UNIT-2 : DIFFERENTIAL CALCULUS

Tutorial Sheet – I

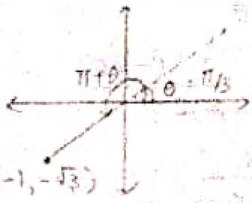
1. If $(-1,-\sqrt{3})$ are Cartesian coordinates of a point in plane, the corresponding polar coordinates are $(2, 4\pi/3)$

⇝ $(x,y)=(-1,-\sqrt{3})$

$r=\sqrt{x^2+y^2}=\sqrt{1+3}=\sqrt{4} \Rightarrow r=2$

$\theta=\tan^{-1}(y/x)=\tan^{-1}(\sqrt{3}) \Rightarrow \theta=\pi/3$

$\therefore \theta=\pi+\pi/3 \Rightarrow \theta=4\pi/3$



$\therefore$ Polar coordinates, $(r,\theta)=(2, 4\pi/3)$

2. If $(\sqrt{2}, 5\pi/4)$ are the polar coordinates of a point in a plane, the corresponding Cartesian coordinates are $(-1,-1)$.

⇝ $(r,\theta)=(\sqrt{2}, 5\pi/4)$

$x=r\cos\theta=\sqrt{2}\cos(5\pi/4)=\sqrt{2}\left[-\frac{\sqrt{2}}{2}\right] \Rightarrow x=-1$

$y=r\sin\theta=\sqrt{2}\sin(5\pi/4)=\sqrt{2}\left[-\frac{\sqrt{2}}{2}\right] \Rightarrow y=-1$

$\therefore$ Cartesian coordinates, $(x,y)=(-1,-1)$

3. The circle $x^2+y^2-2ax=0$ in polar form is $r=2a\cos\theta$.

⇝ $x^2+y^2-2ax=0$

$x^2+y^2=2ax$

We have, $x=r\cos\theta$ and $y=r\sin\theta$ in polar form.

$\Rightarrow r^2\cos^2\theta+r^2\sin^2\theta=2ar\cos\theta$

$r^2\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}=2a\cancel{r}\cos\theta$

$r=2a\cos\theta$

4. The polar equation $\theta-k=0$, geometrically represents straight lines.

⇝ $\theta=\tan^{-1}(y/x)$

Then, $\theta-k=\tan^{-1}(y/x)-k=0 \Rightarrow k=\tan^{-1}(y/x) \Rightarrow \tan k(x)=y$ is straight line.

5. If two polar curves C_1 and C_2 are orthogonal then value of $\cot\varphi_1 \cot\varphi_2 = \underline{-1}$

6. Find the angle of intersection between the polar curves $r = k\theta/1+\theta$ and $r = k/1+\theta^2$

→ For $r = \dfrac{k\theta}{1+\theta} \longrightarrow (1)$

$$\frac{dr}{d\theta} = \frac{(1+\theta)k + (-k\theta)}{(1+\theta)^2}$$

$$\frac{dr}{d\theta} = \frac{k}{(1+\theta)^2}$$

$$\tan\varphi_1 = r\, d\theta/dr$$

$$\tan\varphi_1 = \frac{k\theta}{(1+\theta)} \times \frac{(1+\theta)^2}{k}$$

$$\underline{\tan\varphi_1 = \theta + \theta^2}$$

For $r = \dfrac{k}{1+\theta^2} \longrightarrow (2)$

$$\frac{dr}{d\theta} = \frac{-2k\theta}{(1+\theta^2)^2}$$

$$\frac{dr}{d\theta} = \frac{-2\theta k}{(1+\theta^2)^2}$$

$$\tan\varphi_2 = r\, d\theta/dr$$

$$\tan\varphi_2 = \frac{k}{(1+\theta^2)} \times \frac{(1+\theta^2)^2}{-2\theta k}$$

$$\underline{\tan\varphi_2 = \frac{-(1+\theta^2)}{2\theta}}$$

From (1) and (2), $\dfrac{k\theta}{1+\theta} = \dfrac{k}{1+\theta^2}$

$$\theta + \theta^3 = 1 + \theta$$

$$\underline{\theta^3 = 1}$$

Now, $\tan|\varphi_1 - \varphi_2| = \left|\dfrac{\tan\varphi_1 - \tan\varphi_2}{1 + \tan\varphi_1 \tan\varphi_2}\right|$

$$= \left|\frac{\theta + \theta^2 + \left[\frac{1+\theta^2}{2\theta}\right]}{1 - (\theta+\theta^2)\left[\frac{1+\theta^2}{2\theta}\right]}\right|$$

$$= \left|\frac{2\theta^2 + 2\theta^3 + 1 + \theta^2}{2\theta - \theta - \theta^3 - \theta^2 - \theta^4}\right|$$

$$= \left|\frac{3\theta^2 + 1 + 2\theta^3}{\theta - \theta^2 - \theta^3 - \theta^4}\right|$$

$$= \left|\frac{3\theta^2 + 3}{\theta - \theta^2 - 1 - \theta}\right| \qquad [\because \theta^3 = 1;\ \theta^4 = \theta^3 \cdot \theta = \theta]$$

$$= \left|\frac{3(\cancel{\theta^2+1})}{-1(\cancel{\theta^2+1})}\right|$$

$\tan|\varphi_1 - \varphi_2| = 3$

$|\varphi_1 - \varphi_2| = \tan^{-1}(3)$

7. Show that the angle made by the tangent and the normal at any point $P(r,\theta)$ on the curve lemniscate $r^2 = a^2\cos(2\theta)$ with the initial line is 3θ.

