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Survey in operations research and management science

Knapsack problems — An overview of recent advances. Part I: Single knapsack problems

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ABSTRACT

After the seminal books by Martello and Toth (1990) and Kellerer, Pferschy, and Pisinger (2004), knapsack problems became a classical and rich research area in combinatorial optimization. The purpose of this survey, which is structured in two parts, is to cover the developments that appeared in this field after the publication of the latter volume. *Part I* is devoted to problems whose goal is to optimally assign items to a single knapsack. Besides the classical knapsack problems (binary, subset sum, bounded, unbounded, change-making), we review problems with special constraints (setups, multiple-choice, conflicts, precedences, sharing, compartments) as well as relatively recent fields of investigation, like robust and bilevel problems. The subsequent *Part II* covers multiple, multidimensional, and quadratic knapsack problems, and includes a succinct treatment of online and multiobjective knapsack problems.

1. Introduction

The area of knapsack problems is one of the most active research areas of combinatorial optimization. Two specific monographs have been dedicated to this field. In 1990, Martello and Toth published the first book (Martello and Toth, 1990), now available online at http://www.or.deis.unibo.it/knapsack.html explicitly dedicated to algorithms and computer implementations for knapsack problems, in which about 200 results that appeared in the previous thirty years were thoroughly described and commented. In 2004, Kellerer, Pferschy, and Pisinger published the second book (Kellerer et al., 2004) specifically dedicated to this area. Quoting their introduction, *"Thirteen years have passed since the seminal book on knapsack problems by Martello and Toth appeared. On this occasion a former colleague exclaimed back in 1990: "How can you write 250 pages on the knapsack problem?"... However, in the last decade a large number of research publications contributed new results for the knapsack problem in all areas of interest such as exact algorithms, heuristics and approximation schemes."* Indeed, this new volume included about 500 bibliographic references, two thirds of which appeared after 1990. Seventeen more years have passed, during which the intense research activity on these topics has continued. The purpose of this survey is thus to report the many results that appeared in this period. As all the basic algorithmic approaches have been fully described in the two monographs, they will not be repeated here. We will concentrate instead on a succinct description of the main recent results appeared after 2003 (and until Summer 2021), with the objective of completing the overview of this topic.

The knapsack problem has been known since over a century, see Mathews (1896), and, according to folklore, the name was suggested by Tobias Dantzig (1884–1956), father of George Dantzig. (According to some authors, the suggestion was included in Dantzig (2007), which is wrong: the term 'knapsack' does not appear in this book.) While the first algorithmic studies were published in the Fifties (by Dantzig, 1957 and Bellman, 1957), an intense research activity started in the Sixties. The success of this topic in the subsequent decades is also shown by the fact that a study by Skiena (1999) ranked it as the 18th most popular algorithmic problem, and the second among the $\mathcal{NP}$-hard problems (after traveling salesman).

The ancestor problem is known as the *0-1 Knapsack Problem* (KP01). Using Dantzig's 1957 words, *"In this problem a person is planning a hike and has decided not to carry more than 70 lb of different items, such as bed roll, geiger counters (these days), cans of food ...."*. Formally, we are given a *capacity* c and a set of n items, each with a *weight* w_j and a *profit* p_j. We want to determine a subset of items such that its total weight does not exceed the capacity and its total profit is a maximum. The problem can then be formulated as the *Integer Linear Programming* (ILP) model:

$$\max \sum_{j=1}^{n} p_j x_j \tag{1}$$

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$$\text{s.t.} \sum_{j=1}^{n} w_j x_j \le c \tag{2}$$

$$x_j \in \{0,1\} \qquad (j = 1,\dots,n), \tag{3}$$

where x_j takes the value 1 if and only if item j is selected. The first study on the KP01, see Dantzig (1957), concerned the *Linear Programming* (LP) relaxation of this model, in which (3) is replaced by

$$0 \le x_j \le 1 \qquad (j = 1,\dots,n).$$

In the following, we will generally assume, without loss of generality, that all input values are positive, that $w_j \le c$ $(j = 1,\dots,n)$, and that $\sum_{j=1}^{n} w_j > c$.

The number of knapsack problem variants addressed in recent years is huge. In this survey, we mostly concentrate on those classical problems that were addressed in the main chapters of Kellerer et al. (2004) (which also correspond to chapters in Martello and Toth, 1990) and on variants that received significant attention in the recent literature.

Most results on knapsack problems were developed by the combinatorial optimization community, to which the authors of monographs Martello and Toth (1990) and Kellerer et al. (2004) dedicated to these problems belong. This survey is mainly aimed at this community, and hence our choice of subjects privileges problems with a clear combinatorial aspect. The stochastic knapsack problems are not included, while we address in a succinct way knapsack problems with a clear continuous, non-linear flavor. Problems closer to the computer science community (*online* knapsack problems) or belonging to the area of multiple criteria decision aiding (*multi-objective* knapsack problems) are discussed in the last section of Part II (Cacchiani et al., 2022). Polyhedral aspects of knapsack problems are briefly mentioned in the text but not covered in a dedicated section: in Section 2 we provide pointers to a recent survey and to successive updates.

The decision version of a special case of the KP01 (similar to the *subset sum problem* treated in Section 4) is one of the famous Karp's 21 $\mathcal{NP}$-complete problems (see Karp, 1972). In general, all optimization problems considered in this survey are $\mathcal{NP}$-hard. The "simplest" single knapsack problems (basically those reviewed in Sections 3–8 of Part I) are $\mathcal{NP}$-hard in the weak sense, i.e., they may be solved in pseudo-polynomial time through *Dynamic Programming* (DP). Most variants and generalizations considered in the subsequent sections, as well as most problems treated in Part II, are instead $\mathcal{NP}$-hard in the strong sense (i.e., they cannot be solved by pseudo-polynomial time algorithms unless $\mathcal{P} = \mathcal{NP}$), as specifically stated in the corresponding sections. $\mathcal{NP}$-completeness considerations and several detailed complexity proofs for knapsack problems can be found in Chapter 1 of Martello and Toth (1990) and in Appendix A of Kellerer et al. (2004).

Monograph Kellerer et al. (2004) is updated to the end of June 2003. Apart from a handful of earlier publications (mostly in the present and in the next section), all references in this work are from 2003 onwards. With very few exceptions, we restricted our review to contributions appeared in peer reviewed journals. The survey is subdivided into two parts. Part I covers the classical single knapsack problems, while Part II is devoted to the multiple, multidimensional, and quadratic cases, and to a succinct treatment of online and multi-objective knapsack problems. As we review several variants and generalizations, the survey includes many acronyms: each of the two parts provides an Appendix with the list of the abbreviations used. The contents of Part I are the following:

2. Books and surveys

We already mentioned the two monographs dedicated to knapsack problems. Besides the KP01, the book by Martello and Toth (1990) includes specific chapters on the following basic variants: *bounded* (and *unbounded*) *knapsack*, *subset-sum*, *change-making*, *multiple knapsack*. In addition, there are two chapters on two companion problems: generalized assignment and bin packing. The book by Kellerer et al. (2004) includes chapters on the same basic variants (but change-making), plus chapters on *multidimensional*, *multiple-choice*, and *quadratic knapsack problems*.

