Integrations formula

1. $\int (ax+b)^n \, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c$

2. $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln(ax+b) + c$

3. $\int \sin x \, dx = -\cos x + c$

4. $\int \cos x \, dx = \sin x + c$

5. $\int \sec^2 x \, dx = \tan x + c$

6. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

7. $\int e^x \, dx = e^x + c$

8. $\int \ln x \, dx = x \ln x - x + c$

9. $\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

10. $\int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{1-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

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Unit 10
Partial derivatives and multiple integrals.

- → Limit & Continuity
- → Partial derivative
- → Tangent and normal
- → Maxima and minima
- → Evaluation of double integrals
- → Area and volume using double integral

Evaluation of double integrals

Q) Evaluate: (i) $\int_1^2\int_0^1 (x^2y + 2xy^2)\,dx\,dy$

$\iint_R f(x,y)\,dx\,dy$

$R: 0 \le x \le 1$

$1 \le y \le 2$

$f(x,y) = x^2y + 2xy^2$

$= \int_1^2 \left[y\cdot\frac{x^3}{3} + 2y^2\cdot\frac{x^2}{2}\right]_0^1 dy$

$= \int_1^2 \left[\frac{y}{3} + y^2\right] dy$

$= \left[\frac{1}{3}\cdot\frac{y^2}{2} + \frac{y^3}{3}\right]_1^2$

$= \frac{2}{3} + \frac{8}{3} - \frac{1}{6} - \frac{1}{3}$

$= \frac{17}{6}.$

ii) Evaluate: $\int_0^1\int_1^2 (x^2y + 2xy^2)\,dy\,dx$

$= \int_0^1 \left[x^2\frac{y^2}{2} + 2x\frac{y^3}{3}\right]_1^2 dx$

$= \int_0^1 \left[2x^2 + \frac{16x}{3} - \frac{x^2}{2} - \frac{2x}{3}\right] dx$

$= \int_0^1 \frac{3x^2}{2} + \frac{14x}{3}\,dx$

$= \left[\frac{3}{2}\cdot\frac{x^3}{3} + \frac{14}{3}\frac{x^2}{2}\right]_0^1$

$= \frac{1}{2} + \frac{7}{3}$

$= \frac{17}{6}$

- Fubini's theorem: $\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$

Evaluate :-

$\int_0^1\int_1^2 (2y-x^2)\,dx\,dy$

$= \int_0^1 \left[2y\cdot x - \frac{x^3}{3}\right]_1^2 dy$

$= \int_0^1 \left[4y - \frac{8}{3} - 2y + \frac{1}{3}\right] dy$

$\int_0^1 \left(2y - \frac{7}{3}\right) dy$

$= \left[2\cdot\frac{y^2}{2} - \frac{7y}{3}\right]_0^1$

$= 1 - \frac{7}{3}$

$= \cancel{-2}$

$= \frac{-4}{3}$

(2) $\int_0^x \int_1^2 (xy^2 - 2x)\, dx\, dy$

$= \int_1^2 \int_0^x (xy^2 - 2x)\, dy\, dx$ (Using Fubini's theorem)

