Asymptotic limit of the flow field

Semester Project Report
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1 Introduction

The exact flow field of any microswimmer is a complex equation which high computational time to simulate. For a large number of microswimmers it is not feasible to use the exact solution and one must approximate the equation. A good approximation reduces the computational time and keep the field flow acceptable. In this project, we will calculate the far field limit of the exact solution. We will further compare both the flow and find the deviation from original solution.

2 Model

We have a microswimmer with spherical rigid body with radius a and suspended in a fluid of viscosity η . We assume that no external torque is acting on the swimmer and it is self-propelled with an internal force $f^{sp} = f^{sp}\hat{n}$, where $\hat{n}$ is an arbitrary self propulsion direction.

In the real case of a flagellated bacterium, f^{spc} is interpreted as the time average of the force exerted by the flagella on the fluid and f^{sp} being the resulting thrust. But in this model, we will consider a effective hydrodynamic force acting at a single point on the fluid at a distance l from the r_{cm} and we neglect the force distribution to a first order approximation.

Since the speed and size of the microswimmer it is in low Reynolds number condition, we can apply the incompressible Stokes equation to calculate the flow field. The microswimmer dynamics is controlled by balanced equations of force and torque

$$F^h + f^{sp} + f^{ext} = 0 \tag{1}$$

$$L^h + L^{ext} = 0 \tag{2}$$

F^h and L^h are the total force and torque exerted by the surrounding fluid on the swimmer respectively. f^{ext} and L^{ext} are, respectively, the net external force and torque experienced by it.

2.1 Flow Field

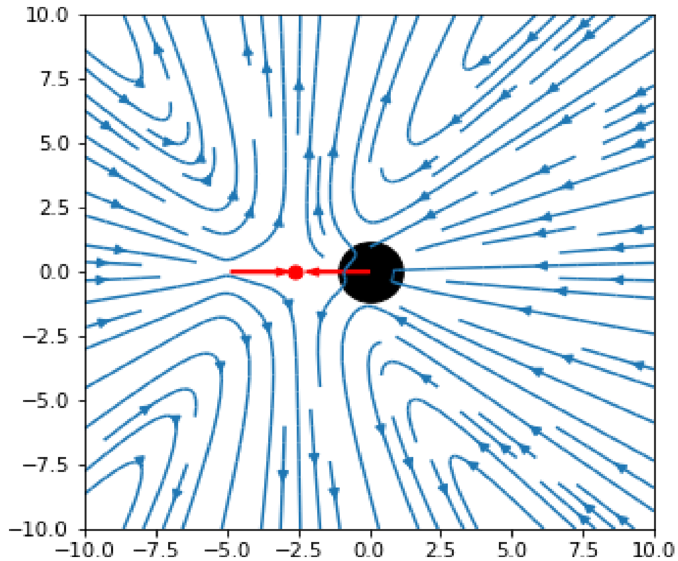
The exact flow field of the swimmer contains two parts: (i) the u_a generated because of the f^{spc} that lies near a sphere. (ii) the translation of the sphere with velocity v^{sp} also contributes to the flow field u_b. From paper [1], the exact flow field 3 are taken.

$$\begin{aligned}
\mathbf{u}^{\mathrm{sp}}(\mathbf{r}) =& \mathbf{u}_a(\mathbf{r}) + \mathbf{u}_b(\mathbf{r}) \\
=& \Big\{ \left(1 - g_{a/\ell}\right) \left[\mathbf{T}(\mathbf{r}) - \frac{a^2}{3} \widetilde{\mathbf{T}}(\mathbf{r}) \right] - \mathbf{T}(\mathbf{r} \pm \ell \hat{\mathbf{n}}) \\
& + g_{a/\ell} \mathbf{T}\left(\mathbf{r} \pm \ell^* \hat{\mathbf{n}}\right) - 2 j_{a/\ell} a^2 \widetilde{\mathbf{T}}\left(\mathbf{r} \pm \ell^* \hat{\mathbf{n}}\right) \Big\} \cdot \mathbf{f}^{\mathrm{sp}} \\
& + h_{a/\ell} a f^{\mathrm{sp}} \mathbf{H}\left(\mathbf{r} \pm \ell^* \hat{\mathbf{n}}, \hat{\mathbf{n}}\right) \cdot \left(\mathbf{r} \pm \ell^* \hat{\mathbf{n}}\right).
\end{aligned} \tag{3}$$

