Multivariable Functions and Partial Differentiation

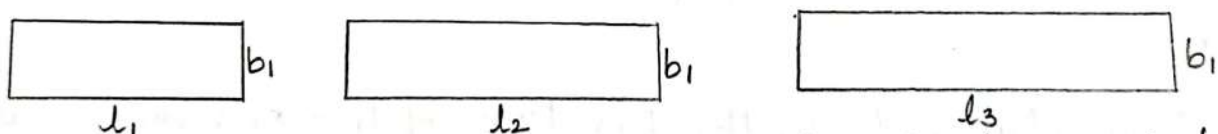
Introduction :

The physical quantities like displacement, density, temperature of metal plates etc. depend on more than one variable. These quantities vary with space and time. Hence, rate of change of these lead to concepts of partial derivatives.

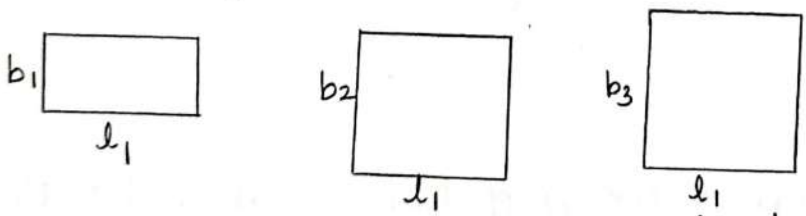
For example, the area of a rectangle of length 'l' and breadth 'b' is given by $A = lb$. This can be represented as a general function $u = f(l, b)$. There are two types of changes possible with such a function resulting in two first order partial changes and one total differential.

The partial change in area of rectangle is due to changes in any one of the variables either length or breadth. When there are changes in both variables at a time, it results in total change in area.

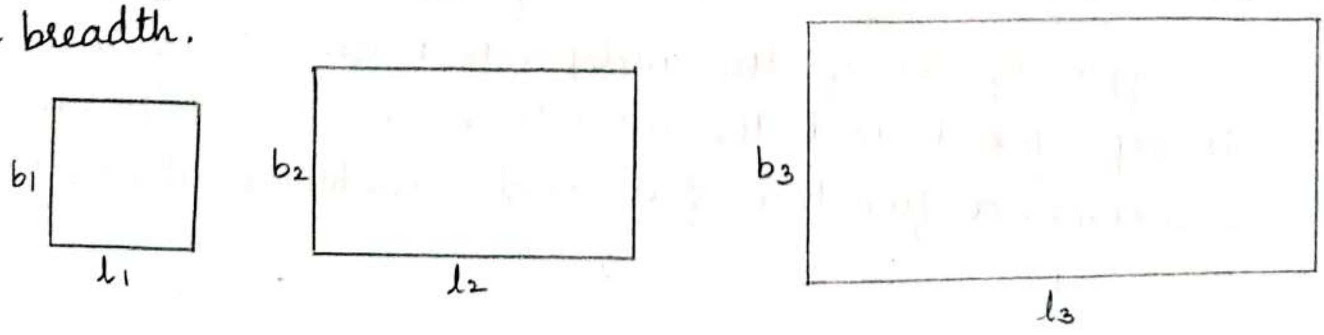
Partial change in area of rectangle due to change in length only.



Partial change in area of rectangle due to change in breadth only.



Total change in area of rectangle due to changes in both length and breadth.



* ## Functions of several variables :

<u>Definition</u> : If for each pair of real values of the variables x and y a unique real number is associated to the variable z, then we say z is a function of two variables x and y. This we denote it by $z=f(x,y)$.

If $z=f(x,y)$ is a function of two variables, x and y are independent variables and z is the dependent variable.

In a similar manner, a function of several variables can be defined.

Consider a function $z=f(x,y)$ of two variables. The number 'l' is said to be the limit of the function $f(x,y)$ as $x\to a$ and $y\to b$, if the following condition holds.

$\forall\, \varepsilon>0,\ \exists\, \delta>0$ such that $|f(x,y)-l|<\varepsilon$ for $|x-a|<\delta$ and $|y-b|<\delta$.

This we denote by $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = l$.

If this limit is equal to the value of the function $f(x,y)$ at $x=a, y=b$, then we say $f(x,y)$ is continuous at the point $x=a$, $y=b$.

* <u>Note</u> : In what follows, the functions of two or more variables are assumed to be continuous functions.

* ## Partial Differentiation

Consider a function $z=f(x,y)$ of two variables. Let this function be continuous in the domain of the definition.

Supposing one of the independent variables x or y, say y is kept fixed and the variable x is allowed to vary, then z becomes a function of a single variable x alone. If this function possess the derivative then this derivative is called the partial derivative of z w.r.t x and is denoted by z_x / f_x / p / $\frac{\partial z}{\partial x}$ / $\frac{\partial f}{\partial x}$.

More explicitly, $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$

provided the right hand limit exists.

Similarly, the partial derivative of $z = f(x,y)$ w.r.t y is also defined and is denoted by z_y / f_y / q / $\frac{\partial z}{\partial y}$ / $\frac{\partial f}{\partial y}$.

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

These derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are called the first order partial derivatives of $z = f(x,y)$ w.r.t x and y respectively.

The partial derivatives f_x, f_y if it exists, of $f(x,y)$ are once again the functions of the variables x and y. If these function are again differentiated partially, we get the higher order partial derivatives of the function $f(x,y)$. These are denoted as below.

(i) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ or f_{xx} or $\frac{\partial^2 z}{\partial x^2}$ or z_{xx} or r.

(ii) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ or f_{yx} or $\frac{\partial^2 z}{\partial x \partial y}$ or z_{yx} or s.

(iii) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ or f_{yy} or $\frac{\partial^2 z}{\partial y^2}$ or z_{yy} or t.

(iv) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ or f_{xy} or $\frac{\partial^2 z}{\partial y \partial x}$ or z_{xy} or s.

<u>Note</u>: If $z = f(x,y)$ is continuous and possess continuous partial derivatives at the point (x,y), then at this point

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

i.e., Second order derivatives is independent of order of differentiation w.r.t x and y.

The rules of differentiation of a function $f(x)$ of a single variable are applicable for the partial differentiation of $f(x,y)$ of two variables. While differentiating the function $f(x,y)$ partially w.r.t x, we treat the variable y as constant and differentiate w.r.t x as if the function is of single independent variable. Similarly, while we differentiate $f(x,y)$ partially w.r.t y, we treat x as a constant and differentiate w.r.t y.

1. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

(a) $z = x^2 y - x \sin xy$

Sol:
$$\frac{\partial z}{\partial x} = 2xy - (\sin xy + x(\cos xy).y)$$
$$= 2xy - \sin xy - xy\cos xy$$

$$\frac{\partial z}{\partial y} = x^2 - x^2\cos xy = x^2(1-\cos xy)$$

(b) $x + y + z = \log z$

Sol:
$$\log z - z = x + y \rightarrow (1)$$

Differentiating (1) partially w.r.t x, treating y as constant,
$$\frac{1}{z}\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 1$$
$$\Rightarrow \frac{\partial z}{\partial x}\left(\frac{1}{z} - 1\right) = 1$$
i.e., $$\frac{\partial z}{\partial x} = \frac{z}{1-z}$$

Differentiating (1) partially w.r.t y, treating x as constant,
$$\frac{1}{z}\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 1$$
$$\Rightarrow \frac{\partial z}{\partial y}\left(\frac{1}{z} - 1\right) = 1 \Rightarrow \frac{\partial z}{\partial y} = \frac{z}{1-z}.$$

(C) $z = (\sin x)^y$

$$\frac{\partial z}{\partial x} = y(\sin x)^{y-1}\cdot \cos x$$

$$\frac{\partial z}{\partial y} = (\sin x)^y \log_e(\sin x)$$

2. If $\theta = t^n e^{-r^2/4t}$, find the value of n such that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

Soln Consider $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\cdot\left(\frac{-2r}{4t}\right) = \frac{-r}{2} t^{n-1} e^{-r^2/4t}$$

$$\therefore r^2\frac{\partial \theta}{\partial r} = -\frac{r^3}{2} t^{n-1} e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2} t^{n-1} e^{-r^2/4t} + \left(-\frac{r^3}{2}\right) t^{n-1} e^{-r^2/4t}\left(\frac{-2r}{4t}\right)$$

$$= \frac{-3}{2} r^2 t^{n-1} e^{-r^2/4t} + \frac{r^4}{4t} t^{n-1} e^{-r^2/4t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t} \longrightarrow (1)$$

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right)$$

$$= \left(n t^{n-1} + \frac{1}{4} r^2 t^{n-2}\right) e^{-r^2/4t} \longrightarrow (2)$$

Given $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$. Therefore equations (1) and (2) gives

$$\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t} = \left(n t^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)e^{-r^2/4t}$$

$$\Rightarrow \left(n + 3/2\right) t^{n-1} = 0$$

$$\Rightarrow \left(n + 3/2\right) = 0$$

$$\Rightarrow n = -3/2$$

3. If $z = e^{ax+by} f(ax-by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Sol: $z = e^{ax+by} f(ax-by) \rightarrow (1)$

Differentiating (1) partially w.r.t x using product and chain rules,

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by) \cdot a + f(ax-by)\, e^{ax+by} \cdot a$$

$$\frac{\partial z}{\partial x} = a e^{ax+by} \{ f(ax-by) + f'(ax-by) \}$$

$$b\frac{\partial z}{\partial x} = ab e^{ax+by} \{ f(ax-by) + f'(ax-by) \} \rightarrow (2)$$

|||ly $$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by) e^{ax+by} \cdot b$$

$$= b e^{ax+by} \{ f(ax-by) - f'(ax-by) \}$$

$$a\frac{\partial z}{\partial y} = ab e^{ax+by} \{ f(ax-by) - f'(ax-by) \} \rightarrow (3)$$

From (2) and (3), we have

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ab e^{ax+by} [f(ax-by) + f'(ax-by) + f(ax-by) - f'(ax-by)]$$

$$= 2ab e^{ax+by} f(ax-by)$$

$$= \underline{\underline{2abz}}$$

4. If $u = e^{r\cos\theta} \cos(r\sin\theta)$, $v = e^{r\cos\theta} \sin(r\sin\theta)$, prove that $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.

Sol: $u = e^{r\cos\theta} \cos(r\sin\theta)$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta} \cdot \cos\theta \cos(r\sin\theta) + e^{r\cos\theta}(-1)\sin(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta} [\cos(r\sin\theta)\cos\theta - \sin(r\sin\theta)\sin\theta]$$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta} [\cos(r\sin\theta + \theta)] \rightarrow (1)$$

$$\frac{\partial u}{\partial \theta} = e^{r\cos\theta} r(-\sin\theta)\cos(r\sin\theta) + e^{r\cos\theta}(-1)\sin(r\sin\theta) \cdot (+r\cos\theta)$$

$$\frac{\partial u}{\partial \theta} = -r e^{r\cos\theta}\left[\sin(r\sin\theta)\cos\theta + \cos(r\sin\theta)\sin\theta\right]$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = e^{r\cos\theta}\sin(r\sin\theta+\theta) \longrightarrow (2)$$

$$v = e^{r\cos\theta}\sin(r\sin\theta)$$

$$\frac{\partial v}{\partial r} = e^{r\cos\theta}\cos\theta\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta).\sin\theta$$

$$= e^{r\cos\theta}\left[\sin(r\sin\theta)\cos\theta + \cos(r\sin\theta)\sin\theta\right]$$

$$\frac{\partial v}{\partial r} = e^{r\cos\theta}\sin(r\sin\theta+\theta) \longrightarrow (3)$$

$$\frac{\partial v}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta)\ r\cos\theta$$

$$= r e^{r\cos\theta}\left[\cos(r\sin\theta)\cos\theta - \sin(r\sin\theta)\sin\theta\right]$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = e^{r\cos\theta}\left[\cos(r\sin\theta+\theta)\right] \rightarrow (4)$$

Thus, from (1) and (4), we have $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

From (2) and (3), we have $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.

* For the following functions verify that, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ where $u = f(x,y)$.

1. $u = x^y$

Sol:

$$\frac{\partial u}{\partial x} = y x^{y-1}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = y x^{y-1}\log x + x^{y-1} = x^{y-1}(1+y\log x) \rightarrow (1)$$

$\because \frac{d}{dx}a^x = a^x \log a$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^y.\frac{1}{x} + \log x . y x^{y-1} = x^{y-1}(1+y\log x) \rightarrow (2)$$

From (1) and (2), we have $u_{xy} = u_{yx}$.

2. $u = \sin^{-1}(y/x)$

Sol: $$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(y^2/x^2)}} \cdot \left(-y/x^2\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = -\frac{x\sqrt{x^2-y^2}\,(+1) - y\left\{x \frac{1}{2\sqrt{x^2-y^2}} \cdot (-2y)\right\}}{x^2(x^2-y^2)}$$

$$= -\frac{x\sqrt{x^2-y^2} + \frac{xy^2}{\sqrt{x^2-y^2}}}{x^2(x^2-y^2)} = -\frac{x(x^2-y^2)+xy^2}{x^2(x^2-y^2)^{3/2}}$$

$$= -\frac{x}{(x^2-y^2)^{3/2}} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\frac{-1}{2\sqrt{x^2-y^2}} \cdot (2x)}{(x^2-y^2)} = \frac{-x}{(x^2-y^2)^{3/2}} \rightarrow (2)$$

From (1) and (2), we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

3. $u = e^x(x \sin y - y \sin y)$

Sol: $$\frac{\partial u}{\partial x} = e^x(x\sin y - y \sin y) + e^x \sin y = e^x\left[(1+x)\sin y - y \sin y\right]$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = e^x\left[(1+x)\cos y - \sin y - y\cos y\right] \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = e^x\left[x\cos y - \sin y - y\cos y\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = e^x(x\cos y - \sin y - y\cos y) + e^x \cos y$$

$$= e^x\left[(1+x)\cos y - \sin y - y\cos y\right] \rightarrow (2)$$

From (1) and (2),

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

*1. If $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Soln Given $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left\{\frac{(x^2-y^2)2y - 2xy(2x)}{(x^2-y^2)^2}\right\}$$

$$= \frac{1}{(x^2-y^2)^2+2^2x^2y^2}\left\{2x^2y - 2y^3 - 4x^2y\right\}$$

$$= \frac{-2y(x^2+y^2)}{x^2+y^2-2x^2y^2+4x^2y^2} = \frac{-2y\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$$\frac{\partial u}{\partial x} = \frac{-2y}{x^2+y^2} \longrightarrow (1)$$

