Double Integration in polar Coordinates :-

Q $$\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

let $a^2-r^2=t$

$-2r\,dr=dt$

$r=0 \longrightarrow t=a^2$

$r=a\cos\theta \longrightarrow t\,a^2\sin^2\theta$

$$=\frac{1}{2}\int_0^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} t^{1/2}\,dt\,d\theta \quad = \frac{1}{2}\int_0^{\pi/2}\left[\frac{t^{3/2}}{3/2}\right]_{a^2}^{a^2\sin^2\theta} d\theta$$

$$=\frac{-1}{3}\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right)d\theta$$

$$=\frac{-1}{3}\left(\frac{a^3\,\Gamma 4/2\cdot\Gamma 1/2}{2\,\Gamma 5/2} - a^3\frac{\pi}{2}\right)$$

$$=\frac{-a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right]$$ Ans

Q $\displaystyle\int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr\, d\theta$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta \left[a^3(1-\cos\theta)^3 - 0\right] d\theta$$

Let $1-\cos\theta = t$

$\cos\theta = 1-t$

$\cancel{-}\sin\theta\, d\theta = \cancel{-}dt$

$d\theta = \dfrac{dt}{\sin\theta}$

$\theta = 0 \rightarrow t = 0$

$\theta = \pi \rightarrow t = -1$

$$= \frac{1}{3}\int_0^{\cancel{\theta}-1} a^3 \cdot t^3 \cdot dt$$

$$= \frac{1}{3} a^3 \left[\frac{t^4}{4}\right]_0^{\cancel{\pi}-1}$$

$$= \frac{1}{3} a^3 \cdot \frac{1}{4} = \frac{a^3}{12}$$

Q Evaluate $\iint r^3 dr\,d\theta$ over the area bounded b/w the circle $r = 2\cos\theta$ & $r = 4\cos\theta$.

we know,

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre $(-g, -f)$, radius $\sqrt{g^2 + f^2 - c^2}$

