MULTIPLE INTEGRALS

- A definite integral $\int_a^b f(x)dx$ is said to be proper integral if the limits of integration are finite and the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$. If atleast one of these conditions is violated, then the integral is known as improper integral.

Ex: $\int_1^2 x^2\,dx$ – Proper integral

$\int_4^\infty \frac{1}{x^4}\,dx$ – Improper integral of first kind

$\left.\begin{array}{l} \int_0^4 \frac{1}{x^4}\,dx \quad \text{Not defined at } x=0 \\ \int_0^2 \frac{1}{x-1}\,dx \quad \text{Not defined at } x=1 \end{array}\right\}$ Improper integrals of second kind.

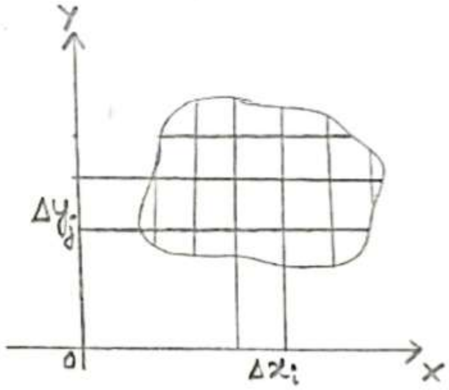
- Simple closed curve : A closed curve which doesnot cross itself is called a simple closed curve.
  Eg: Circle, Ellipse, Triangle, Rectangle.

- Simply connected region : A region bounded by a simple closed curve is called a simply connected region.

* <u>Double Integral</u>

Let $f(x,y)$ be a continuous function of x and y within a region R bounded by a simple closed curve C and upon the boundary C. Let the region R be subdivided into n subregions of areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. Let (x_K, y_K) be any point in the sub region of area ΔA_K. Consider the sum $\sum_{K=1}^{n} f(x_K, y_K)\Delta A_K$. The limit of this sum as $n \to \infty$ and $\Delta A_K \to 0$ is defined as the double integral of $f(x,y)$ over the region R and is written as $\iint_R f(x,y)\,dA$.

Thus, $$\lim_{\substack{n\to\infty \\ \Delta A_K \to 0}} \sum_{K=1}^{n} f(x_K, y_K)\Delta A_K = \iint_R f(x,y)\,dA$$



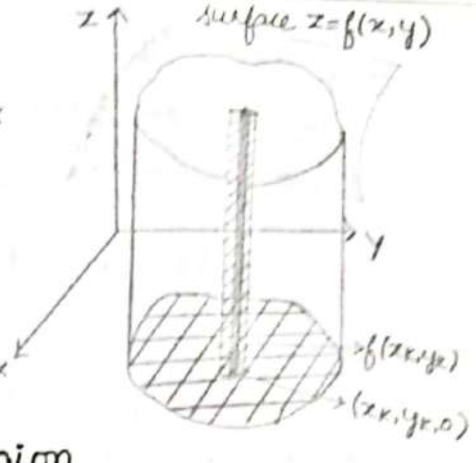
The region R is called the region of integration and this corresponds to the interval of integration (a,b) in the case of the definite integral.

In order to simplify the evaluation of the double integral, we consider sub regions R_i in the form of rectangular sub regions or rectangular grids obtained by dividing R with lines parallel to coordinate axes as shown in the figure. Since the area of a rectangle is $\Delta x_i \Delta y_j$, it follows that

$$\iint_R f(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i, y_j)\Delta x_i \Delta y_j$$

- <u>Area</u> : When $f(x,y) = 1$ on R, then $\lim_{\substack{n\to\infty \\ \Delta A_K\to 0}} \sum_{K=1}^{n} \Delta A_K$ simply gives the area A of the region R. That is $A = \iint_R dA$.

Note: $dA = dx\,dy$ is called the area element.



- Volume : If $f(x,y) \geq 0$ on R, then as shown in figure, the product $f(x_k, y_k)\Delta A_k$ can be interpreted as the volume of a rectangular prism of height $f(x_k,y_k)$ and base of area ΔA_k. The summation of volumes $\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k$ is an approximation to the volume V of the solid above the region R and below the surface $z=f(x,y)$.

$$V = \iint_R f(x,y)\,dA .$$

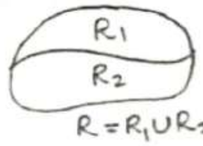
* Properties of double integrals :

Let f and g be functions of two variables that are integrable over a region R. Then

(i) $\iint_R K f(x,y)\,dA = K \iint_R f(x,y)\,dA$, where K is a constant.

(ii) $\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

(iii) $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$, where R_1 and R_2 are subregions of R that donot overlap and $R = R_1 \cup R_2$.



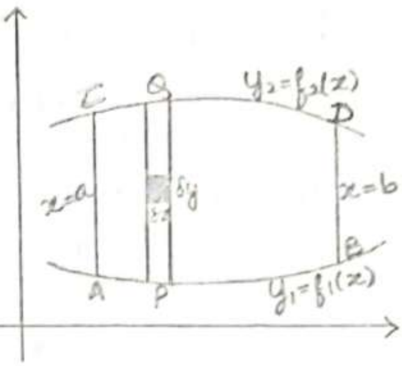
* Evaluation of double integrals :

Double integral can be evaluated by expressing it in terms of two single integrals called iterated or repeated integral. i.e., Double integral over region R may be evaluated by two successive integrations.

* Type-I :

If R is described as $f_1(x) \le y \le f_2(x)$ i.e., $y_1 \le y \le y_2$ and $a \le x \le b$, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{y_1}^{y_2} f(x,y)\,dy \right] dx$$



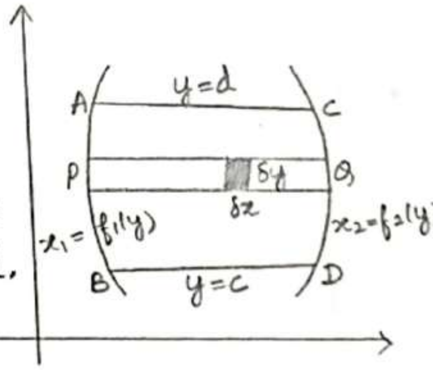
$f(x,y)$ is first integrated w.r.t y treating x as constant between the limits y_1 and y_2 and then the resulting function is integrated w.r.t x between limits a and b.

In the region we take an elementary area $\delta x \delta y$. Then integration w.r.t y (keeping x constant) converts small rectangle $\delta x \delta y$ into a strip PQ, while the integration of the result w.r.t x corresponds to sliding the strip PQ from AC to BD covering the whole region ABCD.

* Type - II :

If R is described as $g_1(y) \le x \le g_2(y)$ i.e., $x_1 \le x \le x_2$ and $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{x_1}^{x_2} f(x,y)\,dx \right] dy$$



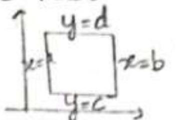
Here $f(x,y)$ is first integrated w.r.t x, keeping y constant between the limits x_1 and x_2 and then the resulting expression is integrated w.r.t y between limits c and d.

Take a small area $\delta x \delta y$ in the region.

The integration w.r.t x between the limits x_1, x_2 keeping y constant converts small rectangle into a horizontal strip PQ, while the integration of resulting function w.r.t y corresponds to sliding the strip from BD to AC covering the whole region ABCD.

* Type - 3 :

If the region R is a rectangle bounded by the lines $x=a$, $x=b$, $y=c$, $y=d$, then



$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx.$$

For constant limits, it does not matter whether we first integrate w.r.t x and then w.r.t y or vice versa.

