Exercise:

1) Find the total derivative of u wrt 't' when $u = e^x \sin y$, where $x = \log t$, $y = t^2$ Ans: $\frac{du}{dt} = \sin t^2 + 2t^2 \cdot \cos t^2$

2) If $u = e^x \sin(yz)$, where $x = t^2$, $y = t-1$, $z = \frac{1}{t}$, find $\frac{du}{dt}$ at $t = 1$. Ans: $\frac{du}{dt} = e$

3) Find $\frac{du}{dt}$ if $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ Ans: $\frac{du}{dt} = 4e^{2t}$

4) If $z = f(u,v)$, where $u = x^2 - y^2$, $v = 2xy$, P.T.

a) $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2+y^2)\frac{\partial z}{\partial u}$

b) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

5) If $u = e^x \sin y$, $v = e^x \cos y$ and $w = f(u,v)$, P.T.
$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right)$

6) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, P.T. $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

7) If $u = f(r,s,t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$ and $t = \frac{z}{x}$, S.T.
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

8) If $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$ Ans: $\frac{2y}{x}$

9) If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

((r,θ,z) defined above are cylindrical polar coordinates of the point (x, y, z) in space) Ans: r

10) If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, S.T.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (x-y)(y-z)(z-x)$$

11) If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$, S.T. $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

12) If $u = z - x$, $v = y - z$, $w = x + y + z$, find Jacobians

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} \quad \text{and} \quad J' = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$ Also verify $JJ' = 1$.

13) Determine the extreme values of following functions $f(x, y)$:

a) $2xy - 5x^2 - 2y^2 + 4x + 4y - 6$ Ans: $\left(\frac{2}{3}, \frac{4}{3}\right)$ is point of maximum and $f\left(\frac{2}{3}, \frac{4}{3}\right) = -2$

b) $x^3 y^2 (1 - x - y)$ Ans: $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$ are critical points, $\left(\frac{1}{2}, \frac{1}{3}\right)$ is point of maximum and $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$

c) $x^3 + y^3 - 3xy$ Ans: $(0, 0)$ and $(1, 1)$ are critical points, $(0, 0)$ is a saddle point, $(1, 1)$ is a point of minimum and $f(1, 1) = -1$

d) $x^4 + 2x^2 y - x^2 + 3y^2$ Ans: $(0, 0)$, $\left(\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$ are critical points, $(0, 0) \rightarrow$ saddle point, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) \rightarrow$ minimum point,

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) \rightarrow$ minimum point, $f\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) = f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) = -\frac{3}{8}$

e) $2x^3 + x^2y + 5x^2 + y^2$ Ans: min. value $= 0$ at $(0, 0)$, max. value $= \frac{125}{7}$ at $\left(-\frac{5}{3}, 0\right)$ and $(-1, 2), (-1, -2) \rightarrow$ saddle points

f) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ Ans: max. value $= 112$ at $(4, 0)$

14) A rectangular box without a lid is to be made from $12\,m^2$ cardbox. Find the maximum value of such a box using Lagrange's method of multipliers

Ans: $V(2, 2, 1) = 4\,m^3$

15) Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$ Ans: Max. value is $f(0, \pm 1) = 2$, Min. value $f(\pm 1, 0) = 1$.

16) Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$

Ans: Closest point is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$, Farthest point is $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

17) A rectangular box open at the top is to have volume of 32 cubic feet. Find dimensions of box requiring least material for construction.

Ans: Minimum surface area, represents least material required for construction and dimensions are $x = y = 4$ ft, $z = 2$ ft