Gamma function (Integration)

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \quad \text{where } n>0$$

Gamma (n)

$$\Gamma 1 = \int_0^\infty e^{-x} x^0 dx = [-e^{-x}]_0^\infty = 0-(-1) = 1$$

$$\boxed{\Gamma 1 = 1}$$

for $n=2$

$$\Gamma 2 = \int_0^\infty e^{-x} . x dx \qquad \therefore \int u.v dx = u\int v dx - \int \left\{\frac{d(u)}{dx}\int v dx\right\}dx$$

$$= \left[x\frac{e^{-x}}{-1} - 1.e^{-x}\right]_0^\infty$$

$$\Gamma 2 = 1$$

shortcut

$$\int \underset{x^n}{u} v dx$$

$$= uv_1 - u'v_2 + u''v_3 - u'''v_4 + - - - - - -$$

$$\int x^3 e^{2x} dx$$

$$= x^3\frac{e^{2x}}{2} - 3x^2\frac{e^{2x}}{4} + 6x.\frac{e^{2x}}{8} - 6\frac{e^{2x}}{16}$$

$$\Gamma 3 = \int_0^\infty e^{-x} x^2 \, dx$$

$$= \left[x^2 \frac{e^{-x}}{-1} - 2x e^{-x} + 2 \frac{e^{-x}}{-1} \right]_0^\infty$$

$$= [0 - (-2)] = 2$$

$$\boxed{\Gamma 3 = 2}$$

Recurrence Relation

$$\boxed{\Gamma n+1 = n \Gamma n}$$

$$\Gamma n+1 = \int_0^\infty e^{-x} x^n \, dx$$

$$= \left[x^n \cdot \frac{e^{-x}}{-1} \right]_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} \, dx$$

$$= 0 + n \int_0^\infty x^{n-1} \cdot e^{-x} \, dx$$

$$\Gamma n+1 = n \Gamma n = \underline{\underline{\text{RHS}}}$$

Hence proved

(OR)

$$\{ \Gamma n = (n-1) \Gamma n-1 \} ; n > 1$$

* If n = natural no.

$$[\Gamma n = (n-1)!]$$

Q $\int_0^\infty e^{-x} x^{10} dx$

$\cancel{n=}$ $n-1=10$

$n=11$

$\Rightarrow \Gamma 11 = 10!$ Ans

Q $\int_0^\infty e^{-x} x^{5/2} dx$; $\left(\Gamma \frac{1}{2} = \sqrt{\pi}\right)$ **

$n-1 = \frac{5}{2}$

$n = 7/2$

$= \Gamma 7/2$

$(7-2)$ $\Gamma \frac{7}{2} = \frac{5}{2} \Gamma \frac{5}{2}$

$= \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \cdot \Gamma \frac{1}{2}$

$= \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}$

$= \frac{15}{8} \sqrt{\pi}$

Q $\Gamma \frac{15}{2} = \frac{13}{2} \times \frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \Gamma \frac{1}{2}$

Q $\Gamma \frac{8}{5} = \cancel{\frac{8}{5}} \cancel{\times} \frac{3}{5} \Gamma \frac{3}{5}$

Q $\Gamma\frac{17}{3} = \frac{14}{3}\cdot\frac{11}{3}\cdot\frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\Gamma\frac{2}{3}$

Q $\int_0^\infty e^{-x}x^{11/2}\,dx = \frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

Q $\int_0^\infty e^{-x}x^{15/4}\,dx = \frac{15}{4}\cdot\frac{11}{4}\cdot\frac{7}{4}\cdot\frac{3}{4}\Gamma\frac{3}{4}$

