Questions

① Evaluate (i) $\int_0^1 x^5(1-x^3)^{10}dx$ (ii) $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2}dx$

(iii) $\int_0^\infty x^{1/4}e^{-\sqrt{x}}\,dx$ (iv) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$

(v) $\int_0^2 x(8-x^3)^{1/3}dx$ (vi) $\int_0^\infty \frac{dx}{1+x^4}$ (vii) $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$

Ans. (ii) $\frac{\Gamma 5/6\,\sqrt{\pi}}{\Gamma 1/3}$ (i) $\frac{1}{396}$ (iii) $\frac{3}{2}\sqrt{\pi}$ (iv) 0 (v) $\frac{16\pi}{9\sqrt{3}}$

(vi) $\frac{\pi\sqrt{2}}{4}$ (vii) $\frac{\pi}{\sqrt{2}}$

② Show that

(i) $\int_0^1 \frac{dx}{\sqrt{-\log_e x}} = \sqrt{\pi}$ (ii) $\int_0^1 \frac{1}{\sqrt{1-x^4}}dx = \frac{1}{4\sqrt{2\pi}}\left(\Gamma\frac{1}{4}\right)^2$

(iii) $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma c+1}{(\log_e c)^{c+1}}$; $c>1$

③ Evaluate $\iint e^{2x+3y}dx\,dy$ over the triangle bounded by $x=0, y=0$ and $x+y=1$. Ans. $-\frac{1}{3}\left[\frac{3}{2}e^2-e^3-\frac{1}{2}\right]$

④ Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$ Ans. $\pi^2/4$

⑤ Evaluate $\iint_R y\,dx\,dy$ where R is bounded by $y^2=4x$ & $x^2=4y$. Ans. $48/5$

⑥ Evaluate $\iint y\,dx\,dy$ over the part of the plane bounded by $y=x$ & $y=4x-x^2$. Ans. $54/5$

⑦ Evaluate (i) $\int_0^1 dx\int_0^x e^{y/x}dy$ & (ii) $\int_0^1\int_{x^2}^x (x^2+3y+2)\,dy\,dx$

Ans. (i) $\frac{1}{2}(e-1)$ (ii) $7/12$

⑧ Evaluate $\iint_A xy\,dx\,dy$ where A is bounded by x-axis, $x=2a$ & $x^2=4ay$. Ans. $\frac{a^4}{3}$

(9) Evaluate by changing the order of integration

(i) $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ Ans. $\frac{3}{8}$

(ii) $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$ Ans. $e-1$

(iii) $\int_1^4 \int_1^{\sqrt{x}} (x+y^2)\,dy\,dx$ Ans. $\frac{241}{30}$

(iv) $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ Ans. $1-\frac{1}{\sqrt{2}}$

(v) $\int_0^\infty \int_0^y y\,e^{-y^2/x}\,dx\,dy$ Ans. $\frac{1}{2}$

(vi) $\int_0^1 \int_{4y}^{4} e^{x^2}\,dx\,dy$ Ans. $\frac{e^{16}-1}{8}$

(vii) $\int_0^\infty \int_0^1 \frac{dy\,dx}{1+yx^2}$

$= \int_0^\infty dx \int_0^1 \frac{dy}{1+yx^2}$ Ans.

(10) Change the order of integration:

(i) $\int_0^{2a} dx \int_0^{x^2/4a} (x+y)^3\,dy$ Ans. $\int_0^a \int_{\sqrt{4ay}}^{2a} (x+y)^3\,dx\,dy$

(ii) $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)\,dy\,dx$ Ans. $\int_0^a \int_{y^2/a}^{a/2-\sqrt{a^2/4-y^2}} f(x,y)\,dx\,dy + \int_0^{a/2} \int_{\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a} f(x,y)\,dx\,dy + \int_{a/2}^{a} \int_{y/a}^{a} f(x,y)\,dx\,dy$

(11) Evaluate by changing to polar coordinates:

(i) $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$ Ans. $\frac{\pi a^5}{20}$

(ii) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ Ans. $4/3$

(iii) $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$ Ans. $8\left(\frac{\pi}{2}-\frac{5}{3}\right)a^2$

(iv) $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ Ans. $\frac{\pi a}{4}$

(v) $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$ Ans. πa^2

(12) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar coordinates.

Ans $\pi^2/8$

(13) Using the transformation $x+y=u$, $y=uv$, show that
$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \frac{1}{2}(e-1)$.

