UNIT - 1

ELEMENTARY LINEAR ALGEBRA

→ Topic Learning Objectives :

With completion of this unit, students will be able to :

- Understand the fundamentals of the rank of a matrix, echelon form of the matrix.
- Check the consistency of system of linear equations.
- Apply elementary operations to solve homogenous and non-homogenous linear equations.
- Solve system of linear equations by using Gauss elimination, Gauss Jordan and Gauss Seidel methods.
- Find eigenvalues and eigenvectors of a given square matrix.
- Know properties of eigenvalues and eigenvectors.
- Apply power method to obtain largest eigenvalue.

→ **Definition** : A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m×n matrix (read as "m by n" matrix). A matrix is also denoted by a capital letter.

Thus,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

is a matrix of order $m \times n$. In the double subscripts of an element a_{ij}, the first subscript 'i' determines the row and the second subscript 'j' determines the column(i.e., element a_{ij} lies in i^{th} row and j^{th} column.

→ **Elementary row transformations** :

The following three operations which refer to rows are known as elementary row transformations.

1. Interchanging any two rows. This transformation is indicated by R_{ij}, if the i^{th} and j^{th} rows are interchanged.

2. Multiplication of the elements of any row R_i by a non-zero scalar quantity k is denoted by (kR_i).

3. Addition of constant multiplication of the elements of any row R_j to the corresponding elements of any other row R_i is denoted by $(R_i + kR_j)$.

Example: Let $A = \begin{bmatrix} 2 & 8 & 6 & 4 \\ 1 & 5 & 6 & 7 \\ 3 & 1 & 4 & 2 \end{bmatrix}$

Performing row operation R_{23} (i.e., $R_2 \leftrightarrow R_3$) of A, we get

$$A \sim \begin{bmatrix} 2 & 8 & 6 & 4 \\ 3 & 1 & 4 & 2 \\ 1 & 5 & 6 & 7 \end{bmatrix}$$

Next, by performing row operation on R_1 by multiplying $\frac{1}{2}$ to it (i.e., $R_1 \rightarrow \frac{1}{2} R_1$), we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 1 & 4 & 2 \\ 1 & 5 & 6 & 7 \end{bmatrix}$$

The row operation $R_2 \rightarrow R_2 - R_3$ gives,

$$A \sim \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & -4 & -2 & -5 \\ 1 & 5 & 6 & 7 \end{bmatrix}$$

→ Equivalent matrices: Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.

→ Elementary matrices: A matrix obtained from a unit matrix by a single elementary transformation is called elementary matrix.

→ Theorem: Elementary row transformations of a matrix A can be obtained by pre-multiplying A by the corresponding elementary matrices.

→ **Minor** : If we select any r rows and r columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the minor of A of order r.

Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

→ **Rank of a matrix** :

A matrix is said to be of rank r if

(i) it has at least one non-zero minor of order r and

(ii) every minor of order higher than r vanishes.

The rank of a matrix A shall be denoted by $\rho(A)$.

Observations :

1. Rank of a matrix and its transpose is the same.
2. Rank of a null matrix is zero.
3. Rank of a non-singular square matrix of order r is r.
4. If a matrix has a non-zero minor of order r, its rank is $\geq r$.
5. If all minors of a matrix of order $r+1$ are zero, its rank is $\leq r$.
6. Two equivalent matrices have the same order and the rank.

→ **Echelon form or Row Echelon form** :

A non-zero matrix A is an echelon matrix, if the number of zeros preceding the first non-zero entry of a row increases row by row until zero rows remain.

Ex :

$$B = \begin{bmatrix} 1 & 3 & 1 & 5 & 0 \\ 0 & 1 & 5 & 1 & 5 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is in row-echelon form.

* **Note** : The rank of an echelon matrix is the number of non-zero rows in it. i.e., $\rho(B) = 3$

→ Problems :

1. Determine the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Sol:
Since the given matrix A is of order 3×4, $\rho(A) \leq 3$.
Consider all the minors of order 3.
They are,

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 1(-8) - 2(8-3) + 3(6) = 10 - 10 = 0$$

$$\begin{vmatrix} 2 & 3 & -2 \\ -2 & 1 & 3 \\ 0 & 4 & 1 \end{vmatrix} = 2(1-12) - 3(-2) - 2(-8) = -22 + 6 + 16 = 0$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 3 & 4 & 1 \end{vmatrix} = 1(1-12) - 3(2-9) - 2(8-3) = -11 + 21 - 10 = 0$$

$$\begin{vmatrix} 1 & 2 & -2 \\ 2 & -2 & 3 \\ 3 & 0 & 1 \end{vmatrix} = 1(-2) - 2(2-9) - 2(6) = -2 + 14 - 12 = 0$$

Therefore, the rank is less than 3.
Now, consider minors of order 2.

$$\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -2 - 4 = -6 \neq 0$$

Since this minor of order 2 is non-zero, $\rho(A) = 2$.

* **Note** : The method of finding the rank of a matrix by using the definition of the rank of a matrix is very tedious. However, it would be better to apply the definition to find the rank, after bringing the given matrix to echelon form.

2. Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Soln The rank of a matrix can be obtained by reducing it to row echelon form.

Given, $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ gives

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to 5R_3 - 4R_2$; $R_4 \to 5R_4 - 9R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

$45 - 12 = 33$
$50 - 28 = 22$
$60 - 27 = 33$
$85 - 63 = 22$

$R_4 \to R_4 - R_3$ gives $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

As there are no elements below the fourth diagonal element the process is complete.

$\therefore \rho(A) =$ number of non-zero rows $= 3$.

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3. Reduce the following matrix to echelon form and hence find the rank of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Sol:

$R_1 \longleftrightarrow R_2$ gives

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$; $R_4 \rightarrow R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$; $R_4 \rightarrow R_4 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Above matrix is in the echelon form, therefore rank of matrix $= \rho(A) = 3$.

4. Using the elementary transformations find the rank of the matrix

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

Sol: Given

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$; $R_3 \to R_3 + 3R_1$; $R_4 \to R_4 + 5R_1$

$$B \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$; $R_4 \to R_4 - 2R_2$

$$B \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \ \rho(B) = 2$

5. Find the value of b in the matrix $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$ given that its rank is 2.

Sol: Given

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$$

$R_3 \to R_3 - bR_1$

$$\sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 13-5b & 10-4b \end{bmatrix}$$

$R_3 \to 3R_3 - (13-5b)R_2$

$$\sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 4-2b \end{bmatrix}$$

$39 - 15b - 39 + 15b$
$30 - 12b - 26 + 10b = 4 - 2b$

For the rank to be $\underline{\underline{2}}$, $4 - 2b = 0 \Rightarrow b = 4/2 = 2$

* **Applications:**

1. Rank helps ascertain the uniqueness and feasibility of solutions. It can indicate whether a system has a single unique solution, infinitely many solutions or no solution at all.
2. Rank provides a measure for the dimension of the image of a linear map.
3. In the realm of network analysis, the rank of the adjacency matrix delivers insights into the network's structure, even helping identify potential redundancies in the system.
4. In the area of source enumeration.
5. In the classification of an image.

→ **Normal form (Canonical form)**

By performing elementary transformation, any non-zero matrix A can be reduced to one of the following four forms, called the Normal form of A :

(i) I_r (ii) $\begin{bmatrix} I_r & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained is called the rank. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called first canonical form of A.

Augmented matrix :

The matrix obtained by appending to A the column matrix B is called the augmented matrix associated with the system and is denoted by $[A|B]$ or $[A:B]$.

i.e., $$[A:B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$ is called the augmented matrix.

Solution of simultaneous linear equations :

Types of linear equations :

(i) Consistent : A system of equations is said to be consistent if they have one or more solution.

Ex :

(a) $x + 2y = 4$; $3x + 2y = 2$ → unique solution

(b) $x + 2y = 4$; $3x + 6y = 12$ → infinite solution

(ii) Inconsistent : If a system of equations has no solution, it is said to be inconsistent.

Ex : $x + 2y = 4$; $3x + 6y = 5$

* Solution of non-homogenous system of linear equations :

(a) Consistent equations if $\rho(A) = \rho([A:B])$

(i) Unique Solution if $\rho(A) = \rho([A:B]) = n$ (the number of unknowns)

(ii) Infinite Solution if $\rho(A) = \rho([A:B]) < n$

(b) Inconsistent equations if $\rho(A) \neq \rho([A:B])$

→ **Solution of simultaneous linear equations:**

A linear system of m simultaneous equations in n unknowns can be expressed as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad & \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

