g(x)$

then $y_p = \frac{1}{F(D)}\, g(x) \longrightarrow (1)$

i.e. $P.I. = \frac{1}{F(D)}\, g(x)$ where $\frac{1}{F(D)}$ is called 'inverse differential operator.'

Rules for finding particular integral:

$m^3+1=0$
$m=-1$

1) Let $g(x) = K e^{ax}$ where $a \in R$ then (1) becomes

$$y_p = \frac{K e^{ax}}{F(D)} = \frac{K e^{ax}}{F(a)}, \text{ provided } F(a) \neq 0$$

If $F(a) = 0$ then $y_p = \frac{K x e^{ax}}{F'(a)}$, provided $F'(a) \neq 0$

If $F'(a) = 0$ then $y_p = \frac{K x^2 e^{ax}}{F''(a)}$, provided $F''(a) \neq 0$ and so on.

Ex: $P.I. = \frac{e^{-x}}{D^3+1} = \frac{x e^{-x}}{3D^2} = \frac{x e^{-x}}{3}$

2) Let $g(x) = \sin(ax+b)$ or $\cos(ax+b)$ then (1) becomes

$$y_p = \frac{1}{F(D)} \sin(ax+b) \text{ or } \frac{1}{F(D)} \cos(ax+b)$$

We have 3 cases:

Case 1: Suppose $F(D)$ contains only even powers of D

i.e. $F(D) = F(D^2)$

Then $y_p = \frac{1}{F(D^2)} \sin(ax)$ or $\frac{1}{F(D^2)} \cos(ax)$

It can be shown that

$$y_p = \frac{1}{F(-a^2)} \sin ax \text{ or } \frac{1}{F(-a^2)} \cos ax, \text{ provided } F(-a^2) \neq 0$$

ex:- $\frac{1}{D^4+D^2+1} \sin 2x = \frac{\sin 2x}{(D^2)^2+D^2+1} = \frac{\sin 2x}{(-4)^2+(-4)+1} = \frac{\sin 2x}{16-4+1}$

$$= \frac{\sin 2x}{13}$$

If $F(-a^2) = 0$, we proceed as follows:

ex: 1) $\frac{1}{D^2+4} \sin 2x = \frac{x}{2D} \sin 2x = \frac{x}{2}\left\{\frac{1}{D} \sin 2x\right\}$

$$= \frac{x}{2}\left\{-\frac{\cos 2x}{2}\right\} = -\frac{x}{4} \cos 2x$$

2) $\frac{1}{D^2+9} \cos 3x = \frac{x}{2D} \cos 3x = \frac{x}{2}\left\{\frac{1}{D} \cos 3x\right\}$

$$= \frac{x}{2}\left\{\frac{1}{3} \sin 3x\right\} = \frac{x}{6} \sin 3x$$

Case 2: Suppose $F(D)$ also contains odd powered terms.

Then we proceed as in the foll example:

ex: $y_p = \frac{1}{D^2+2D+1} \sin 3x = \frac{1}{-9+2D+1} \sin 3x = \frac{\sin 3x}{2D-8}$

$$= \frac{2D+8}{(2D-8)(2D+8)} \sin 3x$$

$$= \frac{2D+8}{4D^2-64} \sin 3x$$

$$= \frac{(2D+8)}{-100}\sin 3x$$

$$= \frac{6\cos 3x}{-100} - \frac{8\sin 3x}{100}$$

$$= -\frac{1}{100}\left\{6\cos 3x + 8\sin 3x\right\}$$

$$\therefore\ y_p = -\frac{1}{50}\left\{3\cos 3x + 4\sin 3x\right\}$$

Case 3: If $F(D) = D^2 + a^2$

Then $F(D)\,y = \sin ax$ or $\cos ax$

$$\Rightarrow y_p = \frac{1}{D^2+a^2}\sin ax \text{ or } \frac{1}{D^2+a^2}\cos ax$$

Now if we replace D^2 by $(-a^2)$ then the denominator becomes zero, so in this case we cannot replace D^2 by $-a^2$

It can be shown that

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax$$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$$

3) If $g(x) = x^n$ then $y_p = \frac{1}{F(D)}x^n$

In order to evaluate P.I., we write $\frac{1}{F(D)}$ as $[F(D)]^{-1}$ and expand it in ascending powers of D and then each term in R.H.S. is operated on x^n to get the required P.I.

Note : 1) WKT $D^k x^n = \frac{d^k}{dx^k}(x^n) = 0$ if $k > n$

so the expansion is carried out only upto the term containing D^n

2) The following series expansions are useful:

a) $(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$

b) $(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$

c) $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$

d) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$

ex: $\frac{1}{D^2-1} \cdot x^3$

$= -\frac{1}{(1-D^2)} x^3$

$= -(1 + D^2 + D^4 + \dots) x^3$

$= -(x^3 + 6x)$

4) If $g(x) = e^{ax} V(x)$ where $V(x)$ is a function of x

$$y_p = \frac{1}{F(D)} e^{ax} V(x) = e^{ax} \frac{1}{F(D+a)} V(x)$$

The R.H.S. can be evaluated by earlier methods.