Monograph Kellerer et al. (2004) only briefly mentions bin packing and generalized assignment, which are instead extensively treated in monograph Martello and Toth (1990). In the *Bin Packing Problem* (BPP), one is given n items with weights w_j $(j = 1,\dots,n)$ and an unlimited number of identical knapsacks (*bins*) of capacity c, and the objective is to pack *all* the items into the minimum number of bins without exceeding their capacities. In the *Generalized Assignment Problem* (GAP), one is given n items and m knapsacks of capacity c_i $(i = 1,\dots,m)$: inserting item j into knapsack i $(i = 1,\dots,m; j = 1,\dots,n)$ produces a profit p_{ij} and assigns a weight w_{ij} to knapsack i. The objective is to pack *each* item into exactly one knapsack so as to maximize the overall profit without assigning to any knapsack a total weight greater than its capacity. In the last decades, these two topics where subject to intense investigation, which made them autonomous areas of research. For this reason, they will not be treated in the present survey. We refer the reader to some studies specifically devoted to them, namely:

- for the BPP, a review of exact approaches has recently been presented by Delorme et al. (2016), while an exhaustive treatment of approximation algorithms can be found in Coffman et al. (2013);
- for the GAP, three surveys have been published in the last fifteen years, by Morales and Romeijn (2005), Öncan (2007), and Wu et al. (2018b).

Coming to knapsack problems, a number of surveys devoted to specific methodologies or problem variants appeared in the last two decades:

- Pisinger (2005) presented an overview of exact solution approaches for the KP01 using classical and new benchmark tests. Also see the recent article by Smith-Miles et al. (2021) for a revisiting of this study;
- Bretthauer and Shetty (2002) presented an exhaustive review of algorithms for various classes of *nonlinear knapsack problems*, see Part II (Cacchiani et al., 2022): continuous, integer, convex, nonconvex, separable, and nonseparable. In addition, they discussed some interesting applications arising in production planning, health care, and computer networks. An updated survey was

later presented by Li and Sun (2006). We also refer the reader to Ibaraki and Katoh (1988) and Lin (1998) for previous surveys;
- Fréville (2004) provided a survey on the *multidimensional knapsack problem* (see Part II) reviewing exact, heuristic, approximation, and metaheuristic algorithms, as well as commercial software products; the work was later extended in Fréville and Hanafi (2005);
- Pisinger (2007) reviewed the literature on the *quadratic knapsack problem* (see Part II), with special emphasis on methods for computing upper bounds. The study includes an extensive experimental analysis, comparing tightness and computational effort of the various bounds;
- Wilbaut et al. (2008) discussed heuristic algorithms for a number of knapsack problem variants;
- Hu et al. (2009) reviewed exact and approximation approaches to the *unbounded knapsack problem* (see Section 5.2). Recently, Becker and Buriol (2019) reported the results of extensive computational experiments on several exact algorithms from the literature for this problem;
- Lust and Teghem (2012) considered the *multiobjective* version of single and multidimensional knapsack problems, reviewing exact, approximation, heuristic and metaheuristic algorithms (see Part II);
- Kellerer and Strusevich (2012) reviewed the main results on the *symmetric quadratic knapsack problem* (see Part II), focusing on approximation algorithms and their application to scheduling problems;
- Laabadi et al. (2018) reviewed heuristic algorithms for variants of the *multidimensional knapsack problem* (see Part II);
- recently, Hojny et al. (2019) provided a comprehensive overview of theoretical results on the polytopes of knapsack problems, to which the reader is referred for a deep analysis of these topics. After the publication of Hojny et al. (2019), relevant studies on the knapsack polytopes have been presented by Hojny (2020) (polynomial-size formulations), Letchford and Souli (2020) (knapsack cover inequalities), and Bienstock et al. (2022) (polytope of the minimization version of the problem);
- the *geometric knapsack problem* (see Part II) has been treated in a number of recent surveys, namely, Christensen et al. (2017) (mainly devoted to the BPP), Silva et al. (2019) (focused on exact methods for the three-dimensional version of the problem), Leao et al. (2020) (devoted to the case of irregular shapes), Iori et al. (2021) (focused on exact algorithms and mathematical models for the two-dimensional version of the problem).

We finally mention the special issue of *Computers & Operations Research* on knapsack problems and applications, edited by Hifi and M'Hallah (2012).

3. 0-1 knapsack problem

The KP01 is the most popular among knapsack problems and it has been the subject of intense research for decades. These investigations have produced a rich variety of theoretical, practical, and algorithmic results which have, to a certain extent, saturated this specific field. The most widely used approach to the exact solution of the problem is still the `Combo` algorithm, developed by Martello et al. (1999), whose C code is available at http://hjemmesider.diku.dk/~pisinger/codes.html.

New branching strategies for *Branch-and-Bound* (B&B) approaches were developed by Morales and Martínez (2020) and Yang et al. (2021). The sensitivity analysis to perturbations of item profits or weights was studied by Hifi et al. (2005b, 2008) and by Belgacem and Hifi (2008b), while Pisinger and Saidi (2017) investigated a particular sensitivity analysis (*tolerance analysis*) that can be performed in amortized time $O(c \log n)$ for each item. Improvements over existing *Fully Polynomial Time Approximation Schemes* (FPTAS) were recently developed by Chan (2018) and by Jin (2019).

In the next sections, we review recent results on relevant variants of the basic KP01.

4. Subset sum problem

When the profit and the weight of each item are identical, the problem, given by

$$\max \sum_{j=1}^{n} w_j x_j \tag{4}$$

s.t. (2)–(3),

is denoted as the *Subset Sum Problem* (SSP). In the special case in which $c = \sum_{j=1}^{n} w_j/2$, the SSP is called the *Partition Problem* and is regarded as the "simplest" $\mathcal{NP}$-hard problem (see Garey and Johnson, 1979). Differently from the KP01, the SSP has been the subject of intensive research in recent years too, probably also due to its connection to cryptography (see, e.g., Kate and Goldberg, 2011).

Exact solution. The B&B algorithms for the exact solution of the SSP are normally initialized by sorting the items according to decreasing weight. Kolpakov and Posypkin (2018) showed that this policy is the one requiring, in the worst case, the smallest number of iterations. Kolpakov et al. (2017) proposed a variation of the B&B method that decreases the number of iterations by a factor of two. Curtis and Sanches (2019) presented an improved version of a DP algorithm called `Balsub` (see Kellerer et al. (2004), Section 4.1.5), and computationally compared it with other approaches from the literature on benchmarks of difficult SSP instances.

Approximation. The classical greedy algorithm for the SSP has worst-case performance ratio equal to 0.5. Using a multiple-pass variant of the algorithm, Martello and Toth improved it to 0.75 (see Kellerer et al. (2004), Section 4.5). More recently, Ye and Borodin (2008) studied greedy variants in which decisions may be revoked, obtaining a performance ratio between 0.8 and $\delta \approx 0.893$. Gál et al. (2016) adopted an unusual computation model (*input stream*) to obtain an FPTAS having space requirement $O(1/\varepsilon)$ (while that of previous FPTASs depends on n), where ε is the required accuracy. Pseudopolynomial-time algorithms for the SSP were presented at computer science conferences by Koiliaris and Xu (2019) and by Bringmann (2017).

Heuristics. Since most instances of the SSP can be exactly solved in very short computing times (see Martello and Toth, 1990; Kellerer et al., 2004), no relevant results have been recently achieved by heuristic or metaheuristic algorithms. Ghosh et al. (2006) presented a sensitivity analysis of greedy heuristics for the KP01 and the SSP.

Variants and generalizations

In recent years, a number of variants of the SSP has been considered. Rohlfshagen and Yao (2011) empirically analyzed the *dynamic* version of the SSP (in which the parameters change over time), investigating the correlation between the parameter change and the movement of the optimum.