Given, $r^2 = a^2\cos 2\theta \longrightarrow (*)$

$$2r\frac{dr}{d\theta} = -a^2\sin 2\theta\,(2)$$

$$\frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$$

Now, $\tan\varphi = r\,{d\theta}/{dr}$

$$\tan\varphi = \frac{r^2}{-a^2\sin 2\theta}$$

$$\tan\varphi = \frac{a^2\cos 2\theta}{-a^2\sin 2\theta} \quad [\text{From } (*)]$$

$$\tan\varphi = -\cot 2\theta$$

$$\tan\varphi = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\varphi = \frac{\pi}{2} + 2\theta$$

Given, normal at point $P(r,\theta) \Rightarrow \varphi = 2\theta$

Then, $\psi = \varphi + \theta$

$$\psi = 2\theta + \theta$$

$$\psi = 3\theta$$

8. Show that the tangents to the cardiod $r = a(1+\cos\theta)$ at $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively parallel and perpendicular to the initial line.

Given, $r = a + a\cos\theta$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$\tan\varphi = r\, {d\theta}/{dr}$

$\tan\varphi = \dfrac{a(1+\cos\theta)}{-a\sin\theta} = \dfrac{2\cos^2\theta/2}{-2\sin\theta/2\cos\theta/2}$

$\tan\varphi = -\cot\theta/2$

$\tan\varphi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\varphi = \frac{\pi}{2}+\frac{\theta}{2}$

Then, $\psi_1 = \varphi+\theta_1$ where $\theta_1 = \pi/3$

$\psi_1 = \frac{\pi}{2}+\frac{\pi}{6}+\frac{\pi}{3}$

$\psi_1 = \pi \Rightarrow$ Tangent is parallel to initial line

And, $\psi_2 = \varphi+\theta_2$ where $\theta_2 = 2\pi/3$

$\psi_2 = \frac{\pi}{2}+\frac{\pi}{3}+\frac{2\pi}{3}$

$\psi_2 = \frac{3\pi}{2} \Rightarrow$ Tangent is perpendicular to initial line.

9. Show that the circle $r=b$ intersects the curve $r^2=a^2\cos(2\theta)+b^2$ at an angle given by $\tan^{-1}(a^2/b^2)$.

Given, $r^2 = a^2\cos(2\theta)+b^2 \rightarrow$ (ii) and $r=b$ — (i)

$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta$

$\frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$

$\frac{dr}{d\theta} = 0$

$\tan\varphi_1 = \infty$

$\varphi_1 = \frac{\pi}{2}$

$\tan\varphi_2 = r\, {d\theta}/{dr}$

$\tan\varphi_2 = \frac{r^2}{-a^2\sin 2\theta}$

$\tan\varphi_2 = -\frac{b^2}{a^2\sin 2\theta}$

Given, $r=b$; $r^2 = a^2\cos 2\theta+b^2$

$a^2\cos 2\theta = 0$

$\cos 2\theta = 0 \Rightarrow \sqrt{1-\sin^2 2\theta} = 0 \quad \therefore \sin 2\theta = 0$

$$\tan\varphi_2 = \frac{-b^2}{a^2} \Rightarrow \varphi_2 = \tan^{-1}\left(\frac{-b^2}{a^2}\right)$$

$$\tan|\varphi_1 - \varphi_2| = \frac{\tan\varphi_1 - \tan\varphi_2}{1 - \tan\varphi_1 \tan\varphi_2}$$

$$|\varphi_1 - \varphi_2| = \left|\frac{\pi}{2} - \tan^{-1}\left(-\frac{b^2}{a^2}\right)\right|$$

$$|\varphi_1 - \varphi_2| = \left|\cot^{-1}\left(-\frac{b^2}{a^2}\right)\right|$$

$$|\varphi_1 - \varphi_2| = \left|\tan^{-1}\left(-\frac{a^2}{b^2}\right)\right|$$

$$|\varphi_1 - \varphi_2| = \left|-\tan^{-1}\left(\frac{a^2}{b^2}\right)\right|$$

$$|\varphi_1 - \varphi_2| = \tan^{-1}\left(\frac{a^2}{b^2}\right)$$

10. Find the angle of intersection between the curves $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$

→ For $r = a(1+\sin\theta)$

$$\frac{dr}{d\theta} = a\cos\theta$$

$$\tan\varphi_1 = r\,{}^{d\theta}/_{dr}$$

$$\tan\varphi_1 = \frac{a(1+\sin\theta)}{a\cos\theta}$$

$$\tan\varphi_1 = \frac{(\sin\theta/2 + \cos\theta/2)^2}{(\cos^2\theta/2 - \sin^2\theta/2)}$$

$$\tan\varphi_1 = \frac{(\sin\theta/2 + \cos\theta/2)^{\cancel{2}}}{\cancel{(\cos\theta/2 + \sin\theta/2)}(\cos\theta/2 - \sin\theta/2)}$$

$$\tan\varphi_1 = \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}$$

$$\tan\varphi_1 = \frac{1 + \tan\theta/2}{1 - \tan\theta/2} \Rightarrow \tan\varphi_1 = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