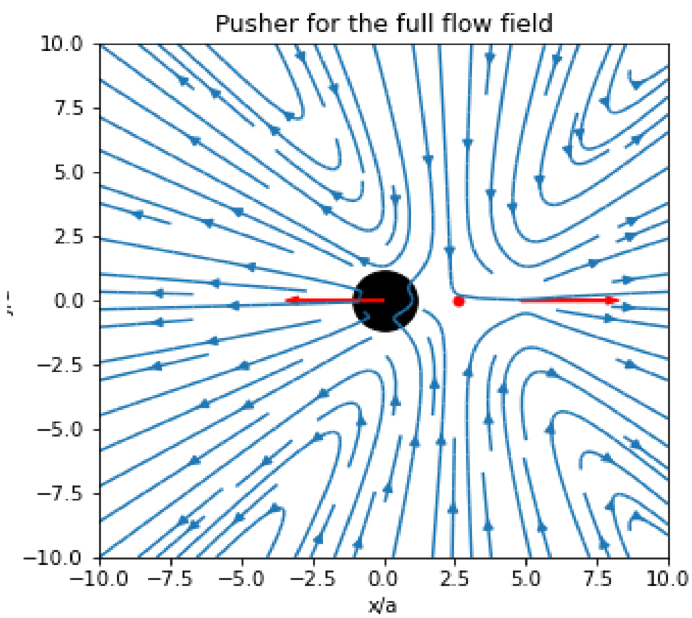
$$\tilde{T} = -\frac{1}{2}\nabla^2 T(r) = \frac{1}{8\pi\eta r^3}(-I + \frac{3rr}{r^2}) \tag{4}$$

$$H(r, \hat{n}) = \frac{1}{8\pi\eta r^3}(-\boldsymbol{I} + \frac{3(r.\hat{n})^2}{r^2}) \tag{5}$$

Here, + and - are for the pusher and puller types respectively. 3 represents the exact form of the flow field. Here $T(r)$ is the **oseen matrix** given as $T(r) = (1/8\pi\eta r)(\boldsymbol{I} + \boldsymbol{rr/r^2})$. In Fig 3, we have plotted the field flow lines of the exact solution. We have taken $\hat{n} = -\hat{x} atl = 5a$ and have plotted in the x-y plane. The red dot indicates the r_c^{eff} of the far-field point force. We can see that for the pusher we have outward flow and for the puller inward flow in the $\hat{n}$ direction.



(a) Puller at l = 5a



(b) Pusher at l = 5a

Figure 1: Exact field flow lines for the pusher and puller

2.2 Far-Field Approximation

The in the previous section, we had considered the exact equation for the flow field. Now to study the far field behaviour of the model we take the Taylor expansion of equation 3 till $O(1/r^4)$, we get

$$\begin{aligned} u_{sp} =& f_{sp}\left[\nabla^2 T(\boldsymbol{r})\cdot\hat{n}\right]\left(\left(1-g_{a/l}\right)\frac{a^2}{6}+j_{a/l}a^2\right) \\ & - f_{sp}(\nabla\cdot\hat{n})[T(\boldsymbol{r})\cdot\hat{n}]\left(\mp l \pm l^* - h_{a/l}a\right) \\ & + f_{sp}\left[(\hat{n}\cdot\nabla)^2 T(\boldsymbol{r})\cdot\hat{n}\right]\left(\frac{1}{2}l^{*2}g_{a/l}-\frac{l^2}{2}\right) \\ & \pm \frac{h_{a/l}a^3 f^{sp}}{lr^3}[\frac{1}{8\pi\eta}(9(\hat{n}\cdot\hat{r})\hat{r}-15(\hat{n}\cdot\hat{r})^3\hat{r})+H(r,\hat{n})\cdot\hat{n}]+O\left(\frac{1}{r^4}\right) \end{aligned} \tag{6}$$