Differentiating this partially w.r.t x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2+y^2}\right) = \frac{(x^2+y^2)(0) - (-2y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left\{\frac{(x^2-y^2)2x - 2xy(-2y)}{(x^2-y^2)^2}\right\}$$

$$= \frac{(x^2-y^2)^2}{x^2+y^2-2x^2y^2+4x^2y^2}\left\{\frac{2x^3-2xy^2+4xy^2}{(x^2-y^2)^2}\right\}$$

$$= \frac{2x(x^2+y^2)}{(x^2+y^2)^2} = \frac{2x}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) = \frac{(x^2+y^2)(0) - 2x(2y)}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2} \rightarrow (2)$$

From (1) and (2), we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2}$

$$= 0$$

* **Note:**

(i) The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as Laplace's equation in two dimensions which has variety of applications in field theory.

(ii) The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is known as Laplace's equation in three dimensions where $u = u(x,y,z)$.

(iii) Any function $u(x,y)$ satisfying Laplace equation is called harmonic function.

2. If $u = f(x+ay) + g(x-ay)$, show that $\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

Sol: $\frac{\partial u}{\partial x} = f'(x+ay)(1) + g'(x-ay)(1) = f'(x+ay) + g'(x-ay)$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+ay) + g''(x-ay)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = a^2 f''(x+ay) + a^2 g''(x-ay) \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = f'(x+ay)(a) + g'(x-ay)(-a) = af'(x+ay) - ag'(x-ay)$$

$$\frac{\partial^2 u}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay) \rightarrow (2)$$

From (1) and (2), we have $\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

* **Note:**

(i) The equation $\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ or the equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is known as one-dimensional wave equation.

(ii) The equation $\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ is known as three dimensional wave equation.

(iii) The solution of the type $u = f(x+ay) + g(x-ay)$ is called D'Alembert's solution of wave equation.

3. If $u = e^{-2t}\cos 3x$, find the value of 'c' such that $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Sol: Given $u = e^{-2t}\cos 3x$

$$\frac{\partial u}{\partial t} = -2e^{-2t}\cos 3x$$

$$\frac{\partial u}{\partial x} = -3e^{-2t}\sin 3x$$

$$\frac{\partial^2 u}{\partial x^2} = -9e^{-2t}\cos 3x$$

$$\therefore \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \Rightarrow -2e^{-2t}\cos 3x = c^2(-9e^{-2t}\cos 3x)$$

$$\Rightarrow c^2 = \frac{2}{9}$$

$$c = \sqrt{2/9}$$

* Note: The equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ is called the one-dimensional heat equation and the equation $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ is known as three-dimensional heat equation. The constant c in the equation is called dissipation coefficient.

4. If $u = \log(x^3+y^3+z^3-3xyz)$, prove that

(a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{3}{x+y+z}\right)$

(b) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Sol: Given $u = \log(x^3+y^3+z^3-3xyz)$,

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz) = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

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$$\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$ (u is symmetric function)

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3zx+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+c^2-xy-yz-zx)+3xyz-3xyz}$$

$\because [a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc]$

$$= \frac{3}{(x+y+z)}$$

×[5. If $u=f(r)$, where $r=\sqrt{x^2+y^2+z^2}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = f''(r)+\frac{2}{r}f'(r)$.] ×

(b) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

5. If $u = f(r)$, where $r = \sqrt{x^2+y^2+z^2}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

Soln Given $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t x, we get

$$2r \frac{\partial r}{\partial x} = 2x \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now, differentiating $u = f(r)$ partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \frac{x}{r} = x\left(\frac{f'(r)}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = x\left[\frac{r f''(r)\frac{\partial r}{\partial x} - f'(r)\cdot\frac{\partial r}{\partial x}}{r^2}\right] + \frac{f'(r)}{r}$$

$$= x\left[\frac{r f''(r)\cdot\frac{x}{r} - f'(r)\cdot\frac{x}{r}}{r^2}\right] + \frac{f'(r)}{r}$$

$$= \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r) + \frac{f'(r)}{r}$$

$$= \frac{x^2}{r^2} f''(r) + \frac{f'(r)}{r}\left[1 - \frac{x^2}{r^2}\right]$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f''(r) + \frac{f'(r)}{r}\left\{1 - \frac{y^2}{r^2}\right\}$

$$\frac{\partial^2 u}{\partial z^2} = \frac{z^2}{r^2} f''(r) + \frac{f'(r)}{r}\left\{1 - \frac{z^2}{r^2}\right\}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left\{\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right\} f''(r) + \frac{f'(r)}{r}\left\{3 - \left(\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right)\right\}$$

$$= \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{f'(r)}{r}\left\{3 - \frac{x^2+y^2+z^2}{r^2}\right\}$$

$$= f''(r) + \frac{f'(r)}{r}(3-1)$$

$$= f''(r) + \frac{2}{r} f'(r)$$

* **Total derivative and total differential**

Let $z = f(x,y)$ be a differentiable function of two variables, x and y. Then,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

The term dz is called the total differential of the function $z = f(x,y)$.

Consider a function $z = f(x,y)$ and let the variables x and y be functions of a single variable 't'. That is let $x = \phi(t)$ and $y = \psi(t)$. Thus z is a composite function of t.

Now we have, $\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

Here, $\frac{dz}{dt}$ is called the total derivative of z w.r.t t.

If u is a function of three variables x, y and z, then the total differential and total derivatives are respectively given by

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

and

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Supposing $z = f(x,y)$ and $x = \phi(u,v)$ and $y = \psi(u,v)$, then

z - (composite function)

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

* Implicit function : $f(x,y) = 0$

Explicit function : $y = f(x)$

The function which can be easily written as $y = f(x)$ with the y variable on one side and the function of x on the other side, is called an explicit function.

The implicit function is of the form $f(x,y) = 0$ and can have more than one variables, which cannot be separated as a dependent variable and independent variable for differentiation.

Let $z = f(x,y)$. We have $\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

Let y be a function of x. Then $t = x$ and therefore $\frac{dx}{dt} = 1$.

$$\Rightarrow \quad \frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \quad \longrightarrow (*)$$

· Consider an implicit function $f(x,y)=0$.

$$\Rightarrow \frac{df}{dx} = 0$$

$$\therefore (*) \Rightarrow \quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

* Suppose $H = f(u,v,w)$ and u,v,w are functions of x,y,z. Then

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial z}$$

The above formulae can be extended to functions of more variables and expressions are called chain rule for partial differentiation.

