Vector Calculus

Scalar point function (or scalar field)

$$f : R^3 \longrightarrow R$$

Ex. Temperature
speed

$$f(x, y, z) = xyz$$
$$f(x, y, z) = 2$$

Vector point function (vector field)

$$f : R^3 \longrightarrow R^3$$

Ex:- $f(x, y, z) = x^2\hat{i} + y\hat{j} + z^2\hat{k}$

Ex:- velocity, pressure.

Gradient :-

Let ϕ be a scalar field Then gradient of ϕ is given by

$$\text{grad}\,\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \nabla \phi$$

where $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Q If $\phi = x^2y + y^2z$, then find grad ϕ at $(1,2,1)$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + y^2z)$$

$$= \hat{i}(2xy) + \hat{j}(x^2 + 2yz) + \hat{k}(y^2)$$

At $(1,2,1)$ grad ϕ $= 4\hat{i} + 5\hat{j} + 4\hat{k}$

Uses of grad ϕ

→ If ~~grad~~ $\phi = c$ $\rightarrow \phi(x,y,z)$ a level surface, then grad ϕ represents the normal vector to the surface at any point

Q If $x^2y + y^2z + z^2x = 2$ is a surface, then find the unit normal vector at $(1,0,1)$.

$$\phi = x^2y + y^2z + z^2x - 2 = 0$$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + y^2z + z^2x)$$

$$\vec{n} = \hat{i}(2xy + z^2) + \hat{j}(x^2 + 2yz) + \hat{k}(y^2 + 2zx)$$

$$\vec{n}\big|_{(1,0,1)} = \hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{n}| = \sqrt{6}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$$

Q Find the angle b/w the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$

$$\phi_1 = x^2+y^2+z^2-9$$
$$\phi_2 = x^2+y^2-z-3$$

$$\text{grad}\,\phi_1 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$\text{grad}\,\phi_1 = \hat{i}(2x)+\hat{j}(2y)+\hat{k}(2z)$$

$$\text{grad}\,\phi_1\Big|_{2,-1,2} = 4\hat{i}-2\hat{j}+4\hat{k}$$

$$\text{grad}\,\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$

at $(2,-1,2)$ $= 4\hat{i}-2\hat{j}-\hat{k}$

$$|\vec{n}_1| = 6$$
$$|\vec{n}_2| = \sqrt{21}$$

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \cancel{(2x\hat{i}}$$

$$= \frac{(4\hat{i}-2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{6\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Directional Derivative

Let ϕ be a scaler field. Then directional derivative of ϕ in the direction of $\vec{b}$ is given by $D_b(\phi) = \nabla\phi \cdot \hat{b}$

Q Find the directional Derivative of $f = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in the Direction of the Line PQ, where Q is the point $(5,0,4)$

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2 - y^2 + 2z^2\right)$$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

$$\vec{n} = \nabla f\Big|_{(1,2,3)} = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

$$\vec{b} = \vec{PQ} = (5-1)\hat{i} + (0-2)\hat{j} + (4-3)\hat{k}$$

$$\vec{B} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$|\vec{b}| = \sqrt{16+4+1} = \sqrt{21}$$

$$\hat{b} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$$D_f(\phi) = \nabla f \cdot \hat{b}$$

$$= \frac{1}{\sqrt{21}}(2\hat{i} - 4\hat{j} + 12\hat{k})(4\hat{i} - 2\hat{j} + \hat{k})$$

$$= \frac{28}{\sqrt{21}} \quad \text{Ans}$$

Q Find the directional Derivative of $\phi = (x^2+y^2+z^2)^{-1/2}$ at $(3,1,2)$ in the Direction of $yz\hat{i} + zx\hat{j} + xy\hat{k}$.

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$\nabla\phi = \cancel{\frac{-1}{2}[(x^2}= \frac{-(x^2+y^2+z^2)^{-3/2}}{2}\left[2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right]$$

$$\vec{n} = \nabla\phi\Big|_{(3,1,2)} = \frac{-(9+1+4)^{-3/2}}{2}(6\hat{i}+2\hat{j}+4\hat{k})$$

$$\vec{n} = -(14)^{-3/2}(3\hat{i}+\hat{j}+2\hat{k})$$

$$\vec{b} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{b}\Big|_{(3,1,2)} = 2\hat{i} + 6\hat{j} + 3\hat{k}$$

$$|\vec{b}| = \sqrt{4+36+9} = \sqrt{49} = 7$$

$$\hat{b} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7}$$

Directional Derivative

$$\vec{n}\cdot\hat{b} = \frac{-(3\hat{i}+\hat{j}+2\hat{k})}{14\sqrt{14}} \cdot \frac{(2\hat{i}+6\hat{j}+3\hat{k})}{7}$$