On collecting terms, taking $r >> a$ and expanding the terms with $O\left(\frac{1}{r^4}\right)$, 6 becomes

$$\begin{aligned} u^{sp}_{r>>a} = S_{eff}(\nabla\cdot\hat{n})[T(r)\cdot\hat{n}] - \frac{1}{2}D_{eff}\left[\nabla^2 T(r)\cdot\hat{n}\right] \\ +Q_{eff}\left[(\hat{n}\cdot\nabla)^2 T(r)\cdot\hat{n}\right]+O\left(\frac{1}{r^4}\right) \end{aligned} \tag{7}$$

$$D_{eff} = -[2j_{a/l}a^2 + (1-g_{a/l})a^2/3]f_{sp} \tag{8}$$

$$S_{eff} = \pm f_{sp}l[1-\left(\frac{a}{l}\right)^2 g_{a/l} \pm h_{a/l}\left(\frac{a}{l}\right)] \tag{9}$$

$$Q_{eff} = -f_{sp}l^2\left(\frac{1}{2}-\frac{1}{2}\left(\frac{a}{l}\right)^4 g_{a/l} \pm \left(\frac{a}{l}\right)^3 h_{a/l}\right) \tag{10}$$

Here, $\nabla\cdot\hat{n})[T(r)\cdot\hat{n}]$ is the force dipole and S_{eff} is the strength of this dipole. $[\nabla^2 T(r)\cdot\hat{n}]$ is the source dipole and the strength is given by D_{eff}. The last term $[(\hat{n}\cdot\nabla)^2 T(r)\cdot\hat{n}]$ gives the force quadrapole and strength depends on Q_{eff}. The force dipole decays as $1/r^2$ and the other two quadrapole decay as $1/r^3$.

For the far field calculation, we expand the u_{sp} 3 in terms of power of

1/r along the centre of mass of the body. The leading order term we will get is

$$\mathbf{u}^{\mathrm{sp}}_{r\gg\ell}(\mathbf{r}) = \pm\frac{1}{8\pi\eta}\frac{S^{\mathrm{eff}}}{r^2}\left[-1+3(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}})^2\right]\hat{\mathbf{r}} \tag{11}$$

Here we have the far field case $r >> l > a$. It implies that equation 7, will give same field flow lines as force dipole for a large r.