Note : The second and higher order partial derivatives of $z=f(x,y)$ can be obtained by repeated applications of the above expressions

Examples :

1. Find the total differential of
(i) $e^x[x\sin y + y\cos y]$
(ii) e^{xyz}

Sol. (i) Let $z = f(x,y) = e^x[x\sin y + y\cos y]$

Total differential $dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy$

$$\therefore \frac{\partial z}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y)$$

$$= e^x[(1+x)\sin y + y\cos y]$$

$$\frac{\partial z}{\partial y} = e^x[x\cos y + \cos y - y\sin y]$$

$$= e^x[(1+x)\cos y - y\sin y]$$

$$\therefore dz = e^x[(1+x)\sin y + y\cos y]\,dx + e^x[(1+x)\cos y - y\sin y]\,dy$$

(ii) Let $u = f(x,y,z) = e^{xyz}$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= (yz)e^{xyz}\,dx + (xz)e^{xyz}\,dy + (xy)e^{xyz}\,dz$$

$$du = e^{xyz}(yz\,dx + xz\,dy + xy\,dz)$$

2. Find dz/dt if

(i) $z = xy^2 + x^2y$, where $x = at^2, y = 2at$

Sol $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

$$= (4a^2t^2 + 4a^2t^3)2at + (4a^2t^3 + a^2t^4)2a$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$= 16a^3t^3 + 10a^3t^4$$

$$= 2a^3t^3(8 + 5t)$$

(ii) $z = \tan^{-1}(y/x)$, where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

Sol $\frac{\partial z}{\partial x} = \frac{1}{1 + y^2/x^2}\cdot\left(-y/x^2\right) = \frac{-y}{x^2+y^2}$

$$\frac{\partial z}{\partial y} = \frac{1}{1+y^2/x^2}(1/x) = \frac{x}{x^2+y^2}$$

$$\frac{dx}{dt} = e^t + e^{-t} \quad ; \quad \frac{dy}{dt} = e^t - e^{-t}$$
$$= y \qquad\qquad = x$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \left(\frac{-y}{x^2+y^2}\right)y + \left(\frac{x}{x^2+y^2}\right)x$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$= \frac{e^{2t}+e^{-2t}-2-e^{2t}-e^{-2t}-2}{e^{2t}+e^{-2t}-2+e^{2t}+e^{-2t}+2}$$

$$= \frac{-2}{e^{2t}+e^{-2t}}$$

(iii) $z = e^x \sin y$ where $x = \log t$, $y = t^2$.

Sol

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^x \sin y \times \frac{1}{t} + e^x \cos y\cdot(2t)$$

$$= e^{\log t}\sin t^2\cdot\frac{1}{t} + e^{\log t}\cos t^2\,(2t)$$

$$= \sin t^2 + 2t^2\cos t^2$$

3. If $u = xyz$ where $x = e^{-t}$, $y = e^{-t}\sin^2 t$, $z = \sin t$, then find $\frac{du}{dt}$.

Sol

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= -yze^{-t} + xz(-e^{-t}\sin^2 t + 2e^{-t}\sin t\cos t) + xy\cos t$$

$$= -e^{-2t}\sin^3 t - e^{-2t}\sin^3 t + 2e^{-2t}\sin^2 t\cos t + e^{-2t}\sin^2 t\cos t$$

$$= -2e^{-2t}\sin^3 t + 3e^{-2t}\sin^2 t\cos t = e^{-2t}\sin^2 t(3\cos t - 2\sin t)$$

4. Find $\frac{dy}{dx}$ if

(i) $x^y + y^x = c$

Let $z = f(x,y) = x^y + y^x - c$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{+\partial f/\partial y}$$

$$= -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

(ii) $e^x + e^y = 2xy$

$z = f(x,y) = e^x + e^y - 2xy$

$$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$= -\frac{e^x - 2y}{e^y - 2x}$$

(iii) $x^3 + y^3 = 6xy$

$z = f(x,y) = x^3 + y^3 - 6xy$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

(iv) $e^x - e^y = 2xy$

$z = f(x,y) = e^x - e^y - 2xy$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(e^x - 2y)}{-e^y - 2x}$$

$$= \frac{e^x - 2y}{e^y + 2x}$$

5.(i) If $z = f(x,y)$, where $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Soln

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(ii) If $z = f(x,y)$, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Soln

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) + \frac{\partial z}{\partial y}(-e^{-u} + e^v)$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

6.(i) If $u = f(xz, y/z)$ then show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$.

Soln Let $u = f(v, w)$ where $v = xz$, $w = \frac{y}{z}$.

Using Chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}(z) + \frac{\partial u}{\partial w}(0) = z\frac{\partial u}{\partial v}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z}\right) = \frac{1}{z}\frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(x) + \frac{\partial u}{\partial w}\left(-y/z^2\right) = x\frac{\partial u}{\partial v} - \frac{y}{z^2}\frac{\partial u}{\partial w}$$

$$\therefore x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = xz\frac{\partial u}{\partial v} - \frac{y}{z}\frac{\partial u}{\partial w} - xz\frac{\partial u}{\partial v} + \frac{yz}{z^2}\cdot\frac{\partial u}{\partial w}$$

$$= 0$$

(ii) If $H = f(x-y, y-z, z-x)$, show that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$.

Soln Let $H = f(u, v, w)$, where $u = x-y$, $v = y-z$, $w = z-x$.

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial H}{\partial u}(1) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(-1) = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w}$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$= \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}\cdot(1) + \frac{\partial H}{\partial w}(0) = \frac{\partial H}{\partial v} - \frac{\partial H}{\partial u}$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$= \frac{\partial H}{\partial u}(0) + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w}(1) = \frac{\partial H}{\partial w} - \frac{\partial H}{\partial v}$$

$$\therefore \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} + \frac{\partial H}{\partial v} - \frac{\partial H}{\partial u} + \frac{\partial H}{\partial w} - \frac{\partial H}{\partial v} = 0$$

7. The length and breadth of a rectangle are increasing at the rate of 1.5cm/sec and 0.5 cm/sec respectively. Find the rate at which the area is increasing at the instant when length is 40 cm and breadth is 30cm respectively.

Soln Let l be the length and b be the breadth. Also s be the area at any time t.

$\therefore \quad s = lb$

It is required to find $\frac{ds}{dt}$ when $l = 40$ cm and $b = 30$ cm,

given that $\frac{dl}{dt} = 1.5$cm/sec , $\frac{db}{dt} = 0.5$cm/sec.

$$\frac{ds}{dt} = \frac{\partial s}{\partial l} \cdot \frac{dl}{dt} + \frac{\partial s}{\partial b} \cdot \frac{db}{dt}$$

$$= b \cdot \frac{dl}{dt} + l \frac{db}{dt}$$

$$= 30 \times 1.5 + 40 \times 0.5$$

$$= 65.$$

$\therefore$ Area is increasing at the rate of 65 cm²/sec.

8. If $z = f(u,v)$ and $u = ax+by$, $v = ay-bx$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (a^2+b^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Soln $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}$$

Apply chain rule $z \rightarrow \frac{\partial z}{\partial x}$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}\left(a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}\right) \cdot a + \frac{\partial}{\partial v}\left(a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}\right)(-b)$$

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial u^2} - ab\frac{\partial^2 z}{\partial u \partial v} - ab\frac{\partial^2 z}{\partial u \partial v} + b^2\frac{\partial^2 z}{\partial v^2}$$

Similarly, $\frac{\partial^2 z}{\partial y^2} = b^2 \frac{\partial^2 z}{\partial u^2} + ab \frac{\partial^2 z}{\partial u \partial v} + ab \frac{\partial^2 z}{\partial v \partial u} + a^2 \frac{\partial^2 z}{\partial v^2}$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial u^2} + b^2 \frac{\partial^2 z}{\partial v^2} + b^2 \frac{\partial^2 z}{\partial u^2} + a^2 \frac{\partial^2 z}{\partial v^2}$$

$$= (a^2+b^2) \frac{\partial^2 z}{\partial u^2} + (a^2+b^2) \frac{\partial^2 z}{\partial v^2}$$

$$= (a^2+b^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$$

9. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz} \right)$, prove that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.