For $r = a(1-\sin\theta)$

$$\frac{dr}{d\theta} = -a\cos\theta$$

$$\tan\varphi_2 = r\,{}^{d\theta}/_{dr}$$

$$\tan\varphi_2 = \frac{a(1-\sin\theta)}{-a\cos\theta}$$

$$\tan\varphi_2 = \frac{(\cos\theta/2 - \sin\theta/2)^2}{-(\cos^2\theta/2 - \sin^2\theta/2)}$$

$$\tan\varphi_2 = \frac{(\cos\theta/2 - \sin\theta/2)^{\cancel{2}}}{-\cancel{(\cos\theta/2 - \sin\theta/2)}(\cos\theta/2 + \sin\theta/2)}$$

$$\tan\varphi_2 = -\left[\frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}\right]$$

$$\tan\varphi_2 = -\left[\frac{1 - \tan\theta/2}{1 + \tan\theta/2}\right] = -\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$|\varphi_1-\varphi_2| = \left|\frac{\pi}{4}+\frac{\theta}{2}-\left(-\frac{\pi}{4}+\frac{\theta}{2}\right)\right|$$

$$|\varphi_1-\varphi_2| = \frac{\pi}{2}$$

Tutorial Sheet - II

1. The curvature of a circle $S = a\psi$ at any point is $K = 1/a$

$$K = \left|\frac{d\psi}{dS}\right|$$

$$S = a\psi$$

$$dS = a\,d\psi$$

$$\frac{1}{a} = \frac{d\psi}{dS}$$

$$\therefore K = 1/a$$

2. The radius of curvature for straight line $y = mx + c$ is $\rho = \infty$ (not defined).

For straight line, $y_2 = m = 0$

$$\therefore \rho = \infty \text{ (not defined)}$$

3. The curvature of the curve $y = e^x$ at any point where it crosses the y-axis is $K = 1/2^{3/2}$

$$K = \frac{y_2}{(1+y_1^2)^{3/2}}$$

$$y = e^x \;;\; y_1 = e^x \;;\; y_2 = e^x$$

$$K = \frac{e^x}{(1+e^{2x})^{3/2}}$$

At y axis, $x = 0 \implies K = \dfrac{1}{2^{3/2}}$

4. The Taylor series expansion of $\log x$ about $x=1$ upto second degree term is $\log x = (x-1) - \frac{(x-1)^2}{2} + - \dots \infty$

→ Taylor's series : $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$

$f(x) = \log x \Rightarrow f(a) = f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(a) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(a) = -1$

$\therefore \log x = (x-1) - \frac{(x-1)^2}{2} + - \dots \infty$

5. The Maclaurin series expansion of $\cos x$ is $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - + \dots \infty$

→ Maclaurin series: $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$

$f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$\therefore \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - + \dots \infty$

6. Show that the radius of curvature of the Folium $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is given by $-\frac{3a}{8\sqrt{2}}$.

→ Radius of curvature, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \longrightarrow (*)$

Given, $x^3 + y^3 = 3axy$

$$3x^2 + 3y^2 y_1 = 3a[y + xy_1]$$

$$[3y^2 - 3ax] y_1 = 3ay - 3x^2$$

$$y_1 = \frac{\cancel{3}[ay - x^2]}{\cancel{3}[y^2 - ax]}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y_1 = \frac{a(3a/2) - (3a/2)^2}{(3a/2)^2 - a(3a/2)} \Rightarrow y_1 = -1$

$$y_2 = \frac{(ay_1-2x)(y^2-ax)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

$$y_2 = \frac{[a(-1)-2(\frac{3a}{2})((\frac{3a}{2})^2-a(\frac{3a}{2}))]-[a(\frac{3a}{2})-(\frac{3a}{2})^2((2(\frac{3a}{2})(-1))-a)]}{[(\frac{3a}{2})^2-a(\frac{3a}{2})]^2} \text{ at } (\frac{3a}{2},\frac{3a}{2})$$

$$y_2 = \frac{[(-a-3a)(9a^2/4-3a^2/2)]-[(\frac{3a^2}{2}-\frac{9a^2}{4})(-3a-a)]}{[\frac{9a^2}{4}-\frac{3a^2}{2}]^2}$$

$$y_2 = \frac{[(-4a)(3a^2/4)]-[(-\frac{3a^2}{4})(-4a)]}{[3a^2/4]^2}$$

$$y_2 = \frac{-3a^3-3a^3}{9a^4/16} = \frac{-6a^3}{9a^4} \times 16$$

$$y_2 = \frac{-32}{3a}$$

Sub $y_1=-1$ and $y_2=-32/3a$ in (*),

$$\rho = \frac{[1+(-1)^2]^{3/2}}{-32/3a}$$

$$\rho = -\frac{(\sqrt{2})^3\, 3a}{32}$$

$$\rho = -\frac{2\sqrt{2}\,3a}{32}$$

$$\rho = \frac{-3a}{8\sqrt{2}}$$

7. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ where the curve meets the x-axis.