2.3 Plots

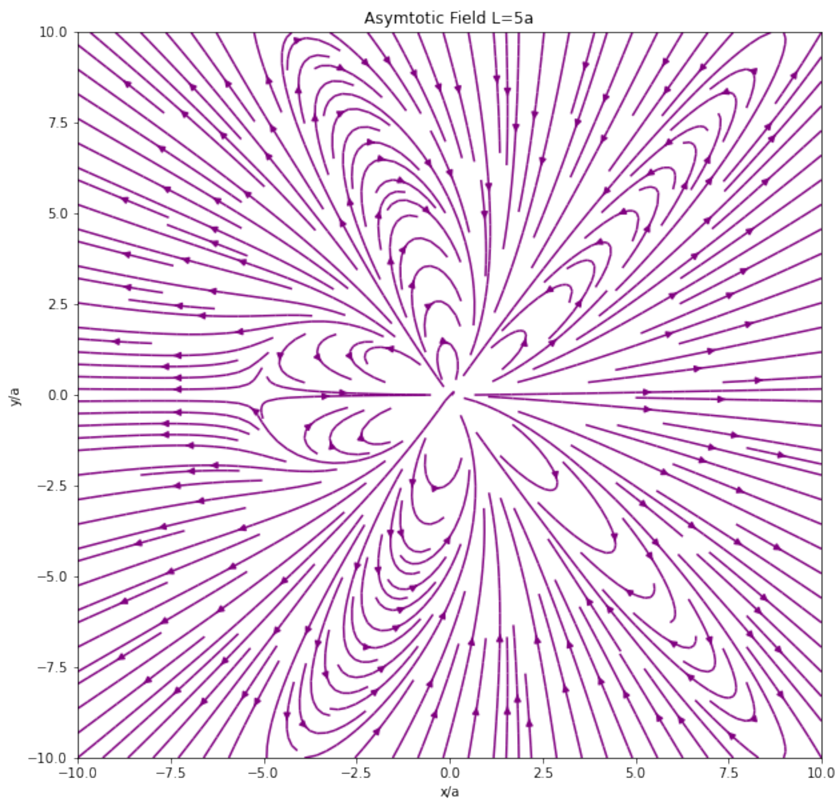
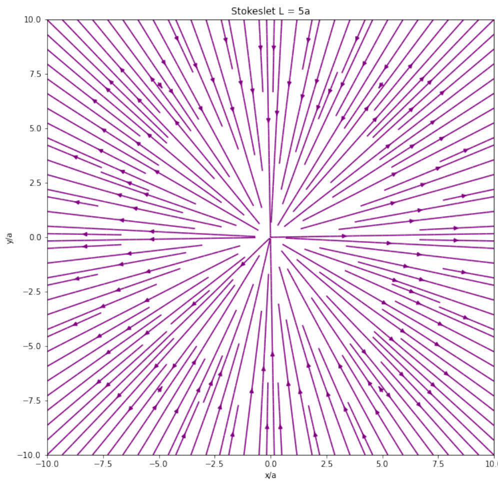
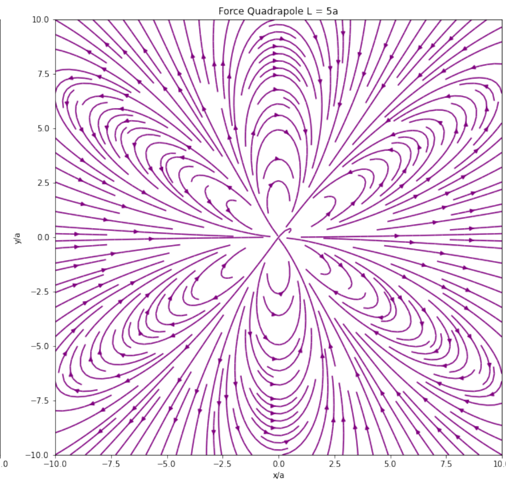


Figure 2: Asymptotic flow field

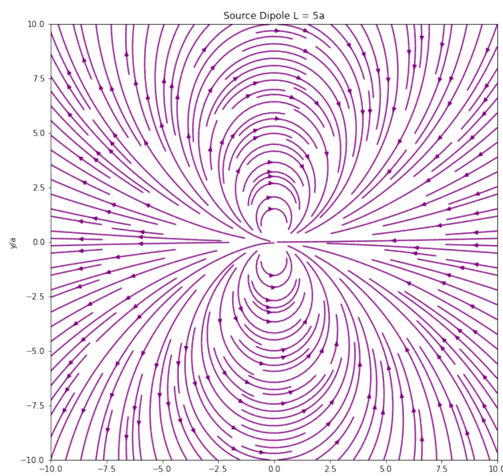
Fig.2 shows the approximated flow field. Here we have taken l = 5a. The field have contribution form all the three terms in eq.7. Fig. 3 represents the flow field of each of the terms.



