FUNDAMENTALS OF LINEAR ALGEBRA, CALCULUS & DIFFERENTIAL EQUATIONS (MAT211BT)

UNIT-V

LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Differential equations arise in an attempt to describe physical phenomena in mathematical terms. A single differential equation can serve as a mathematical model for many different processes. In the study of mechanics – spring/mass systems, the free, underdamped and critically damped motion can be expressed as linear differential equation. Similar to this, LRC series circuits can be described by a linear second- order differential equation.

Differential Equation:
An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be differential equation.
The ‘order’ of a differential equation is the order of the highest derivative in the equation and the power of highest order derivative after clearing fractional powers from derivatives in the equation gives the ‘degree’ of differential equation.

Classification of differential equation:
Differential equations are classified as:
i) Ordinary differential equation (ODE)
ii) Partial differential equations (PDE)

If the differential equation contains ordinary derivatives of one or more dependent variables with respect to single independent variable, equation is known as ODE.

Examples:

i) $\ddot{x} = -n^2 x$

ii) $\frac{[1+(y\prime)^2]^{3/2}}{y"} = a$

If the differential equation contains partial derivatives of one or more dependent variables of two or more independent variables, equation is known as PDE.

Examples:

i) $u_{xx} + u_{yy} = 0$

ii) $c^2 u_{xx} = u_t$

Linear Differential Equation:

A differential equation is said to be linear if dependent variable, all its derivatives are of first degree and they are not multiplied together.

A non-linear differential equation is one that is not linear.

Linear ODE of higher order:

A linear ODE of n[th] order is of the form:

$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = g(x)$ ... (1),

where $a_0, a_1, \cdots a_n$ are either constants or function of 'x' alone, $D = \frac{d}{dx}$ differential operator and $g(x)$ is a known function of 'x'.

Let $F(D) = (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)$ be a differential operator, then above equation can be written as:

$$F(D) y = g(x) \qquad \text{... (2)}$$

If $g(x) = 0$, equation is said to be 'homogeneous' otherwise it is known as 'non-homogenous'. If $y_1, y_2, \cdots, y_n$ are linearly independent solutions of the nth order homogeneous differential equation then $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, where $c_1, c_2, \cdots, c_n$ are arbitrary constants is also a solution of homogeneous equation.

If y_p be a particular function which satisfies non-homogeneous linear differential equation then the general solution of the equation (2) is given by:

$$y = y_c + y_p$$

Where $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is called as complementary function and y_p is called particular integral.

Homogeneous Linear ODE with constant coefficients:

Consider a n[th] order homogeneous ODE,

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)y = 0$$

Let $y = e^{mx}$ be a trial solution, obtaining the derivatives and using above equation we obtain:

$$e^{mx}(a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n) = 0$$

Since $e^{mx} \neq 0$, above equation vanishes if 'm' is root of algebraic equation. This algebraic equation is known as 'auxiliary' or 'characteristic' equation. The general solution of homogeneous equation depends on nature of roots of this equation.

Consider the second order ODE of the form:

$$(a_0D^2 + a_1D + a_n)y = 0$$

The auxiliary equation is given by $a_0m^2 + a_1m + a_n = 0$.

Case (i) Let the roots of auxiliary equation be real and distinct say $m = m_1,\ m_2$. The general solution of above equation is given by

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

Case (ii) Let the roots of auxiliary equation be real and repeated say $m = m_1,\ m_1$. The general solution of above equation is given by
$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

Case (iii) Let the roots of auxiliary equation be complex say $m = \alpha + i\beta, m = \alpha - i\beta$ .The general solution of above equation is given by $y = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x)$

Case (iv) If the ODE is fourth order and the roots of auxiliary equation are complex repeated
say $m = \alpha \pm i\beta, m = \alpha \pm i\beta$ .The general solution of equation is given by

$$y = e^{\alpha x}[(c_1 + c_2 x)cos\beta x + (c_3 + c_4 x)sin\beta x]$$

The above procedure can be extended to higher order linear homogenous ODE in a similar manner.

