Moksha Shah bp4199
Aryan Bhagat sl5310

CS 611

Homework-1

Problem-1

given, $P(n) = 2^n < n!$ w $\forall n$ where $n \geq 4$

- Let $P(n)$ be the proposition that $2^n < n!$
Prove that $p(n)$ for all $n \geq 4$.

- Prove that base case i.e $P(4)$ is true

for $n = 4$, $P(4) = 2^4 < 4!$
$= 16 < 24$

$\therefore P(4)$ is true

- Inductive Hypothesis

Let $P(n)$ be true for $n = k$.
$\therefore$ We can say that $P(k)$ is true.

$\therefore P(k) = 2^k < k!$ is true

- Prove for $P(k+1)$ assuming that inductive hypothesis is true.

$(k+1)! = (k+1).k!$
$= k.k! + k!$

from inductive hypothesis,

$\cancel{k.k! + k!} \cancel{\geq k \times 2^k + 2^k}$
$\cancel{> 4 \times 2^k + 2^k}$ $\cancel{[\forall k \geq 4]}$

$2^k < k!$

We have, $2^{k+1} = 2 \cdot 2^k$

$\therefore 2^{k+1} < 2 \cdot k!$

$< (k+1) \cdot k!$ $\quad (2 < k+1)$

$< (k+1)!$

$\therefore \boxed{2^{k+1} < (k+1)!}$

This shows that $P(k+1)$ is true when $P(k)$ is true

Problem 2 :

$A = \{1, 2, 3, 4, 5\}$, $R \subseteq A \times A$ be the relation $\{(a,b) \mid a-b$ is a multiple of $2\}$

1. To show that R is an equivalence relation on A, we need to show that R satisfies 3 properties according to equivalence relation defination

- Reflexive :

Since $a-a = 0 = 2 \cdot 0$ is a multiple of 2, then $(a,a) \in R$ for all $a \in A$, hence R satisfies this property.

- Symmetric :

Symmetricity is : if $(a,b) \in R$, then $(b,a) \in R$.
if $(a,b) \in R$ then $a-b = i = 2k$ is a multiple of 2 where k is an integer, then $b-a = -i = 2(-k)$ is also a multiple of 2. Hence $(b,a) \in R$

- Transitive :

For $a, b, c \in A$, if $(a,b) \in R$, ~~then $(b,a) \in R$~~. and $(b,c) \in R$ then $(a,c) \in R$.

if $(a,b) \in R \Rightarrow a-b = i = 2k$
if $(b,c) \in R \Rightarrow b-c = j = 2k'$
(i & j are multiples of 2)

$a-c = (a-b)+(b-c) = i+j = 2k+2k'$
$= 2(k+k')$ is a multiple of 2.

$\therefore (a,c) \in R$.

Because R satisfies all three properties, R is an equivalence relation according to definition.

2. $[1]\_R$ is the set of b related to 1. that is $\{b \mid (1,b) \in R\}$.
Hence, $[1]\_R = \{1,3,5\}$. Similarly, $[2]\_R = \{2,4\}$, $[3]\_R = \{1,3,5\}$, $[4]\_R = \{2,4\}$, $[5]\_R = \{1,3,5\}$.

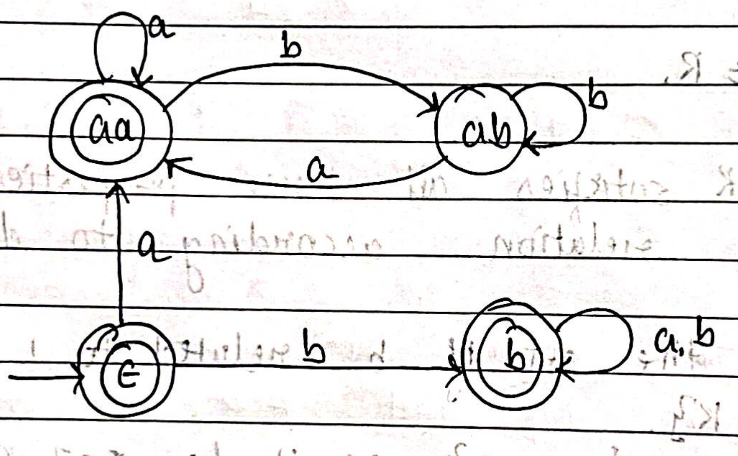
3. There are two distinct equivalence classes for R - $\{1,3,5\}$, $\{2,4\}$.