Darmann et al. (2014) studied a game theoretic variant (the *Subset Sum Game*), in which two decision makers compete for a common resource (the capacity), showed that finding an optimal sequence of decisions is an $\mathcal{NP}$-hard problem, and analyzed the worst-case performance of two natural heuristic strategies. Another game-theoretic issue (the *Fair Subset Sum Problem*) was studied by Nicosia et al. (2017).

Kothari et al. (2005) introduced the *Interval Subset Sum Problem*: given a set of intervals and an integer target T, find a set of integers, at most one from each interval, such that their sum is closest to, without exceeding, T. They defined an efficient FPTAS for its approximate solution. Later, Diao et al. (2017) proposed an improved FPTAS having almost the same time complexity but a significantly lower space complexity. Gourvès et al. (2018) considered a node-weighted digraph in which one has to select a subset of vertices with total weight not exceeding a given capacity and added additional precedence and maximality constraints, whose combination leads to four problem variants: they proved that all problems are $\mathcal{NP}$-hard and gave approximation results for special classes of digraphs.

We conclude this section by observing that the simple and neat structure of the SSP also attracted researchers outside the operations research community. In mathematics, properties of the SSP over finite fields were investigated by Li and Wan (2008), Wang and Nguyen (2018), and Choe and Choe (2019). The computer science community developed parallel algorithms for different architectures (Sanches et al., 2007, 2008; Curtis and Sanches, 2017).

5. Knapsack problems with item types

In this section, we examine variants of the KP01 in which a number of identical copies of each item is available. In these contexts, the term 'item' is normally replaced by *item type*.

5.1. Bounded knapsack problem

The generalization of the KP01 in which b_j identical copies of item type j are available ($j = 1, \dots, n$) is known as the *Bounded Knapsack Problem* (BKP), formally defined by

(1)–(2)

$$0 \le x_j \le b_j, \ x_j \text{ integer} \qquad (j = 1, \dots, n), \tag{5}$$

where x_j represents the number of selected copies of item type j.

We are only aware of two recent results on the BKP. Tamir (2009) proposed a pseudo-polynomial algorithm having time complexity $O(n^3 \max_j \{w_j\}^2)$, to be compared with the $O(nc)$ algorithm by Kellerer et al. (see Kellerer et al. (2004), Section 7.2.2). Deineko and Woeginger (2011b) showed that all instances satisfying a set of special inequalities that relate weight ratios to profit ratios (*cross ratio ordered instances*) can be solved in $O(n)$ time.

Most results appeared in recent years concern the following special case of the BKP:

5.2. Unbounded knapsack problem

In this case, an unlimited number of copies of each item type are available. The *Unbounded Knapsack Problem* (UKP) is defined by

(1)–(2)

$$x_j \ge 0 \text{ and integer} \qquad (j = 1, \dots, n). \tag{6}$$

The UKP has a well-known characteristic: most of the contribution to the optimal solution profit comes from the item type, say s, with highest profit-to-weight ratio, provided the knapsack capacity is sufficiently large (see Hu et al., 2009). In particular, a number of studies has been devoted to finding, for a given set of item types, the capacity bound c_0 such that, for all $c \ge c_0$, the optimal solution value is $z^*(c) = z^*(c - w_s) + p_s$ (*periodicity property*; see Kellerer et al. (2004), Section 8.2, for a more detailed treatment.) The survey by Hu et al. (2009) (see Section 2) also includes a DP approach for evaluating the periodicity property. Improved periodicity bounds were proposed by Huang et al. (2011) and Huang and Tang (2012), although the latter, which requires $O(n^2)$ time, can be time-consuming when $c < n$.

Poirriez et al. (2009) proposed an algorithm based on a combination of DP, dominance rules, and B&B for the exact solution of the UKP. A hybrid approach, that also includes the generation of valid inequalities, was presented by He et al. (2016a), who also extended it to the multidimensional version of the problem, see Part II (Cacchiani et al., 2022). Becker and Buriol (2019) reported on an extensive computational experimentation (on classical and new benchmarks) of seven old and recent exact algorithms for the UKP, including the algorithm in Poirriez et al. (2009) and commercial solvers CPLEX and Gurobi: quite surprisingly, a DP algorithm developed in the Sixties by Gilmore and Gomory (1966) (slightly improved by the authors) achieved the best results among all DP algorithms.

Jansen and Kraft (2018) proposed an FPTAS for the UKP, whose time and space complexities improve those of classical FPTASs from the literature (see Kellerer et al. (2004), Section 8.5). Deineko and Woeginger (2011a) investigated a special case of the UKP in which the item weights form an arithmetic sequence and proposed an exact $O(n^8)$ time algorithm.

5.3. Change-making problems

The name of this problem comes from an interpretation in which a cashier needs to make change for a certain sum using the minimum number of coins from a given set of coin denominations. Using the knapsack terminology, given n (different) item types, each having weight w_j ($j = 1, \dots, n$), and a knapsack of capacity c, the *Unbounded Change-Making Problem* (UCMP) is:

$$\min \sum_{j=1}^{n} x_j \tag{7}$$

$$\text{s.t.} \ \sum_{j=1}^{n} w_j x_j = c \tag{8}$$

$$x_j \ge 0 \text{ and integer} \qquad (j = 1, \dots, n). \tag{9}$$

It is normally assumed that $w_1 < w_2 < \cdots < w_n$ and that $w_1 = 1$ (so a solution to the problem always exists). The case in which there is a limited number of items of each type is known as the *Bounded Change-Making Problem* (BCMP).

A greedy algorithm for the UCMP iteratively selects the item type (coin) whose weight is no larger than the remaining capacity. Most recent results for the UCMP studied characterizations of coin systems for which the greedy algorithm is optimal. Pearson (2005) gave an $O(n^3)$ algorithm to determine, for a given coin system, whether the greedy algorithm is optimal. Cowen et al. (2008) characterized *totally greedy* coin sets, i.e., instances of the UCMP such that, for each j, the greedy algorithm is optimal for the coin set $\{w_1 = 1, w_2, \dots, w_j\}$. Adamaszek and Adamaszek (2010) provided necessary conditions that must be satisfied by a coin system in order to ensure that the greedy algorithm is optimal. Goebbels et al. (2017) studied approximate solutions for a generalization of the problem in which the sum of the weights may be larger than c.

6. Knapsack problems with setup

Although the first definition of setup knapsack problems dates back to the Nineties (see Chajakis and Guignard, 1994) these problems have attracted more attention in the last decade. A number of different (although similar) definitions can be encountered in the literature, the most frequent one being the following. Consider a generalization of the KP01 in which items belong to F disjoint families and can be selected only if the corresponding family is activated. Family $i \in \{1, \dots, F\}$ contains n_i items and has positive activation (setup) cost f_i and weight d_i. The profit and weight of an item j of family i are p_{ij} and w_{ij}, respectively. The *Knapsack Problem with Setup* (KPS) is then

$$\max \sum_{i=1}^{F} \sum_{j=1}^{n_i} (p_{ij} x_{ij} - f_i y_i) \tag{10}$$

$$\text{s.t.} \ \sum_{i=1}^{F} \sum_{j=1}^{n_i} (w_{ij} x_{ij} + d_i y_i) \le c \tag{11}$$

$$x_{ij} \le y_i \qquad (i = 1, \dots, F; j = 1, \dots, n_i) \tag{12}$$

$$x_{ij} \in \{0, 1\} \qquad (i = 1, \dots, F; j = 1, \dots, n_i) \tag{13}$$

$$y_i \in \{0, 1\} \qquad (i = 1, \dots, F), \tag{14}$$

where x_{ij} takes the value one iff item j of family i is selected and y_i takes the value one iff family i is activated. Constraints (12) impose the activation of a family if one of its items is selected. The KPS has a number of real world applications, e.g., in make-to-order production contexts in the management of different product categories.