⇒ Examples :

1. Evaluate $\int_1^4 \int_3^5 x^2 y\,dy\,dx$.

Sol: Here, the order of integration is first w.r.t y and then w.r.t x. Therefore, the limits for x are $x: 1 \to 4$ and that of y are $y: 3 \to 5$.

$$\therefore \int_1^4 \int_3^5 x^2 y\,dy\,dx = \int_1^4 x^2 \left[\int_3^5 y\,dy\right] dx$$

$$= \int_1^4 x^2 \left(\frac{y^2}{2}\right)_3^5 dx = \int_1^4 \left(\frac{25}{2} - \frac{9}{2}\right) x^2\,dx$$

$$= 8\left(\frac{x^3}{3}\right)_1^4 = \frac{8}{3}(64-1) = \underline{168}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$

Sol: Here, the limits of y are variables i.e., $y: x \to \sqrt{x}$ and limits of x are constants i.e., $x: 0 \to 1$. Therefore, integrate first w.r.t y and then w.r.t x.

$$\int_0^1 \left[\int_x^{\sqrt{x}} xy\,dy\right] dx = \int_0^1 x\left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \underline{\frac{1}{24}}$$

3. $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$

Soln

$= \int_0^1 y \left[\int_0^{\sqrt{1-y^2}} x^3 \, dx \right] dy$

$= \int_0^1 y \left[\frac{x^4}{4} \right]_0^{\sqrt{1-y^2}} dy \quad = \frac{1}{4} \int_0^1 y (1-y^2)^2 \, dy$

$= \frac{1}{4} \int_0^1 y (1+y^4-2y^2) \, dy \quad = \frac{1}{4} \int_0^1 (y+y^5-2y^3) dy$

$= \frac{1}{4} \left[\frac{y^2}{2} + \frac{y^6}{6} - \frac{2y^4}{4} \right]_0^1 \quad = \frac{1}{4} \left(\frac{1}{2} + \frac{1}{6} - \frac{1}{2} \right)$

$= \frac{1}{24}$

4. Evaluate $\int_0^{\pi} \int_0^{\cos\theta} r \sin\theta \, dr \, d\theta$

Soln

$= \int_0^{\pi} \left[\int_0^{\cos\theta} r \sin\theta \, dr \right] d\theta \quad = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{\cos\theta} d\theta$

$= \frac{1}{2} \int_0^{\pi} \sin\theta \cos^2\theta \, d\theta$

put $\quad t = \cos\theta \; . \Rightarrow \; dt = -\sin\theta \, d\theta$

$\theta = 0 \Rightarrow t = 1 \quad , \; \theta = \pi \Rightarrow t = -1$

$= \frac{1}{2} \int_1^{-1} -t^2 \, dt$

$= -\frac{1}{2} \left[\frac{t^3}{3} \right]_1^{-1}$

$= \frac{-1}{2} \left(-\frac{1}{3} - \frac{1}{3} \right) \; = -\frac{1}{2} \left(-\frac{2}{3} \right)$

$= \frac{1}{3}$

5. Show that $\int_0^\infty \int_y^\infty x e^{-x^2/y}\, dx\, dy = 1/2$

Sol: Limits of integration are : $y: 0 \to \infty$ and $x: y \to \infty$.

$$\int_0^\infty \left[\int_y^\infty x e^{-x^2/y} dx\right] dy = \int_0^\infty \left[\int_y^\infty \left(\frac{y}{2} \cdot \frac{2x}{y} e^{-x^2/y}\right) dx\right] dy$$

Let $\frac{x^2}{y} = t \Rightarrow \frac{2x}{y} dx = dt$

$x = y \Rightarrow t = y$

$x \to \infty \Rightarrow t \to \infty$

$$\therefore = \int_0^\infty \int_y^\infty \frac{y}{2} e^{-t}\, dt\, dy = \int_0^\infty \frac{y}{2}\left(\frac{e^{-t}}{-1}\right)_y^\infty dy$$

$$= \int_0^\infty \frac{y e^{-y}}{2} dy = \frac{1}{2}\left[y \frac{e^{-y}}{-1} - \int 1 \cdot \frac{e^{-y}}{-1} dy\right]_0^\infty$$

$$= \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^\infty = \underline{\underline{\frac{1}{2}}}$$

6. Evaluate $\iint_R \frac{dx\,dy}{x+y+1}$ over the space $R: 0 \le x \le 1, 0 \le y \le 1, 0 \le x, y \le 1$.

Sol: Here, the limits of integration are given by $x: 0 \to 1, y: 0 \to 1$.

$$\therefore \iint_R \frac{dx\,dy}{(x+y+1)} = \int_0^1 \int_0^1 \frac{dx\,dy}{(x+y+1)}$$

$$= \int_0^1 \left[\log(x+y+1)\right]_0^1 dy$$

$$= \int_0^1 \left[\log(y+2) - \log(y+1)\right] dy$$

$$= \Big[(y+2)\left[\log(y+2) - 1\right] - (y+1)\left[\log(y+1) - 1\right]\Big]_0^1 \qquad \left(\int \log x = x(\log x - 1)\right)$$

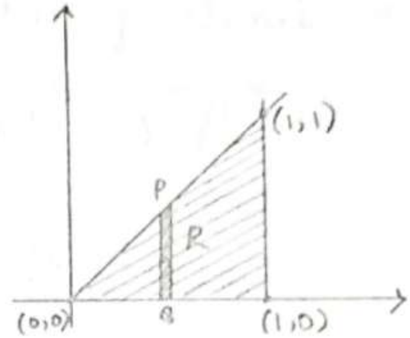
$$= 3\{\log 3 - 1\} - 2\{\log 2 - 1\} - 2\{\log 2 - 1\} + (-1)$$

$$= \log 3^3 - 3 - \log 2^2 + 2 - \log 2^2 + 2 - 1$$

$$= \log 27 - 2\log 4 = \log 27 - \log 16 = \underline{\underline{\log\left(\frac{27}{16}\right)}}$$

7. Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the triangle bounded by the lines $y=0$, $y=x$ and $x=1$.

Soln In the region R, y varies from 0 to a point on the line $y=x$.



$\therefore\ y: 0 \to x.$

$x: 0 \to 1.$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_0^x (x^2+y^2)\,dy \right] dx$$

$$= \int_0^1 \left[x^2y + y^3/3\right]_0^x dx = \int_0^1 (x^3 + x^3/3)\,dx$$

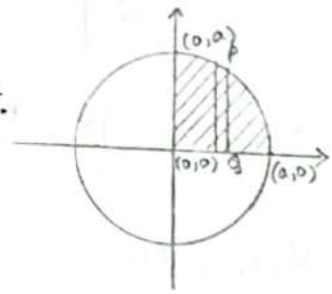
$$= \frac{4}{3}\int_0^1 x^3\,dx = \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{3}$$

8. Evaluate $\iint_A xy\,dx\,dy$ where A is the area bounded by the circle $x^2+y^2=a^2$ in the first quadrant.

Soln We have taken a strip parallel to Y-axis.

y varies from 0 to point on circle $x^2+y^2=a^2$.

x varies from 0 to a.



$\therefore\ y: 0 \to \sqrt{a^2-x^2}$; $x: 0 \to a.$

$$\therefore \iint_A xy\,dx\,dy = \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} xy\,dy \right] dx$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \frac{x}{2}(a^2-x^2)\,dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{1}{2}\left(a^2\frac{x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{a^4}{8}$$

9. Evaluate $\int_3^4 \int_1^2 \frac{dy\,dx}{(x+y)^2}$

Sol: $= \int_3^4 \left[\int_1^2 (x+y)^{-2}\, dy\right] dx = \int_3^4 \left[\frac{(x+y)^{-1}}{-1}\right]_1^2 dx$

$= -\int_3^4 \left\{\frac{1}{(x+2)} - \frac{1}{(x+1)}\right\} dx$

$= -\left[\log(x+2) - \log(x+1)\right]_3^4$

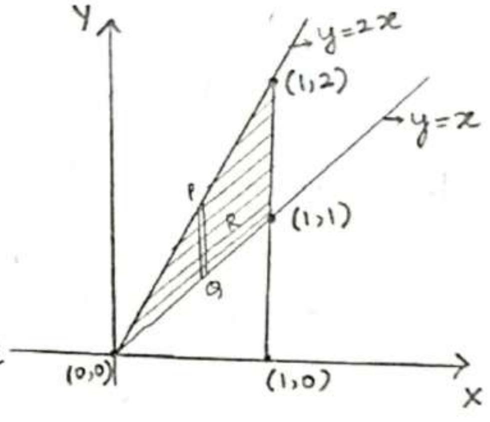
$= -\left[\log 6 - \log 5 - \log 5 + \log 4\right]$

$= \log 25 - \log 24$

$= \log\left(\frac{25}{24}\right)$

10. Evaluate $\iint_R (x^2+y^2)\, dy\, dx$ where R is the region bounded by $y=x$, $y=2x$ and $x=1$ in the first quadrant.