Examples:

1) Solve : $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

Sol: The auxiliary equation is $(m-2)^2 = 0$, the roots are $m = 2, 2$

$$\therefore y_c = c_1 e^{2x} + c_2 x e^{2x}$$

The particular integral is obtained as follows:

$$y_p = 8\left[\frac{e^{2x}}{(D-2)^2} + \frac{\sin 2x}{(D-2)^2} + \frac{x^2}{(D-2)^2}\right]$$

$$\frac{1}{(D-2)^2} e^{2x} = \frac{x^2 e^{2x}}{2}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-4-4D+4}\sin 2x = -\frac{1}{4D}\sin 2x = \frac{\cos 2x}{8}$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{\left\{-2\left(1-\frac{D}{2}\right)\right\}^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2} = \frac{1}{4}\left[1+\frac{2D}{2}+3\left(\frac{D}{2}\right)^2+\dots\right]x^2$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{4}\left\{x^2+D(x^2)+\frac{3}{4}D^2(x^2)\right\} = \frac{1}{4}\left\{x^2+2x+\frac{3}{2}\right\}$$

$$\therefore y_p = 4x^2e^{2x}+\cos 2x+2x^2+4x+3$$

$\therefore$ The general solution is $y = y_c + y_p$

$$\therefore y = c_1e^{2x}+c_2xe^{2x}+4x^2e^{2x}+\cos 2x+2x^2+4x+3$$

2) $y''-4y = \cosh(2x-1)+3^x$

sol: The auxiliary equation is : $m^2-4=0$

$$\Rightarrow m = \pm 2$$

$$\therefore y_c = c_1e^{2x}+c_2e^{-2x}$$

The particular integral is:

$$y_p = \frac{1}{(D^2-4)}\cosh(2x-1) + \frac{1}{(D^2-4)}3^x$$

$$\frac{1}{(D^2-4)}\cosh(2x-1) = \frac{1}{(D^2-4)}\left\{\frac{e^{2x-1}+e^{-2x+1}}{2}\right\} = \frac{xe^{2x-1}}{2\cdot 2D}+\frac{xe^{-2x+1}}{2\cdot 2D}$$

$$= \frac{1}{8}\left\{xe^{2x-1}-xe^{-2x+1}\right\}$$

$$\frac{1}{(D^2-4)}3^x = \frac{1}{(D^2-4)}e^{x\ln 3} = \frac{1}{[(\ln 3)^2-4]}3^x$$

$\therefore$ The general solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{8}\left(xe^{2x-1} - xe^{-2x+1}\right) + \frac{1}{[(\log 3)^2 - 4]} 3^x$$

3) $y'' - 2y' + y = xe^x \sin x$

Sol: The A.E. is $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0$$

$$\therefore m = 1, 1$$

$$y_c = c_1 e^x + c_2 x e^x$$

$$y_p = \frac{1}{(D-1)^2} e^x x \sin x = e^x \frac{1}{(D-1+1)^2} x \sin x = e^x \frac{1}{D^2} x \sin x$$

Integrating $x \sin x$ twice using by parts

$$\int x \sin x \, dx = x(-\cos x) - \int 1 \cdot (-\cos x)\, dx = -x\cos x + \sin x$$

$$\int (-x\cos x + \sin x)\, dx = \int (-x\cos x)\, dx + \int \sin x \, dx$$

$$= -\left\{x \sin x - \int 1 \cdot \sin x \, dx\right\} - \cos x$$

$$= -\left\{x \sin x + \cos x\right\} - \cos x = -x\sin x - 2\cos x$$

$$\therefore y_p = -e^x(x\sin x + 2\cos x)$$

The general solution is $y = y_c + y_p$

$$\therefore y = c_1 e^x + c_2 x e^x - e^x(x \sin x + 2\cos x)$$

4) $(D^3+3D^2+3D+1)y = e^{-x}$

Sol: The auxiliary equation is $m^3+3m^2+3m+1=0$

$$\Rightarrow (m+1)^3 = 0$$

$$\therefore m = -1, -1, -1$$

$$\therefore y_c = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

$$y_p = \frac{1}{(D+1)^3} e^{-x} = \frac{x^3 e^{-x}}{6}$$

$$\frac{xe^{-x}}{3(D+1)^2} \Downarrow \frac{x^2 e^{-x}}{6(D+1)} \Downarrow \frac{x^3 e^{-x}}{6}$$

The general solution is $y = y_c + y_p$

$$\therefore y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + \frac{x^3 e^{-x}}{6}$$

5) Find the particular integral of $(D^3+4D)y = 2\sin x \cos x$

Sol:

$$y_p = \frac{1}{D^3+4D} \sin(2x) = \frac{1}{D(D^2+4)} \sin 2x$$

$$\therefore y_p = \frac{1}{3D^2+4} x \sin 2x = \frac{x}{3(-4)+4} \sin 2x = \frac{-x}{8} \sin 2x$$