Exact solution. The first exact approach to the KPS was proposed by Yang and Bulfin (2009), who presented a B&B algorithm and experimentally tested its performance. Ichimura et al. (2012) proposed an extension of such method, especially aimed at solving large-scale instances. Chebil and Khemakhem (2015) developed a DP approach with an original space reduction technique. Della Croce et al. (2017a) presented an exact method based on the identification of sub-problems, tackled through a commercial solver (CPLEX). Furini et al. (2018) studied alternative ILP formulations and implemented a parallel approach that executes, on a multi-thread computer, three algorithms (a B&B, a B&P, and a DP procedure) in parallel, and halts execution as soon as one of the three terminates. Although a precise computational comparison with other approaches from the literature is not immediate, this method appears to be the current state-of-the-art for the KPS.

Approximation. Pferschy and Scatamacchia (2018) proved that no polynomial time approximation algorithm can exist for the KPS (unless $\mathcal{P} = \mathcal{NP}$) and analyzed three special cases of the problem that admit an FPTAS. They also proposed and computationally evaluated an improved DP algorithm for its exact solution.

Heuristics. Most exact algorithms solve large instances of the KPS within very short computing times. For example, the computational experiments in Furini et al. (2018) refer to instances with up to 30 families and 10,000 items. Few heuristic approaches were also proposed: Khemakhem and Chebil (2016) presented a method based on a truncated tree search approach and compared it with CPLEX on instances of the same size. Amiri (2020) proposed an iterative approach based on a Lagrangian relaxation of the problem, followed by a greedy-type heuristic guided by the current Lagrangian multipliers to construct feasible solutions. The computational experiments in Amiri (2020) show that the method produces good quality solutions for very large instances with up to 500 families and 2,000,000 items.

Variants and generalizations

A number of variants of the KPS can be found in the literature. Altay et al. (2008) considered a mixed-integer variant of the KPS where fractions of items are allowed to be selected. They developed exact and heuristic solution methods as well as a Benders decomposition approach for the continuous relaxation of the problem.

Akinc (2006) studied the *fixed-charge knapsack problem*, a special case of the KPS with no setup capacity consumption, i.e., $d_i = 0$ ($i = 1, \dots, F$). He developed several algorithmic components to improve the efficiency of B&B, such as effective procedures to obtain good candidate solutions and a set of rules to peg the set-up variables y_i to 1 or 0.

McLay and Jacobson (2007a) considered a generalization of the BKP (see Section 5.1) where each item type has a set-up weight and a (positive) set-up value that are activated if at least one copy of that item type is selected. They proposed three DP algorithms and an FPTAS. Al-Maliky et al. (2018) developed a sensitivity analysis of this problem to the perturbation of item profits or weights.

McLay and Jacobson (2007b) studied additional variants, one involving an unbounded number of copies of each item type, and one imposing that exactly k items are inserted into the knapsack. For each of these two variants, they proposed DP approaches, heuristics, and an FPTAS. Another unbounded variant of the problem, considered in McLay and Jacobson (2007a), was studied by Caserta et al. (2008), who proposed a metaheuristic algorithm based on a "cross entropy" scheme.

Michel et al. (2009) defined the *multiple-class integer knapsack problem with setups*, a variant of the KPS where there are multiple copies of each item, the item weights are multiples of a value associated with the class, and each class has an upper and a lower bound on the number of items to select. They studied this problem, as well as a number of its variants, showing how specialized B&B procedures derived from the classical Horowitz–Sahni algorithm for the KP01 (see Martello and Toth (1990), Section 2.5.1) can be extended to deal with them.

7. Multiple-choice knapsack problem

The *Multiple-Choice Knapsack Problem* (MCKP), also known as the *knapsack problem with generalized upper bound constraints*, is a generalization of the KP01 in which the item set is partitioned into ℓ *classes* $N_1, \dots, N_\ell$ and it is requested to select exactly one item per class. Formally,

(1)–(3)

$$\sum_{j \in N_i} x_j = 1 \qquad (i = 1, \dots, \ell). \tag{15}$$

The problem is sometimes modeled with the '$\le$' sign in (15). Such formulation can be transformed in an equivalent MCKP formulation by adding a dummy item, with null profit and weight, to each class. To the best of our knowledge, no relevant result on the exact solution of the MCKP appeared after the publication of monograph Kellerer et al. (2004).

He et al. (2016b) presented an approximation algorithm that, for a prefixed positive integer parameter t, guarantees a worst-case ratio of $3 + (\frac{1}{2})^t$ and runs in $O(n(t + \log \ell))$ time. Bednarczuk et al. (2018) presented a heuristic approach that removes the capacity constraint (2) and solves a bi-objective problem that maximizes the total profit and minimizes the total weight: the corresponding solution set is then searched for Pareto efficient solutions which are feasible for the MCKP. Sbihi (2013) developed a reactive Tabu search algorithm for a variant of the MCKP arising in budget planning over discrete periods, in which periods correspond to classes and have individual capacities.

Agra and Requejo (2009) studied a special case of the MCKP (the *linking set problem*) which, under certain conditions, can be solved exactly in polynomial time. Zhong and Young (2010) described a real-world case in which the decision on the allocation of funds to alternative projects was solved through an MCKP model.

Kozanidis and Melachrinoudis (2004) and Kozanidis et al. (2005) discussed continuous and mixed-integer knapsack problems that include multiple-choice type constraints imposing, for each class, an upper bound on the sum of the corresponding continuous variables. They proved that the former problem can be exactly solved by a two-phase greedy algorithm and developed a B&B approach for the exact solution of the latter.

8. Knapsack sharing problem

Another generalization of the KP01 is the *Knapsack Sharing Problem* (KSP). As in the MCKP, the item set is partitioned into ℓ *classes* $N_1, \dots, N_\ell$ but the objective is to select a set of items that maximizes the minimum total profit of a class. Formally,

$$\max \min_{1 \le i \le \ell} \left\{ \sum_{j \in N_i} p_j x_j \right\} \tag{16}$$

s.t. (2)–(3).

Hifi et al. (2005a) proposed an exact algorithm based on a decomposition of the problem into ℓ KP01s, which are solved through DP for different tentative values of the capacity assigned to each subproblem. Similar decompositions and a sensitivity analysis have been studied by Hifi and Sadfi (2002), Hifi and Wu (2014), Belgacem and Hifi (2008a), Hifi and Mhalla (2013).

Haddar et al. (2015) proposed a hybrid heuristic based on the combination of an LP-based heuristic and a metaheuristic approach, and evaluated its average performance through extensive computational experiments.

In a relevant variant of the problem, the *Generalized Knapsack Sharing Problem* (GKSP) (also referred to as the *knapsack sharing problem with common items*), there is an additional class N_0 and its profit is summed to the profit of each class N_i ($i = 1, \dots, \ell$). Fujimoto and Yamada (2006), who first defined this problem, proposed an exact algorithm based on the decomposition of the problem into a KP01 and a KSP and the enumeration of the possible capacities of the two subproblems. The method was improved by Dahmani et al. (2016) through upper bound computations and reduction procedures. A metaheuristic approach for the approximate solution of the GKSP was proposed by Haddar et al. (2016).