$= \int_1^2 \left[\frac{xy^3}{3} - 2xy\right]_0^x dx$

$= \int_1^2 \left[\frac{x^4}{3} - 2x^2\right] dx$

$= \left[\frac{x^5}{5 \cdot 3} - \frac{2x^3}{3}\right]_1^2$

$= \frac{32}{15} - 2 \cdot \frac{8}{3} - \frac{1}{15} + \frac{2}{3}$

$= \frac{32 - 80 - 1 + 10}{15}$

$= \frac{-39}{15} = -\frac{13}{5}$

(3) $\int_0^1 \int_1^x (2x + 3y)\, dy\, dx$

$= \int_0^1 \left[2x \cdot y + \frac{3y^2}{2}\right]_1^x dx$

$= \int_0^1 \left(2x^2 + \frac{3x^2}{2} - 2x - \frac{3}{2}\right) dx$

$= \int_0^1 \left(\frac{7x^2}{2} - 2x - \frac{3}{2}\right) dx$

$= \left(\frac{7}{2}\frac{x^3}{3} - \frac{2x^2}{2} - \frac{3}{2}x\right)_0^1$

$= \left(\frac{7}{6} - 1 - \frac{3}{2}\right)$

$= \frac{7 - 6 - 9}{6} = \frac{-8}{6} = -\frac{4}{3}$

TU asked questions

(1) $\int_{-\pi}^{2\pi}\int_{0}^{\pi} (\sin x + \cos y)\, dx\, dy$

$= \int_{-\pi}^{2\pi} \left[\cos x + x\cos y\right]_0^{\pi} dy$

$= \int_{-\pi}^{2\pi} \left[-\cos\pi + \pi\cos y + \cos 0 - 0\cos y\right] dy$

$= \int_{-\pi}^{2\pi} \left[1 + \pi\cos y + 1 - 0\right] dy$

$= \int_{-\pi}^{2\pi} (2 + \pi\cos y)\, dy$

$= \left[2y + \pi\sin y\right]_{-\pi}^{2\pi}$

$= \left[2\cdot 2\pi + \pi\sin 2\pi - (2(-\pi) + \pi\sin(-\pi))\right]$

$= 4\pi + 0 + 2\pi + \cancel{\pi}\sin 0$

$= 6\pi$

(2) $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\, dx$

$= \int_{-1}^{1} [y]_0^{\sqrt{1-x^2}}\, dx$

$= \int_{-1}^{1} \sqrt{1-x^2}\, dx$

$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{-1}^{1}$

$= \frac{1\cdot 0}{2} + \frac{1}{2}\sin^{-1} 1 - \frac{(-1)\cdot 0}{2} - \frac{1}{2}\sin^{-1}(-1)$

$= \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\left(\frac{-\pi}{2}\right)$

$\frac{\pi}{4} + \frac{\pi}{4}$

$\frac{\pi}{2}$

$\int_0^3 \int_0^2 (4-y^2)^2 \, dy \, dx$

$\int_0^3 \left[\frac{(4-y)^3}{3} \cdot \frac{1}{(-1)} \right]_0^2 dx$

$\int_0^3 \left[\frac{-8}{3} + \frac{64}{3} \right] dx$

$\int_0^3 \frac{56}{3} \, dx$

$\left[\frac{56}{3} x \right]_0^3$

56

$\int_{x^2}^{2x} \int_0^2 (x^2+y^2) \, dx \, dy$

$\int_0^2 \int_{x^2}^{2x} (x^2+y^2) \, dy \, dx$

$\int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{2x} dx$

$\int_0^2 \left(2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$

$\int_0^2 \left(\frac{14x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$

$\left[\frac{14}{3} \frac{x^4}{4} - \frac{x^5}{5} - \frac{1}{3} \cdot \frac{x^7}{7} \right]_0^2$

$\frac{14}{3} 4 - \frac{32}{5} - \frac{128}{3 \cdot 7} = \frac{56}{3} - \frac{32}{5} - \frac{128}{21} = \frac{216}{35}$

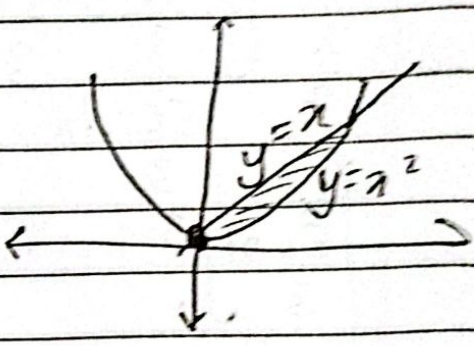
Area using double integer:-

The area of region R bounded by given curves is,

$$A = \iint_R dA$$

(1) Find the area of the region bounded by $y = x^2$ and $y = x$ (Using double integer)

→ Solⁿ: We sketch the region of integration as shown in fig.



For y-limits

Imagine a vertical line through the region which enters at $y = x^2$ and leaves at $y = x$.

For x-limits

Let us move vertical line throughout the region parallel to y-axis. It ~~ente~~ leave the region at $x = 0$.