Radius of curvature, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \longrightarrow (*)$

Given, $y^2 = \frac{4a^2(2a-x)}{x}$

$$2yy_1 = 4a^2\left[\frac{(-x)-(2a-x)}{x^2}\right]$$

$$y_1 = \frac{2a^2(-2a)}{x^2y}$$

$$y_1 = \frac{-4a^3}{x^2y}$$

$$y_2 = -4a^3\left[-\frac{[2xy+x^2y_1]}{x^4y^2}\right] = \frac{4a^3}{x^4y^2}\left[2xy + x^2\left(\frac{-4a^3}{x^2y}\right)\right]$$

$$y_2 = \frac{4a^3}{x^4y^3}[2xy^2 - 4a^3]$$

Substituting in (*), $\rho = \dfrac{\left[1 + \dfrac{16a^6}{x^4y^2}\right]^{3/2}}{\dfrac{4a^3}{x^4y^3}[2xy^2-4a^3]}$

$$\rho = \frac{[x^4y^2+16a^6]^{3/2}}{x^6y^3} \times \frac{x^4y^3}{4a^3[2xy^2-4a^3]} \longrightarrow (1)$$

At x-axis, $y=0$; Equation (1) becomes,

$$\rho = \frac{[16a^6]^{3/2}}{x^2(-16a^6)} = \left|\frac{64a^9}{-16a^6x^2}\right|$$

$$\rho = \frac{4a^3}{x^2}$$

When $y=0$, $y^2 = \dfrac{4a^2(2a-x)}{x}$

$$2a - x = 0$$

$$x = 2a$$

Then; $\rho = \dfrac{4a^3}{4a^2}$

$$\therefore \rho = a$$

8. For the curve $y = \dfrac{ax}{a+x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$.

Given, $y = \dfrac{ax}{a+x}$

Then, $y_1 = \dfrac{a(a+x)-ax}{(a+x)^2}$

$$y_1 = \frac{a^2}{(a+x)^2}$$

And, $y_2 = \dfrac{-2a^2(a+x)}{(a+x)^4}$

$$y_2 = \frac{-2a^2}{(a+x)^3}$$

Radius of curvature, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{\left[1 + \dfrac{a^4}{(a+x)^4}\right]^{3/2}}{\dfrac{-2a^2}{(a+x)^3}}$$

$$\rho = \frac{\left[(a+x)^4 + a^4\right]^{3/2}(a+x)^3}{-2a^2(a+x)^{\cancel{6}\,3}}$$

$$\rho = \left|\frac{\left[(a+x)^4 + a^4\right]^{3/2}}{-2a^2(a+x)^3}\right|$$

$$LHS = \left[\frac{2\rho}{a}\right]^{2/3} = \left[\frac{\left[(a+x)^4 + a^4\right]^{3/2}}{+\cancel{2}a^2(a+x)^3}\cdot\frac{\cancel{2}}{a}\right]^{2/3}$$

$$= \left[\frac{(a+x)^4 + a^4}{+a^2(a+x)^2}\right]$$

$$LHS = \left[\frac{(a+x)^{\cancel{4}\,2}}{a^2\cancel{(a+x)^2}}\right] + \left[\frac{a^{\cancel{4}\,2}}{a^2(a+x)^2}\right] \longrightarrow (1)$$

$$RHS = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 = \left[\frac{x}{\left(\frac{ax}{a+x}\right)}\right]^2 + \left[\frac{\left(\frac{ax}{a+x}\right)}{x}\right]^2$$

$$= \left[\frac{\cancel{x^2}(a+x)^2}{a^2\cancel{x^2}}\right] + \left[\frac{a^2\cancel{x^2}}{\cancel{x^2}(a+x)^2}\right]$$

$$RHS = \left[\frac{(a+x)^2}{a^2}\right] + \left[\frac{a^2}{(a+x)^2}\right] \longrightarrow (2)$$

From (1) *and* (2), <u>LHS = RHS</u>

9. Find the radius of curvature of $x = a\log(\sec t + \tan t)$ and $y = a\sec t$.

⇝ Given, $x = a\log(\sec t + \tan t)$

$$x' = a\left[\frac{\sec t\tan t + \sec^2 t}{(\sec t + \tan t)}\right]$$

$$x' = a\sec t\left[\frac{\tan t + \sec t}{\sec t + \tan t}\right]$$

$$x' = a\sec t$$

$$x'' = a\sec t\tan t$$

Given, $y = a\sec t$

$$y' = a\sec t\tan t$$

$$y'' = a[\sec t\tan^2 t + \sec^3 t]$$

$$y'' = a\sec t(\tan^2 t + \sec^2 t)$$

Radius of curvature, $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \longrightarrow (*)$

$$(x'^2 + y'^2)^{3/2} = [a^2\sec^2 t + a^2\sec^2 t\tan^2 t]^{3/2}$$

$$= \left[\sqrt{a^2\sec^2 t(1 + \tan^2 t)}\right]^3$$

$$= \left[\sqrt{a^2\sec^2 t\sec^2 t}\right]^3$$

$$(x'^2 + y'^2)^{3/2} = a^3\sec^6 t \longrightarrow (1)$$

$$x'y'' - y'x'' = [(a\sec t)(a\sec t)(\tan^2 t + \sec^2 t) - (a\sec t\tan t)(a\sec t\tan t)]$$

$$= [\cancel{a^2\sec^2 t\tan^2 t} + a^2\sec^4 t - \cancel{a^2\sec^2 t\tan^2 t}]$$

$$x'y'' - y'x'' = a^2\sec^4 t \longrightarrow (2)$$

Substituting (1) and (2) in (*), $\rho = \dfrac{a^3\sec^6 t}{a^2\sec^4 t}$

$$\rho = a\sec^2 t$$

10. Show that the curvature of the tractrix $x = a[\cos t + \log\tan(t/2)]$, $y = a\sin t$ at any point is given by $K = \dfrac{\tan t}{a}$

