UNIT-5

LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Differential equations arise in an attempt to describe physical phenomena in mathematical terms. Linear ordinary differential equations are the most important one because of their applications in the study of electrical, mechanical and other linear systems. In fact such equations play a dominant role, in unifying the theory of electrical and oscillatory systems, eg: SHM, simple pendulum, oscillations of a spring, oscillatory electrical circuit, electro-mechanical analogy, deflections of beams etc. LCR series circuits can be described by a linear second-order differential equation.

* Differential Equation :

An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables, is said to be differential equation.

The order of a differential equation is the order of the highest derivative appearing in it.

The degree of a differential equation is the degree of the highest derivative occuring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Eg: (i) $\frac{d^2x}{dt^2} + n^2x = 0 \rightarrow$ Order - 2, Degree - 1

(ii) $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$

$\Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x \rightarrow$ order : 1, Degree : 2

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2} = c$

$\Rightarrow \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2 \quad \rightarrow$ Order : 2
Degree : 2

* **Classification of differential equations :**

Differential equations are classified as :

(i) Ordinary differential equations (ODE)

(ii) Partial differential equations (PDE)

If the differential equation contains ordinary derivatives of one or more dependent variables with respect to single independent variable, equation is known as ODE.

(or)

An ordinary differential equation is that in which all the differential coefficients have reference to a single independent variable.

Ex : (i) $e^x\,dx + e^y\,dy = 0$

(ii) $\frac{[1+(y')^2]^{3/2}}{y''} = c$

If the differential equation contains partial derivatives of one or more dependent variables of two or more independent variables, equation is known as PDE.

(or)

A PDE is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.

Ex : (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ (ii) $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

* Linear Differential Equation :

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

An ordinary linear differential equation of order n is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \quad \rightarrow (1)$$

where, $a_1, a_2, \dots, a_{n-1}, a_n$ and $f(x)$ are functions of x only. (constant also)

If $f(x)=0$, then the equation (1) is called a homogenous equation. If $f(x) \neq 0$, it is a non-homogenous equation.

* Operators :

The operator method is an important and time saving tool of solving many differential equations.

Denoting $D \equiv \frac{d}{dx}$, we define $Dy = \frac{dy}{dx}$, $D^2 y = \frac{d^2 y}{dx^2}, \dots, D^n y = \frac{d^n y}{dx^n}$

Now, in terms of the operators equation (1) can be written as

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = f(x)$$

i.e., $F(D) y = f(x)$ so that

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) \equiv F(D)$ is the operator now.

The indices can also be negative. We define $D^{-1} y = \frac{1}{D} y = z$ so that $Dz = y$.

In other words, an operator with a negative indese is equivalent to integration.

* $DD^{-1} = 1$

Homogenous linear ODE with constant coefficients :

If $y_1, y_2, \dots, y_n$ are 'n' linearly independent solutions of $F(D)y = 0$, where $F(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n$, then $c_1y_1 + c_2y_2 + \dots + c_ny_n$ is also a solution of $F(D)y = 0$. The general solution of the n^{th} order differential equation consists of 'n' arbitrary constants $c_1, c_2, \dots, c_n$.

Denoting $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$, we have $F(D)y_c = 0$.
y_c is called the complimentary function (CF).

Method of finding the complimentary function :

Consider a homogenous differential equation of order 'n'.

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n = 0$$

i.e., $(D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n)y = 0$ or $F(D)y = 0 \longrightarrow (*)$

Taking $y = e^{mx}$, we have $Dy = me^{mx}$, $D^2y = m^2e^{mx}$, $\dots$,
$D^ny = m^ne^{mx}$

Hence $(*)$ becomes $(m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_{n-1}m + a_n)e^{mx} = 0$.
Since $e^{mx} \neq 0$, above equation vanishes if 'm' is root of algebraic equation.
The equation, $m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_{n-1}m + a_n = 0$ or $F(m) = 0$ is called the Auxiliary equation (or) the characteristic equation.