6) Find the P.I. of $(D^2+9)y = x\cos x$

Sol:

$$y_p = \frac{1}{D^2+9} \mathrm{Re}(x e^{ix}) = \mathrm{Re} \frac{1}{D^2+9} (x e^{ix})$$

$$= \mathrm{Re}(e^{ix}) \frac{1}{(D+i)^2+9} (x) = \mathrm{Re}(e^{ix}) \frac{1}{D^2+i^2+2iD+9} \cdot x$$

$$= \mathrm{Re}(e^{ix}) \frac{1}{D^2+2iD+8} x = \mathrm{Re}(e^{ix}) \frac{1}{8\left(1+\frac{D^2+2iD}{8}\right)} x$$

$$= \frac{1}{8} Re\left(e^{ix}\right) \left\{1 + \frac{D^2+2iD}{8}\right\}^{-1} x$$

$$= \frac{1}{8} Re\left(e^{ix}\right) \left\{1 - \frac{D^2+2iD}{8} + \left(\frac{D^2+2iD}{8}\right)^2 - \dots\right\} x$$

$$= \frac{1}{8} Re\left(e^{ix}\right) \left\{x - \frac{i}{4}\right\} = \frac{1}{8} Re\left(\cos x + i\sin x\right)\left(x - \frac{i}{4}\right)$$

$$= Re\left\{\frac{1}{8}(\cos x + i\sin x)\left(\frac{4x-i}{4}\right)\right\} = \frac{1}{32}(4x\cos x + \sin x)$$

7) Find P.I. of $(D^2 - 4D + 4)y = x^2 e^{3x} + \sin^2 x$

Sol:

$$y_p = \frac{1}{(D-2)^2}\left(x^2 e^{3x} + \sin^2 x\right)$$

Consider $\frac{1}{(D-2)^2} x^2 e^{3x} = e^{3x} \frac{1}{\{(D+3)-2\}^2} x^2 = e^{3x} \frac{1}{(D+1)^2} x^2$

$$= e^{3x} (1+D)^{-2} x^2 = e^{3x}\left\{1 - 2D + 3D^2 - \dots\right\} x^2$$

$$\frac{1}{(D-2)^2} x^2 e^{3x} = e^{3x}\left\{x^2 - 4x + 6\right\}$$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$

$$\frac{1}{(D-2)^2} \sin^2 x = \frac{1}{(D-2)^2}\left\{\frac{1-\cos 2x}{2}\right\}$$

$$= \frac{1}{2} \frac{1}{(D^2-4D+4)}(1 - \cos 2x)$$

$$= \frac{1}{2}\left\{\frac{1}{D^2-4D+4}(1) - \frac{1}{D^2-4D+4}(\cos 2x)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{D^2-4D+4}\left(e^{0x}\right) - \frac{1}{(D^2-4D+4)}\cos 2x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{4} - \frac{1}{-4-4D+4}\cos 2x\right\}$$

$$\frac{1}{(D-2)^2}\sin^2 x = \frac{1}{8} + \frac{1}{2}\cdot\frac{1}{4D}(\cos 2x) = \frac{1}{8} + \frac{1}{8}\frac{\sin 2x}{2}$$

$$\therefore\ y_p = e^{3x}(x^2-4x+6) + \frac{1}{8} + \frac{1}{16}\sin 2x$$

8) Find P.I. of $(D^2+1)y = \sin x \sin 2x$

Sol: P.I. is

$$y_p = \frac{1}{D^2+1}\frac{1}{2}\left[\cos(-x) - \cos(3x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x\cos x}{2D} - \frac{1}{(-9+1)}\cos 3x\right]$$

$$\therefore\ y_p = \frac{1}{2}\left[\frac{x\sin x}{2} + \frac{1}{8}\cos 3x\right]$$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

Exercise Problems: Solve the foll DEs:

1) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} \cdot \sin x$ Ans: $y = c_1 e^{-x} + c_2 e^{-2x} - \dfrac{e^{2x}}{170}(7\cos x - 11\sin x)$

2) $(D^3 + D^2 - D - 1)y = \cos 2x$ Ans: $y = c_1 e^{x} + (c_2 + c_3 x)e^{-x} - \dfrac{1}{25}(2\sin x + \cos 2x)$

3) $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$ Ans: $y = c_1 e^{x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{x}{3}e^{-x} + \dfrac{5}{9}e^{2x}$

4) $(D^2 - 2D + 5)y = \sin 3x$ Ans: $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \dfrac{1}{26}(3\cos 3x - 2\sin 2x)$

5) $(D^2 - 6D + 9)y = e^{3x}(x^2 + 7x + 5)$ Ans: $y = (c_1 + c_2 x)e^{3x} + \dfrac{e^{3x}}{12}(x^4 + 14x^3 + 30x^2)$

6) $\dfrac{d^2y}{dx^2} - y = 2 + 5x$ Ans: $y = c_1 e^{x} + c_2 e^{-x} - 2 - 5x$

7) $(D^2 - 3D + 2)y = x^2 e^{3x}$ Ans: $y = c_1 e^{2x} + c_2 e^{x} - \dfrac{3e^{3x}}{4}(2x + 3)$