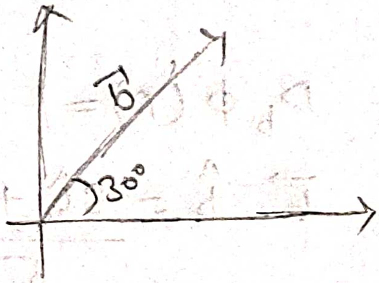
$$= \frac{-9}{49\sqrt{14}} \text{ Ans}$$

Q For $\phi = \frac{y}{x^2+y^2}$, find D.D. making an angle of 30° with the x-axis at (0,1)

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{y}{x^2+y^2}\right)$$

$$= \hat{i}\left(\frac{-y(2x)}{(x^2+y^2)^2}\right) + \hat{j}\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)$$

$$\nabla\phi\Big|_{(0,1)} = 0\hat{i} - \hat{j}$$
$$= -\hat{j}$$



$$\vec{b} = |\vec{b}|\cos 30°\,\hat{i} + |\vec{b}|\sin 30°\,\hat{j}$$

$$\hat{b} = \cos 30°\,\hat{i} + \sin 30°\,\hat{j}$$

$$\hat{b} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$\nabla\phi \cdot \hat{b} = (-\hat{j})\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$= -\frac{1}{2}$$ Ans

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* Max. value of D.D. (Directⁿ Derivative)
  $= |\nabla\phi|$

* Min. value of D.D.
  $= -|\nabla\phi|$

* D.D. is max. in the Drxn of grad ϕ

* D.D. is min. in the Drxn of $-\nabla\phi$

Q what is the greatest rate of increase of $u = xyz^2$ at $(1,0,3)$

$$\nabla u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz^2)$$

$$\nabla u = yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$$

at $(1,0,3)$

$$\nabla u = 9\hat{j}$$

$$|\nabla u| = \sqrt{81} = 9 \quad \text{Ans}$$

Q $D_b\phi(P) = 1$, $D_u\phi(P) = 3$

$$\hat{b} = \frac{3\hat{i} + 4\hat{j}}{5}, \quad \hat{u} = \frac{4\hat{i} - 3\hat{j}}{5}$$

find $\nabla\phi(P)$

$$\nabla\phi(P) = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}$$

$$D_b\phi(P) = 1$$

$$\nabla\phi(P)\cdot\hat{b} = 1$$

$$\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}\right)\cdot\left(\frac{3\hat{i}+4\hat{j}}{5}\right) = 1$$

$$\frac{3}{5}\frac{\partial\phi}{\partial x} + \frac{4}{5}\frac{\partial\phi}{\partial y} = 1 \quad \text{--- (1)}$$

$$D_u\phi(P) = 3$$

$$\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}\right)\left(\frac{4\hat{i}-3\hat{j}}{5}\right) = 3$$

$$\frac{4}{5}\frac{\partial\phi}{\partial x} - \frac{3}{5}\frac{\partial\phi}{\partial y} = 3 \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial x} = 3 \quad , \quad \frac{\partial \phi}{\partial y} = -1$$

$$\nabla \phi = 3\hat{i} - \hat{j} \quad \text{Ans}$$

Divergence of a vector field

Let $\vec{F}$ be a vector field. Then divergence of $\vec{F}$ is a given by

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} \quad , \text{ where } \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial F_3}{\partial z}$$

- It represents the rate of outward flow per unit volume.
- For incompressible fluid, $\text{div}\,\vec{F} = 0$
- If $\text{div}\,\vec{F} = 0$, then vector field is called solenoidal.

curl

curl of vector field $\vec{F}$ is given by

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- curl $\vec{F}$ tells about the rotation of the vector field.
- If curl $\vec{F} = \vec{0}$ then vector field is irrotational.

Q A fluid motion is given by

$\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

(i) Is this motion irrotational

(ii) Is this motion possible for an incompressible fluid?

