change of variables :-

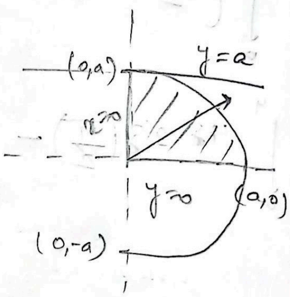
(I) Cartesian coordinates (x, y) to polar coordinates (r, θ)

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Q Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$ by changing into polar coordinates

$x = 0,\ x = \sqrt{a^2-y^2}$

$y = 0,\ y = a$



$0 \le r \le a$

$0 \le \theta \le \frac{\pi}{2}$

$x^2 + y^2 = a^2$

$r^2 = a^2$

$r = a$

$$\int_0^{\pi/2}\int_0^a r^2\sin^2\theta \cdot r\,dr\,d\theta$$

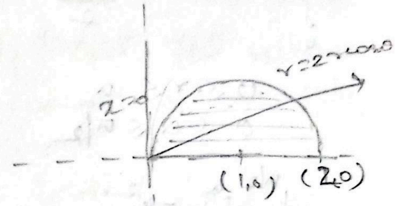
$$= \int_0^{\pi/2} \sin^2\theta \left[\frac{r^5}{5}\right]_0^a d\theta = \int_0^{\pi/2} \frac{a^5}{5}\sin^2\theta\,d\theta$$

$$= \frac{a^5}{5}\,\frac{\Gamma 3/2 \cdot \Gamma 1/2}{2\,\Gamma 4/2} = \frac{a^5}{5}\cdot\frac{\frac{1}{2}\pi}{2}$$

$$= \frac{a^5\pi}{20}$$

Q Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing into polar coordinates.

$y=0,\ y=\sqrt{2x-x^2}$
$x=0,\quad x=2$



$y^2 = 2x-x^2$
$x^2+y^2-2x=0$
$2g=-2 \Rightarrow g=-1$
$f=0,\ c=0$
$(-g,-f) = (1,0)$
$R=\sqrt{g^2+f^2-c}$
$= 1$

$x^2+y^2-2x=0$
$r^2 = 2r\cos\theta$
$[r=2\cos\theta]$

$0 \le r \le 2\cos\theta$
$0 \le \theta \le \pi/2$

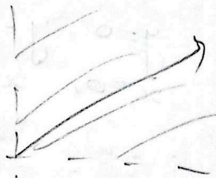
$$= \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r}}\, r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{2\cos\theta} \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{2\,\Gamma 4/2 \cdot \Gamma 1/2}{2\,\Gamma 5/2} = \frac{\Gamma 1/2}{\frac{3}{2}\cdot\frac{1}{2}\Gamma 1/2}$$

$$= 4/3$$

Q Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$ by changing into polar coordinates. Hence show that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$



$$0 \leq r \leq \infty$$
$$0 \leq \theta \leq \pi/2$$

$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

let $r^2 = t$

$2r\,dr = dt$

$$= \int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2}\, d\theta = \frac{1}{2} \int_0^{\pi/2} \Gamma 1\, d\theta$$

$$I = \frac{\pi}{4}$$

let $I_1 = \int_0^\infty e^{-x^2} dx$

$$I_1 = \int_0^\infty e^{-y^2} dy$$

$$I_1^2 = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$$

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\, dy = \frac{\pi}{4}$$

$I = \frac{\sqrt{\pi}}{2}$ Hence Proved

(V) Cartesian coordinates (x,y,z) to cylindrical coordinates (r,θ,z)

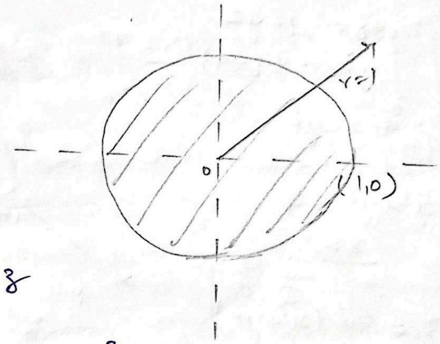
$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta, r\sin\theta, r\,dr\,d\theta\,dz$$

Evaluate $\iiint z^2(x^2+y^2)\,dx\,dy\,dz$ over the volume of cylinder $x^2+y^2=1$ intercepted by the planes $z=2$ and $z=3$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$2 \le z \le 3$