(14) Transform the integral $\iiint (x+y+z)\,x^2y^2z^2\,dx\,dy\,dz$ taken over the volume bounded by $x=0$, $y=0$, $z=0$, $x+y+z=1$, substituting $u=x+y+z$, $x+y=uv$, $y=uvw$ and then evaluate it.

Ans. $\frac{1}{50400}$

(15) Evaluate $\iint_R (x-y)^4 e^{x+y}\,dx\,dy$ where R is the square with vertices at $(1,0)$, $(2,1)$, $(1,2)$ and $(0,1)$.

Ans. $\frac{e(e^2-1)}{5}$

(16) Find the directional derivative of the function

(i) $\phi = 4xz^3 - 3x^2yz^2$ at $(2,-1,2)$ along z-axis.

(ii) $f = xy^2 + yz^3$ at $(2,-1,1)$ in the direction of $\hat{i}+2\hat{j}+2\hat{k}$.

(iii) $\phi = xy^2 + yz^3$ at $(2,-1,1)$ in the direction of the normal to the surface $x\log_e z - y^2 + 4 = 0$ at $(2,-1,1)$

Ans. (i) 144, (iii) $-3\sqrt{2}$ (ii) $-11/3$

(17) Find the directional derivative of $f = xyz$ at $P(1,1,3)$ in the direction of the outward drawn normal of the surface $x^2+y^2+z^2=11$ through the point P.

Ans. $\frac{9}{\sqrt{11}}$

(18) Find the divergence and curl of the vector field $\vec{V} = x^2y^2\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$.

Ans. $2xy^2 + 2x$, $(2y - x)\hat{i} + y\hat{j} + 2y(1 - x^2)\hat{k}$

(19) A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

(i) Is this motion irrotational? If so, find the velocity potential.

(ii) Is the motion possible for an incompressible fluid?

Ans. (i) Yes, $\phi = xy + yz + zx + c$

(ii) Yes

(20) Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

(21) Prove that $\text{curl}(r^n\vec{r}) = \vec{0}$ & $\text{div}(r^n\vec{r}) = (n+3)r^n$

(22) Evaluate the line integrals $\int_c (x^2 + xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

Ans. 0

(23) Find the circulation of $\vec{F}$ round the curve c, where $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ and c is the rectangle whose vertices are $(0,0)$, $(1,0)$, $(1,\pi/2)$, $(0,\pi/2)$.

(24) If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_c \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along following c

(i) $x = t$; $y = t^2$, $z = t^3$

(ii) the straight line joining $(0,0,0)$ to $(1,1,1)$.

(iii) the straight line from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

Ans. (i) 5 , (ii) $\frac{13}{3}$, (iii) $\frac{23}{3}$

(25) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve $x^2+y^2=1$, $z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$ where $\vec{F} = (yz+2x)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}$. Ans. 1.

(26) The vector field $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface S, evaluate $\iint_S \vec{F}\cdot d\vec{s}$. Ans. $abc\left(a+\frac{b}{2}\right)$

(27) Verify divergence theorem for $\vec{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

(28) For any closed surface S, prove that $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 0$.

(29) Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$, where S is a closed surface and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

(30) Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.

(31) Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$. Ans. $1/3$.

(32) ~~Use~~ Verify Stoke's theorem for $\vec{F} = xy^2\hat{i} + y\hat{j} + z^2x\hat{k}$ for the surface of a rectangular lamina bounded by $x=0$, $y=0$, $x=1$, $y=2$, $z=0$.

(33) A vector field is given by $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$. Evaluate $\int_c \vec{F}\cdot d\vec{r}$ where c is the circular path given by $x^2+y^2=a^2$.

Ans. πa^2

(34) Use Green's theorem to evaluate $\int_c (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $y=\pm 1$, $x=\pm 1$.

Ans. 0

~~(35) By the use of Green's theorem, show that area bounded by a simple closed curve is given by $\frac{1}{2}\int (x\,dy - y\,dx)$. Hence find the area of an ellipse.~~

~~πab.~~

(35) Use Green's theorem to evaluate $\int_c 2y^2\,dx + 3x\,dy$ where c is the boundary of the region bounded by $y=x$ & $y=x^2$.

$7/30$

(36) State Green's theorem and verify it for $\int_c (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where c is the boundary of the region ~~et~~ enclosed by the lines $x=0$, $y=0$, $x+y=1$.