Sol: We have considered a strip parallel to Y-axis.
y varies from line $y=x$ to $y=2x$.
x varies from 0 to 1.



$\therefore \iint_R (x^2+y^2)\, dy\, dx = \int_0^1 \int_x^{2x} (x^2+y^2)\, dy\, dx$

$= \int_0^1 \left[x^2 y + y^3/3\right]_x^{2x} dx$

$= \int_0^1 \left[2x^3 + \frac{8x^3}{3} - x^3 - \frac{x^3}{3}\right] dx$

$= \int_0^1 \left(x^3 + \frac{7x^3}{3}\right) dx = \left(\frac{x^4}{4} + \frac{7x^4}{12}\right)_0^1$

$= \frac{1}{4} + \frac{7}{12} = \frac{10}{12} = \frac{5}{6}$

* Change of order of integration

(i) When the limits are constants, then the order of integration is immaterial. i.e., $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$.

Here we have to keep it in mind that the limits of x are to be used for x and those of y used for y only.

(ii) When the limits of integration are variables, on changing the order of integration, the limits of integration change. To find the new limits, a rough sketch of the region of integration is essential. This helps in fixing the new limits of integration.

Some of the problems connected with double integrals, which seem to be complicated can be made easy to handle by a change in the order of integration.

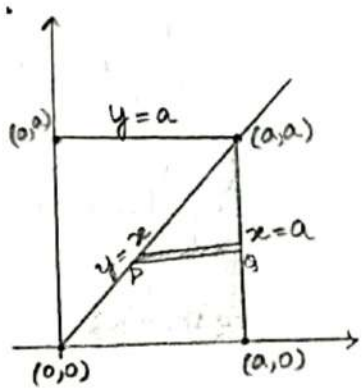
Examples

1. Change the order of integration and hence evaluate
$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy.$$

Sol: It is clear that the region of integration is bounded between $x=y$, $x=a$, $y=0$, $y=a$.

The region is divided into horizontal strips.

For changing order of integration, consider a vertical strip.



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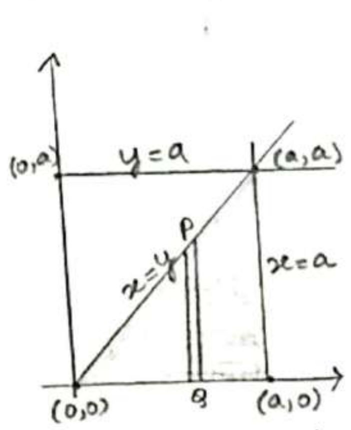


(Change of order)

y varies from 0 to $y=x$.
x varies from 0 to a.

$$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \int_0^a \int_0^x \frac{x}{(x^2+y^2)}\, dy\, dx$$

$\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= \int_0^a \left(x \cdot \frac{1}{x} \tan^{-1} \frac{y}{x}\right)_0^x dx$$

$$= (\tan^{-1} 1 - \tan^{-1} 0) \int_0^a 1 . dx$$

$$= \pi/4 \, (x)_0^a$$

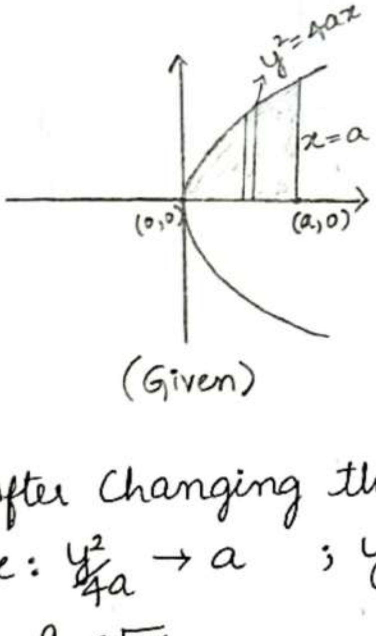
$$= \frac{\pi a}{4}$$

2. Change the order of integration in $\int_0^a \int_0^{2\sqrt{ax}} x^2\, dy\, dx$ and then evaluate it.

Soln Given limits of integration are : $y : 0 \to 2\sqrt{ax}$

$\Rightarrow y = 0$ to $y^2 = 4ax$.

$x : 0 \to a$.



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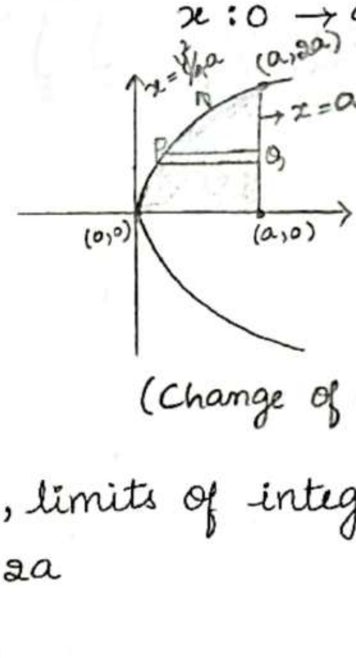


(Change of order)

$y^2 = 4ax$
$x = a$
$\Rightarrow y^2 = 4a^2$
$y = 2a$

After Changing the order, limits of integration are given by,

$x : \frac{y^2}{4a} \to a \quad ; y : 0 \to 2a$

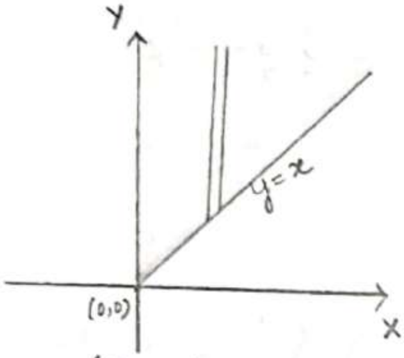
$$\therefore \int_0^a \int_0^{2\sqrt{ax}} x^2 dy dx = \int_0^{2a} \int_{y^2/4a}^{a} x^2\, dx\, dy = \int_0^{2a} \left[\frac{x^3}{3}\right]_{y^2/4a}^{a} dy$$

$$= \frac{1}{3}\int_0^{2a} \left[a^3 - \frac{y^6}{64a^3}\right] dy = \frac{1}{3}\left[a^3 y - \frac{y^7}{64 \times 7 a^3}\right]_0^{2a}$$

$$= \frac{1}{3}\left[2a^4 - \frac{\cancel{2^6} \times 2 a^7}{\cancel{64} \times 7 a^3}\right] = \frac{1}{21}(14a^4 - 2a^4) = \frac{12}{21} a^4 = \frac{4}{7} a^4$$

3. Change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ and hence evaluate it.

Sol: Given limits of integration are : $y : x \to \infty$
$x : 0 \to \infty$



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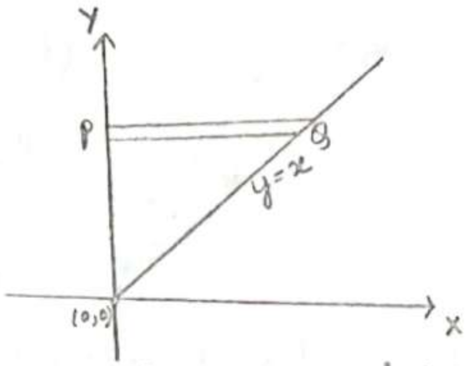


(Change of order)

After changing the order, limits of integration are given by $x : 0 \to y$ and $y : 0 \to \infty$.