(i) curl $\vec{V} = \nabla \times \vec{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial(x+y)}{\partial y} - \frac{\partial(z+x)}{\partial z}\right) - \hat{j}\left(\frac{\partial(x+y)}{\partial x} - \frac{\partial(y+z)}{\partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial(z+x)}{\partial x} - \frac{\partial(y+z)}{\partial y}\right)$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$= \vec{0}$ (yes) this motion is irrotational

(ii) div $\vec{V} = \nabla \cdot \vec{V}$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y)$$

$= 0$ (yes) this motion is incompressible fluid

Q Find div $\vec{F}$ and curl $\vec{F}$ where

$$\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$\vec{F} = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

$$\text{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}\right]$$

$$= 6x + 6y + 6z$$

$$\text{curl}\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$$

$$= \hat{i}(-3x+3x) - \hat{j}(-3y+3y) + \hat{k}(-3z+3z)$$

$$= \vec{0}$$

Gradient in polar coordinates

(F) For $f = \phi(r)$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} \quad \because |\vec{r}| = r$$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$\nabla f = \frac{\partial f}{\partial r} \cdot \frac{\vec{r}}{r}$$

Q $f = \frac{1}{r^2}$ Find grad f or ∇f?

$$\nabla f = \frac{\partial f}{\partial r} \cdot \frac{\vec{r}}{r}$$

$$= \frac{-2}{r^3} \cdot \frac{\vec{r}}{r}$$

$$= \frac{-2}{r^4}\vec{r}$$

Q $f = |\vec{r}|^2$ Find $\nabla f = ?$

$$f = r^2$$

$$\nabla f = 2\cancel{r} \cdot \frac{\vec{r}}{\cancel{r}}$$

$$= 2\vec{r}$$

Q $f = e^{r^2}$ Find ∇f

$\nabla f = 2e^{r^2} \cdot \cancel{r} \cdot \frac{\vec{r}}{\cancel{r}}$

$= 2e^{r^2} \cdot \vec{r}$

(ii) $f = \phi(r, \theta)$

$\nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta$

$f = r\cos\theta + \tan\theta$ Find ∇f

$\nabla f = \cos\theta \, \hat{e}_r + \frac{1}{r}(-r\sin\theta + \sec^2\theta) \hat{e}_\theta$

Vector Identities

1. $\text{div}(u\vec{a}) = u \, \text{div} \vec{a} + (\text{grad } u) \cdot \vec{a}$
2. $\text{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl } \vec{a} - \vec{a} \text{ curl } \vec{b}$
3. $\text{curl}(u\vec{a}) = u \, \text{curl } \vec{a} + \text{grad } u \times \vec{a}$
4. $\text{div}(\text{grad } \phi) = \nabla^2 \phi$; $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$
5. $\text{curl}(\text{grad } \phi) = \nabla \times \nabla \phi = \vec{0}$
6. $\text{div}(\text{curl } \vec{V}) = \nabla \cdot (\nabla \times \vec{V}) = 0$

Q If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$

show that

(i) $\text{div}(\vec{r}\phi) = 3\phi + \vec{r} \cdot \text{grad } \phi$

(ii) $\text{div}(\hat{r}) = \frac{2}{r}$

(i) $\longrightarrow$ $div(\vec{r}\phi)$

$= \phi \, div \vec{r} + grad\phi \cdot \vec{r}$

$div \vec{r} = \nabla \cdot \vec{r}$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$

$= 1+1+1 = 3$

so $= 3\phi + grad\,\phi \cdot \vec{r}$

(ii) $div(\hat{r})$

$= div\left(\frac{\vec{r}}{r}\right)$

$= \frac{1}{r} \cdot div(\vec{r}) + grad\left(\frac{1}{r}\right) \cdot \vec{r}$

$= \frac{3}{r} + \left(\frac{-1}{r^2} \cdot \frac{\vec{r}}{r} \cdot \vec{r}\right)$

$= \frac{3}{r} + \left(\frac{-1}{\cancel{r^2}} \cdot \frac{\cancel{r^2}}{r}\right)$

$= \frac{2}{r}$

Q Show that $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal.

$curl \vec{F} = curl\left(\frac{\vec{r}}{r^3}\right)$

$= \frac{1}{r^3} curl(\vec{r}) + grad\left(\frac{1}{r^3}\right) \times \vec{r}$

$$curl(\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

$= \frac{1}{r^3}\vec{0} + \left(\frac{-3}{r^4}\cdot\frac{\vec{r}}{r}\right)\times\vec{r}$ $\left[\because \vec{r}\times\vec{r} = 0\right]$

$= \vec{0} + \vec{0} = \vec{0}$ (Irrotational)

Q. prove that ~~div(∇φ × ∇ψ)~~ $\operatorname{div}(\nabla\phi\times\nabla\psi) = 0$

$\operatorname{div}(\vec{a}\times\vec{b}) = b\cdot\operatorname{curl}\vec{a} - \vec{a}\cdot\operatorname{curl}\vec{b}$

$\vec{a} = \nabla\phi,\ \vec{b} = \nabla\psi$

$\operatorname{div}(\nabla\phi\times\nabla\psi)$

$= \nabla\psi\ \operatorname{curl}(\nabla\phi) - \nabla\phi\cdot\operatorname{curl}(\nabla\psi)$