8) $(4D^2 - 1)y = e^{x/2} + 12e^{x}$ Ans: $y = c_1 e^{x/2} + c_2 e^{-x/2} + \dfrac{x}{2}e^{x/2} + 4e^{x}$

9) $(D^2 + 2D + 2)y = 2e^{-x}\sin x$ Ans: $y = e^{-x}(c_1 \cos x + c_2 \sin x) - xe^{-x}\cos x$

10) $(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$ Ans: $y = (c_1 + c_2 x)e^{2x} + \dfrac{1}{36}e^{-4x} - \dfrac{5}{169}(12\sin x + 5\cos 3x)$

11) $(D^3 + 8)y = x^4 + 2x + 1$ Ans: $y = c_1 e^{-2x} + e^{x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x) + \dfrac{1}{8}(x^4 - x + 1)$

Method of variation of parameters : It is useful to find particular integral (PI) for higher order differential equation (provided C.F. is known)

This method is more general method in which the function $g(x)$ is not restricted to any particular form.

The working rule is:

Consider a second order ODE $a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=g(x) \rightarrow (*)$

The C.F. of (*) is $y_c = c_1 y_1(x) + c_2 y_2(x). \rightarrow (1)$

We replace c_1 and c_2 in (1) by respective functions $A(x)$ and $B(x)$ (which will be determined later), we obtain

$$P.I. = A(x)\, y_1(x) + B(x)\, y_2(x)$$

(hence the name variation of parameters.), where

$$\boxed{A(x) = -\int \frac{y_2(x)\, g(x)}{a_0(x)\, W(y_1,y_2)}\, dx = -\int \frac{y_2\, g(x)}{a_0\, W}\, dx}$$

$$\boxed{B(x) = \int \frac{y_1(x)\, g(x)}{a_0(x)\, W(y_1,y_2)}\, dx = \int \frac{y_1\, g(x)}{a_0\, W}\, dx}$$

with the Wronskian of y_1 and y_2 $W(y_1,y_2) = W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Solve the foll ODEs by method of variation of parameters:

i) $\frac{d^2y}{dx^2} + y = \tan x$ ——— (1)

Soln: The G.S is

$$y = y_c + y_p$$

To find y_c :

Consider $\frac{d^2y}{dx^2} + y = 0$

$(D^2+1)y = 0$

AE: $m^2+1=0$

AE : $m^2 + 1 = 0$

Roots are $m = \pm\sqrt{-1}$

$= \pm i$

indep solns are $\cos x$, $\sin x$.

Thus, $\boxed{y_c = C_1 \cos x + C_2 \sin x}$ ——— (2)

To find y_p:

Assume $y_p = A(x)\cos x + B(x)\sin x$ ——— (3)

here $y_1(x) = \cos x$, $y_2(x) = \sin x$, $a_0(x) = 1$

non homo. term $g(x) = \tan x$,

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore A(x) = \int \frac{-g(x)\, y_2(x)}{a_0(x)\, W(y_1, y_2)}$$

$$= \int -\frac{\tan x \cdot \sin x}{1}\, dx$$

$$= \int -\frac{\sin^2 x}{\cos x}\, dx$$

$$= \int -\frac{(1-\cos^2 x)}{\cos x}\, dx$$

$$= \int (-\sec x + \cos x)\, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{a_0(x)\, W(y_1, y_2)}\, dx$$

$$= \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \sin x\, dx = -\cos x$$

Sub for $A(x)$ and $B(x)$ in ③,

$$y_p = \left(-\log(\sec x + \tan x) + \sin x\right)\cos x - \cos x \sin x$$

$$\Rightarrow \boxed{y_p = -\log(\sec x + \tan x)\cos x} \quad \text{——— ④}$$

The G.S is

$$y = y_c + y_p \quad (\text{sub.})$$

ii) $y'' - 4y' + 4y = (x+1)e^{2x}$ ——— (1)

Soln The G.S is

$$y = y_c + y_p.$$

To find y_c:

Consider $y'' - 4y' + 4y = 0$

AE: $m^2 - 4m + 4 = 0$

Roots are $m = 2, 2$ (repeated)

Indep. solns are e^{2x}, xe^{2x}

$\therefore \boxed{y_c = c_1 e^{2x} + c_2 x e^{2x}}$ ——— (2)

To find y_p:

$$y_p = A(x)\, e^{2x} + B(x)\, x e^{2x} \quad \text{——— (3)}$$

Here

$y_1(x) = e^{2x}$

$y_2(x) = xe^{2x}$

$a_0(x) = 1$

$g(x) = (x+1)e^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + x\,2e^{2x} \end{vmatrix}$$

$$= e^{4x} + \cancel{x\,2e^{4x}} - \cancel{2x\,e^{4x}}$$

$$= e^{4x}$$

$$A(x) = \int -\frac{g(x)\, y_2(x)}{a_0\, W(y_1, y_2)}\, dx$$

$$= \int -\frac{(x+1)\cancel{e^{2x}}\, x\, \cancel{e^{2x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (-x^2 - x)\, dx$$

$$= -\frac{x^3}{3} - \frac{x^2}{2}$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(y_1, y_2)\, a_0}\, dx$$

$$= \int \frac{(x+1)\cancel{e^{2x}}\,\cancel{e^{2x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (x+1)\, dx$$

$$= \frac{x^2}{2} + x$$

Sub for A and B in ③,

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right) e^{2x} + \left(\frac{x^2}{2} + x\right) x e^{2x}.$$

$$\Rightarrow \boxed{y_p = \left(\frac{1}{6} x^3 + \frac{1}{2} x^2\right) e^{2x}}$$

The G.S

$$y = y_c + y_p \qquad (\text{sub.})$$

Exercise:

Solve the following differential eqns by the method of variation of parameters.