For another limit of x, solving given equations we get,

$$x^2 = x$$

$$\text{or, } x(x-1) = 0$$

$$\Rightarrow x = 1, 0$$

∴ Required area is,

$$A = \iint_R dA$$

$$= \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 [y]_{x^2}^{x}\, dx$$

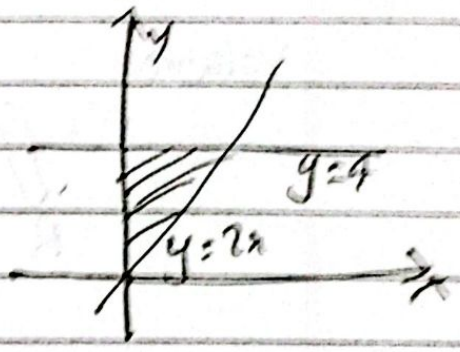
$$= \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$

② Find the area of the region bounded by lines $x=0$, $y=2x$ and $y=4$.

→ Soln: We sketch the region of integration as shown in fig.



For y-limits,
Imagine a vertical line through the region which enters at $y=2x$ and leaves at $y=4$.

For x-limits,
Let us move vertical line throughout the region parallel to y-axis and it leaves the region at $x=0$.
For another limit of x, solving given equations we get,

$$2x=4$$
$$x=2$$

$\therefore$ Required area is

$$A = \iint_R dA$$

$$= \int_0^2 \int_{2x}^4 dy\, dx$$

$$= \int_0^2 [y]_{2x}^4\, dx$$

$$= \int_0^2 (4 - 2x)\, dx$$

$$= \left[4x - \frac{2x^2}{x}\right]_0^2$$

$$= 8 - 4$$

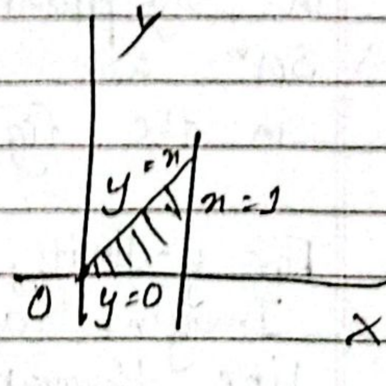
$$= 4 \text{ sq. unit}$$

Volume:-

$$V = \iint_R z \cdot dA \quad , \text{where } z = f(x,y)$$

(9) Find the volume of the prism whose base is the triangle in the xy-plane bounded by x-axis, lines $y = x$ and $x = 1$ and whose top lies on plane $z = f(x,y) = 3 - x - y$.

→ Solⁿ: We sketch the base of triangular prism in x-y-plane as shown in fig.



For y-limits,
Imagine a vertical line through the region which enters at $y = 0$ and leaves at $y = x$.

For x-limits,
Let us move vertical line throughout the region parallel to y-axis. It leaves the region at $x = 0$ and $x = 1$

Now,
Volume of prism,

$$V = \int_0^1 \int_0^x z \, dy \, dx$$

$$= \int_0^1 \int_0^x (3 - x - y) \, dy \, dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx$$

$$= \int_0^1 \left[3x - x^2 - \frac{x^2}{2} \right] dx$$

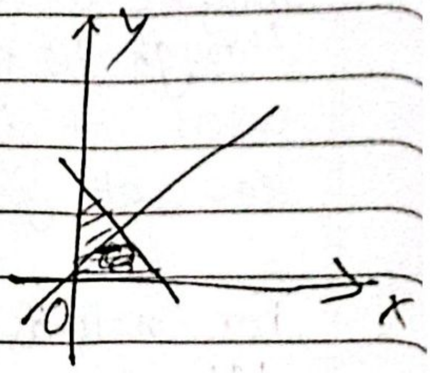
$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} - \frac{1}{2}\frac{x^3}{3} \right]_0^1$$

$$= \frac{3}{2} - \frac{1}{3} - \frac{1}{6}$$

$$= \frac{9 - 2 - 1}{6} = \frac{6}{6} = 1 \text{ cu. units.}$$

Find the volume of region under the paraboloid $z = x^2+y^2$ and triangle $y=x$, $x=0$ and $y = x+y=2$ in xy-plane.

