

Question #3

3. A random variable with **geometric distribution** is a discrete variable that can attain values of 1, 2, It is used to model the number of trials of a Bernoulli (binary) test is repeated until the first success occurs. So, if a Bernoulli trial can have "success" with probability p and "failure" with probability $1 - p$, then the number of iid¹ trials before a success happens

¹The term 'iid' means identically and independently distributed.

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Algorithmic Machine Learning, Fall 2024
Homework 1 Due date: 9/29/24 at 11:50PM

follows the geometric distribution. The pmf for this distribution is given by

$$f(k) = (1 - p)^{k-1} p$$

This is because if the first success is seen at the k^{th} trial, it means that we have seen $k - 1$ failures in a row and then saw success in the k^{th} trial.

$$f(k) = (1-p)^{k-1} p$$

3a) Given $\begin{cases} \text{fair coin} = 50\% \text{ of heads} \\ \text{false coin} = 20\% \text{ of heads} \end{cases}$
 Toss coin 3 times all tail with heads on 4th toss

$$\text{fair win} = (1 - 0.50)^{4-1} 0.50 = \underline{0.0625}$$

6.25% probability of 3 tail then 1 head with fair coin

$$\text{False win} = (1 - 0.20)^{4-1} 0.20 = 0.1024$$

10.24% probability of 3 tail then 1 head
with fake coin

3q) Using the geometric distribution we are able to see probability of 3 tails, then 1 head with fair coin is 6.25% . While 3 tails, then 1 head with fake coin is 10.24% . Since it is more likely that it is a fake coin than fair coin I would say that based on the current number of trials we cannot definitively say if it is fair or fake but currently there is a 10.24% chance that we got 3 tails and then 1 head with the fair coin. which is -4% increase in probability than that occurring with real coin which was 6.25% . Therefore, it is around 4% more likely this was fake coin compared to fair but we cannot guarantee it is fake. We need more trials to confirm if it is fake.

3q Answer

→ (More likely it's fake but no guarantee)

3b

- 3b) Suppose we have two scenarios: the probability of success is p_1 , or probability of success is p_2 . Given values of $0 \leq p_1 < p_2 \leq 1$, what is the minimum number of times failures should occur before the first success in order for you to select scenario 1 over scenario 2 using Bayes decision rule?

Based on Bayes we will choose the hypothesis that maximizes the posterior prob.

H_1 = coin has probability of tails p_1

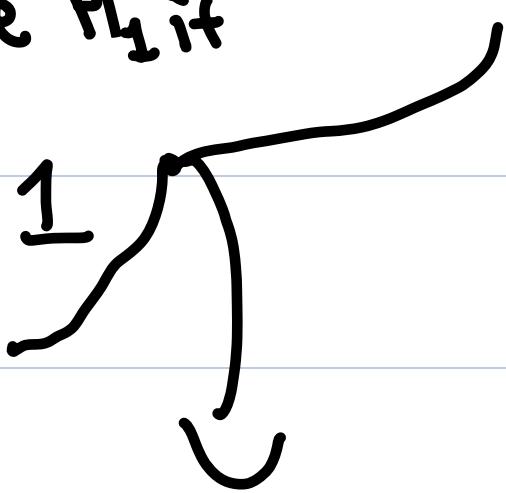
H_2 = coin has probability of heads p_2

Assuming equal priors $P(H_1) = P(H_2)$, we will only need to evaluate the likelihood ratio

$$\frac{P(\text{data} | H_1)}{P(\text{data} | H_2)} = \frac{(1-p_1)^{n-1} p_1}{(1-p_2)^{n-1} p_2} = \frac{p_1}{p_2} * \left(\frac{1-p_1}{1-p_2} \right)^{n-1}$$

We should choose H_1 if

$$\frac{P(\text{data} | H_1)}{P(\text{data} | H_2)} > 1$$



$$\frac{P_1}{P_2} \cdot \left(\frac{1-P_1}{1-P_2} \right)^{n-1} > 1$$

$$\left(\frac{1-P_1}{1-P_2} \right)^{n-1} > \frac{P_2}{P_1}$$

$$(n-1) \log \left(\frac{1-P_1}{1-P_2} \right) > \log \left(\frac{P_2}{P_1} \right)$$

$$n-1 > \frac{\log \left(\frac{P_2}{P_1} \right)}{\log \left(\frac{1-P_1}{1-P_2} \right)}$$

$$\frac{\log \left(\frac{P_2}{P_1} \right)}{\log \left(\frac{1-P_1}{1-P_2} \right)}$$

3b)
Answer

$$n > \frac{\log \left(\frac{P_2}{P_1} \right)}{\log \left(\frac{1-P_1}{1-P_2} \right)} + 1$$

This gives the smallest number of trials n , where the difference between the two hypothesis becomes distinguishable using Bayes Decision Rule, in order to select scenario 1 over scenario 2.