Examples:

1. Solve: $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$

Solution: The auxiliary equation is given by $2m^2 - 5m - 3 = 0$

The roots are $m = -\frac{1}{2}, m = 3$. The roots are real and distinct the general solution of given ODE is

$y = c_1 e^{-(\frac{1}{2})x} + c_2 e^{3x}$

2. Solve: $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

Solution: The auxiliary equation is given by $(m - 5)^2 = 0$

The roots are $m = 5, 5$ as the roots are real and repeated the general solution of given ODE is $y = c_1 e^{5x} + c_2 x e^{5x}$

3. Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 7y = 0$

Solution: The auxiliary equation is given by $m^2 + 4m + 7 = 0$

The roots are $m = -2 \pm \sqrt{3}i$ which are complex, hence the general solution is given by

$$y=e^{-2x}[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)]$$

4. Solve: $(D^3 + 1)y = 0$

Solution: The auxiliary equation is given by $m^3 + 1 = 0$

The roots are $m = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, hence the general solution is given by

$$y=c_1 e^{-x} + e^{\left(\frac{1}{2}\right)x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

5. Solve: $(D^4 + 8D^2 + 16)y = 0$

Solution: The auxiliary equation is given by $m^4 + 8m^2 + 16 = 0$, $(m^2 + 4)^2 = 0$

The roots are complex repeated $m = \pm 2i, \pm 2i$, hence the general solution is given by

$$y = [(c_1 + c_2 x)\cos(2x) + (c_3 + c_4 x)\sin(2x)]$$

6. Solve the initial value problem $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, given that $x(0) = 0, \frac{dx}{dt}(0) = 15$.

Solution: The auxiliary equation is given by $m^2 + 5m + 6 = 0$

The roots are $m = -3, -2$, hence the general solution is given by

$$x = c_1 e^{-3t} + c_2 e^{-2t}$$

Using the given initial conditions, we find that $c_1 + c_2 = 0$ and $2c_1 + 3c_2 = -15$

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Solving these we get $c_1 = 15$ and $c_2 = -15$.

The particular solution of the given differential equation is

$$x = 15(e^{-3t} - e^{-2t})$$

Exercise:

1 Solve $\left(D^3 - 8\right)y = 0$, where $D \equiv \frac{d}{dx}$.

Answer: $y = c_1e^{2x} + e^{-x}\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$.

2 Solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0;\ \ y(0) = 0,\ \frac{dy(0)}{dt} = 15.$

Answer: $y = 15\left(e^{-2t} - e^{-3t}\right)$.

3 Solve $\left(D^4 - 4D^3 + 8D^2 - 8D + 4\right)y = 0$, where $D \equiv \frac{d}{dx}$.

Answer: $y = e^x\left[\left(c_1 + c_2x\right)\cos x + \left(c_3 + c_4x\right)\sin x\right]$.

4 Solve $\left(D^3 - D^2 - D + 1\right)y = 0$, where $D \equiv \frac{d}{dx}$.

Answer: $y = \left(c_1 + c_2x\right)e^x + c_3e^{-x}$.

5 Solve $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$.

Answer: $y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x$.

6 Find the solution of the boundary value problem $y'' + y = 0$ given that $y(0) = 1,\ y\left(\pi/2\right) = 2$.

Answer: $y = \cos x + 2\sin x$.

7 Find the solution of the initial value problem:

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0;\ \ y(0)=0,\ \frac{dy(0)}{dt}=15.$$

Answer: $y=15\left(e^{-2t}-e^{-3t}\right)$.