Q $\int_0^\infty e^{-2x}x^2\,dx$

let $2x = t$

$x = t/2$

$dx = \frac{dt}{2}$

$x=0 \rightarrow t=0$

$x=\infty \rightarrow t=\infty$

$\int_0^\infty e^{-t}\frac{t^2}{4}\frac{dt}{2}$

$= \frac{1}{8}\cdot 2! = \frac{2}{8} = \frac{1}{4}$

Q $\int_0^\infty e^{-x^2}x^3\,dx$

let $x^2 = t$

$x = t^{1/2}$

$dx = \frac{1}{2}t^{-1/2}dt$

$= \int_0^\infty e^{-t}t^{3/2}\frac{1}{2}t^{-1/2}dt$

$\frac{1}{2}\int_0^\infty e^{-t}t^{1+1-1}\,dt$

$= \frac{1}{2}\Gamma 2 = \frac{1}{2}$

let $\sqrt{x} = t$
$x = t^2$
$dx = 2t\,dt$

Q $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\,dx$

$\int_0^\infty e^{-t}\, t^{2/4} \cdot 2t\,dt$

$= 2\int_0^\infty e^{-t}\, t^{3/2}\,dt = 2\Gamma\frac{5}{2}$

$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{3}{2}\sqrt{\pi}$

Transformations of Gamma function :-

① $\Gamma n = K^n \int_0^\infty e^{-Kx} x^{n-1}\,dx$

RHS = let $Kx = t$
$x = t/K$
$dx = dt/K$

$= \cancel{K^n} \int_0^\infty e^{-t} \cdot \frac{t^{n-1}}{\cancel{K^{n-1}}} \frac{dt}{\cancel{K}} = \Gamma n = \text{LHS}$

② $\Gamma n = \int_0^1 \left[\log_e\left(\frac{1}{x}\right)\right]^{n-1} dx \quad ; n>0$

let $\log\left(\frac{1}{x}\right) = t$

$\frac{1}{x} = e^t$

$x = e^{-t}$

$dx = -e^{-t}\,dt$

$x = 0 \rightarrow t = \infty$
$x = 1 \rightarrow t = 0$

RHS

$= \int_{\infty}^{0} t^{n-1} \cdot -e^{-t}\, dt$

$= \int_{0}^{\infty} t^{n-1} e^{-t}\, dt = \Gamma n = LHS$

(3) $\Gamma n = \frac{1}{n}\int_{0}^{\infty} e^{-x^{1/n}}\, dx$

let $x^{1/n} = t$

$x = t^{n}$

$dx = n t^{n-1}\, dt$

RHS $= \frac{1}{\cancel{n}}\int_{0}^{\infty} e^{-t}\, \cancel{n} t^{n-1}\, dt$

$= \Gamma n = LHS$

Q $\int_{0}^{\infty} \sqrt{x}\, e^{-x^{3}}\, dx$

let $x^{3} = t$

$x = t^{1/3}$

$dx = \frac{1}{3} t^{-2/3}\, dt$

$= \int_{0}^{\infty} e^{-t}\, t^{1/6} \cdot \frac{1}{3} t^{-2/3}\, dt$

$= \frac{1}{3}\int_{0}^{\infty} e^{-t}\, t^{-1/2}\, dt \quad = \frac{1}{3}\int_{0}^{\infty} e^{-t}\, t^{-1/2+1-1}\, dt$

$= \frac{\sqrt{\pi}}{3}$

Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; \text{ where } m,n>0$$

Q $\int_0^1 x^{2+1-1}(1-x)^{4+1-1}\,dx$

$= \beta(3,5)$

$*\left[\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$

Q $\int_0^1 x^{3/2+1-1}(1-x)^{1/2+1-1}\,dx$

$= \beta(5/2, 3/2) = \dfrac{\Gamma\frac{5}{2}\cdot\Gamma\frac{3}{2}}{\Gamma \cancel{8}/\cancel{2}}$

$= \dfrac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3!} = \dfrac{\pi}{16}$

Q $\int_0^1 x^3(1-x^2)^4\,dx$

let $x^2 = t$
$x = t^{1/2}$
$dx = \frac{1}{2}t^{-1/2}dt$

$= \int_0^1 t^{3/2}(1-t)^4\cdot\frac{1}{2}t^{-1/2}dt$

$= \frac{1}{2}\int_0^1 t^{1+1-1}(1-t)^{4+1-1}dt = \frac{1}{2}\beta(2,5) = \frac{1}{2}\left[\frac{1!\cdot 4!}{6!}\right]$