1) $y'' - 2y' + y = x e^x \sin x$

Ans: $y = (c_1 + c_2 x)e^x - e^x (x \sin x + 2\cos x)$.

2) $y'' - 2y' = e^x \sin x$

Ans. $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$.

3) $y'' - 2y' + y = x e^x \log x$

Ans: $y = c_1 e^x + c_2 x e^x + \frac{1}{6} x^3 e^x \log x - \frac{5}{36} x^3 e^x$.

4) $y'' - 2y' + 2y = e^x \tan x$

Ans: $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$.

5) $y'' + 4y = \cot 2x$

Ans : $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2} \log(\tan x)$.

6) $y'' - y = \cosh x$

Ans : $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} x \sinh x$.

7) $4y'' + 36y = \operatorname{cosec} 3x$

Ans: $y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12} \cos 3x + \frac{1}{36} \log(\sin 3x) \sin 3x$

Cauchy-Euler equations

A. D.E. of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n = g(x) \quad \to (1)$$

where a_i's are constants and $g(x)$ is a function of x is called Cauchy-Euler D.E. of order n.

(1) can be reduced to ODE with constant coefficients by a suitable substitution as follows.

(1) can be written as

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_n) y = g(x) \to (2) \text{ where } D = \frac{d}{dx}$$

Put $x = e^{z} \Rightarrow z = \log_e x$

$$\frac{dz}{dx} = \frac{1}{x}$$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow \boxed{xDy = D_1 y} \quad \text{where } D_1 = \frac{d}{dz}$$

Again, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right)$

$$\left\{\frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx} + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)\right\}$$

$$= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow \boxed{x^2D^2y = D_1^2y - D_1y = D_1(D_1-1)y} \quad \Big| \quad \text{where } D_1^2 = \frac{d^2}{dz^2}$$

$|||^{ly}$ $\boxed{x^3D^3y = D_1(D_1-1)(D_1-2)y}$ and so on.

$\therefore$ (2) becomes $F(D_1)y = g_1(z) \rightarrow$ (3) where $g_1(z)$ is the value of $g(x)$ in terms of z.

(3) is now clearly LDE with constant coefficients and can be solved by the methods already discussed.

1) Solve : $4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$

Sol: The given ODE is 2^{nd} order Cauchy-Euler eq.

Let $x = e^z$

$$\Rightarrow xDy = D_1y,\ x^2D^2y = D_1(D_1-1)y \quad \text{where } D_1 = \frac{d}{dz}$$

The given eq reduces to

$$4D_1(D_1-1)y + 8D_1 y + y = 0$$

$$(4D_1^2 - 4D_1 + 8D_1 + 1)y = 0$$

$$\Rightarrow (4D_1^2 + 4D_1 + 1)y = 0$$

which is a LDE with constant coefficients

Auxiliary eq is $4m^2 + 4m + 1 = 0$

$$\Rightarrow (2m+1)^2 = 0$$

$$\therefore m = -\frac{1}{2}, -\frac{1}{2}$$

The general solution is : $y = c_1 e^{-\frac{1}{2}z} + c_2 z e^{-\frac{1}{2}z}$

$$\therefore y = c_1 e^{-\frac{1}{2}\log x} + c_2 \log x \, e^{-\frac{1}{2}\log x} = \frac{c_1}{\sqrt{x}} + c_2 \frac{\log x}{\sqrt{x}}$$

2) Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol: Let $x = e^z \Rightarrow z = \log x$

$x^2 D^2 y = D_1(D_1-1)y$, $xDy = D_1 y$ where $D_1 = \frac{d}{dz}$

The given eq reduces to

$$(D_1(D_1-1) - D_1 + 1)y = z$$

$$(D_1^2 - D_1 - D_1 + 1)y = z$$

$\therefore (D_1^2 - 2D_1 + 1)y = z$ is linear ODE with constant coefficients

A.E : $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \quad \therefore m = 1, 1$

$$\therefore\ y_c = c_1 e^{z} + c_2 z e^{z}$$

$$y_p = \frac{1}{(D_1-1)^2} z = \frac{1}{(1-D_1)^2} z = (1-D_1)^{-2} z = (1+2D_1+3D_1^2+\dots)z$$

$$\therefore\ y_p = z+2$$

$\therefore$ The complete solution is

$$y = y_c + y_p = c_1 e^{z} + c_2 z e^{z} + z + 2 \quad \text{where } z = \log x$$