⇝ Given, $x = a[\cos t + \log\tan(t/2)]$ and $y = a\sin t$

$$x' = -a\sin t + \frac{a\sec^2(t/2)}{2\tan(t/2)}$$

$$x' = -a\sin t + \frac{a}{2\sin(t/2)\cos(t/2)}$$

$$x' = -a\sin t + a/\sin t \Rightarrow x' = -a\sin t + a\operatorname{cosec} t$$

$$y' = a\cos t$$

$$y'' = -a\sin t$$

$x'' = -a\cos t - a\,\mathrm{cosec}\,t\cot t$

Curvature, $K = \dfrac{x'y'' - y'x''}{(x'^2+y'^2)^{3/2}} \longrightarrow (*)$

Then, $(x'y''-y'x'') = [a(\mathrm{cosec}\,t - \sin t)(-a\sin t) - (a\cos t)(-a\cos t - a\,\mathrm{cosec}\,t\cot t)]$

$= -\dfrac{a^2\sin t}{\sin t} + a^2\sin^2 t + a^2\cos^2 t + \dfrac{a^2\cos t\cos t}{\sin t\sin t}$

$= a^2(\sin^2 t + \cos^2 t) - a^2 + \dfrac{a^2\cos^2 t}{\sin^2 t}$

$(x'y''-y'x'') = \underline{a^2\cot^2 t} \longrightarrow (1)$

And, $(x'^2+y'^2)^{3/2} = \left[a^2(\mathrm{cosec}\,t - \sin t)^2 + a^2\cos^2 t\right]^{3/2}$

$= \left[a^2\left[\sin^2 t + \mathrm{cosec}^2 t - 2\sin t\left(\frac{1}{\sin t}\right) + \cos^2 t\right]\right]^{3/2}$

$= \left[a^2[1 + \mathrm{cosec}^2 t - 2]\right]^{3/2}$

$= [a^2[\mathrm{cosec}^2 t - 1]]^{3/2}$

$= [a^2\cot^2 t]^{3/2}$

$(x'^2+y'^2)^{3/2} = \underline{a^3\cot^3 t} \longrightarrow (2)$

Substituting (1) and (2) in $(*)$, $K = \dfrac{a^2\cot^2 t}{a^3\cot^3 t}$

$$K = \frac{\tan t}{a}$$

11. Find the coordinates of the centre of curvature at $(at^2, 2at)$ on the parabola $y^2 = 4ax$.

$\rightsquigarrow$ Given, $y^2 = 4ax$

$2yy_1 = 4a$

$y_1 = \dfrac{2a}{y}$

$y_2 = \dfrac{-2a}{y^2}y_1 = \dfrac{-4a^2}{4axy} = \dfrac{-a}{xy}$

At $(at^2, 2at)$, $y_1 = \dfrac{2a}{2at} \Rightarrow \underline{y_1 = \dfrac{1}{t}}$

$y_2 = \dfrac{-a}{at^2(2at)} \Rightarrow \underline{y_2 = \dfrac{-1}{2at^3}}$

Coordinates of centre of curvature are (α, β) where,

$\alpha = x - \dfrac{y_1(1+y_1^2)}{y_2}$ and $\beta = y + \dfrac{(1+y_1^2)}{y_2}$

$$\alpha_{(at^2, 2at)} = at^2 - \frac{\left(\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$$= at^2 + \left[\frac{1}{\cancel{t}}\left(\frac{t^2+1}{t^2}\right)(2at^3)\right]$$

$$= at^2 + 2at^2 + 2a$$

$$\underline{\underline{\alpha = 3at^2 + 2a}}$$

$$\beta_{(at^2, 2at)} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= 2at - \left[\frac{(t^2+1)(2at^3)}{t^2}\right]$$

$$= 2\cancel{at} - 2at^3 - 2\cancel{at}$$

$$\underline{\underline{\beta_{(at^2, 2at)} = -2at^3}}$$

$$\therefore \underline{\underline{(\alpha, \beta) = [(2a+3at^2), -2at^3]}}$$

12. Find the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ for the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Given, $\sqrt{y} + \sqrt{x} = \sqrt{a}$

$$\frac{y_1}{2\sqrt{y}} + \frac{1}{2\sqrt{x}} = 0$$

$$y_1 = -\sqrt{\frac{y}{x}}$$

$$y_2 = \frac{\left(\frac{y_1}{2\sqrt{y}}\right)(\sqrt{x}) - (\sqrt{y})\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