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

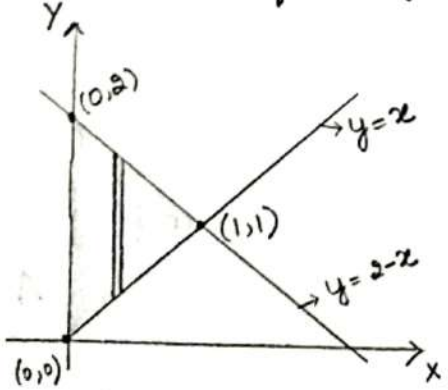
$$= \int_0^\infty \frac{e^{-y}}{y} \left[x\right]_0^y dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y}\, dy$$

$$= \left[-e^{-y}\right]_0^\infty$$

$$= 1$$

4. Evaluate the integral $\int_0^1 \int_x^{2-x} \frac{x}{y}\, dx\, dy$ by changing the order of integration.

Sol: Given limits of integration are : $y : x \to 2-x$; $x : 0 \to 1$



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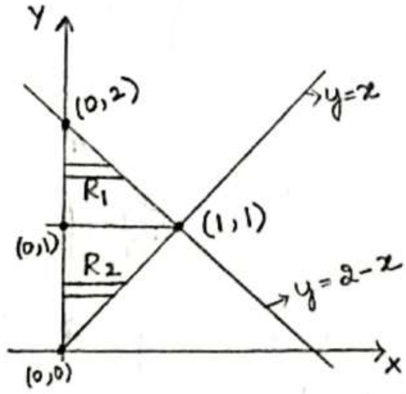


(Change of order)

$y = 2-x$
$y = x$
$\Rightarrow x = 2-x$
$2x = 2$
$x = 1$
$y = 1$

After changing the order,

$$\int_0^1\int_x^{2-x} \frac{x}{y}\,dx\,dy = \iint_{R_1} \frac{x}{y}\,dx\,dy + \iint_{R_2} \frac{x}{y}\,dx\,dy.$$

In R_1, $x : 0$ to $2-y$; $y : 1 \to 2$

In R_2, $x : 0 \to y$; $y : 0 \to 1$.

$$\therefore \int_0^1\int_x^{2-x} \frac{x}{y}\,dy\,dx = \int_1^2\int_0^{2-y} \frac{x}{y}\,dx\,dy + \int_0^1\int_0^{y} \frac{x}{y}\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2y}\right]_0^{2-y} dy + \int_0^1 \left[\frac{x^2}{2y}\right]_0^{y} dy$$

$$= \int_1^2 \frac{(2-y)^2}{2y}\,dy + \int_0^1 \frac{y}{2}\,dy$$

$$= \int_1^2 \left(\frac{4+y^2-4y}{2y}\right) dy + \left(\frac{y^2}{4}\right)_0^1$$

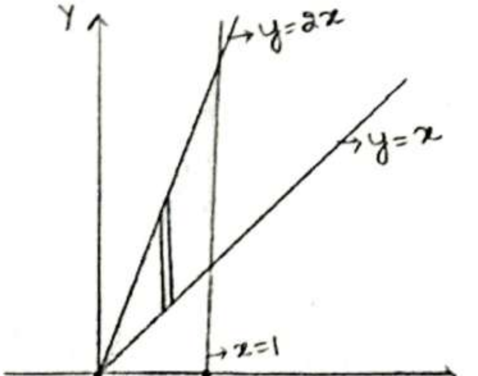
$$= \int_1^2 \left(\frac{2}{y} + \frac{y}{2} - 2\right) dy + \frac{1}{4} = \left(2\log y + \frac{y^2}{4} - 2y\right)_1^2 + \frac{1}{4}$$

$$= \log 4 + 1 - 4 - \cancel{\log 1} - \cancel{\tfrac{1}{4}} + 2 + \cancel{\tfrac{1}{4}}$$

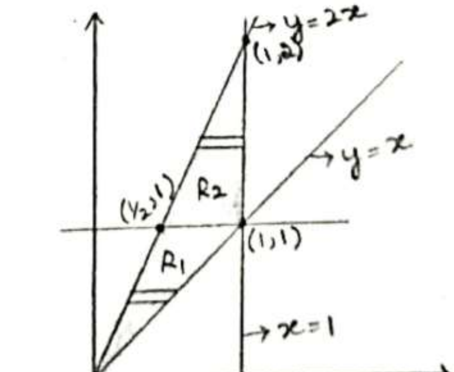
$$= \underline{\log 4 - 1}$$

5. Evaluate $\int_0^1\int_x^{2x} (x^2+y^2)\,dy\,dx$ by changing the order of integration.

Sol: Given limits of integration are : $y : x \to 2x$; $x : 0 \to 1$



(Given)



(Change of order)

After changing the order,

$$\int_0^1\int_x^{2x}(x^2+y^2)\,dy\,dx = \iint_{R_1}(x^2+y^2)\,dx\,dy + \iint_{R_2}(x^2+y^2)\,dx\,dy$$

In R_1, $x : y/2 \to y$ and $y : 0 \to 1$

In R_2, $x : y/2 \to 1$ and $y : 1 \to 2$

$$\therefore \int_0^1\int_x^{2x}(x^2+y^2)\,dy\,dx = \int_0^1\int_{y/2}^{y}(x^2+y^2)\,dx\,dy + \int_1^2\int_{y/2}^{1}(x^2+y^2)\,dx\,dy$$

$$= \int_0^1\left(\frac{x^3}{3}+y^2x\right)_{y/2}^{y}dy + \int_1^2\left(\frac{x^3}{3}+y^2x\right)_{y/2}^{1}dy$$

$$= \int_0^1\left(\frac{y^3}{3}+y^3-\frac{y^3}{24}-\frac{y^3}{2}\right)dy + \int_1^2\left(\frac{1}{3}+y^2-\frac{y^3}{24}-\frac{y^3}{2}\right)dy$$

$$= \int_0^1 \frac{19}{24}y^3\,dy + \int_1^2\left(\frac{1}{3}+y^2-\frac{13}{24}y^3\right)dy$$

$$= \frac{19}{24}\left[\frac{y^4}{4}\right]_0^1 + \left[\frac{1}{3}y+\frac{y^3}{3}-\frac{13}{24}\cdot\frac{y^4}{4}\right]_1^2$$

$$= \frac{19}{24}\times\frac{1}{4} + \left[\frac{2}{3}+\frac{8}{3}-\frac{13}{24}\times\frac{16}{4}-\frac{1}{3}-\frac{1}{3}+\frac{13}{24\times 4}\right]$$

$$= \underline{5/6}$$

* Change of variables in a double integral:

In the evaluation of repeated (iterated) integrals, the computational work can often be reduced by changing the variables from one system of coordinates to another coordinate system.

1. uv-plane: Consider the double integral $\iint_R f(x,y)\,dx\,dy$.

Let the variables be changed from x and y to u and v by the transformations $x=\phi(u,v)$ and $y=\psi(u,v)$, where

where $\phi(u,v)$ and $\psi(u,v)$ are continuous and have continuous first order derivatives in some region R^* in the uv-plane which corresponds to the region R in the xy-plane. Then,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv.$$

Here, R is the region in which (x,y) vary and R^* is the corresponding region in which (u,v) vary.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(ii) In polar co-ordinates : If $x = r\cos\theta$ and $y = r\sin\theta$ with $r^2 = x^2+y^2$, then

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$\because J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

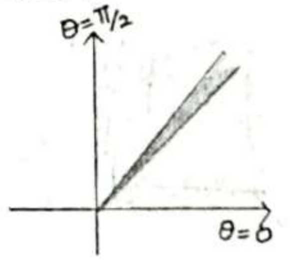
* <u>Examples</u>

1. Transform to polar coordinates and hence evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$

<u>Soln</u> In polar form, $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

Given : $x : 0 \to \infty$; $y : 0 \to \infty$

The region of integration is first quadrant.