Soln Let $u = f(v, w)$, where $v = \frac{y-x}{xy}$, $w = \frac{z-x}{xz}$

$$v = \frac{1}{x} - \frac{1}{y}, \quad w = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{-1}{x^2} \cdot \frac{\partial u}{\partial v} + \frac{-1}{x^2} \frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{1}{y^2} \frac{\partial u}{\partial v}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z} = \frac{1}{z^2} \frac{\partial u}{\partial w}$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$= 0$$

* ## Jacobians

Carl Gustav Jacob Jacobi (10 December 1804 - 18 February 1851) was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations and number theory.

Let u and v are functions of two independent variables x and y, with continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ $\frac{\partial v}{\partial y}$. Then the second order determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian or the functional determinant of u and v w.r.t x and y and is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\frac{(u,v)}{(x,y)}.$$

Similarly, if u, v and w are functions of x, y and z, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

is the Jacobian of u, v and w w.r.t x, y and z.

Note : The Jacobian of n functions of n independent variables can be defined in similar way.

These have an important application in coordinate transformation from one system to another.

Properties of Jacobians :

1. If u and v are functions of two independent variables s and t and s and t themselves are functions of two independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)}$$

2. If $u = f(x,y)$ and $v = g(x,y)$, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

In other words, if J is the Jacobian of u and v w.r.t x and y and J' is the Jacobian of x and y w.r.t u and v, then $JJ' = 1$. In view of this result, J' is called the inverse of J.

3. Definition : Let u and v be functions of two independent variables x and y such that u_x, u_y, v_x and v_y exists and are not all zero simultaneously. If the functions u and v are connected by a relation $\phi(u,v) = 0$, where ϕ is a differentiable function, then u and v are said to be functionally dependent.

If the functions u and v are functionally dependent, then

$$J = \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

4. If u and v are defined as functions of x and y by the equations $f(u,v,x,y) = 0$ and $g(u,v,x,y) = 0$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f,g)}{\partial(x,y)} \div \frac{\partial(f,g)}{\partial(u,v)}$$

Examples:

1. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Soln
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r$$

The above are the transformations between cartesian and polar coordinates. The inverse transforms are $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. It can be verified that $J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$ and therefore $JJ' = 1$.

2. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Soln
$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right)$$
$$+ r\sin\theta\left(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right)$$
$$= r^2\sin\theta\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin^3\theta\left(\sin^2\phi + \cos^2\phi\right)$$
$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$$
$$= r^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right)$$
$$= r^2\sin\theta$$

The above are the transformations between cartesian and spherical coordinates. The inverse transforms are
$r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\left(\sqrt{\frac{x^2+y^2}{z^2}}\right)$ and $\phi = \tan^{-1}\left(y/x\right)$.

3. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$.

Sol: $$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$J_{(1,-1,0)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = -(-4)(-1+6) = \underline{\underline{20}}$$

4. If $x = u(1-v)$, $y = uv$ then compute J and J' and verify that $JJ' = 1$.

Sol: $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv = \underline{\underline{u}}$$

Given $x = u - uv$, $v = \frac{y}{u}$

$x = u - y$

$u = x + y$, $\quad = \frac{y}{x+y}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

$$= \frac{1}{u}.$$

$$\therefore JJ' = u \cdot \frac{1}{u} = \underline{\underline{1}}$$

5. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$.

Sol: $u_x = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$

$|||^{ly}$ $u_y = \frac{1+x^2}{(1-xy)^2}$

$v_x = \frac{1}{1+x^2}$, $v_y = \frac{1}{1+y^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{\cancel{(1+y^2)}}{(1-xy)^2} \times \frac{1}{\cancel{(1+y^2)}} - \frac{\cancel{(1+x^2)}}{(1-xy)^2} \times \frac{1}{\cancel{(1+x^2)}}$$

$$J = 0$$

$\therefore$ u and v are functionally dependent.

$v = \tan^{-1}x + \tan^{-1}y$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$v = \tan^{-1}(u)$

$u = \tan v$ is the relation between u and v.

6. If $x = e^u \sin v$, $y = e^u \cos v$, then find $J\left(\frac{x,y}{u,v}\right)$.

Sol: $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^u \sin v & e^u \cos v \\ e^u \cos v & -e^u \sin v \end{vmatrix}$

$$= -e^{2u}\sin^2 v - e^{2u}\cos^2 v$$

$$= -e^{2u}$$

7. If $\frac{\partial(x,y)}{\partial(u,v)} = v\sin 2u$ and $\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{4}$, then compute $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Sol: $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \frac{1}{4v\sin 2u}$$

8. If $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$, verify if u,v are functionally dependent. If so find the relation between them.

Sol: $$u_x = \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$u_y = \frac{(x-y)-(x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$v_x = \frac{(x-y)^2(y) - xy\cdot 2(x-y)}{(x-y)^4} = \frac{(x-y)\left[xy-y^2-2xy\right]}{(x-y)^4}$$

$$= \frac{-y(x+y)}{(x-y)^3}$$

$$v_y = \frac{(x-y)^2(x) + xy\,2(x-y)}{(x-y)^4} = \frac{(x-y)\left[x^2-xy+2xy\right]}{(x-y)^4}$$

$$= \frac{x(x+y)}{(x-y)^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{-y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix}$$

$$J = \frac{-2x^2y - 2xy^2 + 2x^2y + 2xy^2}{(x-y)^5} = 0$$

$\therefore$ u and v are functionally dependent.

Relation between u and v :

$$(x+y)^2 - (x-y)^2 = 4xy$$

$$\frac{(x+y)^2}{(x-y)^2} - 1 = \frac{4xy}{(x-y)^2}$$

$$\therefore \quad u^2 - 1 = 4v$$

9. If $u = x^2 - y^2$ and $v = 2xy$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$ and $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Soln

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2 = 4(x^2+y^2) = 4r^2$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = 4r^2 \times r = 4r^3$$

10. Let $x+y = u$ and $y = uv$. Find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln

$f = x+y-u$ and $g = y - uv$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f,g)}{\partial(x,y)} \div \frac{\partial(f,g)}{\partial(u,v)}$$

$$\therefore \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -v & -u \end{vmatrix} = u$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{u} = \frac{1}{x+y} \quad //$$

* Maxima and minima of a function of two variables i.e., $f(x,y)$

Many problems in practical situations are concerned with maximizing or minimizing a quantified variable. For example at what angle should a missile be fired in order to give the maximum range? Maximization of profit and minimization of cost of production in an industry. In geometry we come across problems of the type what is the shortest length joining two points on the surface (Geodesics). The problem of quickest descent from one point to another in the shortest time. Finding a closed curve of given perimeter and maximum area. These situations are discussed under the optimization problems.