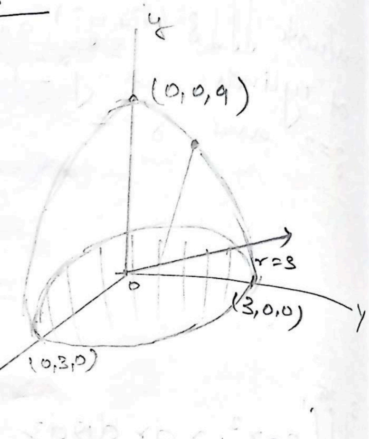
$$\int_2^3\int_0^{2\pi}\int_0^1 z\, r^2 \cdot r\,dr\,d\theta\,dz$$

$$= \int_2^3\int_0^{2\pi} z\left[\frac{r^4}{4}\right]_0^1 d\theta\,dz = \int_2^3\int_0^{2\pi} z\left(\frac{1}{4}\right) d\theta\,dz$$

$$= \int_2^3 \frac{z}{4}\left[\theta\right]_0^{2\pi} dz = \int_2^3 \frac{\pi z}{2}\,dz = \frac{\pi}{2}\left[\frac{z^2}{2}\right]_2^3$$

$$= \frac{\pi}{2}\cdot\frac{5}{2} = \frac{5\pi}{4}$$ Ans

Q use cylindrical coordinates to evaluate $\iiint (x^2+y^2)\,dx\,dy\,dz$ taken over the region (V) bounded by the paraboloid $z = 9-x^2-y^2$ and the plane $z=0$.



$z = 9-x^2-y^2$

at $x=0$

$z = 9-y^2$

$y^2 = -(z-9)$

$z = 9-x^2-y^2$

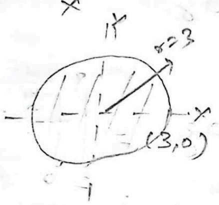
at $z=0$

$x^2+y^2 = 9$

$r=3$

$z = 9-(x^2+y^2)$

$z = 9-r^2$



X-Y (Plane)

$$\left\{\begin{array}{l} 0\le z\le 9-r^2 \\ 0\le r\le 3 \\ 0\le \theta\le 2\pi \end{array}\right\}$$

$$\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2.r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r^3\,[z]_0^{9-r^2}\,dr\,d\theta = \int_0^{2\pi}\int_0^{3} r^3(9-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} 9\left[\frac{r^4}{4}\right]_0^3 d\theta - \int_0^{2\pi}\left[\frac{r^6}{6}\right]_0^3 d\theta$$

$$= \int_0^{2\pi} 9\left(\frac{3^4}{4}\right) d\theta - \int_0^{2\pi} \frac{3^6}{6}\, d\theta$$

$$= 9\left(\frac{3^4}{\cancel{4}_2}\right)\cdot(\cancel{2}\pi) - \frac{3^6}{\cancel{6}_3}(\cancel{2}\pi)$$

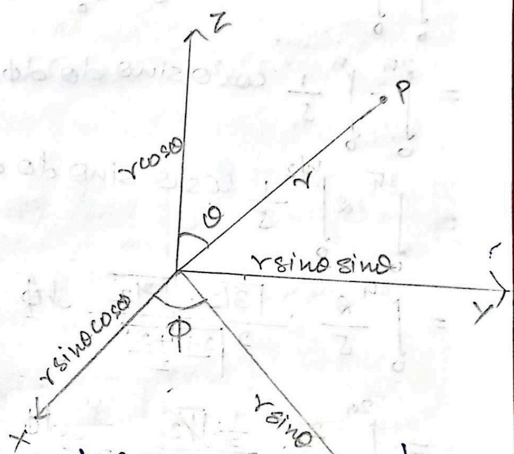
$$= \frac{729\pi}{2} - \frac{729\pi}{3} = \frac{729\pi}{6} \text{ Ans}$$

iii) Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ).

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$



$x^2+y^2+z^2 = r^2$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$= \left(\iiint_{V'} f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi\right)$$

Limits:-

For complete sphere:- $(x^2+y^2+z^2=a^2)$ $\boxed{r=a}$	For Hemisphere	For the octant
$0 \le r \le a$	$0 \le r \le a$	$0 \le r \le a$
$0 \le \theta \le \pi$	$0 \le \theta \le \pi/2$	$0 \le \theta \le \pi/2$
$0 \le \phi \le 2\pi$	$0 \le \phi \le 2\pi$	$0 \le \phi \le \pi/2$

Q Evaluate $\iiint z^2\,dx\,dy\,dz$ over the volume of the sphere $x^2+y^2+z^2=1$