9. Knapsack problem with conflict graph

The *Knapsack Problem with Conflict Graph* (KPCG), also referred to as the *knapsack problem with conflicts* or the *disjunctively constrained knapsack problem*, is a generalization of the KP01 in which a given undirected graph $G = (V, E)$ defines the pairs of incompatible items that cannot be simultaneously selected. Formally,

(1)–(3)

$$x_i + x_j \le 1 \qquad (i, j) \in E. \tag{17}$$

An alternative formulation replaces constraints (17) with

$$\sum_{j \in V(i)} x_j \le |V(i)|\,(1 - x_i) \qquad i \in V, \tag{18}$$

where $V(i)$ denotes the set of neighboring vertices of vertex $i \in V$. To possibly obtain a stronger LP-relaxation bound, constraints (17) can also be replaced by

$$\sum_{j \in C} x_j \le 1 \qquad C \in \mathcal{C}, \tag{19}$$

where $\mathcal{C}$ denotes the family of cliques of G such that, for each edge $(i, j) \in E$, items i and j belong to some clique $C \in \mathcal{C}$.

The KPCG is a generalization of the *stable set problem* (given an undirected graph, find a maximal set of vertices no two of which are connected by an edge) and hence it is strongly $\mathcal{NP}$-hard, see Pferschy and Schauer (2009).

Exact solution. Hifi and Michrafy (2007) proposed a three-phase algorithm: a reactive local search algorithm, adapted from Hifi and Michrafy (2006), computes a lower bound, reduction strategies are applied to fix some decision variables, and the reduced problem is solved by the commercial solver CPLEX. The algorithm was tested on instances with very sparse conflict graphs. Bettinelli et al. (2017) proposed a B&B algorithm based on model (1)–(3), (19): the branching phase makes use of DP for pruning decision nodes, and upper bounds are computed as an extension of the *weighted clique cover bound*, proposed in Held et al. (2012). The algorithm was extensively tested on instances with conflict graph densities between 0.1 and 0.9, exhibiting better performance than the B&B approach proposed in Sadykov and Vanderbeck (2013), in which KPCG arises as a subproblem of the *bin packing problem with conflicts*. Salem et al. (2018) studied polyhedral aspects of the KPCG, presented new families of valid inequalities, and determined necessary and sufficient conditions for these inequalities to be facet defining. They also developed a *Branch-and-Cut* (B&C) algorithm that employs exact and heuristic separation procedures for the valid inequalities. Recently, Coniglio et al. (2021) proposed a B&B algorithm which adopts a branching scheme similar to the one in Bettinelli et al. (2017), but makes use of bounds based on clique partition and on transformation to an MCKP. Extensive computational results show that this algorithm compares favorably with the one in Bettinelli et al. (2017). Facet defining cutting planes for the KPCG were recently studied by Luiz et al. (2021).

Approximation. Pferschy and Schauer (2009) presented algorithms with pseudo-polynomial time and space complexity for special classes of conflict graphs (graphs with bounded tree-width and chordal graphs), and derived from these algorithms FPTASs for the same classes of graphs. In addition, they showed that the KPCG remains strongly $\mathcal{NP}$-hard for perfect graphs. Pferschy and Schauer (2017) proposed, for special classes of graphs (bounded tree-width, chordal, weakly chordal, planar, perfect), several complexity results and approximation algorithms, both for the KPCG and for the *knapsack problem with forcing graph* (a KP01 with constraints requiring that, for each edge of the graph, at least one of the two items be selected). The same two problems were tackled by Gurski and Rehs (2019), who provided pseudo-polynomial algorithms, based on DP, for the case of co-graphs as conflict and forcing graphs, and FPTASs for the case of graphs of bounded clique-width.

Heuristics. All heuristic algorithms in the literature were tested on instances with sparse graphs (density at most 0.4). The reactive local search algorithm in Hifi and Michrafy (2006) applies a greedy algorithm followed by a swapping procedure and a diversification strategy. Other heuristics (local branching), metaheuristics (scatter search, local search, ant colony), and hybrid approaches have been examined by Akeb et al. (2011), Hifi and Otmani (2012), Hifi (2014), Hifi et al. (2015). Quan and Wu (2017) introduced a cooperative parallel adaptive neighborhood search algorithm, in which the cooperation stage collects and shares information on local optima found by subprocesses. Salem et al. (2017) presented a probabilistic Tabu search heuristic with multiple neighborhood structures, a variant of Tabu search where the move is chosen probabilistically from a pool. The algorithm provided better results than those in Hifi (2014) and Hifi et al. (2015).

10. Precedence constrained knapsack problem

Given a directed graph $G = (V, A)$ of n vertices, the *Precedence Constrained Knapsack Problem* (PCKP) is a generalization of the KP01 in which, for each arc $(i, j) \in A$, item j can only be selected if item i has been selected. Formally,

(1)–(3)

$$x_i \ge x_j \qquad \forall\,(i, j) \in A. \tag{20}$$

If G contains a cycle, the vertices of the cycle must either all be selected or all be excluded. It follows that the items in a cycle can be replaced by a single item, with cumulative profit and weight, so G can be assumed to be acyclic.

The PCKP is $\mathcal{NP}$-hard in the strong sense, as it can be shown (see, e.g., Kellerer et al., 2004) by reduction from the *clique problem* (given an undirected graph, find a maximal complete subgraph).

You and Yamada (2007) presented a reduction procedure (pegging test) based on Lagrangian relaxation of the precedence constraints (20) and subgradient optimization. Computational experiments showed that most randomly generated instances with up to 2000 items can be optimally solved, after reduction, by a standard commercial solver.

Boland et al. (2012) presented methods for determining facets of the PCKP polyhedron based on clique inequalities, and tested their effectiveness in reducing solution times when applied at the root node of a cutting plane approach and within a B&C framework. Espinoza et al. (2015) provided a partial characterization of maximally violated inequalities and computationally evaluated their usefulness in improving the performance of a cutting plane algorithm.

Precedence constrained covering problems, which include the PCKP as a special case, were studied by McCormick et al. (2017), who presented a strongly polynomial primal–dual approximation algorithm. A multi-period generalization of the PCKP, arising in the mining industry, was considered by Samavati et al. (2017), who strengthened the LP relaxation of the problem so as to improve the efficiency of a classical sequencing heuristic for mine production scheduling.

11. Robust knapsack problems

Robust optimization is an approach to uncertain optimization, frequently adopted as an alternative to stochastic optimization. Loosely speaking, it consists in finding a solution that is "robust" (according to some criterion) against variations in the input data. For the knapsack problem, a set of data that defines an instance (profits, weights, capacity) is called a *scenario*. There is a vast literature on scenario-based robust optimization: we refer the interested reader to the classical books by Kouvelis and Yu (2013) and Kasperski (2008). Three main research streams have been followed in the recent literature, as illustrated in the next sections.

11.1. Max–min knapsack problem

Consider a generalization of the KP01 in which we are given S scenarios, each characterized by a set of profits p_j^s ($j = 1, \dots, n$; $s = 1, \dots S$). It is assumed that weights and capacity do not vary. The *Max-Min Knapsack Problem* (MMKP) consists in finding a solution that maximizes the worst-case profit over all scenarios, i.e.,

$$\max \min_{1 \le s \le S} \left\{ \sum_{j=1}^{n} p_j^s x_j \right\} \tag{21}$$

s.t. (2)–(3).