$\therefore$ r varies from 0 to ∞.

θ varies from 0 to $\pi/2$

$$\therefore \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}.\,r\,dr\,d\theta$$

Let $r^2 = t$, $2r\,dr = dt$; $r = 0 \Rightarrow t = 0$, $r \to \infty \Rightarrow t \to \infty$.

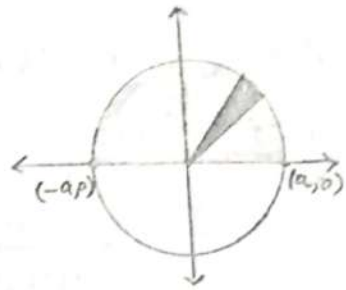
$$= \int_0^{\pi/2}\int_0^\infty e^{-t}.\frac{dt}{2}.\,d\theta = \left[\frac{e^{-t}}{-2}\right]_0^\infty .\left[\theta\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

2. Evaluate $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ by changing to polar coordinates.

Sol: In polar form, $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$,
$dx\,dy = r\,dr\,d\theta$.

Given : $y : 0 \to \sqrt{a^2-x^2}$; $x : -a \to a$

$y^2 = a^2 - x^2 \Rightarrow x^2+y^2 = a^2$



r varies from 0 to point on curve

$x^2+y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a.$

$\therefore\ r : 0 \to a$

$\theta : 0 \to \pi$

$$\therefore \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_{0}^{\pi}\int_{0}^{a} \sqrt{r^2}\cdot r\, dr \cdot d\theta$$

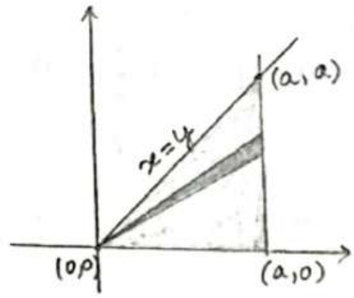
$$= \int_{0}^{\pi}\int_{0}^{a} r^2\, dr\, d\theta = \int_{0}^{\pi} \left(\frac{r^3}{3}\right)_0^a d\theta$$

$$= \frac{a^3}{3}(\theta)_0^{\pi} = \frac{\pi a^3}{3}$$

3. Change to polar coordinates and hence evaluate $\int_{0}^{a}\int_{y}^{a} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$

Sol: In polar form, $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$, $dx\,dy = r\,dr\,d\theta$.

Given : $x : y \to a$; $y : 0 \to a$



r varies from 0 to point on curve $x = a$.

$\therefore\ a = r\cos\theta \Rightarrow r = a\sec\theta.$

$\therefore\ r : 0 \to a\sec\theta$

$\theta : 0 \to \pi/4$

$$\therefore \int_{0}^{a}\int_{y}^{a} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy = \int_{0}^{\pi/4}\int_{0}^{a\sec\theta} \frac{r\cos\theta}{\cancel{r}} \cdot r\, dr\, d\theta$$

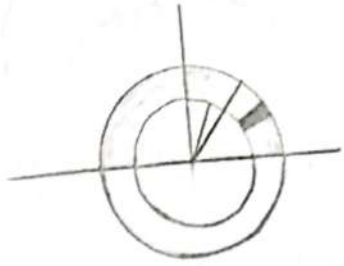
$$= \int_{0}^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sec\theta} \cos\theta\, d\theta = \int_{0}^{\pi/4} \frac{a^2}{2}\cdot\sec^2\theta\cdot\cos\theta\, d\theta = \frac{a^2}{2}\int_{0}^{\pi/4} \sec\theta\, d\theta$$

$$= \frac{a^2}{2}\Big[\log(\sec\theta + \tan\theta)\Big]_0^{\pi/4} = \frac{a^2}{2}\log(\sqrt{2}+1).$$

4. Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Soln: In polar form, $x = r\cos\theta$, $y = r\sin\theta$,

$dx\,dy = r\,dr\,d\theta$.



In the region of integration, r varies from $x^2+y^2=1$ to $x^2+y^2=4$.

i.e., $r^2=1$ to $r^2=4$

$\therefore\ r : 1 \rightarrow 2$.

$\theta : 0 \rightarrow \pi$

$$\therefore \iint_R (3x+4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^\pi \left[3.\frac{r^3}{3}\cos\theta + 4\frac{r^4}{4}\left(\frac{1-\cos 2\theta}{2}\right)\right]_1^2 d\theta$$

$$= \int_0^\pi \left[(8-1)\cos\theta + \left(\frac{16-1}{2}\right)(1-\cos 2\theta)\right] d\theta$$

$$= \int_0^\pi \left\{7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right\} d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^\pi$$

$$= \frac{15}{2}\pi$$

* Computation of plane areas:

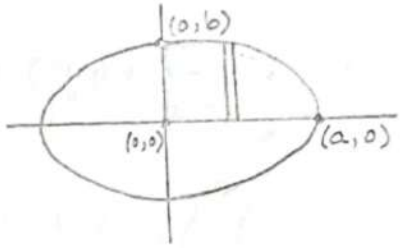
In Cartesian form, area of the region R is given by

$$\text{Area} = \iint_R dx\,dy$$

In polar form, $\text{Area} = \iint_{R^*} r\,dr\,d\theta$.

Examples

1. Find the area bounded by one quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Sol: In the given region, y varies from 0 to point on the ellipse.

i.e., $y : 0 \to \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$

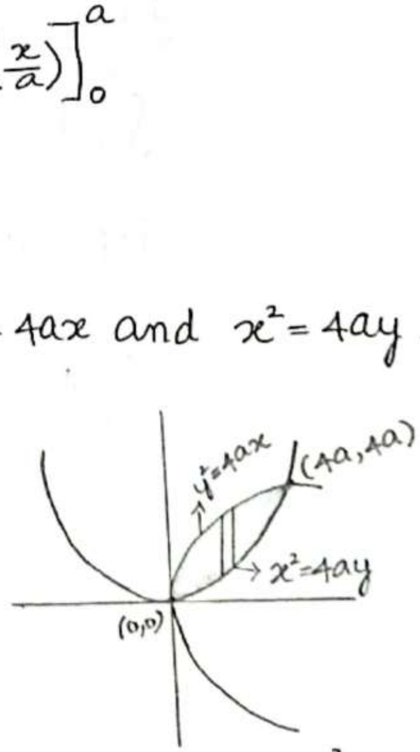
$x : 0 \to a$

$$\therefore A = \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} dy\,dx = \int_0^a [y]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)\right] = \frac{\pi}{4}ab$$

2. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.



Sol: In the given region, $y : \frac{x^2}{4a} \to \sqrt{4ax}$

$x : 0 \to 4a$

$$\therefore A = \int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dy\,dx = \int_0^{4a} [y]_{x^2/4a}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a}\left[\sqrt{4ax} - \frac{x^2}{4a}\right]dx = \left[\sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{2}{3}\cdot\sqrt{4a}\cdot 4a\cdot\sqrt{4a} - \frac{64a^3}{12a} = \frac{2}{3}\times 16a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}$$

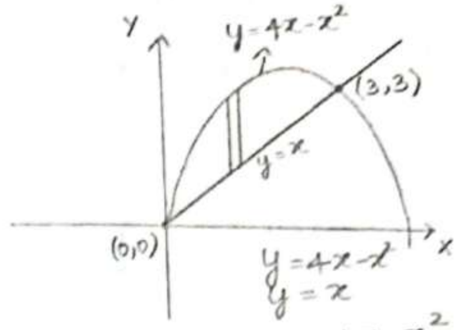
$y^2 = 4ax$; $x^2 = 4ay$

$\Rightarrow \frac{x^4}{16a^2} = 4ax$

$\Rightarrow x^3 = 64a^3$

$x = 4a, y = 4a$.