$$y_2 = \frac{(1+\sqrt{y/x})}{2x}$$

$$y_2 = \frac{\sqrt{x}+\sqrt{y}}{2x^{3/2}}$$

At $\left(\frac{a}{4}, \frac{a}{4}\right)$, $\underline{y_1 = -1}$

$$y_2 = \frac{2\sqrt{\frac{a}{4}}}{2\left(\frac{\sqrt{a}}{\sqrt{4}}\right)^{3}}$$

$$\underline{\underline{y_2 = \frac{4}{a}}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{(1+y/x)^{3/2}}{\frac{\sqrt{x}+\sqrt{y}}{2x^{3/2}}} = \frac{\left(\frac{x+y}{x}\right)^{3/2}}{\frac{\sqrt{x}+\sqrt{y}}{2\cancel{x}^{3/2}}}$$

$$\rho = \frac{(x+y)^{3/2} \cdot 2}{\sqrt{a}}$$

$$\rho_{\left(\frac{a}{4}, \frac{a}{4}\right)} = \frac{2\left(\frac{2a}{4}\right)^{3/2}}{\sqrt{a}}$$

$$\rho_{\left(\frac{a}{4}, \frac{a}{4}\right)} = \frac{\cancel{2}(2\sqrt{a} \cdot a\sqrt{2})}{_2 8\sqrt{a}}$$

$$\underline{\underline{\rho_{\left(\frac{a}{4}, \frac{a}{4}\right)} = \frac{a}{\sqrt{2}}}}$$

Circle of curvature, $(x-\alpha)^2+(y-\beta)^2=\rho^2 \longrightarrow (*)$

$$\alpha = x-\frac{y_1(1+y_1^2)}{y_2} \qquad \beta = y+\frac{(1+y_1^2)}{y_2}$$

$$\alpha = x-\frac{(-1)(1+1)}{4/a} \qquad \beta = y+\frac{(1+1)}{4/a}$$

$$\alpha_{(\frac{a}{4},\frac{a}{4})} = \frac{a}{4}+\frac{2a}{4} \qquad \beta_{(\frac{a}{4},\frac{a}{4})} = \frac{a}{4}+\frac{2a}{4}$$

$$\alpha_{(\frac{a}{4},\frac{a}{4})} = \frac{3a}{4} \qquad \beta_{(\frac{a}{4},\frac{a}{4})} = \frac{3a}{4}$$

Substituting in $(*)$, $\boxed{\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}}$

13. Find the radius of curvature of the curve $r^n = a^n\cos(n\theta)$.

$\rightsquigarrow$ Radius of curvature, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \longrightarrow (*)$

Given, $r^n = a^n\cos(n\theta)$

$$(nr^{n-1})r_1 = a^n(-n\sin(n\theta))$$

$$r_1 = \frac{-a^n\sin(n\theta)}{r^{n-1}} \Longrightarrow r_1 = \frac{-a^n\sin(n\theta)r}{a^n\cos(n\theta)} \Rightarrow r_1 = -r\tan(n\theta)$$

$$r_2 = -a^n\left[\frac{r^{n-1}(n\cos(n\theta))-(\sin(n\theta))r^{n-2}(n-1)r_1}{(r^{n-1})^2}\right]$$

$$r_2 = -a^n\left[\frac{r^{n-1}(n\cos(n\theta))}{(r^{n-1})^2}-\frac{(n-1)(-a^n)\sin^2(n\theta)r^{n-2}}{(r^{n-1})^2(r^{n-1})}\right]$$

$$r_2 = -a^n\left[\frac{n\cos(n\theta)}{r^{n-1}}+\frac{a^n(n-1)\sin^2(n\theta)}{r^{2n-1}}\right]$$

$$r_2 = \frac{(n-1)a^{2n}\sin^2(n\theta)r}{r^{2n}}-\frac{a^n\cos(n\theta)n}{r^{n-1}}$$

$$r_2 = \frac{(n-1)\sin^2(n\theta)r}{\cos^2(n\theta)}-\frac{nr^n}{r^{n-1}}$$

$$r_2 = (n-1)r\tan^2(n\theta)-nr$$

$$r_2 = -r\left[(n-1)\tan^2(n\theta)+n\right]$$

$$(r^2+r_1^2)^{3/2} = [r^2+(r\tan(n\theta))^2]^{3/2}$$

$$= \left[\sqrt{r^2(1+\tan^2(n\theta))}\right]^3$$

$$(r^2+r_1^2)^{3/2} = r^3\sec^3(n\theta) \longrightarrow (1)$$

$$r^2+2r_1^2-rr_2 = r^2+2r^2\tan^2(n\theta)+r^2[(n-1)\tan^2(n\theta)+n]$$

$$= r^2[1+2\tan^2(n\theta)+n\tan^2(n\theta)-\tan^2(n\theta)+n]$$

$$= r^2[1+\tan^2(n\theta)+n[1+\tan^2(n\theta)]]$$

$$r^2+2r_1^2\ rr_2 = r^2\sec^2 n\theta[n+1] \longrightarrow (2)$$

Substituting (1) and (2) in $(*)$, $\rho = \dfrac{r^3\sec^3(n\theta)}{r^2\sec^2(n\theta)(n+1)}$

