power series

The power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots\cdots$$

In general,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots\cdots$$

this is called power series about the point a or centered at a.

where $c_0, c_1, c_2 \cdots\cdots$ are constants called co-efficients of the power series.

Suppose a function $f(x)$ satisfies the following 2 conditions:

1) $f(x)$ and its first $(n-1)$ derivatives are continuous in a closed interval $[a, b]$

2) $f^{n-1}(x)$ is differentiable in open interval (a, b).

Then $\exists$ Taylor series expansion for the given function $f(x)$ in powers of $(x-a)$ or about the point 'a'.

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \quad \text{— (1)}$$

put $x = a$, $\quad c_0 = f(a)$

Diffᵗ (1) w.r.t x $\quad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad \text{— (2)}$

put $x = a$, $\quad c_1 = f'(a)$

Diffᵗ (2) w.r.t x $\quad f''(x) = 2c_2 + 3\cdot 2\cdot c_3(x-a) + 4\cdot 3\, c_4(x-a)^2 + \cdots$

put $x = a$, $\quad 2c_2 = f''(a)$

$$c_2 = \frac{f''(a)}{2!} \qquad |||^{ly} \quad c_3 = \frac{f'''(a)}{3!}$$

$$\cdots\cdots \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

Equation (M) becomes, we get

$$\boxed{\begin{aligned} f(x) = (1+x)^{K} &= 1 + Kx + \frac{K(K-1)}{2!}x^{2} + \frac{K(K-1)(K-2)}{3!}x^{3} \\ &+ \cdots + \frac{K(K-1)\cdots(K-(n-1))}{n!}x^{K-n} + \cdots \end{aligned}}$$

$$(1+x)^{K} = \sum_{n=0}^{\infty} \binom{K}{n} x^{n}$$

where $\binom{K}{n}$ is called binomial co-efficient.

$$\& \quad \binom{K}{n} = \frac{K(K-1)\cdots(K-(n-1))}{n!}$$

Note: 1) $(1+x)^{-1} = 1 - x + x^{2} - x^{3} + \cdots$

2) $(1-x)^{-1} = 1 + x + x^{2} + x^{3} + \cdots$

3) $(1+x)^{-2} = 1 - 2x + 3x^{2} - 4x^{3} + \cdots$

4) $(1-x)^{-2} = 1 + 2x + 3x^{2} + 4x^{3} + \cdots$

5) $\sin x = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \cdots$

6) $\cos x = 1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} - \cdots$

7) $e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$

8) $\tan x = x + \frac{x^{3}}{3} + \frac{2}{15}x^{5} + \cdots$

9) $\sinh x = \frac{e^{x} - e^{-x}}{2} = x + \frac{x^{3}}{3!} + \frac{x^{5}}{5!} + \cdots$

10) $\cosh x = \frac{e^{x} + e^{-x}}{2} = 1 + \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \cdots$

11) $\log(1 \pm x) = \pm x - \frac{x^{2}}{2} \pm \frac{x^{3}}{3} - \frac{x^{4}}{4} \pm \cdots$

12) $\tan^{-1} x = x - \frac{x^{3}}{3} + \frac{x^{5}}{5} - \frac{x^{7}}{7} + \cdots$

Examples

① Obtain the Taylor series expansion for $\cos x$ about the point $x = \frac{\pi}{3}$ up to fourth degree terms. Hence determine the approximate value of $\cos(61^\circ)$.