3. Find the area between the parabola $y = 4x - x^2$ and the line $y = x$, using double integration.



$y = 4x - x^2$
$y = x$
$\Rightarrow x = 4x - x^2$
$x^2 - 3x = 0$
$x(x-3) = 0$
$x = 0 \quad x = 3$
$y = 0 \quad y = 3$

Sol: In the given region,

$$y : x \to 4x - x^2$$
$$x : 0 \to 3$$

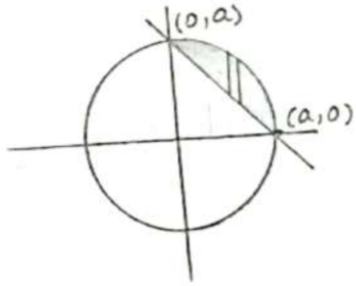
$$\therefore \quad A = \int_0^3 \int_x^{4x-x^2} dy\, dx$$

$$= \int_0^3 [y]_x^{4x-x^2}\, dx$$

$$= \int_0^3 (4x - x^2 - x)\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}$$

4. Find the area bounded by the circle $x^2+y^2=a^2$ and the line $x+y=a$ in the first quadrant.



Sol: In the given region, y varies from point on line $x+y=a$ to point on circle $x^2+y^2=a^2$.

i.e., $y : a - x \to \sqrt{a^2 - x^2}$
$x : 0 \to a$.

$$\therefore \quad \text{Area} = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a [y]_{a-x}^{\sqrt{a^2-x^2}}\, dx$$

$$= \int_0^a \left(\sqrt{a^2-x^2} - a + x\right) dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{x^2}{2}\right]_0^a$$

$$= \frac{a^2}{2}\sin^{-1}(1) - a^2 + \frac{a^2}{2}$$

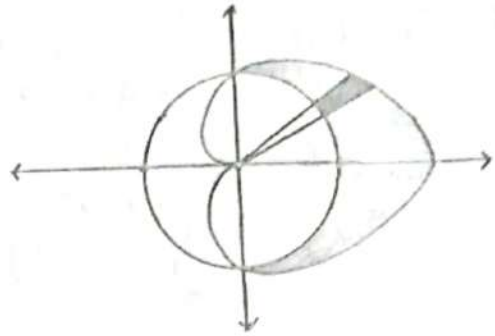
$$= \frac{\pi}{4}a^2 - \frac{a^2}{2}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$$

5. Find the area which is inside the cardioid $r = 2(1+\cos\theta)$ and outside the circle $r = 2$.

Soln $r : 2 \rightarrow 2(1+\cos\theta)$

$\theta : -\pi/2 \rightarrow \pi/2$



$$\therefore \text{Area} = 2\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2}\left[4(1+\cos\theta)^2 - 4\right] d\theta$$

$$= 4\int_0^{\pi/2}(1+\cos^2\theta+2\cos\theta-1)\,d\theta = 4\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}+2\cos\theta\right)d\theta$$

$$= 4\left[\theta/2 + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi/2}$$

$$= 4\left[\pi/4 + 2\right] = \pi + 8$$

$\Rightarrow$ Evaluation of triple integrals :

The triple integrals also known as volume integrals in $\mathbb{R}^3$ is a simple and straight extension of the ideas in respect of double integrals.

For the purpose of evaluation, the triple integral over the region R i.e., $\iiint_R f(x,y,z)\,dv$ can be expressed as an iterated or repeated integral in the form

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_a^b\left[\int_{g(x)}^{h(x)}\left\{\int_{\psi(x,y)}^{\phi(x,y)} f(x,y,z)\,dz\right\}dy\right]dx$$

Where $f(x,y,z)$ is continuous in a region R, bounded by surfaces $z=\psi(x,y)$, $z=\phi(x,y)$, $y=g(x)$, $y=h(x)$, $x=a$, $x=b$. The above integral indicates three successive integrations to be performed in the following order, first w.r.t z, keeping x and y as constants, then w.r.t y keeping x as constant and finally w.r.t x.

* Note :

1. When an integration is performed w.r.t a variable, that variable is eliminated completely from the remaining integral.

2. If the limits are not constants, the integration should be in the order in which dx, dy, dz is given in the integral.

3. Evaluation of the integral may be performed in any order if all the limits are constants.

4. If $f(x,y,z) = 1$ then the triple integral gives the volume of the region.

* Examples

1. Evaluate $\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$

Soln

$$= \int_0^1\int_0^1 \left[\frac{x^2}{2}+yx+zx\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left(\frac{1}{2}+y+z\right) dy\,dz = \int_0^1 \left[\frac{y}{2}+\frac{y^2}{2}+zy\right]_0^1 dz$$

$$= \int_0^1 \left(\frac{1}{2}+\frac{1}{2}+z\right) dz = \left[z+\frac{z^2}{2}\right]_0^1 = 1+\frac{1}{2} = \underline{\underline{3/2}}$$

2. Evaluate $\int_{r=0}^{a}\int_0^{\pi/2}\int_0^{\pi/2} r^2 \sin\theta \, d\phi\, d\theta\, dr$

Soln

$$= \int_0^a r^2\,dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} d\phi$$

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{\pi/2}$$

$$= \frac{a^3}{3} \times 1 \times \pi/2$$

$$= \underline{\underline{\frac{\pi a^3}{6}}}$$

3. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx$

Soln

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x} \left[(x+y+1-x-y+1)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x} \left\{\frac{1}{4} - (x+y+1)^{-2}\right\} dy\,dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{y}{4} - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1-x}{4} + (x+1-x+1)^{-1} - (x+1)^{-1}\right] dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x}\right] dx$$

$$= \frac{-1}{2}\left[\frac{(1-x)^2}{-8} + \frac{x}{2} - \log(1+x)\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{1}{2} - \log(2) + \frac{1}{8}\right]$$

$$= \frac{-1}{2}\left[\frac{5}{8} - \log(2)\right]$$

$$= \frac{\log 2}{2} - \frac{5}{16}$$

4. Evaluate $\int_0^1 \int_0^x \int_0^{x+y} (x+y+z)\, dz\, dy\, dx$

Soln $\int_0^1 \int_0^x \int_0^{x+y} (x+y+z)\, dz\, dy\, dx = \int_0^1 \int_0^x \left[z(x+y) + \frac{z^2}{2} \right]_0^{x+y} dy\, dx$

$$= \int_0^1 \int_0^x \left[(x+y)^2 + \frac{(x+y)^2}{2} \right] dy\, dx$$

$$= \frac{3}{2} \int_0^1 \int_0^x (x+y)^2\, dy\, dx = \frac{3}{2} \int_0^1 \left[\frac{(x+y)^3}{3} \right]_0^x dx$$

$$= \frac{1}{2} \int_0^1 (8x^3 - x^3)\, dx = \frac{7}{2} \left[\frac{x^4}{4} \right]_0^1 = \underline{\underline{\frac{7}{8}}}$$

* Volume using triple integrals :

The volume element is $dV = dx\,dy\,dz$. Summation of all volume elements gives the volume of solid.

i.e., $V = \iiint_R dx\,dy\,dz$

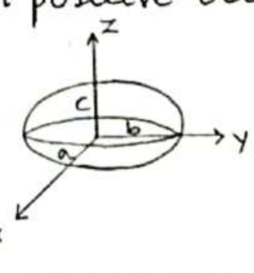
In polar coordinates, $V = \iiint_R r^2 \sin\theta\, dr\, d\theta\, d\phi$. (spherical)

$V = \iiint_R r\, dr\, d\phi\, dz$ (cylindrical)

1. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Soln Required volume $V = \iiint_R dx\, dy\, dz$

Since the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is symmetrical about each of the co-ordinate planes, required volume $V = 8V_1$ where V_1 is the volume bounded by ellipsoid in positive octant.