$= \nabla\psi\cdot\vec{0} - \nabla\phi\cdot\vec{0}$

$= 0$

Scalar Potential

If $\vec{A} = M\hat{i} + N\hat{j} + P\hat{k}$ then Scalar Potential (ϕ)

$\operatorname{curl}\vec{A} = \vec{0}$

$\hookrightarrow \vec{A} = \operatorname{grad}\phi$

$$\phi = \int M\,dx + \int\begin{pmatrix}\text{Terms of } N\\ \text{without } x\end{pmatrix}dy + \int\begin{pmatrix}\text{Terms of } P \text{ without}\\ x \text{ and } y\end{pmatrix}dz + C$$

Q. $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$, $\operatorname{curl}\vec{A} = 0$

Find scalar potential

$$\phi = \int (x^2 + xy^2)\,dx + \int y^2\,dy + \int + C$$

$$= \frac{x^3}{3} + \frac{x^2y^2}{2} + \frac{y^3}{3} + C$$

Line Integral

The Integral along a curve is called Line Integral.

It is defined as $\int \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is the vector field and c is the curve.

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then

$$\int_c \vec{F} \cdot \vec{dr} = \int_c \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \int_c F_1\,dx + F_2\,dy + F_3\,dz$$

* If $\vec{F}$ represents the Force during displacement from A to B, then,

$$\text{work done} = \int_A^B \vec{F} \cdot \vec{dr}$$

* If c is the closed curve and $\vec{V}$ is the velocity,

$$\text{Circulation} = \oint_c \vec{F} \cdot d\vec{r}$$

Q If $\vec{F} = 3xy\,\hat{i} - y^2\,\hat{j}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

$$\vec{F} \cdot d\vec{r} = \left(3xy\,\hat{i} - y^2\,\hat{j}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= 3xy\,dx - y^2\,dy$$

$$y = 2x^2$$

$$dy = 4x\,dx$$

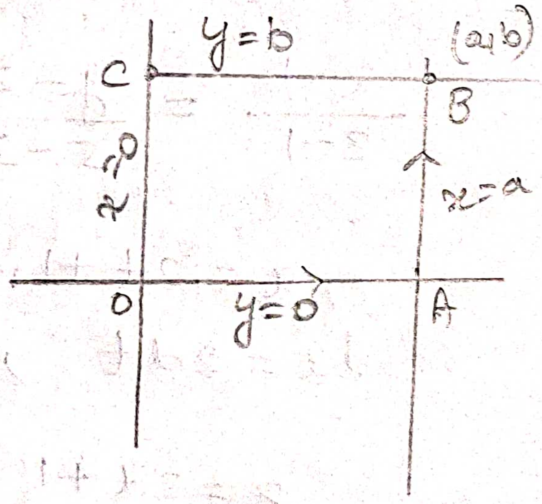
$$\vec{F}\cdot d\vec{r} = 3x(2x^2)\,dx - 4x^4 \cdot 4x\,dx$$
$$= (6x^3 - 16x^5)\,dx$$

$$0 \leq x \leq 1$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\,dx$$
$$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$
$$= \frac{\cancel{6}\,3}{\cancel{4}\,2} - \frac{\cancel{16}\,8}{\cancel{6}\,3} = \frac{9-16}{6} = -7/6$$

Q Find the total work done by a force $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ in moving a point from $(0,0)$ to (a,b) along the rectangle bounded by lines $x=0$, $x=a$, $y=0$, $y=b$.

$$\vec{F}\cdot d\vec{r} = \left((x^2+y^2)\hat{i} - 2xy\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$$
$$= (x^2+y^2)\,dx - 2xy\,dy$$



Along OA :-

$y=0$

$dy=0$

$\vec{F}\cdot d\vec{r} = x^2\,dx$

$0 \leq x \leq a$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

Along AB :-

$x = a$
$dx = 0$

$\vec{F}\cdot d\vec{r} = -2ay\, dy$

$0 \leq y \leq b$

$I_2 = \int_{AB} F\cdot dr = \int_0^b -2ay\, dy = -2a\left[\frac{y^2}{2}\right]_0^b$

$= -ab^2$

Total work done $= I_1 + I_2$
$= \frac{a^3}{3} - ab^2$ Ans

Q $\vec{F} = x^3\hat{i} + y\hat{j} + z\hat{k}$ find work done by $\vec{F}$ along the Line from $(1,2,3)$ to $(3,5,7)$