Soln:- $f(x) = \cos x$, $f(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

Diff^t $f(x)$ w.r.t x

$f'(x) = -\sin(x)$, at $x = \frac{\pi}{3}$ $f'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3})$

$f'(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

Diff^t $f'(x)$ w.r.t x

$f''(x) = -\cos x$, at $x = \frac{\pi}{3}$ $f''(\frac{\pi}{3}) = -\cos(\frac{\pi}{3})$

$f''(\frac{\pi}{3}) = -\frac{1}{2}$

III^{ly} $f'''(x) = \sin(x)$, at $x = \frac{\pi}{3}$ $f'''(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \cos(x)$ at $x = \frac{\pi}{3}$ $f^{(4)}(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

Taylor series expansion upto fourth degree term is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$$

$$f(x) = \cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{2\cdot 2!}(x - \frac{\pi}{3})^2 + \frac{\sqrt{3}}{2}\frac{1}{3!}(x - \frac{\pi}{3})^3$$
$$+ \frac{1}{2\cdot 4!}(x - \frac{\pi}{3})^4 \quad \text{—— ①}$$

$61^\circ = 61 \cdot \frac{\pi}{180}$

put $x = \frac{61\pi}{180}$ in ①

$\cos(61^\circ) = \cos(\frac{61\pi}{180}) = 0.48$

2) Obtain Taylor's expansion for $f(x) = \log_e x$ up to the term containing $(x-1)^4$ and hence find $\log_e(1\cdot 1)$

Sol: The Taylor's series for $f(x)$ about the point 1 is

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{IV}(1)}{4!}(x-1)^4 + \dots \quad \to (1)$$

Given:- $f(x) = \log_e x \Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{IV}(x) = -\frac{6}{x^4} \Rightarrow f^{IV}(1) = -6$

$$(1) \Rightarrow f(x) = \log_e x = 0 + (x-1)1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \dots$$

$$\Rightarrow \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \dots$$

Taking $x = 1\cdot 1$ in the above expansion

$$\log_e(1\cdot 1) = (0\cdot 1) - \frac{(0\cdot 1)^2}{2} + \frac{(0\cdot 1)^3}{3} - \frac{(0\cdot 1)^4}{4} = 0\cdot 0953$$

3) obtain the maclaurin series expansion of the following function

$y = \sin x$

solⁿ :- ⓐ $y = \sin x$, $y(0) = 0$

The maclaurine series is given by

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots + \frac{y^{(n)}(0)}{n!}x^n + \cdots \quad \text{--- (1)}$$

Diffᵗ y w.r.t x

$y' = \cos x$, $y'(0) = \cos 0 = 1$

|||ly $y'' = -\sin x$, $y''(0) = -\sin 0 = 0$

$y''' = -\cos x$, $y'''(0) = -\cos 0 = -1$

$y^{(4)} = \sin x$, $y^{(4)}(0) = \sin 0 = 0$

$y^{(5)} = \cos x$, $y^{(5)}(0) = \cos 0 = 1$

(1) implies that

$$y = \sin x = 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{1}{5!}x^5 - \cdots$$

$$\boxed{y = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

4) Expand $y = \log(\sec(x))$ as a series in powers of x, and hence obtain the series expansion for $\tan(x)$.

Sol: Here $y = \log[\sec(x)]$

$$\Rightarrow y' = \frac{1}{\cancel{\sec(x)}} \cdot \cancel{\sec(x)} \cdot \tan(x)$$

$$y'' = \sec^2 x = 1 + \tan^2 x = 1 + y'^2$$

$$y''' = 2y'y''$$

$$y^{iv} = 2[y'y''' + (y'')^2]$$

Put $x = 0$, $\sec(0) = 1 \Rightarrow \log(1) = 0$

i.e. $y(0) = 0$, $y'(0) = \tan(0) = 0$, $y''(0) = 1 + 0^2 = 1$

$y'''(0) = 2y'(0) \cdot y''(0) = 2(0)(1) = 0$, $y^{iv}(0) = 2[(0)(0) + 1^2] = 2$

Now, $$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$\log[\sec(x)] = 0 + 0(x) + \frac{1}{2}x^2 + 0(x^3) + \frac{2}{24}x^4 + \dots$$

$$\log[\sec(x)] = \frac{x^2}{2} + \frac{x^4}{12} + \dots \rightarrow (1)$$

Differentiate (1) wrt x

$$\frac{1}{\sec x} \cdot \sec x \tan x = \frac{2x}{2} + \frac{4x^3}{12} + \dots$$