$= \frac{1}{2}\left(\frac{1}{30}\right) = \frac{1}{60}$

Imp

Q $\int_0^1 \left(\frac{x^2}{1-x^3}\right)^{1/2} dx$

$= \int_0^1 x^{3/2} (1-x^3)^{-1/2} dx$

Let $x^3 = t$

$x = t^{1/3}$

$dx = \frac{1}{3} t^{-2/3} dt$

$= \int_0^1 t^{1/2} (1-t)^{-1/2} \cdot \frac{1}{3} t^{-2/3} dt$

$= \frac{1}{3}\int_0^1 t^{1/2-\frac{2}{3}} (1-t)^{-1/2} dt = \frac{1}{3}\int_0^1 t^{\frac{-1}{6}+1-1} (1-t)^{-1/2+1-1} dt$

$= \frac{1}{3} \beta\left(\frac{5}{6}, \frac{1}{2}\right)$

$= \frac{1}{3} \frac{\Gamma\frac{5}{6} \cdot \Gamma\frac{1}{2}}{\Gamma\frac{8}{6}}$

Transformations

① $\beta(m,n) = \beta(n,m)$

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$

$= \int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} dx$

$= \int_0^1 x^{n-1} (1-x)^{m-1} dx = \beta(n,m)$

(2) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

let $\frac{1}{1+x} = t$

$1+x = \frac{1}{t}$ $\qquad x=0 \longrightarrow t=1$

$\qquad\qquad\qquad x=\infty \longrightarrow t=0$

$x = \frac{1}{t} - 1$

$dx = \frac{-1}{t^2} dt$

$= -\int_1^0 \left(\frac{1}{t} - 1\right)^{m-1} (t)^{m+n} \cdot \frac{1}{t^2} dt$

$= \int_0^1 \frac{(1-t)^{m-1}}{t^{m-1}} \cdot \frac{t^{\cancel{m}+n}}{t^2} dt$

$= \int_0^1 (1-t)^{m-1}\, t^{n-1}\, dt$

$= \beta(n,m) = \beta(m,n)$

(2) Q $\int_0^\infty \frac{x^{2+1-1}}{(1+x)^4} dx$

$m = 3$

$m+n = 4$

$n = 1$

$\beta(3,1) = \frac{\Gamma 3 \cdot \Gamma 1}{\Gamma 4} \cdot \frac{2\times 1}{3!} = \frac{1}{3}$

Q $\displaystyle\int_0^{\infty} \frac{x^2(1-x^3)}{(1+x)^7}\,dx$

$$= \int_0^{\infty} \frac{x^{2+1-1}}{(1+x)^7} - \frac{x^{6+1-1}}{(1+x)^7}\,dx$$

$$\beta(3,4) - \beta(6,1)$$

$$= \frac{\Gamma 3 \cdot \Gamma 4}{\Gamma 7} - \frac{\Gamma 6\, \Gamma 1}{\Gamma 7}$$

$$= \frac{2\cdot 3!}{6!} - \frac{5!}{6!} = \frac{\cancel{2}}{6\times5\times\cancel{4}_2} - \frac{1}{6}$$

$$= \frac{1}{60} - \frac{1}{6}$$

$$= \frac{1-10}{60} = \frac{-9}{60}$$

Imp

3. $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

Q $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \{\, m = 1/2,\ n = -1/2$$

$$= \frac{\Gamma\frac{3}{4}\cdot\Gamma\frac{1}{4}}{2\,\Gamma\frac{\cancel{2}}{\cancel{2}}\;1} \quad \# \quad \cancel{\frac{3}{4}\Gamma\frac{3}{4}}$$

$$= \frac{1}{2}\Gamma\frac{3}{4}\,\Gamma\frac{1}{4} \quad \text{Ans}$$

Imp

$$\# \quad \boxed{\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}} \quad ; \ 0<n<1 \quad (m+n=1)$$

Q $$\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \frac{\pi}{\sin\frac{2\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