In mathematics, the maximum and minimum (plural: maxima and minima) of a function, known collectively as extrema (singular: extremum), are the largest and smallest value that the function takes at a point within a given neighborhood.

- **Definition**: A function $f(x,y)$ is said to have a maximum value at point (a,b) if there exists a neighbourhood of the point (a,b) say $(a+h, b+k)$ (h and k are small) such that $f(a,b) > f(a+h, b+k)$. Similarly if $f(a,b) < f(a+h, b+k)$ then $f(x,y)$ is said to have a minimum value at (a,b).

  $f(a,b)$ is said to be an extreme value of $f(x,y)$ if it is a maximum value or a minimum value.

- **Note**: An extremum point of $f(x,y)$ is also called a critical point or stationary point.

- Suppose $f(x,y)$ has an extremum at (a,b). The function $f(x,b)$ of a single variable x must also have the corresponding extremum for which the necessary condition is $f_x(x,b)=0$. In particular at $x=a$, $f_x(a,b)=0$. Similarly, $f(a,y)$ being a function of single variable also has the corresponding extremum for which the necessary condition is $f_y(a,y)=0$. In particular at $y=b$, $f_y(a,b)=0$.

* Necessary and sufficient conditions for maxima or minima.

- Necessary conditions :

The necessary conditions for $f(x,y)$ to have an extremum at (a,b) are that $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ at the point (a,b).

- Sufficient conditions :

$f(a,b)$ is a maximum value if $rt-s^2>0$ and $r<0$.

$f(a,b)$ is a minimum value if $rt-s^2>0$ and $r>0$.

where $r=f_{xx}$, $s=f_{xy}$ or f_{yx}, $t=f_{yy}$

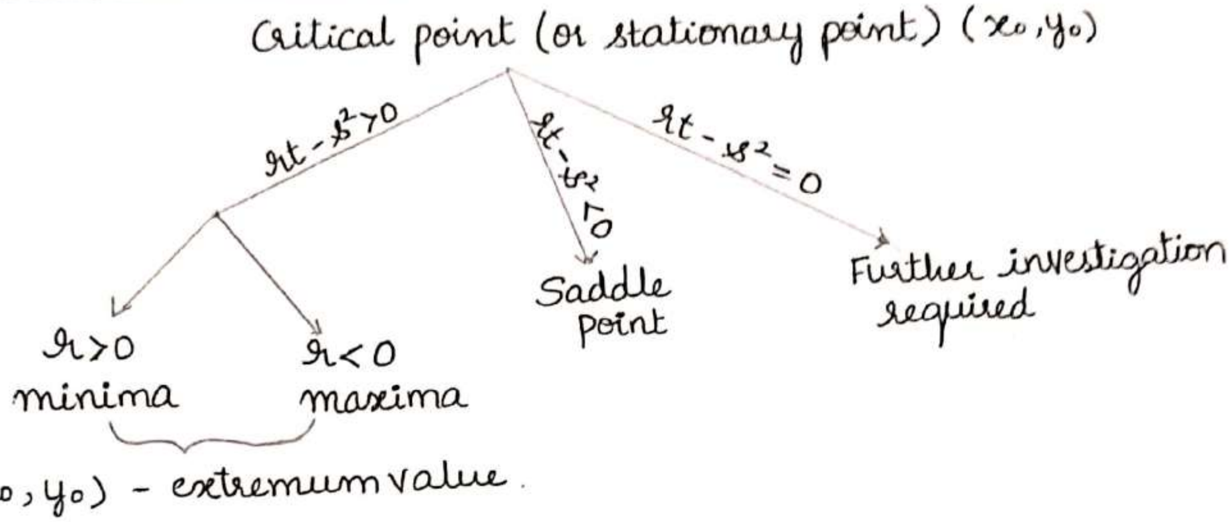
- Note : If $rt-s^2<0$, then $f(a,b)$ is neither maximum nor minimum and (a,b) is called saddle point.

If $rt-s^2=0$, the case is doubtful and further analysis is necessary to decide the nature of (a,b).

* Working rule for finding extreme values of $f(x,y)$

1. Find f_x and f_y and equate these to zero. Solve them simultaneously to obtain the pairs of values of (a,b) of (x,y).
2. Find $r=f_{xx}$, $s=f_{xy}$, $t=f_{yy}$. Evaluate these at each pair of values obtained above. Also compute $rt-s^2$.

3.



* Examples:

1. Show that $z(x,y) = xy(a-x-y)$, $a>0$ is maximum at the point $(a/3, a/3)$.

Sol: Given $z = xy(a-x-y) = axy - x^2y - xy^2$

$$p = z_x = ay - 2xy - y^2$$

$$q = z_y = ax - x^2 - 2xy$$

$$z_x(a/3, a/3) = \frac{a^2}{3} - \frac{2a^2}{9} - \frac{a^2}{9} = \frac{2a^2}{9} - \frac{2a^2}{9} = 0$$

$$z_y(a/3, a/3) = \frac{a^2}{3} - \frac{a^2}{9} - \frac{2a^2}{9} = 0$$

$\therefore$ The necessary conditions are satisfied.

$$r = z_{xx} = -2y \quad ; \quad r_{(a/3, a/3)} = -2a/3$$

$$s = z_{xy} = a - 2x - 2y \quad ; \quad s(a/3, a/3) = a - \frac{4a}{3} = -a/3$$

$$t = z_{yy} = -2x \quad ; \quad t_{(a/3, a/3)} = -2a/3$$

$$rt - s^2 = \frac{4a^2}{3} - \frac{a^2}{3} = a^2 > 0$$

At $(a/3, a/3)$, $rt - s^2 > 0$ and $r < 0$.

It follows that $f(x,y)$ i.e., $z(x,y)$ is maximum at $(a/3, a/3)$.

The maximum value is $z(a/3, a/3) = a^2/9\,(a - a/3 - a/3) = \frac{a^3}{27}$.

2. Examine the following function for extreme values
$f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Sol: Given $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$p = f_x = 4x^3 - 4x + 4y$

$q = f_y = 4y^3 + 4x - 4y$

$r = f_{xx} = 12x^2 - 4$

$s = f_{xy} = 4$

$t = f_{yy} = 12y^2 - 4$

Solving $p=0$ and $q=0$ simultaneously, we get

$4(x^3+y^3) = 0$ $\quad (p=0 + q=0)$

$\Rightarrow x^3+y^3=0$

$\Rightarrow y = -x$

Substituting this in $p=0$, we get

$4x^3 - 4x - 4x = 0$

$\Rightarrow x^3 - 2x = 0$

$\Rightarrow x(x^2-2) = 0$

$\Rightarrow x = 0$ and $x = \pm\sqrt{2}$

$\therefore y = 0$ and $y = \mp\sqrt{2}$

Hence the points are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

Point	r	s	t	$rt - s^2$	Nature of point
$(0,0)$	$-4 < 0$	4	-4	0	Further investigation needed
$(\sqrt{2},-\sqrt{2})$	$20 > 0$	4	20	$384 > 0$	Minimum
$(-\sqrt{2},\sqrt{2})$	$20 > 0$	4	20	$384 > 0$	Minimum

$\Rightarrow$ The function is minimum at $(\pm\sqrt{2}, \mp\sqrt{2})$.