The problem is $\mathcal{NP}$-hard in the strong sense, as it can be shown (see, e.g., Kellerer et al., 2004) by reduction from the *set covering problem* (given a set $\mathcal{S}$ and a family $\mathcal{F}$ of subsets of $\mathcal{S}$, find the minimum cardinality subfamily of $\mathcal{F}$ whose union is $\mathcal{S}$). In an earlier study on the MMKP, Yu (1996) proved that the problem is strongly $\mathcal{NP}$-hard in the case of an unbounded number of scenarios. On the other hand, he showed that it can be solved in pseudo-polynomial time if the number of scenarios is bounded by a constant.

Taniguchi et al. (2008) introduced a particular surrogate relaxation and a reduction (pegging) procedure, and embedded them into a B&B algorithm that was computationally tested on a large set of benchmark instances. Goerigk (2014) presented a new method to compute upper bounds and computationally proved that it considerably improves on the performance of B&B approaches to the MMKP. A different exact solution method, based on column generation and B&P, was developed by Pinto et al. (2015) and computationally tested on large-size benchmark instances (with up to 20,000 items and 1000 scenarios).

Metaheuristic algorithms for the MMKP have been proposed by Sbihi (2010) (greedy solution and Tabu search), Aldouri and Hifi (2018) (hybrid reactive search), and Al-douri et al. (2021) (greedy randomized search and path-relinking).

Taniguchi et al. (2009) adapted their algorithm (Taniguchi et al., 2008) to the special case in which there are just two scenarios. A different exact approach for the two-scenarios case, based on mixed integer programming formulations and reduction procedures, was proposed by Hanafi et al. (2012).

11.2. Min–max regret knapsack problem

In this case, the scenarios are defined by n intervals $[p_j^-, p_j^+]$, and the actual profit of an item j in scenario s can take any integer value, p_j^s, in the corresponding interval. As for the MMKP, weights and capacity do not vary across the scenarios. Each feasible solution x associated with scenario s has a value ($z^s(x) = \sum_{j=1}^n p_j^s x_j$) and a *regret*, $r^s(x) = z_\star^s - z^s(x)$, where $z_\star^s$ denotes the optimal solution value for scenario s. The (interval) *Min–Max Regret Knapsack Problem* (MMRKP) consists in finding a solution that minimizes the worst-case regret over all scenarios, i.e.,

$$\min \max_{1 \le s \le S} \{ r^s(x) \} \tag{22}$$

s.t. (2)–(3).

The MMRKP is extremely challenging both from a theoretical and a practical point of view. Its precise complexity status is unclear. Being a generalization of the KP01 (the special case in which $p_j^- = p_j^+$ for all items) the MMRKP is $\mathcal{NP}$-hard, while it is an open question whether it is strongly $\mathcal{NP}$-hard. Deineko and Woeginger (2010) proved that its decision version is complete for the complexity class Σ_2^p (see Garey and Johnson, 1979), and hence is most likely not in $\mathcal{NP}$. Observe that even computing the regret of a single solution x is an $\mathcal{NP}$-hard problem, as it requires the solution of a KP01.

Furini et al. (2015) evaluated the performance of standard approaches (Benders-like decomposition and B&C) when adapted to the MMRKP, and proposed a Lagrangian-based B&C algorithm, an iterated local search approach and an ILP-based heuristic. Extensive computational experiments showed that the method can solve instances with 50 items to proven optimality.

Kalaï and Vanderpooten (2011) proposed the *lexicographic α-robust knapsack problem*, a variant that is somehow intermediate between the MMKP and the MMRKP, and presented an algorithm to determine the set of all α-robust solutions in pseudo-polynomial time.

In another variant, the *discrete min–max regret knapsack problem*, the profits do not vary in intervals but according to a discrete number of scenarios. Approximation results for this variant have been studied by Aissi et al. (2009) and later surveyed by Candia-Véjar et al. (2011).

Wu et al. (2016, 2018a) presented exact and heuristic algorithms for the extension of the MMRKP to the multiple knapsack problem, see Section 2 of Part II (Cacchiani et al., 2022), and to a further generalization of the problem (the GAP). Metaheuristics for another variant of the problem were recently presented by Wang et al. (2021).

We finally mention that Conde (2005) developed a linear-time algorithm for the min–max regret version of the *continuous* variant of the UKP (see Section 5.2).

11.3. Γ-robust knapsack problem

In this case, profits and capacity are constant, while the weight of each item j has a *nominal value* w_j and a variability range $[w_j - \underline{w}_j, w_j + \overline{w}_j]$. At most Γ weights can change from their nominal value to an arbitrary value in the interval. A solution is *Γ-robust* if it satisfies the capacity constraint (2) for any possible set of weights. The *Γ-Robust Knapsack Problem* (ΓRKP) is to find the maximum profit Γ-robust solution. Differently from the MMKP and the MMRKP, the ΓRKP is weakly $\mathcal{NP}$-hard.

Monaci et al. (2013) presented an FPTAS and a DP algorithm (with special techniques to reduce the space complexity). They computationally tested the resulting algorithm on a large set of randomly generated instances (up to 5000 items and $\Gamma = 50$). Monaci and Pferschy (2013) analyzed the worst case ratio between optimal solution values of the KP01 and the ΓRKP, and extended their analysis to the fractional version of the problem (in which fractions of items may be packed).

Claßen et al. (2015) considered a generalization of the ΓRKP, the *multi-band robust knapsack problem*, in which the variability range is subdivided into several smaller intervals (bands): they presented two DP algorithms and compared their performance by focusing on benchmarks with 2 bands. Büsing et al. (2019) studied a different variant, the *recoverable* ΓRKP, in which it is possible to remove at most k items in order to restore the feasibility of any scenario: they presented different ILP formulations and computationally evaluated them on a large testbed. A further recoverable ΓRKP variant, in which both weights and profits can vary, was studied by Büsing et al. (2011).

Goerigk et al. (2015) considered a variant in which one is allowed to perform Q *queries* ($Q < n$): each query returns the actual weight of an item. The *robust knapsack problem with queries* consists in deciding which items have to be queried so as to maximize the optimal solution value to the resulting instance (in which the weight of the any non-queried item j is set to $w_j + \overline{w}_j$). They studied the *query competitiveness* (derived from *online optimization*, see Part II) to evaluate the quality of an algorithm by comparing its solution value with that produced by the best possible choice of items to query (if the real weights were known).

12. Compartmentalized knapsack problems

The problems treated in this section come from a number of real world applications, mostly related to two-phase steel roll cutting problems, in which the items to be produced are subdivided according to their thickness. As we will see, these problems were defined in various ways, induced by specific constraints appearing in different applications, and formalized through various modeling techniques, ranging from linear to nonlinear, from integer to mixed integer.

Consider a BKP (or an UKP, see Section 5) in which the item types are partitioned into ℓ classes $\{N_1, \dots, N_\ell\}$ (corresponding to different typologies, e.g., steel thickness) and one is requested to build compartments inside the knapsack so that only items of the same class are loaded into a compartment. Building a compartment has a cost (depending on the specific application), the capacity of each compartment has a lower bound l_i and an upper bound u_i ($i = 1, \dots, \ell$), and the creation of each compartment may produce a fixed loss of capacity of the original knapsack. The *Compartmentalized Knapsack Problem* (CKP) is to build compartments and assign the items in such a way that the overall profit (item profits minus building costs) is maximized. In the *Constrained Compartmentalized Knapsack Problem* (CCKP), the number of items of type j in the overall knapsack cannot exceed a given value β_j.