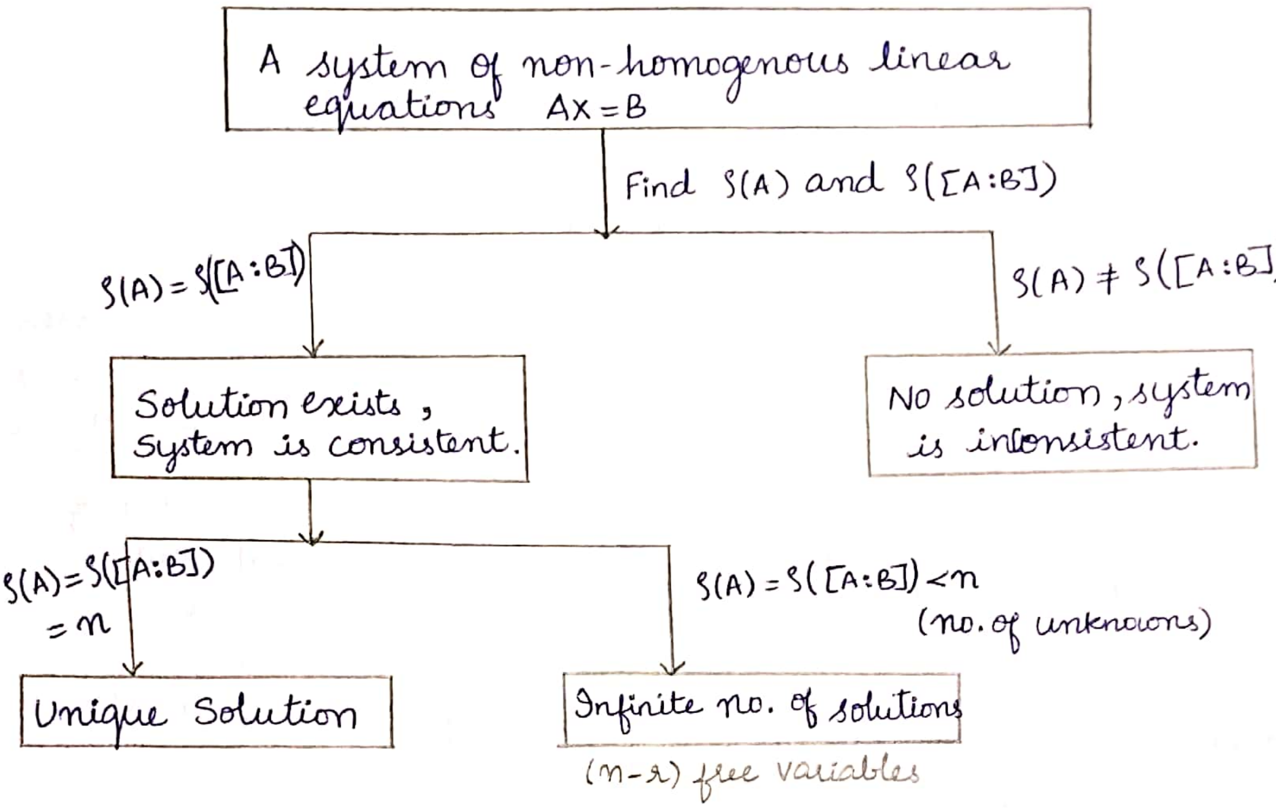
The above system in the matrix equivalent form can be expressed as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$ is called the coefficient matrix,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$ is called the matrix of unknowns and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$ is the column matrix of constants.

If all b_i's for $i = 1, 2, \ldots, m$ are zero i.e., $b_1 = b_2 = \cdots = b_m = 0$, then the system is said to be homogenous and is said to be non-homogenous if at least one of the b_i's is non-zero.



* Solution of homogenous system of linear equations :

For a system of homogenous linear equations $AX = 0$,

(i) $X = 0$ is always a solution. This solution in which each unknown has the value zero is called the Null solution or the Trivial solution. Thus a homogenous system is always consistent.

(ii) If $\rho(A) = n$ (the number of unknowns), the system has only the trivial solution.

(iii) If $\rho(A) < n$, the system has an infinite number of non-trivial solutions.

(A non-trivial solution exists to a system if and only if $|A| = 0$).

A system of homogenous linear equations $AX = 0$

↓

Always has a solution

↓ Find $\rho(A)$

$\rho(A) = n$ → Unique or Trivial Solution (each unknown equal to zero)

$\rho(A) < n$ → Infinite number of non-trivial solutions

$(n-r)$ free variables

$r = \rho(A)$

1. Test the consistency of the following system and solve,

$$2x - x_2 + 3x_3 = 1$$
$$-3x_1 + 4x_2 - 5x_3 = 0$$
$$x_1 + 3x_2 - 6x_3 = 0$$

Soln: Consider the augmented matrix

$$[A : B] = \left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ -3 & 4 & -5 & 0 \\ 1 & 3 & -6 & 0 \end{array}\right]$$

$R_2 \to 2R_2 + 3R_1$; $R_3 \to 2R_3 - R_1$

$$[A : B] \sim \left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 7 & -15 & -1 \end{array}\right]$$

$R_3 \to 5R_3 - 7R_2$

$$[A : B] \sim \left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & -68 & -26 \end{array}\right]$$

$\rho(A) = \rho([A:B]) = 3 =$ number of unknowns.

Thus, the system of linear equations is consistent and has a unique solution.

To find the unknowns, consider the rows of [A : B] in the last step in terms of its equivalent equations,

$$2x_1 - x_2 + 3x_3 = 1$$
$$5x_2 - x_3 = 3$$
$$-68x_3 = -26$$

Here, we make use of back substitution in order to find the unknowns by considering last equation to find x_3, next second to find x_2 and finally first equation to find x_1.

$$\therefore \quad -68x_3 = -26$$
$$x_3 = \frac{-26}{-68} = \frac{13}{34}$$

$$5x_2 - x_3 = 3 \Rightarrow 5x_2 = 3 + \frac{13}{34} ; x_2 = \frac{102+13}{34 \times 5} = \frac{23}{34}$$

$$2x_1 - x_2 + 3x_3 = 1 \Rightarrow 2x_1 = 1 + \frac{23}{34} - \frac{3 \times 13}{34}$$

$$\Rightarrow 2x_1 = \frac{34 + 23 - 39}{34} \Rightarrow x_1 = \frac{9}{34}$$

Hence the solution is,

$$x_1 = \frac{9}{34}, \quad x_2 = \frac{23}{34}, \quad x_3 = \frac{13}{34}$$

2. Check the following system of equations for consistency and solve if consistent.

$x + 2y + 2z = 1$, $2x + y + z = 2$, $3x + 2y + 2z = 3$, $y + z = 0$

Sol: The augmented matrix is given by

$$[A : B] = \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 2 & 1 & 1 & : & 2 \\ 3 & 2 & 2 & : & 3 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & -3 & -3 & : & 0 \\ 0 & -4 & -4 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$R_3 \to 3R_3 - 4R_2$; $R_4 \to 3R_4 + R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & -3 & -3 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = \rho([A:B]) = 2 < 3$, no. of unknowns.