Non-homogeneous Linear ODE with constant coefficients:

The general solution of non-homogeneous linear ODE has two parts – Complementary function and Particular integral

The complementary function is solution of homogeneous ODE

The rules for finding particular integral are given below:

The kinds of functions that make up the input function $g(x)$are generally constants, exponentials, sine, cosine and polynomials.

$$\text{Consider } F(D)y=g(x),$$

If $y = y_p$ is a particular integral of this ODE, then

$$F(D)y_p=g(x)$$

$$\text{then } y_p=\frac{1}{F(D)}g(x),$$

where $\frac{1}{F(D)}$ is called as 'inverse differential operator'.

Rules for finding particular integral:

i) Let $g(x) = ke^{ax}$

$$\text{then } y_p = k\frac{1}{F(D)}e^{ax} = k\frac{1}{F(a)}e^{ax}, \text{ provided } F(a) \neq 0$$

If $F(a) = 0$ then $y_p = k\frac{x}{[F'(D)]_{D=a}}e^{ax}$, provided $F'(a) \neq 0$

If $F'(a) = 0$ then $y_p = k\frac{x^2}{[F''(D)]_{D=a}}e^{ax}$, provided $F''(a) \neq 0$ and so on

ii) Let $g(x) = \sin(ax + b)\ or \cos\,(ax + b)$

$$y_p = \frac{1}{F(D^2)}\sin(ax + b)\ or\ \frac{1}{F(D^2)}\cos(ax + b)$$
$$= \frac{1}{F(-a^2)}\sin(ax + b)\ or\ \frac{1}{F(-a^2)}\cos\,(ax + b),$$

provided $F(-a^2) \neq 0$

If $F(-a^2) = 0$, $y_p = \frac{x}{F'(-a^2)}\sin(ax + b)\ or\ \frac{x}{F'(-a^2)}\cos\,(ax + b)$, provided $F'(-a^2) \neq 0$

Particular case: $\frac{1}{(D^2+a^2)}\cos\,(ax + b) = \frac{x}{2a}\sin\,(ax + b)$

$$\frac{1}{(D^2+a^2)}\sin(ax + b) = -\frac{x}{2a}\cos\,(ax + b)$$

iii) Let $g(x) = x^m$ then $y_p = \frac{1}{F(D)}x^m = [F(D)]^{-1}x^m$

Expanding the right hand side as a binomial series, the particular integral can be obtained. The following series expansions are useful:

$$(1 + D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$
$$(1 - D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$
$$(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$
$$(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

iv) Let $g(x) = e^{ax}V(x)$ where $V(x)$ is function of 'x'

$$y_p = \frac{1}{F(D)}e^{ax}V(x) = e^{ax}\frac{1}{F(D+a)}V(x)$$

The right hand side can be evaluated by earlier methods.

Examples:

1. Solve: $(D - 2)^2y = 8(e^{2x} + sin2x + x^2)$

Solution: The auxiliary equation is $(m-2)^2 = 0$, the roots are $m = 2, 2$

$$\therefore y_c = c_1 e^{2x} + c_2 x e^{2x}$$

The particular integral is obtained as follows: $y_p = 8\left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} sin2x + \frac{1}{(D-2)^2} x^2\right]$

Now $\frac{1}{(D-2)^2} e^{2x} = x^2 \frac{1}{[2(1-0)]_{D=2}} e^{2x} = \frac{x^2 e^{2x}}{2}$

$$\frac{1}{(D-2)^2} sin2x = \frac{1}{(D^2-D+4)} sin2x = \frac{1}{-4-4D+4} \sin 2x$$

$$=-\frac{1}{4D}\sin 2x=-\frac{1}{4}\int \sin(2x)\,dx=\frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}x^2=\frac{1}{4}\left(-\frac{D}{2}\right)^{-2}x^2=\frac{1}{4}\left(1+\frac{2D}{2}+\frac{3D^2}{4}+\ldots\right)x^2=\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