For r

$r^2 =$

$x^2+y^2 =$

x^2+y^2

Centre

radi

For $r = 2\cos\theta$

$r^2 = 2r\cos\theta$

$x^2+y^2 = 2x$

$x^2+y^2-2x = 0$

centre $= (1,0)$

radius $(\sqrt{1}) = 1$

For $r = 4\cos\theta$

$r^2 = 4r\cos\theta$

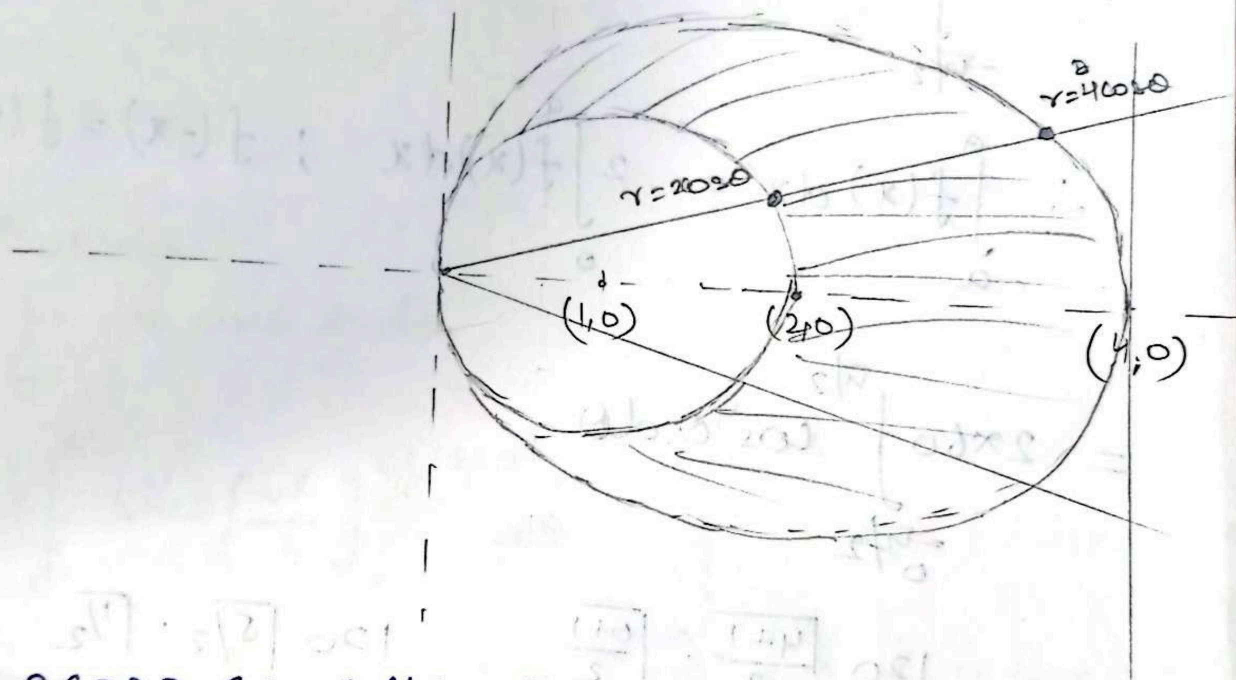
$x^2+y^2 = 4x$

$x^2+y^2-4x = 0$

centre $(2,0)$

radius $= \sqrt{4+0-0}$

$= 2$



$2\cos\theta \leq r \leq 4\cos\theta$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$= \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r^3 \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(4^4\cos^4\theta - 2^4\cos^4\theta\right) d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} 240\cos^4\theta \, d\theta$$

$$= 60\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \; ; \; f(-x) = f(x)$$

$$= 2\times 60 \int_{0}^{\pi/2} \cos^4\theta \, d\theta$$

$$= 120 \frac{\Gamma\left(\frac{4+1}{2}\right)\cdot\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{4+0+2}{2}\right)} = \frac{120\,\Gamma(5/2)\cdot\Gamma(1/2)}{2\cdot 3\,\Gamma(6/\cancel{2})}$$

$$= \overset{15}{\cancel{30}}\cdot\frac{3}{\cancel{2}}\cdot\frac{1}{2}\cdot\left(\sqrt{\pi}\right)^2$$

$$= \frac{45\pi}{2} \quad \underline{\text{Ans}}$$

Q Ev
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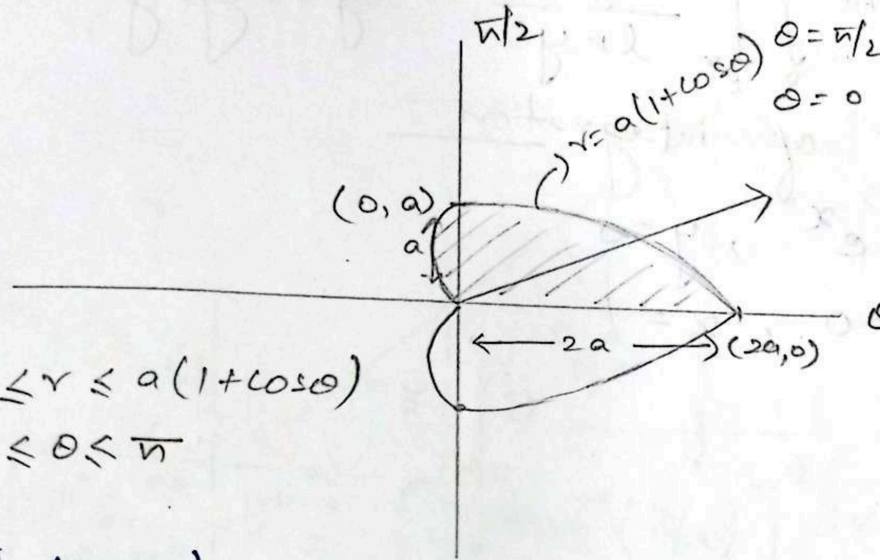
Q Evaluate $\iint r\sin\theta \, dr \, d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ above the initial line.