Q $$\underbrace{\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}}_{I_1} \times \underbrace{\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta}_{I_2}$$

$$I_1 = \int_0^{\pi/2}\sin^{-1/2}\theta\,\cos^0\theta\,d\theta = \frac{\Gamma\frac{1}{4}\cdot\Gamma\frac{1}{2}}{2\,\Gamma\frac{3}{4}}$$

$$I_2 \int_0^{\pi/2}\sin^{1/2}\theta\,\cos^0\theta\,d\theta$$

$$= \frac{\Gamma\frac{3}{4}\cdot\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{4}}$$

$$I = I_1\times I_2 = \frac{\Gamma\frac{1}{4}\cdot\Gamma\frac{1}{2}}{2\,\cancel{\Gamma\frac{3}{4}}} \times \frac{\cancel{\Gamma\frac{3}{4}}\cdot\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{4}}$$

$$= \frac{\Gamma\frac{1}{4} \cdot \Gamma\frac{1}{2} \cdot \Gamma\frac{1}{2}}{4\,\Gamma\frac{5}{4}}$$

$$= \frac{\Gamma\frac{1}{4} \cdot \pi}{4\,\Gamma\frac{5}{4}} = \frac{\cancel{\pi}}{\cancel{4}} \frac{\cancel{\Gamma\frac{1}{4}}}{\cancel{4} \cdot \frac{1}{\cancel{4}} \cancel{\Gamma\frac{1}{4}}}$$

$$= \underline{\pi}$$

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} = \beta(m,n)$$

$m+n=1 \;;\; 0<n<1$

$$\int_0^\infty \frac{x^{1/4-1}}{(1+x)^1}\,dx = \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\# \left\{ \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)} \;,\; 0<n<1 \right.$$

$$\left. = \frac{\pi}{\sin n\pi} \right\}$$

Q $\displaystyle \int_0^\infty \frac{dx}{1+x^4}$

let $x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

$$= \frac{1}{4} \int_0^\infty \frac{t^{(-3/4)+1-1}\,dt}{1+t}$$

$$= \frac{1}{4} \cdot \sqrt{2}\,\pi = \frac{\pi}{2\sqrt{2}}$$

Q $\int_0^2 (8-x^3)^{-1/3}\,dx$

Let $\frac{x^3}{8} = y$

$x = 8^{1/3} y^{1/3}$

$x = 2y^{1/3}$

$dx = \frac{2}{3} y^{-2/3} dy$

$x \to 0,\ y \to 0$

$x \to 2,\ y \to 1$

$= \int_0^2 (8)^{-1/3} \left(1 - \frac{x^3}{8}\right)^{-1/3} dx$

$= \frac{1}{\cancel{2}} \int_0^1 (1-y)^{-1/3+1-1} \cdot \frac{\cancel{2}}{3} y^{-2/3+1-1} dy$

$= \frac{1}{3} \beta\left(\frac{2}{3}, \frac{1}{3}\right)$

$= \frac{1}{3} \frac{\Gamma\frac{2}{3}\,\Gamma\frac{1}{3}}{\Gamma 1}$

$= \frac{1}{3} \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$

Q $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx =$

Let $\left(1-\frac{x}{2}\right)^{1/2} = t$

$1 - \frac{x}{2} = t^2$

$\frac{x}{2} = 1 - t^2$

$x = 2(1-t^2)$

$= \int_1^0 \frac{x^2}{(2)^{1/2}\sqrt{1-\frac{x}{2}}}\,dx$

$= \frac{1}{\sqrt{2}} \int_0^2 \frac{4(1-t^2)^2}{t}\,dx$

$= \frac{4}{\sqrt{2}} \int$

Q $\int_0^\infty \frac{x^c}{c^x}\,dx \; ; \; c>1$

$= \int_0^\infty c^{-x} x^c\,dx$

$= \int_0^\infty e^{\log_e c^{-x}} x^c\,dx$

Let
$x \ln c = t$
$x = \frac{t}{\ln c}$
$dx = \frac{dt}{\ln c}$

$= \int_0^\infty e^{-x\log_e c} x^c\,dx$

$= \int_0^\infty e^{-t}\left(\frac{t}{\ln c}\right)^c \frac{dt}{\ln c}$

$= \frac{1}{(\ln c)^{c+1}} \int_0^\infty e^{-t}\, t^{(c+1)-1}\,dt$

$= \frac{\Gamma(c+1)}{(\log_e c)^{c+1}}$

Q $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Let $x^2 = \tan\theta$

$x = \sqrt{\tan\theta}$

$dx = \frac{1 \sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}$

$= \frac{1}{2} \sin^{-1/2}\theta \cos^{1/2}\theta \cos^{-2}\theta\,d\theta$

$= \frac{1}{2} \sin^{-1/2}\theta \cos^{-3/2}\theta\,d\theta$

Double Integration

For a bounded region R & cont. fn. (x, y), we ~~want~~ write the double Integration as