$$\therefore \tan x = x + \frac{x^3}{3} + \dots$$

5) Obtain the Maclaurin series expansion of $\tan^{-1}x$ and hence obtain series for $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol:- Let $y = \tan^{-1}x$

$$\therefore y' = \frac{1}{1+x^2}$$

i.e. $$\frac{dy}{dx} = (1+x^2)^{-1}$$

We have binomial series expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$\therefore \frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + \dots$$

$$dy = [1 - x^2 + x^4 - x^6 + \dots]dx$$

Integrating both sides we get

$$y = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right] + c$$

Put $x = 0$, $y(0) = \tan^{-1}(0) = 0$ $\therefore c = 0$

$$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Consider $$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Put $x = \tan\theta$

then $$\frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$$

$$f(x) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right]$$

6) Expand $y = \log(1+x)$ in ascending powers of 'x' and hence S.T. $\log\left(\sqrt{\frac{1+x}{1-x}}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

Sol:

$f(x) = \log(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{IV}(x) = -\frac{6}{(1+x)^4}$ $\qquad f^{IV}(0) = -6$

By Maclaurin series,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$f(x) = 0 + (1)x + \frac{(-1)}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \ldots$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \rightarrow (1)$$

Replace x by $-x$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \rightarrow (2)$$

Now $\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots\right]$$

$$\therefore \log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

7) Obtain the expansion of the function $\frac{x}{\sin x}$ in ascending powers of x.

Sol: $f(x) = \frac{x}{\sin x}$ and $f(0)$ is indeterminate

But we have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$f(x) = \frac{x}{\sin x} = \frac{x}{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]} = \frac{1}{\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right]}$$

$$= \frac{1}{\left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right)\right]}$$

$$= \frac{1}{1-t} \quad \text{where } t = \frac{x^2}{6} - \frac{x^4}{120} + \dots$$

$$= (1-t)^{-1} = 1 + t + t^2 + \dots \quad \text{(by binomial series)}$$

$$= 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right) + \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right)^2 + \dots$$

$$= 1 + \frac{x^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right) x^4 + \dots$$

$$\therefore \frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7}{360} x^4 + \dots$$

8) Obtain the Maclaurin series expansion of the function $f(x) = \frac{x}{e^x - 1}$ up to 4^{th} degree term

sol:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\frac{x}{e^x - 1} = \frac{x}{x\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right]} = \frac{1}{1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)}$$

WKT $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$$\frac{x}{e^x - 1} = \left[1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)\right]^{-1}$$

$$= 1 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right) + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)^2 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)^3 + \dots$$

$$= 1 - \frac{x}{2} + \left(-\frac{1}{6} + \frac{1}{4}\right)x^2 + \left(-\frac{1}{120} + \frac{1}{36} + \frac{2}{48} - \frac{3}{24} + \frac{1}{16}\right)x^4 + \dots$$

$$\therefore \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots$$

$$\left(-\frac{1}{24} + \frac{2x^3}{12} - \frac{x^3}{8}\right)x^3$$

$$\left(\frac{-1 + 4 - 3}{24}\right)x^3 = 0$$

$(a+b+c)^3 =$
$a^3 + b^3 + c^3 + 6abc + 3ab(a+b) + 3bc(b+c) + 3ca(c+a)$

9) Use Maclaurin series to evaluate the approximate value of integral $\int_0^1 e^{\sin x}\, dx$

Sol^n:- Consider $y = f(x) = e^{\sin x}$, $f(0) = e^{\sin 0} = e^0 = 1$

The Maclaurin Series is given by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \quad \text{(i)}$$