$0 \le r \le 1$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} z^2\, r^2\cos^2\theta \cdot r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \cdot \left[\frac{r^5}{5}\right]_0^1 d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{5} \cos^2\theta \sin\theta \, d\theta\, d\phi$$

$$= \int_0^{2\pi} 2\int_0^{\pi/2} \frac{1}{5} \cos^2\theta \sin\theta \, d\theta\, d\phi$$

$$= \int_0^{2\pi} \frac{2}{5} \cdot \frac{\Gamma 3/2 \cdot \Gamma 2/2}{2\,\Gamma \frac{2+1+2}{2}} \, d\phi$$

$$= \int_0^{2\pi} \frac{\cancel{2}}{5} \cdot \frac{\frac{1}{\cancel{2}}\cancel{\Gamma 1/2}}{\cancel{2}\,\Gamma 3/2 \cdot \frac{1}{\cancel{2}} \cdot \cancel{\Gamma 1/2}} \, d\phi = \int_0^{2\pi} \frac{2}{15}\, d\phi$$

$$= \frac{2}{15}(2\pi) = \frac{4\pi}{15}$$

Q Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$ taken over the region V in the I octant bounded by the cones $\theta = \frac{\pi}{4}$ and $\theta = \tan^{-1} 2$ and the sphere $x^2+y^2+z^2 = 6$

$$0 \le r \le \sqrt{6}$$
$$\frac{\pi}{4} \le \theta \le \tan^{-1} 2$$
$$0 \le \phi \le \pi/2$$

$\therefore\ x^2+y^2+z^2 = r^2$

$$= \int_0^{\pi/2}\int_{\pi/4}^{\tan^{-1}2}\int_0^{\sqrt{6}} \frac{1}{\cancel{r}}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\pi/4}^{\tan^{-1}2}\int_0^{\pi/2}\int_0^{\sqrt{6}} r\sin\theta\, dr\, d\phi\, d\theta = \int_{\pi/4}^{\tan^{-1}2}\int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_0^{\sqrt{6}} d\phi\, d\theta$$

$$= \frac{3\pi}{2}\int_{\pi/4}^{\tan^{-1}2} \sin\theta\, d\theta = -\frac{3\pi}{2}\left[\cos\theta\right]_{\pi/4}^{\tan^{-1}2}$$

$$= -\frac{3\pi}{2}\left[\cos(\tan^{-1}2) - \frac{1}{\sqrt{2}}\right]$$ Ans

Q Evaluate $\iiint xyz\,(x^2+y^2+z^2)^{n/2}\,dx\,dy\,dz$ taken through the ~~the~~ octant of the sphere $x^2+y^2+z^2=b^2$ bounded $n+5>0$.

IV Change of variable from ~~coordinate~~ Cartesian coordinates to (u,v) coordinates

Imp Imp

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f'(u,v)\,|J|\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

Similarly

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f'(u,v,w)\cdot|J|\,du\,dv\,dw$$

where $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Q Let D be the region in I quadrant bounded by $x=0$, $y=0$ and $x+y=1$ change the xy to u,v, where $x+y=u$, $y=uv$. and evaluate $\iint xy(1-x-y)^{1/2}\,dx\,dy$.

I) Find x and y in terms of u and v.

$y = uv$

$x = u-y$

$= u-uv = u(1-v)$

II) $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} (1-v) & -u \\ v & u \end{vmatrix} = (1-v)u - vu$

$J = u$

III) Convert $f(x,y)\,dx\,dy$ to $f'(u,v)\,|J|\,du\,dv$

$xy(1-x-y)^{1/2}\,dx\,dy$

$= u(1-v)\,uv\,(1-u)^{1/2}\,u\,du\,dv$

$= u^3(1-u)^{1/2}\,v(1-v)\,du\,dv$

IV) Find limits for u and v

$x=0$, $y=0 \Rightarrow uv=0 \Rightarrow u=0, v=0$

$u(1-v)=0$

$u=0, v=1$ $\quad x+y=1 \Rightarrow u=1$

$0 \leq u \leq 1$

$0 \leq v \leq 1$

V) Integrate

$\int_0^1\int_0^1 u^{3+1-1}(1-u)^{\frac{1}{2}+1-1}\,v^{1+1-1}(1-v)^{1+1-1}\,du\,dv$

