DIFFERENTIAL CALCULUS

→ Recapitulation : Functions of single variable :

The concept of functions is important in calculus because they play a key role in describing the real-world problems in mathematical terms. The temperature at which water boils depends on the elevation above sea level (the boiling point drops as the height increases). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels from an initial location along a straight line path depends on its speed.

In each case, the value of one variable quantity, which might be called as y, depends on the value of another variable quantity, which might be called x. Since the value of y is completely determined by the value of x, it is said that y is a function of x. Often the value of y is given by a rule or formula that says how to calculate it from the variable x. For instance, $A = \pi r^2$, the equation is a rule that calculates the area A of a circle from its radius r.

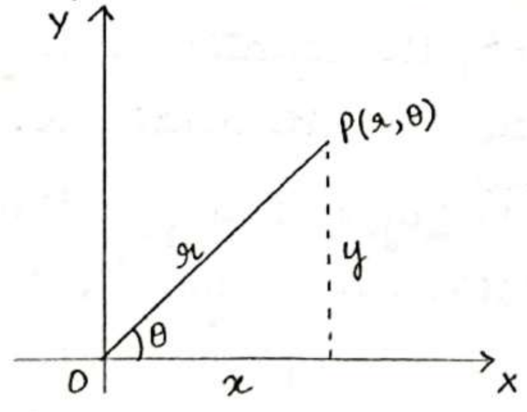
A symbolic way to say 'y' is a function of x is by writing $y = f(x)$. In this notation, the symbol f represents the function. The letter x, called the independent variable represents the input value of f and y, the dependent variable, represents the corresponding output value of f at x.

→ Requirement of new coordinate systems :

There is already a familiarity with Cartesian coordinate system for specifying a point in the XY-plane in two-dimensional geometry and XYZ-space in three-dimensional geometry. The requirement to define any new coordinate system is two-fold. One is based on geometry of the problem of practical situation wherein a more suitable coordinate system has to be chosen. For example, the study of dispersion of a medicine injected in blood flow requires cylindrical co-ordinate system as the veins are cylindrical in nature. Use of cartesian system may not be very suitable as it represents a rectangular channel and the corner effects have to be taken care. The second requirement is more of theoretical in nature. A mathematical expression which cannot be simplified in one coordinate system may be solved in simple way by transforming to other coordinate systems. For example, $\log(x+y)$ cannot be further simplified in Cartesian system whereas it is easier to solve in Polar co-ordinates.

Polar Co-ordinates

Consider a point P in the XY-plane. Join the point O (origin) and P. Let r be the length of OP and θ be the angle which OP makes with the (positive) axes. Then (r, θ) are called the polar coordinates of the point P and we write $P = (r, \theta)$ or $P(r, \theta)$



In particular, r is called the radial distance and θ is called polar angle. Also, O is called the pole, the x-axis is called the initial line and OP is called the radius vector.

Let (x, y) be the Cartesian coordinates of the point P. Then we find that

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x) \qquad \longrightarrow (1)$$

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \longrightarrow (2)$$

Relation (1) enable us to find the polar coordinates (r,θ) when the cartesian coordinates (x,y) are known. Conversely, the relation (2) enable us to find the cartesian coordinates when the polar coordinates are known.

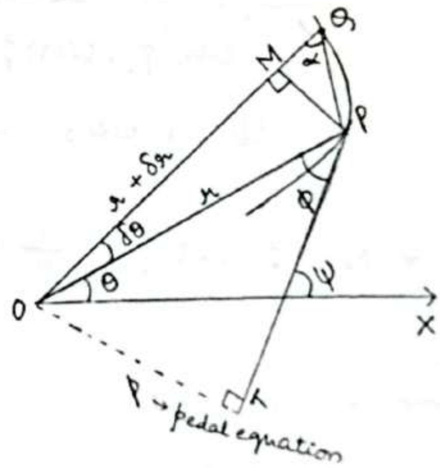
If P is a variable point on a plane curve C, then the equation of the curve in the cartesian form is a relationship of the form $y=f(x)$.

Similarly, the equation of curve C in the polar form (or the polar equation of C) is a relationship of the form $r=f(\theta)$, a curve specified by a polar equation referred to as a polar curve.

Angle between Radius Vector and Tangent

Let ϕ be the angle between the radius vector and the tangent at any point of the curve $r=f(\theta)$. Then $\tan\phi = r\dfrac{d\theta}{dr}$.

{Let $P(r,\theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points on the curve as shown in figure. Join PQ and draw $PM \perp OQ$. Then from the right angled triangle OMP,



$MP = r\sin\delta\theta$, $OM = r\cos\delta\theta$.

$$\therefore \quad MQ = OQ - OM = r+\delta r - r\cos\delta\theta$$

$$= \delta r + r(1-\cos\delta\theta) = \delta r + 2r\sin^2\frac{\delta\theta}{2}$$

If $\angle MQP = \alpha$, then $\tan\alpha = \dfrac{MP}{MQ} = \dfrac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$

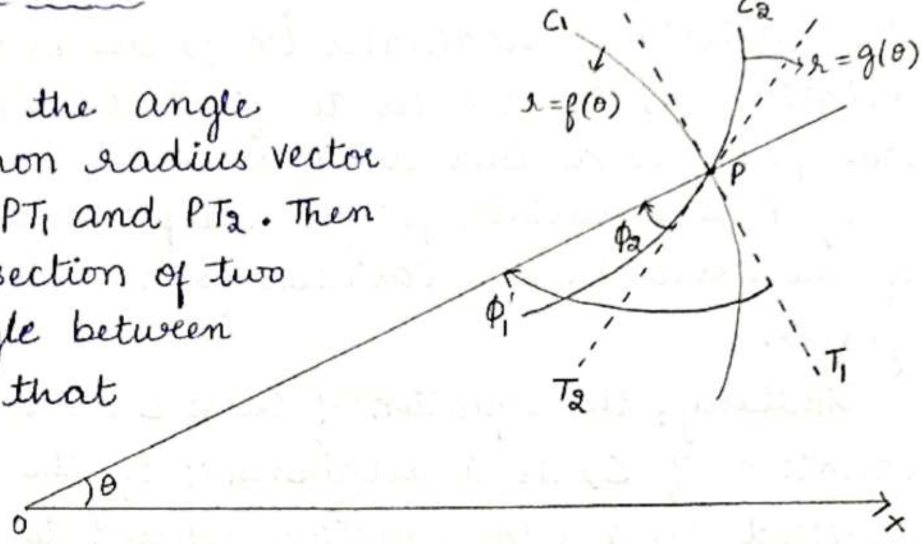
In the limit as $Q\to P$ (i.e., $\delta\theta \to 0$), the chord PQ turns about P and becomes the tangent at P and $\alpha \to \phi$.