$\vec{F}\cdot d\vec{r} = (x^3\hat{i} + y\hat{j} + z\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$
$= x^3 dx + y\,dy + z\,dz$

$\frac{x-1}{3-1} = \frac{y-2}{5-2} = \frac{z-3}{7-3} = t$

$x = 2t+1, \quad y = 3t+2, \quad z = 4t+3$
$dx = 2dt, \quad dy = 3dt, \quad dz = 4dt$

$x = 2t+1$
$x = 1 \rightarrow t = 0$
$x = 3 \rightarrow t = 1$

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 2(2t+1)^3\,dt + 3(3t+2)\,dt + 4(4t+3)\,dt$$

$$= \cancel{2}\left[\frac{(2t+1)^4}{4\,\cancel{(2)}}\right]_0^1 + \left[\frac{\cancel{3}(3t+2)^2}{2(\cancel{3})}\right]_0^1 + \left[\frac{\cancel{4}(4t+3)^2}{2(\cancel{4})}\right]_0^1$$

$$= \frac{101}{2}$$

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Green's Theorem

Let C be a closed curve in the plane and R be the region bounded by C, then

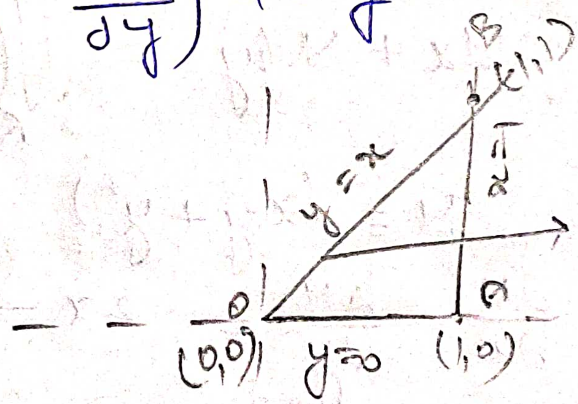
$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

where M and N are continiously differentiable functions.

Using Green's Theorem, evaluate $\int_C x^2y\,dx + x^2\,dy$, where C is the boundary described counter clockwise of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.

By Green's Theorem:-

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$$M = x^2y, \quad N = x^2$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x^2$$

Limits

$y \leq x \leq 1$

$0 \leq y \leq 1$

$$\int_0^1 \int_y^1 (2x - x^2)\, dx\, dy = \int_0^1 2\left[\frac{x^2}{2}\right]_y^1 - \left[\frac{x^3}{3}\right]_y^1 dy$$

$$= \int_0^1 2\left(\frac{1}{2} - \frac{y^2}{2}\right) - \left(\frac{1}{3} - \frac{y^3}{3}\right) dy$$

$$= \int_0^1 1 - y^2 - \frac{1}{3} + \frac{y^3}{3}\, dy$$

$$= \int_0^1 \frac{2}{3} - y^2 + \frac{y^3}{3}\, dy = \frac{2}{3}[y]_0^1 - \left[\frac{y^3}{3}\right]_0^1 + \left[\frac{y^4}{4 \times 3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{1}{12} = \frac{8 - 4 + 1}{12} = \frac{5}{12}$$

Q verify Green's Theorem in the plane for $\int_c [(xy + y^2)dx + x^2 dy]$ where c is the ~~bounde~~ closed curve of the region bounded by $y = x$ & $y = x^2$

By Green's Theorem:-

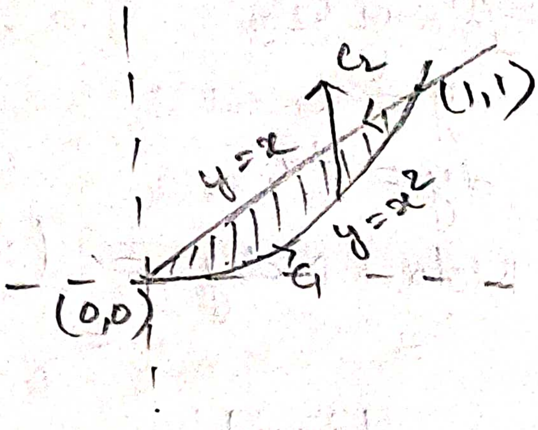
$$\int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$M = (xy + y^2), \quad N = x^2$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - (x + 2y) = x - 2y$$

$x^2 \le y \le x$

$0 \le x \le 1$



RHS :- $\int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$

$= \int_0^1 [xy]_{x^2}^{x} - \cancel{2}\left[\frac{y^2}{\cancel{2}}\right]_{x^2}^{x} dx$