$$\text{i.e. } y = c_1 x + c_2 \log x \cdot x + \log x + 2$$

3) $$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$$

sol: Let $x = e^{z}$ i.e. $z = \log x$

The given eq becomes

$$[D_1(D_1-1) - 3D_1 + 5]y = e^{2z}\sin z$$

$$\Rightarrow (D_1^2 - 4D_1 + 5)y = e^{2z}\sin z$$

A. E: $m^2 - 4m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

C.F: $y_c = e^{2z}(c_1 \cos z + c_2 \sin z)$

P.I: $$y_p = \frac{1}{D_1^2 - 4D + 5} e^{2z}\sin z = e^{2z}\frac{1}{(D_1+2)^2 - 4(D_1+2)+5}\sin z$$

$$= e^{2z}\frac{1}{D_1^2+4+4D_1-4D_1-8+5}\sin z = e^{2z}\frac{1}{D_1^2+1}\sin z$$

$$\therefore y_p = e^{2z}\frac{z\sin z}{2D_1} = \frac{e^{2z}}{2} z \cdot \frac{1}{D_1}\sin z = -\frac{z}{2}e^{2z}\cos z$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow y = e^{2z}(c_1\cos z + c_2 \sin z) - \frac{z}{2}e^{2z}\cos z$$

$$\therefore y = x^2\{c_1\cos(\log x) + c_2\sin(\log x)\} - \frac{x^2}{2}\log x \cos(\log x)$$

Exercise : Solve the following Differential equations:

1) $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ Ans: $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2}e^x$

2) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$

Ans: $\frac{c_1}{x} + \sqrt{x}\left[c_2\cos\frac{\sqrt{3}}{2}(\log x) + c_3\sin\frac{\sqrt{3}}{2}(\log x)\right] + \frac{x}{2} + \log x$

3) $4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - y = 4x^2$ Ans: $c_1\sqrt{x} + c_2\frac{1}{\sqrt{x}} + \frac{4}{15}x^2$

4) $\frac{d^3y}{dx^3} - \frac{4}{x}\frac{d^2y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$ (Hint: Multiply throughout by x^3) Ans: $y = c_1x^2 + x^{5/2}\left(c_2x^{\sqrt{21/2}} + c_3x^{\sqrt{-21/2}}\right) - \frac{1}{5}x^3$

5) $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \frac{1}{x}\right)$

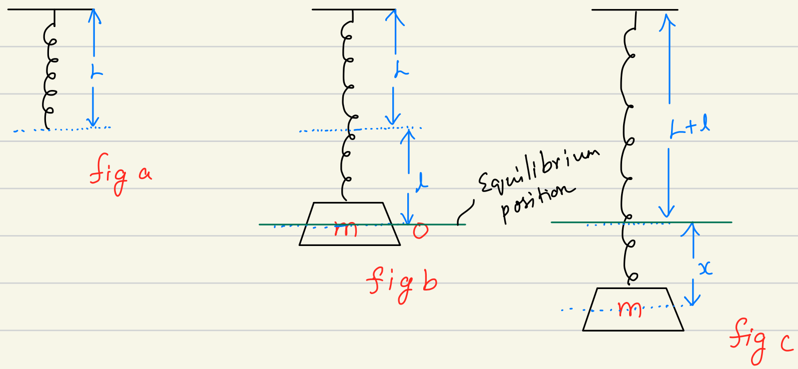
Ans: $y = \frac{c_1}{x} + x\left[c_2\cos(\log x) + c_3\sin(\log x)\right] + 5x + \frac{2}{x}\log x$

Applications of second order ODE

Mass and spring system

The differential equation of the vibrations of a mass on a spring.

The Basic problem:



- fig a shows natural length L of a spring, i.e, unstretch length of a spring.
- Now we attach a mass to the lower end of the spring and allow it to come to rest in an equilibrium position as given in fig b.
- The system is then set in motion. Let $x(t)$ denote the displacement of the mass from O along the line as shown in fig c.
- x is considered to be +ve, zero or −ve according to whether the mass is below, at or above its equilibrium position.

Problem is to determine the resulting motion of the mass on the spring.

We set up differential eqn to the problem, in order to do so we consider two laws of physics

1) Hooke's law
2) Newton's second law of motion

Hooke's law states that the magnitude of the force needed to produce certain elongation is directly proportional to the amount of its elongation

i.e $\boxed{|F| \propto l \Rightarrow |F| = kl,}$

where F is the force, l is the amount of elongation and k is a constant of proportionality called spring constant.

Newton's 2nd law states that net forces acting on an object is equal to the mass times acceleration of the object.

i.e $\boxed{\sum F = m\frac{d^2x}{dt^2}}$.

Here, we assume that forces tending to pull the mass downward are +ve, while those tending to pull it upward are negative.

In fig b The mass is in equibrium, Thus net forces on the mass is equal to zero.