The general solution of homogenous equation depends on nature of roots of this equation.

Case (i) : The roots of the auxiliary equation are real and distinct.

Suppose the roots of AE are $m_1, m_2, \ldots, m_n$ which are real and distinct associated with an n^{th} order equation, then the general solution is of the form

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x},$$

where $c_1, c_2, \ldots, c_n$ are 'n' arbitrary constants.

Case (ii) : The roots of the auxiliary equation are real and coincident (i.e., repeated).

Let $m_1 = m_2 = m_3 = \ldots = m_n = m$ be the roots of AE associated with an n^{th} order equation. Then

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_n x^{n-1}) e^{mx}$$

is the general solution.

Note : If $m_1 = m_2 = m_3 = m$ and the remaining roots are real and distinct, then the general solution is of the form

$$y = (c_1 + c_2 x + c_3 x^2) e^{mx} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}.$$

Case (iii) Auxiliary equation having complex roots.

For the equation $(D^2 + a_1 D + a_2) y = 0$, the AE is given by $m^2 + a_1 m + a_2 = 0$.

The complex roots always occur in conjugate pairs say $\alpha \pm i\beta$. Then

$$y = c_1 \cos \beta x + c_2 \sin \beta x$$

is the general solution.

Note : (i) If the root is purely imaginary i.e., $\pm i\beta$ ($\alpha = 0$) then $y = c_1 \cos \beta x + c_2 \sin \beta x$.

(ii) If the complex root $(\alpha \pm i\beta)$ is repeated n times, then the general solution is given by

$$y = e^{\alpha x}\left[(c_1 + c_2 x + c_3 x^2 + \dots + c_{n-1} x^{n-1})\cos\beta x + (c_1' + c_2' x + c_3' x^2 + \dots + c_{n-1}' x^{n-1})\sin\beta x\right]$$

* Examples :

1. Solve : $2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 3y = 0$

Sol Given $(2D^2 - 5D - 3)y = 0$

The auxiliary equation is given by $2m^2 - 5m - 3 = 0$.

$$\Rightarrow 2m^2 - 6m + m - 3 = 0$$
$$2m(m-3) + 1(m-3) = 0$$
$$(2m+1)(m-3) = 0$$
$$\therefore m = -\tfrac{1}{2}, 3.$$

The roots are real and distinct.

$\therefore$ The general solution is given by $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$.

2. Solve : $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 25y = 0$

Sol Given $(D^2 - 10D + 25)y = 0$

Auxiliary equation is given by $m^2 - 10m + 25 = 0$

$$\Rightarrow (m-5)^2 = 0$$
$$m = 5, 5.$$

The roots are real and repeated.

$\therefore$ The general solution is given by $y = (c_1 + c_2 x)e^{5x}$

3. Solve : $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 7y = 0.$

Sol Given $(D^2 + 4D + 7)y = 0$

Auxiliary equation is given by $m^2+4m+7=0$

$$\therefore\ m = \frac{-4 \pm \sqrt{16-4\times7}}{2}$$

$$= \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm \sqrt{3}\,i$$

The roots are complex.

$\therefore$ The general solution is given by $y = e^{-2x}[c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x]$

4. Solve : $(D^3+1)y=0$

Sol: The auxiliary equation is given by $m^3+1=0$

$m=-1$ is a root by inspection.

$\therefore\ (m+1)(m^2-m+1)=0$

-1	1	0	0	1
	0	-1	1	-1
	1	-1	1	0

$m=-1$ $\qquad m^2-m+1=0$

$$m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore$ The general solution is given by,

$$y = c_1e^{-x} + \left[c_2\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_3\sin\left(\tfrac{\sqrt{3}}{2}x\right)\right]e^{\frac{1}{2}x}$$

5. Solve : $(D^4+8D^2+16)y=0$

Sol: Auxiliary equation is $m^4+8m^2+16=0$

$$\Rightarrow (m^2+4)^2=0$$

$$m = \pm 2i, \pm 2i$$

$\therefore$ General solution is $y = (c_1+c_2x)\cos(2x) + (c_3+c_4x)\sin(2x)$.