$$\iint_R f(x,y)\,dx\,dy \quad (\text{or}) \quad \iint_R f(x,y)\,dy\,dx$$

Evaluation of Double Integration

① $\displaystyle\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$, where $x_1 = g_1(y)$
$x_2 = g_2(y)$

Q $\displaystyle\int_1^2\int_y^{2y} (x^2+y^2)\,dx\,dy$

$$= \int_1^2 \left[\frac{x^3}{3} + y^2x\right]_y^{2y} dy = \int_1^2 \left(\frac{8y^3}{3} + 2y^3\right) - \left(\frac{y^3}{3} + y^3\right) dy$$

$$= \int_1^2 \frac{10}{3} y^3\,dy = \frac{10}{3}\left[\frac{y^4}{4}\right]_1^2 = \frac{10}{3\times 4}\left(2^4 - 1^4\right)$$

$$= \frac{\cancel{10}^{5}}{\cancel{12}_{\cancel{4}\,2}} \times \cancel{15}^{5} = \frac{25}{2}$$

Q $\int_0^1 \int_0^y e^{x+y} \, dx \, dy$

$$\int_0^1 [e^{x+y}]_0^y \, dy = \int_0^1 e^{2y} - e^y \, dy$$

$$= \left[\frac{e^{2y}}{2} - e^y\right]_0^1 = \frac{e^2}{2} - e^1 - \frac{e^0}{2} + e^0$$

$$= \frac{e^2}{2} - e^1 + \frac{1}{2} \quad \underline{\text{Ans}}$$

Q② $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y) \, dy \, dx$; $y_1 = g_1(x)$, $y_2 = g_2(x)$

Q $\int_1^2 \int_x^{x^2} (2y - x + 1) \, dy \, dx$

$$= \int_1^2 [y^2 - xy + y]_x^{x^2} \, dx$$

$$= \int_1^2 [x^4 - x^3 + x^2 - (x^2 - x^2 + x)] \, dx = \int_1^2 (x^4 - x^3 + x^2 - x) \, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \left[\frac{2^5}{5} - \frac{2^4}{4} + \frac{2^3}{3} - \frac{2^2}{2} - \left(\frac{1}{5} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}\right)\right]$$

Q $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\, dy\,dx$

$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\Big]_0^{\sqrt{1+x^2}} dx$

$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - \frac{1}{\sqrt{1+x^2}} \tan^{-1}(0)\, dx$

$= \int_0^1 \frac{\pi}{4\sqrt{1+x^2}}\, dx = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$

$= \frac{\pi}{4} \log_e\left(x+\sqrt{1^2+x^2}\right)$

Q $\int_1^2 \int_0^x \frac{dy\,dx}{x^2+y^2} = \int_1^2 \frac{1}{x} \tan^{-1}(y/x)\Big]_0^x dx$

$= \int_{\cancel{0}\,1}^{\cancel{1}\,2} \frac{1}{x}\tan^{-1}(\cancel{x/x}) - \frac{1}{x}\tan^{-1}(0)\Big] dx$

$= \frac{\pi}{4}\int_{1\,\cancel{0}}^{2\,\cancel{1}} \frac{1}{x}\, dx = \frac{\pi}{4}(\ln x)_{\cancel{0}\,1}^{\cancel{1}\,2}$