Thus, the given system is consistent and possesses infinite number of solutions by assigning arbitrary values to $(n-r) = 3-2 = 1$ free variable.

$$\Rightarrow \quad x + 2y + 2z = 1$$
$$-3y - 3z = 0$$

Let z be the free variable. i.e., $z = k$ (arbitrary value)

$$\therefore \quad -3y - 3z = 0 \quad \Rightarrow \quad \underline{y = -z = -k}$$

$$x + 2y + 2z = 1 \quad \Rightarrow \quad \underline{x} = 1 - 2y - 2z \quad \Rightarrow x = 1 + 2k - 2k = 1$$

$\therefore$ The solution is given by $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -k \\ k \end{bmatrix}$

3. Show that the following system of equations is not consistent $x + 2y + 3z = 6$, $3x - y + z = 4$, $2x + 2y - z = -3$, $-x + y + 2z = 5$.

Sol: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 3 & -1 & 1 & : & 4 \\ 2 & 2 & -1 & : & -3 \\ -1 & 1 & 2 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$; $R_4 \to R_4 + R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -7 & -8 & : & -14 \\ 0 & -2 & -7 & : & -15 \\ 0 & 3 & 5 & : & 11 \end{bmatrix}$$

$R_3 \to 7R_3 - 2R_2$; $R_4 \to 7R_4 + 3R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -7 & -8 & : & -14 \\ 0 & 0 & -33 & : & -77 \\ 0 & 0 & 11 & : & 35 \end{bmatrix}$$

$R_4 \to 3R_4 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -7 & -8 & : & -14 \\ 0 & 0 & -33 & : & -77 \\ 0 & 0 & 0 & : & 28 \end{bmatrix}$$

$\rho(A) = 3$ and $\rho([A:B]) = 4$

$\rho(A) \neq \rho([A:B])$

$\therefore$ The given system is inconsistent and has no solution.

4. Determine whether the following system of equations possesses a non-trivial solution.

$$x_1 + 2x_2 - x_3 = 0 \; ; \; 4x_1 - x_2 + x_3 = 0 \; ; \; 5x_1 + x_2 - 2x_3 = 0.$$

Soln: Since the given system of linear equations is homogenous for which the rank of the coefficient matrix is same as rank of the augmented matrix, we consider only the coefficient matrix and reduce it to row echelon form and solve the system as we did in the case of non-homogenous system.

Method-1: Consider $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 5 & 1 & -2 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$; $R_3 \to R_3 - 5R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & -9 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & 0 & -2 \end{bmatrix}$$

Thus, the given system is equivalent to

$$x_1 + 2x_2 - x_3 = 0$$
$$-9x_2 + 5x_3 = 0$$
$$-2x_3 = 0$$

$\Rightarrow x_3 = 0. \quad \therefore x_2 = 0.$

$x_1 + 2x_2 - x_3 = 0 \quad \Rightarrow x_1 = 0$

i.e., $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which is a trivial solution.

Hence the system doesnot possess a non-trivial solution.

<u>Method 2</u> : $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 5 & 1 & -2 \end{vmatrix} = 1(2-1) - 2(-8-5) - 1(+4+5)$

$= 1 + 26 - 9 = 18 \neq 0$

Hence, the system doesnot possess non-trivial solution.

5. Find the values of λ for which the system has a solution and solve it in each case.

$$x + y + z = 1 \; ; \; x + 2y + 4z = \lambda \; ; \; x + 4y + 10z = \lambda^2$$

<u>Soln</u> The augmented matrix is given by

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \lambda \\ 1 & 4 & 10 & : & \lambda^2 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \; ; \; R_3 \to R_3 - R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 3 & 9 & : & \lambda^2-1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 0 & 0 & : & \lambda^2-3\lambda+2 \end{bmatrix}$$

We observe that $\rho(A) = 2$ and $\rho([A:B]) = 2$ iff $\lambda^2 - 3\lambda + 2 = 0$.
$(\lambda-1)(\lambda-2)$

i.e., for $\lambda = 1$ or $\lambda = 2$.

$\Rightarrow$ System will possess a solution if $\lambda = 1$ or 2 and in both the cases the system will have infinite number of solutions as $\rho(A) = \rho([A:B]) = 2 < 3$, number of unknowns and hence has 1 free variable.

Let us consider these cases one after the other.

Case (i) : When $\lambda = 1$, the reduced system gives

$x + y + z = 1$; $y + 3z = 1 - 1 = 0$.

Let $z = k$ be arbitrary.

Then $y = -3z = -3k$.