$$\tan\phi = \lim_{Q\to P}(\tan\alpha) = \lim_{\delta\theta\to 0}\frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$

$$= \lim_{\delta\theta\to 0}\frac{r\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + r\sin\dfrac{\delta\theta}{2}\left(\dfrac{\sin\frac{\delta\theta}{2}}{\delta\theta/2}\right)} = \frac{r.1}{\dfrac{dr}{d\theta}+r.0.1} = r\frac{d\theta}{dr} \Bigg\}$$

extra

→ <u>Angle between polar curves</u>



Let ϕ_1 and ϕ_2 be the angle between the common radius vector OP and tangents PT_1 and PT_2. Then the angle of intersection of two curves is the angle between their tangents at that point which is given by $|\phi_1-\phi_2|$.

This angle is determined using the formula

$$\tan|\phi_1-\phi_2| = |\tan(\phi_1-\phi_2)| = \left|\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1 . \tan\phi_2}\right|$$

→ <u>Orthogonal Curves</u>

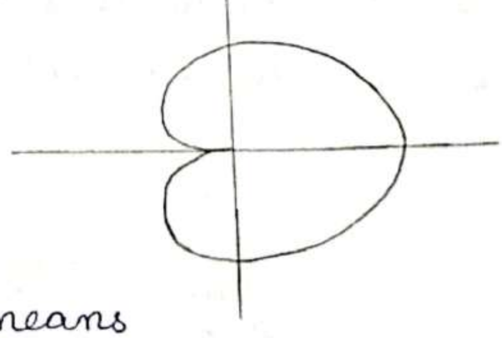
If $\tan\phi_1 . \tan\phi_2 = -1$, then $\tan(\phi_1-\phi_2) = \infty \Rightarrow \phi_1-\phi_2 = \pi/2$.
In this case, we say that the curves intersect orthogonally.

* Note: $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

→ <u>Examples</u>:

* Find the angle between the radius vector and the tangent to the following polar curves:

1) $r = a(1+\cos\theta)$



[Cardioid $r=a(1+\cos\theta)$ is a curve that is the locus of a point on the circumference of circle rolling round the circumference of a circle of equal radius. Ofcourse the name means heart shaped. Curve is symmetrical about the initial line.]

Soln Consider $r = a(1+\cos\theta)$

Differentiating w.r.t to θ, we have

$$\frac{dr}{d\theta} = -a\sin\theta$$

S | A
T | C

$$r\frac{d\theta}{dr} = -\frac{\cancel{a}(1+\cos\theta)}{\cancel{a}\sin\theta}$$

$$\tan\phi = -\frac{\cancel{2}\cos^{\cancel{2}}\theta/2}{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}} = -\cot\frac{\theta}{2}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta + 1 = 2\cos^2\theta$

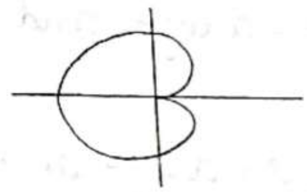
$$\tan\phi = \tan\left(\pi/2 + \theta/2\right)$$

$$\Rightarrow \phi = \pi/2 + \theta/2$$

2) $r = b(1-\cos\theta)$ [cardioid with other orientation]

Soln Consider $r = b(1-\cos\theta)$

$$\frac{dr}{d\theta} = b\sin\theta$$



$$r\frac{d\theta}{dr} = \frac{\cancel{b}(1-\cos\theta)}{\cancel{b}\sin\theta} = \frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\,\cancel{\sin\theta/2}\cos\theta/2}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$$\tan\phi = \tan\theta/2$$

$$\Rightarrow \phi = \theta/2$$

3) Circle : $r = \sin\theta + \cos\theta$

[This is a circle centered at $(1/2, 1/2)$ with a radius of $\sqrt{1/2}$.

Soln Consider $r = \sin\theta + \cos\theta$

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} \qquad (\div \text{ by } \cos\theta)$$

$$\tan\left(\pi/4 + A\right) = \frac{1+\tan A}{1-\tan A}$$

$$\tan\phi = \frac{\tan\theta + 1}{1 - \tan\theta} = \tan\left(\pi/4 + \theta\right)$$

$$\Rightarrow \phi = \pi/4 + \theta$$

4) $r = 16\sec^2 \theta/2$

$\frac{d}{dx}\sec x = \sec x \tan x$

Sol: $\frac{dr}{d\theta} = 16 \times 2\sec \theta/2 . \sec\theta/2 \tan \theta/2 \times 1/2$

$\frac{dr}{d\theta} = \frac{32}{2}\sec^2\theta/2 \tan\theta/2 = 16\sec^2\theta/2\tan\theta/2$

$r\frac{d\theta}{dr} = \frac{16\sec^2\theta/2}{16\sec^2\theta/2 \tan\theta/2} = \cot\theta/2 = \tan\phi$

$\tan\phi = \tan(\pi/2 - \theta/2)$

$\Rightarrow \phi = \pi/2 - \theta/2$

* Find the angle of intersection for the following pairs of curves :

1) $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

[$r = a\log\theta$ is some kind of logarithmic 
spiral . The graph comes from negative x - infinity,
goes through the origin at $\theta=1$, and then spirals outwards.
It looks like it is heading to a definite limit of the radius
but this is an illusion].

Sol: consider $r = a\log\theta$

$\frac{dr}{d\theta} = \frac{a}{\theta}$

$r\frac{dr}{d\theta} = \frac{a\log\theta}{a}\times\theta$

$\Rightarrow \tan\phi_1 = \theta\log\theta$

consider $r = \frac{a}{\log\theta}$

$\log r = \log a - \log(\log\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log\theta . \theta}$

$\cot\phi_2 = -\frac{1}{\theta\log\theta}$

$\Rightarrow \tan\phi_2 = -\theta\log\theta$

$\therefore \tan(\phi_1 - \phi_2) = \frac{\theta\log\theta + \theta\log\theta}{1+(\theta\log\theta)(-\theta\log\theta)} = \frac{2\theta\log\theta}{1-(\theta\log\theta)^2}$

We have to eliminate θ between the given curves, $r = a\log\theta$ and $r = \frac{a}{\log\theta}$.