6. Solve the initial value problem $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, given that $x(0)=0$, $\frac{dx}{dt}(0)=15$.

Sol: Given $(D^2+5D+6)x=0$

Auxiliary equation is $m^2+5m+6=0$

$$\Rightarrow (m+3)(m+2)=0$$

$$m = -3, -2$$

Hence the general solution is given by

$$x = c_1 e^{-3t} + c_2 e^{-2t}$$

Given $x(0) = 0$

$$\Rightarrow \quad 0 = c_1 + c_2 \quad \to (1)$$

and $\frac{dx}{dt}(0) = 15 \quad \Rightarrow \quad \frac{dx}{dt} = -3c_1 e^{-3t} - 2c_2 e^{-2t}$

$$\Rightarrow \quad -015 = 3c_1 + 2c_2$$

i.e., $\quad 3c_1 + 2c_2 = -15 \quad \to (2)$

From (1) & (2), we have $c_1 = 15$ and $c_2 = -15$.

$\therefore$ The particular solution is given by

$$x = 15(e^{-3t} - e^{-2t})$$

* Non-homogenous linear ODE with constant coefficients :

Consider a linear non-homogenous equation of order n.

$$(D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n)y = f(x). \quad \to *$$

Theorem: If $y = y_c$ is the general solution of the homogenous equation $D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n = 0$ and y_p is a function that satisfies the equation (*), then $y = y_c + y_p$ is the general solution of equation (*).

Theorem : If $f(x) = f(x_1) + f(x_2)$, then the particular integral of equation (*) is $y_p = y_{p_1} + y_{p_2}$, where y_{p_1} is a particular integral corresponding to $f(x_1)$ and y_{p_2} is PI of $f(x_2)$.

Note: In the general solution $y = y_c + y_p$, y_c is called the complementary function (CF) and y_p is called the particular integral (PI). The general solution is also referred to as the complete solution.

→ Inverse operator and particular integral :

Consider $F(D)y = f(x)$.
WKT the particular integral y_p satisfies the equation.

$$\therefore \quad F(D)\, y_p = f(x).$$

$$y_p = \frac{1}{F(D)} f(x),$$

where $\frac{1}{F(D)}$ is the inverse of the linear differential operator $F(D)$.

* Specific forms of particular integrals :

1. Type I : When $f(x) = e^{ax}$

$$y_p = \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax} \quad \text{where } F(a) \neq 0$$

[D is being replaced by a]

Suppose $F(a) = 0$, then $\frac{1}{F(D)} e^{ax} = \frac{x}{F'(a)} e^{ax}$, provided $F'(a) \neq 0$.

Suppose $F'(a) = 0$, then $\frac{1}{F(D)} e^{ax} = \frac{x^2}{F''(a)} e^{ax}$, provided $F''(a) \neq 0$,

and so on.

Note : If $f(x)$ is of the form $\sinh x$ or $\cosh x$, use the above results by writing

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

2. Type - 2 : When $f(x) = \sin(ax+b)$ or $\cos(ax+b)$.

Formula 1 : Express $F(D)$ as a function of D^2, say $\phi(D^2)$ and then replace D^2 by $-a^2$. If $\phi(-a^2) \neq 0$, then we use the result,

$$y_p = \frac{1}{F(D)} \sin(ax+b) = \frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b)$$

Similarly $y_p = \frac{1}{F(D)} \cos(ax+b) = \frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b)$

Formula 2 : Some times we cannot form $\phi(D^2)$. Then we shall try to get $F(D, D^2)$, that is a function of D and D^2.