$$\rho = \frac{r}{(n+1)\cos(n\theta)}$$

$$\rho = \frac{r\quad a^n}{(n+1)\,r^n}$$

$$\boxed{\rho = \frac{a^n r^{1-n}}{(n+1)}}$$

14. Show that the radius of curvature at any point (r,θ) on the Cardiod $r=a(1-\cos\theta)$ varies as $\sqrt{r}$.

Given, $r = a(1-\cos\theta)$

$r_1 = a\sin\theta$

$r_2 = a\cos\theta$

Radius of curvature, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \longrightarrow (*)$

$$(r^2+r_1^2)^{3/2} = [a^2(1+\cos^2\theta-2\cos\theta)+a^2\sin^2\theta]^{3/2}$$

$$= [a^2(\sin^2\theta+\cos^2\theta-2\cos\theta+1)]^{3/2}$$

$$= [a^2(2(1-\cos\theta))]^{3/2}$$

$$(r^2+r_1^2)^{3/2} = a^3 2\sqrt{2}\,(1-\cos\theta)^{3/2} \longrightarrow (1)$$

$$r^2+2r_1^2-rr_2 = a^2(1+\cos^2\theta-2\cos\theta)+2a^2\sin^2\theta-a^2\cos\theta+a^2\cos^2\theta$$

$$= a^2[1+\cos^2\theta-2\cos\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]$$

$$= a^2[1+2-3\cos\theta]$$

$r^2+2r_1^2-rr_2 = 3a^2(1-\cos\theta) \longrightarrow (2)$

Substituting (1) and (2) in (*), $\rho = \dfrac{2\sqrt{2}\,a^3(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$

$$\rho = \frac{2\sqrt{2}}{3}a(1-\cos\theta)^{1/2}$$

$$\rho = \frac{2\sqrt{2}}{3}\sqrt{a(1-\cos\theta)}\,\sqrt{a}$$

$$\rho = \frac{2\sqrt{2}\sqrt{a}}{3}\sqrt{r}$$

$\therefore$ $\rho \propto \sqrt{r}$ (or) ρ varies as $\sqrt{r}$

15. Find the radius of curvature for the parabola $\frac{2a}{r} = 1-\cos\theta$ at any point (r,θ).

Given, $r = \dfrac{2a}{(1-\cos\theta)}$

$$r_1 = \frac{2a(\sin\theta)}{-(1-\cos\theta)^2}$$

$$r_1 = \frac{-2a\sin\theta}{(1-\cos\theta)^2} \Rightarrow r_1 = \frac{-r\sin\theta}{(1-\cos\theta)} = -r\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$r_1 = -r\cot\theta/2$$

$$r_2 = -r_1\cot\theta/2 + r\,\mathrm{cosec}^2\theta/2\,(1/2)$$

$$r_2 = r\cot^2\theta/2 + \frac{r}{2}\mathrm{cosec}^2\theta/2$$

Radius of Curvature, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$$\rho = \frac{(r^2+r^2\cot^2\theta/2)^{3/2}}{r^2+2r^2\cot^2\theta/2 - r^2\cot^2\theta/2 - \frac{r^2}{2}\mathrm{cosec}^2\theta/2}$$

$$\rho = \frac{[r^2(1+\cot^2\theta/2)]^{3/2}}{r^2+r^2\cot^2\theta/2 - \frac{r^2}{2}\mathrm{cosec}^2\theta/2}$$

$$\rho = \frac{[r^2\mathrm{cosec}^2\theta/2]^{3/2}}{r^2(1+\cot^2\theta/2) - \frac{r^2}{2}\mathrm{cosec}^2\theta/2}$$

$$\rho = \frac{r^3\mathrm{cosec}^3\theta/2}{r^2(\mathrm{cosec}^2\theta/2 - \frac{1}{2}\mathrm{cosec}^2\theta/2)}$$

$$\rho = \frac{r^3 \operatorname{cosec}^3 \theta/2}{r^2\left(\frac{1}{2}\operatorname{cosec}^2 \theta/2\right)}$$

$$\rho = 2r \operatorname{cosec} \theta/2 \longrightarrow (*)$$

Given, $2a = r(1-\cos\theta)$

$$2a = r(2\sin^2\theta/2)$$

$$\sin^2\theta/2 = \frac{a}{r}$$

$$\operatorname{cosec}^2\theta/2 = \frac{r}{a}$$

$$\operatorname{cosec}\theta/2 = \sqrt{\frac{r}{a}}$$

Substituting in $(*)$, $\rho = 2r\sqrt{\frac{r}{a}}$

$$\rho = 2(\sqrt{r})^2\sqrt{\frac{r}{a}}$$

$$\rho = 2\sqrt{\frac{r^3}{a}}$$

Tutorial Sheet → III

1. Match the following:

(i) The angle between radius vector and tangent for the polar curve at any point $P(r,\theta)$ is $\rightarrow$ (c) $\frac{1}{r}\frac{dr}{d\theta}$

(ii) The angle between radius vector and tangent for the Cartesian curve at anypoint (x,y) is $\rightarrow$ (e) $\tan\phi = \frac{xy'-y}{x+yy'}$

(iii) The radius of curvature at any point $P(x,y)$ on the catenary $y = c.\cosh\left(\frac{x}{c}\right)$ is $\rightarrow$ (a) $\rho \propto y^2$

$\rightsquigarrow$ (i) $\tan\phi = r\frac{d\theta}{dr} \Rightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