$\Rightarrow (\log\theta)^2 = 1.$

$\Rightarrow \log\theta = \pm 1$

$\rightarrow \theta = e$ or $1/e$

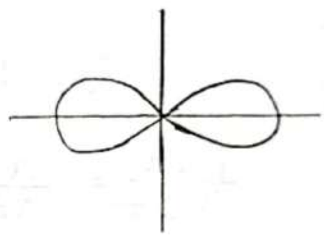
$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1-e^2}$

$$\frac{2\times 1/e}{1-(1/e)^2} = \frac{2}{e\left(\frac{e^2-1}{e^2}\right)} = \frac{2e}{e^2-1}$$

2) $r = 2(1+\cos\theta)$ and $r^2 = 4\cos 2\theta$

[$r^2 = 4\cos 2\theta$ is lemniscate which is any of several figure-eight or ∞-shaped curves]. Curve is symmetrical about both the axes.

Soln Consider



$r = 2(1+\cos\theta)$

$\log r = \log 2 + \log(1+\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi_1 = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = -\tan\theta/2 = \cot(\pi/2 + \theta/2)$

$\Rightarrow \phi_1 = \pi/2 + \theta/2$

Consider $r^2 = 4\cos 2\theta \quad\Rightarrow\quad 2r\frac{dr}{d\theta} = -4\sin 2\theta \times 2$

$\Rightarrow \frac{dr}{d\theta} = -\frac{4\sin 2\theta}{r} \quad\Rightarrow\quad r\frac{d\theta}{dr} = \frac{-r^2}{4\sin 2\theta} = \frac{-4\cos 2\theta}{4\sin 2\theta}$

$\Rightarrow \tan\phi_2 = -\cot 2\theta = \tan(\pi/2 + 2\theta)$

$\Rightarrow \phi_2 = \pi/2 + 2\theta$

Now, we have to eliminate θ between the given curves, $r = 2(1+\cos\theta)$ and $r^2 = 4\cos 2\theta$

$$4(1+\cos\theta)^2 = 4\cos 2\theta$$

$$1+\cos^2\theta + 2\cos\theta = 2\cos^2\theta - 1$$

$$\cos^2\theta - 2\cos\theta - 2 = 0$$

$$\cos\theta = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

$\cos\theta$ always lies between $[-1, 1]$. $\therefore \cos\theta = 1-\sqrt{3}$

$$\theta = \cos^{-1}(1-\sqrt{3}) = 2.3921.$$

$$\therefore |\phi_1 - \phi_2| = |2\theta - \theta/2| = \frac{3\theta}{2} = \frac{3}{2}\cos^{-1}(1-\sqrt{3})$$

3) $r = \dfrac{a\theta}{1+\theta}$ and $r = \dfrac{a}{1+\theta^2}$

Soln

Consider $r = \dfrac{a\theta}{1+\theta}$

$$\log r = \log(a\theta) - \log(1+\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a}{a\theta} - \frac{1}{1+\theta} = \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1+\theta-\theta}{\theta(1+\theta)} = \frac{1}{\theta(1+\theta)}$$

$$r\frac{d\theta}{dr} = \theta(1+\theta) = \tan\phi_1$$

consider $r = \dfrac{a}{1+\theta^2} \Rightarrow \log r = \log a - \log(1+\theta^2)$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2} \Rightarrow \tan\phi_2 = \frac{-(1+\theta^2)}{2\theta}$$

To eliminate θ, consider $\dfrac{a\theta}{1+\theta} = \dfrac{a}{1+\theta^2} \Rightarrow \theta + \theta^3 = 1+\theta$

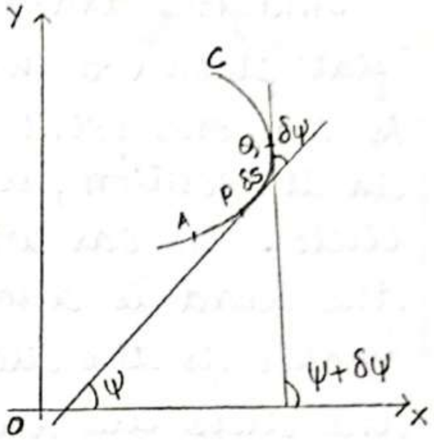
$$\Rightarrow \theta^3 = 1 \Rightarrow \theta = 1.$$

$$\therefore \tan\phi_1 = 2 \quad ; \quad \tan\phi_2 = -1$$

$$\therefore \tan(\phi_1 - \phi_2) = \left|\frac{2-(-1)}{1+(-2)}\right| = |-3| = 3$$

$$|\phi_1 - \phi_2| = |\tan^{-1}(3)|$$

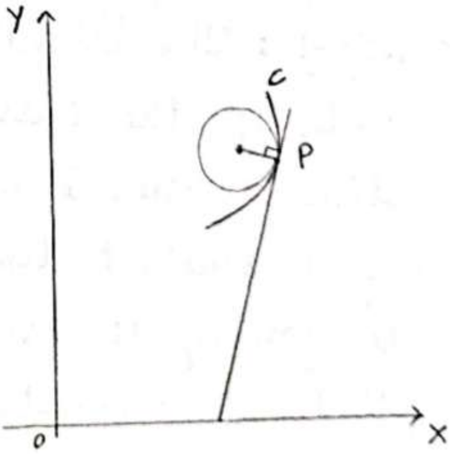
→ Curvature and Radius of Curvature :



Consider a curve C in the XY-plane. Let P be any point on the curve and Q another point in the neighborhood of P. Let A be a fixed point on the curve and the length of the arc be measured from A so that arc AP = S and arc AQ = S+δS. Then arc PQ = δS.

Let the tangents at P and Q make angles ψ and $\psi+\delta\psi$ with the x-axis respectively, so that the angle between these two tangents is $\delta\psi$. The angle $\delta\psi$ between the tangents at P and Q is called the total curvature of the arc PQ. The ratio $\frac{\delta\psi}{\delta s}$ is called the average curvature of the arc PQ.

The limiting value of $\frac{\delta\psi}{\delta s}$ when $Q \to P$ ultimately coincides with P is called the curvature at the point P. Thus the curvature of the curve at a point P is $\kappa = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$. So the curvature of a curve at a given point is a measure of the rate of change of bending of the curve at that point.



The circle having the same curvature as the curve at P touching the curve C, is called the circle of curvature. It is also called the osculating circle. The centre of the circle of the curvature is called the centre of curvature. The radius of the circle of curvature is the radius of curvature and is denoted by 'ρ'.

