✱ Differential or Total differential or Exact differential

For a differentiable function of one variable, $y = f(x)$ we define the differential dx to be an independent variable, i.e. dx can be given the value of any real no. The differential of y is defined as

$$dy = f'(x)\,dx$$

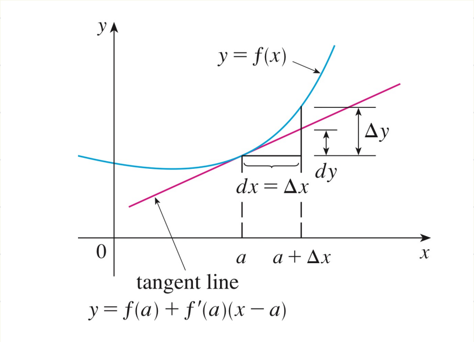


Fig: 1

Figure 1 shows the relationship b/w the increment Δy and differential dy. Δy represents the change in height of curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a function of 2 variables, $z = f(x, y)$ we define dx and dy to be independent variables, i.e. they can be given any values. Then the differential dz, also called the total differential, is defined by

$$\boxed{dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

If $w = f(x,y,z)$ is a function of 3 variables then the differential dw is given by

$$\boxed{dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz}$$

Chain rule:

Recall that chain rule for functions of a single variable gives the rule for differentiating a composite function. If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions then y is indirectly a differentiable function of t and

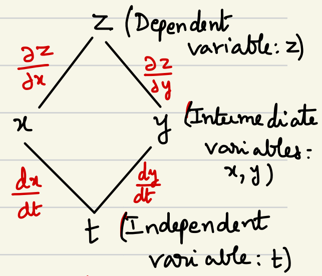
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For functions of more than one variable, the Chain rule has several versions, each of them giving a rule for differentiating a composite function.

※ Chain rule (Case 1): If $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are functions of t. Then z is a differentiable function of t called as total derivative of z given by

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

(To remember the chain rule it's helpful to draw tree diagram)



Tree diagram

※ Chain rule (Case 2): If $z = f(x, y)$ is a differentiable function of x and y where $x = g(s, t)$ and $y = h(s, t)$ are functions of s and t then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \& \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

Tree diagram for the above case:

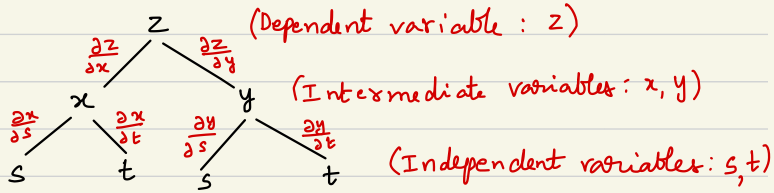


Fig: 2

resembles the one-dimensional Chain Rule in Equation 1.

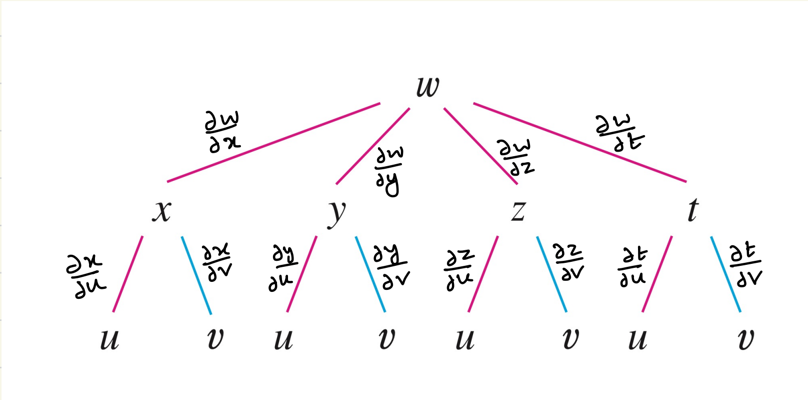
To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t. On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Ex: Write out the chain rule for the case $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and $t = t(u, v)$.

Sol:

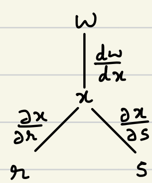


$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

If w is a function of x alone i.e. $w = f(x)$ and $x = \phi(u, v)$ then our equations are even simpler.

$$\boxed{\frac{\partial w}{\partial r} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial r} \quad \& \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}}$$



∗ Implicit differentiation or differentiation of implicit functions

The chain rule can be used for the process of implicit differentiation. An implicit function with x as an independent variable and y as the dependent variable is generally of the form $z = f(x, y) = 0$.

This gives $\frac{dz}{dx} = 0$.

Using case ① of chain rule, take $t = x$

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$0 = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-f_x}{f_y}}, \text{ if } \frac{\partial f}{\partial y} \neq 0.$$

ex: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

sol: $f(x, y) = x^3 + y^3 - 6xy = 0$

$f_x = 3x^2 - 6y$, $f_y = 3y^2 - 6x$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(3x^2 - 6y)}{3y^2 - 6x} = -\frac{(x^2 - 2y)}{y^2 - 2x}$$

Consider $x^3 + y^3 - 6xy = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} - 6y = 0$

$\Rightarrow \frac{dy}{dx}(3y^2 - 6x) = -3x^2 + 6y$

$\therefore \frac{dy}{dx} = \frac{-3x^2 + 6y}{3y^2 - 6x} = -\frac{x^2 + 2y}{y^2 - 2x}$

Now, if z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.
We can use chain rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

$$\therefore \frac{\partial F}{\partial x}(1) + 0 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-F_x}{F_z}, \quad |||^{ly} \quad \frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-F_y}{F_z}$$

Ex: Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

Sol: Let $F(x,y,z) = x^3+y^3+z^3+6xyz-1=0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2+6yz)}{3z^2+6xy} = \frac{-(x^2+2yz)}{z^2+2xy} \to (1)$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(3y^2+6xz)}{3z^2+6xy} = -\frac{(y^2+2xz)}{z^2+2xy}$$

or

Differentiating the given eq. partially wrt x

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(3z^2+6xy) = -(3x^2+6yz)$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{(3x^2+6yz)}{3z^2+6xy} = -\frac{(x^2+2yz)}{z^2+2xy} \to (2)$$

(1) and (2) are equal but evaluating $\dfrac{\partial z}{\partial x}$ by implicit differentiation formula is easier.