(a) Force Dipole

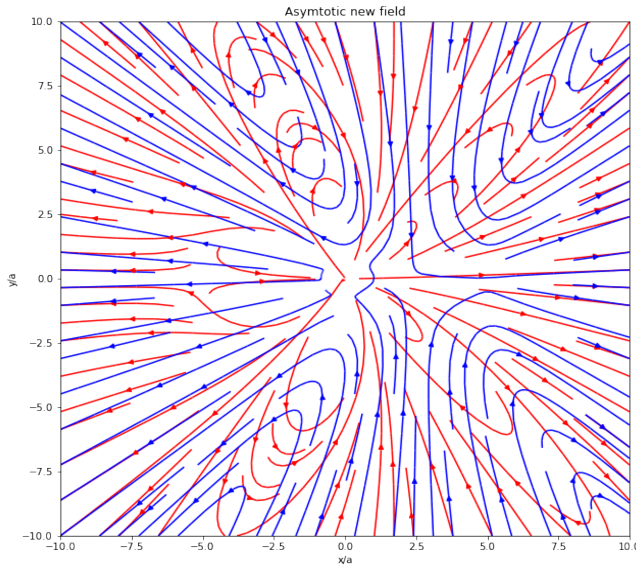


(b) Force Quadrapole

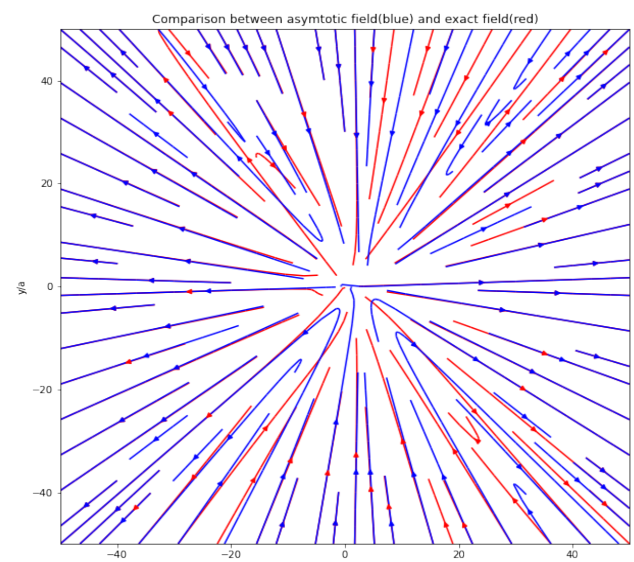


(c) Source Dipole

Figure 3: Velocity field cross- sections of (a) a Stokeslet (force) dipole which decays as $\frac{1}{R^2}$; (b) a source dipole which decays as $\frac{1}{R^3}$; and (c) a force quadrupole, which decays as $\frac{1}{R^3}$.



(a) At close distance: the red shows asymptotic field and blue, the exact field)



(b) At large distance

Figure 4: Comparison between Exact field and Asymptotic field

Fig. 4(b) shows that at large distances our flow fields look the same but near the body, we can see the difference in the flow field. This also concludes that our flow field approximation is correct.

We have plotted u_x and u_y as a function of x and y respectively for both the fields.

In fig.5, (a) and (b) are almost same but with different range. In (b) we can see the singularity because of the l = 5a we have taken. We can see that at large distances both the values of the exact field and asymptotic field match.

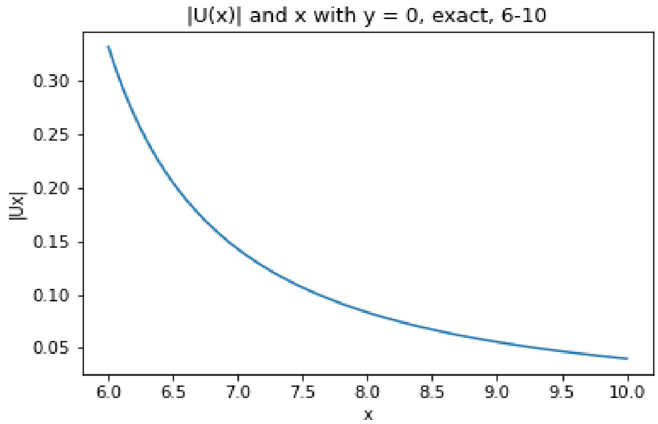
3 Conclusion

In the project we have calculated the far-field of the microswimmer from the exact flow field. We also worked out the leading order terms which contributed in flow field. Then we compared the exact flow field and asymptotic flow field, its streamline flows and their magnitude.