$$\therefore y_p = 4x^2 e^{2x} + cos2x + 2x^2 + 4x + 3$$

Hence, the general solution of given equation is $y = y_c + y_p$

$$y = c_1 e^{2x} + c_2 x e^{2x} + 4x^2 e^{2x} + cos2x + 2x^2 + 4x + 3$$

2. Solve: $y'' - 4y = \cosh(2x-1) + 3^x$

Solution: The auxiliary equation and its roots are: $m^2 - 4 = 0$ $m = +2, -2$

$$\therefore y_c = c_1 e^{-2x} + c_2 e^{2x}$$

The particular integral is obtained as follows:

$$y_p = \frac{1}{(D^2-4)}\cosh(2x-1) + \frac{1}{(D^2-4)}3^x$$

Now $\frac{1}{(D^2-4)}\cosh(2x-1) = \frac{1}{2}\left[\frac{e^{2x-1}+e^{-2x+1}}{D^2-4}\right] = \frac{1}{8}(xe^{2x-1} - xe^{-2x+1})$

$$\frac{1}{(D^2-4)}3^x = \frac{1}{(D^2-4)}e^{x\log(3)} = \frac{1}{[(\log 3)^2-4]}3^x$$

Hence, the solution of given equation is $y = y_c + y_p$

$$\therefore y=c_1e^{-2x}+c_2e^{2x}+\frac{1}{8}(xe^{2x-1}-xe^{-2x+1})+\frac{1}{[(\log 3)^2-4]}3^x$$

3. Solve: $y'' - 2y' + y = xe^x \sin x$

Solution: The auxiliary equation and its roots are: $m^2 - 2m + 1 = 0$

$m = 1, 1$

$$\therefore y_c = c_1e^x + c_2xe^x$$

The particular integral is obtained as follows:

$$y_p = \left[\frac{1}{(D-1)^2}\right]e^x x\sin x = e^x\frac{1}{[D+1-1]^2}x\sin x = e^x\frac{1}{D^2}x\sin x$$

Integrating right hand side twice by using the rule of parts we obtain:

$$y_p = -e^x(x\sin x + 2\cos x)$$

Hence, the solution of given equation is $y = y_c + y_p$

$$\therefore y = c_1e^x + c_2xe^x - e^x(x\sin x + 2\cos x)$$

4. Solve: $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$

Solution: The auxiliary equation and its roots are:

$m^3 + 3m^2 + 3m + 1 = 0$

$$m = -1, -1, -1$$
$$\therefore y_c = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}$$

Now $\frac{1}{(D+1)^3}e^{-x} = x^2\frac{1}{[6(D+1)]_{D=-1}}e^{-x} = \frac{x^3e^{-x}}{6}$

Hence, the solution of given equation is $y = y_c + y_p$

$$\therefore y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + \frac{x^3e^{-x}}{6}$$

5. Find the particular integral of $(D^3+1)y = \cos(2x-1)$.

Solution: The particular integral is given by:

$$y_p = \left[\frac{1}{D^3+1}\right]\cos(2x-1) = \frac{1}{D(-4)+1}\cos(2x-1)$$
$$= \frac{(1+4D)}{1-16D^2}\cos(2x-1)$$
$$y_p = \frac{(1+4D)}{1-16(-4)}\cos(2x-1) = \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$$

6. Find the particular integral of $(D^3+4D)y = 2\sin x\cos x$.

Solution: The particular integral is given by:

$$y_p = \left[\frac{1}{D^3+4D}\right]2\sin x\cos x = \frac{1}{(D^2+4)D}\sin(2x)$$
$$y_p = x\frac{1}{3D^2+4}\sin(2x) = x\frac{1}{3(-4)+4}\sin(2x) = -\frac{x}{8}\sin(2x)$$