$= \int_0^1 x(x-x^2) - (x^2-x^4)\,dx$

$= \int_0^1 (\cancel{x^2} - x^3 - \cancel{x^2} + x^4)\,dx = \left[\frac{-x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1$

$= \frac{-1}{4} + \frac{1}{5} = \frac{-1}{20}$

LHS :-

$Mdx + Ndy = (xy+y^4)\,dx + x^2\,dy$

Along C_1

$y = x^2$

$dy = 2x\,dx$

$Mdx + Ndy = (x^3+x^4)\,dx + x^2(2x)\,dx$

$= (3x^3 + x^4)\,dx$

Limit :- $0 \le x \le 1$

$I = \int_0^1 (3x^3+x^4)\,dx = 3\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1 dx$

$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$

Along C_2

$y = x$

$dy = dx$

$$Mdx + Ndy = (x^2+x^2)dx + x^2dx$$
$$= 3x^2dx$$

Limit!

x : 1 to 0

$$I_2 = -\int_0^1 3x^2dx = -\cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_0^1 = -1$$

LHS:- $I_1 + I_2$

$$= \frac{19}{20} - 1 = \frac{-1}{20} = RHS$$

Hence, Green's Theorem is verified.

Q If C is a simple closed curve in the xy-plane not containing the origin, evaluate $\int_c \vec{F}\cdot d\vec{r}$; where $\vec{F} = \dfrac{-y\hat{i} + \hat{j}x}{x^2+y^2}$

By Green's Theorem:-

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$M = \frac{-y}{x^2+y^2}, \quad N = \frac{x}{x^2+y^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2) - x(2x)}{x^2+y^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 0$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \underline{0}.$$

Stoke's Theorem

If S is an open surface bounded by closed curve C, then

$$\int_C \vec{F}\cdot d\vec{r} = \iint \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

where $\hat{n}$ is the unit normal vector to the surface (S) drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C.

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} = \frac{dz\,dx}{\hat{n}\cdot\hat{j}}$$

Q Verify stoke's theorem for $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x=0$, $y=0$, $x=a$, $y=a$. in the plane $z=0$.

By stoke's Theorem,

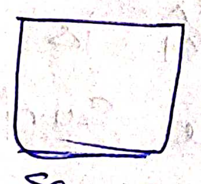
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\, ds$$

$$\text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(y)$$

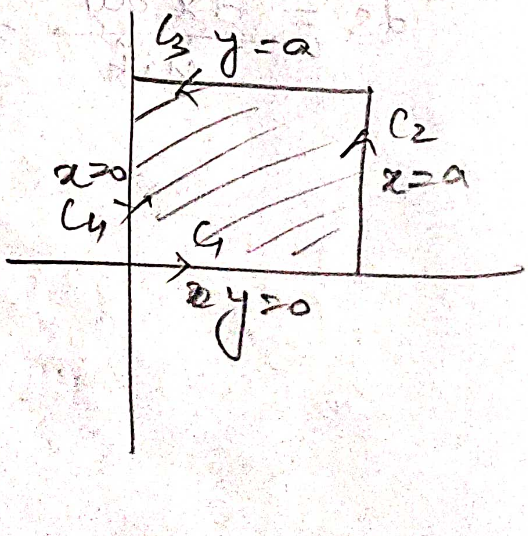
$$= y\hat{k}$$

$$\hat{n} = \hat{k}$$



$$\text{curl}\vec{F}\cdot\hat{n} = y\hat{k}\cdot\hat{k}$$
$$= y$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$$