$f_{min} = f(\pm\sqrt{2}, \mp\sqrt{2}) = \sqrt{16} + \sqrt{16} - 8 + 4(\sqrt{2})(-\sqrt{2})$

$= -8$

3. Find maximum and minimum of $2(x^2-y^2)-x^4+y^4$.

Sol: Let $f(x,y) = 2(x^2-y^2)-x^4+y^4$

$p = f_x = 4x - 4x^3$

$q = f_y = -4y + 4y^3$

$r = f_{xx} = 4 - 12x^2$

$s = f_{xy} = 0$

$t = f_{yy} = -4 + 12y^2$

$p = 0 \Rightarrow 4x - 4x^3 = 0$
$\Rightarrow 4x(1-x^2) = 0$
$\Rightarrow x = 0, 1, -1$

$q = 0 \Rightarrow -4y + 4y^3 = 0$
$\Rightarrow 4y(-1+y^2) = 0$
$\Rightarrow y = 0, 1, -1$

$\therefore$ Stationary points are $(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1)$, $(-1,0), (-1,1)$ and $(-1,-1)$.

Point	r	s	t	$rt - s^2$	Nature of point
$(0,0)$	$4 > 0$	0	-4	$-16 < 0$	Saddle point
$(0,1)$	$4 > 0$	0	8	$32 > 0$	Minimum point
$(0,-1)$	$4 > 0$	0	8	$32 > 0$	Minimum point
$(1,0)$	$-8 < 0$	0	-4	$32 > 0$	Maximum point
$(1,1)$	$-8 < 0$	0	8	$-64 < 0$	Saddle point
$(1,-1)$	$-8 < 0$	0	8	$-64 < 0$	Saddle point
$(-1,0)$	$-8 < 0$	0	-4	$32 > 0$	Maximum point
$(-1,1)$	$-8 < 0$	0	8	$-64 < 0$	Saddle point
$(-1,-1)$	$-8 < 0$	0	8	$-64 < 0$	Saddle point

$\Rightarrow$ f is maximum at $(1,0)$ and $(-1,0)$.

f is minimum at $(0,1)$ and $(0,-1)$.

Maximum value is $f(1,0) = f(-1,0) = 2 - 1 = 1$

Minimum value is $f(0,1) = f(0,-1) = -2 + 1 = -1$.

4. A rectangle box, open at the top to have a volume of 32 cubic units, what must be the dimensions so that total surface area of the box is minimum.

Soln Let x (length), y (breadth) and z (height) be the dimensions of the box.

Then Volume $V = xyz$.

Surface area $S = xy + 2yz + 2zx$

Given $V = 32$.

$$\Rightarrow xyz = 32 \quad \Rightarrow z = \frac{32}{xy}$$

$$\therefore S = f(x,y) = xy + 2y \times \frac{32}{xy} + 2x \times \frac{32}{xy}$$

$$f(x,y) = xy + \frac{64}{x} + \frac{64}{y}$$

$$f_x = y - \frac{64}{x^2}$$

$$f_y = x - \frac{64}{y^2}$$

$f_x = 0$, $f_y = 0$ $\Rightarrow x^2y - 64 = 0$ and $xy^2 - 64 = 0$

$$\Rightarrow x^2y = y^2x$$

$$\Rightarrow x = y$$

$$\therefore x^2 . x = 64 \quad \Rightarrow x^3 = 64 \quad \Rightarrow x = y = 4.$$

Hence $(4,4)$ is the critical point.

$$r = f_{xx} = \frac{128}{x^3} \quad ; \quad r_{(4,4)} = 2$$

$$s = f_{xy} = 1$$

$$t = f_{yy} = \frac{128}{y^3} \quad ; \quad t_{(4,4)} = 2.$$

$rt - s^2 = 4 - 1 = 3 > 0$, $r > 0$.

$\therefore$ $(4,4)$ is minimum point.

Thus the dimensions of box which make surface area minimum are $x = 4$, $y = 4$ and $z = 2$.

Minimum surface area $= f(4,4) = 16 + 16 + 16 = \underline{48}$

5. Find maximum and minimum of $x^3+y^3-63(x+y)+12xy$.

Sol: Let $f(x,y) = x^3+y^3-63(x+y)+12xy$

$p = f_x = 3x^2-63+12y$

$q = f_y = 3y^2-63+12x$

$r = f_{xx} = 6x$

$s = f_{xy} = 12$

$t = f_{yy} = 6y$

Solving $f_x=0$ and $f_y=0$, we get

$x^2+4y=21 \rightarrow (1)$ and $y^2+4x=21 \rightarrow (2)$

$\Rightarrow x^2+4y = y^2+4x$

$x^2-y^2+4y-4x=0$

$(x+y)(x-y)-4(x-y)=0$

$(x-y)(x+y-4)=0$

$\Rightarrow x=y, \quad x+y=4$

Putting $x=y$ in (1), we get $y^2+4y-21=0$

$\Rightarrow (y+7)(y-3)=0$

$\Rightarrow y=3,-7$

Since $x=y$, stationary points are $(3,3)$ and $(-7,-7)$.

Put $x=4-y$ in (1). We obtain $x^2+16-4x=21$
or $y=4-x$

$x^2-4x-5=0$

$(x-5)(x+1)=0$

$x=-1,5$.

Since $y=4,x$, stationary points are $(-1,5)$ and $(5,-1)$.

Points	r	s	t	$rt-s^2$	Nature of the point
$(3,3)$	$18>0$	12	18	$180>0$	Minimum point
$(-7,-7)$	$-42<0$	12	-42	$1620>0$	Maximum point
$(-1,5)$	$-6<0$	12	30	$-324<0$	Saddle point
$(5,-1)$	$30>0$	12	-6	$-324<0$	Saddle point

$\Rightarrow f(x,y)$ is minimum at $(3,3)$ and maximum at $f(-7,-7)$.

Minimum value is $f(3,3) = 27+27-378+108 = -216$.

Maximum value is $f(-7,-7) = -343-343+882+588 = 784$.

* **Lagrange's method of undetermined multipliers or constrained extrema.**

This is a method of obtaining stationary values of a function of three independent variables subject to a certain specified constraints (conditions).

Consider a function $f(x,y,z)$ of three independent real variables x,y,z. Suppose x,y,z are subject to a constraint of the form $g(x,y,z)=0$. Suppose we want to find stationary values of x,y,z for which $f(x,y,z)$ is stationary (maximum or minimum). For this consider an auxiliary function $F(x,y,z)$ defined by $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$ where λ is a constant (parameter).

Determine F_x, F_y, F_z and equate it to zero. Thus we obtain three equations of the form $F_1(x,y,z,\lambda)=0$, $F_2(x,y,z,\lambda)=0$, $F_3(x,y,z,\lambda)=0$. These equations together with the constraints give a value for λ, say $\lambda=\lambda_1$. For this value λ, solve the equations $F_1=0$, $F_2=0$, $F_3=0$ for x,y,z.

Suppose a solution is $(x,y,z)=(x_1,y_1,z_1)$. Then (x_1,y_1,z_1) is a stationary point for the function $f(x,y,z)$ and $f(x_1,y_1,z_1)$ is a stationary value.