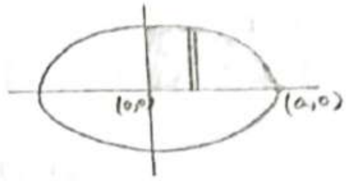


$z : 0 \rightarrow$ point on ellipsoid

i.e., $z : 0 \rightarrow c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

The projection in XY plane is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore$ y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$



$x : 0 \to a$

Hence, Volume $V_1 = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$

$$V_1 = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\,dx$$

$$= \frac{c}{b}\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\left[\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\right]^2 - y^2}\, dy\,dx.$$

WKT $\int\sqrt{a^2-y^2}\,dy = \frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}(y/a)$

$= b\sqrt{1-x^2/a^2}$

$$\therefore V_1 = \frac{c}{b}\int_0^a \left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{1-x^2/a^2}}\right)\right]_0^{\boxed{\frac{b}{a}\sqrt{a^2-x^2}}} dx$$

$$= \frac{c}{b}\int_0^a \frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\sin^{-1}(1)\, dx$$

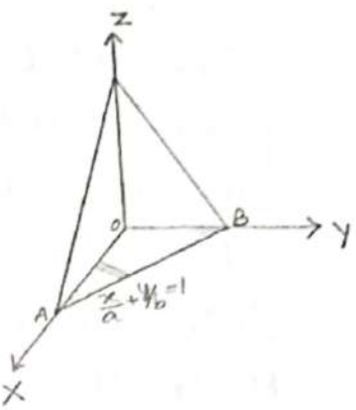
$$= \frac{\pi bc}{4}\int_0^a \left(1-\frac{x^2}{a^2}\right) dx$$

$$= \frac{\pi bc}{4}\left(x-\frac{x^3}{3a^2}\right)_0^a = \frac{\pi bc}{4}\left(a-\frac{a^3}{3a^2}\right) = \frac{\pi bc}{4}\times\frac{2a}{3}$$

$$= \frac{\pi abc}{6}$$

$$\therefore V = 8V_1 = 8\left(\frac{\pi abc}{6}\right) = \frac{4}{3}\pi abc$$

2. Find the volume of the tetrahedron $x \geq 0, y \geq 0, z \geq 0,$ $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$.



Soln Required volume $V = \iiint_R dx\,dy\,dz$

z varies from 0 to point on tetrahedron.

$\therefore\ z : 0 \to c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

The projection in XY plane is the ΔOAB with AB as $\frac{x}{a} + \frac{y}{b} = 1$.

$\therefore\ y : 0 \to b\left(1 - \frac{x}{a}\right)$

$x : 0 \to a$

For $z=0$, $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow y = b\left(1 - \frac{x}{a}\right)$

For $z=0, y=0$, $\frac{x}{a} = 1 \Rightarrow x = a$

$$\therefore\ V = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} [z]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a \left\{\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right\}_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a b\left(1 - \frac{x}{a}\right)^2 - \frac{b^2\left(1 - \frac{x}{a}\right)^2}{2b}\, dx$$

$$= c\left(b - \frac{b}{2}\right) \int_0^a \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{(1 - x/a)^3}{-3/a}\right]_0^a$$

$$= \frac{abc}{6}(+1)$$

$$= \frac{abc}{6}$$

⇒ Center of gravity :

The total weight of the object concentrated in a single point is called the object's centre of gravity or it is a point on which the object is in balance.

Let $f(x,y)$ be the density (mass per unit area) of a distribution of mass in the xy-plane. Then the total mass M in the region R is given by

$$M = \iint_R f(x,y)\,dx\,dy$$

The center of gravity of a mass in R has the coordinates $(\bar{x},\bar{y})$ where,

$$\bar{x} = \frac{1}{M}\iint_R x\,f(x,y)\,dx\,dy$$

and

$$\bar{y} = \frac{1}{M}\iint_R y\,f(x,y)\,dx\,dy$$

For a solid if the density at the point $P(x,y,z)$ be $f(x,y,z)$ then total mass of the solid is given by,

$$M = \iiint_R f(x,y,z)\,dx\,dy\,dz.$$

The center of gravity of a mass in R has the coordinates $(\bar{x},\bar{y},\bar{z})$, where

$$\bar{x} = \frac{1}{M}\iiint_R x\,f(x,y,z)\,dx\,dy\,dz$$

$$\bar{y} = \frac{1}{M}\iiint_R y\,f(x,y,z)\,dx\,dy\,dz$$

and

$$\bar{z} = \frac{1}{M}\iiint_R z\,f(x,y,z)\,dx\,dy\,dz$$

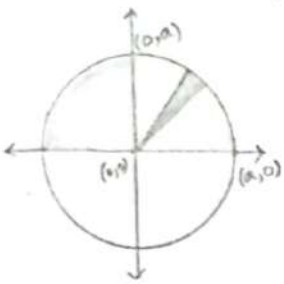
1. Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x,y)=1$ in the region R of the semidisc $x^2+y^2 \le a^2$, $y \geq 0$.

Soln Given region is a circle with center at origin and radius 'a'. It is easy to evaluate if we use polar coordinates.

$\therefore \; x = r\cos\theta \;, \; y = r\sin\theta \;, \; x^2+y^2 = r^2 \;, \; dx\,dy = r\,dr\,d\theta.$

Here, $r : 0 \to a$ and $\theta : 0 \to \pi$ ($\because$ semidisc)

Mass is given by $M = \iint_R f(x,y)\,dx\,dy$



$$= \int_0^{\pi}\int_0^{a} 1 \cdot r\,dr\,d\theta = [\theta]_0^{\pi}\left[\frac{r^2}{2}\right]_0^a$$

$$M = \frac{\pi a^2}{2}$$

$$\bar{x} = \frac{1}{M}\iint_R x\, f(x,y)\,dx\,dy$$

$$= \frac{2}{\pi a^2}\int_0^{\pi}\int_0^{a} r\cos\theta \cdot r\,dr\,d\theta$$

$$= \frac{2}{\pi a^2}\int_0^{\pi}\cos\theta\,d\theta \int_0^{a} r^2\,dr = \frac{2}{\pi a^2}[\sin\theta]_0^{\pi}\left[\frac{r^3}{3}\right]_0^a$$

$$= 0$$

$$\bar{y} = \frac{1}{M}\iint_R y\, f(x,y)\,dx\,dy$$

$$= \frac{2}{\pi a^2}\int_0^{\pi}\int_0^{a} r\sin\theta \; r\,dr\,d\theta$$

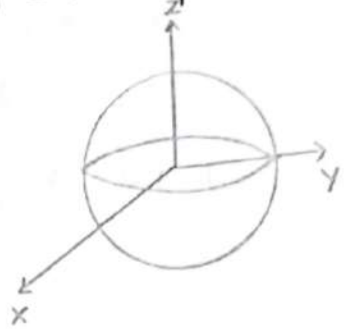
$$= \frac{2}{\pi a^2}[-\cos\theta]_0^{\pi}\left[\frac{r^3}{3}\right]_0^a$$

$$= \frac{2}{\pi a^2}(1+1)\cdot\frac{a^3}{3} = \frac{4a}{3\pi}$$

$\therefore$ center of gravity $(\bar{x}, \bar{y})$ is given by $\left(0, \frac{4a}{3\pi}\right)$.

2. Find the center of gravity in a volume of solid, which is in the form of positive octant in the sphere $x^2+y^2+z^2=1$, the density ρ at any point (x,y) is given by $\rho = \mu xyz$, where μ is a constant.