$x + y + z = 1 \Rightarrow x = 1 - y - z = 1 + 3k - k = 1 + 2k$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2k \\ -3k \\ k \end{bmatrix}$$

Case (ii) : When $\lambda = 2$, the reduced system gives

$x + y + z = 1$; $y + 3z = 2 - 1 = 1$.

Let $z = k$, then $y = 1 - 3z = 1 - 3k$

$x = 1 - y - z = 1 - 1 + 3k - k = 2k$, where k is arbitrary

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ 1-3k \\ k \end{bmatrix}$$

6. Find the values of λ and μ for which the system $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ has (i) a unique solution (ii) infinitely many solutions and (iii) no solution

Sol: Consider the augmented matrix

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Here, we observe that,

(a) if $\lambda-3=0$ and $\mu-10\neq 0$, i.e., $\lambda=3$ and $\mu\neq 10$, then the system will be inconsistent and possess no solution.

(b) if $\lambda-3=0$ and $\mu-10=0$ i.e., $\lambda=3$ and $\mu=10$, the system will reduce to $x+y+z=6$

$$y+2z=4$$

Hence in this case, the system possess infinite solutions.

(c) if $\lambda-3\neq 0$ i.e., $\lambda\neq 3$, the system will possess a unique solution, irrespective of the value of μ.

7. Show that the equations

$$-2x+y+z=a\ ,\ x-2y+z=b\ ,\ x+y-2z=c$$

have a solution only if $a+b+c=0$. Find all possible solutions when $a=1, b=1, c=-2$.

Sol: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} -2 & 1 & 1 & : & a \\ 1 & -2 & 1 & : & b \\ 1 & 1 & -2 & : & c \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + R_1$

$$\sim \begin{bmatrix} -2 & 1 & 1 & : & a \\ 0 & -3 & 3 & : & 2b+a \\ 0 & 3 & -3 & : & 2c+a \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} -2 & 1 & 1 & : & a \\ 0 & -3 & 3 & : & 2b+a \\ 0 & 0 & 0 & : & 2a+2b+2c \end{bmatrix}$$

The above system of equations will be consistent if $2a+2b+2c=0$. i.e., if $a+b+c=0$.

To find the solution when $a=1, b=1, c=-2$.

The given system is equivalent to

$$-2x+y+z=1 \quad ; \quad -3y+3z = 2+1 = 3$$

Let $z=k$ be arbitrary.

Then $y = -\dfrac{3-3z}{3} = z-1 = k-1$

Also, $x = \dfrac{y+z-1}{2} = \dfrac{k-1+k-1}{2} = k-1$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k-1 \\ k-1 \\ k \end{bmatrix}$$

Gauss elimination method :

The Gauss elimination method is known as the row reduction algorithm for solving system of linear equations. It consists of a sequence of elementary row operations performed on the corresponding coefficient matrix so that it is reduced to row echelon form. Since the system is reduced to upper triangular system, the unknowns are found by back substitution.

The steps of the Gauss elimination method are :

(i) Write the given system of linear equations in matrix form $AX = B$, where A is the coefficient matrix, X is a column matrix of unknowns and B is the column matrix of the constants.

(ii) Reduce the augmented matrix $[A : B]$ by elementary row operations to get $[A' : B']$.

(iii) We get A' as an upper triangular matrix.

(iv) By the backward substitution in $A'X = B'$, we get the solution of the given system of linear equations.

1. Solve the following system by Gauss elimination method.
$x + y - z = 0$; $2x - 3y + z = -1$; $x + y + 3z = 12$; $y + z = 5$.

Sol: The augmented matrix is given by

$$[A : B] = \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 2 & -3 & 1 & : & -1 \\ 1 & 1 & 3 & : & 12 \\ 0 & 1 & 1 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 0 & -5 & 3 & : & -1 \\ 0 & 0 & 4 & : & 12 \\ 0 & 1 & 1 & : & 5 \end{bmatrix}$$

$R_4 \to 5R_4 + R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

$R_4 \to R_4 - 2R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By back substitution, we get

$4z = 12 \Rightarrow z = 3$

$-5y + 3z = -1 \Rightarrow y = \frac{3z+1}{5} = \frac{10}{5} = 2$

$x + y - z = 0 \Rightarrow x = z - y = 1$

2. Solve the following system by Gauss elimination method.

$2x_1 - x_2 + 3x_3 = 1$; $-3x_1 + 4x_2 - 5x_3 = 0$; $x_1 + 3x_2 - 6x_3 = 0$.

Sol: Consider the augmented matrix

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ -3 & 4 & -5 & 0 \\ 1 & 3 & -6 & 0 \end{array}\right]$$

$R_2 \to 2R_2 + 3R_1$; $R_3 \to 2R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 7 & -15 & -1 \end{array}\right]$$

$R_3 \to 5R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & -68 & -26 \end{array}\right]$$

By back substitution, we have

$$-68x_3 = -26 \quad \Rightarrow x_3 = \frac{13}{34}$$

$$5x_2 - x_3 = 3 \quad \Rightarrow \quad x_2 = \frac{3+x_3}{5} = \frac{3+\frac{13}{34}}{5} = \frac{23}{34}$$

$$2x_1 - x_2 + 3x_3 = 1 \quad \Rightarrow 2x_1 = 1 + x_2 - 3x_3 \quad \Rightarrow 2x_1 = 1 + \frac{23}{34} - \frac{39}{34}$$

$$\Rightarrow x_1 = \frac{9}{34}$$

→ **Gauss - Jordan elimination method :**

The procedure for Gauss-Jordan elimination is as follows :

(i) Write the augmented matrix.