In such cases we proceed as illustrated in the following example.

$$y_p = \frac{1}{D^2+2D+1} \sin 3x = \frac{1}{-9+2D+1} \sin 3x$$

$$= \frac{1}{2D-8} \sin 3x = \frac{1}{2(D-4)} \sin 3x.$$

(To get D^2 in denominator, multiply Nr and Dr by $D+4$)

$$= \frac{1}{2} \frac{D+4}{D^2-16} \sin 3x = \frac{1}{2} \cdot \frac{D+4}{-9-16} \sin 3x = \frac{-1}{50} (D+4) \sin 3x$$

$$= \frac{-1}{50} \left\{ \frac{d}{dx} \sin 3x + 4 \sin 3x \right\} = \frac{-1}{50} \left\{ 3\cos 3x + 4 \sin 3x \right\}$$

Formula III : If $\phi(-a^2) = 0$, we have exceptional cases.

(i) $\frac{1}{D^2+a^2} \cos(ax+b) = \frac{x}{2a} \sin(ax+b)$

(ii) $\frac{1}{D^2+a^2} \sin(ax+b) = -\frac{x}{2a} \cos(ax+b)$

3. Type-3 : When $f(x)$ is a polynomial in x i.e., $f(x) = x^n$, where n is a positive integer.

In this case, the non-homogenous equation takes the form $F(D)y = x^n$ where n is a positive integer. The particular integral in this case is $\frac{1}{F(D)}[x^n]$. In order to evaluate the PI, we write $\frac{1}{F(D)}$ as $[F(D)]^{-1}$ and expand it in ascending powers of D to get an expression of the form

$[F(D)]^{-1} = \{a_0 + a_1 D + a_2 D^2 + \dots\}$. Then we operate x^n with various terms of the expansion thus obtained. The result so obtained will be the particular integral.

Note (i) : The expansion is carried out upto the term containing D^n since the higher order derivatives when operated on x^n vanish. Some times the index may be even -2 or -3 in which case the appropriate binomial expansion must be used.

(ii) The following are some standard binomial expansions.

- $(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$
- $(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$
- $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$
- $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$

4. Type-4 : When $f(x) = e^{ax} V(x)$ where $V(x)$ is function of x

$$y_p = \frac{1}{F(D)} e^{ax} V(x) = e^{ax} \frac{1}{F(D+a)} V(x)$$

<u>Note</u>: If e^{ax} is multiplied with some function of x, first shift D to $(D+a)$, then compute PI using Type-2 or Type-3.

<u>Examples</u>:

1. Solve : $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

<u>Soln</u> AE is $(m-2)^2 = 0$

$\Rightarrow m = 2, 2$

$\therefore y_c = (c_1 x + c_2)e^{2x}$

$$y_p = \frac{1}{(D-2)^2} \cdot 8\left[e^{2x} + \sin 2x + x^2\right]$$

$$y_p = 8\left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{D^2+4-2D} \sin 2x + \frac{1}{(D-2)^2} \cdot x^2\right] \rightarrow (1)$$

NOW, $$\frac{1}{(D-2)^2} e^{2x} = \frac{x}{2(D-2)} e^{2x} = \frac{x^2}{2} e^{2x}$$

$$\frac{1}{D^2-4D+4} \sin 2x = \frac{1}{-2^2-4D+4} \sin 2x = \frac{-1}{4D} \sin 2x$$

$$= \frac{-1}{4} \int \sin 2x \, dx = +\frac{1}{4} \frac{\cos 2x}{2} = \frac{1}{8} \cos 2x$$

$$\frac{1}{(D-2)^2} x^2 = \frac{(-1)^2}{(2-D)^2} x^2 = \frac{(-1)^2}{2^2(1-\frac{D}{2})^2} x^2 = \frac{+1}{2^2}(1-D/2)^{-2} x^2$$

$$= \frac{+1}{2^2}\left[1 + 2\frac{D}{2} + 3\frac{D^2}{4} + \cdots\right] x^2$$

$$= \frac{1}{4}\left(x^2 + 2x + 3/2\right)$$

$$\therefore y_p = 8\left[\frac{x^2}{2} e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}(x^2 + 2x + 3/2)\right]$$

$$= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence, general solution is given by $y = y_c + y_p$.