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

Imagine driving a car on a curvy road on a completely flat plain (so that the geographic plain is a geometric plane). At any one point along the way, lock the steering wheel in its position, so that the car thereafter follows a perfect circle. The car will, of course, deviate from the road, unless the road is also a perfect circle. The circle that the car makes is the circle of curvature, radius and the centre of the circle are radius of curvature and centre of curvature of the curvy road at the point at which the steering wheel was locked. The more sharply curved the road is at the point you locked the steering wheel, the smaller the radius of curvature.

* Applications :

1. Radius of curvature is applied to measurements of the stress in the semiconductor structures.

2. When engineers design trains track, they need to ensure the curvature of the track to be safe and provide a comfortable ride for the given speed of the trains.

→ Note-1 : The curvature of a curve at a point is zero if and only if the curve does not bend at that point. A straight line doesnot bend at any point on it. Therefore the curvature of a straight line is zero (at every point on it). A circle bends uniformly at every point on it. Therefore the curvature of a circle is a constant $\neq 0$.

Note 2 : If for an arc of a curve ψ decreases as s increases, then $\frac{d\psi}{ds}$ is negative. The sign of $\frac{d\psi}{ds}$ determines the convexity or the concavity of the curve. If we are not concerned about the convexity or concavity of a curve, we take the absolute value of $\frac{d\psi}{ds}$ while determining k and ρ.

Thus $k = \left|\frac{d\psi}{ds}\right|$ and $\rho = \left|\frac{ds}{d\psi}\right|$

Note-3 : The equation connecting s and ψ is called the intrinsic equation of the curve. The pair (s, ψ) are called the intrinsic coordinates of the point.

→ Radius of curvature for different forms of curves :

1. Radius of curvature for cartesian curves :

We know that, $\tan\psi = \frac{dy}{dx}$; $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

↳ arc length

Differentiating both sides w.r.t x, we get

$$\sec^2\psi \, \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds}$$

$$(1+\tan^2\psi)\,\frac{d\psi}{ds}\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (1+\tan^2\psi)\cdot\frac{1}{\rho}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]\cdot\frac{1}{\rho}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\Rightarrow \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+y_1^2\right]^{3/2}}{y_2}$$

where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

2. Radius of Curvature in Parametric form :

Let $x=f(t)$ and $y=g(t)$ be the parametric equations of a curve C and $P(x,y)$ be a given point on it.

Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\rho = \frac{ds}{d\psi} = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$$

where, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, $x'' = \frac{d^2x}{dt^2}$, $y'' = \frac{d^2y}{dt^2}$

3. Radius of Curvature in Polar form :

Let $r=f(\theta)$ be the equation of a curve in the polar form and $P(r,\theta)$ be a point on it. Then,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

where $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

* Examples

1. Find the curvature at any point on the curve $y=x^3$.

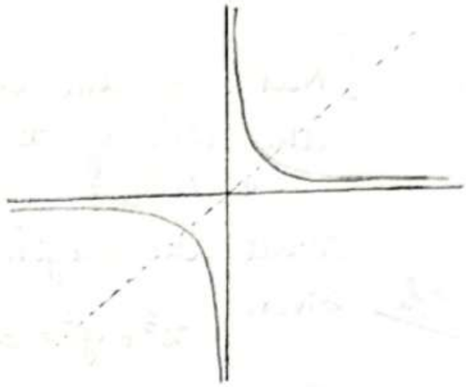
Soln. curvature $\kappa = \frac{y_2}{(1+y_1^2)^{3/2}}$

Here, $y=x^3$, $y_1 = \frac{dy}{dx} = 3x^2$, $y_2 = \frac{d^2y}{dx^2} = 6x$

$$\therefore \kappa = \frac{6x}{[1+(3x^2)^2]^{3/2}}$$

$$= \frac{6x}{(1+9x^4)^{3/2}}$$

2. Find the curvature at any point on the rectangular hyperbola $xy = c^2$.



Soln Curvature $K = \dfrac{y_2}{(1+y_1^2)^{3/2}}$

Here $xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$

$\therefore y_1 = -\dfrac{c^2}{x^2}$ and $y_2 = \dfrac{2c^2}{x^3}$

$$K = \frac{\frac{2c^2}{x^3}}{\left[1+\left(-\frac{c^2}{x^2}\right)^2\right]^{3/2}} = \frac{\frac{2c^2}{x^3}}{\left[\frac{x^4+c^4}{x^4}\right]^{3/2}} = \frac{2c^2 \times x^6}{x^3(x^4+c^4)^{3/2}}$$

$$= \frac{2c^2x^3}{(x^4+x^2y^2)^{3/2}} \quad \because c^2 = xy$$

$$= \frac{2x^4y}{x^3(x^2+y^2)^{3/2}} = \frac{2xy}{(x^2+y^2)^{3/2}}$$

3. Find the radius of curvature at the origin on $y = x(x-a)^2$.

Soln We are suppose to find $\rho(0,0)$.

WKT $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Given $y = x(x^2+a^2-2ax) = x^3+xa^2-2ax^2$

$y_1 = 3x^2+a^2-4ax$

$y_2 = 6x-4a$

$y_1(0,0) = a^2 \quad ; \quad y_2(0,0) = -4a.$

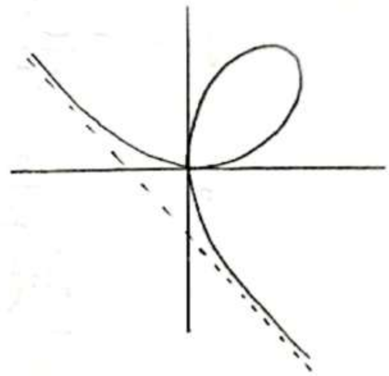
$\therefore \rho(0,0) = \dfrac{(1+a^4)^{3/2}}{-4a}$

Taking the magnitude, we have $\rho(0,0) = \dfrac{(1+a^4)^{3/2}}{4a}$

4. Find the radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on $x^3+y^3=3axy$.

[Name of the curve is Folium Descartes; it is symmetrical about the line $y=x$.
It is not symmetric about any other line, nor it is symmetric about the origin].