(ii) Interchange rows if necessary to obtain a non-zero number in the first row, first column.

(iii) If the first row first column entry is a, then multiply the top row by 1/a to form a leading 1 in that row.

(iv) Use row operations to make all other entries as zeros in column one.

(v) Interchange rows if necessary to obtain a non zero number in the second row, second column. Use a row operation to make this entry one. Use row operations to make all other entries as zeros in column two.

(vi) Repeat step 5 for row three, column three.

Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row echelon form. The process of obtaining a 1 in a location, and then making all other entries zeros in that column is called pivoting.

1. Solve for x, y and z in :

$2y - 3z = 2$; $x + z = 3$; $x - y + 3z = 1$

Sol: The augmented matrix of given system of equations is

$$[A : B] = \left[\begin{array}{ccc:c} 0 & 2 & -3 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & -1 & 3 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 2 & -3 & 2 \\ 1 & -1 & 3 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 2 & -3 & 2 \\ 0 & -1 & 2 & -2 \end{array}\right]$$

$R_2 \rightarrow R_2/2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3/2 & 1 \\ 0 & -1 & 2 & -2 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3/2 & 1 \\ 0 & 0 & 1/2 & -1 \end{array}\right]$$

$R_3 \rightarrow 2R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3/2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$R_1 \rightarrow R_1 - R_3$; $R_2 \rightarrow R_2 + \frac{3}{2} R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This is now in reduced row-echelon form.

$\therefore$ $x = 5$

$y = -2$

$z = -2$

2. Use Gauss - Jordan method to find the inverse of the following matrix :

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Sol: Consider the matrix $[A|I]$ and apply elementary row operations on both A and I until A gets transformed to I.

Consider

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$R_2 \to \frac{1}{2} R_2$; $R_3 \to -\frac{1}{4} R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$R_1 \to R_1 - 3R_3$; $R_2 \to R_2 + 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{7}{4} & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

* Diagonally dominant form :

A system of n linear equations in n unknowns given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

is said to be in diagonally dominant form if

$$|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$$
$$|a_{22}| > |a_{21}| + |a_{23}| + \dots + |a_{2n}|$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \dots + |a_{n(n-1)}|.$$

Gauss Seidel Method :

The Gauss Seidel method is an iterative method used to solve a system of linear equations based on the idea of successive approximations. To find the solution to system of equations $AX = B$, we assume that the system of equations have a unique solution and the system of equations is to be arranged in the diagonally dominant form.

Now, we shall begin to solve equations for $x_1, x_2, \dots, x_n$ as

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n\right]$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn-1}x_{n-1}\right]$$

By making an initial guess for the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ and substituting these values only to the RHS of the above equations, we get first approximations $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$. But the value is substituted in the next equation as and when it is calculated. i.e.,

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \dots - a_{1n}x_n^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{13}x_3^{(0)} - \dots - a_{1n}x_n^{(0)}\right]$$

$$\vdots$$

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \dots - a_{nn-1}x_{n-1}^{(1)}\right]$$

This process continues until two successive iterations have the same values upto certain decimal points.

1. Solve the following system of equations using Gauss-Seidel method.

$6x + 15y + 2z = 72$; $x + y + 54z = 110$; $27x + 6y - z = 85$.

Soln In the above equations, we have

$|15| > |6| + |2|$, $|54| > |1| + |1|$, $|27| > |6| + |-1|$.

Hence the equations are arranged in the diagonally dominant form as :

$27x + 6y - z = 85$; $6x + 15y + 2z = 72$; $x + y + 54z = 110$.

$\therefore\ x = \dfrac{85 - 6y + z}{27}$; $y = \dfrac{1}{15}[72 - 6x - 2z]$; $z = \dfrac{1}{54}[110 - x - y]$

These equations are used to find sequentially x, y and z in each of the iterations.

Let us choose $[x^{(0)}, y^{(0)}, z^{(0)}] = [0, 0, 0]$ as the initial solution

First iteration :

$$x^{(1)} = \frac{1}{27}[85 - 0 + 0] = 3.1481$$

$$y^{(1)} = \frac{1}{15}[72 - 6(3.1481) - 0] = 3.5408$$

$$z^{(1)} = \frac{1}{54}[110 - 3.1481 - 3.5408] = 1.9132$$

Note that in finding $y^{(1)}$, the latest value of $x^{(1)}$ is used and not $x^{(0)}$. Similarly for $z^{(1)}$.