$\beta\left(4,\frac{3}{2}\right)\ \beta(2,2)$

$$= \frac{\Gamma 4 \cdot \Gamma 3/2}{\Gamma 4+3/2} \cdot \frac{\Gamma 2 \; \Gamma 2}{\Gamma 4}$$

$$= \frac{3! \; \frac{1}{2} \Gamma 1/2}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma 1/2} \cdot \frac{1! \, 1!}{3!}$$

$$= \frac{16}{945}$$

Q. Evaluate $\iint_R (x+y)^2 \, dx\,dy$ where R is the ||gm in the xy-plane with vertices (1,0), (3,1), (2,2), (0,1), using the transformation $u = x+y$, $v = x-2y$

$$J_1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

$$J = \frac{1}{J_1} = \frac{1}{-3}$$

$$|J| = \frac{1}{3}$$

$$(x+y)^2 \, dx\,dy = \frac{u^2}{3} \, du\,dv$$

Coordinates of (u,v)

	$u = x+y$	$v = x-2y$	
$(1,0)$	$u=1$	$v=1$	$(1,1)$
$(3,1)$	$u=4$	$v=1$	$(4,1)$
$(2,2)$	$u=4$	$v=-2$	$(4,-2)$
$(0,1)$	$u=1$	$v=-2$	$(1,-2)$

$1 \le u \le 4$

$-2 \le v \le 1$

$$= \int_{-2}^{1}\int_{1}^{4} \frac{u^2}{3}\, du\, dv = \int_{-2}^{1} \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv$$

$$= \int_{-2}^{1} \frac{1}{3}\left(\frac{4^3}{3} - \frac{1}{3}\right) dv = \frac{\cancel{63}^{\,\cancel{21}\,7}}{\cancel{9}_{\cancel{3}}}(1+2)$$

$$= 7 \times 3 = 21$$

Q Evaluate $\iint_R (x+y)^2\, dx\, dy$ where R is the region bounded by llgm, $x+y=0$, $x+y=2$, $3x-2y=0$, $3x-2y=3$

let $u = x+y$

$v = 3x-2y$

$$J\frac{(u,v)}{(x,y)} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2-3 = -5$$

$$|J| = \left|\frac{1}{J_1}\right| = \frac{1}{5}$$

$$(x+y)^2\,dx\,dy = \frac{u^2}{5}\,du\,dv$$

$\Rightarrow$ Limits:- $0 \le u \le 2$
$0 \le v \le 3$

$$\int_0^3\int_0^2 \frac{u^2}{5}\,du\,dv = \int_0^3 \frac{1}{5}\left[\frac{u^3}{3}\right]_0^2 dv$$

$$= \int_0^3 \frac{1}{5}\left(\frac{8}{3}-0\right)dv = \frac{8}{15}\int_0^3 dv$$

$$= \frac{8}{15}(3) = \frac{8}{5}$$

Q 10 V.V.V. Imp. Evaluate $\iiint (x+y+z)\,x^2y^2z^2\,dx\,dy\,dz$ ~~taken~~ over the volume bounded by $x=0, y=0$, $x+y+z=1$, substituting $u=x+y+z$, $x+y=uv$, $y=$ ~~uvz~~ uvw

$y = uvw$
$x = uv - y \Rightarrow uv - uvw$
$= uv(1-w)$

$z = u-(x+y) = u - uv = u(1-v)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} v(1-w) & u(1-w) & -uv \\ vw & uw & uv \\ 1-v & -u & 0 \end{vmatrix}$$

$R_2 \longrightarrow R_1 + R_2$

$$= \begin{vmatrix} v(1-w) & u(1-w) & -uv \\ v & u & 0 \\ (1-v) & -u & 0 \end{vmatrix}$$

$= -uv\,(-uv - u(1-v))$

$= -uv(-u) = u^2 v$

$|J| = \underline{u^2 v}$

$(x+y+z)\, x^2 y^2 z^2\, dx\,dy\,dz = u\, u^2 v^2 (1-w)^2 u^2 v^2 w^2 u^2 (1-v)^2 u^2 v\, du\,dv\,dw$

$= \underline{u^9 v^5 (1-v)^2 w^2 (1-w)^2\, du\,dv\,dw}$

$x=0 \Rightarrow uv(1-w)=0 \Rightarrow u=0, v=0, w=1$

$y=0 \Rightarrow uvw=0 \Rightarrow u=0, v=0, w=0$

$z=0 \Rightarrow u(1-v)=0 \Rightarrow u=0, v=1$

$x+y+z=1 \Rightarrow u=1.$

$\underline{0 \le u, v, w \le 1}$