• **Note:**

1. λ being a multiplier, the term undetermined multiplier refers to λ.
2. The disadvantage of Lagrange's method is that, the method serves to determine only stationary values and further analysis is needed to decide maxima or minima.

1. The temperature at any point (x,y,z) in space is $400xy^2z$. Find the highest temperature at the surface of unit sphere $x^2+y^2+z^2=1$ using Lagrange's method of undetermined multipliers.

Sol: Let $f(x,y,z) = 400xy^2z$

$$g(x,y,z) = x^2+y^2+z^2-1$$

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$= 400xy^2z + \lambda(x^2+y^2+z^2-1)$$

$$F_x = 400y^2z + 2\lambda x = 0 \Rightarrow -\lambda = \frac{200y^2z}{x}$$

$$F_y = 800xyz + 2\lambda y = 0 \Rightarrow -\lambda = \frac{400xyz}{y} = 400xz$$

$$F_z = 400xy^2 + 2\lambda z = 0 \Rightarrow -\lambda = \frac{200xy^2}{z}$$

$$\Rightarrow \quad \underset{①}{\frac{200y^2z}{x}} = \underset{②}{400xz} = \underset{③}{\frac{200xy^2}{z}}$$

Solving (1) and (2), we get $200y^2z = 400x^2z$

$$\Rightarrow y^2 = 2x^2 \longrightarrow (4)$$

Solving (1) and (3), we get $200y^2z^2 = 200x^2y^2$

$$\Rightarrow z^2 = x^2 \longrightarrow (5)$$

Substituting (4) and (5) in $g(x,y,z)=0$, we get

$$x^2+2x^2+x^2 = 1$$

$$4x^2 = 1 \Rightarrow x^2 = 1/4$$

$$x = \pm 1/2$$

$$\Rightarrow y = \pm\frac{1}{\sqrt{2}} \text{ and } z = \pm 1/2$$

$\therefore$ The highest temperature is $T = 400xy^2z = 400 \times 1/2 \times \frac{1}{2} \times \frac{1}{2}$

$$T = 50$$

2. A tank is in the form of a rectangular box open at the top. Find its dimensions if its inner surface area is $192\, m^2$ and maximum capacity.

Sol: Let the length, breadth and height of the box be x, y, z meters respectively. Then volume $V = xyz$.

It is given that $xy + 2yz + 2zx = 192$.

$\therefore f(x,y,z) = xyz$

$g(x,y,z) = xy + 2yz + 2zx - 192$

$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$= xyz + \lambda(xy + 2yz + 2zx - 192)$

$F_x = yz + \lambda(y + 2z) = 0 \Rightarrow -1/\lambda = \dfrac{y+2z}{yz}$

$F_y = xz + \lambda(x + 2z) = 0 \Rightarrow -1/\lambda = \dfrac{x+2z}{xz}$

$F_z = xy + \lambda(2y + 2x) = 0 \Rightarrow -1/\lambda = \dfrac{2y+2x}{xy}$

$$\therefore \underset{①}{\frac{y+2z}{yz}} = \underset{②}{\frac{x+2z}{xz}} = \underset{③}{\frac{2y+2x}{xy}}$$

Solving (1) and (2), we get $\cancel{xyz} + 2xz^2 = \cancel{xyz} + 2yz^2$

$\Rightarrow x = y \rightarrow (4)$

Solving (2) and (3), we get $x^2y + \cancel{2xyz} = \cancel{2xyz} + 2x^2z$

$\Rightarrow y = 2z \rightarrow (5)$

$\therefore$ Substituting (4) and (5) in $g(x,y,z) = 0$, we get

$x^2 + 2x \cdot \frac{x}{2} + 2 \cdot \frac{x}{2} \cdot x = 192$

$3x^2 = 192 \Rightarrow x^2 = 64 \Rightarrow x = 8$

($x \neq -8$, because dimension cannot be negative).

$\therefore y = 8, \; z = 4$

Thus, for V to be maximum, length = 8m, breadth = 8m and height = 4m.

Max Volume $= 256\, m^3$.

3. A rectangle box open at the top to have volume of 32 cubic units, what must be dimension so that total surface area of the box is a minimum?

Soln

$$f(x,y,z) = xy + 2yz + 2zx$$

$$g(x,y,z) = xyz - 32$$

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$= xy + 2yz + 2zx + \lambda(xyz - 32)$$

$$F_x = y + 2z + \lambda(yz) = 0 \Rightarrow -\lambda = \frac{y+2z}{yz}$$

$$F_y = x + 2z + \lambda(xz) = 0 \Rightarrow -\lambda = \frac{x+2z}{xz}$$

$$F_z = 2y + 2x + \lambda(xy) = 0 \Rightarrow -\lambda = \frac{2x+2y}{xy}$$

$$\therefore \underset{(1)}{\frac{y+2z}{yz}} = \underset{(2)}{\frac{x+2z}{xz}} = \underset{(3)}{\frac{2x+2y}{xy}}$$

Solving (1) and (2), we get $\cancel{xyz} + 2xz^2 = \cancel{xyz} + 2yz^2$

$$\Rightarrow x = y \rightarrow (4)$$

Solving (2) and (3), we get $x^2y + \cancel{2xyz} = \cancel{2xyz} + 2x^2z$

$$\Rightarrow y = 2z \rightarrow (5)$$

Substituting (4) and (5) in $g(x,y,z) = 0$, we get

$$x \times x \times \frac{x}{2} = 32$$

$$\Rightarrow x^3 = 64$$

$$\Rightarrow x = 8/2 = 4$$

$$\therefore y = 4, \quad z = 2.$$

Hence the dimensions of the box are length = 4 units, breadth = 4 units and height = 2 units.

Minimum surface area = $f(4,4,2) = 16 + 16 + 16 = 48$ sq. units

4. In economics, a cobb-Douglas equation is used to compute total production Y as a function of labor input L and capital input K. One such equation is $Y = 32L^{0.6}K^{0.4}$. Maximize production subject to the constraint $4L + 2K = 50$.

Soln

$$f(L,K) = 32L^{0.6}K^{0.4}$$

$$g(L,K) = 4L + 2K - 50$$

$$F(L,K) = f(L,K) + \lambda g(L,K)$$

$$= 32L^{0.6}K^{0.4} + \lambda(4L + 2K - 50)$$

$$F_L = 19.2L^{-0.4}K^{0.4} + 4\lambda = 0 \Rightarrow -\lambda = 4.8L^{-0.4}K^{0.4}$$

$$F_K = 12.8L^{0.6}K^{-0.6} + 2\lambda = 0 \Rightarrow -\lambda = 6.4L^{0.6}K^{-0.6}$$

$$\Rightarrow 4.8L^{-0.4}K^{0.4} = 6.4L^{0.6}K^{-0.6}$$

$$K^{0.4+0.6} = \frac{6.4}{4.8}L^{0.6+0.4}$$

$$K = \frac{4}{3}L$$

Substituting this in $g(L,K) = 0$, we get

$$4L + \frac{8}{3}L = 50$$

$$20L = 150$$

$$L = 7.5$$

$$\Rightarrow K = \frac{4 \times 7.5}{3} = 10$$

Thus production is maximum when $L = 7.5$ and $K = 10$.

Maximum production $f(7.5, 10) = 32(7.5)^{0.6}(10)^{0.4}$

$= 269.2692$ units