Second iteration

$$x^{(2)} = \frac{1}{27}[85 - 6(3.5408) + 1.9132] = 2.4322$$

$$y^{(2)} = \frac{1}{15}[72 - 6(2.4322) - 2(1.9132)] = 3.5720$$

$$z^{(2)} = \frac{1}{54}[110 - 2.4322 - 3.5720] = 1.9258$$

Third iteration

$$x^{(3)} = \frac{1}{27}[85 - 6(3.5720) + 1.9258] = 2.4257$$

$$y^{(3)} = \frac{1}{15}[72 - 6(2.4257) - 2(1.9258)] = 3.5729$$

$$z^{(3)} = \frac{1}{54}[110 - 2.4257 - 3.5729] = 1.9259$$

Fourth iteration

$$x^{(4)} = \frac{1}{27}[85 - 6(3.5729) + 1.9259] = 2.4255$$

$$y^{(4)} = \frac{1}{15}[72 - 6(2.4255) - 2(1.9259)] = 3.5730$$

$$z^{(4)} = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9259$$

Fifth iteration

$$x^{(5)} = \frac{1}{27}[85 - 6(3.5730) + 1.9259] = 2.4255$$

$$y^{(5)} = \frac{1}{15}[72 - 6(2.4255) - 2(1.9259)] = 3.5730$$

$$z^{(5)} = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9259$$

Since the solutions in 4^{th} and 5^{th} iterations agree upto 4 decima places, the solution can be taken as

$$[x, y, z] = [2.4255, 3.5730, 1.9259]$$

Eigenvalues and Eigenvectors

* **Vectors** : Any quantity having n-components is called a vector of order n. Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any n numbers $x_1, x_2, \ldots, x_n$ written in a particular order, constitute a vector X.

The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are said to be linearly dependent if there exists numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ not all zero, such that

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_r \mathbf{x}_r = 0. \longrightarrow (1)$$

If no such numbers, other than zero exist, the vectors are said to be linearly independent.

If $\lambda_1 \neq 0$, transposing $\lambda_1 \mathbf{x}_1$ to the other side and dividing by $-\lambda_1$, we write (1) in the form

$$\mathbf{x}_1 = \mu_2 \mathbf{x}_2 + \mu_3 \mathbf{x}_3 + \ldots + \mu_r \mathbf{x}_r .$$

Then the vector x_1 is said to be a linear combination of the vectors $x_2, x_3, \ldots, x_r$.

* **Eigen Values** :

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the n^{th} order unit matrix. The determinant of this matrix equated to zero,

i.e.,

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the characteristic equation of A. On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots + k_n = 0,$$

where k's are expressible in terms of the elements a_{ij}.

The roots of this equation are called the eigenvalues or latent roots or characteristic roots of the matrix A.

* **Eigen Vectors :**

The linear transformation

$$Y = AX \quad \longrightarrow (i)$$

carries the column vector X into the column vector Y by means of the square matrix A. In practice, it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (i).

Then, $\lambda X = AX$ or $AX - \lambda I X = 0$ or $[A - \lambda I]X = 0$. $\longrightarrow$ (ii)

This matrix equation represents n homogenous linear equations which will have a non-trivial solution only if the coefficient matrix is singular i.e., if $|A - \lambda I| = 0$.

This is called the characteristic equation of the transformation and it has n roots. Corresponding to each root, the equation (ii) will have a non-zero solution $X = [x_1, x_2, \ldots, x_n]^T$ which is known as the eigen vector or latent vector.

* **Properties of eigenvalues and eigenvectors :**

1. Square matrix of order n will always possess n eigenvalues which may be distinct or not.

2. If all the n eigenvalues of A are distinct, then there exists exactly one eigenvector corresponding to each one of them.

3. For eigenvalues that are repeated, there may be exactly one or more than one eigenvector.

4. A^{-1} exists if and only if 0 is not an eigenvalue of A.
5. If λ is the eigenvalue of A, then $1/\lambda$ is the eigenvalue of A^{-1}.
6. The same characteristic vector cannot correspond to two distinct eigenvalues.
7. Any square matrix A and its transpose A^T have the same eigenvalues.
8. The eigenvalues of a triangular matrix are just the diagonal elements of the matrix.
9. The eigenvalues of an idempotent matrix are either zero or unity.
10. The sum of the eigen values of a matrix is its trace. i.e., the sum of the elements of principal diagonal.
11. The product of the eigenvalues of a matrix A is equal to its determinant.
12. If λ is an eigenvalue of an orthogonal matrix, then $1/\lambda$ is also its eigenvalue.
13. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then A^m has the eigenvalues $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer).

* Example:
Let $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Consider the product $AX = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= (-1) X$

The above is of the form $AX = \lambda X$ where $\lambda = -1$ and $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue -1.

→ **Working procedure :**

Given a square matrix A, we form $|A-\lambda I| = 0$. On expanding we get the characteristic equation of A. By solving it we get all the eigenvalues.

We then form the system of homogenous equations $[A-\lambda I][X] = [0]$ and solve for (x, y, z) corresponding to every value of λ.

Simple techniques of solving or the rule of cross multiplication (for any pair of equations) can be employed. The values x, y, z obtained by the rule of cross multiplication satisfy simultaneously all the three equations.