Forces acting are i) Force of gravity (mg, g is acceleration due to gravity)

ii) Restoring force of the spring, equal to kl from Hooke's law

i.e $mg - kl = 0$ (mg acts downward (+ve) and kl acts upward (−ve))

$\Rightarrow \boxed{mg = kl}$ —— (1)

In fig c, the mass is in motion, x is the displacement of the mass at time $t>0$

1) F_1, the force of gravity

$$\boxed{F_1 = mg} \quad (2) \qquad (\text{it acts downward } (+ve))$$

2) F_2, the restoring force of the spring

$$F_2 = -k(x+l)$$

$$\Rightarrow \boxed{F_2 = -kx - mg} \quad (3) \qquad (\because kl = mg \text{ from } (1))$$

3) F_3, the resisting force of the medium, called the damping force

$$\boxed{F_3 = -a\frac{dx}{dt}} \quad (4) \qquad (-ve \text{ because it is against the motion of mass})$$

Here $a>0$, is called damping constant.

4) F_4, any external forces that act upon the mass.

Let $f(t)$ be the resultant of all external forces.

$$\boxed{F_4 = f(t)} \quad (5)$$

From Newton's law

$$F_1 + F_2 + F_3 + F_4 = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \cancel{mg} - kx - \cancel{mg} - a\frac{dx}{dt} + f(t) = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \boxed{m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t)} \quad (6)$$

This is a second order non-homogenous LDE with constant coefficients.

This we take as the DE for the motion of the mass on the spring.

Note: If $a=0$ the motion is called undamped. Otherwise it is called damped.

If $f(t)=0 \ \forall t$ the motion is called free, otherwise it is called forced.

Ex1: An 8-lb weight is placed upon the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. The weight is then pulled down 3 in below its equilibrium position and released at $t=0$ with an initial velocity of 1 ft/sec, directed downward. Neglecting the resistance of the medium and assuming that no external forces are present, determine the displacement of the weight and hence determine amplitude, period and frequency of the resulting motion.

Soln: Weight $W = 8$ lb

8 lb weight streches the spring by 6 in $= \frac{1}{2}$ ft $(= l)$

If $x(t)$ is the displacement of the mass at time t,

$x(0) = 3 \text{ in} = \frac{1}{4}$ ft,

$\frac{dx}{dt}(0) = 1$ ft/sec, (+ve because directed downward)

accelaration due to gravity, $g = 32$ ft/sec^2

$W = mg \Rightarrow m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}$

Given: $a=0$, $f(t)=0$ (external forces)

Spring constant $k = \frac{mg}{l} = \frac{8}{1/2} = 16$ lb/ft ($mg = kl$)

This is free and undamped motion, thus the DE is

$$m\frac{d^2x}{dt^2} + kx = 0$$

or

$$\frac{1}{4}\frac{d^2x}{dt^2} + 16x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + 64x = 0 \qquad (\lambda^2 = 64)$$

AE : $m^2 + 64 = 0$

Roots are $m = \pm 8i$

Independent solns are $\cos 8t$, $\sin 8t$

G.S is $\boxed{x = c_1 \cos 8t + c_2 \sin 8t}$ —— (1)

IC's are $x(0) = \frac{1}{4}$ ft, $\frac{dx}{dt}(0) = 1$ ft/sec

Diff (1) wrt t,

$$\frac{dx}{dt} = -c_1 8 \sin 8t + 8 c_2 \cos 8t \quad \text{—— (2)}$$

At $t=0$,

(1) $\Rightarrow$ $\boxed{\frac{1}{4} = c_1}$

(2) $\Rightarrow$ $1 = 8 c_2$ $\Rightarrow$ $\boxed{c_2 = \frac{1}{8}}$

Sub for c_1 and c_2 in (1),

$$\boxed{x(t) = \frac{1}{4} \cos 8t + \frac{1}{8} \sin 8t}$$

Amplitude is $\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2} = \frac{\sqrt{5}}{8}$ ft.

frequency is $\frac{8}{2\pi} = \frac{4}{\pi}$ osc / sec.

Time period is $\frac{\pi}{4}$ sec.

Free, damped motion

Let $a>0$ be the damping constanst and $f(t)=0 \ \forall t$. The basic DE reduces to

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + Kx = 0$$

$$r = \frac{-a \pm \sqrt{a^2-4mK}}{2m}$$

Depending on nature of these roots, 3 distinct cases occur.

Case 1: $a^2 - 4mK > 0$ (over damping)

Case 2: $a^2 - 4mK = 0$ (critical damping)

Case 3: $a^2 - 4mK < 0$ (under damping)

Ex2: A 32 lb - weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 2 ft. The weight is then pulled down 6 in below its equilibrium position and released at $t=0$. No external forces are present; but the resistance of the medium in pounds is equal to $4\frac{dx}{dt}$, where $\frac{dx}{dt}$ is the instantaneous velocity in feet per sec.

Detemine the resulting motion of the weight on the spring. Interpret the motion.