Sol^n:. We sketch the triangle as shown in the fig below



For y-limits,

Imagine a vertical line through the region which enters at $y=x$ and leaves at $y=2-x$

For x-limits,

Let us move vertical line throughout the region parallel to y-axis. It leaves the region at $x=0$

For another limit of x, solving the given equations we get,

$$x = 2-x$$
$$2x = 2$$
$$x = 1$$

Now, Volume of the parabolid

$$V = \iint_R z \cdot dA$$

$$= \int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx$$

$$= \int_0^1 \left[x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \left[2x^2 - x^3 + \frac{(2-x)^3}{3} - \frac{4x^3}{3} \right] dx$$

$$= \int_0^1 \left(2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right) dx$$

$$= \left[\frac{2x^3}{3} - \frac{7}{3}\frac{x^4}{4} + \frac{1}{3} \cdot \frac{(2-x)^4}{4} \cdot \frac{1}{(-1)} \right]_0^1$$

$$= \left(\frac{2x^3}{3} - \frac{7}{12}x^4 - \frac{1}{12}(2-x)^4 \right]_0^1$$

$$= \frac{2}{3} - \frac{7}{12} - \frac{1}{12} + \frac{16}{12}$$

$$= \frac{2}{3} + \frac{\cancel{8}\,2}{\cancel{12}\,3}$$

$$= \frac{4}{3} \text{ cu units.}$$

① Evaluate the following integrals.

a) $\int_0^1 \int_2^{4-2x} dy \, dx$

$$= \int_0^1 [y]_2^{4-2x} \, dx$$

$$= \int_0^1 [4-2x-2] \, dx$$

$$= \int_0^1 2-2x \, dx$$

$$= \left[2x - \frac{2x^2}{2} \right]_0^1 = 2 - 1 = 1$$

b) $\int_{0}^{1}\int_{y-2}^{0} dx\ dy$

$= \int_{0}^{1} [x]_{y-2}^{0}\ dy$

$= \int_{0}^{1} -y+2\ dy$

$= \left[-\frac{y^2}{2} + 2y\right]_{0}^{1}$

$= -\frac{1}{2} + 2$

$= \frac{3}{2}$

c) $\int_{-1}^{0}\int_{-1}^{1} (x+y+1)\,dx\ dy$

$= \int_{-1}^{0} \left[\frac{x^2}{2} + xy + x\right]_{-1}^{1} dy$

$= \int_{-1}^{0} \left[\frac{1}{2} + y + 1 + \frac{1}{2} - y + 1\right] dy$

$= \int_{-1}^{0} 1\ dy$

$= [y]_{-1}^{0}$

$= 1$

d) $\int_{1}^{\ln 8}\int_{0}^{\ln x} e^{x+y}\,dy\,dx$

$= \int_{1}^{\ln 8}\int_{0}^{\ln x} e^{x}.e^{y}\,dy\,dx$

$= \int_{1}^{\ln 8} [e^{x}.e^{y}]_{0}^{\ln x}\,dx$

$= \int_{1}^{\ln 8} [e^{x}.e^{\ln x} - e^{x}.e^{0}]\,dx$

$= \int_{0}^{\ln 8} (e^{x}.x - e^{x})\,dx$

e) $\int_{0}^{1}\int_{2}^{4-2y} dx\ dy$

$= \int_{0}^{1} [x]_{2}^{4-2y}\ dy$

$= \int_{0}^{1} (4-2y-2)\ dy$

$= \int_{0}^{1} [2-2y]\,dy$

$= [2y - y^2]_{0}^{1}$

$= 2-1$

$= 1$

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1) $\int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$

$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x} dx$

$= \int_0^2 \left[2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3}\right] dx$

$= \int_0^2 \left[\frac{14x^3}{3} - x^4 - \frac{x^6}{3}\right] dx$

$= \left[\frac{14}{3}\frac{x^4}{4} - \frac{x^5}{5} - \frac{x^7}{3\cdot 7}\right]_0^2$

$= \frac{14\times \cancel{16}^{4}}{3\times\cancel{4}} - \frac{32}{5} - \frac{128}{3\cdot 7}$