Hoto et al. (2007) proposed exact algorithms for the CKP, based on the solution of a large number of knapsack problems (one for each feasible combination of weights of each class), and a heuristic in which the knapsack solutions are replaced by the computation of the Martello–Toth upper bound (see Martello and Toth, 1990). They also proposed a decomposition heuristic for the CCKP, in which feasible promising compartments are generated and a KP01 is solved to choose the compartments. Marques and Arenales (2007) proposed various heuristics and an upper bound for the CCKP and performed extensive computational experiments. Leão et al. (2011) presented an ILP formulation and a number of effective heuristics. Modified versions of the heuristics in Marques and Arenales (2007) were proposed and computationally tested by Hoto and Bressan (2017). Other ILP models for the CCKP have been recently proposed by Inarejos et al. (2019) and Quiroga-Orozco et al. (2019). Approximation schemes for the CCKP (under a different name) were studied by Xavier and Miyazawa (2006).

13. Bilevel knapsack problem

Bilevel programs model a hierarchical relationship between two decision-makers, a leader and a follower, who take their decisions in a sequential way: first, the leader (upper level), who has perfect knowledge of the follower's problem (objective function, constraints, and follower's decisions), takes an action to optimize her own objective, and then the follower (lower level) reacts, thereby influencing the decision of the leader (as the leader's objective depends on the follower's decision). These problems, taking into account two agents, each with her own individual objective, are known as *two-player Stackelberg games*. Although they belong to Game Theory, in the *Bilevel Knapsack Problems* (BLKP) the leader and the follower solve a combinatorial optimization problem, so we provide a description of the main results. The interested reader is referred to the survey on bilevel programming by Labbé and Violin (2016).

Several different definitions of the BLKP can be found in the literature. Brotcorne et al. (2009) proposed a two-phase DP algorithm for the BLKP originally introduced by Dempe and Richter (2000), in which item profits vary with respect to the leader and the follower, and the leader first determines the knapsack capacity (within given lower and upper bound values), while the follower decides the subset of items to be selected by solving a KP01.

In a second version of the BLKP, introduced by Mansi et al. (2012), the item set is split into two sets, and both players can select items from their respective sets to be inserted in a single knapsack: hence, the leader's decision interferes with the residual capacity available to the follower. In Mansi et al. (2012), a DP algorithm was presented, while Brotcorne et al. (2013) proposed, for a more general version, a two-step exact algorithm that (i) in the first step applies a DP procedure taking into account only the follower problem, and (ii) in the second step reformulates the bilevel problem as a one-level model, based on the optimal solutions computed in the first step.

Chen and Zhang (2013) introduced another version with two knapsacks, one for each player: items can be selected simultaneously by both players, but each item profit depends on whether the item is selected by both the leader and the follower or by only one of them. Approximation algorithm were derived in Chen and Zhang (2013) and improved by Qiu and Kern (2015).

The most intensively studied version of the BLKP, introduced by DeNegre (2011), is known as the *Bilevel Knapsack Problem with Interdiction Constraints* (BLKPI). In the BLKPI, two knapsacks, one for each player, with capacity c_1 and c_2, respectively, are considered, and there is a common set of items having the same profits but different weights for the two players: the item weight is v_j for the leader and w_j for the follower ($j = 1, \dots, n$). The interdiction constraints impose that if an item is selected by the leader, it cannot be selected by the follower. The objective of the follower is to maximize the total profit, while the objective of the hostile leader is to minimize it. Binary variables x_j and y_j represent the item selection for the leader and the follower, respectively. Formally,

$$\min \sum_{j=1}^{n} p_j y_j \tag{23}$$

$$\sum_{j=1}^{n} v_j x_j \le c_1 \tag{24}$$

$$x_j \in \{0,1\} \qquad (j = 1, \dots, n), \tag{25}$$

where variables $y_1, \dots, y_n$ solve the follower's problem:

$$\max \sum_{j=1}^{n} p_j y_j \tag{26}$$

$$\sum_{j=1}^{n} w_j y_j \le c_2 \tag{27}$$

$$y_j \le 1 - x_j \qquad (j = 1, \dots, n) \tag{28}$$

$$y_j \in \{0,1\} \qquad (j = 1, \dots, n). \tag{29}$$

Caprara et al. (2014) showed that the computational complexity of the decision version of three variants studied in Dempe and Richter (2000), Mansi et al. (2012), and DeNegre (2011) is complete for the complexity class Σ_2^p, and hence there is no way of formulating each problem as a single-level integer program of polynomial size unless the polynomial hierarchy collapses. In addition, they studied these variants under so-called unary encodings, showing that the first two become polynomially solvable while the third one becomes $\mathcal{NP}$-hard. For the third variant, a *Polynomial Time Approximation Scheme* (PTAS) was derived, providing the first approximation scheme for a Σ_2^p-hard problem. Caprara et al. (2016a) proposed an exact algorithm that iteratively computes lower and upper bounds until optimality is reached. Upper bounds are derived by solving a single-level MILP model amended by *nogood cuts*. The algorithm was able to solve to optimality instances with up to 50 items. Fischetti et al. (2019) studied a family of mixed integer linear bilevel problems known as *interdiction games*, which include the BLKPI, and proposed a Benders-like algorithm, in which the problem is reformulated as a single-level problem with an exponential number of constraints, called *interdiction cuts*. Additional families of modified or lifted interdiction cuts were presented, for which exact and heuristic separation procedures were developed. The algorithm was extensively tested on benchmark instances of the BLKPI showing significant better performance with respect to the method in Caprara et al. (2016a). Carvalho et al. (2018) proposed a polynomial algorithm with worst-case complexity $O(n^2)$ for the continuous relaxation of the BLKPI. Fischer and Woeginger (2020) derived a faster algorithm that improves the worst-case complexity to $O(n \log n)$. Recently, Della Croce and Scatamacchia (2020) proposed an exact algorithm that relies on an effective lower bound and computes leader's solutions by exploring the follower's subproblems that have better lower bounds. Computational experiments showed that the algorithm can solve instances with up to 500 items in very short computing times, thus significantly improving the results in Caprara et al. (2016a) and Fischetti et al. (2019), even though the tests were carried out on different machines. The algorithm was extended to the MMRKP (see Section 11.2) and computational results showed that it outperforms the approach in Furini et al. (2015) on most instances.

Pferschy et al. (2021a) considered another variant in which the item set is partitioned into two sets, one for each player, and the leader can decide to assign an incentive to each of her own items, with the aim of influencing the follower's selection. The goal of the leader is to maximize the profit of the items selected from her set, reduced by the incentives, while the follower's goal is to maximize the profit of all selected items, whichever set they are taken from, increased by the incentives. The complexity is analyzed when the KP01 of the follower is solved to optimality, or when it is solved with greedy heuristics, and algorithms and ILP models are provided. In a companion paper, Pferschy et al. (2019) considered the case where the weights can be modified by the leader (instead of the profits) and analyzed the complexity of the problem for three different solution strategies of the follower.

14. Extensions, generalizations, and research directions

This section lists a selection of less studied variants for which relatively few results have been published, and hence they could be promising research areas. For these problems we provide a concise description and references to the latest works.