Limits

$0 \leq x \leq a$
$0 \leq y \leq a$

RHS :- $\iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds$

$$= \int_0^a \int_0^a y\, dx\, dy = \frac{a^3}{2}$$

$\vec{F} \cdot d\vec{r} = x^2 dx + xy\, dy$

Along C_1 :-

$y = 0 \longrightarrow dy = 0$

$\vec{F} \cdot d\vec{r} = x^2 dx$

$0 \leq x \leq a$

$I_1 = \int_0^a x^2 dx = \frac{a^3}{3}$

Along C_3

$y = a$

$dy = 0$

$\vec{F} \cdot d\vec{r} = x^2 dx$

$I_3 = -\int_0^a x^2 dx$

$= -a^3/3$

Along C_2 :-

$x = a$

$dx = 0$

$\vec{F} \cdot d\vec{r} = ay\, dy$

$0 \leq y \leq a$

$I_2 = \int_0^a ay\, dy = \frac{a^3}{2}$

Along C_4 :-

$x = 0$

$dx = 0$

$y :- a$ too

$I_4 \int_0^a 0\, dy = 0$

LHS :- $I_1 + I_2 + I_3 + I_4$

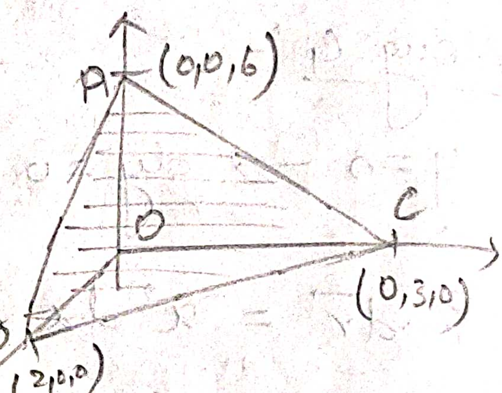
$\cancel{\frac{a^3}{3}} - \cancel{\frac{a^3}{3}} + \frac{a^3}{2} + 0 = \frac{a^3}{2}$

Q Apply stoke's theorem to evaluate

$$\int_C [(x+y)dx + (2x-3)dy + (y+z)dz]$$

where C is the boundary of the Triangle with vertices (2,0,0), (0,3,0) & (0,0,6)

By stoke's Theorem



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-3 & y+z \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(0-0) + \hat{k}(2-1)$$

$$\text{curl}\vec{F} = 2\hat{i} + \hat{k}$$

Eq. of plane is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$\phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1$$

$$\vec{n} = \Delta\phi$$

$$= \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}$$

$$|\vec{n}| = \sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}} = \frac{\sqrt{14}}{6}$$

$$\hat{n} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$$\text{curl}\,\vec{F}\cdot\hat{n} = (2\hat{i} + \hat{k})\cdot\frac{(3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{1/\sqrt{14}} = \sqrt{14}\,dx\,dy$$

$$\iint_s \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

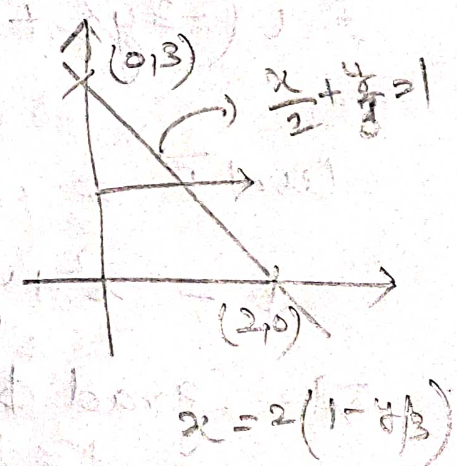
$$= \iint \frac{7}{\cancel{\sqrt{14}}}\cdot\cancel{\sqrt{14}}\,dx\,dy = 7\iint_s dx\,dy$$

If no function then it represents Area

$$= 7(\text{Area of Triangle})$$

$$= 7\left(\frac{1}{2}\times 2\times 3\right)$$

$$= 21$$



Q verify stoke's Theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the x-y p

By stoke's Theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}(-2yz+2yz) - \hat{j}(0-0) + \hat{k}(0-(-1))$$

$$\text{curl}\,\vec{F} = \hat{k}$$

$$\phi = x^2+y^2+z^2-1$$

$$\vec{n} = \text{grad}\,\phi$$

$$\vec{n} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\vec{n}| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2}$$

$$= 2$$

$$\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{curl}\,\vec{F}\cdot\hat{n} = (\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= z$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{z}$$

$$= \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds = \iint_S \cancel{z}\,\frac{dx\,dy}{\cancel{z}}$$

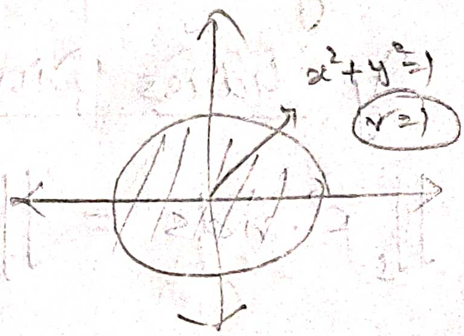
$$= \iint dx\,dy = \pi(1)^2$$

$$= \pi$$

LHS :-

$$\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

In x-y Plane



$z=0$

$dz=0$

$$\vec{F}\cdot d\vec{r} = (2x-y)\,dx$$

let $x = \cos t$, $y = \sin t$

$dx = -\sin t\,dt$

$0 \leq t \leq 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = -\int_0^{2\pi} (2\cos t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} (\sin 2t - \sin^2 t)\,dt$$

$$= -\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + 4\int_0^{\pi/2} \sin^2 t\,dt$$

$$= \frac{4\,\Gamma 3/2\,\Gamma 1/2}{2\,\Gamma 2} = 2\cdot\frac{1}{2}\pi = \pi$$

Gauss Divergence Theorem

If $\vec{F}$ is a vector point function having Ist order continuous partial derivatives and S is the closed surface enclosing the volume V, then:-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\cdot dv$$