$\theta = \pi/2$, $r = a$

$\theta = 0$, $r = 2a$

$0 \le r \le a(1+\cos\theta)$

$0 \le \theta \le \pi$

$$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{a^2(1+\cos\theta)^2}{2}\right] d\theta$$

Let $1+\cos\theta = dt$

$-\sin\theta d\theta = dt$

$d\theta = \frac{-dt}{\sin\theta}$

$$= \frac{1(-)}{2}\int_{2}^{0} a^2 t^2 \, dt$$

$$= \frac{+1}{2} a^2 \left[\frac{t^3}{3}\right]_0^2$$

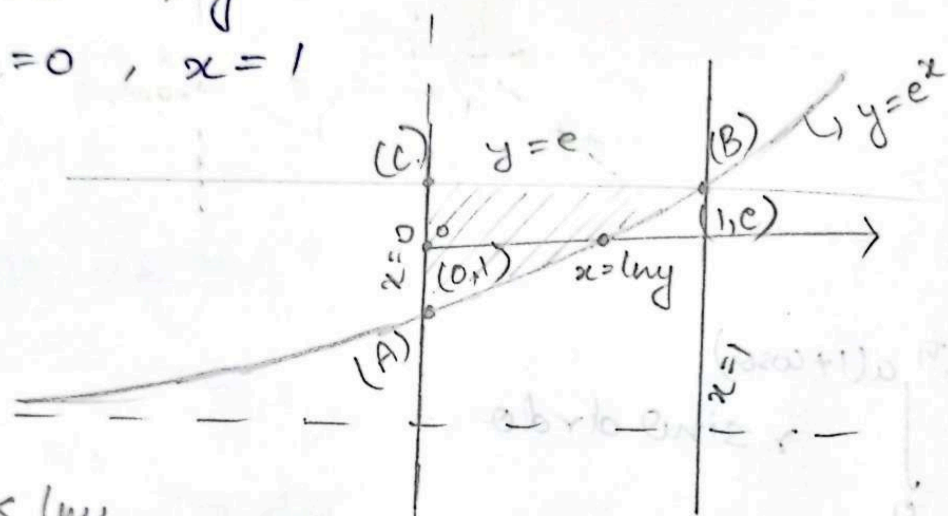
$$= \frac{1}{2} a^2 \left[\frac{8}{3} - 0\right] = \frac{4a^2}{3} \text{ Ans}$$

Change of order of Integration

Q Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\ln y}$ by changing the order of integration.

$y = e^x$, $y = 0$

$x = 0$, $x = 1$



$0 \le x \le \ln y$

$1 \le y \le e$

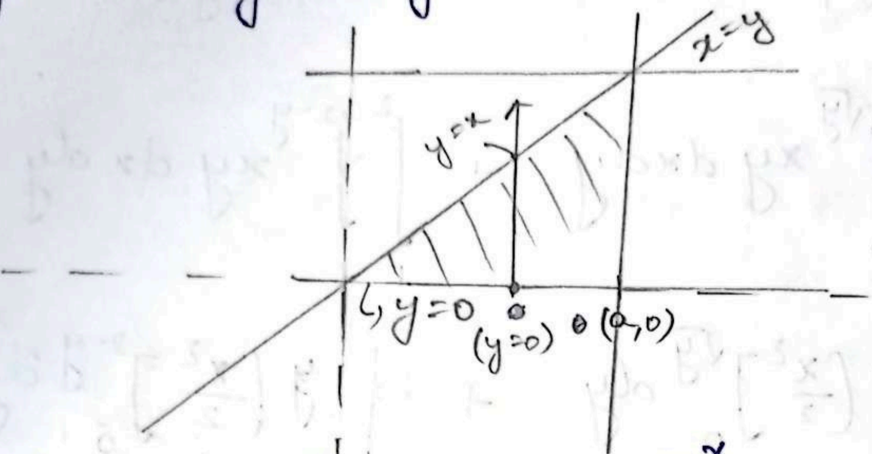
$$\int_1^e \int_0^{\ln y} \frac{dx\,dy}{\ln y}$$