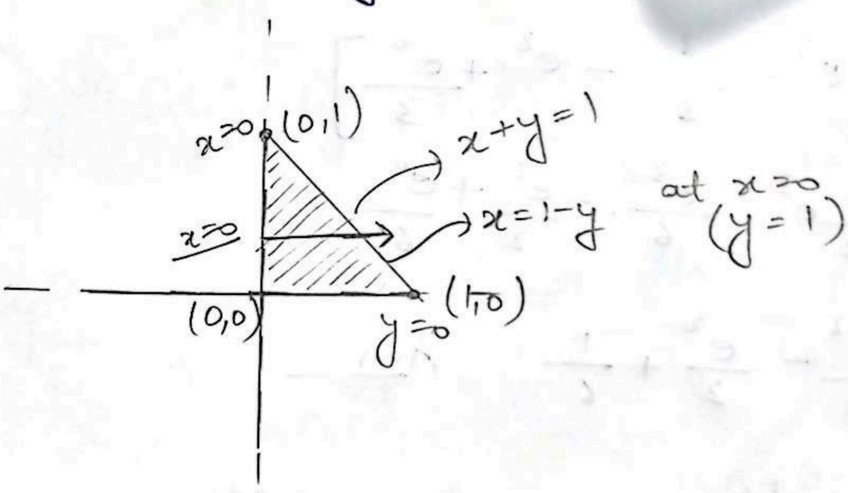
$= \frac{\pi}{4}\ln(2) - \frac{\pi}{4}\ln(\phi)$

$= \frac{\pi}{4}\ln(2)$

Q Evaluate $\iint e^{2x+3y}\,dx\,dy$

over the Triangle bounded by $x=0, y=0$ and $x+y=1$.

I) Draw the region.



II) Find Limits

Draw a ray ∥ to x-axis (or y-axis)

$(0 \le x \le 1-y) \longrightarrow$ Find the ~~xaty~~ value of x and y at enter and exit points

$(0 \le y \le 1)$

III) Put Limits and Integrate

$$\int_0^1 \int_0^{1-y} e^{2x+3y}\,dx\,dy$$

$$= \int_0^1 \left[\frac{e^{2x+3y}}{2}\right]_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{e^{2-2y+3y}}{2} - \frac{e^{3y}}{2}\right] dy$$

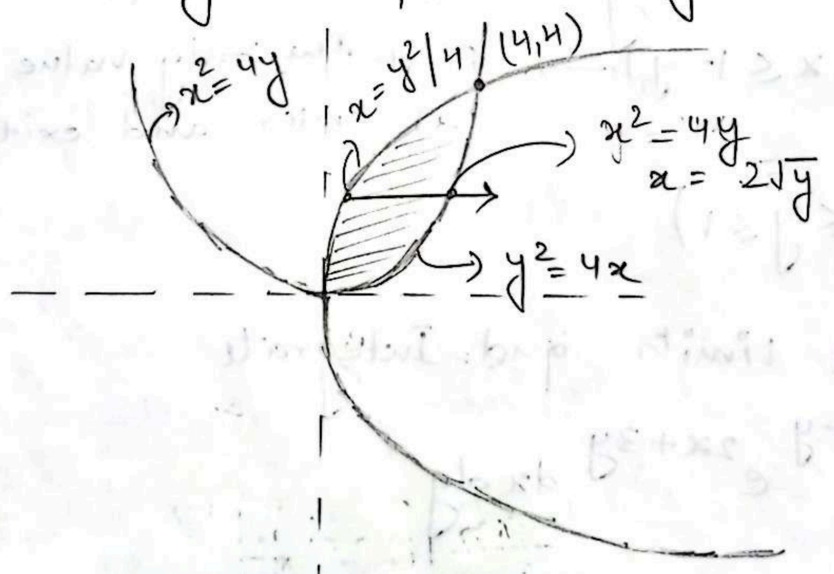
$$= \frac{1}{2}\int_0^1 e^{2+y} - e^{3y}\, dy = \frac{1}{2}\left[e^{2+y} - \frac{e^{3y}}{3}\right]_0^1$$

$$= \frac{1}{2}\left[e^3 - \frac{e^3}{3} - e^2 + \frac{e^0}{3}\right]$$

$$= \frac{1}{2}e^3 - \frac{e^3}{6} - \frac{e^2}{2} + \frac{e^0}{6}$$

$$= \frac{e^3}{3} - \frac{e^2}{2} + \frac{1}{6} \quad \text{Ans}$$

Q Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabola $y^2 = 4x$ & $x^2 = 4y$



$$\frac{y^2}{4} \leq x \leq 2\sqrt{y}$$

$$0 \leq y \leq 4$$

$$= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} y\,dx\,dy$$

$$= \int_0^4 [yx]_{y^2/4}^{2\sqrt{y}}\,dy = \int_0^4 y\left(2\sqrt{y} - y^2/4\right) dy$$