Soln Given $x^3+y^3=3axy$



Differentiating w.r.t x, we get

$$3x^2+3y^2y_1 = 3a(xy_1+y)$$

$$\Rightarrow x^2+y^2y_1 = axy_1+ay$$

$$\Rightarrow \quad y_1 = \frac{ay-x^2}{y^2-ax} \rightarrow (1)$$

$$\therefore \quad y_2 = \frac{(y^2-ax)(ay_1-2x) - (ay-x^2)(2yy_1-a)}{(y^2-ax)^2} \rightarrow (2)$$

From (1), we have $y_1\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)} = -1$

From (2), we have

$$y_2\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a-3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a-a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\frac{3}{4}a^2 \times -4a - \left(-\frac{3a^2}{4}\right)(-4a)}{\left(\frac{3}{4}a^2\right)^2} = \frac{-3a^3-3a^3}{\frac{9}{16}a^4}$$

$$= \frac{-\cancel{6}^{2}a^3 \times 16}{{}_{3}\cancel{9}a^4} = \frac{-32}{3a}$$

$$\therefore \rho\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{(1+1)^{3/2}}{-32/3a} = \frac{2\sqrt{2} \times 3a}{-32} = \frac{-3a}{8\sqrt{2}}$$

$\therefore$ Radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{3a}{8\sqrt{2}}$

5. Find the radius of curvature at $b^2x^2+a^2y^2=a^2b^2$ at its point of intersection with y-axis.

Sol: It is required to find $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ at $x=0$ on $b^2x^2+a^2y^2=a^2b^2$

when $x=0$, $b^2x^2+a^2y^2=a^2b^2$ reduces to $a^2y^2=a^2b^2 \Rightarrow y=\pm b$

$\therefore$ Points of intersection are $(0,b)$ and $(0,-b)$.

Given $b^2x^2+a^2y^2=a^2b^2$

Differentiating w.r.t x, we get

$$2b^2x+2a^2yy_1=0$$

$$\Rightarrow y_1 = \frac{-b^2x}{a^2y}$$

$$y_1(0,b)=0 \quad , \quad y_1(0,-b)=0$$

Differentiating w.r.t x, we get

$$y_2 = \frac{-b^2}{a^2}\left(\frac{y-xy_1}{y^2}\right)$$

$$y_2(0,b) = \frac{-b^2}{a^2}\left(\frac{1}{b}\right) = \frac{-b}{a^2}$$

$$y_2(0,-b) = \frac{-b^2}{a^2}\left(\frac{1}{-b}\right) = \frac{b}{a^2}$$

$$\therefore \rho_{(0,b)} = \left|\frac{(1+0^2)^{3/2}}{-b/a^2}\right| = \frac{a^2}{b}$$

$$\rho_{(0,-b)} = \left|\frac{(1+0^2)^{3/2}}{b/a^2}\right| = \frac{a^2}{b}$$

6. Show that the radius of curvature at any point (x,y) on $x^{2/3}+y^{2/3}=a^{2/3}$ is $3a^{1/3}x^{1/3}y^{1/3}$.

Sol: we know that $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $x^{2/3}+y^{2/3}=a^{2/3}$

Differentiating w.r.t x, we get

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y_1=0 \Rightarrow y_1=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

Differentiating again w.r.t x, we get

$$y_2=\frac{-\left\{x^{1/3}\left(\frac{1}{3}y^{-2/3}y_1\right)-y^{1/3}\cdot\frac{1}{3}x^{-2/3}\right\}}{(x^{1/3})^2}$$

$$=\frac{-\frac{1}{3}x^{1/3}y^{-2/3}\left(\frac{-y^{1/3}}{x^{1/3}}\right)+\frac{1}{3}y^{1/3}x^{-2/3}}{x^{2/3}}=\frac{\frac{1}{3}\left[\frac{1}{y^{1/3}}+\frac{y^{1/3}}{x^{2/3}}\right]}{x^{2/3}}$$

$$=\frac{1}{3}\frac{x^{2/3}+y^{2/3}}{x^{4/3}y^{1/3}}=\frac{a^{2/3}}{3x^{4/3}y^{1/3}}$$

Now, $1+y_1^2=1+\frac{y^{2/3}}{x^{2/3}}=\frac{x^{2/3}+y^{2/3}}{x^{2/3}}=\frac{a^{2/3}}{x^{2/3}}$

$$\therefore \rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{\left(\frac{a^{2/3}}{x^{2/3}}\right)^{3/2}}{\frac{a^{2/3}}{3x^{4/3}y^{1/3}}}=\frac{a\times 3x^{4/3}y^{1/3}}{x a^{2/3}}$$

$$=3a^{1/3}x^{1/3}y^{1/3}$$

7. Show that for ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\rho=\frac{a^2b^2}{p^3}$ where p is the length of the perpendicular from the center upon the tangent at (x,y) to the ellipse.

Sol: The ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Differentiating w.r.t x, we get

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2} = 0 \quad \Rightarrow y_1 = -\frac{2x}{a^2} \times \frac{b^2}{2y} = \frac{-b^2}{a^2}\frac{x}{y}$$

Differentiating again, we get

$$y_2 = \frac{-b^2}{a^2}\left(\frac{y - xy_1}{y^2}\right) = \frac{-b^2}{a^2}\left\{\frac{y + x\cdot\frac{b^2}{a^2}\cdot\frac{x}{y}}{y^2}\right\} = \frac{-b^2}{a^2y^2}\left\{\frac{y^2a^2+x^2b^2}{a^2y}\right\}$$

$\div a^2b^2$

$$= \frac{-b^2}{a^2y^2}\left\{\frac{y^2}{b^2} + \frac{x^2}{a^2}\right\}\cdot\frac{b^2}{y} = \frac{-b^4}{a^2y^3} \qquad \because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Now, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{b^4}{a^4}\frac{x^2}{y^2}\right)^{3/2}}{\frac{-b^4}{a^2y^3}} = -\frac{(a^4y^2+b^4x^2)^{3/2}}{a^6y^3} \times \frac{a^2y^3}{b^4}$$

$$\therefore \rho = \frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}$$

The equation to the tangent at any point (x_0, y_0) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$.

length of perpendicular from the origin upon the tangent is given by $\frac{1}{\sqrt{\left(\frac{x_0}{a^2}\right)^2+\left(\frac{y_0}{b^2}\right)^2}} = \frac{a^2b^2}{\sqrt{a^4y_0^2+b^4x_0^2}}$

$$\therefore p = \frac{a^2b^2}{\sqrt{a^4y^2+b^4x^2}} \quad \text{(by replacing } x_0 \text{ by } x \text{ and } y_0 \text{ by } y\text{)}$$

$$\Rightarrow \frac{1}{p^3} = \frac{(a^4y^2+b^4x^2)^{3/2}}{a^6b^6} = \frac{\rho}{a^2b^2}$$

$$\Rightarrow \rho = \frac{a^2b^2}{p^3}$$

8. Find the radius of curvature at the point θ on the curve

$$x = a\log\sec\theta \;;\; y = a(\tan\theta - \theta)$$

Sol: $x = a\log\sec\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{a}{\sec\theta}\cdot\sec\theta\tan\theta = a\tan\theta$