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Sol: The characteristic equation of A is $|A-\lambda I| = 0$.

i.e., $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$

$$\Rightarrow \lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow (\lambda-4)(\lambda+1) = 0$$

Thus, the eigenvalues are $\lambda = 4, -1$.

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ denote the characteristic vector. Then

(i) Characteristic vector corresponding to $\lambda = 4$ is the solution of the system

$$(1-4)x + 2y = 0 \qquad \Rightarrow \qquad -3x + 2y = 0$$

$$3x + (2-4)y = 0 \qquad\qquad 3x - 2y = 0$$

Note that the above two equations are identical. Let $x = k$ $(k \neq 0)$

then, $y = \frac{3k}{2}$.

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ 3k/2 \end{bmatrix}$ is the eigenvector corresponding to eigenvalue 4.

Characteristic vector corresponding to $\lambda=-1$ is the solution of the system

$$\begin{aligned}(1+1)x + 2y &= 0 \\ 3x + (2+1)y &= 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}2x+2y&=0\\ 3x+3y&=0\end{aligned}$$

Note that the above two equations are identical as in earlier case.

Set $x=k$ (a non-zero number), then $y=-k$.

Thus $\begin{bmatrix} k \\ -k \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue -1.

2. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol: The characteristic equation is $|A-\lambda I|=0$

i.e., $$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 01 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + 1(-2+\lambda) = 0$$

$$\Rightarrow 8-\lambda^3-6\lambda(2-\lambda) - 2+\lambda = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$\lambda=1$ is a root of the above equation (by inspection).

Let us use synthetic division to find the other two roots.

$$\begin{array}{c|cccc} \lambda=1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$\therefore$ The other two roots are determined by $\lambda^2-5\lambda+6=0$.

$$\Rightarrow \lambda = 2, 3$$

Thus the eigenvalues are $\lambda=1, 2$ and 3.

(i) Eigenvector corresponding to $\lambda = 1$ is the solution of the system of equations

$$x_1 + x_3 = 0$$
$$x_2 = 0$$
$$x_1 + x_3 = 0.$$

$\Rightarrow x_1 = -x_3$

Let $x_1 = k_1 \ (k_1 \neq 0)$.

$\therefore [x_1 \ x_2 \ x_3]^T = [k_1 \ 0 \ -k_1]^T$ is the eigenvector corresponding to eigenvalue 1.

(ii) Eigenvector corresponding to $\lambda = 2$ is the solution of the system of equations

$$x_3 = 0 \ ; \ 0 = 0 \ ; \ x_1 = 0.$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0} = k_2 \ (k_2 \neq 0)$$

$\therefore [x_1 \ x_2 \ x_3]^T = [0 \ k_2 \ 0]^T$ is the eigenvector corresponding to eigenvalue 2.

(iii) Eigenvector corresponding to $\lambda = 3$ is the solution of the system of equations

$$-x_1 + x_3 = 0$$
$$-x_2 = 0$$
$$x_1 - x_3 = 0$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1} = k_3 \ (k_3 \neq 0)$$

$\therefore [x_1 \ x_2 \ x_3]^T = [k_3 \ 0 \ k_3]^T$ is the eigenvector corresponding to eigenvalue 3.

Rayleigh's Power Method :

This is an iterative method to determine the numerically largest eigenvalue (dominant eigenvalue) and the corresponding eigenvector of a square matrix.

Suppose A is the given square matrix, we assume initially an eigenvector (column matrix) X_0 in a simple form like $[1\ 0\ 0]^T$ or $[0\ 1\ 0]^T$ or $[0\ 0\ 1]^T$ or $[1\ 1\ 1]^T$ and find the matrix product AX_0 which will also be a column matrix.

We take out the largest element as the common factor (this technique is called normalization) to obtain $AX_0 = \lambda_1 X_1$. We then compute AX_1 and again put it in the form $AX_1 = \lambda_2 X_2$ by normalization.

This iterative process is continued till two consecutive iterative values of λ and X are same upto a desired accuracy. The values so obtained are respectively the largest eigenvalue and the corresponding eigenvector of the given square matrix A.

1. Find the largest eigenvalue and the corresponding eigenvector of the matrix A by the power method given that

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Soln: Let $X_0 = [1\ 0\ 0]^T$ be the initial eigen vector.

$$AX_0 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2.98 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix}$$

$\therefore$ We conclude that the largest eigenvalue is approximately 3 and the corresponding eigenvector is $[1 \ 0 \ 1]^T$.

2. Using Rayleigh's power method find numerically the largest eigen value and the corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

Sol: Let $X_0 = [1 \ 0 \ 0]^T$ be the initial eigen vector.

$$AX_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \lambda_3 X_3$$

Since $X_2 = X_3$ and λ_2 and λ_3 are approximately equal, we conclude that the largest eigen value of A is 25.18 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$.

→ Applications :

1. Google page rank algorithm : The largest eigenvector of the graph of the internet is how the pages are ranked.
2. Dimensionality reduction : This is extremely useful in machine learning and data analysis as it allows one to understand where most of the variation in data comes from.