$$= \int_1^e \frac{1}{\ln y}[x]_0^{\ln y}\,dy = \int_1^e dy = [y]_1^e$$

$= (e-1)$ Ans

Q Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$

$x=y,\ x=a,\ y=0,\ y=a$



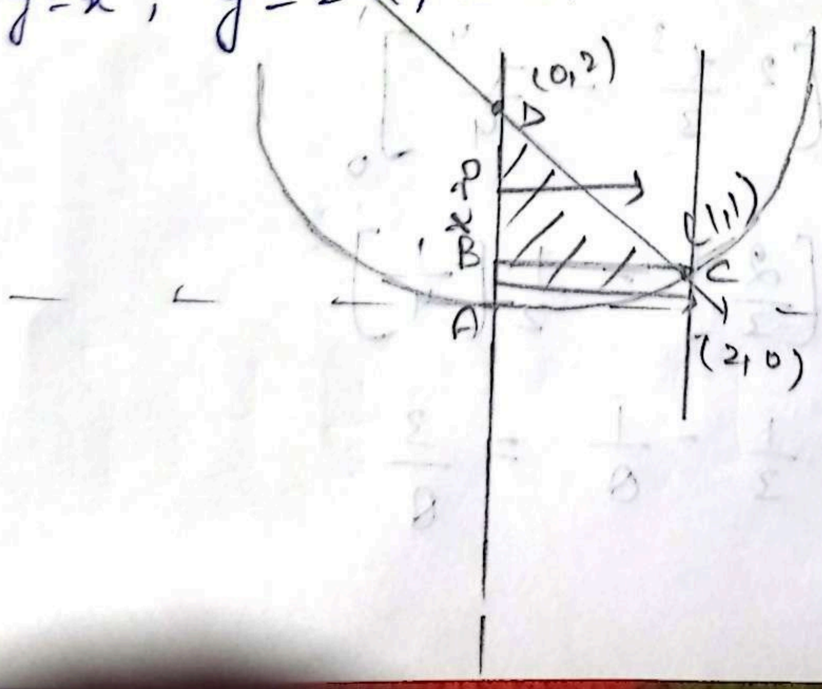
$$\int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a \frac{1}{\cancel{x}} \cancel{x} \tan^{-1}(y/x) \Big]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\left(\frac{1}{x}\right) - 0\,dx$$

$$= \cancel{\ } \frac{\pi}{4}[x]_0^a = \frac{\pi}{4}a \quad \text{Ans}$$

Q Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

$y=x^2,\ y=2-x,\ x=0,\ x=1$



In ABC

$0 \leq x \leq \sqrt{y}$

$0 \leq y \leq 1$

In BCD

$0 \leq x \leq 2-y$

$1 \leq y \leq 2$

$$= \int_0^1 \int_0^{\sqrt{y}} xy \, dx\, dy + \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 \frac{y}{2} (2-y)^2 dy$$

$y = 2-t$

Let $2-y = t$

$-dy = dt$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 - \frac{1}{2}\int_1^0 (2-t)\, t^2 \, dt$$

$$= \frac{1}{2}\cdot\frac{1}{3} - \frac{1}{2}\int_1^0 2t^2 - t^3 \, dt$$

$$= \frac{1}{6} + \frac{1}{2}\left[2\frac{t^3}{3} - \frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{2}{3}\right] - \frac{1}{2}\left[\frac{1}{4}\right]$$