$y = a(\tan\theta - \theta) \Rightarrow \dfrac{dy}{d\theta} = a(\sec^2\theta - 1) = a\tan^2\theta$

$$\therefore y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\tan^2\theta}{a\tan\theta} = \tan\theta$$

$$y_2 = \frac{d}{dx}(\tan\theta) = \frac{d}{d\theta}(\tan\theta)\cdot\frac{d\theta}{dx}$$

$$= \sec^2\theta \times \frac{1}{a\tan\theta} = \frac{\sec^2\theta}{a\tan\theta}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta)^{3/2}}{\left(\frac{\sec^2\theta}{a\tan\theta}\right)} = \frac{\sec^3\theta}{\sec^2\theta}\times a\tan\theta$$

$$= a\sec\theta\tan\theta$$

9. Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos(\theta/2)$.

Sol: $\dfrac{dx}{d\theta} = a(1+\cos\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}(\tan\theta/2)\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\theta/2\cdot\frac{1}{a\cdot 2\cos^2\theta/2}$$

$$= \frac{1}{4a}\sec^4\theta/2$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\tan^2\theta/2\right)^{3/2}}{\frac{1}{4a}\sec^4\theta/2} = \frac{4a\left(\sec^2\theta/2\right)^{3/2}}{\sec^4\theta/2}$$

$$= 4a\cos\theta/2$$

10. Find radius of curvature at a point for the curves

$x = 6t^2 - 3t^4$, $y = 8t^3$

Sol:
$x' = \frac{dx}{dt} = 12t - 12t^3$ $\qquad$ $y' = \frac{dy}{dt} = 24t^2$

$x'' = \frac{d^2x}{dt^2} = 12 - 36t^2$ $\qquad$ $y'' = \frac{d^2y}{dt^2} = 48t$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{\left\{[12t(1-t^2)]^2 + [24t^2]^2\right\}^{3/2}}{12t(1-t^2)48t - 24t^2 \cdot 12(1-3t^2)} = \frac{\left[144t^2(1+t^4-2t^2) + 576t^4\right]^{3/2}}{576t^2 - 576t^4 - 288t^2 + 864t^4}$$

$$= \frac{\left[144t^2 + 144t^6 - 288t^4 + 576t^4\right]^{3/2}}{288t^2 + 288t^4}$$

$$= \frac{(144)^{3/2}(t^2)^{3/2}\left[1+t^4+2t^2\right]^{3/2}}{288t^2(1+t^2)}$$

$$= \frac{144 \times 12 \times t^3\left\{[1+t^2]^2\right\}^{3/2}}{288t^2(1+t^2)}$$

$$= 6t(1+t^2)^2$$

11. Show that the radius of curvature at any point of the cardioid $r = a(1-\cos\theta)$ varies as $\sqrt{r}$.

Soln: Given $r = a(1-\cos\theta)$

$$r_1 = \frac{dr}{d\theta} = a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\therefore \quad \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(a^2(1-\cos\theta)^2+a^2\sin^2\theta)^{3/2}}{a^2(1-\cos\theta)^2+2a^2\sin^2\theta-a^2(1-\cos\theta)\cos\theta}$$

$$= \frac{a^3(1+\cos^2\theta-2\cos\theta+\sin^2\theta)^{3/2}}{a^2[1+\cos^2\theta-2\cos\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a^3[2(1-\cos\theta)]^{3/2}}{a^2[3-3\cos\theta]} = \frac{a^3\sqrt{8}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{2a\sqrt{2}}{3}(1-\cos\theta)^{1/2} = \frac{2\sqrt{2}\,a}{3}\left(\frac{r}{a}\right)^{1/2}$$

$$= \frac{2\sqrt{2a}}{3}\cdot\sqrt{r} \quad \propto \sqrt{r}$$

12. Find the radius of curvature at any point for the curve

$$r^n = a^n\cos n\theta$$

Soln: Taking log on both sides, we get

$$n\log r = n\log a + \log\cos n\theta$$

Differentiating w.r.t θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}\cdot n$$

$$r_1 = \frac{dr}{d\theta} = -r\tan n\theta$$

$$r_2 = \frac{d^2 r}{d\theta^2} = -\frac{dr}{d\theta}\tan n\theta - nr\sec^2 n\theta$$

$$= r\tan^2 n\theta - nr\sec^2 n\theta$$

Now, $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\tan^2 n\theta)^{3/2}}{r^2+2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

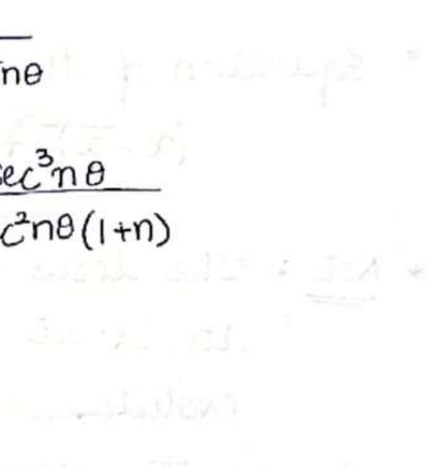
$$= \frac{r^3(\sec^2 n\theta)^{3/2}}{r^2(1+\tan^2 n\theta)+nr^2\sec^2 n\theta} = \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta(1+n)}$$

$$= \frac{r\sec n\theta}{(1+n)} = \frac{ra^n}{r^n(1+n)}$$

$$= \frac{a^n r^{1-n}}{n+1}$$

→ Centre of curvature and circle of curvature.