7. Find the particular integral of $(D^2+9)y = x\cos x$

Solution: The particular integral is given by:

$$y_p = \frac{1}{D^2+9} Re\left(xe^{ix}\right)$$

$$= Re \frac{1}{D^2+9}\left(xe^{ix}\right)$$

$$= Re\ e^{ix} \frac{1}{(D+i)^2+9}(\mathrm{x})$$

$$y_p = Re\ e^{ix} \frac{1}{D^2+2iD+8}(\mathrm{x})$$

$$= \ Re\ e^{ix} \frac{1}{8\left(1+\frac{iD}{4}\right)}(\mathrm{x})$$

$$= Re \frac{e^{ix}}{8}\left(1+\frac{iD}{4}\right)^{-1}(\mathrm{x})$$

$$y_p = Re \frac{e^{ix}}{8}\left(1-\frac{iD}{4}\right)(\mathrm{x})$$

$$= Re \frac{\cos x + i\sin x}{8}\left(x-\frac{i}{4}\right)$$

$$= \frac{1}{32}(4x\cos x + \sin x\ )$$

8. Find the particular integral of $(D^2-4D+4)y = x^2e^{3x} + \sin^2 x$

Solution: The particular integral is given by:

$$y_p = \frac{1}{(D-2)^2}[x^2e^{3x} + \sin^2 x]$$

$$= \frac{1}{(D-2)^2}[x^2e^{3x}] + \frac{1}{(D-2)^2}[\sin^2 x]$$

Now, $\frac{1}{(D-2)^2}[x^2e^{3x}]$

$$= e^{3x}\frac{1}{(D+3-2)^2}[x^2]$$

$$= e^{3x}\frac{1}{(D+1)^2}[x^2]$$
$$= e^{3x}(1+D)^{-2}[x^2]$$
$$= e^{3x}(1-2D+3D^2)[x^2]$$
$$= e^{3x}(x^2-4x+6)$$

and

$$\frac{1}{(D-2)^2}[\sin^2 x]$$
$$= \frac{1}{(D-2)^2}\left[\frac{1-\cos 2x}{2}\right]$$
$$= \frac{1}{(D-2)^2}\left[\frac{1}{2}e^{0x}\right] - \frac{1}{(D-2)^2}\left[\frac{1}{2}\cos 2x\right]$$
$$= \frac{1}{(0-2)^2}\left[\frac{1}{2}e^{0x}\right] - \frac{1}{D^2-4D+4}\left[\frac{1}{2}\cos 2x\right]$$
$$= \frac{1}{8} - \frac{1}{2}.\frac{1}{2^2-4D+4}[\cos 2x]$$
$$= \frac{1}{8} + \frac{1}{8D}[\cos 2x] = \frac{1}{8} + \frac{1}{16}\sin 2x$$
$$\therefore y_p = e^{3x}(x^2-4x+6) + \frac{1}{8} + \frac{1}{16}\sin 2x$$

Exercise:

1. Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2e^{2t}$.

   Answer: $y = (c_1 + c_2 t)e^{-t} + \frac{2}{9}e^{2t}$

2. Solve $y''' - 12y' + 16y = (e^x + e^{-2x})^2$.

   Answer: $y = (c_1 + c_2 x)e^{2x} + c_3 e^{-4x} + \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}$

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3. Solve $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=\cos 2x$.

   Answer: $y=ae^{x}+(b+cx)e^{-x}-\frac{1}{25}(2\sin 2x+\cos 2x)$

4. Solve $\left(D^3-D^2-6D\right)x=1+t^2$.

   Answer: $x=a+be^{3t}+ce^{-2t}-\frac{1}{108}(25t-3t^2+6t^3)$

5. Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{2x}\sin x$.

   Answer: $y=ae^{-x}+be^{-2x}-\frac{1}{170}e^{2x}(7\cos x-11\sin x)$

6. Solve $(D^4+4)y=\sin^2 x$.

   Answer: $y=a\cos 2x+b\sin 2x+\frac{1}{8}(1-x\sin 2x)$

7. Solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=e^{3x}(x^2+7x+5)$.

   Answer: $y=\left(a+bx+\frac{1}{12}(x^4+14x^3+30x^2)\right)e^{3x}$

8. Solve $\frac{d^2y}{dx^2}-4y=x\sinh x$.

   Answer: $y=ae^{2x}+be^{-2x}-\frac{x}{3}\sinh x-\frac{2}{9}\cosh x$

9. Solve $(D^2-4D+4)y=8x^2e^{2x}\sin 2x$.

   Answer: $y=\left(a+bx(3-2x^2)\sin 2x-4x\cos 2x\right)e^{2x}$

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10. Solve $y'' - y = x\sin x + (1 + x^2)e^x$.