$$\therefore y = (c_1 x + c_2)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

2. Solve : $y'' - 4y = \cosh(2x-1) + 3^x$

Sol: Given $(D^2-4)y = \cosh(2x-1) + 3^x$

$\therefore$ The auxiliary equation is $m^2 - 4 = 0$
$\Rightarrow m = 2, -2$

$$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{D^2-4}\cosh(2x-1) + \frac{1}{D^2-4}3^x$$

Now, $$\frac{1}{D^2-4}\cosh(2x-1) = \frac{1}{2}\left[\frac{e^{2x-1}+e^{-2x+1}}{D^2-4}\right]$$

$$= \frac{1}{2}\left[\frac{x}{2D}e^{2x-1} + \frac{x}{2D}e^{-2x+1}\right]$$

$$= \frac{1}{2}\left[\frac{x}{4}e^{2x-1} + \frac{x}{-4}e^{-2x+1}\right]$$

$$= \frac{1}{8}\left[xe^{2x-1} - xe^{-2x+1}\right]$$

$$\frac{1}{D^2-4}3^x = \frac{1}{D^2-4}e^{x\log 3} = \frac{1}{(\log 3)^2-4}3^x$$

$$\therefore y_p = \frac{1}{8}\left[xe^{2x-1} - xe^{-2x+1}\right] + \frac{1}{(\log 3)^2-4}3^x$$

Hence the general solution is $y = y_c + y_p$.

i.e., $$y = c_1e^{2x} + c_2e^{-2x} + \frac{1}{8}\left[xe^{2x-1} - xe^{-2x+1}\right] + \frac{1}{(\log 3)^2-4}3^x$$

3. Solve : $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$

Sol: AE is $m^3 + 3m^2 + 3m + 1 = 0$

$\Rightarrow (m+1)^3 = 0$

$\Rightarrow m = -1, -1, -1.$

$$\therefore y_c = (c_1 + c_2x + c_3x^2)e^{-x}$$

$$y_p = \frac{1}{D^3+3D^2+3D+1}e^{-x}$$

$$= \frac{x}{3D^2+6D+3}e^{-x} = \frac{x^2}{6D+6}e^{-x} = \frac{x^3}{6}e^{-x}$$

Hence, the general solution is $y = y_c + y_p$

$$\therefore y = (C_1 + C_2x + C_3x^2)e^{-x} + \frac{x^3 e^{-x}}{6}$$

4. Find the particular integral of $(D^3+1)y = \cos(2x-1)$

Sol: $y_p = \frac{1}{D^3+1}\cos(2x-1)$

$$= \frac{1}{D^2 \cdot D+1}\cos(2x-1) = \frac{1}{(-4)D+1}\cos(2x-1)$$

$$= \frac{(1+4D)}{1-16D^2}\cos(2x-1)$$

$$= \frac{1+4D}{1-16(-4)}\cos(2x-1) = \frac{1}{65}\left[\cos(2x-1) + 4\frac{d}{dx}\cos(2x-1)\right]$$

$$= \frac{1}{65}\left[\cos(2x-1) - 8\sin(2x-1)\right]$$

5. Solve $y'' - 2y' + y = xe^x \sin x$

Sol: Given $(D^2 - 2D + 1)y = xe^x \sin x$

AE : $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$\therefore y_c = C_1e^x + C_2xe^x$ i.e., $y_c = (C_1 + C_2x)e^x$

$$y_p = \frac{1}{(D-1)^2} \cdot e^x x\sin x$$

$$= e^x \frac{1}{(D+1-1)^2} x\sin x = e^x \cdot \frac{1}{D^2} x\sin x$$

$$= e^x \cdot \frac{1}{D}\int x\sin x = e^x \cdot \frac{1}{D}\left[-x\cos x - \int(+1)(-\cos x)dx\right]$$