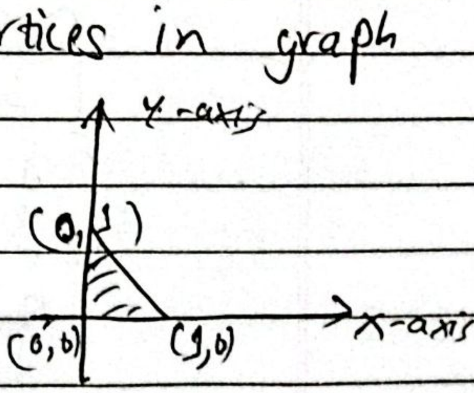
$= \frac{56}{3} - \frac{32}{5} - \frac{128}{21}$

$= \frac{56\times 35 - 32\times 21 - 128\times 5}{105}$

$= \frac{648}{105} = \frac{216}{35}$

(3) Integrate $f(x,y) = x^2 + y^2$ over the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,1)$

→ Here, we plot the given vertices in graph



To ~~dot~~ obtain equation of one side,

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - 0 = \frac{1-0}{0-1}(\overset{x}{\cancel{0}} - 1)$

The other two sides are

$-y = x - 1$ $\quad$ $x = 0$ and $y = 0$

$y = 1 - x$

For y-limits

Imagine a vertical line passing through the region entering at $y = 0$ and leaving at $y = 1 - x$

For x-limits

Let us move the vertical line throughout the region parallel to y-axis, It leaves the region at $x = 0$ and $x = 1$.

Integrating,

$$I = \int_0^1 \int_0^{1-x} x^2 + y^2 \, dy \, dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} \right) dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3} \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(-1)}{3} \frac{(1-x)^4}{4} \right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} - 0 + \frac{1}{12} \right]$$

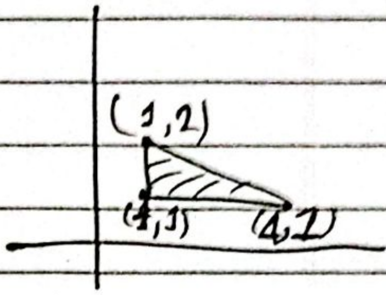
$$= \frac{4 - 3 + 1}{12}$$

$$= \frac{2}{12} = \frac{1}{6}$$

(4) Find the volume of solid tir that lies under the surface $z = xy$ and above the triangle with vertices $(1,1), (4,1)$ and $(1,2)$

→ Soln:

Plotting the points in graph



The two sides forming the triangle are $y=1$ and $x=1$.
For third side,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $$y - 2 = \frac{1-2}{4-1}(x-1)$$

or, $$3y - 6 = -x + 1$$

or, $$x + 3y - 7 = 0$$

$\therefore$ $$y = \frac{7-x}{3}$$

For y-limits
Imagine a vertical line passing through the region entering at $y = 1$ and leaving at $y = \frac{7-x}{3}$

For x-limits,
Let us move the vertical line throughout the region parallel to y-axis. It leaves the region at $x=1$ and $x=4$.

Volume required is given by

$$\text{Volume (V)} = \int_1^4 \int_1^{\frac{7-x}{3}} xy \, dy \, dx$$

$$= \int_1^4 \left[\frac{xy^2}{2}\right]_1^{\frac{7-x}{3}} dx$$

$$= \int_1^4 \left[\frac{x}{2}\left(\frac{7-x}{3}\right)^2 - \frac{x}{2}\right] dx$$

$$= \int_1^4 \left[\frac{x}{2}\left(\frac{49-14x+x^2-9}{9}\right)\right] dx$$

$$= \int_1^4 \left[\frac{40x-14x^2+x^3}{18}\right] dx$$

$$= \frac{1}{18}\left[\frac{40x^2}{2} - \frac{14x^3}{3} + \frac{x^4}{4}\right]_1^4$$

$$= \frac{1}{18}\left[\frac{\overset{20}{\cancel{40}}\cdot 16}{\cancel{2}} - \frac{14 \cdot 64}{3} + \frac{256}{4} - \frac{40}{2} + \frac{14}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{18} \times \frac{279}{4}$$

$$= \frac{31}{8} \text{ cu·unit.}$$

> Calculate the double integral $\iint_R (3x+5)\, dA$, where R is the region given by:
> $R: 0 \le x \le 2,\ x^2 \le y \le 2x$