Answer: $y = c_1e^{-x} + c_2e^x - \frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^x}{12}\left(\frac{3x}{2} - \frac{x^2}{2} + \frac{x^3}{3}\right)$

Method of Variation of Parameters:

The method of variation of parameter is more general method, in which the function g(x) is not restricted to any particular form. The working rule for finding solution of the given ODE is given below:

Consider the second order ODE of the form $y'' + P(x)y' + Q(x)y = g(x)$

Let $y = c_1y_1 + c_2y_2$ be solution of the equation with $g(x) = 0$, the complete solution of given ODE is $y = A(x)y_1 + B(x)y_2$, where

$$A(x) = -\int \frac{y_2 g(x)}{W} dx + k_1$$

$$B(x) = \int \frac{y_1 g(x)}{W} dx + k_2$$

With the 'Wronskian' given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

Examples:

1. **Solve:** $y'' + a^2y = \sec(ax)$

Solution: The auxiliary equation and its roots are $m^2 + a^2 = 0$, $m = \pm ai$

$$\therefore y_c = c_1\cos(ax) + c_2\sin(ax)$$

Let $y_1 = \cos(ax)$, $y_2 = \sin(ax)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = a[\cos^2(ax) + \sin^2(ax)] = a$$

$$A(x)=-\int \frac{y_2 g(x)}{W}dx+k_1=-\int \frac{\tan(ax)}{a}dx+k_1=-\frac{1}{a^2}\log[\cos(ax)]+k_1$$

$$B(x)=\int \frac{y_1 g(x)}{W}dx+k_2=\frac{1}{a}\int dx+k_2=\frac{x}{a}+k_2$$

The complete solution of given ODE is $y = A(x)y_1 + B(x)y_2$,

$$\therefore y=k_1\cos(ax)+k_2\sin(ax)-\frac{1}{a^2}\cos(ax)\log[\cos(ax)]+\frac{x}{a}\sin(ax)$$

2. Solve: $y'' - y = e^{-2x}\cos(e^{-x})$

Solution: The auxiliary equation and its roots are $m^2 - 1 = 0$, $m = \pm 1$

$$\therefore y_c = c_1 e^{-x} + c_2 e^{x}$$

$$\text{Let } y_1 = e^{x},\ y_2 = e^{-x}$$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=-2$$

$$A(x)=-\int \frac{y_2 g(x)}{W}dx+k_1$$
$$=-\frac{1}{2}\int z^2\cos(z)\,dz+k_1,$$
$$\textit{where } z = e^{-x}, \text{and } dz = -e^{-x}dx$$

Integrating by rule of parts obtain:

$$A(x)=-\frac{1}{2}e^{-2x}\sin(e^{-x})-e^{-x}\cos(e^{-x})+\sin(e^{-x})+k_1$$

$$B(x)=\int \frac{y_1 g(x)}{W}dx+k_2$$

$$=\frac{1}{2}\int \cos(z)dz+k_2=\frac{1}{2}\sin(e^{-x})+k_2\text{, where } z=e^{-x}$$

The complete solution of given ODE is $y = A(x)y_1 + B(x)y_2$

$$\therefore y= k_1e^{-x}+k_2e^{x}-\cos(e^{-x})+e^{x}\sin(e^{-x})$$

Exercise:

1. Using the method of variation of parameters, solve $(D^2+4)x=\tan 2t$.

   Answer: $x=a\cos 2t+b\sin 2t-\frac{1}{4}\cos 2t\log(\sec 2t+\tan 2t)$

2. Solve $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$ by the method of variation of parameters.

   Answer: $y=ae^x+be^{-x}-1+e^x\log(e^{-x}+1)-e^{-x}\log(e^x+1)$.