Soln: Given Weight $W = 32 lb$

It stretches the spring by 2 ft. By Hooke's law

$W = kl$ or $32 = k\,2$ $\Rightarrow$ $\boxed{k = 16\ lb/ft}$

mass $= \dfrac{W}{g} = \dfrac{32}{32} = 1$ $\qquad | \ g = 32\ ft/sec^2$

$\Rightarrow$ $\boxed{m = 1}$

Damping constant $\boxed{a = 4}$

External forces $\boxed{F(t) = 0}$

The DE of the problem:

$$\boxed{\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 16x = 0}$$

where $x(t)$ is a displacement of the mass at time t.

IC's $\boxed{\begin{array}{l} x(0) = \tfrac{1}{2} \\ x'(0) = 0 \end{array}}$ Since it is pulled 6 in $= \tfrac{1}{2}$ ft below its equilibrium position at $t=0$ and released it with no initial velocity.

Here $a^2 - 4mk = 16 - (4 \times 1 \times 16) = 16 - 64 = -48 < 0$

Since $a^2 - 4mk < 0$

So the motion is damped oscillatory (under damping)

AE of the DE is

$$r^2 + 4r + 16 = 0$$

Roots are $\dfrac{-4 \pm \sqrt{16-64}}{2} = -2 \pm 2\sqrt{3}$

indep. solns are $e^{-2t}\cos 2\sqrt{3}t$, $e^{-2t}\sin 2\sqrt{3}t$

G.S is

$$\boxed{x = e^{-2t}\left(c_1 \cos(2\sqrt{3}t) + c_2 \sin(2\sqrt{3}t)\right)}$$

Given $x(0) = \frac{1}{2}$

$$\therefore \quad \boxed{\frac{1}{2} = c_1}$$

$$x'(t) = e^{-2t}\left(-2\sqrt{3}\, c_1 \sin(2\sqrt{3}\, t) + 2\sqrt{3}\, c_2 \cos(2\sqrt{3}\, t)\right)$$
$$+ (-2)\, e^{-2t}\left(c_1 \cos(2\sqrt{3}\, t) + c_2 \sin(2\sqrt{3}\, t)\right)$$

But $x'(0) = 0$

$$\therefore \quad 0 = 2\sqrt{3}\, c_2 - 2c_1$$

$$\Rightarrow \quad \boxed{c_2 = \frac{1}{2\sqrt{3}}}$$

Sub for c_1 and c_2 in the G.S,

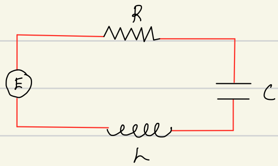
$$x(t) = e^{-2t}\left(\frac{1}{2}\cos(2\sqrt{3}\, t) + \frac{1}{2\sqrt{3}}\sin(2\sqrt{3}\, t)\right)$$

Exercise:

1) A 16 lb weight is attached to the lower end of a coil spring suspended from the ceiling the spring constant of the spring being 10 lb/ft. The weight comes to rest in its equilibrium position. Beginning at $t=0$ an external force given by $F(t) = 5\cos 2t$ is applied to the system. Determine the resulting motion if the damping force in pounds is equal to $2\frac{dx}{dt}$, where $\frac{dx}{dt}$ is the instantaneous velocity in ft. per second.

Ans: $\boxed{x = e^{-2t}\left(-\frac{1}{2}\cos 4t - \frac{3}{8}\sin 4t\right) + \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t}$

Electric circuit problems



Let E be emf, let i be the current in the circuit

Voltage drop across a resistor is iR

Voltage drop across an inductor is $L\frac{di}{dt}$

Voltage drop across a capacitor is $\frac{1}{C}q = \frac{1}{C}\int i\,dt$

Kirchhoff's voltage law (KVL) :

The sum of the voltage drops across resistors, inductors and capacitors is equal to the total emf in a closed circuit.

That is $$L\frac{di}{dt} + iR + \frac{1}{C}q = E(t) \text{ —— } (1)$$

WKT $\frac{dq}{dt} = i$

Thus, (1) $\Rightarrow$ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Also $$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$$

Ex: 1) Find the charge on the capacitor in a LCR series circuit when $L = \frac{5}{3}$ H, $R = 10\Omega$, $C = \left(\frac{1}{30}\right)f$, $E(t) = 300V$, $q(0) = 0$, $\frac{dq(0)}{dt} = 0$

Sol: $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t)$$

$$\frac{5}{3}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 30q = 300$$

$$\Rightarrow 5\frac{d^2q}{dt^2} + 30\frac{dq}{dt} + 90q = 900$$

A.E: $5m^2 + 30m + 90 = 0$

$\therefore m = -3 \pm 3i$

$$q_c = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$q_p = \frac{1}{5D^2+30D+90} 900 \cdot e^{0t} = \frac{1}{0+0+90} \times 900 = 10$$

G.S. is

$$q(t) = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t) + 10 \rightarrow (1)$$

Using $q(0) = 0$, $(1) \Rightarrow 0 = c_1 + 10 \Rightarrow c_1 = -10$

$\frac{dq}{dt} = 0$, $(1) \Rightarrow c_2 = -10$

$\therefore$ Charge on capacitor is:

$$q(t) = 10 - 10e^{-3t}[\cos 3t + \sin 3t]$$