Sol^n:- Let,

$$I = \iint_R (3x+5)\, dA$$

$= \int_0^2 \int_{x^2}^{2x} (3x+5)\, dy \; dx$

$= \int_0^2 \left[3xy + 5y\right]_{x^2}^{2x} dx$

$= \int_0^2 \left[6x^2 + 10x - 3x^3 - 5x^2\right] dx$

$= \int_0^2 \left[x^2 - 3x^3 + 10x\right] dx$

$= \left[\frac{x^3}{3} - \cancel{3x}\, \frac{3x^4}{4} + \frac{10x^2}{2}\right]_0^2$

$= \left[\frac{8}{3} - 3 \cdot \frac{\overset{4}{\cancel{16}}}{\cancel{4}} + 10 \cdot \frac{4^2}{2}\right]$

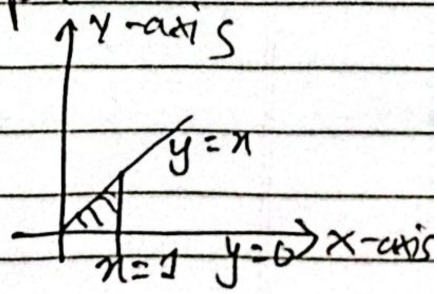
$= \frac{8}{3} - \cancel{12 + 12} - 12 + 20$

$= \frac{8}{3} + 8$

$= \frac{24+8}{3} = \frac{32}{3}$

(6) Calculate $\iint_R \frac{\sin x}{x}\, dA$; where R is triangle in the xy plane bounded by the x-axis, line $y=x$ and line $x=1$.

→ Sol^n.. We sketch the triangle in graph.
Imagine a vertical line passing through the region entering at $y=0$ and leaving at $y=x$



For x limits,
We move the vertical line throughout the region parallel to y-axis. It leaves region at $x=0$ and $x=1$.

Now,

$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1\int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \left[\frac{\sin x}{x}\cdot y\right]_0^x dx$$

$$= \int_0^1 \frac{\cancel{x}\sin x}{\cancel{x}}\, dx$$

$$= \left[-\cos x\right]_0^1$$

$$= -\cos 1 + \cos 0$$

$$= 1 - \cos 1$$

(7) Find the volume of region under the plane $z = 4-x-y$ over rectangular region $R: 0 \le x \le 2$, $0 \le y \le 1$ in xy-plane.

→ Soln:

volume is given by,

$$V = \iint_R z\, dA$$

$$= \int_0^2\int_0^1 4-x-y\, dy\, dx$$

$$= \int_0^2 \left[4y - xy - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^2 4 - x - \frac{1}{2} \, dx$$

$$= \int_0^2 \frac{7}{2} - x \, dx$$

$$= \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2$$

$$= \frac{7\times 2}{2} - \frac{4}{2}$$

$$= 7 - 2 = 5 \text{ cu cunit}$$

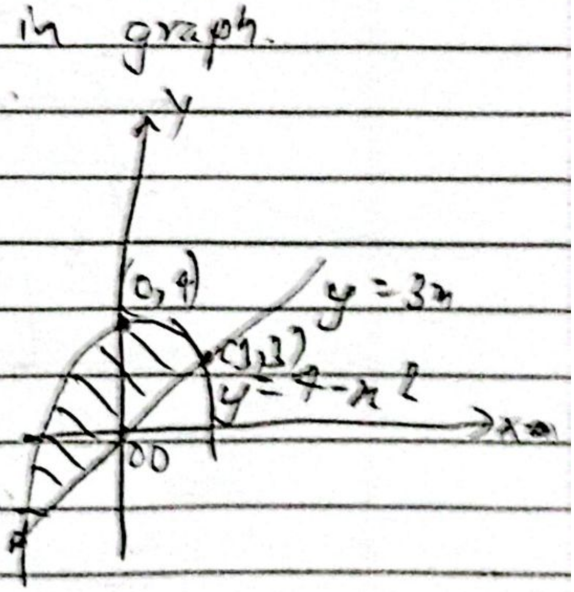
⑧ ~~Find the volume of region bounded by parabolid $z = x^2 + y^2$ and below by triangled~~

⑨ Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$ while top of the solid is bounded by plane $z = x + 4$

→ Sol^n:

Plotting the given curves in graph.