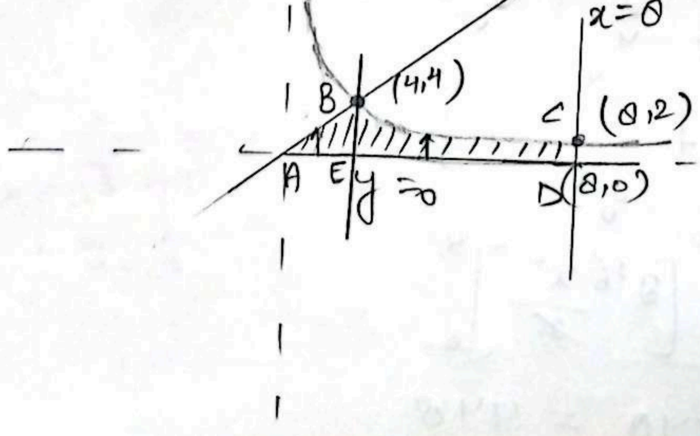
$$= \int_0^4 2y^{3/2} - y^3/4\,dy = \left[2\frac{y^{5/2}}{5/2} - \frac{y^4}{16}\right]_0^4$$

$$= \left[\frac{4y^{5/2}}{5} - \frac{y^4}{16}\right]_0^4 = \frac{4(4)^{5/2}}{5} - \frac{(4)^4}{16}$$

$$= \frac{4\times 32}{5} - 16 = \frac{128-80}{5}$$

$$= \frac{48}{5} \quad \text{Ans}$$

Q If D is the region in the 1st quadrant bounded by the curves $xy=16$, $x=y$, $y=0$, $x=8$ Evaluate $\iint_D x^2\,dx\,dy$



Q Evaluate $\iint_S \sqrt{xy - y^2}\, dx\, dy$

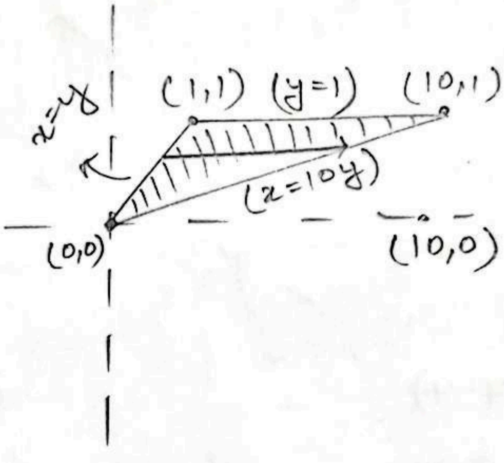
where S is a triangle with vertices $(0,0)$, $(10,1)$ and $(1,1)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{1-0}{10-0}(x - 0)$$

$$y = \frac{x}{10}$$

$$\boxed{x = 10y}$$



$$y \le x \le 10y$$
$$0 \le y \le 1$$

$$= \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\, dx\, dy$$

$$= \int_0^1 \left[\frac{(xy - y^2)^{3/2}}{\frac{3}{2} y^{\cancel{3/2}}} \right]_y^{10y} dy = \int_0^1 \frac{2(9y^2)^{3/2}}{3y^{\cancel{3/2}}} \cdot dy$$

$$= \frac{2}{\cancel{3}} \int_0^1 \frac{\cancel{9}\,27\, y^{\cancel{3}\,2}}{\cancel{y}}\, dy$$

$$= 18 \left[\frac{y^3}{3} \right]_0^1 = \overset{6}{\cancel{18}} \cdot \frac{1}{\cancel{3}} = 6 \text{ Ans}$$