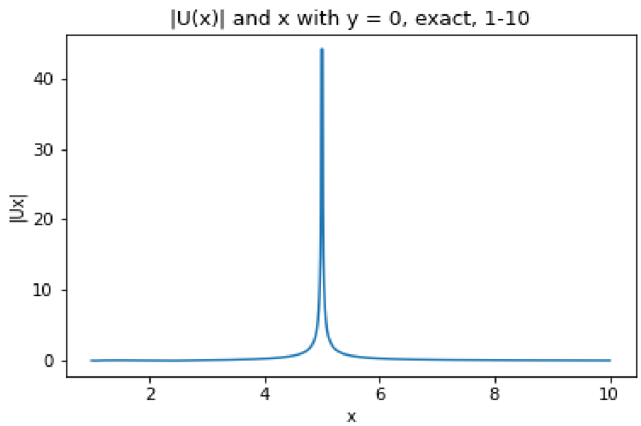
4 References

[1] Tapan Chandra Adhyapak and Sara Jabbari-Farouji Phys. Rev. E 96, 052608

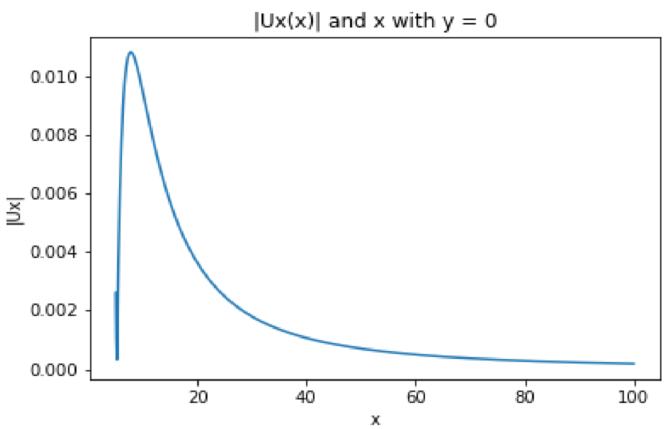
[2] S. E. Spagnolie and E. Lauga, J. Fluid Mech. 700, 105 (2012).



(a) $U_x(x)$ and x with y = 0, exact

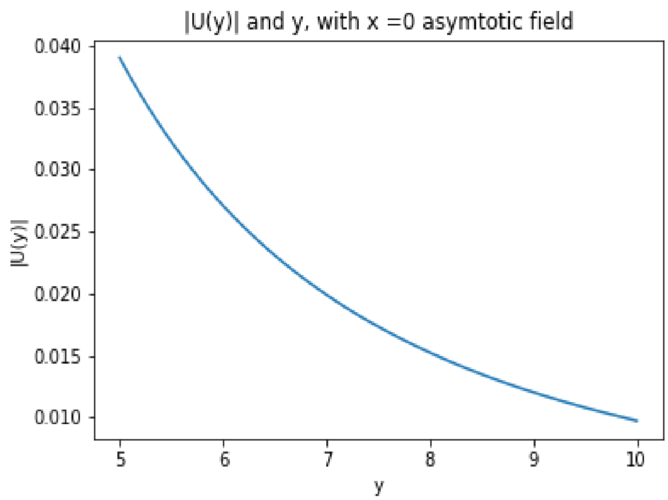


(b) $U_x(x)$ and x with y = 0, exact flow field

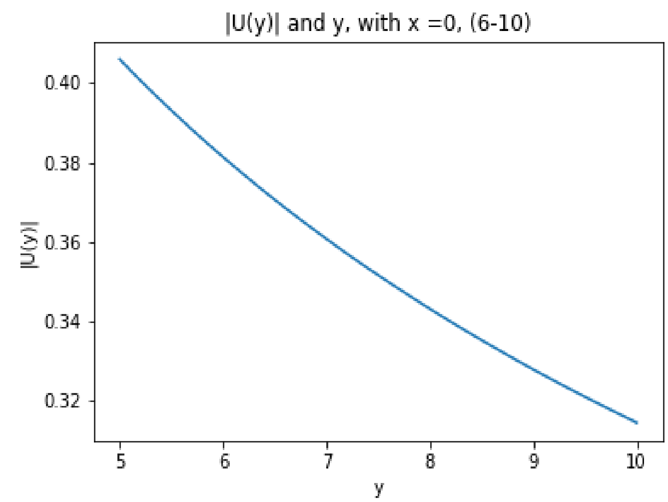


(c) $U_x(x)$ asymptotic field flow

Figure 5: Comparison between Exact field and Asymptotic field



(a) $U_y(y)$ asymptotic field flow



(b) $U_y(y)$ exact field flow

Figure 6: Comparison between exact field and asymptotic field