For y-limits,
we imagine a vertical line passing through the region entering at ~~y=0~~ $y = 3x$ and leaving at $y = 4 - x^2$



For x-limits.

let us move the vertical line throughout the region parallel to y-axis. It leaves the region at $x = 1$.

For another limit of x, solving given equations,

$$4 - x^2 = 3x$$

$$\text{or, } x^2 + 3x - 4 = 0$$

$$\text{or, } x^2 + 4x - x - 4 = 0$$

$$\text{or, } x(x+4) - 1(x+4) = 0$$

$$\therefore (x+4)(x-1) = 0$$

$\therefore x = 1$ and $x = -4$

Now Required volume is,

$$V = \iint_R z \, dA$$

$$= \int_{-4}^{1} \int_{3x}^{4-x^2} x + 4 \, dy \, dx$$

$$= \cancel{\int_{-4}^{1} \left[\frac{x^2 + 4x}{2}\right]_{3x}^{4-x^2} dx}$$

$$= \cancel{\int_{-4}^{1} \left[\frac{(4-x^2)^2}{2} + 4(4-x^2) - 9x\right.}$$

$$= \int_{-4}^{1} [xy + 4y]_{3x}^{4-x^2} \, dx$$

$$= \int_{-4}^{1} [x(4-x^2) + 4(4-x^2) - x\cdot 3x - 4\cdot 3x]\, dx$$

$$= \int_{-4}^{1} [4x - x^3 + 16 - 4x^2 - 3x^2 - 12x]\, dx$$

$$= \int_{-4}^{1} [16 - x^3 - 7x^2 - 8x]\, dx$$

$$= \left[16x - \frac{x^4}{4} - \frac{7x^3}{3} - \frac{8x^2}{2}\right]_{-4}^{1}$$

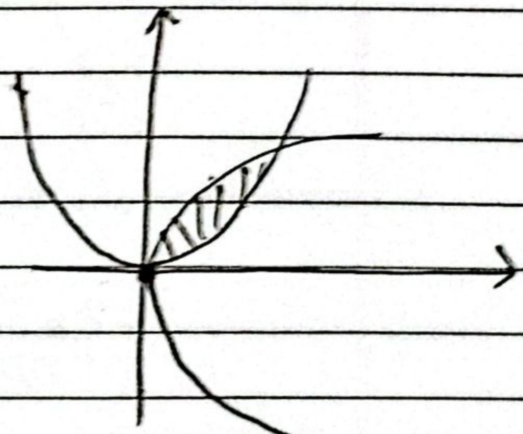
$$= \left[16 - \frac{1}{4} - \frac{7}{3} - 4 + 64 + \frac{256}{4} - \frac{7\times 64}{3} + \frac{8\times 16}{2}\right]$$

$= \frac{625}{12}$ cu. units.

(10) Find the area of the region R bounded by $y = 2x^2$ and $y^2 = 4x$

→ Soln: We plot the given equations in graph.

For y limits,
we imagine a vertical line passing through the region entering at $y = 2x^2$ and leaving at $y = \sqrt{4x} = 2\sqrt{x}$



For x-limits,

Let us move the vertical line throughout the region parallel to y-axis. It leaves the region at $x = 0$.

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For another limit of x, solving given equations we get,

$$\cancel{(2x^2)\sqrt{2} = 4x}$$

$$2x^2 = 2\sqrt{x}$$

$$x^2 = \sqrt{x}$$

$$x^4 = x$$

$$x(x^3 - 1) = 0$$

$$\therefore x = 0$$

$$x = 1$$

Now, area is given by,

$$A = \iint_R dA$$

$$= \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\, dx$$

$$= \int_0^1 [y]_{2x^2}^{2\sqrt{x}} dx$$

$$= \int_0^1 [2\sqrt{x} - 2x^2]\, dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{2x^3}{3}\right]_0^1$$

$$= \left[\frac{4}{3} - \frac{2}{3}\right]$$

$$= \frac{2}{3} \text{ sq. units.}$$

(12) ~~Change the~~ .

