Homework 1
CS611: Theory of Computation

Problem 1. [Category: Proof] Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1$, 2, and 3.) **[10 points]**
We will use induction to prove the above statement:

1. Formulate the problem in terms of proposition $P(n)$;
2. Prove the base case, $P(4)$ is true;
3. Write the induction hypothesis;
4. Prove that $P(k+1)$ is true assuming the induction hypothesis is true.

Problem 2. [Category: Comprehensive] Let $A = \{1, 2, 3, 4, 5\}$, $R \subseteq A \times A$ be the relation $\{(a, b)|a - b$ is a multiple of $2\}$. **[5 points]**

1. Show that R is an equivalence relation. Recall that R is an equivalence relation if it is reflexive ($\forall a \in A, (a, a) \in R$), symmetric ($\forall a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$), and transitive ($\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$).
2. Given $a \in A$, let $[a]\_R$, called an equivalence class, be the set of all elements related to a through R, that is, $[a]\_R = \{b|(a, b) \in R\}$. What is $[1]\_R, [2]\_R, [3]\_R, [4]\_R, [5]\_R$?
3. How many distinct equivalence classes are there for R?
4. If A is the set of all natural numbers and R is defined as above, then how many distinct equivalence classes does it have?

Problem 3. [Category: Design] Design a DFA for the language $L_{A3} = \{w \in \{a, b\}^* \mid$ if w starts with an a then it does not end with a $b\}$. **[5 points]**

Problem 4. [Category: Design] Design a DFA for the language $L_{A1} = \{w \in \{a, b\}^* |$ number of a's in w is not divisible by $3\}$. **[5 points]**

Problem 5. [Category: Design] Design a DFA for the language $L_{A4} = \{w \in \{a, b\}^* \mid ba$ appears exactly twice as a substring$\}$. **[5 points]**

Problem 6. [Category: Design+Proof] Let $A_k \subseteq \{a, b\}^*$ be the collection of strings w where there is a position i in w such that the symbol at position i (in w) is a, and the symbol at position $i + k$ is b. For example, consider A_2 (when $k = 2$). $baab \in A_2$ because the second position ($i = 2$) has an a and the fourth position has a b. On the other hand, $bb \notin A_2$ (because there are no as) and $aba \notin A_2$ (because none of the as are followed by a b 2 positions away).

1. Design a DFA for language A_k, when $k = 2$, you just need to draw the transition diagram. **[5 points]**

2. Design an NFA for language A_2 that has at most 4 states. You need not prove that your construction is correct, but the intuition behind your solution should be clear and understandable. **[5 points]**

Problem 7. [Category: Design] Design a DFA for the language $L = \{w \in \{a, b\}^* \mid \text{number of } a\text{s in } w \text{ is at least 2 and number of } b\text{s in } w \text{ is exactly one}\}$. You can just draw the diagram. **[5 points]**

Hint: Think about a DFA accepts strings that has at least two as in it and a DFA accepts strings have exactly one b in it.