Diff^t $f(x)$ w.r.t x

$y' = f'(x) = e^{\sin x} \cdot \cos x$

$y' = y\cos x$, $y'(0) = y(0)\cos 0 = 1$

$y'' = y'\cos x - y\sin x$, $y''(0) = y'(0)\cos 0 - y(0)\sin 0$

$y''(0) = 1$

$y''' = y''\cos x - y'\sin x - y\cos x - y'\sin x = y''\cos x - 2y'\sin x - y\cos x$

$y'''(0) = y''(0)\cos 0 - y'(0)\sin 0 - y(0)\cos 0 - y'(0)\sin 0$

$y'''(0) = 1 - 1 = 0$

$y^{(4)} = y'''\cos x - y''\sin x - 2y'\cos x - 2y''\sin x - y'\cos x + y\sin x$

$y^{(4)}(0) = y'''(0)\cos 0 - y''(0)\sin 0 - 2y'(0)\cos 0 - 2y''(0)\sin(0) - y'(0)\cos 0 + y(0)\cdot\sin(0)$

$y^{(4)}(0) = -2 - 1 = -3$

Equation (i) implies that

$$e^{\sin x} = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + 0 - \frac{3}{4!}x^4 + \cdots$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3}{4!}x^4 + \cdots$$

$$\therefore \int_0^1 e^{\sin x}\,dx = \int_0^1 \left(1 + x + \frac{x^2}{2!} - \frac{3}{4!}x^4 + \cdots\right)dx$$

$$= \left[x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2!} - \frac{3}{4!}\cdot\frac{x^5}{5} + \cdots\right]_0^1$$

$$= 1 + \frac{1}{2!} + \frac{1}{3!} - \frac{3}{5!} + \cdots - 0$$

$$= 1 + \frac{1}{2!} + \frac{1}{3!} - \frac{3}{5!} + \cdots$$

$$\int_0^1 e^{\sin x}\,dx \approx 1 + \frac{1}{2!} + \frac{1}{3!} - \frac{3}{5!} \simeq 1.641 \text{ (Approximate value)}$$

Exercise:

1) Expand $y = \sqrt{x}$ as Taylor series about the point $x = 1$, up to 4^{th} degree term.

Ans: $\sqrt{x} \simeq 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{(x-1)^3}{16} - \frac{5}{128}(x-1)^4$

2) Obtain the series expansion of $\sqrt{1+\sin(2x)}$ in ascending powers of x.

Ans: $\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

3) Find the first 3 non-zero terms in the Maclaurin series for $e^x \sin x$ Ans: $e^x \sin x = x + x^2 + \frac{x^3}{3} + \ldots$

4) Obtain the Maclaurin series expansion of the function $y = \ln(\sec x + \tan x)$ Ans: $x + \frac{x^3}{6} + \frac{x^5}{24} + \ldots$

5) Find the binomial expansion of the function $\frac{1}{\sqrt{1-x^2}}$ and deduce the power series of $\sin^{-1} x$.

Ans: $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots$

$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5}{112}x^7 + \ldots$

6) Expand $\log(1+\sin x)$ up to the term containing x^4 by using Maclaurin series.

Ans: $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$

7) Expand by Maclaurin series $\frac{e^x}{1+e^x}$ up to the term containing x^3 Ans: $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots$

8) Expand $\sin x$ in ascending powers of $x = \frac{\pi}{2}$ using Taylor series and hence evaluate $\sin 91^\circ$ correct to 4 decimal places. Ans: $\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \pi/2\right)^4 - \dots$,
$\sin 91^\circ = 0.9998$.

9) Determine the maclaurin series for the function $y = \frac{1}{1+x^2}$.
Hence show that $\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5}$.
By using maclaurin series for $\tan^{-1} x$ upto 5^{th} degree evaluate $\int_0^{1/\sqrt{3}} \tan^{-1} x \, dx$, verify the same with exact value.

Ans: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$

$$\int_0^{1/\sqrt{3}} \tan^{-1}(x)\, dx \simeq 0.1584$$

10) Expand $\log\left(x + \sqrt{x^2+1}\right)$ using Maclaurin series up to term containing x^3.