* $\hat{n}$ is the outward unit normal vector

Q Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$

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By Gauss Divergence Theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv$$

RHS:- $\text{div}.\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

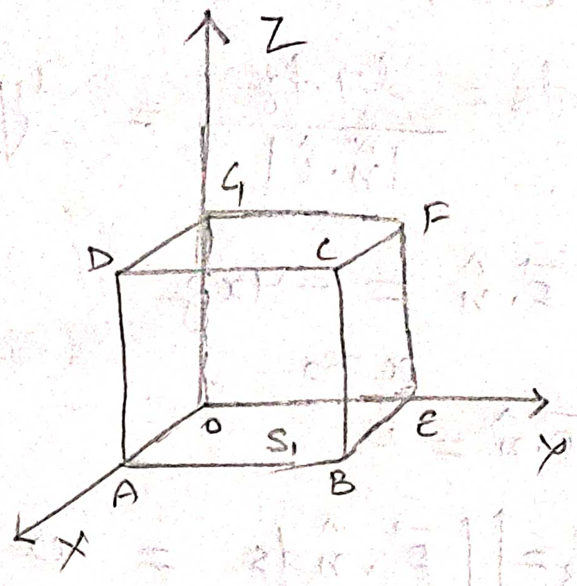
$$\iiint_V \text{div}\,\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 4z[x]_0^1 - y[x]_0^1\,dy\,dz = \int_0^1\int_0^1 4z - y\,dy\,dz$$

$$= \int_0^1 4z[y]_0^1 - \left[\frac{y^2}{2}\right]_0^1 dz = \int_0^1 4z - \frac{1}{2}\,dz$$

$$= \left[2z^2\right]_0^1 - \frac{1}{2}\left[z\right]_0^1$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$



LHS :

For AOEB,

$z = 0$, $\hat{n} = -\hat{k}$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$$

$$\vec{F}\cdot\hat{n} = -yz$$

$\because z = 0$

$$\vec{F}\cdot\hat{n} = 0$$

$$I_1 = \iint_{AOEB} \vec{F}\cdot\hat{n}\,ds = \iint_{AOEB} 0\,ds = 0$$

For DCFG

$z = 1$, $\hat{n} = \hat{k}$

$ds = dx\,dy$

$\vec{F}\cdot\hat{n} = yz$

$\because z = 1$

$\vec{F}\cdot\hat{n} = y$

$$I_2 = \iint_{DCFG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y\,dx\,dy$$

$$= \frac{1}{2}$$

For OEFG

$x=0, \quad \hat{n} = -\hat{i}$

$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = dy\,dz$

$\vec{F}\cdot\hat{n} = -4xz$

$x=0$

$\vec{F}\cdot\hat{n} = 0$

$I_3 = \iint_{OEFG} \vec{F}\cdot\hat{n}\,ds = 0$

For AOGD

$y=0, \quad \hat{n} = -\hat{j}$

$ds = dx\,dz$

$\vec{F}\cdot\hat{n} = y^2$

$y=0$

$\vec{F}\cdot\hat{n} = 0$

$I_5 = \iint_{ABCD} 0\,dx\,dy = 0$

For ABCD

$x=1, \quad \hat{n} = \hat{i}$

$ds = dy\,dz$

$\vec{F}\cdot\hat{n} = 4xz \; (x=1) = 4z$

$I_4 = \int_0^1\int_0^1 4z\,dy\,dz$

$= 2$

For BEFC

$y=1, \quad \hat{n} = \hat{j}$

$ds = dx\,dz$

$\vec{F}\cdot\hat{n} = -y^2 \; (y=1)$

$= -1$

$I_6 = \int_0^1\int_0^1 -1\,dx\,dz$

$= -1$

LHS:- $I_1 + I_2 + I_3 + I_4 + I_5 + I_6$

$= \frac{3}{2} = \text{RHS}$

Hence Gauss Divergence Theorem verified

Q Find $\iint_S \vec{F}\cdot\hat{n}\,ds$ where

$\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and

S is the surface of the sphere having centre at $(3,-1,2)$ and radius 3.

By Gauss Divergence Theorem

$$\iint \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left((2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}\right)$$

$$= 3$$

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iiint 3\,dv = 3\times \text{vol. of sphere}$$

$$= 3\cdot\frac{4}{3}\pi r^3 = 4\pi(3)^3$$

$$= 108\pi$$