(iii) $y = c.\cosh\left(\frac{x}{c}\right)$

$y_1 = \sinh(x/c)$

$y_2 = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{(\cosh^2(x/c))^{3/2}c}{\cosh(x/c)} \Rightarrow \rho = c.\cosh^2(x/c) \Rightarrow \rho = \frac{y^2}{c} \Rightarrow \rho \propto y^2$$

2. *Find the Taylor series expansion of the function $y = \log(\cos x)$ about the point $x = \pi/3$.*

⇝ Taylor's series expansion:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

Now, $f(x) = \log(\cos x)$; $f(\pi/3) = \log(2)^{-1} = -\log 2$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$; $f'(\pi/3) = -\sqrt{3}$

$f''(x) = -\sec^2 x$; $f''(\pi/3) = -4$

$f'''(x) = -2\sec^2 x \tan x$; $f'''(\pi/3) = -8\sqrt{3}$

Then,

$$\log(\cos x) = -\log 2 - \sqrt{3}(x-\pi/3) - \frac{4(x-\pi/3)^2}{2} - \frac{8\sqrt{3}(x-\pi/3)^3}{6} - \dots$$

$$\boxed{\log(\cos x) = -\log 2 - \sqrt{3}(x-\pi/3) - 2(x-\pi/3)^2 - \frac{4\sqrt{3}(x-\pi/3)^3}{3} - \dots}$$

3. *Obtain the expression of the function $e^{\sin x}$ in ascending powers of 'x' upto terms containing x^4.*

⇝ Maclaurin's expansion:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{IV}(0) + \dots$$

Now, $f(x) = e^{\sin x}$; $f(0) = 1$

$f'(x) = e^{\sin x}\cdot\cos x$; $f'(0) = 1$

$f''(x) = f'(x)\cos x - \sin x\, e^{\sin x}$; $f''(0) = 1$

$f''(x) = f'(x)\cos x - f(x)\sin x$

$f'''(x) = f''(x)\cos x - f'(x)\sin x - f'(x)\sin x - f(x)\cos x$

$f'''(x) = \cos x\,[f''(x) - f(x)] - \sin x\,[f''(x) + f'(x)]$; $f'''(0) = 0$

$f^{IV}(x) = -\sin x\,[f''(x) - f(x)] + \cos x\,[f'''(x) - f'(x)] - \cos x\,[f''(x) + f'(x)]$
$\quad - \sin x\,[f'''(x) + f''(x)]$

$f^{IV}(0) = -1 - 2 = -3$

Then,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{3x^4}{24} + \ldots$$

$$\boxed{e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} \ldots}$$

4. Obtain the Maclaurin series expansion for the function $f(x) = \tan^{-1}(x)$ and hence deduce that $\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - + \cdots]$

<u>Maclaurin's expansion</u>:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

Now, $f(x) = \tan^{-1}(x)$; $f(0) = 0$

$f'(x) = \frac{1}{(1+x^2)} = (1+x^2)^{-1}$; $f'(0) = 1$

$f'(x) = 1 - x^2 + x^4 - x^6 + \ldots$ [Binomial theorem]

$f''(x) = -2x + 4x^3 - 6x^5 + \ldots$; $f''(0) = 0$

$f'''(x) = -2 + 12x^2 - 30x^4 + \ldots$; $f'''(0) = -2$

$f^{iv}(x) = 24x - 120x^3 + \ldots$; $f^{iv}(0) = 0$

$f^{v}(x) = 24 - 360x^2 + \ldots$; $f^{v}(0) = 24$

Then,

$$\tan^{-1} x = x - \frac{2x^3}{6} + \frac{24x^5}{120} - + \ldots$$

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots}$$

If $x = 1$, then,

$$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - + \ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \ldots$$

$$\underline{\pi = 4[1 - 1/3 + 1/5 - + \ldots]}$$

5 Using Maclaurin's series, prove that, $\sqrt{1+\sin 2x} = 1+x-\frac{x^2}{2}-\frac{x^3}{6}+\dots$

→ Maclaurin's series : $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$

Now, $f(x) = \sqrt{1+\sin 2x}$; $f(0) = 1$

$f(x) = \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x} = \sqrt{(\sin x+\cos x)^2} = \sin x + \cos x$

$f'(x) = \cos x - \sin x$; $f'(0) = 1$

$f''(x) = -\sin x - \cos x$; $f''(0) = -1$

$f'''(x) = -\cos x + \sin x$; $f'''(0) = -1$

Then, $\boxed{\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \dots}$

6. Show that $\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$

→ We know that $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - + \dots$

Then, $f(x) = \frac{x}{\sin x} = x\left[x - \frac{x^3}{6} + \frac{x^5}{120} - + \dots\right]^{-1}$

$f(x) = \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + - \dots\right)\right]^{-1}$

$f(x) = 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + - \dots\right) + \left(\frac{x^2}{6} - \frac{x^4}{120} + - \dots\right)^2 + \dots$

$f(x) = 1 + \frac{x^2}{6} - \frac{x^4}{120} + - \dots + \frac{x^4}{36} + \frac{x^8}{(120)^2} + - \dots$

$f(x) = 1 + \frac{x^2}{6} + \frac{10x^4 - 3x^4}{360} + \dots$

$$\boxed{\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots}$$