$$= e^x \cdot \frac{1}{D}\left[-x\cos x + \sin x\right]$$

$$= e^x \int(-x\cos x + \sin x)$$

$$= e^{x}\left[-x\sin x - \int(-1)\sin x - \cos x\right]$$

$$= e^{x}\left[-x\sin x - \cos x - \cos x\right]$$

$$y_p = -e^{x}(x\sin x + 2\cos x)$$

Hence, the general solution is given by $y = y_c + y_p$

$$\therefore \quad y = (C_1 + C_2 x)e^{x} - e^{x}(x\sin x + 2\cos x)$$

6. Find the particular integral of $(D^3 + 4D)y = 2\sin x\cos x$

Soln

$$y_p = \frac{1}{D^3 + 4D} 2\sin x\cos x$$

$$= \frac{1}{D^2 \cdot D + 4D} \sin 2x$$

$$= \frac{x}{3D^2 + 4} \sin 2x$$

$$= \frac{x}{3(-4) + 4} \sin 2x$$

$$= -\frac{x}{8} \sin 2x$$

7. Find the particular integral of $(D^2 + 9)y = x\cos x$.

Soln

$$y_p = \frac{1}{D^2 + 9} Re(xe^{ix}) \qquad \because \left[e^{ix} = \cos x + i\sin x\right]$$

$$= Re\left[\frac{1}{D^2+9} xe^{ix}\right] = Re\left[e^{ix} \cdot \frac{1}{(D+i)^2 + 9} x\right]$$

$$= Re\left[e^{ix} \cdot \frac{1}{D^2 + 2iD + 8} \cdot x\right] = Re\left[e^{ix} \frac{1}{8\left[1 + \frac{D^2 + 2iD}{8}\right]} x\right]$$

$$= Re\left[\frac{e^{ix}}{8}\left(1 + \frac{D^2 + 2iD}{8}\right)^{-1} x\right] = Re\left[\frac{e^{ix}}{8}\left(1 + \left(-\frac{D^2}{8}\right) - \frac{iD}{4}\right)x\right]$$

$$= Re\left(\frac{\cos x + i\sin x}{8}\right)\left(x - \frac{i}{4}\right)$$

$$= \frac{x\cos x}{8} + \frac{\sin x}{32}$$

8. Find the particular integral of $(D^2-4D+4)y = x^2e^{3x} + \sin^2 x$

Soln $y_p = \frac{1}{(D-2)^2} x^2 e^{3x} + \frac{1}{(D-2)^2} \sin^2 x$

Now, $\frac{1}{(D-2)^2} x^2 e^{3x} = e^{3x} \frac{1}{(D+3-2)^2} x^2$

$= +e^{3x}(1+D)^{-2} x^2$

$= e^{3x}(1-2D+3D^2-4D^3)x^2$

$= e^{3x}(x^2-4x+6)$

$\frac{1}{(D-2)^2}\sin^2 x = \frac{1}{(D-2)^2}\left(\frac{1-\cos 2x}{2}\right)$

$= \frac{1}{(D-2)^2} \cdot \frac{1}{2}(e^{0x} - \cos 2x)$

$= \frac{1}{2}\left[\frac{1}{(D-2)^2}e^{0x} - \frac{1}{D^2-4D+4}\cos 2x\right]$

$= \frac{1}{2}\left[\frac{1}{4} - \frac{1}{-4-4D+4}\cos 2x\right]$

$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{4}\int \cos 2x\right]$

$= \frac{1}{8} + \frac{1}{8}\cdot\frac{\sin 2x}{2}$

$= \frac{1}{8} + \frac{1}{16}\sin 2x$

$\therefore y_p = e^{3x}(x^2-4x+6) + \frac{1}{8} + \frac{\sin 2x}{16}$