3. Solve $y''-2y'+y=e^x\log x$ by the method of variation of parameters.

   Answer: $y=(a+bx)e^x+\frac{1}{4}x^2e^x(2\log x-3)$.

Cauchy-Euler equation:

The linear ODE of the form

$$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + a_2x^{n-2}D^{n-2} + \cdots + a_{n-1}xD + a_n)y = g(x)$$

where $a_0, a_1, \cdots a_n$ are constants, is known as 'Cauchy-Euler' or equidimensional equation.

This equation can be reduced to ODE with constant coefficients by changing the independent variable as follows –

$$\text{Take } x = e^z, \text{ then } xDy = D_1 y,$$
$$x^2D^2y = D_1(D_1 - 1)y\ ,$$
$$x^3D^3y = D_1(D_1 - 1)(D_1 - 2)y$$
$$\text{where } D_1 = \frac{d}{dz} \text{ and so on.}$$

The resulting ODE can be solved using the earlier methods.

Examples:

1. Solve: $4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$

Solution: The given ODE is second order Euler equation.

Put $x = e^z$, then $xDy = D_1y$, $x^2D^2y = D_1(D_1 - 1)y$

The given equation reduces to $({D_1}^2 + 4D_1 + 1)y = 0$

The auxiliary equation and its roots are $m^2 + 4m + 1 = 0$,

$$m = -\frac{1}{2}, -\frac{1}{2}$$

The general solution of given ODE is $y = c_1e^{-(\frac{1}{2})z} + c_2ze^{-(\frac{1}{2})z}$, where $z = \log(x)$

2. **Solve**: $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log(x)$

Solution: The given ODE is second order Euler equation.

Put $x = e^z$, then $xDy = D_1y$, $x^2D^2y = D_1(D_1 - 1)y$

The given equation reduces to $({D_1}^2 - 2D_1 + 1)y = z$

The auxiliary equation and its roots are $m^2 - 2m + 1 = 0$,

$$m = 1, 1$$
$$\therefore y_c = c_1e^z + c_2ze^z$$

Now $y_p = \frac{1}{(D_1-1)^2}z = (1 - D_1)^{-2}z = \left(1 + 2D_1 + 3{D_1}^2 + \ldots\right)z = z + 2$

FUNDAMENTALS OF LINEAR ALGEBRA, CALCULUS & DIFFERENTIAL EQUATIONS (MAT211BT)

Hence, the solution of given equation is $y = y_c + y_p$

$$y = c_1 e^z + c_2 z e^z + z + 2, \text{ where } z = \log(x)$$

Exercise:

1. Solve $4x\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - \frac{y}{x} = 4x.$

   Answer: $y = a\sqrt{x} + b\frac{1}{\sqrt{x}} + \frac{4}{15}x^2$

2. Solve $x^3\frac{d^3y}{dx^3} + 2x\frac{dy}{dx} - 2y = x^2\log x + 3x$

   Answer: $y = ax + x(b\cos\log x + c\sin\log x) - \frac{1}{2}x^2 + 3x\log x$

3. Solve $\left(x^2D^2 + xD + 9\right)y = 3x^2 + \sin(3\log x)$, where $D \equiv \frac{d}{dx}$.

   Answer: $y = a\cos(3\log x) + b\sin(3\log x) + \frac{3}{13}x^2 - \frac{1}{6}\log x\cos(3\log x).$

FUNDAMENTALS OF LINEAR ALGEBRA, CALCULUS & DIFFERENTIAL EQUATIONS (MAT211BT)

Applications of differential equations:

A flexible spring is suspended vertically from a rigid support and then a mass is attached to its free end. Assuming that the mass vibrates, free of other external forces the displacement describes SHM. Fig (1)