a) $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

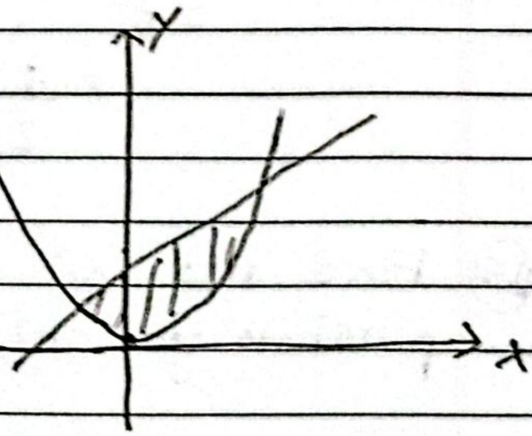
$= \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_0^{\sqrt{1-y^2}} dy$

$= \int_0^1 \left[\frac{(\sqrt{1-y^2})^3}{3} + y^2\sqrt{1-y^2}\right] dy$

(13) Find the area of the region R enclosed by parabola $y = x^2$ and $y = x+2$ by double integral.

→ Soln.:

we sketch the given curves in graph.



For y-limits
we imagine a vertical line throughout the region entering at $y = x^2$ and leaving at $y = x+2$.

For x-limits, we solve the given equations,

$x^2 = x+2$

or, $x^2 - x - 2 = 0$

or, $x^2 - 2x + x - 2 = 0$

or, $x(x-2) + 1(x-2) = 0$

or, $(x-2)(x+1) = 0$

$\therefore x = -1, 2$

Now, Required area is

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx$$

$$= \int_{-1}^{2} [y]_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} (x+2-x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \cancel{9} - \frac{9}{3}\; 6 - 3 + \frac{3}{2}$$

$$= \cancel{9-3}\;\; 3 + \frac{3}{2}$$

$$= \cancel{6}$$

$$= \frac{9}{2}$$

8) $\int_1^{\ln 8}\int_0^{\ln x} e^{x+y}\,dy\,dx$

→ Soln: $\int_1^{\ln 8}\int_0^{\ln x} e^{x}\cdot e^{y}\,dy\,dx$

$= \int_1^{\ln 8} e^{x}\cdot\left[e^{y}\right]_0^{\ln x}\,dx$

$= \int_1^{\ln 8} e^{x}\cdot\left[e^{\ln x} - e^{0}\right]dx$

$= \int_1^{\ln 8} e^{x}(x-1)\,dx$

$= \left[(x-1)\int e^{x}dx - \int\left(\frac{d(x-1)}{dx}\int e^{x}dx\right)dx\right]_1^{\ln 8}$

$= \left[(x-1)e^{x} - e^{x}\right]_1^{\ln 8}$

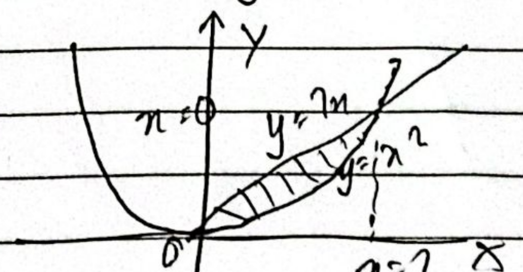
$= (\ln 8 - 1)e^{\ln 8} - e^{\ln 8} - (0-e)$

$= (\ln 8 - 1)\cdot 8 - 8 + e$

$= 8\ln 8 - 16 + e$

Sketch the region of integration of the integral $\int_0^2\int_{x^2}^{2x}(4x+2)\,dx\,dy$ and write the equivalent integral with order of integration reversed.

a) Sketch the region of integration.



$y = x^2$, $y = 2x$

b) Write the equivalent integral with order of Integral reversed.

Ans :- Imagine a horizontal line instead of a vertical line. enters at $x = \frac{y}{2}$ and leaves at $x = \sqrt{y}$

move the horizontal line parallel to x-axis. Leaves at $y=0$ and $y=4$.

$\therefore$ Equivalent integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy$$