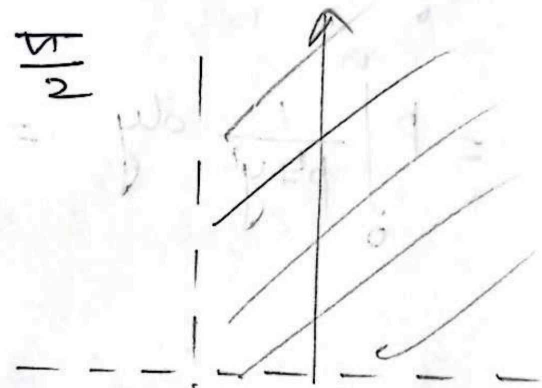
$$= \frac{1}{6} + \frac{1}{3} - \frac{1}{8} = \frac{3}{8}$$

$$= \frac{1}{a}\int_0 y^2 \left(\sin^{-1}\frac{ax}{y^2}\right]_0 dy = \frac{1}{a}\int_0 y \frac{\pi}{2} y$$

$$= \frac{a^2\pi}{6} \quad \text{Ans}$$

Q By changing the order of Integration of Integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy$

show that $\int_0^\infty \frac{\sin px \, dx}{x} = \frac{\pi}{2}$



$x=0,\ x=\infty,\ y=0,\ y=\infty$

$0 \leq y \leq \infty$

$0 \leq x \leq \infty$

$$= \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dy\, dx$$

$$= \int_0^\infty \left[\frac{e^{-xy}}{-x}\right]_0^\infty \sin px \, dx = \int_0^\infty \frac{1}{x} \sin px \, dx \quad \text{——— (1)}$$

$$= \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\because \int e^{ax} \cos bx \cdot dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$= \int_0^\infty \left\{ \frac{e^{-xy}}{y^2+p^2}\left[-y \sin px - p \cos px\right] \right\}_0^\infty dy$$

$$= \int_0^\infty \left[0 - \frac{1}{y^2+p^2}(-p) \right] dy$$

$$= p \int_0^\infty \frac{1}{p^2+y^2} \, dy = \cancel{p} \cdot \frac{1}{\cancel{p}} \left[\tan^{-1}(y/p)\right]_0^\infty$$

$$= \frac{\pi}{2} \quad \text{—— (2)}$$

From (1) and (2)

$$\int_0^\infty \frac{1}{x} \sin px \, dx = \frac{\pi}{2}$$

Triple Integration

① $$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[e^{x+y+z}\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left(e^{1+y+z} - e^{y+z}\right) dy\,dz$$

$$= \int_0^1 \left[e^{1+y+z} - e^{y+z}\right]_0^1 dz$$

$$= \int_0^1 \left(e^{2+z} - e^{1+z}\right) - \left(e^{1+z} - e^{z}\right) dz$$

$$= \left[e^{2+z} - 2e^{1+z} + e^{z}\right]_0^1$$

$$= \left(e^3 - 2e^2 + e\right) - \left(e^2 - 2e + 1\right)$$

$$= e^3 - 3e^2 + 3e - 1$$

$$= \underline{(e-1)^3}$$

Q $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}}\, dz\, dy\, dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2}\right) dy\, dx$

$= \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2} \int_0^1 \left[\sqrt{1-x^2}\right] dx$

$\int \sqrt{a^2-x^2}\, dx = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)\right]$

$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{8}$

Q $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

$= \int_0^4 \int_0^{2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}} dx\, dz$

$= \int_0^4 \int_0^{2\sqrt{z}} \left[\sqrt{4z-x^2}\right] dx\, dz = \int_0^4 \int_0^{2\sqrt{z}} \left[\sqrt{(2\sqrt{z})^2 - x^2}\right] dx\, dz$

$= \int_0^4 \left(\frac{x}{2}\sqrt{4z-x^2} + \frac{4z}{2}\sin^{-1}(x/2\sqrt{z}) \right]_0^{2\sqrt{z}} \cancel{dx}\, dz$

$= \int_0^4 \cancel{4z}\, \cancel{2}z\left(\frac{\pi}{\cancel{2}}\right) - x\sqrt{z}$