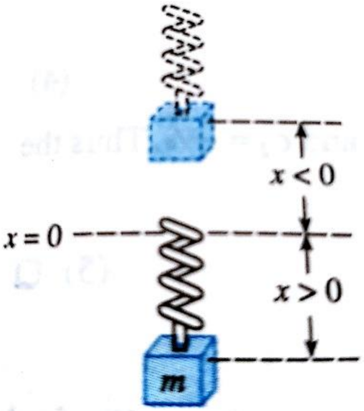

Fig (1)

1. The displacement 'x' of a particle executing simple harmonic motion (SHM) is given by $\ddot{x} = -n^2 x$. Where n^2, is a force constant. Solve this equation under the initial conditions $x = a$ and $\frac{dx}{dt} = 0$, at $t = 0$.

Solution: The given ODE is $({D_1}^2 + n^2)x = 0$, where $D_1 = \frac{d}{dt}$

The auxiliary equation and its roots are $m^2 - n^2 = 0$

$$m = \pm ni$$

The general solution is given by $x = c_1 \cos(nt) + c_2 \sin(nt)$, using the given conditions $x = a$ and $\frac{dx}{dt} = 0$, at $t = 0$ we find that $c_1 = a$ and $c_2 = 0$

The displacement at any instant of time is given by $x = a\cos(nt)$

2. The charge $q = q(t)$ on the capacitor in a LRC series circuit (Fig (2)) is governed by the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$. Find the charge on the capacitor $q = q(t)$,

when L=0.25h, R=10Ω, C=0.001f, E(t) = 0, $q(0) = q_0$, $\frac{dq}{dt} = 0, for\ t = 0$.

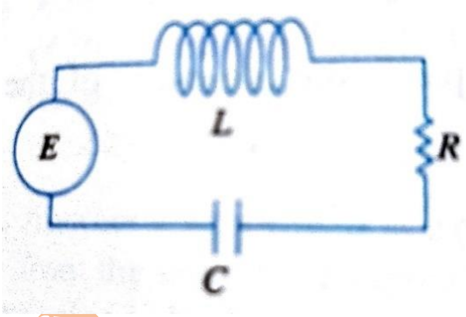

Fig (2)

Solution: The given ODE is $\frac{1}{4}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 1000q = 0$

or $\frac{d^2q}{dt^2} + 40\frac{dq}{dt} + 4000q = 0$

The auxiliary equation and its roots are $m^2 + 40m + 4000 = 0$,

$$m = -20 \pm 60i$$

The general solution is given by $q = q(t)$

$$= e^{-20t}[c_1\cos(60t) + c_2\sin(60t)]$$

Using the given conditions, we get $c_1 = q_0$ and $c_2 = q_0/3$

The charge on the capacitor is given by:

$q = q(t) = e^{-20t}[q_0 \cos(60t) + \frac{q_0}{3} \sin(60t)]$

3. Find the charge on the capacitor $q = q(t)$, in a LRC series circuit from $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

when L=(5/3)h, R=10Ω, C=(1/30)f, E(t) = 300V, $q(0) = 0$, $\frac{dq}{dt} = 0$

Solution: The given ODE is $\frac{5}{3}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 30q = 300$ or

$$5\frac{d^2q}{dt^2} + 30\frac{dq}{dt} + 90q = 900$$

The auxiliary equation and its roots are $5m^2 + 30m + 90 = 0$,

$$m = -3 \pm 3i$$

$$q_p = \left[\frac{1}{5D_1^2 + 30D_1 + 90}\right] 900e^{0t} = \frac{900}{90} = 10$$

The general solution is given by:

$q = q(t) = e^{-3t}[c_1 \cos(3t) + c_2 \sin(3t)] + 10$

Using the given conditions, we get $c_1 = -10$ and $c_2 = -10$

The charge on the capacitor is given by:

$q = q(t) = 10 - e^{-20t}[\cos(3t) + \sin(3t)]$

Video links:

https://www.youtube.com/watch?v=V9bl02Ffo_o