Let $C(\bar{x},\bar{y})$ be the centre of curvature and ρ the radius of curvature of the curve at $P(x,y)$. Draw $PL \perp OX$, $CM \perp OX$ and $PN \perp CM$. Let the tangent at P make an angle ψ with the x-axis. Then



$\angle NCP = 90^\circ - \angle NPC = \angle NPT = \psi$

$\therefore \bar{x} = OM = OL - ML = OL - NP$

$= x - \rho\sin\psi$

$= x - \dfrac{(1+y_1^2)^{3/2}}{y_2} \cdot \dfrac{y_1}{\sqrt{1+y_1^2}}$

$\bar{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$\tan\psi = y_1$

$\sec\psi = \sqrt{1+\tan^2\psi} = \sqrt{1+y_1^2}$

$\therefore \sin\psi = \dfrac{y_1}{\sqrt{1+y_1^2}}$

and $\bar{y} = MC = MN + NC = LP + \rho\cos\psi$

$$= y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{1+y_1^2}}$$

$$= y + \frac{(1+y_1^2)}{y_2}$$

* Equation of the circle of curvature at $P(x,y)$ is given by

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2.$$

×[* Note : The locus of the centre of curvature for a curve is called its evolute and the curve is called an involute of its evolute.

Obs-1 : The normal to the curve (i.e., involute) is the tangent to the evolute.

Obs-2 : The length of the arc of the evolute between two points C_1 and C_2 is equal to the difference between the radii of curvature at P_1 and P_2 on the curve corresponding to C_1 and C_2.] ×

1. Find the coordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$.

Sol: Given $y^2 = 4ax$

$$\Rightarrow 2yy_1 = 4a \quad \Rightarrow y_1 = \frac{2a}{y}$$

$$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{\frac{2a}{y}\left(1 + \frac{4a^2}{y^2}\right)}{-\frac{4a^2}{y^3}} = x + \frac{y^2}{2a}\left(\frac{y^2+4a^2}{y^2}\right)$$

$$= x + \left(\frac{y^2+4a^2}{2a}\right) = x + \frac{4ax+4a^2}{2a} = x + 2x + 2a = 3x + 2a$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = y + \frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{-4a^2}{y^3}} = y - \frac{y^3}{4a^2}\left(\frac{y^2+4a^2}{y^2}\right)$$

$$= y - \frac{y}{4a^2}(4ax+4a^2)$$

$$= y - \frac{yx}{a} - y = -\frac{\sqrt{4ax}\cdot x}{a}$$

$$= -\frac{2x^{3/2}}{\sqrt{a}}$$

2. Find circle of curvature at $(1,0)$ on $y = x^3 - x^2$.

Sol: The equation of circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

Given $y = x^3 - x^2$

$$y_1 = \frac{dy}{dx} = 3x^2 - 2x$$

$$y_{1(1,0)} = 1$$

$$y_2 = \frac{d^2y}{dx^2} = 6x - 2$$

$$y_{2(1,0)} = 4$$

$$\bar{x}_{(1,0)} = x - \frac{y_1(1+y_1^2)}{y_2} = 1 - \frac{1(1+1)}{4} = \frac{1}{2}$$

$$\bar{y}_{(1,0)} = y + \frac{(1+y_1^2)}{y_2} = 0 + \frac{(1+1)}{4} = \frac{1}{2}$$

$$\rho_{(1,0)} = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

$\therefore$ circle of curvature at $(1,0)$ on $y = x^3 - x^2$ is

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2} \Rightarrow x^2 + \frac{1}{4} - \frac{2x}{2} + y^2 + \frac{1}{4} - \frac{2y}{2} = \frac{1}{2}$$

$$\Rightarrow x^2 + y^2 - x - y = 0$$

* Continuous Function :

A function $f(x)$ defined in a neighbourhood of a point 'a' and also at 'a' is said to be continuous at $x=a$, if $\lim_{x\to a} f(x) = f(a)$.

From the above definition it follows that the following three conditions are necessary for the function $f(x)$ to be continuous at $x=a$.

(i) $f(x)$ is defined at $x=a$. i.e., $f(a)$ exists.

(ii) $\lim_{x\to a} f(x)$ exists. i.e., both LHL and RHL exists as x tends to a.

(iii) $\lim_{x\to a} f(x) = f(a)$.

The above definition can be restated by using the definition of limit of the function :

A function $f(x)$ is said to be continuous at $x=a$, if for every $\varepsilon>0$, there exists a real number $\delta>0$, such that,

$$|f(x)-f(a)| < \varepsilon, \text{ whenever } |x-a|<\delta.$$

A function $f(x)$ is said to be continuous in an interval $[a,b]$, if it is continuous at every point of the interval.

Note : The continuity of a function $f(x)$ at the end points of the closed interval $[a,b]$ is defined as below.

(i) $f(x)$ is continuous at $x=a$, if $\lim_{x\to a^+} f(x) = f(a)$.

(ii) $f(x)$ is continuous at $x=b$, if $\lim_{x\to b^-} f(x) = f(b)$.

* Differentiability :

Derivative of a function at a point : Let $f(x)$ be a function defined in a domain $D \subset R$ and 'x_0' be any point in D. Then

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

if it exists, is called the derivative of $f(x)$ at $x=x_0$.

If the $\lim_{x \to x_0^-} \dfrac{f(x)-f(x_0)}{x-x_0}$ exists, then it is called the left hand derivative of $f(x)$ at $x = x_0$. It is denoted by $Lf'(x_0)$.

If the $\lim_{x \to x_0^+} \dfrac{f(x)-f(x_0)}{x-x_0}$ exists, then it is called the right hand derivative of $f(x)$ at $x = x_0$. It is denoted by $Rf'(x_0)$.

It is obvious that f is differentiable at x_0 if and only if $Lf'(x_0)$ and $Rf'(x_0)$ both exists and are equal.

The function $f(x)$ is said to be differentiable in $[a,b]$ if and only if it is differentiable at every point of $[a,b]$.

* Taylor's mean value theorem :

Let $f(x)$ be a function defined on $[a,b]$ such that

(i) $f^{n-1}(x)$ is continuous on $[a,b]$

(ii) $f^{n-1}(x)$ is differentiable on (a,b).

Then there exists a real number $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^n(c).$$

If $f(x)$ is a function such that $f^{(n-1)}(x)$ are continuous on $[a,a+h]$ and differentiable in $(a,a+h)$, then there exists a real number $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{(h)^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+ \frac{h^n}{n!}f^{(n)}(a+\theta h).$$

The $(n+1)^{th}$ term in the expansion i.e., $\frac{h^n}{n!}f^{(n)}(a+\theta h)$ is known as Lagrange's form of remainder.

<u>Note</u>: If $n=1$, in the final expansion of theorem, we get

$$f(b) = f(a) - (b-a)f'(c)$$

i.e., $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\rightarrow$ Lagrange's mean value theorem.