Sol: This gives the region of the positive octant of unit sphere.



$$\text{Mass } M = \iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \mu xyz\,dz\,dy\,dx$$

$$= \mu \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{\mu}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy\,dx$$

$$= \frac{\mu}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\,dy\,dx$$

$$= \frac{\mu}{2} \int_0^1 \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\mu}{2} \int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right] dx$$

$$= \frac{\mu}{2.2} \int_0^1 \left[x - x^3 - x^3 + x^5 - \frac{x(1+x^4-2x^2)}{2}\right] dx$$

$$= \frac{\mu}{4}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} - \frac{x^2}{4} - \frac{x^6}{12} + \frac{2x^4}{8}\right]_0^1$$

$$= \frac{\mu}{4}\left[\cancel{\frac{1}{2}} - \cancel{\frac{2}{4}} + \frac{1}{6} - \cancel{\frac{1}{4}} - \frac{1}{12} + \cancel{\frac{2}{8}}\right]$$

$$= \frac{\mu}{12 \times 4} = \frac{\mu}{48}$$

$$\bar{x} = \frac{1}{M}\iiint_R x\, f(x,y,z)\, dx\, dy\, dz$$

$$= \frac{48}{\mu}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \mu x^2 y z\, dz\, dy\, dx$$

$$= \frac{48}{\mu}\cdot\mu \int_0^1\int_0^{\sqrt{1-x^2}} x^2 y \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= 24\int_0^1\int_0^{\sqrt{1-x^2}} x^2 y(1-x^2-y^2)\, dy\, dx$$

$$= 24\int_0^1 \left[\frac{x^2y^2}{2} - \frac{x^4y^2}{2} - \frac{x^2y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{24}{2}\int_0^1 \left\{x^2(1-x^2) - x^4(1-x^2) - \frac{x^2(1+x^4-2x^2)}{2}\right\} dx$$

$$= 12\left[\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7} - \frac{x^3}{6} - \frac{x^7}{14} + \frac{2x^5}{10}\right]_0^1$$

$$= 12\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{14} + \frac{1}{5}\right]$$

$$= \frac{16}{35}$$

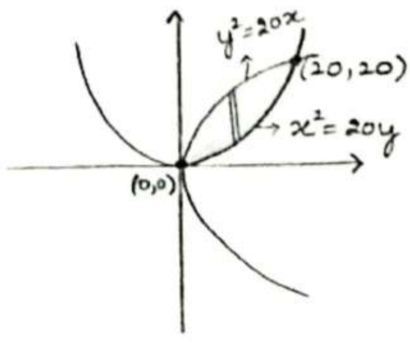
IIIly

$$\bar{y} = \frac{1}{M}\iiint_R y\, f(x,y,z)\, dx\, dy\, dz = \frac{16}{35}$$

$$\bar{z} = \frac{1}{M}\iiint_R z\, f(x,y,z)\, dx\, dy\, dz = \frac{16}{35}$$

$\therefore$ Center of gravity $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{16}{35}, \frac{16}{35}, \frac{16}{35}\right)$

3. Find the centroid of the area bounded by parabolas $y^2 = 20x$, $x^2 = 20y$.



Soln $y : x^2/20 \to \sqrt{20x}$; $x : 0 \to 20$

Mass $M = \iint_R dA = \int_0^{20}\int_{x^2/20}^{\sqrt{20x}} dy\, dx$

$$= \int_0^{20} [y]_{x^2/20}^{\sqrt{20x}} = \int_0^{20}\left\{\sqrt{20x} - \frac{x^2}{20}\right\} dx$$

$$= \left[\sqrt{20} \cdot \frac{x^{3/2}}{3/2} - \frac{x^3}{60}\right]_0^{20} = \frac{\sqrt{20} \cdot 20\sqrt{20}}{3} \times 2 - \frac{20 \times 20 \times 20}{60} = \underline{\underline{\frac{400}{3}}}$$

$$\bar{x} = \frac{1}{M} \iint_R x \, dA = \frac{3}{400} \int_0^{20} \int_{y^2/20}^{\sqrt{20y}} x \, dx \, dy = \frac{3}{400} \int_0^{20} \left[\frac{x^2}{2}\right]_{y^2/20}^{\sqrt{20y}} dy$$

$$= \frac{3}{800} \int_0^{20} \left[20y - \frac{y^4}{400}\right] dy$$

$$= \frac{3}{800} \left[20\frac{y^2}{2} - \frac{y^5}{2000}\right]_0^{20}$$

$$= \frac{3}{800} \left[\frac{20 \times (20)^2}{2} - \frac{(20)^5}{2000}\right]$$

$$= \underline{\underline{9}}$$

$$\bar{y} = \frac{1}{M} \iint_R y \, dA = \frac{3}{400} \int_0^{20} \int_{y^2/20}^{\sqrt{20y}} y \, dx \, dy$$

$$= \frac{3}{400} \int_0^{20} y \left[x\right]_{y^2/20}^{\sqrt{20y}} dy$$

$$= \frac{3}{400} \int_0^{20} y \left(\sqrt{20y} - y^2/20\right) dy$$

$$= \frac{3}{400} \int_0^{20} \left[\sqrt{20}\, y^{3/2} - \frac{y^3}{20}\right] dy$$

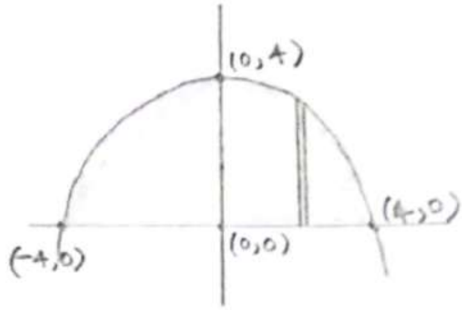
$$= \frac{3}{400} \left[\sqrt{20} \frac{y^{5/2}}{5/2} - \frac{y^4}{80}\right]_0^{20}$$

$$= \frac{3}{400} \left[\frac{\sqrt{20}\,(20)^2\sqrt{20}}{5} \times 2 - \frac{(20)^4}{80}\right]$$

$$= \underline{\underline{9}}$$

$\therefore$ Center of gravity $(\bar{x}, \bar{y}) = (9, 9)$.

4. Calculate the centroid of the area bounded by the parabola $x^2+4y-16=0$ and x-axis.



Soln $y : 0 \to \frac{16-x^2}{4}$; $x : -4 \to 4$

$$M = \iint_R dA = \int_{-4}^{4}\int_{0}^{\frac{16-x^2}{4}} dx\,dy$$

$$= \int_{-4}^{4} [y]_0^{\frac{16-x^2}{4}} dx$$

$$= \int_{-4}^{4}\left(4-\frac{x^2}{4}\right)dx = \left[4x-\frac{x^3}{12}\right]_{-4}^{4}$$

$$= 16-\frac{64}{12}+16-\frac{64}{12}$$

$$= \frac{64}{3}$$

$$\bar{x} = \frac{1}{M}\int_{-4}^{4}\int_{0}^{\frac{16-x^2}{4}} x\,dy\,dx = \frac{3}{64}\int_{-4}^{4}[xy]_0^{\frac{16-x^2}{4}}dx$$

$$= \frac{3}{64}\int_{-4}^{4} x\left(4-\frac{x^2}{4}\right)dx = \frac{3}{64}\left[\frac{4x^2}{2}-\frac{x^4}{16}\right]_{-4}^{4}$$

$$= \frac{3}{64}\left\{\left[\frac{64}{2}-\frac{64\times4}{16}\right]-\left[\frac{64}{2}-\frac{64\times4}{16}\right]\right\} = 0$$

(or) Since the region is symmetric about y-axis, $\bar{x}=0$

$$\bar{y} = \frac{1}{M}\int_{-4}^{4}\int_{0}^{\frac{16-x^2}{4}} y\,dy\,dx = \frac{3}{64}\int_{-4}^{4}\left[\frac{y^2}{2}\right]_0^{\frac{16-x^2}{4}}dx$$

$$= \frac{3}{64}\int_{-4}^{4}\frac{(16-x^2)^2}{32}dx = \frac{3}{64}\int_{-4}^{4}\frac{(256+x^4-32x^2)\,dx}{32}$$

$$= \frac{3}{64\times32}\left[256x+\frac{x^5}{5}-32\frac{x^3}{3}\right]_{-4}^{4} = \frac{3\times2}{64\times32}\left[256(4)+\frac{4^5}{5}-32\frac{(4)^3}{3}\right]$$

$$= 8/5$$

$\therefore$ Center of gravity $(\bar{x},\bar{y}) = (0, 8/5)$