4. If A is the ~~same~~ set of all natural numbers with same R defined, then there are still two distinct equivalence classes for R. - $\{1,3,5\}$, $\{2,4\}$.
We can write these two classes as
even = $\{2k \mid k \in N\}$
odd = $\{2k+1 \mid k \in N\}$ where N is set of natural numbers.

Problem 3 :

$L_{A3} = \{ w \in \{a,b\}^* \mid$ if w starts with an a then it does not end with $b\}$.

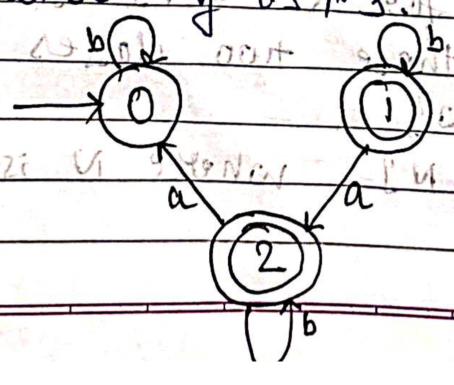
- DFA will remember what the first symbol of the input is, and if the first symbol is an a, it will also remember the last symbol read. Thus, the states will be ε (initial state), b (input began with b), aa (input began with a and the last symbol read is a) and ab (began with a and end with b).



Problem - 4

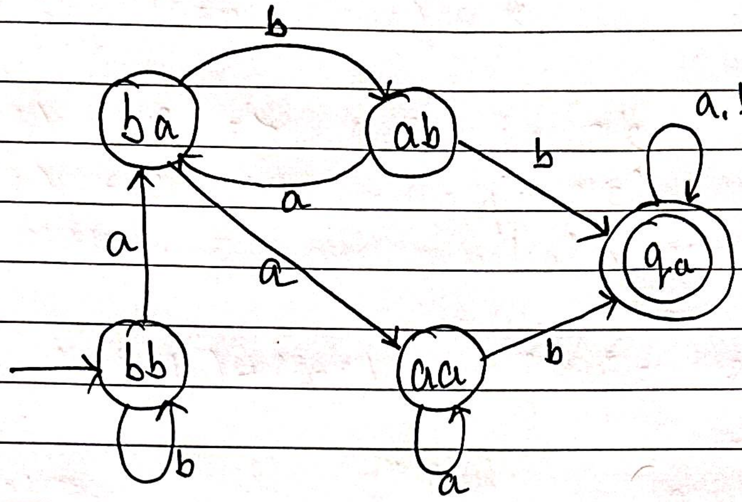
$L_{A1} = \{ w \in \{a,b\}^* \mid$ number of a's in w is not divisible by $3\}$.

- L_{A1} will remember how a's % 3 it has seen so far. States will be 0, 1, 2 where state i denotes the number of a's % 3.



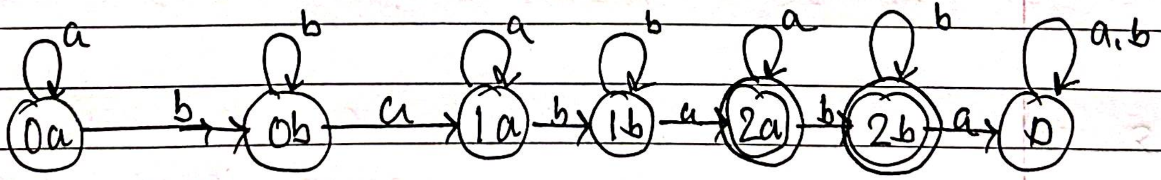
Problem 6 :

1. when $k = 2$, automaton M_2 can be drawn as



Problem 5 :

- DFA will remember how many times ba has appeared and what the last symbol read is. States are of the form ia (i ba substrings and ends in a) or ib (i ba substrings and ends in b), where i is 0, 1, 2. Dead state will remember that substring has been seen more than twice.



Problem 7 :