Taking $b = x$ in final expansion of theorem, we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n$$

where $R_n = \frac{(x-a)^n}{n!}f^{(n)}(c)$.

If R_n tends to zero as $n \to \infty$, we can express a function by means of an infinite series.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots$$

This series is known as Taylor's infinite series for the expansion of $f(x)$ as a power series in powers of $(x-a)$ or about the point 'a'.

* Maclaurin's Series :

When $a = 0$, Taylor's series reduces to Maclaurin's series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!}f^{(n)}(0).$$

* <u>Advantages of using Taylor's series and Maclaurin's series</u> :

(i) Taylor series are studied because polynomial functions are easy. If one can find a way to represent complicated functions as series (infinite polynomials), then they can easily study the properties of difficult functions.

(ii) Evaluating definite integrals.

(iii) Understanding asymptotic behaviour: Taylor's series gives us an information about how a function behaves in an important part of its domain.

(iv) Understanding the growth of functions.

(v) Solving differential equations.

1. Obtain a Taylor's expansion for $f(x) = \sin x$ in the ascending powers of $(x - \pi/4)$ upto the fourth degree term.

Sol: The Taylor's expansion for $f(x)$ about $\pi/4$ is

$$f(x) = f(\pi/4) + (x-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) + \frac{(x-\pi/4)^3}{3!} f'''(\pi/4) + \cdots \quad \to (1)$$

$f(x) = \sin x \quad \Rightarrow f(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \quad \Rightarrow f'(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \quad \Rightarrow f''(\pi/4) = -\sin(\pi/4) = \frac{-1}{\sqrt{2}}$

$f'''(x) = -\cos x \quad \Rightarrow f'''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}$

$f^{iv}(x) = \sin x \quad \Rightarrow f^{iv}(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

Substituting these in (1), we get

$$f(x) = \frac{1}{\sqrt{2}} + (x-\pi/4)\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^3}{3!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + (x-\pi/4) - \frac{(x-\pi/4)^2}{2} - \frac{(x-\pi/4)^3}{6} + \frac{(x-\pi/4)^4}{24} + \cdots\right]$$

2. Obtain a Taylor's expansion for $f(x) = \log_e x$ upto the term containing $(x-1)^4$ and hence find $\log_e(1.1)$.

Sol: The Taylor's series for $f(x)$ about the point 1 is given by

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^{iv}(1) + \cdots \quad \to (1)$$

$f(x) = \log x \quad \Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \quad \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \quad \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -\frac{2 \cdot 3}{x^4} = \frac{-6}{x^4}$

$\Rightarrow f^{iv}(1) = -6$

$\therefore (1) \Rightarrow f(x) = \log x = (x-1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$

$= (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{6} - \frac{(x-1)^4}{4} + \cdots$

Taking $x = 1.1$ in the above expansion, we get

$\log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$

$= 0.0953$

3. Using Taylor's theorem, expand $\log(\sin x)$ in ascending powers of $(x-3)$.

Soln The Taylor's series expansion about the point 3 is given by

$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \cdots$

$f(x) = \log(\sin x) \Rightarrow f(3) = \log(\sin 3)$

$f'(x) = \frac{1}{\sin x}\cdot\cos x = \cot x \Rightarrow f'(3) = \cot(3)$

$f''(x) = -\operatorname{cosec}^2 x \Rightarrow f''(3) = -\operatorname{cosec}^2(3)$

$f'''(x) = +2\operatorname{cosec}^2 x \cot x \Rightarrow f'''(x) = 2\operatorname{cosec}^2(3)\cot(3)$

$\therefore f(x) = \log(\sin x) = \log(\sin(3)) + (x-3)\cot(3) + \frac{(x-3)^2}{2!}(-\operatorname{cosec}^2(3))$

$+ \frac{(x-3)^3}{3!}\cdot 2\operatorname{cosec}^2(3)\cot(3) + \cdots$

$= \log(\sin(3)) + (x-3)\cot(3) - \frac{(x-3)^2}{2}\operatorname{cosec}^2(3) + \frac{(x-3)^3}{3}\operatorname{cosec}^2(3)\cot(3) + \cdots$

4. Obtain a Maclaurin's series for $f(x) = \sin x$ upto the term containing x^5.

Soln The Maclaurin's series for $f(x)$ is

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0.$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 01$

$$\therefore f(x) = \sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

5. Obtain expansion of $f(x) = \frac{x}{\sin x}$ upto the term containing x^4.

Soln

Maclaurin's series expansion of $\sin x$ is given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\therefore f(x) = \frac{x}{\sin x} = x\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right\}^{-1}$$

$$= x x^{-1}\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right\}$$

$$= \left\{1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \dots\right)\right\}^{-1} = 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} - \dots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} - \dots\right)^2 + \dots$$

Binomial expansion :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!}\right)^2$$, terms of order > x^4 are neglected.

$$= 1 + \frac{x^2}{6} - \frac{x^4}{120} + \frac{x^4}{36} + \dots$$

$$= 1 + \frac{x^2}{6} - \frac{7x^4}{360} + \dots$$

6. Obtain Taylor's expansion of the function $\cos\left(\frac{\pi}{4}+h\right)$ in ascending powers of h upto the terms containing h^4.

Soln Taylor's expansion of $f(x+h)$ is given by

$$f(x+h) = f(x) + \sum_{n=1}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

$$= f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} h^{iv}(x) + \cdots$$

At $x = \pi/4$,

$$f(\pi/4 + h) = f(\pi/4) + h f'(\pi/4) + \frac{h^2}{2!} f''(\pi/4) + \frac{h^3}{3!} f'''(\pi/4) + \cdots$$

$f(x) = \cos x \quad \Rightarrow f(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(x) = -\sin x \quad \Rightarrow f'(\pi/4) = -\sin(\pi/4) = \frac{-1}{\sqrt{2}}$

$f''(x) = -\cos x \quad \Rightarrow f''(\pi/4) = -\cos(\pi/4) = \frac{-1}{\sqrt{2}}$

$f'''(x) = \sin x \quad \Rightarrow f'''(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$

$f^{iv}(x) = \cos x \quad \Rightarrow f^{iv}(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$

$$\therefore \cos(\pi/4 + h) = \frac{1}{\sqrt{2}} + h\left(\frac{-1}{\sqrt{2}}\right) + \frac{h^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{h^3}{3!}\left(\frac{1}{\sqrt{2}}\right) + \frac{h^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 - h - \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots\right]$$

* Note : Maclaurin's series of some functions :

(i) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(iii) $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$