In the *knapsack problem with minimum filling constraint* a minimum total weight of the selected items is imposed. Xu and Lai (2011) developed an FPTAS for this problem, which finds applications in auction clearing. In the *incremental knapsack problem*, the capacity is increasing over time periods, an item selected in a period cannot be removed afterwards and contributes with its profit for all time periods in which it is included, with different time multipliers for different periods. Della Croce et al. (2018, 2019b) provided approximation results for the problem and for some variants. A PTAS for a variant of this problem was presented by Faenza and Malinovic (2018).

In contrast to the previous variants, in the *temporal knapsack problem* the capacity does not change, but each item is active in a given time interval, and the goal is to select the maximum profit subset of items whose total weight respects the capacity at any point in time. Caprara et al. (2013, 2016b) studied B&P algorithms, based on a Dantzig–Wolfe reformulation of the problem. Gschwind and Irnich (2017) derived two types of column-generation stabilization methods. A DP algorithm for its exact solution was recently presented by Clautiaux et al. (2021). Slight variants of this problem can be encountered in the literature under different names. For example, Darmann et al. (2010) studied the case where all profits are one (under the name *resource allocation with time intervals*), obtaining a $(\frac{1}{2} - \varepsilon)$ approximation algorithm.

Two knapsack problems with neighbor constraints, in which dependencies between items are represented by adjacencies in a graph, were studied by Borradaile et al. (2012) and, more recently, by Goebbels et al. (2022): the 1-*neighbor knapsack problem*, in which an item can be selected only if at least one of its neighbors is also selected, and the *all-neighbors knapsack problem*, in which an item can be selected only if all its neighbors are also selected. Approximation and hardness results are provided in both works for several classes of graphs.

In the *penalized knapsack problem*, besides profit and weight, each item has a *penalty*, and the goal is to maximize the sum of the profits, decreased by the largest penalty value of the selected items. The problem was introduced by Ceselli and Righini (2006), who presented an exact algorithm that performs an exhaustive search to identify the item with the largest penalty among items in the optimal solution, and then solves a corresponding KP01. Della Croce et al. (2019a) proposed a DP algorithm based on a core problem and on narrowing the relevant range of penalties.

In the *discounted knapsack problem*, a set of item groups is given: each group consists of three items where the third item represents a discounted offer, i.e., its weight is smaller than the sum of the weights of the first two items, while its profit coincides with the sum of the profits of the first two items. At most one item of each group can be selected, and the goal is to maximize the total profit while respecting the knapsack capacity. In Rong et al. (2012), an alternative core concept is proposed to partition the original problem into three sub-problems that are solved through DP. A DP algorithm with lower complexity was proposed by He et al. (2016c), who also derived an FPTAS, a 2-approximation algorithm, and a metaheuristic.

Malaguti et al. (2019) defined the *fractional knapsack problem with penalties*, in which an item can be split at the expense of a penalty that depends on the fractional quantity, and presented mathematical models, an FPTAS, DP algorithms, and various heuristics. An improved FPTAS was developed by Kovalev (2022).

Furini et al. (2017) introduced the *minimum-cost maximal knapsack packing problem*: it consists in finding a maximal knapsack packing that minimizes the cost of the selected items. The authors proposed a DP algorithm, and showed that the problem is equivalent to a "dual" problem (*the maximum-cost minimal knapsack cover*).

In the *parametric knapsack problem*, profits are affine-linear functions of a parameter, and the goal is to compute the optimal solutions for all values of the parameter on the real line (or within a given interval). Approximation schemes for various cases were derived by Giudici et al. (2017), and Holzhauser and Krumke (2017). Halman et al. (2018) studied the case in which weights are affine-linear functions of a parameter, and presented an FPTAS for this problem.

In the *knapsack problem with qualitative levels,* each item has a "qualitative" (imprecise or vague) *level* instead of a numerical profit. Schäfer et al. (2021) presented a DP approach to compute non-dominated solutions and two greedy algorithms to compute a single efficient solution.

We conclude with two variants which, although formally non-linear, have been tackled through combinatorial optimization techniques.

In the *collapsing knapsack problem*, the capacity is a non-increasing function of the number of selected items, i.e., it decreases when the number of selected items increases. Originally introduced as a nonlinear knapsack problem (see Wu and Srikanthan, 2006), it has been solved: (i) by transformation into an equivalent KP01 (see Kellerer et al., 2004, Section 13.3.7), with the drawback of the introduction of very large coefficients, and of a high correlation between profits and weights; (ii) more recently, through a very effective ILP formulation (see Della Croce et al., 2017b, who also present a reduction procedure and an exact algorithm that can be extended to the multidimensional case).

In the *product knapsack problem,* the profits can have positive or negative value, and the goal is to maximize the product of the profits of the selected items. Halman et al. (2019) showed that the problem is weakly $\mathcal{NP}$-hard. D'Ambrosio et al. (2018) presented effective ILP models and a DP algorithm for its exact solution. The first FPTAS for this problem was recently presented by Pferschy et al. (2021b).

15. Conclusions

We have examined over two hundred results on single knapsack problems, mostly appeared in the last seventeen years.

The contributions we have examined show that knapsack problems frequently appear in real-world applications. The introduction to the special issue edited by Hifi and M'Hallah (2012) enumerates applications "encountered in numerous industrial sectors such as transportation, logistics, cutting and packing, telecommunication, reliability, advertisement, investment, budget allocation, and production management". In the previous sections, we have encountered other interesting application areas like, e.g., *make-to-order production* (the KPS), *finance* (the MCKP), *mine production* (the PCKP), and *steel industry* (the CKP).

Apart from being studied as standalone problems, knapsack problems frequently appear as subproblems in some relevant optimization problems from different areas, which further underlines the importance of investigating them. Notably, set covering formulations for bin packing and cutting stock problems are typically tackled through column generation, which requires to solve, at each iteration, a knapsack problem (see, e.g., Delorme et al., 2016). Similarly, multidimensional cutting and packing problems frequently require the solution of knapsack problems. In addition, they often appear in algorithms to separate valid inequalities for optimization problems involving capacity constraints (see, e.g., Kaparis and Letchford, 2010 and Letchford and Souli, 2020).

Further pointers to this huge literature will be provided in Part II (Cacchiani et al., 2022), devoted to multiple, multidimensional, and quadratic knapsack problems, as well as to a succinct treatment of online and multiobjective knapsack problems.

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Appendix. Acronyms

ΓRKP	Γ-Robust Knapsack Problem
B&B	Branch-and-Bound
B&C	Branch-and-Cut
BCMP	Bounded Change-Making Problem
BKP	Bounded Knapsack Problem
BLKP	Bilevel Knapsack Problem
BLKPI	Bilevel Knapsack Problem with Interdiction Constraints
BPP	Bin Packing Problem
CCKP	Constrained Compartmentalized Knapsack Problem
CKP	Compartmentalized Knapsack Problem
DP	Dynamic Programming
FPTAS	Fully Polynomial Time Approximation Scheme
GAP	Generalized Assignment Problem
GKSP	Generalized Knapsack Sharing Problem
ILP	Integer Linear Programming
KP01	0-1 Knapsack Problem
KPCG	Knapsack Problem with Conflict Graph
KPS	Knapsack Problem with Setup
KSP	Knapsack Sharing Problem
LP	Linear Programming
MCKP	Multiple-Choice Knapsack Problem
MMKP	Max-Min Knapsack Problem
MMRKP	Min–Max Regret Knapsack Problem
PCKP	Precedence Constrained Knapsack Problem
PTAS	Polynomial Time Approximation Scheme
SSP	Subset Sum Problem
UCMP	Unbounded Change-Making Problem
UKP	Unbounded Knapsack Problem

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